Multivariable Calculus

Homework #2

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Problem 1

Are the following true, or false

1. There is no subset $A \subseteq \mathbb{R}^2$ such that $Bdy(A)$ contains exactly four points.
2. There is a subset $C \subseteq \mathbb{R}^3$ such that C is not the empty set and $Int(C)$ is the empty set.

Part 1

This statement is false. We can prove this by counter example.

Take $A = \{(0,0), (1,1), (2,2), (3,3)\}$. A has only 4 elements, an open ball with a center at any of those elements will contain at least one point inside it (the center), as well as points outside it, meaning that they are boundary points.

Part 2

This statement is true, an example would be $C = \{(1,1,1)\}$. This set is not empty, and any open ball with center $(1,1,1)$ will contain points outside C, meaning that $(1,1,1)$ is not an interior point.

Therefore, there exists $C \subseteq \mathbb{R}^3$ where $Int(C) = \emptyset$

Problem 2

Suppose the directional derivative of $g(x, y)$ at $(1, 2)$ in the direction of $\vec{i} + \vec{j}$ is $2\sqrt{2}$ and the directional derivative of $g(x, y)$ at $(1, 2)$ in the direction of $-2\vec{j}$ is -3. Find the directional derivative of $g(x, y)$ at $(1, 2)$ in the direction of $-\vec{i} - 2\vec{j}$

$$\nabla g_{|(1,2)} = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j}$$

To make the solution clearer, let $a = \frac{\partial g}{\partial x}$, and $b = \frac{\partial g}{\partial y}$

First, let's find the unit vectors in the directions given:

$$u = \frac{\vec{i} + \vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$$

$$w = \frac{0\vec{i} - 2\vec{j}}{\sqrt{(-2)^2}} = 0\vec{i} - 1\vec{j}$$

$$z = \frac{-1\vec{i} - 2\vec{j}}{\sqrt{(-1)^2 + (-2)^2}} = -\frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}$$

We know that the gradient in the direction of w is given by:

$$(D_w g)_{|(1,2)} = -3$$

Therefore

$$-3 = (a\vec{i} + b\vec{j}) \cdot (0\vec{i} - 1\vec{j}) = -b \implies 3 = b$$

We also know that the gradient in the direction of u is given by:

$$(D_u g)_{|(1,2)} = 2\sqrt{2}$$

Therefore

$$2\sqrt{2} = (a\vec{i} + b\vec{j}) \cdot (\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}) = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = \frac{a}{\sqrt{2}} + \frac{3}{\sqrt{2}}$$

Now, we can solve for a

Problem 2 continued on next page. . .

$$\begin{aligned} 2\sqrt{2} &= \frac{a}{\sqrt{2}} + \frac{3}{\sqrt{2}} \\ 2 \cdot 2 &= a + 3 \\ 4 - 3 &= a = 1 \end{aligned}$$

And so now we now that

$$\nabla g_{|(1,2)} = 1\vec{i} + 3\vec{j}$$

With this, we can now find the directional derivative of g at $(1, 2)$ in the direction of $-\vec{i} - 2\vec{j}$

$$(D_z g)_{|(1,2)} = \nabla g_{|(1,2)} \cdot \underline{w} = (\vec{i} + 3\vec{j}) \cdot (-\frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}) = -\frac{1}{\sqrt{5}} - \frac{3 \cdot 2}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

Problem 3

Part 1

Suppose that $h(x) = \cos\left(y + x^2\right) + \sin\left(x - 2y^2\right)$, find the following:

$$\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial^2 h}{\partial^2 x}, \frac{\partial^2 h}{\partial^2 y}, \frac{\partial^2 h}{\partial x \partial y}$$

For these differentials, I will use the following rules:

$$\frac{\partial}{\partial u} \cos(f(u)) = -f'(u)\sin(f(u)), \frac{\partial}{\partial u} \sin(f(u)) = f'(u)\cos(f(u))$$
$$\frac{\partial}{\partial u} f(u)g(u) = f'(u)g(u) + f(u)g'(u)$$

And also

$$\frac{\partial}{\partial x} y + x^2 = 2x \quad \frac{\partial}{\partial x} x - 2y^2 = 1$$

$$\frac{\partial}{\partial y} y + x^2 = 1 \quad \frac{\partial}{\partial y} x - 2y^2 = -4y$$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \cos\left(y + x^2\right) + \frac{\partial}{\partial x} \sin\left(x - 2y^2\right) = -2x \sin\left(y + x^2\right) + \cos\left(x - 2y^2\right)$$

$$\begin{aligned}\frac{\partial^2 h}{\partial^2 x} &= \frac{\partial}{\partial x} \frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(-2x \sin\left(y + x^2\right) + \cos\left(x - 2y^2\right)) \\ &= -2 \sin\left(y + x^2\right) - 4x^2 cos(y + x^2) - \sin\left(x - 2y^2\right)\end{aligned}$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \cos\left(y + x^2\right) + \frac{\partial}{\partial y} \sin\left(x - 2y^2\right) = -\sin\left(y + x^2\right) - 4y \cos\left(x - 2y^2\right)$$

Problem 3 continued on next page. . .

$$\frac{\partial^2 h}{\partial^2 y} = \frac{\partial}{\partial y}\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(-\sin\left(y + x^2\right) - 4y\cos\left(x - 2y^2\right))$$
$$= -\cos\left(y + x^2\right) - 4\cos\left(x - 2y^2\right) - 16y^2 sin(x - 2y^2)$$

$$\frac{\partial^2 h}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial h}{\partial y} = \frac{\partial}{\partial x}(-\sin\left(y + x^2\right) - 4y\cos\left(x - 2y^2\right))$$
$$= -2x\cos\left(y + x^2\right) - 4y\cos\left(x - 2y^2\right)$$

Problem 3 continued on next page. . .

Part 2

Suppose that $z = 2u - 3w^2$, $u = e^{2r+3s}$, $w = s - r^2$ find the following:

$$\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$$

For these differentials, I will use the chain rule as follows:

$$\frac{\partial}{\partial a} e^{f(a,b)} = (\frac{\partial}{\partial a} f(a,b)) \cdot e^{f(a,b)}$$

$$\frac{\partial}{\partial a} (f(a,b))^2 = 2(f(a,b)) \cdot (\frac{\partial}{\partial a} f(a,b))$$

First, we will find the derivatives of u, and w^2:

$$\frac{\partial}{\partial r} e^{2r+3s} = 2e^{2r+3s}$$
$$\frac{\partial}{\partial s} e^{2r+3s} = 3e^{2r+3s}$$
$$\frac{\partial}{\partial r} (s - r^2)^2 = 2(s - r^2)(-2r) = -4r(s - r^2)$$
$$\frac{\partial}{\partial s} (s - r^2)^2 = 2(s - r^2)(1) = 2(s - r^2)$$

We can use the above derivatives to solve the question

$$\begin{aligned}\frac{\partial z}{\partial r} &= 2\frac{\partial u}{\partial r} - 3\frac{\partial w^2}{\partial r} = 2\frac{\partial}{\partial r} e^{2r+3s} - 3\frac{\partial}{\partial r}(s - r^2)^2 \\ &= 2(2e^{2r+3s}) - 3(-4r(s - r^2)) \\ &= 4e^{2r+3s} + 12r(s - r^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial z}{\partial s} &= 2\frac{\partial u}{\partial s} - 3\frac{\partial w^2}{\partial s} = 2\frac{\partial}{\partial s} e^{2r+3s} - 3\frac{\partial}{\partial s}(s - r^2)^2 \\ &= 2(3e^{2r+3s}) - 3(2(s - r^2)) = 6e^{2r+3s} - 6(s - r^2) \\ &= 6(e^{2r+3s} - s + r^2)\end{aligned}$$

Problem 4

Suppose $f(x, y) = y^2 - \cos(y + x)$. Is f differentiable at every point in $\mathbb{R}^2$? Justify your answer.

We say that $f(x, y)$ is differentiable a point (a, b) if $\frac{\partial f}{\partial x}_{|a,b}$ and $\frac{\partial f}{\partial y}_{|a,b}$ exist.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial x} \cos(y + x) = \sin(y + x)$$

Since $x, y \in \mathbb{R}$, $x + y \in \mathbb{R}$, this means that f is differentiable at any point since $\sin(n)$ is defined for all $n \in \mathbb{R}$.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} y^2 - \frac{\partial}{\partial y} \cos(y + x) = 2y + \sin(y + x)$$

This derivative is also differentiable at any point since $2y$ is defined for all $y \in \mathbb{R}$, and $sin(y + x)$ is also defined for all $x, y \in \mathbb{R}$

Because both $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are differentiable at every $(x, y) \in \mathbb{R}^2$, we have shown that f is differentiable at every point in $\mathbb{R}^2$

Problem 5

Part 1

Find the directional derivative of $g(x, y, z) = x^3y^2 - e^z \sin(yx)$ at $(1, \frac{\pi}{2}, 0)$ in the direction of $\vec{i} - 2\vec{j} + 3\vec{k}$.

Firstly, we will find the unit vector u in the direction of $\vec{i} - 2\vec{j} + 3\vec{k}$. This is given by

$$u = \frac{\vec{i} - 2\vec{j} + 3\vec{k}}{\sqrt{1^2 + (-2)^2 + 3^2}} = \frac{1}{\sqrt{14}}\vec{i} - \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}$$

The directional derivative of g in the direction of $\vec{i} - 2\vec{j} + 3\vec{k}$ at $(1, \frac{\pi}{2}, 0)$ is given by

$$(D_{\underline{u}}f)_{|(1,\frac{\pi}{2},0)} = \nabla g_{|(1,\frac{\pi}{2},0)} \cdot \underline{u}$$

Where

$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k}$$

$$\frac{\partial g}{\partial x} = 3x^2y^2 - e^z y\cos(yx)$$

$$\frac{\partial g}{\partial y} = 2x^3y - e^z x\cos(yx)$$

$$\frac{\partial g}{\partial z} = -e^z \cos(yx)$$

$$\begin{aligned}
(D_{\underline{u}}f)_{|(1,\frac{\pi}{2},0)} &= \nabla g_{|(1,\frac{\pi}{2},0)} \cdot \underline{u} \\
&= [(3x^2y^2 - e^z y\cos(yx))\vec{i} + (2x^3y - e^z x\cos(yx))\vec{j} + -e^z\cos(yx)\vec{k}]_{|(1,\frac{\pi}{2},0)} \cdot \underline{u} \\
&= (\frac{3\pi^2}{4}\vec{i} + \pi\vec{j} - 1\vec{k}) \cdot (\frac{1}{\sqrt{14}}\vec{i} - \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}) \\
&= \frac{\sqrt{14}(3\pi^2 - 8\pi - 12)}{56}
\end{aligned}$$

Problem 5 continued on next page. . .

Part 2

Find the directions in which $g(x, y) = 2y^2x - 4e^{yx}\sin x$ increases and decreases most rapidly at $(0, 1)$. Also, at what rate does g change in these directions?

It will most rapidly increase when the angle between the gradient of g and $\underline{u}$ is 0, and it will most rapidly decrease when it is π.

$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial x}\vec{j}$$

$$\begin{aligned}
\frac{\partial g}{\partial x} &= \frac{\partial}{\partial x}2y^2x - 4\frac{\partial}{\partial x}e^{yx}\sin x \\
&= 2y^2 - 4\frac{\partial}{\partial x}e^{yx}\sin x \\
&= 2y^2 - 4((\frac{\partial}{\partial x}e^{yx})\sin x + e^{yx}(\frac{\partial}{\partial x}\sin x)) \\
&= 2y^2 - 4(ye^{yx}\sin x + e^{yx}\cos x)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g}{\partial y} &= \frac{\partial}{\partial y}2y^2x - 4\frac{\partial}{\partial y}e^{yx}\sin x \\
&= 4yx - 4\sin x\frac{\partial}{\partial y}e^{yx} \\
&= 4yx - 4\sin(x)xe^{yx}
\end{aligned}$$

Therefore,

$$\nabla g = (2y^2 - 4(ye^{yx}\sin x + e^{yx}\cos x))\vec{i} + (4yx - 4\sin(x)xe^{yx})\vec{j}$$

Now, we can evaluate the gradient at $(0, 1)$

$$\nabla g_{|(0,1)} = (2(1)^2 - 4((1)e^0 sin(0) + e^0 cos(0)))\vec{i} + (4(1)(0) - 4\sin(0)(0)e^0)\vec{j} = -2\vec{i} + 0\vec{j}$$

The function will increase the most rapidly when $\underline{u}_{inc}$ is in the direction of the gradient evaluated at $(0, 1)$, meaning that $\underline{u}_{inc}$ is in the direction of $-2\vec{i} + 0\vec{j}$.

Because $\underline{u}_{inc}$ is a unit vector, $\underline{u}_{inc} = \frac{-2\vec{i}+0\vec{j}}{2} = -1\vec{i} + 0\vec{j}$

Problem 5 continued on next page. . .

The change in this direction can be calculated by

$$(D_{u_{inc}} g_{|(0,1)}) = \nabla g_{|(0,1)} \cdot \underline{u}_{inc} = (-2\vec{i} + 0\vec{j}) \cdot (-1\vec{i} + 0\vec{j}) = 2$$

Similarly, it will decrease the most rapid when $\underline{u}_{dec}$ is in the direction of $-\nabla g_{|(0,1)}$. In this case $\underline{u}_{dec} = -\frac{-2\vec{i}+0\vec{j}}{2} = 1\vec{i} + 0\vec{j}$

The change in this direction can be calculated by

$$(D_{\underline{u}_{dec}} g_{|(0,1)}) = \nabla g_{|(0,1)} \cdot \underline{u}_{dec} = (-2\vec{i} + 0\vec{j}) \cdot (1\vec{i} + 0\vec{j}) = -2$$