Abstract Algebra

Notes - Year 1, Semester 2

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Contents

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Definition

Take V, W to be vector spaces over $\mathbb{F}$. Take T, S

$$T : V \to W$$
$$S : V \to W$$

And define $T + S$ to be:

$$(T + S)(v) = T(v) + S(v)$$

and, for any $\lambda \in \mathbb{F}$

$$(\lambda T)(v) = \lambda T(v)$$

Definition

Show that T + S and lambda T are linear maps V to W

Let's call

$$L(V, W) = \{T : V \to W \text{ such that } \text{ T is linear } \}$$

Then, $(L(V, W), \mathbb{F}, +, \cdot)$ is a vector space.

Another operation:

$$T : V \to W$$
$$S : W \to U$$

We can define the composition of T and S to be $(T \circ S)(v) = S(T(v))$.

Claim: S, T being linear $\implies$ $S \circ T$ is linear

$$\begin{aligned} S(T(\lambda v + \mu w)) &= S(\lambda T(v) + \mu T(w)) \\ &= \lambda S(T(v)) + \mu S(T(w)) \\ &= \lambda (S \circ T)(v) + \mu (S \circ T)(w) \end{aligned}$$

If $V = W$, we write $L(V, V) = L(V)$

Properties of composition

1. $R \circ (S \circ T) = (R \circ S) \circ T$
2. $R \circ (S + T) = R \circ S + R \circ T$
3. $S \circ (\lambda \cdot T) = \lambda \cdot (S \circ T) = (\lambda \cdot S) \circ T$

Side note: All of these properties make $L(v)$ an **algebra**

Invertible linear maps

Definition Identity Map

The identity map I is defined by

$$I(v) = v$$

Where $I \in L(V)$

We can see that $\forall T \in L(V), T \circ I = T = I \circ T$

Definition

A linear map $T \in L(V)$ is invertible iff $\exists S \in L(V)$ such that:

$$S \circ T = I = T \circ S$$

Theorem The right inverse and the left inverse are always the same

Suppose that $R, S \in L(v)$, if $ST = I$, and $TR = I$, then $S = R$

To prove this, we can see the following:

$$\begin{aligned}
(ST)R &= STR \\
&= S(TR) \\
&= SI \\
&= S \\
(ST)R &= IR \\
&= R
\end{aligned}$$

Therefore, $S = R$

This also leads us to believe that if T is inversible, then its inverse is unique. To show this, we can assume that there exist two different inverses of T, S_1, S_2.

Then,

$$\begin{aligned} S_1T &= I = TS_1 \\ S_2T &= I = TS_2 \\ S_1T &= I = TS_2 \\ \implies S_1 &= S_2 \end{aligned}$$

Since the inverse is unique, we can denote it by T^{-1}.

Lemma

V, W are vector spaces over $\mathbb{F}$, $T \in L(V, W)$

$$T \text{ is injective } \iff (T(v) = 0 \implies v = 0)$$

To prove this, assume that T is injective. Then, if $v \neq 0$,

$$T(v) \neq T(0) = 0$$

Assume that $T(v) = 0 \Leftrightarrow v = 0$. Suppose that there are $v, u \in V$ such that

$$T(v) = T(u)$$

By linearity, we have

$$0 = T(v) - T(u) = T(v - u) \implies v - u = 0 \text{ meaning that } v = u$$

Lemma

$T \in L(V)$ is injective iff

1. T is injective
2. T is surjective

To prove this, we can assume that T s a bijection. Then we define

$$T^{-1}(w) = \text{ the unique } v \in V \text{ st } T(v) = w$$

We need to show that T^1 is linear.

$$T^{-1}(\lambda_1 w_1 + \lambda_2 w_2) = u \text{ such that } T(u) = \lambda_1 w_1 + \lambda_2 w_2$$

$$u = \lambda_1 w_1 + \lambda_2 w_2$$

Now we need to prove it the other way around.