MT242P - TUTORIAL 3

Tutorial time: Wed 10am (MS1), Week 8.

Question 1: Consider the linear map $T \in L(\mathbb{R}^3)$ given by

$$T(x, y, z) := (x - 2y + z, y - z, x - z).$$

(1) Write the matrix A of T with respect to the standard basis of $\mathbb{R}^3$ ($\mathcal{B} = \{e_1, e_2, e_3\}$).
(2) Write the matrix B of T with respect to the basis $\mathcal{B}' := \{w_1, w_2, w_3\}$, where

$$w_1 := \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad w_2 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad w_3 := \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

(3) Find a matrix P such that $B = P^{-1}AP$.
(4) Find a basis for $\ker T$.
(5) Find a basis for $T(V)$.

Question 2: Let

$$A := \begin{pmatrix} -3 & 1 \\ 3 & -2 \end{pmatrix}$$

and consider the linear map $T \in L(\mathrm{Sym}_2(\mathbb{R}))$ given by

$$T(M) := A^\top M + MA.$$

(1) Write the matrix of T with respect to the basis of $\mathrm{Sym}_2(\mathbb{R})$ given by

$$E_{11} := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_{12} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad E_{22} := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

(2) Is T invertible?
(3) Find a matrix $M \in \mathrm{Sym}_2(\mathbb{R})$ such that $T(M) = -I$.

Question 3: Give an example of a vector space V and a linear map $T \in L(V)$ such that $\ker T \cap T(V) \neq \{0\}$.

Question 4: Show that for any vector space V and linear map $T \in L(V)$ the sets $\ker T$ and $T(V)$ are subspaces of V.

Question 5: Let V be a finite-dimensional vector space (over some field $\mathbb{F}$) and let $\{v_1, \dots, v_n\}$ be a basis of V. Define the linear map $T \in L(V)$ given by

$$\begin{aligned} &T(v_n) = 0 \\ &T(v_j) = v_{j+1} \qquad \text{for any } j = 1, \dots, n-1. \end{aligned}$$

(1) Find a basis of $T(V)$.
(2) Find the kernel of T.