MT242P ASSIGNMENT 2

Question 1: Find bases for the following subspaces of $\mathbb{R}^4$:

$$\begin{aligned} S_1 := \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 - 2x_2 + x_4 = 0,& \\ x_3 + x_4 = 0\},& \end{aligned}$$

$$S_2 := \operatorname{span}\left(\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}5\\6\\5\\-6\end{pmatrix}, \begin{pmatrix}1\\2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\-4\\3\\4\end{pmatrix}\right\}\right).$$

Finally, determine $\dim(S_1 \cap S_2)$.

Question 2: Let $\mathbb{R}_n[X]$ denote the vector space of polynomials in one variable, with real coefficients and of degree at most n. Show that

$$S := \{P \in \mathbb{R}_n[X] : P(1) = 0\}$$

is a subspace of $\mathbb{R}_n[X]$, find a basis for it and determine its dimension.

Question 3: Recall that

$$\mathrm{Sym}_n(\mathbb{R}) = \{A = (a_{ij})_{ij} \in M_{n\times n}(\mathbb{R}) : a_{ij} = a_{ji} \text{ for all } i, j = 1, \ldots, n\}$$

is the subspace of $M_{n\times n}(\mathbb{R})$ matrices that are symmetric; define analogously the subspace of skew-symmetric matrices,

$$\mathrm{Skew}_n(\mathbb{R}) = \{A = (a_{ij})_{ij} \in M_{n\times n}(\mathbb{R}) : a_{ij} = -a_{ji} \text{ for all } i, j = 1, \ldots, n\}.$$

Show that

$$M_{n\times n}(\mathbb{R}) = \mathrm{Sym}_n(\mathbb{R}) \oplus \mathrm{Skew}_n(\mathbb{R}).$$

Question 4: Establish which of the following is a linear functional (justify your answers):

(1) on vector space $\mathbb{R}[X]$ over $\mathbb{R}$,

$$f(P) := 3P(0) + 5P(1);$$

(2) on vector space $\mathbb{R}^3$ over $\mathbb{R}$,

$$f(x_1, x_2, x_3) := x_1x_2 + x_2x_3 + x_3x_1;$$

(3) on vector space $\mathbb{R}^3$ over $\mathbb{R}$,

$$f(x_1, x_2, x_3) := |x_1| + |x_2| + |x_3|;$$

(4) on vector space $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$,

$$f(x_1, x_2, x_3) := x_1^5 + x_2^5 + x_3^5.$$

Question 5: Let us write $\mathbb{R}_{\mathbb{Q}}$ to denote $\mathbb{R}$ as a vector space over the field $\mathbb{Q}$. Let $\mathcal{P}$ denote the set of prime numbers ($\mathcal{P} = \{2, 3, 5, 7, 11, \ldots\}$). Show that the set

$$S := \{\log p : p \in \mathcal{P}\}$$

is $\mathbb{Q}$-linearly independent in $\mathbb{R}_{\mathbb{Q}}$. Deduce that $\mathbb{R}_{\mathbb{Q}}$ is not finite-dimensional.
[hint: reduce to linear combinations with coefficients in $\mathbb{Z}$, then use the properties of logarithm...]

Question 6: Let V be a vector space over field $\mathbb{F}$. Suppose that $f, g \in V^*$ are such that $f(x) = 0$ whenever $g(x) = 0$. Show that there exists a $\lambda \in \mathbb{F}$ such that

$$f = \lambda g.$$

[hint: if $g(x_0) \neq 0$, consider elements $y = x - \alpha x_0$ for α chosen so that $g(y) = 0$...]