

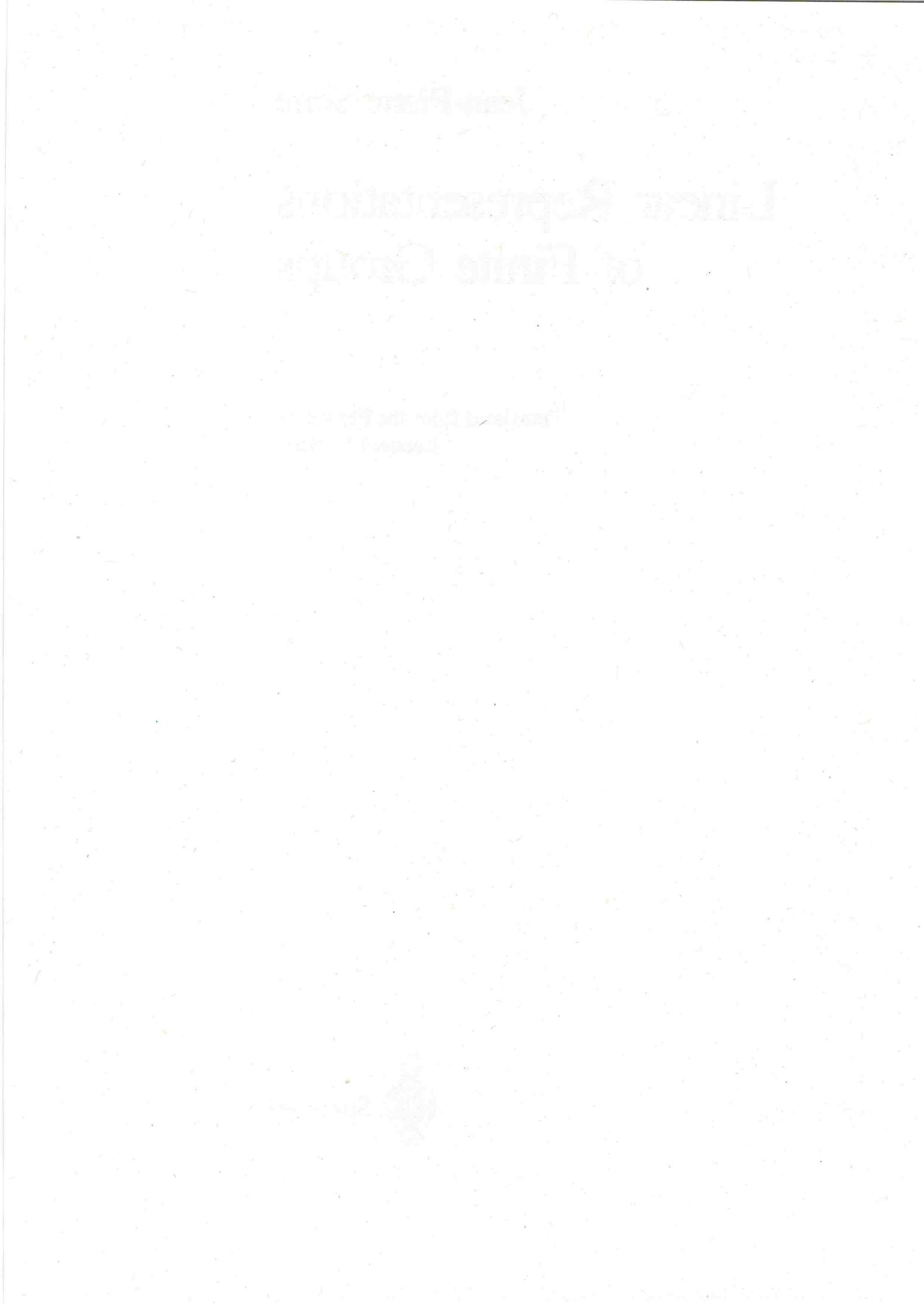
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# Linear Representations of Finite Groups

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# CHAPTER 2

## Character theory

### 2.1 The character of a representation

Let  $V$  be a vector space having a basis  $(e_i)$  of  $n$  elements, and let  $a$  be a linear map of  $V$  into itself, with matrix  $(a_{ij})$ . By the *trace* of  $a$  we mean the scalar

$$\text{Tr}(a) = \sum_i a_{ii}.$$

It is the *sum of the eigenvalues of  $a$*  (counted with their multiplicities), and does not depend on the choice of basis  $(e_i)$ .

Now let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of a finite group  $G$  in the vector space  $V$ . For each  $s \in G$ , put:

$$\chi_\rho(s) = \text{Tr}(\rho_s).$$

The complex valued function  $\chi_\rho$  on  $G$  thus obtained is called the *character* of the representation  $\rho$ ; the importance of this function comes primarily from the fact that it *characterizes* the representation  $\rho$  (cf. 2.3).

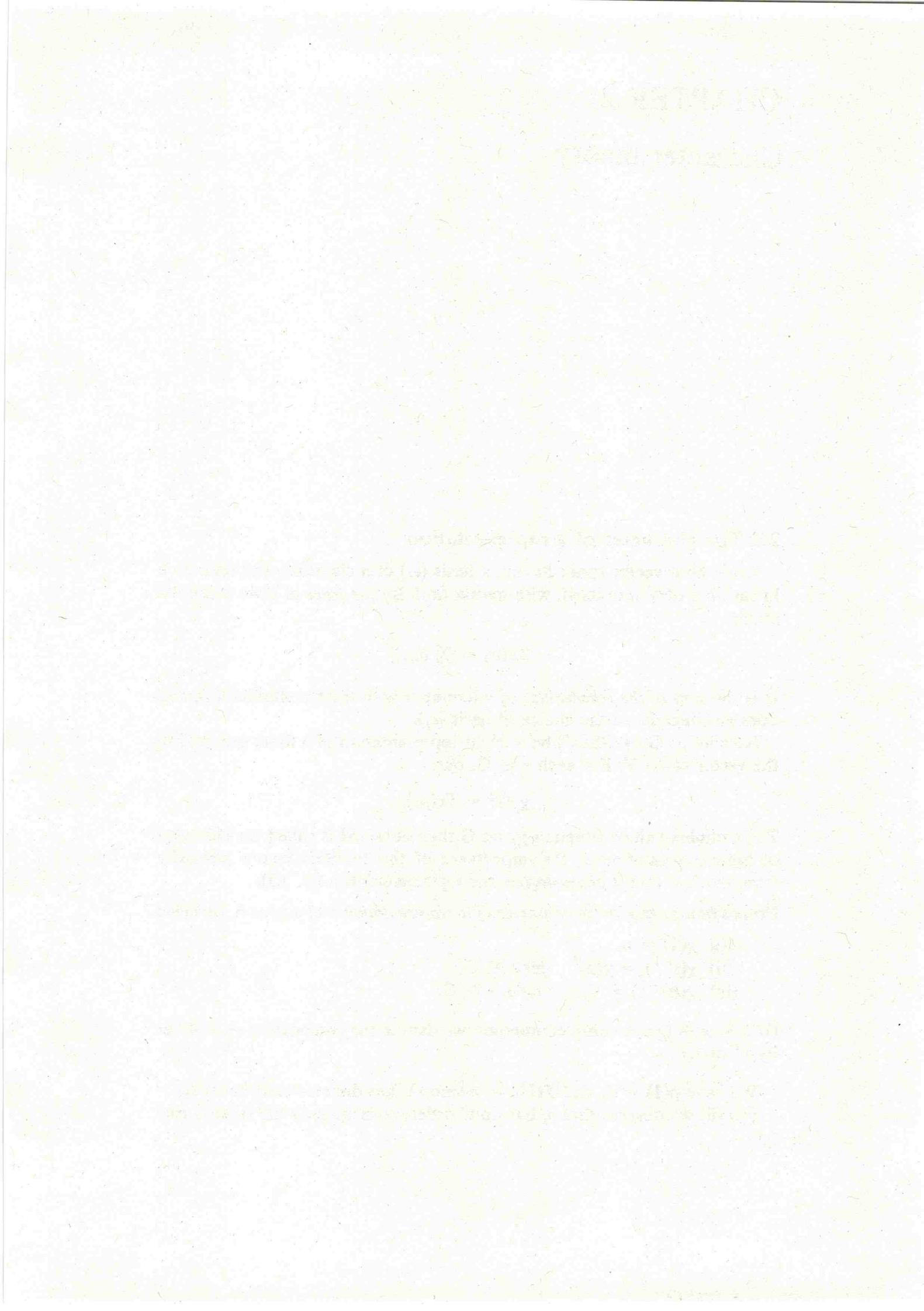
**Proposition 1.** *If  $\chi$  is the character of a representation  $\rho$  of degree  $n$ , we have:*

- (i)  $\chi(1) = n$ ,
- (ii)  $\chi(s^{-1}) = \chi(s)^*$  for  $s \in G$ ,
- (iii)  $\chi(tst^{-1}) = \chi(s)$  for  $s, t \in G$ .

(If  $z = x + iy$  is a complex number, we denote the conjugate  $x - iy$  either by  $z^*$  or  $\bar{z}$ .)

We have  $\rho(1) = 1$ , and  $\text{Tr}(1) = n$  since  $V$  has dimension  $n$ ; hence (i).

For (ii) we observe that  $\rho_s$  has finite order; consequently the same is true



of its eigenvalues  $\lambda_1, \dots, \lambda_n$  and so these have absolute value equal to 1 (this is also a consequence of the fact that  $\rho_s$  can be defined by a unitary matrix, cf. 1.3). Thus

$$\chi(s)^* = \text{Tr}(\rho_s)^* = \sum \lambda_i^* = \sum \lambda_i^{-1} = \text{Tr}(\rho_s^{-1}) = \text{Tr}(\rho_{s^{-1}}) = \chi(s^{-1}).$$

Formula (iii) can also be written  $\chi(vu) = \chi(uv)$ , putting  $u = ts$ ,  $v = t^{-1}$ ; hence it follows from the well known formula

$$\text{Tr}(ab) = \text{Tr}(ba),$$

valid for two arbitrary linear mappings  $a$  and  $b$  of  $V$  into itself.  $\square$

*Remark.* A function  $f$  on  $G$  satisfying identity (iii), or what amounts to the same thing,  $f(uv) = f(vu)$ , is called a *class function*. We will see in 2.5 that each class function is a linear combination of characters.

**Proposition 2.** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two linear representations of  $G$ , and let  $\chi_1$  and  $\chi_2$  be their characters. Then:

- (i) The character  $\chi$  of the direct sum representation  $V_1 \oplus V_2$  is equal to  $\chi_1 + \chi_2$ .
- (ii) The character  $\psi$  of the tensor product representation  $V_1 \otimes V_2$  is equal to  $\chi_1 \cdot \chi_2$ .

Let us be given  $\rho^1$  and  $\rho^2$  in matrix form:  $R_s^1, R_s^2$ . The representation  $V_1 \oplus V_2$  is then given by

$$R_s = \begin{pmatrix} R_s^1 & 0 \\ 0 & R_s^2 \end{pmatrix}$$

whence  $\text{Tr}(R_s) = \text{Tr}(R_s^1) + \text{Tr}(R_s^2)$ , that is  $\chi(s) = \chi_1(s) + \chi_2(s)$ .

We proceed likewise for (ii): with the notation of 1.5, we have

$$\chi_1(s) = \sum_{i_1} r_{i_1 i_1}(s), \quad \chi_2(s) = \sum_{i_2} r_{i_2 i_2}(s),$$

$$\psi(s) = \sum_{i_1, i_2} r_{i_1 i_1}(s) r_{i_2 i_2}(s) = \chi_1(s) \cdot \chi_2(s). \quad \square$$

**Proposition 3.** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$ , and let  $\chi$  be its character. Let  $\chi_\sigma^2$  be the character of the symmetric square  $\text{Sym}^2(V)$  of  $V$  (cf. 1.6), and let  $\chi_\alpha^2$  be that of  $\text{Alt}^2(V)$ . For each  $s \in G$ , we have

$$\chi_\sigma^2(s) = \frac{1}{2}(\chi(s)^2 + \chi(s^2))$$

$$\chi_\alpha^2(s) = \frac{1}{2}(\chi(s)^2 - \chi(s^2))$$

and  $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$ .



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Let  $s \in G$ . A basis  $(e_i)$  of  $V$  can be chosen consisting of *eigenvectors* for  $\rho_s$ ; this follows for example from the fact that  $\rho_s$  can be represented by a *unitary* matrix, cf. 1.3. We have then  $\rho_s e_i = \lambda_i e_i$  with  $\lambda_i \in \mathbb{C}$ , and so

$$\chi(s) = \sum \lambda_i, \quad \chi(s^2) = \sum \lambda_i^2.$$

On the other hand, we have

$$(\rho_s \otimes \rho_s)(e_i \cdot e_j + e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j + e_j \cdot e_i),$$

$$(\rho_s \otimes \rho_s)(e_i \cdot e_j - e_j \cdot e_i) = \lambda_i \lambda_j \cdot (e_i \cdot e_j - e_j \cdot e_i),$$

hence

$$\chi_\sigma^2(s) = \sum_{i < j} \lambda_i \lambda_j = \sum \lambda_i^2 + \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2} (\sum \lambda_i)^2 + \frac{1}{2} \sum \lambda_i^2$$

$$\chi_\alpha^2(s) = \sum_{i < j} \lambda_i \lambda_j = \frac{1}{2} (\sum \lambda_i)^2 - \frac{1}{2} \sum \lambda_i^2.$$

The proposition follows.

(Observe the equality  $\chi_\sigma^2 + \chi_\alpha^2 = \chi^2$ , which reflects the fact that  $V \otimes V$  is the *direct sum* of  $\text{Sym}^2(V)$  and  $\text{Alt}^2(V)$ ).  $\square$

### EXERCISES

**2.1.** Let  $\chi$  and  $\chi'$  be the characters of two representations. Prove the formulas:

$$(\chi + \chi')_\sigma^2 = \chi_\sigma^2 + \chi'^2_\sigma + \chi \chi',$$

$$(\chi + \chi')_\alpha^2 = \chi_\alpha^2 + \chi'^2_\alpha + \chi \chi'.$$

**2.2.** Let  $X$  be a finite set on which  $G$  acts, let  $\rho$  be the corresponding permutation representation [cf. 1.2, example (c)], and  $\chi_X$  be the character of  $\rho$ . Let  $s \in G$ ; show that  $\chi_X(s)$  is the number of elements of  $X$  fixed by  $s$ .

**2.3.** Let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation with character  $\chi$  and let  $V'$  be the dual of  $V$ , i.e., the space of linear forms on  $V$ . For  $x \in V$ ,  $x' \in V'$  let  $\langle x, x' \rangle$  denote the value of the linear form  $x'$  at  $x$ . Show that there exists a unique linear representation  $\rho': G \rightarrow \text{GL}(V')$ , such that

$$\langle \rho_s x, \rho'_s x' \rangle = \langle x, x' \rangle \quad \text{for } s \in G, x \in V, x' \in V'.$$

This is called the *contragredient* (or *dual*) representation of  $\rho$ ; its character is  $\chi^*$ .

**2.4.** Let  $\rho_1: G \rightarrow \text{GL}(V_1)$  and  $\rho_2: G \rightarrow \text{GL}(V_2)$  be two linear representations with characters  $\chi_1$  and  $\chi_2$ . Let  $W = \text{Hom}(V_1, V_2)$ , the vector space of linear mappings  $f: V_1 \rightarrow V_2$ . For  $s \in G$  and  $f \in W$  let  $\rho_s f = \rho_{2,s} \circ f \circ \rho_{1,s}^{-1}$ ; so  $\rho_s f \in W$ . Show that this defines a linear representation  $\rho: G \rightarrow \text{GL}(W)$ , and that its character is  $\chi_1^* \cdot \chi_2$ . This representation is isomorphic to  $\rho_1 \otimes \rho_2$ .



where  $\rho_1'$  is the contragredient of  $\rho_1$ , cf. ex. 2.3.

## 2.2 Schur's lemma; basic applications

**Proposition 4 (Schur's lemma).** Let  $\rho^1: G \rightarrow \text{GL}(V_1)$  and  $\rho^2: G \rightarrow \text{GL}(V_2)$  be two irreducible representations of  $G$ , and let  $f$  be a linear mapping of  $V_1$  into  $V_2$  such that  $\rho_s^2 \circ f = f \circ \rho_s^1$  for all  $s \in G$ . Then:

- (1) If  $\rho^1$  and  $\rho^2$  are not isomorphic, we have  $f = 0$ .
- (2) If  $V_1 = V_2$  and  $\rho^1 = \rho^2$ ,  $f$  is a homothety (i.e., a scalar multiple of the identity).

The case  $f = 0$  is trivial. Suppose now  $f \neq 0$  and let  $W_1$  be its kernel (that is, the set of  $x \in V_1$  such that  $fx = 0$ ). For  $x \in W_1$  we have  $f\rho_s^1 x = \rho_s^2 fx = 0$ , whence  $\rho_s^1 x \in W_1$ , and  $W_1$  is stable under  $G$ . Since  $V_1$  is irreducible,  $W_1$  is equal to  $V_1$  or 0; the first case is excluded, as it implies  $f = 0$ . The same argument shows that the image  $W_2$  of  $f$  (the set of  $fx$ , for  $x \in V_1$ ) is equal to  $V_2$ . The two properties  $W_1 = 0$  and  $W_2 = V_2$  show that  $f$  is an isomorphism of  $V_1$  onto  $V_2$ , which proves assertion (1).

Suppose now that  $V_1 = V_2$ ,  $\rho^1 = \rho^2$ , and let  $\lambda$  be an eigenvalue of  $f$ : there exists at least one, since the field of scalars is the field of complex numbers. Put  $f' = f - \lambda$ . Since  $\lambda$  is an eigenvalue of  $f$ , the kernel of  $f'$  is  $\neq 0$ ; on the other hand, we have  $\rho_s^2 \circ f' = f' \circ \rho_s^1$ . The first part of the proof shows that these properties are possible only if  $f' = 0$ , that is, if  $f$  is equal to  $\lambda$ .  $\square$

Let us keep the hypothesis that  $V_1$  and  $V_2$  are irreducible, and denote by  $g$  the order of the group  $G$ .

**Corollary 1.** Let  $h$  be a linear mapping of  $V_1$  into  $V_2$ , and put:

$$h^0 = \frac{1}{g} \sum_{t \in G} (\rho_t^2)^{-1} h \rho_t^1.$$

Then:

- (1) If  $\rho^1$  and  $\rho^2$  are not isomorphic, we have  $h^0 = 0$ .
- (2) If  $V_1 = V_2$  and  $\rho^1 = \rho^2$ ,  $h^0$  is a homothety of ratio  $(1/n)\text{Tr}(h)$ , with  $n = \dim(V_1)$ .

We have  $\rho_s^2 h^0 = h^0 \rho_s^1$ . Indeed:

$$\begin{aligned} (\rho_s^2)^{-1} h^0 \rho_s^1 &= \frac{1}{g} \sum_{t \in G} (\rho_s^2)^{-1} (\rho_t^2)^{-1} h \rho_t^1 \rho_s^1 \\ &= \frac{1}{g} \sum_{t \in G} (\rho_{ts}^2)^{-1} h \rho_{ts}^1 = h^0. \end{aligned}$$

Applying prop. 4 to  $f = h^0$ , we see in case (1) that  $h^0 = 0$ , and in case (2) that  $h^0$  is equal to a scalar  $\lambda$ . Moreover, in the latter case, we have:



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$$\mathrm{Tr}(h^0) = \frac{1}{g} \sum_{t \in G} \mathrm{Tr}((\rho_t^1)^{-1} h \rho_t^1) = \mathrm{Tr}(h),$$

and since  $\mathrm{Tr}(\lambda) = n \cdot \lambda$ , we get  $\lambda = (1/n) \mathrm{Tr}(h)$ .  $\square$

Now we rewrite corollary 1 assuming that  $\rho^1$  and  $\rho^2$  are given in matrix form:

$$\rho_t^1 = (r_{i_1 j_1}(t)), \rho_t^2 = (r_{i_2 j_2}(t)).$$

The linear mapping  $h$  is defined by a matrix  $(x_{i_2 i_1})$  and likewise  $h^0$  is defined by  $(x_{i_2 i_1}^0)$ . We have by definition of  $h^0$ :

$$x_{i_2 i_1}^0 = \frac{1}{g} \sum_{t, j_1, j_2} r_{i_2 j_2}(t^{-1}) x_{j_2 j_1} r_{j_1 i_1}(t).$$

The right hand side is a linear form with respect to  $x_{j_2 j_1}$ ; in case (1) this form vanishes for all systems of values of the  $x_{j_2 j_1}$ ; thus its coefficients are zero. Whence:

**Corollary 2.** In case (1), we have:

$$\frac{1}{g} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) r_{j_1 i_1}(t) = 0$$

for arbitrary  $i_1, i_2, j_1, j_2$ .

In case (2) we have similarly  $h^0 = \lambda$ , i.e.,  $x_{i_2 i_1}^0 = \lambda \delta_{i_2 i_1}$  ( $\delta_{i_2 i_1}$  denotes the Kronecker symbol, equal to 1 if  $i_1 = i_2$  and 0 otherwise), with  $\lambda = (1/n) \mathrm{Tr}(h)$ , that is,  $\lambda = (1/n) \sum \delta_{j_2 j_1} x_{j_2 j_1}$ . Hence the equality:

$$\frac{1}{g} \sum_{t, j_1, j_2} r_{i_2 j_2}(t^{-1}) x_{j_2 j_1} r_{j_1 i_1}(t) = \frac{1}{n} \sum_{j_1, j_2} \delta_{i_2 i_1} \delta_{j_2 j_1} x_{j_2 j_1}.$$

Equating coefficients of the  $x_{j_2 j_1}$ , we obtain as above:

**Corollary 3.** In case (2) we have:

$$\frac{1}{g} \sum_{t \in G} r_{i_2 j_2}(t^{-1}) r_{j_1 i_1}(t) = \frac{1}{n} \delta_{i_2 i_1} \delta_{j_2 j_1} = \begin{cases} 1/n & \text{if } i_1 = i_2 \text{ and } j_1 = j_2 \\ 0 & \text{otherwise.} \end{cases}$$

**Remarks**

(1) If  $\phi$  and  $\psi$  are functions on  $G$ , set

$$\langle \phi, \psi \rangle = \frac{1}{g} \sum_{t \in G} \phi(t^{-1}) \psi(t) = \frac{1}{g} \sum_{t \in G} \phi(t) \psi(t^{-1}).$$

We have  $\langle \phi, \psi \rangle = \langle \psi, \phi \rangle$ . Moreover  $\langle \phi, \psi \rangle$  is linear in  $\phi$  and in  $\psi$ . With this notation, corollaries 2 and 3 become, respectively

$$\langle r_{i_2 j_2}, r_{j_1 i_1} \rangle = 0 \quad \text{and} \quad \langle r_{i_2 j_2}, r_{j_1 i_1} \rangle = \frac{1}{n} \delta_{i_2 i_1} \delta_{j_2 j_1}.$$



(2) Suppose that the matrices  $(r_{ij}(t))$  are *unitary* (this can be realized by a suitable choice of basis, cf. 1.3). We have then  $r_{ij}(t^{-1}) = r_{ji}(t)^*$  and corollaries 2 and 3 are just *orthogonality relations* for the scalar product  $(\phi|\psi)$  defined in the following section.

### 2.3 Orthogonality relations for characters

We begin with a notation. If  $\phi$  and  $\psi$  are two complex-valued functions on  $G$ , put

$$(\phi|\psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\psi(t)^*, \quad g \text{ being the order of } G.$$

This is a *scalar product*: it is linear in  $\phi$ , semilinear in  $\psi$ , and we have  $(\phi|\phi) > 0$  for all  $\phi \neq 0$ .

If  $\check{\psi}$  is the function defined by the formula  $\check{\psi}(t) = \psi(t^{-1})^*$ , we have

$$(\phi|\psi) = \frac{1}{g} \sum_{t \in G} \phi(t)\check{\psi}(t^{-1}) = \langle \phi, \check{\psi} \rangle,$$

cf. 2.2, remark 1. In particular, if  $\chi$  is the *character* of a representation of  $G$ , we have  $\check{\chi} = \chi$  (prop. 1), so that  $(\phi|\chi) = \langle \phi, \chi \rangle$  for all functions  $\phi$  on  $G$ . So we can use at will  $(\phi|\chi)$  or  $\langle \phi, \chi \rangle$ , so long as we are concerned with characters.

#### Theorem 3

- (i) If  $\chi$  is the character of an irreducible representation, we have  $(\chi|\chi) = 1$  (i.e.,  $\chi$  is "of norm 1").
- ii) If  $\chi$  and  $\chi'$  are the characters of two nonisomorphic irreducible representations, we have  $(\chi|\chi') = 0$  (i.e.  $\chi$  and  $\chi'$  are orthogonal).

Let  $\rho$  be an irreducible representation with character  $\chi$ , given in matrix form  $\rho_t = (r_{ij}(t))$ . We have  $\chi(t) = \sum r_{ii}(t)$ , hence

$$(\chi|\chi) = \langle \chi, \chi \rangle = \sum_{i,j} \langle r_{ii}, r_{jj} \rangle.$$

But according to cor. 3 to prop. 4, we have  $\langle r_{ii}, r_{jj} \rangle = \delta_{ij}/n$ , where  $n$  is the degree of  $\rho$ . Thus

$$(\chi|\chi) = \left( \sum_{i,j} \delta_{ij} \right)/n = n/n = 1,$$

since the indices  $i, j$  each take  $n$  values. (ii) is proved in the same way, by applying cor. 2 instead of cor. 3.  $\square$

*Remark.* A character of an irreducible representation is called an *irreducible character*. Theorem 3 shows that the irreducible characters form an orthonormal system; this result will be completed later (2.5, th. 6).



**Theorem 4.** Let  $V$  be a linear representation of  $G$ , with character  $\phi$ , and suppose  $V$  decomposes into a direct sum of irreducible representations:

$$V = W_1 \oplus \cdots \oplus W_k.$$

Then, if  $W$  is an irreducible representation with character  $\chi$ , the number of  $W_i$  isomorphic to  $W$  is equal to the scalar product  $(\phi|\chi) = \langle \phi, \chi \rangle$ .

Let  $\chi_i$  be the character of  $W_i$ . By prop. 2, we have

$$\phi = \chi_1 + \cdots + \chi_k.$$

Thus  $(\phi|\chi) = (\chi_1|\chi) + \cdots + (\chi_k|\chi)$ . But, according to the preceding theorem,  $(\chi_i|\chi)$  is equal to 1 or 0, depending on whether  $W_i$  is, or is not, isomorphic to  $W$ . The result follows.  $\square$

**Corollary 1.** The number of  $W_i$  isomorphic to  $W$  does not depend on the chosen decomposition.

(This number is called the “number of times that  $W$  occurs in  $V$ ”, or the “number of times that  $W$  is contained in  $V$ .”)

Indeed,  $(\phi|\chi)$  does not depend on the decomposition.  $\square$

*Remark.* It is in this sense that one can say that there is uniqueness in the decomposition of a representation into irreducible representations. We shall return to this in 2.6.

**Corollary 2.** Two representations with the same character are isomorphic.

Indeed, cor. 1 shows that they contain each given irreducible representation the same number of times.

The above results reduce the study of representations to that of their characters. If  $\chi_1, \dots, \chi_h$  are the distinct irreducible characters of  $G$ , and if  $W_1, \dots, W_h$  denote corresponding representations, each representation  $V$  is isomorphic to a direct sum

$$V = m_1 W_1 \oplus \cdots \oplus m_h W_h \quad m_i \text{ integers } \geq 0.$$

The character  $\phi$  of  $V$  is equal to  $m_1 \chi_1 + \cdots + m_h \chi_h$ , and we have  $m_i = (\phi|\chi_i)$ . [This applies notably to the tensor product  $W_i \otimes W_j$  of two irreducible representations, and shows that the product  $\chi_i \cdot \chi_j$  decomposes into  $\chi_i \chi_j = \sum m_{ij}^k \chi_k$ , the  $m_{ij}^k$  being integers  $\geq 0$ .] The orthogonality relations among the  $\chi_i$  imply in addition:

$$(\phi|\phi) = \sum_{i=1}^{i=h} m_i^2,$$

whence:



**Theorem 5.** If  $\phi$  is the character of a representation  $V$ ,  $(\phi|\phi)$  is a positive integer and we have  $(\phi|\phi) = 1$  if and only if  $V$  is irreducible.

Indeed,  $\sum m_i^2$  is only equal to 1 if one of the  $m_i$ 's is equal to 1 and the others to 0, that is, if  $V$  is isomorphic to one of the  $W_i$ .  $\square$

We obtain thus a very convenient *irreducibility criterion*.

### EXERCISES

- 2.5. Let  $\rho$  be a linear representation with character  $\chi$ . Show that the number of times that  $\rho$  contains the unit representation is equal to  $(\chi|1) = (1/g) \sum_{s \in G} \chi(s)$ .
- 2.6. Let  $X$  be a finite set on which  $G$  acts, let  $\rho$  be the corresponding permutation representation (1.2) and let  $\chi$  be its character.
- The set  $Gx$  of images under  $G$  of an element  $x \in X$  is called an *orbit*. Let  $c$  be the number of distinct orbits. Show that  $c$  is equal to the number of times that  $\rho$  contains the unit representation 1; deduce from this that  $(\chi|1) = c$ . In particular, if  $G$  is transitive (i.e., if  $c = 1$ ),  $\rho$  can be decomposed into  $1 \oplus \theta$  and  $\theta$  does not contain the unit representation. If  $\psi$  is the character of  $\theta$ , we have  $\chi = 1 + \psi$  and  $(\psi|1) = 0$ .
  - Let  $G$  act on the product  $X \times X$  of  $X$  by itself by means of the formula  $s(x, y) = (sx, sy)$ . Show that the character of the corresponding permutation representation is equal to  $\chi^2$ .
  - Suppose that  $G$  is *transitive* on  $X$  and that  $X$  has at least two elements. We say that  $G$  is *doubly transitive* if, for all  $x, y, x', y' \in X$  with  $x \neq y$  and  $x' \neq y'$ , there exists  $s \in G$  such that  $x' = sx$  and  $y' = sy$ . Prove the equivalence of the following properties:
    - $G$  is doubly transitive.
    - The action of  $G$  on  $X \times X$  has two orbits, the diagonal and its complement.
    - $(\chi^2|1) = 2$ .
    - The representation  $\theta$  defined in (a) is irreducible.

[The equivalence (i)  $\Leftrightarrow$  (ii) is immediate; (ii)  $\Leftrightarrow$  (iii) follows from (a) and (b). If  $\psi$  is the character of  $\theta$ , we have  $1 + \psi = \chi$  and  $(1|1) = 1$ ,  $(\psi|1) = 0$ , which shows that (iii) is equivalent to  $(\psi^2|1) = 1$ , i.e., to  $(1/g) \sum_{s \in G} \psi(s)^2 = 1$ ; since  $\psi$  is real-valued, this indeed means that  $\theta$  is irreducible, cf. th. 5.]

### 2.4 Decomposition of the regular representation

*Notation.* For the rest of Ch. 2, the irreducible characters of  $G$  are denoted  $\chi_1, \dots, \chi_h$ ; their degrees are written  $n_1, \dots, n_h$ , we have  $n_i = \chi_i(1)$ , cf. prop. 1.

Let  $R$  be the regular representation of  $G$ . Recall (cf. 1.2) that it has a basis  $(e_i)_{i \in G}$  such that  $\rho_s e_t = e_{st}$ . If  $s \neq 1$ , we have  $st \neq t$  for all  $t$ , which

1. *Introducing the new system* - *the first step*

2. *Establishing a common language* - *the second step*

3. *Developing a reporting system* - *the third step*

4. *Establishing a feedback system* - *the fourth step*

5. *Establishing a training and development system* - *the fifth step*

6. *Establishing a reward and recognition system* - *the sixth step*

7. *Establishing a communication system* - *the seventh step*

8. *Establishing a performance management system* - *the eighth step*

9. *Establishing a performance improvement system* - *the ninth step*

10. *Establishing a continuous improvement system* - *the tenth step*

11. *Establishing a culture of continuous improvement* - *the eleventh step*

12. *Establishing a culture of accountability* - *the twelfth step*

13. *Establishing a culture of respect and trust* - *the thirteenth step*

14. *Establishing a culture of learning and growth* - *the fourteenth step*

15. *Establishing a culture of innovation and creativity* - *the fifteenth step*

16. *Establishing a culture of collaboration and teamwork* - *the sixteenth step*

17. *Establishing a culture of integrity and ethics* - *the seventeenth step*

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shows that the diagonal terms of the matrix of  $\rho_s$  are zero; in particular we have  $\text{Tr}(\rho_s) = 0$ . On the other hand, for  $s = 1$ , we have

$$\text{Tr}(\rho_s) = \text{Tr}(1) = \dim(R) = g.$$

Whence:

**Proposition 5.** *The character  $r_G$  of the regular representation is given by the formulas:*

$$\begin{aligned} r_G(1) &= g, && \text{order of } G, \\ r_G(s) &= 0 && \text{if } s \neq 1. \end{aligned}$$

**Corollary 1.** *Every irreducible representation  $W_i$  is contained in the regular representation with multiplicity equal to its degree  $n_i$ .*

According to th. 4, this number is equal to  $\langle r_G, \chi_i \rangle$ , and we have

$$\langle r_G, \chi_i \rangle = \frac{1}{g} \sum_{s \in G} r_G(s^{-1}) \chi_i(s) = \frac{1}{g} g \cdot \chi_i(1) = \chi_i(1) = n_i. \quad \square$$

**Corollary 2.**

- (a) *The degrees  $n_i$  satisfy the relation  $\sum_{i=1}^{h} n_i^2 = g$ .*
- (b) *If  $s \in G$  is different from 1, we have  $\sum_{i=1}^{h} n_i \chi_i(s) = 0$ .*

By cor. 1, we have  $r_G(s) = \sum n_i \chi_i(s)$  for all  $s \in G$ . Taking  $s = 1$  we obtain (a), and taking  $s \neq 1$ , we obtain (b).  $\square$

### Remarks

- (1) The above result can be used in determining the irreducible representations of a group  $G$ : suppose we have constructed some mutually nonisomorphic irreducible representations of degrees  $n_1, \dots, n_k$ ; in order that they be *all* the irreducible representations of  $G$  (up to isomorphism), it is necessary and sufficient that  $n_1^2 + \dots + n_k^2 = g$ .
- (2) We will see later (Part II, 6.5) another property of the degrees  $n_i$ : they divide the order  $g$  of  $G$ .

### EXERCISE

- 2.7. Show that each character of  $G$  which is zero for all  $s \neq 1$  is an *integral* multiple of the character  $r_G$  of the regular representation.

### 2.5 Number of irreducible representations

Recall (cf. 2.1) that a function  $f$  on  $G$  is called a *class function* if  $f(tst^{-1}) = f(s)$  for all  $s, t \in G$ .



**Proposition 6.** Let  $f$  be a class function on  $G$ , and let  $\rho: G \rightarrow \text{GL}(V)$  be a linear representation of  $G$ . Let  $\rho_f$  be the linear mapping of  $V$  into itself defined by:

$$\rho_f = \sum_{t \in G} f(t)\rho_t.$$

If  $V$  is irreducible of degree  $n$  and character  $\chi$ , then  $\rho_f$  is a homothety of ratio  $\lambda$  given by:

$$\lambda = \frac{1}{n} \sum_{t \in G} f(t)\chi(t) = \frac{g}{n}(f|\chi^*).$$

Let us compute  $\rho_s^{-1} \rho_f \rho_s$ . We have:

$$\rho_s^{-1} \rho_f \rho_s = \sum_{t \in G} f(t)\rho_s^{-1} \rho_t \rho_s = \sum_{t \in G} f(t)\rho_{s^{-1}ts}.$$

Putting  $u = s^{-1}ts$ , this becomes:

$$\rho_s^{-1} \rho_f \rho_s = \sum_{u \in G} f(sus^{-1})\rho_u = \sum_{u \in G} f(u)\rho_u = \rho_f.$$

So we have  $\rho_f \rho_s = \rho_s \rho_f$ . By the second part of prop. 4, this shows that  $\rho_f$  is a homothety  $\lambda$ . The trace of  $\lambda$  is  $n\lambda$ ; that of  $\rho_f$  is  $\sum_{t \in G} f(t)\text{Tr}(\rho_t) = \sum_{t \in G} f(t)\chi(t)$ . Hence  $\lambda = (1/n) \sum_{t \in G} f(t)\chi(t) = (g/n)(f|\chi^*)$ .  $\square$

We introduce now the space  $H$  of class functions on  $G$ ; the irreducible characters  $\chi_1, \dots, \chi_h$  belong to  $H$ .

**Theorem 6.** The characters  $\chi_1, \dots, \chi_h$  form an orthonormal basis of  $H$ .

Theorem 3 shows that the  $\chi_i$  form an orthonormal system in  $H$ . It remains to prove that they generate  $H$ , and for this it is enough to show that every element of  $H$  orthogonal to the  $\chi_i^*$  is zero. Let  $f$  be such an element. For each representation  $\rho$  of  $G$ , put  $\rho_f = \sum_{t \in G} f(t)\rho_t$ . Since  $f$  is orthogonal to the  $\chi_i^*$ , prop. 6 above shows that  $\rho_f$  is zero so long as  $\rho$  is irreducible; from the direct sum decomposition we conclude that  $\rho_f$  is always zero. Applying this to the regular representation  $R$  (cf. 2.4) and computing the image of the basis vector  $e_1$  under  $\rho_f$ , we have

$$\rho_f e_1 = \sum_{t \in G} f(t)\rho_t e_1 = \sum_{t \in G} f(t)e_t.$$

Since  $\rho_f$  is zero, we have  $\rho_f e_1 = 0$  and the above formula shows that  $f(t) = 0$  for all  $t \in G$ ; hence  $f = 0$ , and the proof is complete.  $\square$

Recall that two elements  $t$  and  $t'$  of  $G$  are said to be conjugate if there exists  $s \in G$  such that  $t' = sts^{-1}$ ; this is an equivalence relation, which partitions  $G$  into classes (also called conjugacy classes).

**Theorem 7.** The number of irreducible representations of  $G$  (up to isomorphism) is equal to the number of classes of  $G$ .



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Let  $C_1, \dots, C_k$  be the distinct classes of  $G$ . To say that a function  $f$  on  $G$  is a class function is equivalent to saying that it is constant on each of  $C_1, \dots, C_k$ ; it is thus determined by its values  $\lambda_i$  on the  $C_i$ , and these can be chosen arbitrarily. Consequently, the dimension of the space  $H$  of class functions is equal to  $k$ . On the other hand, this dimension is, by th. 6, equal to the number of irreducible representations of  $G$  (up to isomorphism). The result follows.  $\square$

Here is another consequence of th. 6:

**Proposition 7.** Let  $s \in G$ , and let  $c(s)$  be the number of elements in the conjugacy class of  $s$ .

(a) We have  $\sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(s) = g/c(s)$ .

(b) For  $t \in G$  not conjugate to  $s$ , we have  $\sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(t) = 0$ .

(For  $s = 1$ , this yields cor. 2 to prop. 5.)

Let  $f_s$  be the function equal to 1 on the class of  $s$  and equal to 0 elsewhere. Since it is a class function, it can, by th. 6, be written

$$f_s = \sum_{i=1}^{i=h} \lambda_i \chi_i, \quad \text{with } \lambda_i = (f_s | \chi_i) = \frac{c(s)}{g} \chi_i(s)^*$$

We have then, for each  $t \in G$ ,

$$f_s(t) = \frac{c(s)}{g} \sum_{i=1}^{i=h} \chi_i(s)^* \chi_i(t).$$

This gives (a) if  $t = s$ , and (b) if  $t$  is not conjugate to  $s$ .  $\square$

**EXAMPLE.** Take for  $G$  the group of permutations of three letters. We have  $g = 6$ , and there are three classes: the element 1, the three transpositions, and the two cyclic permutations. Let  $t$  be a transposition and  $c$  a cyclic permutation. We have  $t^2 = 1$ ,  $c^3 = 1$ ,  $tc = c^2t$ ; whence there are just two characters of degree 1: the unit character  $\chi_1$  and the character  $\chi_2$  giving the signature of a permutation. Theorem 7 shows that there exists one other irreducible character  $\theta$ ; if  $n$  is its degree we must have  $1 + 1 + n^2 = 6$ , hence  $n = 2$ . The values of  $\theta$  can be deduced from the fact that  $\chi_1 + \chi_2 + 2\theta$  is the character of the regular representation of  $G$  (cf. prop. 5). We thus get the *character table* of  $G$ :

	1	$t$	$c$
$\chi_1$	1	1	1
$\chi_2$	1	-1	1
$\theta$	2	0	-1

