Integration

Homework #1

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Problem 1

$$\begin{aligned}\cos\,(x-\frac{\pi}{2}) &= \cos(x)\cos\left(-\frac{\pi}{2}\right) - \sin(x)\sin\left(-\frac{\pi}{2}\right)\\ &= \cos(x)\cdot 0 - \sin(x)\cdot(-1)\\ &= \sin(x)\end{aligned}$$

$$\begin{aligned}\cos\,(x+\frac{\pi}{2}) &= \cos(x)\cos\left(\frac{\pi}{2}\right) - \sin(x)\sin\left(\frac{\pi}{2}\right)\\ &= \cos(x)\cdot 0 - \sin(x)\cdot(1)\\ &= -\sin(x)\end{aligned}$$

$$\begin{aligned}\sin\,(x-\frac{\pi}{2}) &= \sin(x)cos(-\frac{\pi}{2}) + \cos(x)\sin\left(-\frac{\pi}{2}\right)\\ &= \sin(x)\cdot 0 + \cos(x)\cdot(-1)\\ &= -\cos(x)\end{aligned}$$

$$\begin{aligned}\sin\,(x+\frac{\pi}{2}) &= \sin(x)cos(\frac{\pi}{2}) + \cos(x)\sin\left(\frac{\pi}{2}\right)\\ &= \sin(x)\cdot 0 + \cos(x)\cdot(1)\\ &= \cos(x)\end{aligned}$$

$$\begin{aligned}\cos\,(x-\pi) &= \cos(x)\cos(-\pi) - \sin(x)\sin(-\pi)\\ &= \cos(x)\cdot(-1) - \sin(x)\cdot 0\\ &= -\cos(x)\end{aligned}$$

$$\begin{aligned}\cos\,(x+\pi) &= \cos(x)\cos(\pi) - \sin(x)\sin(\pi)\\ &= \cos(x)\cdot(-1) - \sin(x)\cdot 0\\ &= -\cos(x)\end{aligned}$$

Problem 1 continued on next page. . .

$$
\begin{aligned}
\sin(x-\pi) &= \sin(x)\cos(-\pi) + \cos(x)\sin(-\pi) \\
&= \sin(x)\cdot(-1) + \cos(x)\cdot 0 \\
&= -\sin(x)
\end{aligned}
$$

$$
\begin{aligned}
\sin(x+\pi) &= \sin(x)\cos(\pi) + \cos(x)\sin(\pi) \\
&= \sin(x)\cdot(-1) + \cos(x)\cdot 0 \\
&= -\sin(x)
\end{aligned}
$$

$$
\begin{aligned}
\cos(x-\frac{3\pi}{2}) &= \cos\left(x-\pi-\frac{\pi}{2}\right) \\
&= \cos(x-\pi)\cos\left(-\frac{\pi}{2}\right) - \sin(x-\pi)\sin\left(-\frac{\pi}{2}\right) && * \\
&= -\cos(x)\cdot 0 + \sin(x)\cdot(-1) \\
&= -\sin(x)
\end{aligned}
$$

$$
\begin{aligned}
\cos(x+\frac{3\pi}{2}) &= \cos\left(x+\pi+\frac{\pi}{2}\right) \\
&= \cos(x+\pi)\cos\left(\frac{\pi}{2}\right) - \sin(x+\pi)\sin\left(\frac{\pi}{2}\right) && * \\
&= -\cos(x)\cdot 0 + \sin(x)\cdot(1) \\
&= \sin(x)
\end{aligned}
$$

$$
\begin{aligned}
\sin(x-\frac{3\pi}{2}) &= \sin\left(x-\pi-\frac{\pi}{2}\right) \\
&= \sin(x-\pi)\cos\left(-\frac{\pi}{2}\right) + \cos(x-\pi)\sin\left(-\frac{\pi}{2}\right) && * \\
&= -\sin(x)\cdot 0 - \cos(x)\cdot(-1) \\
&= \cos(x)
\end{aligned}
$$

Problem 1 continued on next page. . .

$$
\begin{aligned}
\sin\left(x+\frac{3\pi}{2}\right) &= \sin\left(x+\pi+\frac{\pi}{2}\right) \\
&= \sin(x+\pi)\cos\left(\frac{\pi}{2}\right)+\cos(x+\pi)\sin\left(\frac{\pi}{2}\right) \qquad * \\
&= -\sin(x)\cdot 0-\cos(x)\cdot(1) \\
&= -\cos(x)
\end{aligned}
$$

* We had found the values of $\cos(x-\pi)$, $\cos(x+\pi)$, $\sin(x-\pi)$, and $\sin(x+\pi)$ earlier in this question, so I replaced it for what we found earlier.

Problem 2

Find the inverse of the following function:

$$f(x) = x^2 - 2bx$$

Where b is a positive constant such that $b \geq x$.

To find the inverse, we will start by manipulating the original function until there is only one x. To do this we will try to find the value of a, c such that $f(x) = (x + a)^2 + c$

$$\begin{aligned} f(x) &= x^2 - 2bx \\ &= (x + a)^2 + c \\ &= x^2 + 2xa + a^2 + c \end{aligned}$$

Therefore:

$$\begin{aligned} 2xa &= -2bx \\ \Rightarrow a &= -b \\ a^2 + c &= 0 \\ \Rightarrow c &= -(b)^2 \end{aligned}$$

Now that we have found the values for a, c, lets verify that they actually work.

$$\begin{aligned} f(x) &= (x - b)^2 - b^2 \\ &= x^2 - 2bx + b^2 - b^2 \\ &= x^2 - 2bx \end{aligned}$$

And so those values work. Now we can find its inverse easier:

Problem 2 continued on next page. . .

$$
\begin{aligned}
f(x) &= (x-b)^2 - b^2 \\
f(x) + b^2 &= (x-b)^2 \\
\sqrt{f(x)+b^2} &= \pm(x-b) \\
\sqrt{f(x)+b^2} &= -(x-b) \qquad\qquad * \\
\sqrt{f(x)+b^2} - b &= -x \\
b - \sqrt{f(x)+b^2} &= x
\end{aligned}
$$

Note that we only keep the negative sign as $b \geq x$, and so $(x-b)$ will always be negative. However, it must equal $\sqrt{f(x)+b^2}$, which cannot be negative number.

Answer

$$f^{-1}(x) = b - \sqrt{x+b^2}$$

To verify that that's actually the inverse, we need to show that $f(f^{-1}(x)) = f^{-1}(f(x)) = x$

$$
\begin{aligned}
f(f^{-1}(x)) &= f(b - \sqrt{x+b^2}) \\
&= (b - \sqrt{x+b^2})^2 - 2b(b - \sqrt{x+b^2}) \\
&= b^2 - 2b\sqrt{x+b^2} + x + b^2 - 2b^2 + 2b\sqrt{x+b^2} \\
&= x - 2b\sqrt{x+b^2} + 2b\sqrt{x+b^2} + 2b^2 - 2b^2 \\
&= x
\end{aligned}
$$

$$
\begin{aligned}
f^{-1}(f(x)) &= b - \sqrt{f(x)+b^2} \\
&= b - \sqrt{x^2 - 2bx + b^2} \\
&= b - \sqrt{(x-b)^2} \\
&= b - x - b \\
&= x
\end{aligned}
$$

And because both $f(f^{-1}(x))$ and $f^{-1}(f(x))$ are equal to x, $f(f^{-1}(x)) = f^{-1}(f(x))$

Problem 3

Find:

$$\lim_{x\to 0} \frac{1-\cos x}{x\sin x}$$

$$\begin{aligned}
\frac{1-\cos x}{x\sin x} &= \frac{1-\cos^2 x}{x\sin x(1+\cos x)} \\
&= \frac{sin^2 x}{x\sin x(1+\cos x)} \\
&= \frac{sinx}{x(1+\cos x)} \\
&= \frac{sinx}{x}\cdot\frac{1}{1+\cos x} \\
\therefore \lim_{x\to 0}\frac{1-\cos x}{x\sin x} &= \lim_{x\to 0}\frac{sinx}{x}\cdot\frac{1}{1+\cos x}
\end{aligned}$$

We know from class that:

$$\begin{aligned}
\lim_{x\to 0}\frac{sinx}{x} &= 1 \\
\lim_{x\to 0}\frac{1}{1+\cos x} &= \frac{1}{1+\cos 0} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{x\to 0}\frac{sinx}{x}\cdot\frac{1}{1+\cos x} &= \lim_{x\to 0}\frac{sinx}{x}\cdot\lim_{x\to 0}\frac{1}{1+\cos x} \\
&= 1\cdot\frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

And so the answer is $\frac{1}{2}$.

Problem 4

Prove the following using $\epsilon - \delta$

$$\lim_{x\to 3} \sqrt{4-x} = 1$$

To show that the limit is true, we need to show that $\forall \epsilon > 0$, $\exists \delta > 0$ such that

$$0 < |x-3| < \delta \implies \left|\sqrt{4-x} - 1\right| < \epsilon$$

$$\begin{aligned} |\sqrt{4-x} - 1| &< \epsilon \\ -\epsilon &< \sqrt{4-x} - 1 < \epsilon \\ -\epsilon + 1 &< \sqrt{4-x} < \epsilon + 1 \\ (-\epsilon+1)^2 &< 4 - x < (\epsilon+1)^2 \\ (-\epsilon+1)^2 - 1 &< 3 - x < (\epsilon+1)^2 - 1 \end{aligned}$$

Therefore:

$$|3-x| = |x-3| < \max(|(-\epsilon+1)^2 - 1|, |(\epsilon+1)^2 - 1|)$$

And so we can let $\delta = \max(|(-\epsilon+1)^2 - 1|, |(\epsilon+1)^2 - 1|)$. Therefore, we have found a value for delta such that the limit definition is satisfied, meaning that $\lim_{x\to 3} \sqrt{4-x} = 1$.

Problem 5

Find the asymptotes for the following function:

$$f(x) = \frac{x^2 - 4}{x^2 - 4x + 3}$$

The given function will have vertical asymptotes when the denominator of the function is 0, as the function will tend towards either positive or negative infinity at these points. $f(x)$ can be re-written as:

$$f(x) = \frac{x^2 - 4}{(x - 1) \cdot (x - 3)}$$

This shows that f has vertical asymptotes at $x = 3$, and at $x = 1$.

To see if there are any horizontal asymptotes we need to find what the function approaches as x goes to infinity.

$$\lim_{x \to \infty} \frac{x^2 - 4}{x^2 - 4x + 3} = \lim_{x \to \infty} \frac{(x - 2) \cdot (x + 2)}{(x - 3) \cdot (x - 1)} = (\lim_{x \to \infty} \frac{x - 2}{x - 3}) \cdot (\lim_{x \to \infty} \frac{x + 2}{x - 1}) = 1 \cdot 1 = 1$$

Because it approaches 1, but it never reaches one, we know that there is a horizontal asymptote at $y = 1$.

Problem 6

At which point/s does the following function fail to be continuous?

$$f(x) = \begin{cases} x+1 & \text{for } x \geq 0 \\ \frac{x}{x^2-x-6} & \text{for } x < 0 \end{cases}$$

$x + 1$ is continuous for all x, therefore, $f(x)$ is continuous $\forall x > 0$.

$f(x)$ will be continuous at $x = 0$ if the limit as x approaches 0 from both sides is the same and if it is defined at $x = 0$.

$f(0) = 1$, however, the limit from the negative side is the following:

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= \lim_{x \to 0} \frac{x}{x^2 - x - 6} \\ &= 0 \end{aligned}$$

Therefore, $f(x)$ is not continuous at $x = 0$. It will also not be continuous at any points where it is undefined.

For any $x < 0$, $f(x)$ will be undefined iff $x^2 - x - 6 = 0$, using the $-b$ formula, we get that $\frac{x}{x^2-x-6}$ is undefined at 3, and at -2, however, $3 > 0$, so it is defined at that point.

Therefore, the function fails to be continuous only at $x = -2$ and at $x = 0$