# MT242P ASSIGNMENT 1

**Question 1:** Consider the set $\mathbb{F} := \{(x_1, x_2) : x_1, x_2 \in \mathbb{Q}\}$ with operations

$$(x_1, x_2) + (y_1, y_2) := (x_1 + y_1, x_2 + y_2),$$
$$(x_1, x_2) \cdot (y_1, y_2) := (x_1 y_1 + 5x_2 y_2, x_1 y_2 + x_2 y_1).$$

The triple $(\mathbb{F}, +, \cdot)$ is a field.

(1) What is the multiplicative identity of $\mathbb{F}$?
(2) For $(x_1, x_2) \neq (0, 0)$, find the multiplicative inverse $(x_1, x_2)^{-1}$ and argue that it always exists.

**Question 2:** Consider the set $S := \mathbb{R}^2$ with summation $\boxplus$ defined by

$$(x_1, x_2) \boxplus (y_1, y_2) := (x_1 + y_2, x_2 + y_1)$$

and scalar multiplication $\cdot$ defined by $(\lambda \in \mathbb{R})$

$$\lambda \cdot (x_1, x_2) := (\lambda x_1, \lambda x_2).$$

State whether the quadruple $(S, \mathbb{R}, \boxplus, \cdot)$ is a vector space or not, and explain why.

**Question 3:** For every set $S$ below, check whether $S$ is a subspace of $\mathbb{R}^3$.

(1) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 = 0\}$;
(2) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2^2 = 0\}$;
(3) $S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 2x_1 - x_2 > 0\}$.

**Question 4:** Show that if $V$ is a finite-dimensional vector space and $U$ is a subspace of $V$ then $U$ is finite-dimensional too.

In your answer, refer to the Theorems/Propositions/facts that you appeal to as they are labelled in the lecture notes on Moodle (e.g. "by Theorem $n$ of the notes, we can find...").
[In the proof of Grassman's Formula seen in class we sneakily used the above fact without comment when we said "*Let* $\{v_1, \ldots, v_k\}$ *be a basis of* $U \cap W$..."]

**Question 5:** Recall that Grassman's formula says that for any subspaces $U, W$ of a common vector space $V$ we have

$$\dim(U + W) = \dim U + \dim V - \dim(U \cap W).$$

There is a formula for the cardinality of sets that closely mirrors this one: if $\#A$ denotes the cardinality of the set $A$, we have for any pair of finite sets $A, B$

$$\#(A \cup B) = \#A + \#B - \#(A \cap B).$$

For three finite sets, we have

$$\#(A \cup B \cup C) = \#A + \#B + \#C$$

$$- \#(A \cap B) - \#(B \cap C) - \#(C \cap A)$$
$$+ \#(A \cap B \cap C).$$

These are called *inclusion-exclusion formulas* – a Venn diagram with three sets should be enough to convince you that the above formula is correct.

Inspired by the analogy, we might be tempted to conjecture that for any three subspaces $T, U, W$ of a common vector space $V$ we have

$$\begin{aligned}\dim(T+U+W) = {} & \dim T + \dim U + \dim W \\ & - \dim(T \cap U) - \dim(U \cap W) - \dim(W \cap T) \\ & + \dim(T \cap U \cap W).\end{aligned}$$

Show that this formula is **false** by exhibiting a counterexample.