Q2)

a) Answer : False.

$A = \{(x,y) \mid y = 0\}$

$f : A \to \mathbb{R}$

$f(x,y) = x^2$

This has no abs max:

Assume $\exists (a,b) \in A$ st $f(a,b) \geq f(x,y)$

for all $(x,y) \in A$. let $k = f(x,y)$

$f(a,b) = a^2$

if $a \geq 0$, then $a^2 \leq (a+1)^2$

$\Rightarrow f(a,b) \leq f(a+1, b)$

$\therefore a \ngeq 0$, as if it were, then its not max

if $a < 0$, then $a^2 \leq (a-1)^2$

$f(a,b) \leq f(a-1, b)$

similarly, $a \nless 0$.

This shows that f does not have an absolute max. Therefore, not every continuous function on a closed unbounded set in $\mathbb{R}^2$ has at least one absolute max.

b) Answer: True

$S = \{(x, y) \mid -1 \leq x \leq 0,\ 3 \leq y \leq 6\}$

Does $\underbrace{\text{the exist a function}}$ $f : S \to \mathbb{R}$, st $(0,3)$ is a saddle point?

Take $f(x, y) = y - x^2$

$f : S \to \mathbb{R}$

We can show that any open ball with a radious $r \in \mathbb{R}_+$ centered at $(0,3)$ contains $(a, b), (c, d) \in S$ st

$f(a,b) < f(0, 3) < f(c, d)$

this would also (by definition) show that $(0,3)$ is a saddle point.

To prove this:

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take some $k \in \mathbb{R}_+$ st:

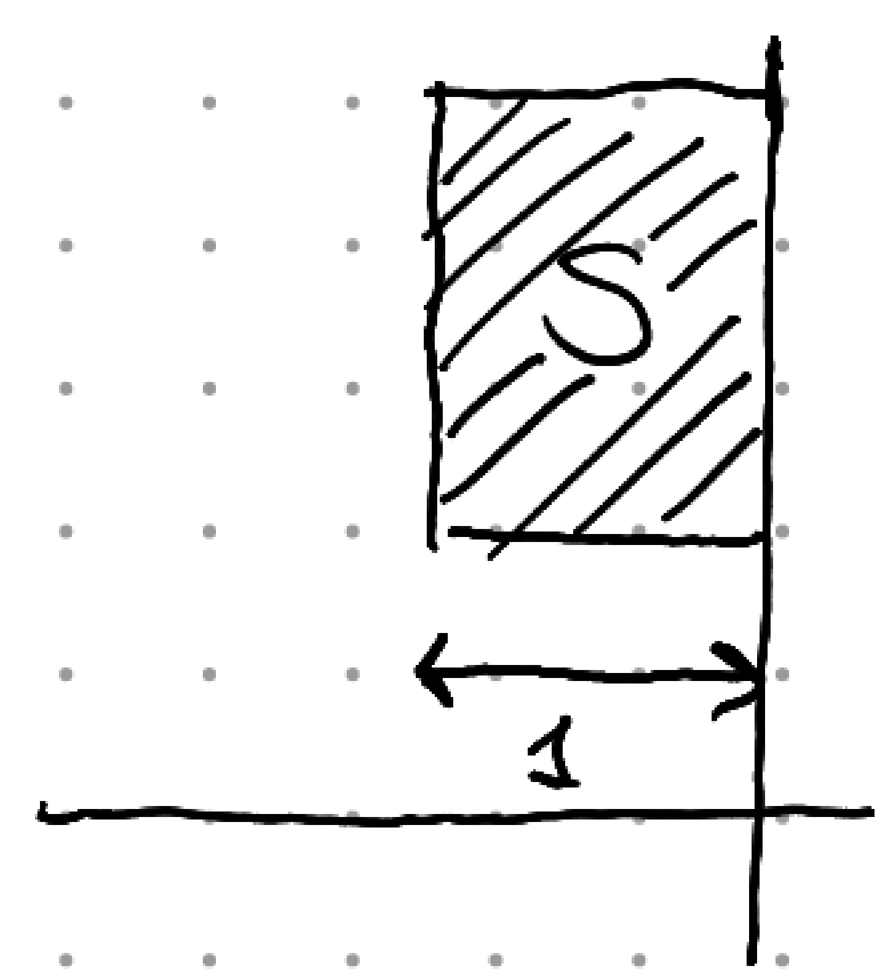
$k < 1$ and $k < r$

(such k will always exist. If $r > 1$, then $k = 0.1$, else, $k = r/2$ is an example)

$f(0,3) = 3$

$f(-k,3) = 3 - (-k)^2 = 3 - k^2$

$f(0,3+k) = 3 + k$



$\therefore f(-k,3) < f(0,3)$

& $f(0,3+k) > f(0,3)$

and $(-k,3), (0,3+k) \in S$ since $k < 1$.

$(a,b), (c,d)$

$\therefore$ There exist points in any open ball centered around $(0,3)$ st

$f(a,b) < f(0,3) < f(c,d)$

This proves f has a saddle p. at $(0,3)$

Q3)

$$g(x,y) = -x^2 - y^2 + 2x + 4y + 5$$

find local max / min / saddles

$$\frac{\partial}{\partial x} g = -2x + 2 \qquad \frac{\partial^2}{\partial x^2} g = -2$$

$$\frac{\partial}{\partial y} g = -2y + 4 \qquad \frac{\partial^2}{\partial y^2} = -2$$

$$\frac{\partial}{\partial x \partial y} g = 0$$

(a, b) is a critical point if:

$$\left.\frac{\partial g}{\partial x}\right|_{(a,b)} = 0 = \left.\frac{\partial g}{\partial y}\right|_{(a,b)}$$

$$\Rightarrow -2(a) + 2 = 0 \Rightarrow a = 1$$

$$\Rightarrow -2(b) + 4 = 0 \Rightarrow b = 2.$$

We can verify that (1, 2) is a max since:

$$\frac{\partial^2 g}{\partial^2 x} = -2 < 0$$

and $\frac{\partial^2 g}{\partial x^2} \cdot \frac{\partial^2 g}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$

$$= (-2)(-2) - 0 = 4 > 0$$

$\therefore$ There are no mins, no saddles, and there is only one local max, which is located at (1, 2)

Q4) $f(x,y) = \frac{1}{9}y^2 - \frac{1}{4}x^2$

$$\frac{\partial f}{\partial x} = -\frac{1}{2}x \qquad \frac{\partial^2 f}{\partial x^2} = -\frac{1}{2} \qquad \frac{\partial f}{\partial x \partial y} = 0$$

$$\frac{\partial f}{\partial y} = \frac{2}{9}y \qquad \frac{\partial^2 f}{\partial y^2} = \frac{2}{9}$$

There are no undefined points, so the only possible critical points are when:

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} \Rightarrow \begin{cases} -\frac{1}{2}a = 0 \Rightarrow a = 0 \\ \frac{2}{9}y = 0 \Rightarrow y = 0 \end{cases}$$

only critical at (0,0)

$$\frac{\partial^2 f}{\partial^2 x} \cdot \frac{\partial^2 f}{\partial y^2} - \frac{\partial f}{\partial x \partial y} = -\frac{2}{18} < 0$$

$\Rightarrow$ there is a saddle point at (0,0), and there are no local maximums or local minimum

Q5)

$$f(x,y) = x^2 + xy + y^2 - 6x + 2$$

$$W = \{(x,y) \mid 0 \le x \le 5, -3 \le y \le 0\}$$

$(a,b) \in W$ is a critical point if:

1) $$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0 = \left.\frac{\partial f}{\partial y}\right|_{(a,b)}$$

2) (a,b) is an exterior point of W

Step 1: $$\frac{\partial f}{\partial x} = 2x + y - 6$$

$$\frac{\partial f}{\partial y} = 2y + x$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} \Rightarrow 2x + y - 6 = 2y + x$$

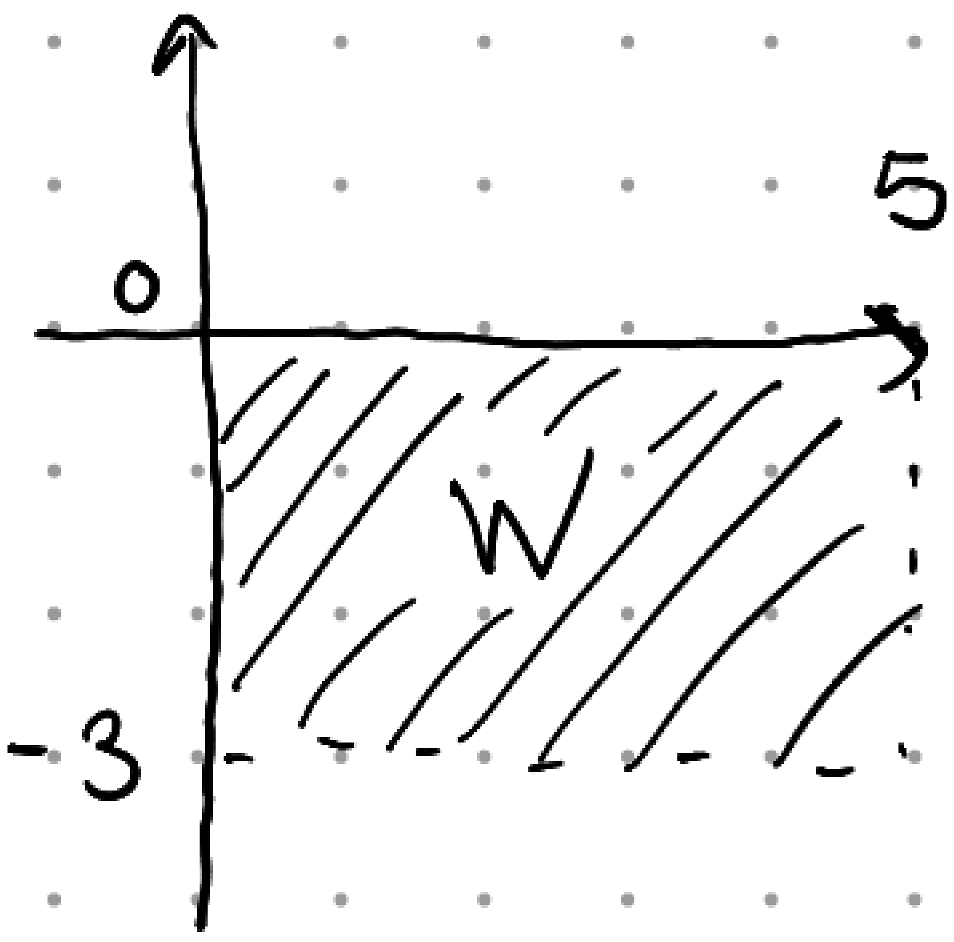
$$x - 6 = y.$$

$$2y + x = 0 \Rightarrow 2x - 12 + x = 0$$

$$x = \frac{12}{3} = 4, \quad y = -2$$

therefore, the only interior point that is a critical point is $(4, -2)$.

Step 2:



top: segment from $(0,0)$ to $(5,0)$
bottom: " " $(0,-3)$ to $(5,-3)$
left: " " $(0,0)$ to $(0,-3)$
right: " " $(5,0)$ to $(5,-3)$

Top: $f(x,0) = x^2 - 6x + 2$

$$\frac{d}{dx} f = 2x - 6 = 0$$

$$\Rightarrow x = 3$$

only critical on top line is $(3,0)$

Bottom: $f(x,-3) = x^2 - 3x + (-3)^2 - 6x + 2$

$$= x^2 - 9x + 11$$

$$\frac{df}{dx} = 2x - 9 = 0$$

$$\Rightarrow x = \frac{9}{2}$$

only critical on bottom line is $\left(\frac{9}{2}, -3\right)$

left : $f(0, y) = y^2 + 2$

$$\frac{df}{dy} = 2y = 0$$

only critical on left is $(0, 0)$

right : $f(5, y) = (5)^2 + 5y + y^2 - 6(5) + 2$

$$= y^2 + 5y + (5^2 - 6(5) + 2)$$

$$\frac{df}{dy} = 2y + 5 = 0$$

$$\Rightarrow y = -\frac{5}{2}$$

only critical point on rigth is $(5, -\frac{5}{2})$

All of the candidates for abs min / abs max:

$$C = \{(3,0), (\tfrac{9}{2}, -3), (0,0), (5, -\tfrac{5}{2})\}$$

we can evaluate those points:

$f(3,0) = 3^2 + (3)(0) + 0^2 - 6(3) + 2 = -7$

$f(\frac{9}{2}, -3) = (\frac{9}{2})^2 + (\frac{9}{2})(-3) + (-3)^2 - 6(\frac{9}{2}) + 2 = -\frac{37}{4}$

$f(0,0) = 2$

$f(5, -\frac{5}{2}) = -37/4$

$f(4, -2) = -10$

absolute max value: 2

absolute maxima: (0,0)

absolute min value: -10

absolute minima: (4,-2)