Integration

Homework #2

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Problem 1

Find the tangent line for

1. $f(x) = \frac{8}{\sqrt{x-2}}$ at $x = 6$
2. $f(x) = 4 + \cot x - 2\csc x$ at $x = \frac{\pi}{2}$

Part 1

First we will find the rate of change of f at $x = 6$

To solve the derivate, we will let $u = \sqrt{x-2}$, then:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx} = (\frac{d}{du}\frac{8}{u})(\frac{d}{dx}\sqrt{x-2})$$

$$\frac{d}{du}\frac{8}{u} = -\frac{8}{u^2} = -\frac{8}{\sqrt{x-2}^2} = -\frac{8}{x-2}$$

Now, all we have left to solve is

$$\frac{d}{dx}\sqrt{x-2}$$

To do this, we will let $h = x - 2$, so $u(h) = \sqrt{h}$

Now, by the chain rule, we have

$$\frac{du}{dx} = \frac{du}{dh}\frac{dh}{dx} = \frac{1}{2\sqrt{h}} \cdot 1 = \frac{1}{2\sqrt{h}} = \frac{1}{2\sqrt{x-2}}$$

So, when putting it all together, we get

$$\frac{df}{dx} = (-\frac{8}{x-2}) \cdot \frac{1}{2\sqrt{x-2}} = -\frac{4}{(x-2)\sqrt{x-2}}$$

To find the slope of the tangent at $x = 6$, we can evaluate the derivative we just calculated

$$\frac{df(6)}{dx} = -\frac{4}{(6-2)\sqrt{6-2}} = -\frac{4}{4\sqrt{4}} = -\frac{1}{2}$$

The tangent line will be in the form:

Problem 1 continued on next page. . .

$$t(x) = mx + c = -\frac{1}{2}x + c$$

To find the value of c, we use the fact that $f(6) = t(6)$

$$f(6) = \frac{8}{\sqrt{6-2}} = 4 = -\frac{1}{2}(6) + c \implies c = 7$$

Therefore, the line tangent to $f(x)$ at $x = 6$ is given by $t(x) = -\frac{1}{2}x + 7$

Problem 1 continued on next page. . .

Part 2

First we will find the rate of change of f at $x = \frac{\pi}{2}$

$$\frac{df}{dx} = \frac{d}{dx}(4 + \cot(x) - 2\csc(x)) = 0 - \csc^2 x + 2\csc x \cot x$$

We can simplify the equation by replacing $\csc x$ by $\frac{1}{\sin x}$ and $\cot x$ by $\frac{\cos x}{\sin x}$

$$\frac{df}{dx} = -\frac{1}{\sin^2 x} + 2\frac{1}{\sin x}\frac{\cos x}{\sin x} = \frac{-1 + 2\cos x}{\sin^2 x}$$

And if we evaluate the derivative at $x = \frac{\pi}{2}$, we get $\frac{df}{dx}(\frac{\pi}{2}) = -1$, this will be the slope of the tangent line $t(x)$.

$$t(x) = -1x + c$$

We can use the fact that $t(x) = f(x)$ at $x = \frac{\pi}{2}$ to find the value of c

$$\begin{aligned} t(\frac{\pi}{2}) &= -\frac{\pi}{2} + c \\ f(\frac{\pi}{2}) &= 4 + \cot\frac{\pi}{2} - 2\csc\frac{\pi}{2} = 2 \\ 2 &= -\frac{\pi}{2} + c \\ c &= 2 + \frac{\pi}{2} \end{aligned}$$

And so the equation of the tangent line of f at $x = \frac{\pi}{2}$ is given by $t(x) = -x + 2 + \frac{\pi}{2}$

Problem 2

Find $\frac{dy}{dx}$ for the following functions by stating how you do substitutions:
- $y = e^{2\cos(\pi x - 1)}$
- $y = (x^{-\frac{3}{4}} + x\sin(x))^{\frac{4}{3}}$

Part 1

To find the derivative of $y = e^{2\cos(\pi x-1)}$, we let $u = \cos(\pi x - 1)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Where

$$\frac{dy}{du} = \frac{d}{du}e^{2u} = 2e^{2u}$$

and

$$\frac{du}{dx} = \frac{d}{dx}\cos(\pi x - 1)$$

to solve this derivative, we will once again use substitution. Let $h = \pi x - 1$

$$\frac{du}{dx} = \frac{du}{dh}\frac{dh}{dx}$$

$$\frac{dh}{dx} = \frac{d}{dx}\pi x - 1 = \pi$$

$$\frac{du}{dh} = \frac{d}{dh}\cos(h) = -\sin(h) = -\sin(\pi x - 1)$$

Putting it all together, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{du}{dh}\frac{dh}{dx} = 2e^{2\cos(\pi x-1)} \cdot (-\sin(\pi x - 1)) \cdot (\pi)$$

Or in a slightly neater way

$$\frac{dy}{dx} = -2\pi e^{2\cos(\pi x-1)}\sin(\pi x - 1)$$

Problem 2 continued on next page...

Part 2

To find the derivative of $y = (x^{-\frac{3}{4}} + x\sin(x))^{\frac{4}{3}}$, we let $u = x^{-\frac{3}{4}} + x\sin(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

Where

$$\frac{dy}{du} = \frac{d}{du}u^{\frac{4}{3}} = \frac{4}{3}u^{\frac{1}{3}} = \frac{4}{3}(x^{-\frac{3}{4}} + x\sin(x))^{\frac{1}{3}}$$

and

$$\begin{aligned}\frac{du}{dx} &= \frac{d}{dx}x^{-\frac{3}{4}} + x\sin(x)\\ &= \frac{d}{dx}x^{-\frac{3}{4}} + \frac{d}{dx}x\sin(x)\\ &= -\frac{3}{4}x^{-\frac{7}{4}} + \sin(x) + x\cos(x)\end{aligned}$$

And so when we put it all together, we get

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\frac{4}{3}(x^{-\frac{3}{4}} + x\sin(x))^{\frac{1}{3}})(-\frac{3}{4}x^{-\frac{7}{4}} + \sin(x) + x\cos(x))$$

Problem 3

Find $\frac{d^2y}{dx^2}$ for $y^2 = e^{x^2} + 2x$.

$$
\begin{align}
y^2 &= e^{x^2} + 2x \tag{1}\\
2yy' &= 2xe^{x^2} + 2 \tag{2}\\
y' &= \frac{2xe^{x^2} + 2}{2y} = \frac{xe^{x^2} + 1}{y} \tag{3}\\
y'' &= \frac{d}{dx}y' = \frac{d}{dx}\frac{xe^{x^2} + 1}{y} \tag{4}\\
&= \frac{y \cdot \frac{d}{dx}(xe^{x^2} + 1) - (xe^{x^2} + 1) \cdot \frac{d}{dx}y}{y^2} \tag{5}\\
&= \frac{y \cdot e^{x^2}(2x^2 + 1) - (xe^{x^2} + 1) \cdot y'}{y^2} \tag{6}\\
&= \frac{y \cdot e^{x^2}(2x^2 + 1) - (xe^{x^2} + 1) \cdot \frac{xe^{x^2}+1}{y}}{y^2} \tag{7}\\
&= \frac{y^2 \cdot e^{x^2}(2x^2 + 1) - (xe^{x^2} + 1)^2}{y^3} \tag{8}
\end{align}
$$

To go from step 5 to step 6, we need the following:

$$
\begin{aligned}
\frac{d}{dx}xe^{x^2} + 1 &= \frac{d}{dx}xe^{x^2}\\
&= x\frac{d}{dx}e^{x^2} - e^{x^2}\frac{d}{dx}x\\
&= 2x^2e^{x^2} - e^{x^2}\\
&= e^{x^2}(2x^2 - 1)
\end{aligned}
$$

Problem 4

Identify the extreme points of the function $f(x) = \frac{x^4}{4} - 2x^2 + 4$, find where the curve is increasing and decreasing, and sketch a rough graph for $f(x)$.

The extreme points of a function are the points where its derivative is equal to 0. Therefore, we must start by finding $\frac{df}{dx}$

$$\frac{df}{dx} = \frac{d}{dx}\frac{x^4}{4} - 2\frac{d}{dx}x^2 + \frac{d}{dx}4 = x^3 - 4x$$

$$x^3 - 4x = 0 \implies x^3 = 4x$$

This means that $x = 0$ or $x^2 = 2$, therefore the possible values for x are 0, 2, and -2.

When $x < -2$, $\frac{dy}{dx} < 0$, we can show this algebraically

$$x^3 - 4x = (x-2)x(x+2)$$

And so, if $x < -2$, then

$$(x-2) < x < (x+2) < 0$$

And therefore, the derivative must be negative, since we will have $x - 2$, which is negative, times x, which is negative, times $x + 2$ which is also negative. This means that the curve is decreasing.

When $-2 < x < 0$, $\frac{dy}{dx} > 0$. Using similar logic, we have that $x - 2, x < 0$, however, $x + 2 > 0$. Therefore, the derivative will in this case be a negative, times a negative, times a positive, which is a positive. This means that the curve is increasing.

When $0 < x < 2$, $\frac{dy}{dx} < 0$. In this case, we have that $x - 2 < 0$, and $x, x + 2 > 0$, therefore, the derivate in that interval will be a negative, times a positive, times a positive, which is a negative number. This means that the curve is decreasing.

When $x > 2$, we have $x - 2, x, x + 2 > 0$, therefore, the derivative in this interval will be a positive, times a positive, times a positive, which is always a positive. This means that the curve is increasing.

Sketch of function $f(x) = \frac{x^4}{4} - 2x^2 + 4$