Abstract Algebra

Homework #1

Angel Cervera Roldan
21319203

Problem 1

Consider $\mathbb{F} := \{(x_1, x_2) \mid x_1, x_2 \in \mathbb{Q}\}$ with the following operations:

$$
\begin{aligned}
(x_1, x_2) + (y_1, y_2) &:= (x_1 + y_1, x_2 + y_2) \\
(x_1, x_2) \cdot (y_1, y_2) &:= (x_1 y_1 + 5x_2 y_2, x_1 y_2 + x_2 y_1)
\end{aligned}
$$

1. What is the multiplicative identity of $\mathbb{F}$
2. Given $(x_1, x_2) \neq (0, 0)$, find $(x_1, x_2)^{-1}$

Part 1

Claim: $(1, 0)$ is the multiplicative identity of $\mathbb{F}$.

To prove this, we need to show 2 things:

1. $(1, 0) \in \mathbb{F}$, we know this holds since $1, 0 \in \mathbb{Q}$.
2. $\forall (x_1, x_2) \in \mathbb{F}$, $(x_1, x_2) \cdot (1, 0) = (x_1, x_2)$, this can be shown algebraically:

$$
\begin{aligned}
(x_1, x_2) \cdot (1, 0) &= (x_1 \cdot (1) + 5x_2 \cdot (0), x_1 \cdot (0) + x_2 \cdot (1)) \\
&= (x_1, x_2)
\end{aligned}
$$

Part 2

Take $(x_1, x_2) \in \mathbb{F}$, by definition, if (a, b) is the multiplicative inverse of (x_1, x_2), then

$$(x_1, x_2) \cdot (a, b) = e_1 = (1, 0)$$

We can solve for a, b algebraically:

$$
\begin{aligned}
(x_1, x_2) \cdot (a, b) &= (1, 0) \\
(x_1, x_2) \cdot (a, b) &= (x_1 \cdot a + 5x_2 \cdot b, x_1 \cdot b + x_2 \cdot a) \\
(1, 0) &= (x_1 \cdot a + 5x_2 \cdot b, x_1 \cdot b + x_2 \cdot a)
\end{aligned}
$$

From that, we get the following two equations:

Problem 1 continued on next page. . .

$$1 = x_1 \cdot a + 5x_2 \cdot b \tag{1}$$
$$0 = x_1 \cdot b + x_2 \cdot a \tag{2}$$

From equation 2, we get

$$a = \frac{-x_1 \cdot b}{x_2} = -\frac{x_1}{x_2} \cdot b \tag{3}$$

Now, we can plug in the value for a we found in equation 3 into equation 1:

$$1 = x_1 \cdot a + 5x_2 \cdot b = x_1 \cdot (-\frac{x_1}{x_2} \cdot b) + 5x_2 \cdot b = -\frac{x_1^2}{x_2} \cdot b + 5x_2 \cdot b = b(-\frac{x_1^2}{x_2} + 5x_2)$$

If we multiply both sides of the equation by x_2,

$$x_2 = b(-x_1^2 + 5x_2^2)$$

From that, we can isolate b in terms of x_1 and x_2.

$$b = \frac{x_2}{5x_2^2 - x_1^2}$$

We can now find a by subbing the value of b we just found into equation 3:

$$a = -\frac{x_1}{x_2} \cdot b = -\frac{x_1}{x_2} \cdot \frac{x_2}{5x_2^2 - x_1^2} = -\frac{x_1}{5x_2^2 - x_1^2}$$

The inverse would be undefined when $5x_2^2 - x_1^2 = 0$, however $5x_2^2 - x_1^2$ cannot be 0 if $x_1, x_2 \in \mathbb{Q} - \{(0)\}$. We can show this by contradiction, assume $x_1, x_2 \in \mathbb{Q} - \{(0)\}$, and $5x_2^2 - x_1^2 = 0$

$$5x_2^2 - x_1^2 = 0 \implies 5x_2^2 = x_1^2 \implies \sqrt{5}x_2 = x_1$$

However, $x_1, x_2 \in \mathbb{Q}$, therefore $\sqrt{5}x_2 \notin \mathbb{Q}$, and so $5x_2^2 - x_1^2$ can not be 0.

The inverse is undefined when $5x_2^2 - x_1^2 = 0$, therefore, the inverse is only undefined at $(0, 0)$, and any other value will have an inverse.

Now, we can verify that the inverse actually works:

$$
\begin{aligned}
(x_1, x_2) \cdot (a, b) &= (x_1, x_2) \cdot (-\frac{x_1}{5x_2^2 - x_1^2}, \frac{x_2}{5x_2^2 - x_1^2}) \\
&= (x_1 \cdot (-\frac{x_1}{5x_2^2 - x_1^2}) + 5x_2 \frac{x_2}{5x_2^2 - x_1^2}, x_1 \cdot \frac{x_2}{5x_2^2 - x_1^2} + x_2 \cdot (-\frac{x_1}{5x_2^2 - x_1^2})) \\
&= (-\frac{x_1^2}{5x_2^2 - x_1^1} + \frac{5x_2^2}{5x_2^2 - x_1^2}, \frac{x_1 x_2}{5x_2^2 - x_1^2} - \frac{x_2 x_1}{5x_2^2 - x_1^2}) \\
&= (\frac{5x_2^2 - x_1^2}{5x_2^2 - x_1^1}, 0) \\
&= (1, 0)
\end{aligned}
$$

Problem 2

Let $S := \mathbb{R}^2$, with the following operations

$$\boxplus \rightarrowtail (x_1, x_2) \boxplus (y_1, y_2) = (x_1 + y_2, x_2 + y_1)$$
$$\cdot \rightarrowtail \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$$

Is $(S, \mathbb{R}, \boxplus, \cdot)$ a vector space?

It is not a vector space, as $\boxplus$ is not commutative, we can show this by counter example:

Take $(1, 3), (3, 4) \in S$. If S is a vector space, then: $(1, 3) \boxplus (3, 4) = (3, 4) \boxplus (1, 3)$, however:

$$(1, 3) \boxplus (3, 4) = (1 + 4, 3 + 3) = (5, 6)$$

$$(3, 4) \boxplus (1, 3) = (3 + 3, 4 + 1) = (6, 5)$$

Therefore, $(1, 3) \boxplus (3, 4) \neq (3, 4) \boxplus (1, 3)$, and so commutativity does not hold.

Problem 3

Which of the following are subspaces of $\mathbb{R}^3$
- $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 = 0\right\}$
- $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2^2 = 0\right\}$
- $S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 > 0\right\}$

Part 1

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 = 0\right\}$$

This is a subspace of $\mathbb{R}^3$

Take $(x_1, x_2, x_3), (y_1, y_2, y_3) \in S$, and $\lambda \in \mathbb{R}$, we need to show that $(x_1, x_2, x_3) + (y_1, y_2, y_3) \in S$, and that $\lambda(x_1, x_2, x_3) \in S$.

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 2y_1 - y_2 &= 0 \\ 2x_1 - x_2 + 2y_1 - y_2 &= 0 \\ 2(x_1 + y_1) - (x_2 + y_2) &= 0 \end{aligned}$$

Therefore, $(x_1 + y_1, x_2 + y_2, x_3 + y_3) \in S$

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ \lambda \cdot (2x_1 - x_2) &= 0 \\ 2(\lambda x_1) - (\lambda x_2) &= 0 \end{aligned}$$

Therefore, $(\lambda x_1, \lambda x_2, \lambda x_3) \in S$

This shows that S is a subspace of $\mathbb{R}^3$.

Part 2

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2^2 = 0\right\}$$

This is not a subspace.

Problem 3 continued on next page. . .

Take $(x_1, x_2, x_3) \in S,$ and $\lambda \in \mathbb{R}$, then, if S is a subspace, $\lambda(x_1, x_2, x_3) \in S$, however, we can find a counter example.

Take $(2, 2, 0) \in \mathbb{R}^3$, this is also an element of S since $2(2) - 2^2 = 4 - 4 = 0$. Take $\lambda = 2$.

Then, if S is a subspace $2 \cdot (2, 2, 0) = (4, 4, 0) \in S$. However, $(4, 4, 0) \notin S$ because $2(4) - 4^2 = 8 - 16 = -8 \neq 0$.

Part 3

$$S = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid 2x_1 - x_2 > 0\right\}$$

This is not a subspace, as is it not closed under multiplication. To prove it, we can assume that S is a subspace, and find a counter example.

Take $(x_1, x_2, x_3) \in S$, we assumed that S is a subspace, therefore, $\forall \lambda \in \mathbb{R} \quad \lambda(x_1, x_2, x_3) \in S$.

Take $\lambda = -1$, then, $-1(x_1, x_2, x_3) = (-x_1, -x_2, -x_3) \in S$, however, if $2x_1 - x_2 > 0$, then

$2(-x_1) - (-x_2) \not> 0$, and so $(-x_1, -x_2, -x_3) \notin S$.

Problem 4

Show that if V is a finite-dimensional vector space and U is a subspace of U, then it also is finite dimensional.

We know that V is finite dimensional. Let $n = dim(V)$. From definition 6.4, we have that any list of n elements in V that is linearly dependent is a basis for V.

We can prove that U is finite dimensional by contradiction. To do this, let's first assume that U is not finite dimensional. This means that there doesn't exist a finite list of linearly independent elements of U that is a basis for U.

Take the list $u_1 \in U$, this is a linearly dependent list. Because U is not finite dimensional, and so u_1 cannot be a basis for it, there must exist an element $u_2 \in U$ st $u_2 \notin \langle u_1 \rangle$.

Now add u_2 to the list, $u_1, u_2 \in U$ must still be linearly independent, since $u_2 \notin \langle u_1 \rangle$. We can repeat the above logic, because U is not finite dimensional, then u_1, u_2 cannot be a basis for it, and there must exist $u_3 \in U$ st $u_3 \notin \langle u_1, u_2 \rangle$ ....

Repeat that process until we have a list $u_1, ..., u_n \in U$ that is linearly independent. Here we will find the contradiction:

From definition 6.4 as stated above, $u_1, ..., u_n \in U$ must be a basis for V since it is linearly independent, and every element is in V (because $U \subseteq V$). This means that $\forall u \in U, u \in \langle u_1, ..., u_2 \rangle$.

However, because U is not finite dimensional, there must exist $u_{n+1} \in U$ st $u_{n+1} \notin \langle u_1, ..., u_2 \rangle$.

This is a contradiction, meaning that u_{n+1} cannot exist, and so U cannot be finite dimensional, and can have at most a dimension of n.

Problem 5

Given that T, U, W are subspaces of some vector space V, prove that the following is false:

$$\begin{aligned} dim(T+U+W) =& dim(T) + dim(U) + dim(W) \\ & - dim(T \cap U) - dim(U \cap W) \\ & - dim(W \cap T) + dim(T \cap U \cap W) \end{aligned}$$

We can prove that this is false by counter example, take $V = \mathbb{R}^3$, and

$$\begin{aligned} T &:= \left\{(x,0,0) \in \mathbb{R}^3\right\} = \langle(1,0,0)\rangle \\ U &:= \left\{(0,x,0) \in \mathbb{R}^3\right\} = \langle(0,1,0)\rangle \\ W &:= \left\{(x,x,0) \in \mathbb{R}^3\right\} = \langle(1,1,0)\rangle \end{aligned}$$

Since T, U, and W all have a basis of just one element, $dim(T) = 1$, $dim(U) = 1$, $dim(W) = 1$.

$T \cap U = \{(0,0,0)\}$, therefore, $dim(T \cap U) = 0$.

$U \cap W = \{(0,0,0)\}$, therefore, $dim(U \cap W) = 0$.

$W \cap T = \{(0,0,0)\}$, therefore, $dim(W \cap T) = 0$.

$T \cap U \cap W = \{(0,0,0)\}$, therefore, $dim(T \cap U \cap W) = 0$

By the equation given, we would expect $dim(T+U+W) = 1+1+1-0-0-0+0 = 3$

However,

$$T + U + W = \{t + u + w \mid t \in T, \text{ and } u \in U, \text{ and } w \in W\}$$

or

$$\left\{(x,y,0) \in \mathbb{R}^3\right\}$$

Therefore, $T + U + W = \langle(1,0,0),(0,1,0)\rangle$, meaning that $dim(T+U+W) = 2$.

Given that the given equation gave us the dimension to be 3, but the actual dimension is 2, this shows that the equation given is false.