Graduate Texts in Mathematics

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Groups and Representations

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If F is any field and U is any FG-module, then we define the character of U to be an F-valued function on U whose value at $g \in G$ is the trace of the F-endomorphism of U induced by g, exactly as was done in this section in the case $F = \mathbb{C}$.

7. Suppose that F has characteristic $p > 0$. Give an example to demonstrate that two FG-modules that have the same character need not be isomorphic.
8. Suppose that F has characteristic $p > 0$. Show that if U is an FG-module then we have $\chi_U(g) = \chi_U(g_{p'})$ for any $g \in G$, where $g_{p'}$ is the p'-part of g as defined in Exercise 1.3.

15. The Character Table

We established in Section 14 that any character of G is a $\mathbb{Z}$-linear combination of the r irreducible characters $\chi_1, \dots, \chi_r$ of G, where r is by Theorem 14.3 equal to the number of conjugacy classes of G. Since each irreducible character is specified by its value on each conjugacy class of G, it follows that the characters of G are completely determined by an $r \times r$ array giving the values of the r irreducible characters on the r conjugacy classes of G. This array is called the *character table* of G. Of course, the character table of G is well-defined only up to reorderings of the rows and columns.

If $\mathcal{X}$ is the character table of G, then $\mathcal{X} = (\chi_i(g_j))_{1 \le i,j \le r}$, where $g_1, \dots, g_r$ are representatives of the r conjugacy classes of G. By convention, we always set $g_1 = 1$, so that the first column of the character table consists of the degrees of G. We will generally write $\mathcal{X}$ in the form

$$\begin{array}{c|cccc} & 1 & k_2 & \dots & k_r \\ & 1 & g_2 & \dots & g_r \\ \hline \chi_1 & 1 & 1 & \dots & 1 \\ \chi_2 & f_2 & \chi_2(g_2) & \dots & \chi_2(g_r) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_r & f_r & \chi_r(g_2) & \dots & \chi_r(g_r) \end{array}$$

where the f_i are the degrees of G, and $k_i = |G : C_G(g_i)|$ is (by Proposition 3.11) the order of the conjugacy class of g_i for each i.

(Recall that we always take χ_1 to be the principal character.)

We now continue to establish some fundamental properties of characters.

ROW ORTHOGONALITY THEOREM. $(\chi_i, \chi_j) = \delta_{ij}$ for any i and j.

PROOF. Let $S_1, \dots, S_r$ be the distinct simple $\mathbb{C}G$-modules. By Theorem 14.12, we have $(\chi_i, \chi_j) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(S_i, S_j)$ for any i and j. For each i, we have $\mathrm{Hom}_{\mathbb{C}G}(S_i, S_i) = \mathrm{End}_{\mathbb{C}G}(S_i) \cong \mathbb{C}$ by Lemma 13.14, and if $i \neq j$ then $\mathrm{Hom}_{\mathbb{C}G}(S_i, S_j) = 0$ by Schur's lemma. ■

In other words, this theorem asserts that

$$\delta_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_j(g)} = \frac{1}{|G|} \sum_{t=1}^{r} k_t \chi_i(g_t)\overline{\chi_j(g_t)}$$

for any i and j, where the g_t are conjugacy class representatives and the k_t are the orders of the conjugacy classes. We can interpret this as saying that the rows of the character table are, when considered as vectors in $\mathbb{C}^r$, orthogonal with respect to an inner product that differs slightly from the standard one, and it is this interpretation that lends its name to the above result.

Row orthogonality has a number of important consequences:

COROLLARY 1. The irreducible characters of G form an orthonormal basis for the vector space of class functions on G.

PROOF. This follows immediately from Proposition 14.10 and the row orthogonality theorem. ■

COROLLARY 2. If $\alpha = \sum_i a_i \chi_i$ and $\beta = \sum_j b_j \chi_j$ are virtual characters of G, then $(\alpha, \beta) = \sum_i a_i b_i$.

PROOF. We have

$$\begin{aligned}(\alpha, \beta) = (\sum_{i=1}^{r} a_i \chi_i, \sum_{j=1}^{r} b_i \chi_j) = \sum_{i=1}^{r}\sum_{j=1}^{r} a_i b_j (\chi_i, \chi_j) &= \sum_{i=1}^{r}\sum_{j=1}^{r} a_i b_j \delta_{ij} \\ &= \sum_{i=1}^{r} a_i b_i\end{aligned}$$

by row orthogonality. ■

COROLLARY 3. If α is a character of G and $n \in \{1, 2, 3\}$, then $(\alpha, \alpha) = n$ iff α is a sum of n irreducible characters.

PROOF. Write $\alpha = \sum_{i=1}^{r} a_i\chi_i$, where the a_i are non-negative integers. Since $(\alpha, \alpha) = \sum_i a_i^2$ by Corollary 2, we see that if $(\alpha, \alpha) = n$, then we must have $a_j = 1$ for exactly n numbers $1 \le j \le r$, and $a_i = 0$ for all other i, in which case α is a sum of n irreducible characters. The converse follows directly from Corollary 2. ■

COROLLARY 4. If α is a virtual character of G, then each χ_j appears with coefficient (α, χ_j) in the unique expression of α as a $\mathbb{Z}$-linear combination of the irreducible characters of G.

PROOF. Write $\alpha = \sum_{i=1}^{r} a_i\chi_i$, where the a_i are integers; then $(\alpha, \chi_j) = a_j$ by Corollary 2. ■

PROPOSITION 5. If α is a linear character of G and χ is an irreducible character of G, then $\alpha\chi$ is an irreducible character of G.

(This result also follows from Exercise 14.3 and part (i) of Proposition 14.8, but here we provide an alternate proof.)

PROOF. Since α is linear, it follows from part (iii) of Proposition 14.4 that $\alpha(g)$ is a root of unity for any $g \in G$, and in particular that $1 = |\alpha(g)| = \alpha(g)\overline{\alpha(g)}$ for every $g \in G$. We now have

$$\begin{aligned}(\alpha\chi, \alpha\chi) &= \frac{1}{|G|}\sum_{g\in G} \alpha(g)\chi(g)\overline{\alpha(g)\chi(g)} \\ &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\chi(g)}\alpha(g)\overline{\alpha(g)} \\ &= \frac{1}{|G|}\sum_{g\in G} \chi(g)\overline{\chi(g)} = (\chi, \chi) = 1\end{aligned}$$

by row orthogonality, and hence $\alpha\chi$ is irreducible by Corollary 3. ■

The following theorem implies that the columns of the character table are orthogonal when considered as vectors in $\mathbb{C}^r$ under the standard inner product.

COLUMN ORTHOGONALITY THEOREM. If $g_1, \dots, g_r$ are a set of conjugacy class representatives of G, and $k_1, \dots, k_r$ are the orders of the conjugacy classes, then for any $1 \le i, j \le r$ we have

$$\sum_{t=1}^{r} \chi_t(g_i)\overline{\chi_t(g_j)} = \frac{|G|}{k_i}\delta_{ij} = |C_G(g_i)|\delta_{ij}.$$

Proof. Let $\mathcal{X} = (\chi_i(g_j))_{1\le i,j\le r}$ be the character table of G, and let K be the $r \times r$ diagonal matrix having $(k_1, \dots, k_r)$ as its main diagonal. Then $(\mathcal{X}K)_{ij} = \sum_{\ell=1}^{r} \chi_i(g_\ell)(K)_{\ell j} = \chi_i(g_j)k_j$ for any i and j, and thus we have

$$\begin{aligned}(\mathcal{X}K\overline{\mathcal{X}}^t)_{ij} = \sum_{\ell=1}^{r} \chi_i(g_\ell)k_\ell(\overline{\mathcal{X}}^t)_{\ell j} &= \sum_{\ell=1}^{r} k_\ell\chi_i(g_\ell)\overline{\chi_j(g_\ell)} \\ &= \sum_{g\in G} \chi_i(g)\overline{\chi_j(g)} \\ &= |G|(\chi_i,\chi_j) = |G|\delta_{ij}\end{aligned}$$

for any i and j by row orthogonality. Therefore $\mathcal{X}K\overline{\mathcal{X}}^t = |G|I$, where I is the identity matrix. We leave it to the reader to verify that if A and B are matrices such that AB is a non-zero scalar matrix, then $BA = AB$. Thus $K\overline{\mathcal{X}}^t\mathcal{X} = |G|I$, and hence for any i and j we have $|G|\delta_{ji} = \sum_{\ell=1}^{r}(K\overline{\mathcal{X}}^t)_{j\ell}\mathcal{X}_{\ell i} = \sum_{\ell=1}^{r} k_j\overline{\chi_\ell(g_j)}\chi_\ell(g_i)$ as required. ■

As we shall see later in this section, the orthogonality relations often make it possible to construct a character table even if little is known about the group in question. This raises the following question, which we now address: What information about a group can be obtained from its character table? We first require some information about the connection between the representation theory of a group and that of its quotient groups.

Lemma 6. Let $N \trianglelefteq G$, and let U be a $\mathbb{C}(G/N)$-module. Then U admits a canonical $\mathbb{C}G$-module structure, with a subspace of U being a $\mathbb{C}G$-submodule iff it is a $\mathbb{C}(G/N)$-submodule. If ψ is the character of the $\mathbb{C}(G/N)$-module U, then the character of the $\mathbb{C}G$-module U is $\psi \circ \eta$, where $\eta\colon G \to G/N$ is the natural map.

Proof. Given $g \in G$ and $u \in U$, we define $gu = (gN)u$; this gives U a $\mathbb{C}G$-module structure in which the $\mathbb{C}G$-submodules of U are exactly the $\mathbb{C}(G/N)$-submodules of U. If $g \in G$, then the linear transformation of U induced by g under the action of G on U is exactly the same as that induced by $\eta(g) = gN$ under the action of G/N on U, which implies the statement concerning characters. ■

We define $K_\chi = \{x \in G \mid \chi(x) = \chi(1)\}$ for any character χ of G; we have $K_\chi \trianglelefteq G$ by part (vi) of Proposition 14.4. We call K_χ the *kernel* of χ, as it is the kernel of the corresponding representation. We write K_i instead of K_{χ_i}.

PROPOSITION 7. The normal subgroups of G are exactly the sets of the form $\bigcap_{i \in I} K_i$ for some $I \subseteq \{1, \ldots, r\}$.

PROOF. Let $N \trianglelefteq G$, and let $U = \mathbb{C}(G/N)$; let ψ be the character of U considered as a $\mathbb{C}(G/N)$-module, and let χ be the character of U considered via Lemma 6 as a $\mathbb{C}G$-module. Since ψ is the regular character of G/N, it follows from Lemma 6 that $\chi(g) = \chi(1)$ iff $g \in N$; therefore $K_\chi = N$. Write $\chi = \sum_i a_i \chi_i$ for some non-negative integers a_i. We observe via part (v) of Proposition 14.4 that we have $|\chi(g)| \leq \sum_i a_i |\chi_i(g)| \leq \sum_i a_i \chi_i(1) = \chi(1)$ for any $g \in G$. It follows from these inequalities and again from part (v) of Proposition 14.4 that $g \in K_\chi$ iff $g \in K_i$ for every i for which $a_i > 0$. Therefore $N = \cap_{i \in I} K_i$, where $I = \{1 \leq i \leq r \mid a_i > 0\}$. Conversely, since each K_i is normal in G, it follows from Proposition 1.7 that $\cap_{i \in I} K_i \trianglelefteq G$ for any $I \subseteq \{1, \ldots, r\}$. ■

COROLLARY 8. *G is simple iff the only irreducible character χ_i for which $\chi_i(g) = \chi_i(1)$ for some $1 \neq g \in G$ is the principal character χ_1.*

PROOF. If G is simple and $\chi_i(g) = \chi_i(1)$ for some $i > 1$ and some $1 \neq g \in G$, then as $g \in K_i \trianglelefteq G$ we obtain a contradiction. Conversely, if G is not simple, then there is some $1 \neq g \in G$ lying in some non-trivial proper normal subgroup N; by Proposition 7 we must have $N \leqslant K_i$ for some $i > 1$, in which case $\chi_i(g) = \chi_i(1)$. ■

COROLLARY 9. The character table of G can be used to determine whether or not G is solvable.

PROOF. It follows from Proposition 7 that the character table of G enables us to determine all of the normal subgroups of G and all of the inclusion relations between the normal subgroups. Hence we can determine all normal series of G and the orders of the terms thereof. In particular, we can determine whether or not G has a normal series whose successive quotients are p-groups, which by Corollary 11.7 is a criterion for solvability. ■

We define $Z_\chi = \{x \in G \mid |\chi(x)| = \chi(1)\}$ for any character χ of G. Again we write Z_i instead of Z_{χ_i}.

LEMMA 10. $Z_\chi \leqslant G$ for any character χ of G, and if in addition χ is irreducible, then $Z_\chi/K_\chi = Z(G/K_\chi)$.

PROOF. Let $g \in G$. We see from part (iv) of Proposition 14.4 that if $g \in Z_\chi$, then $g^{-1} \in Z_\chi$. Since $\chi(g)$ is a sum of $\chi(1)$ roots of unity by part (iii) of Proposition 14.4, we see that $|\chi(g)| = \chi(1)$ iff g has exactly one eigenvalue. If $g \in Z_\chi$, let this eigenvalue be $\lambda(g)$, so that if U is the $\mathbb{C}G$-module corresponding to χ, then we have $gu = \lambda(g)u$ for all $u \in U$. We now see that if $g, h \in Z_\chi$, then $(gh)u = \lambda(g)\lambda(h)u$ for all $u \in U$; hence $\chi(gh) = \chi(1)\lambda(g)\lambda(h)$, and thus $|\chi(gh)| = \chi(1)$, giving $gh \in Z_\chi$. Therefore $Z_\chi \leqslant G$. Now if $\rho\colon G \to \mathrm{GL}(U)$ is the representation corresponding to χ, then for any $g \in Z_\chi$, the matrix of $\rho(g)$ (with respect to any $\mathbb{C}$-basis of U) will be scalar, and hence $\rho(g) \in Z(\rho(G))$. Since $\rho(G) \cong G/K_\chi$, it follows that $Z_\chi/K_\chi \leqslant Z(G/K_\chi)$.

Now suppose that χ is irreducible. If $gK_\chi \in Z(G/K_\chi)$, then $\rho(g)$ commutes with $\rho(x)$ for every $x \in G$, and consequently the map sending $u \in U$ to gu is a $\mathbb{C}G$-endomorphism of U. But U is simple, so we have $\mathrm{End}_{\mathbb{C}G}(U) \cong \mathbb{C}$ by Schur's lemma. Therefore, there is some complex root of unity μ such that $gu = \mu u$ for all $u \in U$. We now have $\chi(g) = \chi(1)\mu$, which gives $|\chi(g)| = \chi(1)$ and hence $g \in Z_\chi$. Therefore $Z_\chi/K_\chi = Z(G/K_\chi)$. ■

COROLLARY 11. If G is non-abelian and simple, then $Z_i = 1$ for all $i > 1$.

PROOF. If $i > 1$ then $K_i = 1$, so $Z_i = Z(G) = 1$ by Lemma 10. ■

PROPOSITION 12. $Z(G) = \bigcap_{i=1}^{r} Z_i$.

PROOF. If χ is any character of G, then $Z(G)K_\chi/K_\chi \leqslant Z(G/K_\chi)$. Thus by Lemma 10 we have $Z(G)K_i/K_i \leqslant Z_i/K_i$, and consequently $Z(G) \leqslant Z_i$, for every i. Conversely, suppose that $g \in Z_i$ for all i. Since $Z_i/K_i = Z(G/K_i)$ by Lemma 10, it follows that for any $x \in G$ we have $[g, x] \in K_i$ for all i. But it follows from Proposition 7 that $K_1 \cap \ldots \cap K_r = 1$; therefore $[g, x] = 1$ for all $x \in G$, and hence $g \in Z(G)$. ■

LEMMA 13. If $N \trianglelefteq G$, then the irreducible characters of G/N can be determined from those of G.

Proof. Let χ be any irreducible character of G whose kernel contains N, and let U be the $\mathbb{C}G$-module corresponding to χ. Then since N acts on U via the identity, we can give U a $\mathbb{C}(G/N)$-module structure by $(gN)u = gu$ for $gN \in G/N$ and $u \in U$. As U is a simple $\mathbb{C}G$-module, it follows that U is also simple as a $\mathbb{C}(G/N)$-module, and the character of the $\mathbb{C}(G/N)$-module U sends gN to $\chi(g)$. Therefore, those irreducible characters of G whose kernel contains N give rise to irreducible characters of G/N. But by Lemma 6, every irreducible character of G/N gives rise to an irreducible character of G whose kernel contains N.

It now follows that all irreducible characters of G/N can be determined from those of G, in the following sense: What we can determine is the number of irreducible characters of G/N, and their values on any element of G/N. We cannot readily determine the actual conjugacy classes of G/N, or their orders, although it is certainly true that conjugate elements of G have images in G/N that are conjugate, and hence the image in G/N of a class of G is a union of classes of G/N. Nonetheless, the information that can be gained from the character table of G about the characters of G/N is sufficient, for instance, to determine $Z(G/N)$ via Proposition 12. ∎

Corollary 14. The character table of G can be used to determine whether or not G is nilpotent.

Proof. Consider the upper central series $1 \leqslant \mathcal{Z}_1 \leqslant \mathcal{Z}_2 \leqslant \dots$ of G defined in the further exercises to Section 11. Since we have $\mathcal{Z}_i \trianglelefteq G$ and $\mathcal{Z}_i/\mathcal{Z}_{i-1} = Z(G/\mathcal{Z}_{i-1})$ for each i, we see from Proposition 12 that each $\mathcal{Z}_i$ can be determined from the irreducible characters of $G/\mathcal{Z}_{i-1}$, which by Lemma 13 can be determined from the character table of G. Consequently, we can use the character table of G to determine every term $\mathcal{Z}_i$ of the upper central series, and the result follows by Exercise 11.10. (This argument also shows that if G is nilpotent, then the character table of G allows us to determine the nilpotency class of G.) ∎

Observe that in Corollary 9 we assumed that we knew both the irreducible characters of G and the orders of the conjugacy classes, whereas in Corollary 14 we did not need to know the orders of the classes. As evidenced by Lemma 13, there may be circumstances in which we can construct a character table without knowing the orders of the conjugacy classes. There is a theorem of G. Higman (see [15,

p. 136]) which asserts that if we know the irreducible characters of G, but not the orders of the conjugacy classes, then we can at least determine the sets of prime divisors of the orders of the conjugacy classes.

The remainder of this section consists of a series of examples in which we develop methods of finding characters and use these methods to construct the character tables of various groups.

EXAMPLE 1. Let $G = \langle g \rangle \cong \mathbf{Z_n}$ for some $n \in \mathbb{N}$, and let λ be a primitive nth root of unity. For each $1 \leq i \leq n$, let V_i be a one-dimensional $\mathbb{C}$-vector space, and let g act on V_i by multiplication by λ^{i-1}; since G is cyclic, this definition completely determines a $\mathbb{C}G$-module structure on V_i. Each V_i, being one-dimensional, is a simple $\mathbb{C}G$-module. If χ_i is the character of V_i, then $\chi_i(g) = \lambda^{i-1}$, and hence $\chi_i(g^a) = \lambda^{a(i-1)}$ for any a. The characters $\chi_1, \ldots, \chi_n$ are n distinct linear characters of G, and since G can have at most $|G| = n$ irreducible characters, we see that the χ_i are precisely the irreducible characters of G. Thus, the character table of G is

$$\begin{array}{c|ccccc} & 1 & 1 & 1 & \ldots & 1 \\ & 1 & g & g^2 & \ldots & g^{n-1} \\ \hline \chi_1 & 1 & 1 & 1 & \ldots & 1 \\ \chi_2 & 1 & \lambda & \lambda^2 & \ldots & \lambda^{n-1} \\ \chi_3 & 1 & \lambda^2 & \lambda^4 & \ldots & \lambda^{n-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_n & 1 & \lambda^{n-1} & \lambda^{n-2} & \ldots & \lambda \end{array}$$

Observe that the set $\{\chi_1, \ldots, \chi_n\}$ is a cyclic group under multiplication of characters, with generator χ_2; we have $\chi_i = \chi_2^{i-1}$ for any i.

EXAMPLE 2. Let G and H be groups with respective irreducible characters $\chi_1, \ldots, \chi_r$ and $\psi_1, \ldots, \psi_s$. We wish to determine the irreducible characters of $G \times H$. We see that two elements (x, y) and (x', y') of $G \times H$ are conjugate iff x and x' are conjugate in G and y and y' are conjugate in H. Since by Theorem 14.3, G and H have r and s conjugacy classes, respectively, we conclude that $G \times H$ has rs conjugacy classes, with each class of $G \times H$ being the product of a class of G and a class of H; hence by Theorem 14.3, $G \times H$ has rs irreducible characters.

Let $S_1, \ldots, S_r$ and $T_1, \ldots, T_s$ be the distinct simple $\mathbb{C}G$- and $\mathbb{C}H$-modules, respectively. For each i and j, we give $S_i \otimes T_j$ the structure of a $\mathbb{C}(G \times H)$-module by $(g,h)(s \otimes t) = gs \otimes ht$ and linear extension. Let τ_{ij} be the character of $S_i \otimes T_j$ for each i and j. By imitating the proof of part (i) of Proposition 14.8, we find that $\tau_{ij}(g,h) = \chi_i(g)\psi_j(h)$; we adopt the notation $\tau_{ij} = \chi_i \times \psi_j$. We note that the τ_{ij} are distinct and that we can recover the χ_i and ψ_j from the τ_{ij} by appropriate restriction. Now for any i, i', j, j' we have

$$\begin{aligned}
(\tau_{ij}, \tau_{i'j'}) &= \frac{1}{|G \times H|} \sum_{(g,h) \in G \times H} \tau_{ij}(g,h)\overline{\tau_{i'j'}(g,h)} \\
&= \frac{1}{|G||H|} \sum_{g \in G} \sum_{h \in H} \chi_i(g)\psi_j(h)\overline{\chi_{i'}(g)\varphi_{j'}(h)} \\
&= \left(\frac{1}{|G|} \sum_{g \in G} \chi_i(g)\overline{\chi_{i'}(g)}\right)\left(\frac{1}{|H|} \sum_{h \in H} \psi_j(h)\overline{\psi_{j'}(h)}\right) \\
&= (\chi_i, \chi_{i'})(\psi_j, \psi_{j'}) = \delta_{ii'}\delta_{jj'}
\end{aligned}$$

by row orthogonality. In particular, we have $(\tau_{ij}, \tau_{ij}) = 1$ for each i and j, which by Corollary 3 implies that τ_{ij} is an irreducible character. Therefore the τ_{ij} are the rs irreducible characters of $G \times H$; that is, the set of irreducible characters of $G \times H$ is $\{\chi_i \times \psi_j\}_{i,j}$.

EXAMPLE 3. Suppose that G is abelian. Then every conjugacy class of G contains exactly one element, and consequently G has $|G|$ irreducible characters by Theorem 14.3. But $\sum_{i=1}^{|G|} f_i^2 = |G|$ by Corollary 14.2, so we must have $f_i = 1$ for each i. Therefore all irreducible characters of G are linear. By Corollary 8.8, G is a direct product of cyclic p-groups; let these cyclic p-groups be of orders $p_1^{a_1}, \ldots, p_t^{a_t}$ with respective generators $g_1, \ldots, g_t$. We could determine the character table of G from those of the cyclic p-groups using Examples 1 and 2, but there is an alternate method, which we now describe. A linear character χ of G is just a homomorphism from G to $\mathbb{C}^\times$, and hence to define such a χ it suffices to specify $\chi(g_i)$ for each i, with the only restriction on $\chi(g_i)$ being that it is a $p_i^{a_i}$-th root of unity. Therefore, there is a bijective correspondence between irreducible characters of G and ordered t-tuples $(\lambda_1, \ldots, \lambda_t)$ in which each λ_i is a $p_i^{a_i}$-th root of unity.

As a simple example, let $G = \langle a, b \rangle \cong \mathbf{Z_2} \times \mathbf{Z_2}$. Both a and b have order 2, so by the above paragraph we see that the four irreducible characters of G correspond to the ordered pairs $(1, 1), (1, -1), (-1, 1)$, and $(-1, -1)$. Therefore, the character table for G is

$$\begin{array}{c|rrrr} & 1 & 1 & 1 & 1 \\ & 1 & a & b & ab \\ \hline \chi_1 & 1 & 1 & 1 & 1 \\ \chi_2 & 1 & 1 & -1 & -1 \\ \chi_3 & 1 & -1 & 1 & -1 \\ \chi_4 & 1 & -1 & -1 & 1 \end{array}$$

Here the second and third columns correspond to the ordered pairs given above, and the fourth column is wholly determined by the previous two columns.

EXAMPLE 4. Let X be a finite G-set. We saw in Section 12 that the vector space $\mathbb{C}X$ having basis X is a $\mathbb{C}G$-module. We wish to determine the character π of $\mathbb{C}X$. Consider the matrix, with respect to the basis X, of the linear transformation of $\mathbb{C}X$ defined by $g \in G$. Since $gx \in X$ for every $x \in X$, this matrix is a permutation matrix; but $\pi(g)$ is the sum of the diagonal entries of this matrix, so we conclude that $\pi(g)$ equals the number of elements of X which are fixed by g.

Let χ_1 be the principal character of G. We have

$$\begin{aligned}(\pi, \chi_1) = \frac{1}{|G|}\sum_{g\in G}\pi(g)\chi_1(g) &= \frac{1}{|G|}\sum_{g\in G}\pi(g) \\ &= \frac{1}{|G|}\sum_{g\in G}|\{x \in X \mid gx = x\}| \\ &= \frac{1}{|G|}|\{(g, x) \in G \times X \mid gx = x\}| \\ &= \frac{1}{|G|}\sum_{x\in X}|\{g \in G \mid gx = x\}|.\end{aligned}$$

For each $x \in X$, let G_x be the stabilizer of x; using Corollary 3.5, we now have

$$\begin{aligned}(\pi, \chi_1) = \sum_{x\in X}\frac{|G_x|}{|G|} &= \sum_{\text{orbits } \mathcal{O}}\sum_{x\in\mathcal{O}}\frac{1}{|G : G_x|} \\ &= \sum_{\text{orbits } \mathcal{O}}\sum_{x\in\mathcal{O}}\frac{1}{|\mathcal{O}|} = \sum_{\text{orbits } \mathcal{O}}|\mathcal{O}|\cdot\frac{1}{|\mathcal{O}|} = \sum_{\text{orbits } \mathcal{O}} 1,\end{aligned}$$

which allows us to conclude that (π, χ_1) is equal to the number of orbits of X under the action of G. (This result is often attributed to Burnside, although as argued in [21] it would be more accurate to call this result the Cauchy-Frobenius lemma.) In particular, if G acts transitively on X, then we see via Corollary 4 that the unique expression of the character $\pi - \chi_1$ as a linear combination of the irreducible characters of G does not involve χ_1.

EXAMPLE 5. The group G/G' is abelian by Proposition 2.6, and hence (as seen in Example 3) all of its irreducible characters are linear. It follows from Lemma 6 that each linear character of G/G' can be lifted to a linear character of G, and that the lifts of distinct characters are distinct. Now let χ be a linear character of G, so that $\chi \colon G \to \mathbb{C}^\times$ is a homomorphism whose kernel is K_χ. Then G/K_χ is abelian, being isomorphic via χ with a subgroup of $\mathbb{C}^\times$, and hence $G' \leqslant K_\chi$ by Proposition 2.6. We can now define a linear character ψ of G/G' whose lift is χ by $\psi(gG') = \chi(g)$. We conclude that every linear character of G is the lift of a linear character of G/G'.

EXAMPLE 6. Consider Σ_3. The conjugacy classes of Σ_3 are $\{1\}$, $\{(1\,2), (1\,3), (2\,3)\}$, and $\{(1\,2\,3), (1\,3\,2)\}$. (This follows from Proposition 1.10.) We have $\Sigma_3/A_3 \cong \mathbf{Z_2}$, so Σ_3 has two linear characters, which are lifted from the characters of $\mathbf{Z_2}$ as in Lemma 6. Letting these characters be χ_1 and χ_2, we have

	1	3	2
	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3			

Since $6 = |\Sigma_3| = f_1^2 + f_2^2 + f_3^2 = 2 + f_3^2$ by Corollary 14.2, we have $f_3 = 2$. Column orthogonality now gives

$$0 = \sum_{i=1}^{3} f_i \chi_i(\,(1\,2)\,) = 1 \cdot 1 + 1 \cdot (-1) + 2\chi_3(\,(1\,2)\,)$$

and

$$0 = \sum_{i=1}^{3} f_i \chi_i(\,(1\,2\,3)\,) = 1 \cdot 1 + 1 \cdot 1 + 2\chi_3(\,(1\,2)\,),$$

from which we see that the character table of Σ_3 is

	1	3	2
	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

EXAMPLE 7. Consider Σ_4. Recall that Σ_4 has a normal subgroup of order 4, namely $K = \{1, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. The subgroup Σ_3 of Σ_4 intersects K trivially, and $|\Sigma_4| = |\Sigma_3||K|$; therefore $\Sigma_4 = K \rtimes \Sigma_3$. In particular, we have $\Sigma_4/K \cong \Sigma_3$, and thus we can obtain irreducible characters of Σ_4 from the irreducible characters of Σ_3 as in Lemma 6.

By Proposition 1.10, each conjugacy class of Σ_4 consists of all elements having a given cycle structure. Hence Σ_4 has 5 classes, with representatives 1, $(1\ 2)(3\ 4)$, $(1\ 2\ 3)$, $(1\ 2)$, and $(1\ 2\ 3\ 4)$; the orders of the classes are 1, 3, 8, 6, and 6, respectively. Let χ_1, χ_2, and χ_3 be the irreducible characters of Σ_4 obtained from those of Σ_3. To determine these characters, we need information about the images in Σ_4/K of the class representatives. We easily see that the images of 1 and $(1\ 2)(3\ 4)$ are trivial, that the image of $(1\ 2\ 3)$ has order 3, and that the images of $(1\ 2)$ and $(1\ 2\ 3\ 4)$ have order 2. Using this observation and the character table for Σ_3 obtained in Example 6, we obtain the following partial character table for Σ_4:

	1	3	8	6	6
	1	(1 2)(3 4)	(1 2 3)	(1 2)	(1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4					
χ_5					

Now Σ_4 acts transitively on the set $X = \{1, 2, 3, 4\}$; let π be the character of the $\mathbb{C}G$-module $\mathbb{C}X$. Since $\pi(g)$ equals the number of fixed points under the action of g on X by Example 4, we have

	1	3	8	6	6
	1	(1 2)(3 4)	(1 2 3)	(1 2)	(1 2 3 4)
π	4	0	1	2	0
$\pi - \chi_1$	3	-1	0	1	-1

We also have

$$\begin{aligned}(\pi - \chi_1, \pi - \chi_1) &= \frac{1}{|\Sigma_4|}\sum_{i=1}^{5} k_i[(\pi - \chi_1)(g_i)]^2 \\ &= \frac{1}{24}(1 \cdot 9 + 3 \cdot 1 + 8 \cdot 0 + 6 \cdot 1 + 6 \cdot 1) = 1,\end{aligned}$$

and hence $\pi - \chi_1$ is irreducible by Corollary 3. Let $\chi_4 = \pi - \chi_1$. Now $\chi_2\chi_4 \neq \chi_4$, and $\chi_2\chi_4$ is irreducible by Proposition 5; therefore $\chi_5 = \chi_2\chi_4$, and hence the character table of Σ_4 is

	1	3	8	6	6
	1	(1 2)(3 4)	(1 2 3)	(1 2)	(1 2 3 4)
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	2	2	-1	0	0
χ_4	3	-1	0	1	-1
χ_5	3	-1	0	-1	1

EXAMPLE 8. Let G be a non-abelian group of order 8. We will show that this information completely determines the character table of G. Since there are two isomorphism classes of non-abelian groups of order 8 (see page 79), this will imply that the character table does not specify a group up to isomorphism.

We have $|Z(G)| = 2$ by Theorem 8.1 and Lemma 8.2, and hence $Z(G) \cong \mathbf{Z}_2$. Since G is non-abelian and $G/Z(G)$ is abelian (being a group of order 4), this forces $G' = Z(G)$ by Proposition 2.6. Now $|G/G'| = 4$, and by Lemma 8.2 G/G' cannot be cyclic since G is non-abelian and $G' = Z(G)$, so we must have $G/G' \cong \mathbf{Z}_2 \times \mathbf{Z}_2$. By Examples 3 and 5, we now know that G has exactly 4 linear characters. By Corollary 14.2, we have $\sum_{i=1}^{r} f_i^2 = 8$, where $f_i = 1$ for $1 \leq i \leq 4$ and $f_i > 1$ for $i > 4$; this forces $r = 5$ and $f_5 = 2$.

Let x be the generator of G'. As $G' = Z(G)$, we see that 1 and x are the only elements of G lying in single-element conjugacy classes, and hence each of the other three classes must have order 2. Let a,

b, and c be representatives of the multi-element classes. Since G/G' also has four conjugacy classes, we see that the images of a, b, and c in G/G' must lie in distinct conjugacy classes. Using the character table for $\mathbf{Z_2} \times \mathbf{Z_2}$ determined in Example 3, we can now obtain a partial character table for G by lifting:

	1	1	2	2	2
	1	x	a	b	c
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2				

We see via column orthogonality that $\chi_5(a) = \chi_5(b) = \chi_5(c) = 0$ and that $\chi_5(x) = -2$, completing the character table.