

# **Multivariable Calculus**

## **Homework #2**

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## Differentiation rules

Some differentiation rules used during the assignment:

Given  $f(x, y)$ ,  $g(x)$ ,  $h(y)$ , and  $k \in \mathbb{R}$

## Problem 1

Are the following true, or false

1. There is no subset  $A \subseteq \mathbb{R}^2$  such that  $Bdy(A)$  contains exactly four points.
2. There is a subset  $C \subseteq \mathbb{R}^3$  such that  $C$  is not the empty set and  $Int(C)$  is the empty set.

### Part 1

This statement is false. We can prove this by counter example.

Take  $A = \{(0, 0), (1, 1), (2, 2), (3, 3)\}$ .  $A$  has only 4 elements, an open ball with a center at any of those elements will contain at least one point inside it (the center), as well as points outside it, meaning that they are boundary points.

### Part 2

This statement is true, an example would be  $C = \{(1, 1, 1)\}$ . This set is not empty, and any open ball with center  $(1, 1, 1)$  will contain points outside  $C$ , meaning that  $(1, 1, 1)$  is not an interior point.

Therefore, there exists  $C \subseteq \mathbb{R}^3$  where  $Int(C) = \emptyset$

## Problem 2

Suppose the directional derivative of  $g(x, y)$  at  $(1, 2)$  in the direction of  $\vec{i} + \vec{j}$  is  $2\sqrt{2}$  and the directional derivative of  $g(x, y)$  at  $(1, 2)$  in the direction of  $-2\vec{j}$  is  $-3$ . Find the directional derivative of  $g(x, y)$  at  $(1, 2)$  in the direction of  $-\vec{i} - 2\vec{j}$

$$\nabla g|_{(1,2)} = \frac{\partial g}{\partial x} \vec{i} + \frac{\partial g}{\partial y} \vec{j}$$

To make the solution clearer, let  $a = \frac{\partial g}{\partial x}$ , and  $b = \frac{\partial g}{\partial y}$

First, let's find the unit vectors in the directions given:

$$u = \frac{\vec{i} + \vec{j}}{\sqrt{2}} = \frac{1}{\sqrt{2}} \vec{i} + \frac{1}{\sqrt{2}} \vec{j}$$

$$w = \frac{0\vec{i} - 2\vec{j}}{\sqrt{(-2)^2}} = 0\vec{i} - 1\vec{j}$$

$$z = \frac{-1\vec{i} - 2\vec{j}}{\sqrt{(-1)^2 + (-2)^2}} = -\frac{1}{\sqrt{5}} \vec{i} - \frac{2}{\sqrt{5}} \vec{j}$$

We know that the gradient in the direction of  $w$  is given by:

$$(D_w g)|_{(1,2)} = -3$$

Therefore

$$-3 = (a\vec{i} + b\vec{j}) \cdot (0\vec{i} - 1\vec{j}) = -b \implies 3 = b$$

We also know that the gradient in the direction of  $u$  is given by:

$$(D_u g)|_{(1,2)} = 2\sqrt{2}$$

Therefore

$$2\sqrt{2} = (a\vec{i} + b\vec{j}) \cdot \left(\frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}\right) = \frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}} = \frac{a}{\sqrt{2}} + \frac{3}{\sqrt{2}}$$

Now, we can solve for  $a$

$$2\sqrt{2} = \frac{a}{\sqrt{2}} + \frac{3}{\sqrt{2}}$$

$$2 \cdot 2 = a + 3$$

$$4 - 3 = a = 1$$

And so now we now that

$$\nabla g|_{(1,2)} = 1\vec{i} + 3\vec{j}$$

With this, we can now find the directional derivative of  $g$  at  $(1, 2)$  in the direction of  $-\vec{i} - 2\vec{j}$

$$(D_z g)|_{(1,2)} = \nabla g|_{(1,2)} \cdot \underline{w} = (\vec{i} + 3\vec{j}) \cdot \left(-\frac{1}{\sqrt{5}}\vec{i} - \frac{2}{\sqrt{5}}\vec{j}\right) = -\frac{1}{\sqrt{5}} - \frac{3 \cdot 2}{\sqrt{5}} = -\frac{7}{\sqrt{5}}$$

## Problem 3

### Part 1

Suppose that  $h(x) = \cos(y + x^2) + \sin(x - 2y^2)$ , find the following:

$$\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial^2 h}{\partial^2 x}, \frac{\partial^2 h}{\partial^2 y}, \frac{\partial^2 h}{\partial x \partial y}$$

For these differentials, I will use the following rules:

$$\frac{\partial}{\partial u} \cos(f(u)) = -f'(u) \sin(f(u)), \quad \frac{\partial}{\partial u} \sin(f(u)) = f'(u) \cos(f(u))$$

$$\frac{\partial}{\partial u} f(u)g(u) = f'(u)g(u) + f(u)g'(u)$$

And also

$$\frac{\partial}{\partial x} y + x^2 = 2x \quad \frac{\partial}{\partial x} x - 2y^2 = 1$$

$$\frac{\partial}{\partial y} y + x^2 = 1 \quad \frac{\partial}{\partial y} x - 2y^2 = -4y$$

$$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x} \cos(y + x^2) + \frac{\partial}{\partial x} \sin(x - 2y^2) = -2x \sin(y + x^2) + \cos(x - 2y^2)$$

$$\begin{aligned} \frac{\partial^2 h}{\partial^2 x} &= \frac{\partial}{\partial x} \frac{\partial h}{\partial x} = \frac{\partial}{\partial x} (-2x \sin(y + x^2) + \cos(x - 2y^2)) \\ &= -2 \sin(y + x^2) - 4x^2 \cos(y + x^2) - \sin(x - 2y^2) \end{aligned}$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y} \cos(y + x^2) + \frac{\partial}{\partial y} \sin(x - 2y^2) = -\sin(y + x^2) - 4y \cos(x - 2y^2)$$

$$\begin{aligned}\frac{\partial^2 h}{\partial^2 y} &= \frac{\partial}{\partial y} \frac{\partial h}{\partial y} = \frac{\partial}{\partial y} (-\sin(y + x^2) - 4y \cos(x - 2y^2)) \\ &= -\cos(y + x^2) - 4 \cos(x - 2y^2) - 16y^2 \sin(x - 2y^2)\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 h}{\partial x \partial y} &= \frac{\partial}{\partial x} \frac{\partial h}{\partial y} = \frac{\partial}{\partial x} (-\sin(y + x^2) - 4y \cos(x - 2y^2)) \\ &= -2x \cos(y + x^2) - 4y \cos(x - 2y^2)\end{aligned}$$

**Part 2**

Suppose that  $z = 2u - 3w^2$ ,  $u = e^{2r+3s}$ ,  $w = s - r^2$  find the following:

$$\frac{\partial z}{\partial r}, \frac{\partial z}{\partial s}$$

For these differentials, I will use the chain rule as follows:

$$\frac{\partial}{\partial a} e^{f(a,b)} = \left( \frac{\partial}{\partial a} f(a,b) \right) \cdot e^{f(a,b)}$$

$$\frac{\partial}{\partial a} (f(a,b))^2 = 2(f(a,b)) \cdot \left( \frac{\partial}{\partial a} f(a,b) \right)$$

First, we will find the derivatives of  $u$ , and  $w^2$ :

$$\frac{\partial}{\partial r} e^{2r+3s} = 2e^{2r+3s}$$

$$\frac{\partial}{\partial s} e^{2r+3s} = 3e^{2r+3s}$$

$$\frac{\partial}{\partial r} (s - r^2)^2 = 2(s - r^2)(-2r) = -4r(s - r^2)$$

$$\frac{\partial}{\partial s} (s - r^2)^2 = 2(s - r^2)(1) = 2(s - r^2)$$

We can use the above derivatives to solve the question

$$\begin{aligned} \frac{\partial z}{\partial r} &= 2 \frac{\partial u}{\partial r} - 3 \frac{\partial w^2}{\partial r} = 2 \frac{\partial}{\partial r} e^{2r+3s} - 3 \frac{\partial}{\partial r} (s - r^2)^2 \\ &= 2(2e^{2r+3s}) - 3(-4r(s - r^2)) \\ &= 4e^{2r+3s} + 12r(s - r^2) \end{aligned}$$

$$\begin{aligned} \frac{\partial z}{\partial s} &= 2 \frac{\partial u}{\partial s} - 3 \frac{\partial w^2}{\partial s} = 2 \frac{\partial}{\partial s} e^{2r+3s} - 3 \frac{\partial}{\partial s} (s - r^2)^2 \\ &= 2(3e^{2r+3s}) - 3(2(s - r^2)) = 6e^{2r+3s} - 6(s - r^2) \\ &= 6(e^{2r+3s} - s + r^2) \end{aligned}$$



**Problem 4**

Suppose  $f(x, y) = y^2 - \cos(y + x)$ . Is  $f$  differentiable at every point in  $\mathbb{R}^2$ ? Justify your answer.

We say that  $f(x, y)$  is differentiable at a point  $(a, b)$  if  $\frac{\partial f}{\partial x}|_{a,b}$  and  $\frac{\partial f}{\partial y}|_{a,b}$  exist.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} y^2 - \frac{\partial}{\partial x} \cos(y + x) = \sin(y + x)$$

Since  $x, y \in \mathbb{R}$ ,  $x + y \in \mathbb{R}$ , this means that  $f$  is differentiable at any point since  $\sin(n)$  is defined for all  $n \in \mathbb{R}$ .

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} y^2 - \frac{\partial}{\partial y} \cos(y + x) = 2y + \sin(y + x)$$

This derivative is also differentiable at any point since  $2y$  is defined for all  $y \in \mathbb{R}$ , and  $\sin(y + x)$  is also defined for all  $x, y \in \mathbb{R}$ .

Because both  $\frac{\partial f}{\partial x}$  and  $\frac{\partial f}{\partial y}$  are differentiable at every  $(x, y) \in \mathbb{R}^2$ , we have shown that  $f$  is differentiable at every point in  $\mathbb{R}^2$ .

## Problem 5

### Part 1

Find the directional derivative of  $g(x, y, z) = x^3y^2 - e^z \sin(yx)$  at  $(1, \frac{\pi}{2}, 0)$  in the direction of  $\vec{i} - 2\vec{j} + 3\vec{k}$ .

Firstly, we will find the unit vector  $u$  in the direction of  $\vec{i} - 2\vec{j} + 3\vec{k}$ . This is given by

$$u = \frac{\vec{i} - 2\vec{j} + 3\vec{k}}{\sqrt{1^2 + (-2)^2 + 3^2}} = \frac{1}{\sqrt{14}}\vec{i} - \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}$$

The directional derivative of  $g$  in the direction of  $\vec{i} - 2\vec{j} + 3\vec{k}$  at  $(1, \frac{\pi}{2}, 0)$  is given by

$$(D_{\underline{u}}f)|_{(1, \frac{\pi}{2}, 0)} = \nabla g|_{(1, \frac{\pi}{2}, 0)} \cdot \underline{u}$$

Where

$$\nabla g = \frac{\partial g}{\partial x}\vec{i} + \frac{\partial g}{\partial y}\vec{j} + \frac{\partial g}{\partial z}\vec{k}$$

$$\frac{\partial g}{\partial x} = 3x^2y^2 - e^zy \cos(yx)$$

$$\frac{\partial g}{\partial y} = 2x^3y - e^zx \cos(yx)$$

$$\frac{\partial g}{\partial z} = -e^z \cos(yx)$$

$$\begin{aligned} (D_{\underline{u}}f)|_{(1, \frac{\pi}{2}, 0)} &= \nabla g|_{(1, \frac{\pi}{2}, 0)} \cdot \underline{u} \\ &= [(3x^2y^2 - e^zy \cos(yx))\vec{i} + (2x^3y - e^zx \cos(yx))\vec{j} + -e^z \cos(yx)\vec{k}]|_{(1, \frac{\pi}{2}, 0)} \cdot \underline{u} \\ &= (\frac{3\pi^2}{4}\vec{i} + \pi\vec{j} - 1\vec{k}) \cdot (\frac{1}{\sqrt{14}}\vec{i} - \frac{2}{\sqrt{14}}\vec{j} + \frac{3}{\sqrt{14}}\vec{k}) \\ &= \frac{\sqrt{14}(3\pi^2 - 8\pi - 12)}{56} \end{aligned}$$

## Part 2

Find the directions in which  $g(x, y) = 2y^2x - 4e^{yx} \sin x$  increases and decreases most rapidly at  $(0, 1)$ . Also, at what rate does  $g$  change in these directions?

It will most rapidly increase when the angle between the gradient of  $g$  and  $\underline{u}$  is 0, and it will most rapidly decrease when it is  $\pi$ .

$$\nabla g = \frac{\partial g}{\partial x} \vec{i} + \frac{\partial g}{\partial y} \vec{j}$$

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial}{\partial x} 2y^2x - 4 \frac{\partial}{\partial x} e^{yx} \sin x \\ &= 2y^2 - 4 \frac{\partial}{\partial x} e^{yx} \sin x \\ &= 2y^2 - 4 \left( \left( \frac{\partial}{\partial x} e^{yx} \right) \sin x + e^{yx} \left( \frac{\partial}{\partial x} \sin x \right) \right) \\ &= 2y^2 - 4(ye^{yx} \sin x + e^{yx} \cos x) \end{aligned}$$

$$\begin{aligned} \frac{\partial g}{\partial y} &= \frac{\partial}{\partial y} 2y^2x - 4 \frac{\partial}{\partial y} e^{yx} \sin x \\ &= 4yx - 4 \sin x \frac{\partial}{\partial y} e^{yx} \\ &= 4yx - 4 \sin(x) x e^{yx} \end{aligned}$$

Therefore,

$$\nabla g = (2y^2 - 4(ye^{yx} \sin x + e^{yx} \cos x)) \vec{i} + (4yx - 4 \sin(x) x e^{yx}) \vec{j}$$

Now, we can evaluate the gradient at  $(0, 1)$

$$\nabla g|_{(0,1)} = (2(1)^2 - 4((1)e^0 \sin(0) + e^0 \cos(0))) \vec{i} + (4(1)(0) - 4 \sin(0)(0)e^0) \vec{j} = -2\vec{i} + 0\vec{j}$$

The function will increase the most rapidly when  $\underline{u}_{inc}$  is in the direction of the gradient evaluated at  $(0, 1)$ , meaning that  $\underline{u}_{inc}$  is in the direction of  $-2\vec{i} + 0\vec{j}$ .

Because  $\underline{u}_{inc}$  is a unit vector,  $\underline{u}_{inc} = \frac{-2\vec{i} + 0\vec{j}}{2} = -1\vec{i} + 0\vec{j}$

The change in this direction can be calculated by

$$(D_{\underline{u}_{inc}} g|_{(0,1)}) = \nabla g|_{(0,1)} \cdot \underline{u}_{inc} = (-2\vec{i} + 0\vec{j}) \cdot (-1\vec{i} + 0\vec{j}) = 2$$

Similarly, it will decrease the most rapid when  $\underline{u}_{dec}$  is in the direction of  $-\nabla g|_{(0,1)}$ . In this case  $\underline{u}_{dec} = -\frac{-2\vec{i}+0\vec{j}}{2} = 1\vec{i} + 0\vec{j}$

The change in this direction can be calculated by

$$(D_{\underline{u}_{dec}} g|_{(0,1)}) = \nabla g|_{(0,1)} \cdot \underline{u}_{dec} = (-2\vec{i} + 0\vec{j}) \cdot (1\vec{i} + 0\vec{j}) = -2$$