Multivariable Calculus

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Contents

Suppose that f is integrable over a region T in the xy plane, ad suppose $f(x, y) \geq 0$ for al $(x, y) \in T$. Then the volume of the solid lying above T and below the surface $z = f(x, y)$ is given by

$$V = \int_T \int f(x, y) \, \mathrm{d}A$$

Supose that f, g are continuous functions on a closed bounded region T in the xy plane and suppose that k is a constant.

$$\int_T \int k f(x, y) \, \mathrm{d}A = k \int_t \int f(x, y) \, \mathrm{d}A$$

$$\int_T \int (f(x, y) + g(x, y)) \, \mathrm{d}A = \int_T \int f(x, y) \, \mathrm{d}A + \int_T \int g(x, y) \, \mathrm{d}A$$

$$\int_T \int f(x, y) \geq 0 \text{ if } f(x, y) \geq 0$$

$$\int_T \int f(x, y) \, \mathrm{d}A = \int_{T_1} \int f(x, y) \, \mathrm{d}A + \int_{T_2} \int f(x, y) \, \mathrm{d}A$$

where $T = T_1 \cup T_2$, and T_1, T_2 are disjoint sets.

Suppose that f is a continuous function on a region T in the xy plane, then:

$T = \left\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\right\}$ where g_1, g_2 are continuous on $[a, b]$

$$\Longrightarrow \int_{T_1} \int f(x, y) \, \mathrm{d}A = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, \mathrm{d}y \, \mathrm{d}x$$

$T = \left\{(x, y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ h_1(y) \leq x \leq h_2(y)\right\}$ where h_1, h_2 are continuous on $[c, d]$

$$\Longrightarrow \int_{T_1} \int f(x, y) \, \mathrm{d}A = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, \mathrm{d}x \, \mathrm{d}y$$

The two integrals (RHS) above are called itrated integrals. Fubini's theorem says that if T is horizontally simple ($a \leq x \leq b$), then the double integral in the first part can be calculated by performing two single integrals one after the other.

Example

Find the volume of the solid lying below $z = 4 - x - y$ and above the square T given by $0 \leq x \leq 1$ and $1 \leq y \leq 2$.

By definition, the volume (V) is given by:

$$V = \int_T \int 4 - x - y \,\mathrm{d}A$$

the function is both horizontally and vertically simple, so we can use the first or second part of Fubini's Theorem

$$\begin{aligned}
V &= \int_{x=0}^{1} \int_{y=1}^{2} 4 - x - y \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_{x=0}^{1} \left[4y - xy - \frac{y^2}{2}\right]_1^2 \mathrm{d}x \\
&= \int_{x=0}^{1} \left[4(2) - x(2) - \frac{(2)^2}{2}\right] - \left[4(1) - x(1) - \frac{(1)^2}{2}\right] \mathrm{d}x \\
&= \int_{x=0}^{1} \frac{5}{2} - x \,\mathrm{d}x \\
&= \left[\frac{5x}{2} - \frac{x^2}{2}\right]_0^1 = 2
\end{aligned}$$

Example

Find $\int_T \int xy \,\mathrm{d}A$ where T is the triangle with vertices $(0,0), (1,0), (1,1)$.

We can write the set of points in T as $T = \{(x,y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$, this set is horizontally simple, and so we can use Fubini's theorem part 1

$$\begin{aligned}
\int_T \int xy \,\mathrm{d}A &= \int_{x=0}^{1} \int_{y=0}^{x} xy \,\mathrm{d}y \,\mathrm{d}x \\
&= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x \mathrm{d}x \\
&= \int_0^1 \frac{x^3}{2} \,\mathrm{d}x \\
&= \left[\frac{x^4}{8}\right]_0^1 = \frac{1}{8}
\end{aligned}$$

We could have also defined T to be vertically simple, by letting $T = \{(x,y) \mid y \leq x \leq 1, 0 \leq y \leq 1\}$, in which case we would have had to solve

$$\int_T \int xy \,\mathrm{d}A = \int_{y=0}^{1} \int_{x=y}^{1} xy \,\mathrm{d}x \,\mathrm{d}y$$

Example

Find $\int_T \int 5y \, \mathrm{d}A$ where T is the region on the xy plane bounded by the curves $y = x^2 - 3$ and $y = -2x$.

First we will find the points of intersection of the curves $x^2 - 3$ and $-2x$

Answer: