Graduate Texts in Mathematics

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Groups and Representations

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Consider an element $\sum_{g \in G} \lambda_g g$ of Z. For any $h \in G$, we have $(\sum_g \lambda_g g)h = h(\sum_g \lambda_g g)$, giving $\sum_g \lambda_g g = \sum_g \lambda_g h^{-1}gh = \sum_g \lambda_{hgh^{-1}} g$. Therefore, we have $\lambda_g = \lambda_{hgh^{-1}}$ for every $g, h \in G$, and so we conclude that the coefficients of elements of Z are constant on conjugacy classes. It follows that a basis for Z is the set of *class sums*, which are the sums of the form $\sum_{g \in K} g$ where K is a conjugacy class of G. Thus, $\dim_{\mathbb{C}} Z$ is equal to the number of conjugacy classes of G, which completes the proof. ∎

We should mention that, in general, there is no natural bijective correspondence between the conjugacy classes of G and the simple $\mathbb{C}G$-modules. However, if G is a symmetric group, then there is such a correspondence, although we shall not develop it here; for an overview, see [13, Section 4.1].

If U is a $\mathbb{C}G$-module, then each $g \in G$ defines an invertible linear transformation of U that sends $u \in U$ to gu. We define the *character* of U to be the function $\chi_U \colon G \to \mathbb{C}$, where $\chi_U(g)$ is the trace of this linear transformation of U defined by g. For example, $\chi_U(1) = \dim_{\mathbb{C}} U$, since the identity element of G induces the identity transformation on U. If $\rho \colon G \to \mathrm{GL}(U)$ is the representation corresponding to U, then $\chi_U(g)$ is just the trace of the map $\rho(g)$. Isomorphic $\mathbb{C}G$-modules have equal characters. We observe that for any $g, h \in G$, the linear transformations of U defined by g and hgh^{-1} are similar and hence have the same trace. Therefore, any character of G is constant on each conjugacy class of G, meaning that the value of the character on any two conjugate elements is the same.

For example, let $U = \mathbb{C}G$, and let $g \in G$. By considering the matrix of the linear transformation defined by g with respect to the basis G of $\mathbb{C}G$, we see that $\chi_U(g)$ is equal to the number of elements $x \in G$ for which $gx = x$. Therefore, we have $\chi_U(1) = |G|$ and $\chi_U(g) = 0$ for every $1 \neq g \in G$. This character is called the *regular character* of G.

The theory of characters was developed by Frobenius and others, starting in 1896. However, at first nothing was known about linear representations, let alone modules over the group algebra; Frobenius defined characters as being functions from G to $\mathbb{C}$ satisfying certain properties, but it turned out that his characters were exactly the trace functions of finitely generated $\mathbb{C}G$-modules.

We will denote the characters of the r simple $\mathbb{C}G$-modules by

$\chi_1, \ldots, \chi_r$; these characters will be referred to as the *irreducible characters* of G. Whenever we say that $S_1, \ldots, S_r$ are the distinct simple $\mathbb{C}G$-modules, we implicitly order them so that $\chi_{S_i} = \chi_i$ for each i. In keeping with the convention that f_1 denotes the degree of the trivial representation, we let χ_1 be the character of the trivial representation; we call χ_1 the *principal character* of G, and we have $\chi_1(g) = 1$ for all $g \in G$.

A character of a one-dimensional $\mathbb{C}G$-module is called a *linear character*. Since one-dimensional modules are simple, we see that all linear characters are irreducible. Let χ be the linear character arising from the $\mathbb{C}G$-module U, and let $g, h \in G$. Since U is one-dimensional, for any $u \in U$ we have $gu = \chi(g)u$ and $hu = \chi(h)u$, and thus $\chi(gh)u = (gh)u = \chi(g)\chi(h)u$; therefore, χ is a homomorphism from G to the multiplicative group $\mathbb{C}^\times$ of non-zero complex numbers. On the other hand, given a homomorphism $\varphi\colon G \to \mathbb{C}^\times$, we can define a one-dimensional $\mathbb{C}G$-module U by $gu = \varphi(g)u$ for $g \in G$ and $u \in U$, and we then have $\chi_U = \varphi$. Therefore, linear characters of G are exactly the same as group homomorphisms from G to $\mathbb{C}^\times$.

Our next result compiles some basic information about characters.

PROPOSITION 4. Let U be a $\mathbb{C}G$-module, let $\rho\colon G \to \mathrm{GL}(U)$ be the representation corresponding to U, and let $g \in G$ be of order n. Then:

(i) $\rho(g)$ is diagonalizable.
(ii) $\chi_U(g)$ equals the sum (with multiplicities) of the eigenvalues of $\rho(g)$.
(iii) $\chi_U(g)$ is a sum of $\chi_U(1)$ nth roots of unity.
(iv) $\chi_U(g^{-1}) = \overline{\chi_U(g)}$. (Here $\overline{z}$ denotes the complex conjugate of $z \in \mathbb{C}$.)
(v) $|\chi_U(g)| \leq \chi_U(1)$.
(vi) $\{x \in G \mid \chi_U(x) = \chi_U(1)\}$ is a normal subgroup of G.

PROOF. Since $g^n = 1$, $\rho(g)$ satisfies the polynomial $X^n - 1$. But $X^n - 1$ splits into distinct linear factors in $\mathbb{C}[X]$, and so it follows that the minimal polynomial of $\rho(g)$ does also, and hence that $\rho(g)$ is diagonalizable, proving (i). It now follows that the trace of $\rho(g)$ is the sum of its eigenvalues, proving (ii). These eigenvalues are precisely the roots of the minimal polynomial of $\rho(g)$, which divides $X^n - 1$; consequently those roots are nth roots of unity, which (since $\chi_U(1) = \dim_{\mathbb{C}} U$) proves (iii). We see easily that any eigenvector for

$\rho(g)$ is also an eigenvector for $\rho(g^{-1})$, with the eigenvalue for $\rho(g^{-1})$ being the inverse of the eigenvalue for $\rho(g)$. Since the eigenvalues of $\rho(g)$ are roots of unity, it follows that the eigenvalues of $\rho(g^{-1})$ are the conjugates of the eigenvalues of $\rho(g)$, and from this (iv) follows easily. (v) follows immediately from (iii). We have already seen that $\chi_U(g)$ is the sum of its $\chi_U(1)$ eigenvalues, each of which is a root of unity. If this sum equals $\chi_U(1)$, then it follows that each of those eigenvalues must be 1, in which case $\rho(g)$ must be the identity map. Conversely, if $\rho(g)$ is the identity map, then we have $\chi_U(g) = \dim_{\mathbb{C}}(U) = \chi_U(1)$; therefore $\{x \in G \mid \chi_U(x) = \chi_U(1)\} = \ker \rho \trianglelefteq G$, proving (vi). ■

Suppose that χ and ψ are characters of G. We can define new functions $\chi + \psi$ and $\chi\psi$ from G to $\mathbb{C}$ by $(\chi + \psi)(g) = \chi(g) + \psi(g)$ and $(\chi\psi)(g) = \chi(g)\psi(g)$ for $g \in G$. These new functions obtained from characters are not, a priori, characters themselves. We can also, given a scalar $\lambda \in \mathbb{C}$, define a new function $\lambda\chi \colon G \to \mathbb{C}$ by $(\lambda\chi)(g) = \lambda\chi(g)$, and consequently we can view the characters of G as elements of a $\mathbb{C}$-vector space of functions from G to $\mathbb{C}$.

PROPOSITION 5. The irreducible characters of G are, as functions from G to $\mathbb{C}$, linearly independent over $\mathbb{C}$.

PROOF. We have $\mathbb{C}G \cong \mathcal{M}_{f_1}(\mathbb{C}) \oplus \dots \oplus \mathcal{M}_{f_r}(\mathbb{C})$ by Theorem 1. Let $S_1, \dots, S_r$ be the distinct simple $\mathbb{C}G$-modules, and for each i let e_i be the identity element of $\mathcal{M}_{f_i}(\mathbb{C})$. Fix some i. Recall that for any $g \in G$, $\chi_i(g)$ is the trace of the linear transformation on S_i defined by $g \in G$. We linearly extend χ_i to a linear map from $\mathbb{C}G$ to $\mathbb{C}$, so that $\chi_i(a)$ for $a \in \mathbb{C}G$ is the trace of the linear transformation on S_i defined by a. We observe that the linear transformation on S_i given by e_i is the identity map, and hence that $\chi_i(e_i) = \dim_{\mathbb{C}} S_i = f_i$. Moreover, if $j \neq i$, then the linear transformation on S_j given by e_i is the zero map, and hence $\chi_j(e_i) = 0$.

Now let $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ be such that $\sum_{j=1}^{r} \lambda_j \chi_j = 0$. From the above we see that $0 = \sum_{j=1}^{r} \lambda_j \chi_j(e_i) = \lambda_i f_i$ for each i; thus $\lambda_j = 0$ for all j, proving the result. ■

LEMMA 6. $\chi_{U \oplus V} = \chi_U + \chi_V$ for any $\mathbb{C}G$-modules U and V.

PROOF. By considering a $\mathbb{C}$-basis for $U \oplus V$ whose first $\dim_{\mathbb{C}} U$ elements form a $\mathbb{C}$-basis for $U \oplus 0$ and whose remaining elements form a $\mathbb{C}$-basis for $0 \oplus V$, we see easily that $\chi_{U \oplus V}(g) = \chi_U(g) + \chi_V(g)$ for any $g \in G$. ■

We can now show that the characters of $\mathbb{C}G$-modules suffice to distinguish between $\mathbb{C}G$-modules:

THEOREM 7. If $S_1, \ldots, S_r$ are the distinct simple $\mathbb{C}G$-modules, then the character of the $\mathbb{C}G$-module $a_1S_1 \oplus \ldots \oplus a_rS_r$ (where the a_i are non-negative integers) is $a_1\chi_1 + \ldots + a_r\chi_r$. Consequently, two $\mathbb{C}G$-modules are isomorphic iff their characters are equal.

PROOF. The first assertion follows directly from Lemma 6. Now suppose that $\chi_U = \chi_V$ for some $\mathbb{C}G$-modules U and V. Since $\mathbb{C}G$ is semisimple, we can write $U \cong \oplus_i a_i S_i$ and $V \cong \oplus_i b_i S_i$, where the a_i and b_i are non-negative integers. By taking characters, we have $0 = \chi_U - \chi_V = \sum_i (a_i - b_i)\chi_i$, which by Proposition 5 forces $a_i = b_i$ for all i; hence $U \cong V$. ■

Although each character determines a $\mathbb{C}G$-module up to isomorphism by Theorem 7, there is no generic way to construct a module from its corresponding character, and so in some sense information is lost by studying characters in lieu of modules. However, characters turn out to be an efficient means of translating information about the ordinary representation theory of G to information about G itself, and for this reason we will soon turn our attention away from $\mathbb{C}G$-modules and toward their characters.

We have seen in Section 12 that if U and V are $\mathbb{C}G$-modules, then $U \otimes_{\mathbb{C}} V$ and $\mathrm{Hom}_{\mathbb{C}}(U, V)$, which we shall write here simply as $U \otimes V$ and $\mathrm{Hom}(U, V)$, admit natural $\mathbb{C}G$-module structures. We now consider the relationship between the characters of these modules and those of U and V.

PROPOSITION 8. Let U and V be $\mathbb{C}G$-modules. Then:

(i) $\chi_{U \otimes V} = \chi_U \chi_V$.
(ii) $\chi_{U^*} = \overline{\chi_U}$. (Recall that $U^* = \mathrm{Hom}(U, F)$.)
(iii) $\chi_{\mathrm{Hom}(U,V)} = \overline{\chi_U}\chi_V$.

PROOF. (i) Let $g \in G$. By part (i) of Proposition 4, the transformation defined by the action of g on U is diagonalizable; let $\{u_1, \ldots, u_m\}$ be a basis of U consisting of eigenvectors of this transformation, with respective eigenvalues $\lambda_1, \ldots, \lambda_m$. Similarly, let $\{v_1, \ldots, v_n\}$ be a basis of V consisting of eigenvectors of the transformation defined by the action of g on V, with respective eigenvalues $\mu_1, \ldots, \mu_n$. We have $\chi_U(g) = \lambda_1 + \ldots + \lambda_m$ and $\chi_V(g) = \mu_1 + \ldots + \mu_n$ by part (ii) of Proposition 4. Now $\{u_i \otimes v_j\}_{i,j}$ is a basis for $U \otimes V$, and $g(u_i \otimes v_j) = gu_i \otimes gv_j = \lambda_i u_i \otimes \mu_j v_j = \lambda_i \mu_j (u_i \otimes v_j)$ for any i, j. Hence the basis $\{u_i \otimes v_j\}_{i,j}$ consists of eigenvectors for the transformation defined by the action of g on $U \otimes V$, and thus $\chi_{U \otimes V}(g) = \sum_{i,j} \lambda_i \mu_j = (\sum_i \lambda_i)(\sum_j \mu_j) = \chi_U(g)\chi_V(g)$.

(ii) Let $g \in G$, and as in the proof of (i) let $\{u_1, \dots, u_m\}$ be a basis for U of eigenvectors of the transformation of U defined by g, with respective eigenvalues $\lambda_1, \dots, \lambda_m$. Let $\{\varphi_1, \dots, \varphi_m\}$ be the dual basis of U^*, so that $\varphi_i \colon U \to \mathbb{C}$ is defined for each i by $\varphi_i(u_j) = \delta_{ij}$ for each j. Fix some j. Now $gu_j = \lambda_j u_j$, so $g^{-1}u_j = \lambda_j^{-1} u_j$. However, we observed in the proof of Proposition 4 that λ_j is a root of unity, and consequently we have $\lambda_j^{-1} = \overline{\lambda_j}$. For any i, j we now have $(g\varphi_i)(u_j) = \varphi_i(g^{-1}u_j) = \varphi_i(\overline{\lambda_j} u_j) = \overline{\lambda_j}\delta_{ij}$, and this gives $g\varphi_i = \overline{\lambda_i}\varphi_i$ for each i; therefore, the basis $\{\varphi_1, \dots, \varphi_m\}$ of U^* consists of eigenvectors for the transformation of U^* defined by g, with respective eigenvalues $\overline{\lambda_1}, \dots, \overline{\lambda_m}$. Therefore $\chi_{U^*}(g) = \overline{\lambda_1} + \dots + \overline{\lambda_m} = \overline{\lambda_1 + \dots + \lambda_m} = \overline{\chi_U(g)}$.

(iii) By Proposition 12.7, we have $\mathrm{Hom}(U, V) \cong U^* \otimes V$, and so the result follows from (i) and (ii) above. ■

A *virtual character* of G is a $\mathbb{Z}$-linear combination of the irreducible characters of G. (Some authors prefer the term "generalized character.") Characters are virtual characters by Theorem 7.

COROLLARY 9. The virtual characters of G form a ring.

PROOF. By part (i) of Proposition 8, the product of two characters is again a character, and the result easily follows from this observation. ■

A *class function* on G is a function from G to $\mathbb{C}$ whose value within any conjugacy class is constant. For example, characters of $\mathbb{C}G$-modules are class functions. The set of class functions on G forms a $\mathbb{C}$-vector space of dimension r, where r is the number of conjugacy classes of G; an obvious basis for this vector space is the set of functions that attain the value 1 on a single conjugacy class and 0 on all other classes.

PROPOSITION 10. The irreducible characters of G form a basis for the space of class functions on G.

PROOF. By Proposition 5, the irreducible characters of G are linearly independent elements of the space of class functions; but their number is equal by Theorem 3 to the number of conjugacy classes of G, which is equal to the dimension of the space of class functions. ■

If α and β are class functions on G, then their *inner product* is the complex number

$$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g)\overline{\beta(g)}.$$

This function $(\ ,\ )$ is, as the name suggests, an inner product on the space of class functions. That is, we have:

- $(\alpha, \alpha) \geq 0$ for all α, and $(\alpha, \alpha) = 0$ iff $\alpha = 0$.
- $(\alpha, \beta) = \overline{(\beta, \alpha)}$ for all α, β.
- $(\lambda\alpha, \beta) = \lambda(\alpha, \beta)$ for all α, β and all $\lambda \in \mathbb{C}$.
- $(\alpha_1 + \alpha_2, \beta) = (\alpha_1, \beta) + (\alpha_2, \beta)$ for all $\alpha_1, \alpha_2, \beta$.

The following are easy consequences of the above properties:

- $(\alpha, \lambda\beta) = \overline{\lambda}(\alpha, \beta)$ for all α, β and all $\lambda \in \mathbb{C}$.
- $(\alpha, \beta_1 + \beta_2) = (\alpha, \beta_1) + (\alpha, \beta_2)$ for all α, β_1, β_2.

We conclude this section with a result that gives some meaning to the inner product of two characters. We first require some notation and a lemma. If U is a $\mathbb{C}G$-module, then the set of elements of U on which G acts trivially is a $\mathbb{C}G$-submodule of U; we call this submodule U^G, and so $U^G = \{u \in U \mid gu = u \text{ for all } g \in G\}$.

LEMMA 11. If U is a $\mathbb{C}G$-module, then $\dim_{\mathbb{C}} U^G = \frac{1}{|G|} \sum_{g \in G} \chi_U(g)$.

PROOF. Let $a = \frac{1}{|G|} \sum_{g \in G} g \in \mathbb{C}G$. We have $ga = a$ for any $g \in G$, and from this we see that $a^2 = a$. If T is the linear transformation of U defined by a, then T must satisfy the equation $X^2 - X = 0$; consequently T is diagonalizable, and the only eigenvalues of T are 0 and 1. Let $U_1 \subseteq U$ be the eigenspace of T corresponding to the eigenvalue 1. If $u \in U_1$, then we have $gu = gau = au = u$ for any $g \in G$, and therefore $u \in U^G$. Conversely, suppose that $u \in U^G$. Then we see that $|G|au = (\sum_{g \in G} g)u = \sum_g gu = \sum_g u = |G|u$, and hence that $au = u$, which gives $u \in U_1$. Therefore $U^G = U_1$. However, the trace of T is clearly equal to the dimension of U_1, and the result now follows from the linearity of the trace map. ■

THEOREM 12. We have $(\chi_U, \chi_V) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(U, V)$ for any $\mathbb{C}G$-modules U and V.

PROOF. We first observe that $\mathrm{Hom}_{\mathbb{C}G}(U, V)$ is a subspace of the $\mathbb{C}G$-module $\mathrm{Hom}(U, V)$. If $\varphi \in \mathrm{Hom}_{\mathbb{C}G}(U, V)$ and $g \in G$, then $(g\varphi)(u) = g\varphi(g^{-1}u) = gg^{-1}\varphi(u) = \varphi(u)$ for any $u \in U$; hence we have $g\varphi = \varphi$ for all $g \in G$, which shows that $\varphi \in \mathrm{Hom}(U, V)^G$. By reversing the argument we conclude that $\mathrm{Hom}_{\mathbb{C}G}(U, V) = \mathrm{Hom}(U, V)^G$. Therefore

$$\begin{aligned}
\dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(U, V) = \dim_{\mathbb{C}} \mathrm{Hom}(U, V)^G &= \frac{1}{|G|} \sum_{g \in G} \chi_{\mathrm{Hom}(U,V)}(g) \\
&= \frac{1}{|G|} \sum_{g \in G} \overline{\chi_U(g)}\chi_V(g) \\
&= (\chi_V, \chi_U)
\end{aligned}$$

by Lemma 11 and part (iii) of Proposition 8. This implies that $(\chi_U, \chi_V) = \overline{(\chi_V, \chi_U)} = (\chi_V, \chi_U) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}G}(U, V)$ since we now know that (χ_V, χ_U) is real-valued. ■

EXERCISES

Throughout these exercises, G denotes a finite group, and all modules are finitely generated.

1. Let A be a semisimple $\mathbb{C}$-algebra. Let $[A, A]$ be the subspace of A spanned by the set $\{ab - ba \mid a, b \in A\}$. Show that the codimension of $[A, A]$ in A (which is just the dimension of $A/[A, A]$) is equal to the number of isomorphism classes of simple A-modules.
2. (cont.) Let $A = \mathbb{C}G$, and let $g_1, \dots, g_r$ be representatives of the r conjugacy classes of G. Show that $\{g_i + [A, A] \mid 1 \leq i \leq r\}$ is a basis of $A/[A, A]$.
3. Show that if S is a simple $\mathbb{C}G$-module and U is a one-dimensional $\mathbb{C}G$-module, then $S \otimes U$ is simple.
4. Show that the set of isomorphism classes of the one-dimensional $\mathbb{C}G$-modules forms a group under the taking of tensor products.
5. Let χ be an irreducible character of G. Show that if λ is any $|G|$th root of unity, then $\{x \in G \mid \chi(x) = \lambda\chi(1)\} \trianglelefteq G$.
6. Show that a $\mathbb{C}G$-module U is simple iff its dual U^* is simple. Conclude that the complex conjugate of an irreducible character is again an irreducible character.