Linear Algebra Class

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Contents

1. Eigen values

Take V over $\mathbb{F}$, $\lambda \in \mathbb{F}$, $T \in L(V)$, λ is an eigenvalue if there exists a $v \neq 0$ such taht

$$\begin{gathered} Tv = \lambda v = \lambda I v \\ \Longrightarrow Tv - \lambda I v = 0 \\ \Longrightarrow (T - \lambda I) v = 0 \\ \Longleftrightarrow v \in \text{ker } (T - \lambda I) \end{gathered}$$

Therefore, if λ is an eigenvalue, then

$$\begin{gathered} \text{ker } (T - \lambda I) \neq \{0\} \\ \Longleftrightarrow (T - \lambda I) \text{ is not bijective} \\ \Longleftrightarrow (T - \lambda I) \text{ does not have an inverse} \\ \text{the matrix } A \text{ of } (T - \lambda I) \text{ in a chosen basis does not have an inverse} \\ \det A = 0 \end{gathered}$$

If A_T denotes the matrix of T, then $A =$ matrix of $T - \lambda I = A_T - \lambda \begin{pmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{pmatrix}$

Therefore, if λ is an eigenvalue $\Longleftrightarrow \det \left(A_T - \lambda \begin{pmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{pmatrix} \right) = 0$. We write $E_\lambda(T) := \ker(T - \lambda I)$ and we call it the eigenspace of T with respect to λ.

The **spectrum $\boldsymbol{\sigma(T)}$** of T is the set of all eigenvalues of T.

Proposition: Let $T \in L(V)$ and let λ be an eigenvalue of T. Then $E_\lambda(T)$ is an invarient* subspace of V.

Proof:

$\lambda \in \sigma(T) \Longrightarrow \exists v \neq 0, Tv = \lambda v$, therefore, $(T - \lambda I) = 0 \Longrightarrow v \in E_\lambda(T)$. $Tv = \lambda v \in E_\lambda(T)$

Proposition: Let $T \in L(V)$, and $\lambda_1, \dots, \lambda_n$ be distincs eigenvalues of T. Then $E_{\lambda_1}(T), \dots, E_{\lambda_n}(T)$ are linearly independent of each other.

In other words for any selection of $v_1 \in E_{\lambda_1}(T), \dots, v_n \in E_{\lambda_n}(T)$

Proof:

We can prove this by induction, for $k = 1$, there is nothing to prove.

For $k = 2$, take:

$$v_1 \in E_{\lambda_1}(T) \quad v_2 \in E_{\lambda_2}(T)$$

suppose that they are linearly dependent, then there exists α_1, α_2 (not both 0) such that:

$$\begin{gathered} \alpha_1 v_1 + \alpha_2 v_2 = 0 \Longrightarrow \alpha_1 v_1 = -\alpha_2 v_2 \\ T(\alpha_1 v_1) = T(-\alpha_2 v_2) \\ \alpha T(v_1) = -\alpha_2 T(v_2) \\ \lambda_1 \alpha_1 v_1 = -\lambda_2 \alpha_2 v_2 \end{gathered}$$

Also

$$\alpha_1 v_1 = -\alpha_2 v_2 \Longrightarrow \lambda_1 \alpha_1 v_1 = -\lambda_1 \alpha_2 v_2$$
$$-\lambda_1 \alpha_2 v_2 = -\lambda_2 \alpha_2 v_2 \Longrightarrow \underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} \alpha_2 \underbrace{v_2}_{\neq 0}$$
$$\therefore \alpha_2 = 0$$

Assume that the conclusion that it is true for $k - 1$:

Suppose that there exists coefficients $\alpha_1, ..., \alpha_{k-1}$, and assume that they are linearly dependent:

$$\sum_{j=1}^{k} \alpha_j v_j = 0$$

Wlog $\alpha_k \neq 0$

$$-\alpha_k v_k = \sum_{j=1}^{k-1} \alpha_j v_j$$
$$T(-\alpha_k v_k) = T\left(\sum_{j=1}^{k-1} \alpha_j v_j\right)$$
$$-\alpha_k T(v_k) = \sum_{j=1}^{k-1} \alpha_j T(v_j)$$
$$-\alpha_k \lambda_k v_k = \sum_{j=1}^{k-1} \alpha_j \lambda_j v_j$$

Multplying both sides by λ_k:

$$-\alpha_k v_k = \sum_{j=1}^{k-1} \alpha_j v_j$$
$$-\alpha_k \lambda_k v_k = \sum_{j=1}^{k-1} \alpha_j \lambda_k v_j$$
$$= \sum_{j=1}^{k-1} \alpha_j (\lambda_k - \lambda_j) v_j = 0$$

Therefore, $\alpha_j = 0$ for all j, and so $-\alpha_k v_k = \sum_{j=1}^{k-1} \alpha_j v_j = 0$

1.1. Example

Take $\mathbb{F} = \mathbb{R}$

Take $T \in L(\mathbb{R}^2)$ be given by $T(x_1, x_2) = (-x_2, x_1)$. We claim that T has no eigen values. an eigen value is a λ such that:

$$T(x_1, x_2) = \lambda(x_1, x_2) = (\lambda x_1, \lambda x_2)$$
$$-x_2 = \lambda x_1, \quad x_1 = \lambda x_2$$
$$\Longrightarrow x_1 = -\lambda^2 x_2$$

however, there exists no lambda in the real such that the above is true.

1.2. Example

Take $T \in L\left(\mathbb{C}^2\right)$ be given by $T(x_1, x_2) = (-x_2, x_1)$, note that this is the same function as in the example above

$$\begin{cases} x_1 = -\lambda^2 x_1 \\ x_2 = -\lambda^2 x_2 \end{cases}$$

$$\text{Take } \lambda = \pm 1$$

$$\begin{cases} -x_2 = i x_1 \\ x_1 = i x_2 \end{cases}$$

$$x_1 = i x_2 \Longrightarrow \text{solution: } \begin{pmatrix} i x_2 \\ x_2 \end{pmatrix}$$