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Linear Representations of Finite Groups

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CHAPTER 1

Generalities on linear representations

1.1 Definitions

Let V be a vector space over the field **C** of complex numbers and let GL(V) be the group of *isomorphisms* of V onto itself. An element a of GL(V) is, by definition, a linear mapping of V into V which has an inverse a^{-1}; this inverse is linear. When V has a finite basis (e_i) of n elements, each linear map $a\colon \mathrm{V} \to \mathrm{V}$ is defined by a square matrix (a_{ij}) of order n. The coefficients a_{ij} are complex numbers; they are obtained by expressing the images $a(e_j)$ in terms of the basis (e_i):

$$a(e_j) = \sum_i a_{ij} e_i .$$

Saying that a is an isomorphism is equivalent to saying that the determinant $\det(a) = \det(a_{ij})$ of a is not zero. The group GL(V) is thus identifiable with the group of *invertible square matrices of order n*.

Suppose now G is a *finite* group, with identity element 1 and with composition $(s, t) \mapsto st$. A *linear representation* of G in V is a homomorphism ρ from the group G into the group GL(V). In other words, we associate with each element $s \in \mathrm{G}$ an element $\rho(s)$ of GL(V) in such a way that we have the equality

$$\rho(st) = \rho(s) \cdot \rho(t) \quad \text{for } s, t \in \mathrm{G}.$$

[We will also frequently write ρ_s instead of $\rho(s)$.] Observe that the preceding formula implies the following:

$$\rho(1) = 1, \quad \rho(s^{-1}) = \rho(s)^{-1}.$$

When ρ is given, we say that V is a *representation space* of G (or even simply, by abuse of language, a *representation* of G). In what follows, we

restrict ourselves to the case where V has *finite dimension.* This is not a very severe restriction. Indeed, for most applications, one is interested in dealing with a *finite number of elements* x_i of V, and can always find a *subrepresentation* of V (in a sense defined later, cf. 1.3) of finite dimension, which contains the x_i: just take the vector subspace generated by the images $\rho_s(x_i)$ of the x_i.

Suppose now that V has finite dimension, and let n be its dimension; we say also that n is the degree of the representation under consideration. Let (e_i) be a basis of V, and let R_s be the matrix of ρ_s with respect to this basis. We have

$$\det(\mathrm{R}_s) \neq 0, \qquad \mathrm{R}_{st} = \mathrm{R}_s \cdot \mathrm{R}_t \quad \text{if } s, t \in \mathrm{G}.$$

If we denote by $r_{ij}(s)$ the coefficients of the matrix R_s, the second formula becomes

$$r_{ik}(st) = \sum_j r_{ij}(s) \cdot r_{jk}(t).$$

Conversely, given invertible matrices $\mathrm{R}_s = (r_{ij}(s))$ satisfying the preceding identities, there is a corresponding linear representation ρ of G in V; this is what it means to give a representation "in matrix form."

Let ρ and ρ' be two representations of the same group G in vector spaces V and V′. These representations are said to be *similar* (or *isomorphic*) if there exists a linear isomorphism $\tau\colon \mathrm{V} \to \mathrm{V}'$ which "transforms" ρ into ρ', that is, which satisfies the identity

$$\tau \circ \rho(s) = \rho'(s) \circ \tau \quad \text{for all } s \in \mathrm{G}.$$

When ρ and ρ' are given in matrix form by R_s and R'_s respectively, this means that there exists an invertible matrix T such that

$$\mathrm{T} \cdot \mathrm{R}_s = \mathrm{R}'_s \cdot \mathrm{T}, \quad \text{for all } s \in \mathrm{G},$$

which is also written $\mathrm{R}'_s = \mathrm{T} \cdot \mathrm{R}_s \cdot \mathrm{T}^{-1}$. We can *identify* two such representations (by having each $x \in \mathrm{V}$ correspond to the element $\tau(x) \in \mathrm{V}'$); in particular, ρ and ρ' have the same degree.

1.2 Basic examples

(a) A representation *of degree* 1 of a group G is a homomorphism $\rho\colon \mathrm{G} \to \mathbf{C}^*$, where $\mathbf{C}^*$ denotes the multiplicative group of nonzero complex numbers. Since each element of G has finite order, the values $\rho(s)$ of ρ are roots of unity; in particular, we have $|\rho(s)| = 1$.

If we take $\rho(s) = 1$ for all $s \in \mathrm{G}$, we obtain a representation of G which is called the *unit* (or *trivial*) representation.

(b) Let g be the order of G, and let V be a vector space of dimension g, with a basis $(e_t)_{t \in \mathrm{G}}$ indexed by the elements t of G. For $s \in \mathrm{G}$, let ρ_s be

the linear map of V into V which sends e_t to e_{st}; this defines a linear representation, which is called the ***regular representation*** of G. Its degree is equal to the order of G. Note that $e_s = \rho_s(e_1)$; hence note that the images of e_1 form a basis of V. Conversely, let W be a representation of G containing a vector w such that the $\rho_s(w)$, $s \in G$, form a basis of W; then W is isomorphic to the regular representation (an isomorphism τ: $V \to W$ is defined by putting $\tau(e_s) = \rho_s(w)$).

(c) More generally, suppose that G acts on a finite set X. This means that, for each $s \in G$, there is given a permutation $x \mapsto sx$ of X, satisfying the identities

$$1x = x, \; s(tx) = (st)x \quad \text{if } s, t \in G, \; x \in X.$$

Let V be a vector space having a basis $(e_x)_{x \in X}$ indexed by the elements of X. For $s \in G$ let ρ_s be the linear map of V into V which sends e_x to e_{sx}; the linear representation of G thus obtained is called the *permutation representation* associated with X.

1.3 Subrepresentations

Let ρ: $G \to GL(V)$ be a linear representation and let W be a vector subspace of V. Suppose that W is *stable* under the action of G (we say also "invariant"), or in other words, suppose that $x \in W$ implies $\rho_s x \in W$ for all $s \in G$. The restriction ρ_s^W of ρ_s to W is then an isomorphism of W onto itself, and we have $\rho_{st}^W = \rho_s^W \cdot \rho_t^W$. Thus ρ^W: $G \to GL(W)$ is a linear representation of G in W; W is said to be a *subrepresentation* of V.

Example. Take for V the regular representation of G [cf. 1.2 (b)], and let W be the subspace of dimension 1 of V generated by the element $x = \sum_{s \in G} e_s$. We have $\rho_s x = x$ for all $s \in G$; consequently W is a subrepresentation of V, isomorphic to the unit representation. (We will determine in 2.4 all the subrepresentations of the regular representation.)

Before going further, we recall some concepts from linear algebra. Let V be a vector space, and let W and W′ be two subspaces of V. The space V is said to be the *direct sum* of W and W′ if each $x \in V$ can be written uniquely in the form $x = w + w'$, with $w \in W$ and $w' \in W'$; this amounts to saying that the intersection $W \cap W'$ of W and W′ is 0 and that $\dim(V) = \dim(W) + \dim(W')$. We then write $V = W \oplus W'$ and say that W′ is a complement of W in V. The mapping p which sends each $x \in V$ to its component $w \in W$ is called the *projection* of V onto W associated with the decomposition $V = W \oplus W'$; the image of p is W, and $p(x) = x$ for $x \in W$; conversely if p is a linear map of V into itself satisfying these two properties, one checks that V is the direct sum of W and the *kernel* W′ of p

(the set of x such that $px = 0$). A bijective correspondence is thus established between the *projections* of V onto W and the *complements* of W in V.

We return now to subrepresentations:

Theorem 1. *Let* ρ: G $\to$ GL(V) *be a linear representation of* G *in* V *and let* W *be a vector subspace of* V *stable under* G. *Then there exists a complement* W^0 *of* W *in* V *which is stable under* G.

Let W′ be an arbitrary complement of W in V, and let p be the corresponding projection of V onto W. Form the average p^0 of the conjugates of p by the elements of G:

$$p^0 = \frac{1}{g} \sum_{t \in \mathrm{G}} \rho_t \cdot p \cdot \rho_t^{-1} \qquad (g \text{ being the order of G}).$$

Since p maps V into W and ρ_t preserves W we see that p^0 maps V into W; we have $\rho_t^{-1} x \in \mathrm{W}$ for $x \in \mathrm{W}$, whence

$$p \cdot \rho_t^{-1} x = \rho_t^{-1} x, \qquad \rho_t \cdot p \cdot \rho_t^{-1} x = x, \qquad \text{and} \qquad p^0 x = x.$$

Thus p^0 is a projection of V onto W, corresponding to some complement W^0 of W. We have moreover

$$\rho_s \cdot p^0 = p^0 \cdot \rho_s \quad \text{for all } s \in \mathrm{G}.$$

Indeed, computing $\rho_s \cdot p^0 \cdot \rho_s^{-1}$, we find:

$$\rho_s \cdot p^0 \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in \mathrm{G}} \rho_s \cdot \rho_t \cdot p \cdot \rho_t^{-1} \cdot \rho_s^{-1} = \frac{1}{g} \sum_{t \in \mathrm{G}} \rho_{st} \cdot p \cdot \rho_{st}^{-1} = p^0.$$

If now $x \in \mathrm{W}^0$ and $s \in \mathrm{G}$ we have $p^0 x = 0$, hence $p^0 \cdot \rho_s x = \rho_s \cdot p^0 x = 0$, that is, $\rho_s x \in \mathrm{W}^0$, which shows that W^0 is stable under G, and completes the proof. □

Remark. Suppose that V is endowed with a *scalar product* $(x|y)$ satisfying the usual conditions: linearity in x, semilinearity in y, and $(x|x) > 0$ if $x \neq 0$. Suppose that this scalar product is *invariant* under G, i.e., that $(\rho_s x|\rho_s y) = (x|y)$; we can always reduce to this case by replacing $(x|y)$ by $\sum_{t \in \mathrm{G}} (\rho_t x|\rho_t y)$. Under these hypotheses the *orthogonal complement* W^0 *of* W in V is a complement of W stable under G; another proof of theorem 1 is thus obtained. Note that the invariance of the scalar product $(x|y)$ means that, if (e_i) is an orthonormal basis of V, the matrix of ρ_s with respect to this basis is a *unitary matrix*.

Keeping the hypothesis and notation of theorem 1, let $x \in \mathrm{V}$ and let w and w^0 be its projections on W and W^0. We have $x = w + w^0$, whence $\rho_s x = \rho_s w + \rho_s w^0$, and since W and W^0 are stable under G, we have $\rho_s w \in \mathrm{W}$ and $\rho_s w^0 \in \mathrm{W}^0$; thus $\rho_s w$ and $\rho_s w^0$ are the projections of $\rho_s x$. It follows the representations W and W^0 determine the representation V.

We say that V is the direct sum of W and W^0, and write $V = W \oplus W^0$. An element of V is identified with a pair (w, w^0) with $w \in W$ and $w^0 \in W^0$. If W and W^0 are given in matrix form by R_s and R_s^0, $W \oplus W^0$ is given in matrix form by

$$\begin{pmatrix} R_s & 0 \\ 0 & R_s^0 \end{pmatrix}.$$

The direct sum of an arbitrary finite number of representations is defined similarly.

1.4 Irreducible representations

Let $\rho: G \to GL(V)$ be a linear representation of G. We say that it is *irreducible* or *simple* if V is not 0 and if no vector subspace of V is stable under G, except of course 0 and V. By theorem 1, this second condition is equivalent to saying V *is not the direct sum of two representations* (except for the trivial decomposition $V = 0 \oplus V$). A representation of degree 1 is evidently irreducible. We will see later (3.1) that each nonabelian group possesses at least one irreducible representation of degree $\geqslant 2$.

The irreducible representations are used to construct the others by means of the direct sum:

Theorem 2. *Every representation is a direct sum of irreducible representations.*

Let V be a linear representation of G. We proceed by induction on $\dim(V)$. If $\dim(V) = 0$, the theorem is obvious (0 is the direct sum of the *empty family* of irreducible representations). Suppose then $\dim(V) \geqslant 1$. If V is irreducible, there is nothing to prove. Otherwise, because of th. 1, V can be decomposed into a direct sum $V' \oplus V''$ with $\dim(V') < \dim(V)$ and $\dim(V'') < \dim(V)$. By the induction hypothesis V' and V'' are direct sums of irreducible representations, and so the same is true of V. □

Remark. Let V be a representation, and let $V = W_1 \oplus \cdots \oplus W_k$ be a decomposition of V into a direct sum of irreducible representations. We can ask if this decomposition is *unique*. The case where all the ρ_s are equal to 1 shows that this is not true in general (in this case the W_i are lines, and we have a plethora of decompositions of a vector space into a direct sum of lines). Nevertheless, we will see in 2.3 that the *number* of W_i isomorphic to a given irreducible representation does not depend on the chosen decomposition.

1.5 Tensor product of two representations

Along with the direct sum operation (which has the formal properties of an addition), there is a "multiplication": the *tensor product*, sometimes called the *Kronecker* product. It is defined as follows:

To begin with, let V_1 and V_2 be two vector spaces. A space W furnished with a map $(x_1, x_2) \mapsto x_1 \cdot x_2$ of $V_1 \times V_2$ into W, is called the tensor product of V_1 and V_2 if the two following conditions are satisfied:

(i) $x_1 \cdot x_2$ is linear in each of the variables x_1 and x_2.
(ii) If (e_{i_1}) is a basis of V_1 and (e_{i_2}) is a basis of V_2, the family of products $e_{i_1} \cdot e_{i_2}$ is a basis of W.

It is easily shown that such a space exists, and is unique (up to isomorphism); it is denoted $V_1 \otimes V_2$. Condition (ii) shows that

$$\dim(V_1 \otimes V_2) = \dim(V_1) \cdot \dim(V_2).$$

Now let $\rho^1 : G \to \mathbf{GL}(V_1)$ and $\rho^2 : G \to \mathbf{GL}(V_2)$ be two linear representations of a group G. For $s \in G$, define an element ρ_s of $\mathbf{GL}(V_1 \otimes V_2)$ by the condition:

$$\rho_s(x_1 \cdot x_2) = \rho_s^1(x_1) \cdot \rho_s^2(x_2) \quad \text{for } x_1 \in V_1, x_2 \in V_2.$$

[The existence and uniqueness of ρ_s follows easily from conditions (i) and (ii).] We write:

$$\rho_s = \rho_s^1 \otimes \rho_s^2.$$

The ρ_s define a linear representation of G in $V_1 \otimes V_2$ which is called the *tensor product* of the given representations.

The matrix translation of this definition is the following: let (e_{i_1}) be a basis for V_1, let $r_{i_1 j_1}(s)$ be the matrix of ρ_s^1 with respect to this basis, and define (e_{i_2}) and $r_{i_2 j_2}(s)$ in the same way. The formulas:

$$\rho_s^1(e_{j_1}) = \sum_{i_1} r_{i_1 j_1}(s) \cdot e_{i_1}, \qquad \rho_s^2(e_{j_2}) = \sum_{i_2} r_{i_2 j_2}(s) \cdot e_{i_2}$$

imply:

$$\rho_s(e_{j_1} \cdot e_{j_2}) = \sum_{i_1, i_2} r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s) \cdot e_{i_1} \cdot e_{i_2}.$$

Accordingly the matrix of ρ_s is $(r_{i_1 j_1}(s) \cdot r_{i_2 j_2}(s))$; it is the *tensor product* of the matrices of ρ_s^1 and ρ_s^2.

The tensor product of two irreducible representations is not in general irreducible. It decomposes into a direct sum of irreducible representations which can be determined by means of character theory (cf. 2.3).

In quantum chemistry, the tensor product often appears in the following way: V_1 and V_2 are two spaces of functions stable under G, with respective bases (ϕ_{i_1}) and (ψ_{i_2}), and $V_1 \otimes V_2$ is the vector space generated by the *products* $\phi_{i_1} \cdot \psi_{i_2}$, these products being linearly independent. This last condition is essential. Here are two particular cases where it is satisfied: