Integration Assignment 3

Angel Cervera Roldan **21319203**

Problem 1

Part 1

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=1}\frac{\mathrm{d}}{\mathrm{d}y}\Big|_{y=2}\int_0^{xy} xw + e^{w^2x^2}\,\mathrm{d}w$$

To solve this, we will use substitution:

$$\int_0^{xy} xw + e^{w^2x^2}\,\mathrm{d}w = \int_0^{xy} xw\,\mathrm{d}w + \int_0^{xy} e^{w^2x^2}\,\mathrm{d}w$$

$$\int xw\,\mathrm{d}w = \frac{xw^2}{2} + C_1$$

We will use substitution to solve the second integral, let $u = (wx)^2$

$$u = (uw)^2 \Longrightarrow \frac{\mathrm{d}u}{\mathrm{d}w} = 2x^2w \Longrightarrow \mathrm{d}u = 2x^2w\,\mathrm{d}w$$

Part 2

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=2}\int_0^{x}\left(e^{\sqrt{u}} + \frac{\mathrm{d}}{\mathrm{d}v}\Big|_{v=u}e^{uvx}\right)\mathrm{d}u$$

$$\frac{\mathrm{d}}{\mathrm{d}v}\Big|_{v=u}e^{uvx} = \left[uxe^{uvx}\right]_{v=u} = uxe^{u^2x}$$

$$\begin{aligned}\Longrightarrow \int_0^{x}\left(e^{\sqrt{u}} + \frac{\mathrm{d}}{\mathrm{d}v}\Big|_{v=u}e^{uvx}\right)\mathrm{d}u &= \int_0^{x}\left(e^{\sqrt{u}} + uxe^{u^2x}\right)\mathrm{d}u \\ &= \int_0^{x} e^{\sqrt{u}}\,\mathrm{d}u + \int_0^{x} uxe^{u^2x}\,\mathrm{d}u \\ &= 2e^{\sqrt{u}}(\sqrt{u}-1) + \frac{e^{u^2x}}{2}\end{aligned}$$

Above, we needed to solve the integral:

$$\int_0^{x} uxe^{u^2x}\,\mathrm{d}u$$

To solve that integral, let $p = u^2$, then, $\mathrm{d}p = 2u\,\mathrm{d}u$, and so, after substitution:

$$\int xe^{\,px}\frac{1}{2}\,\mathrm{d}p = \frac{x}{2}\int e^{\,px}\,\mathrm{d}p = \frac{x}{2}\frac{e^{\,px}}{x} + C = \frac{e^{\,px}}{2} + C$$

$$
\begin{aligned}
\implies \frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=2}\int_0^x \left(e^{\sqrt{u}} + \frac{\mathrm{d}}{\mathrm{d}v}\Big|_{v=u} e^{uvx}\right)\mathrm{d}u &= \frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=2}\left(2e^{\sqrt{u}}(\sqrt{u}-1) + \frac{e^{u^2x}}{2}\right) \\
&= \left[\frac{3}{2}e^{x^3}x^2 + e^{\sqrt{x}}\right]_{x=2} \\
&= 6e^8 + e^{\sqrt{2}}
\end{aligned}
$$

Problem 2

Part 1

Compute $\int_{-3}^{7} \operatorname{sgn}(x)$ where the sgn function is defined as $\operatorname{sgn} : \mathbb{R} \to \mathbb{R}$

$$\operatorname{sgn}(x) := \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ -1 \text{ if } x < 1 \end{cases}$$

$$\begin{aligned} \int_{-3}^{7} \operatorname{sgn}(x) &= \int_{-3}^{0} \operatorname{sgn}(x) + \int_{0}^{7} \operatorname{sgn}(x) \\ &= \int_{-3}^{0} -1 + \int_{0}^{7} 1 \\ &= -\left. x\right|_{-3}^{0} + \left. x\right|_{0}^{7} \\ &= -(0 - (-3)) + (7 - 0) \\ &= -3 + 7 = 4 \end{aligned}$$

Part 2

Compute $\int_0^5 [x] \,\mathrm{d}x$ where $[x]$ is the largest integer not greater than x, therefore, $[x] = k \in \mathbb{Z}$ if and only if $k \leq x < k + 1$

If $a \in \mathbb{Z}$ st, $[x] = a$ then $x \in [a, a + 1)$, therefore, the integral can be written as follows:

$$\begin{aligned} \int_{0}^{5} [x] &= \int_{0}^{1} [x] \,\mathrm{d}x + \int_{1}^{2} [x] \,\mathrm{d}x + \int_{2}^{3} [x] \,\mathrm{d}x + \int_{3}^{4} [x] \,\mathrm{d}x + \int_{4}^{5} [x] \,\mathrm{d}x \\ &= \int_{0}^{1} 0 \,\mathrm{d}x + \int_{1}^{2} 1 \,\mathrm{d}x + \int_{2}^{3} 2 \,\mathrm{d}x + \int_{3}^{4} 3 \,\mathrm{d}x + \int_{4}^{5} 4 \,\mathrm{d}x \\ &= \left. x\right|_{1}^{2} + \left. 2x\right|_{2}^{3} + \left. 3x\right|_{3}^{4} + \left. 4x\right|_{4}^{5} \\ &= (2 - 1) + 2(3 - 2) + 3(4 - 3) + 4(5 - 4) \\ &= 1 + 2 + 3 + 4 = 10 \end{aligned}$$

Problem 3

Solve the following integrals using substitution

Part 1

$$\int_0^2 \frac{e^x}{\sqrt{1+e^x}} \, \mathrm{d}x$$

To solve this, let $u = e^x + 1$, then

$$\begin{aligned}
\frac{\mathrm{d}u}{\mathrm{d}x} = e^x &\implies \mathrm{d}x \; e^x = \mathrm{d}u \implies \mathrm{d}x = \frac{1}{e^x} \, \mathrm{d}u \\
\int \frac{e^x}{\sqrt{1+e^x}} \, \mathrm{d}x &= \int \frac{e^x}{\sqrt{u}} \frac{1}{e^x} \, \mathrm{d}u \\
&= \int \frac{1}{\sqrt{u}} \, \mathrm{d}u = \int u^{-\frac{1}{2}} \, \mathrm{d}u \\
&= \frac{u^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C \\
&= 2\sqrt{u} + C \\
&= 2\sqrt{e^x + 1} + C
\end{aligned}$$

Therefore:

$$\begin{aligned}
\int_0^2 \frac{e^x}{\sqrt{1+e^x}} \, \mathrm{d}x &= 2\sqrt{e^2+1} - 2\sqrt{e^0+1} \\
&= 2\sqrt{e^2+1} - 2\sqrt{2}
\end{aligned}$$

Part 2

$$\int_1^2 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x$$

Let $u = \sqrt{x} + 1$, then

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2\sqrt{x}} \Longrightarrow 2\,\mathrm{d}u = \frac{1}{\sqrt{x}}\,\mathrm{d}x$$

Therefore

$$\begin{aligned}\int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &= \int \sqrt{u}\frac{1}{\sqrt{x}}\,\mathrm{d}x \\ &= \int \sqrt{u}\ 2\,\mathrm{d}u \\ &= 4\frac{u^{\frac{3}{2}}}{3} + C \\ &= 4\frac{(1+\sqrt{x})^{\frac{3}{2}}}{3} + C\end{aligned}$$

Therefore:

$$\begin{aligned}\int_1^2 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x &= \left(4\frac{\left(1+\sqrt{2}\right)^{\frac{3}{2}}}{3} + C\right) - \left(4\frac{\left(1+\sqrt{1}\right)^{\frac{3}{2}}}{3} + C\right) \\ &= 4\frac{\left(1+\sqrt{2}\right)^{\frac{3}{2}} - 2^{\frac{3}{2}}}{3}\end{aligned}$$

Problem 4

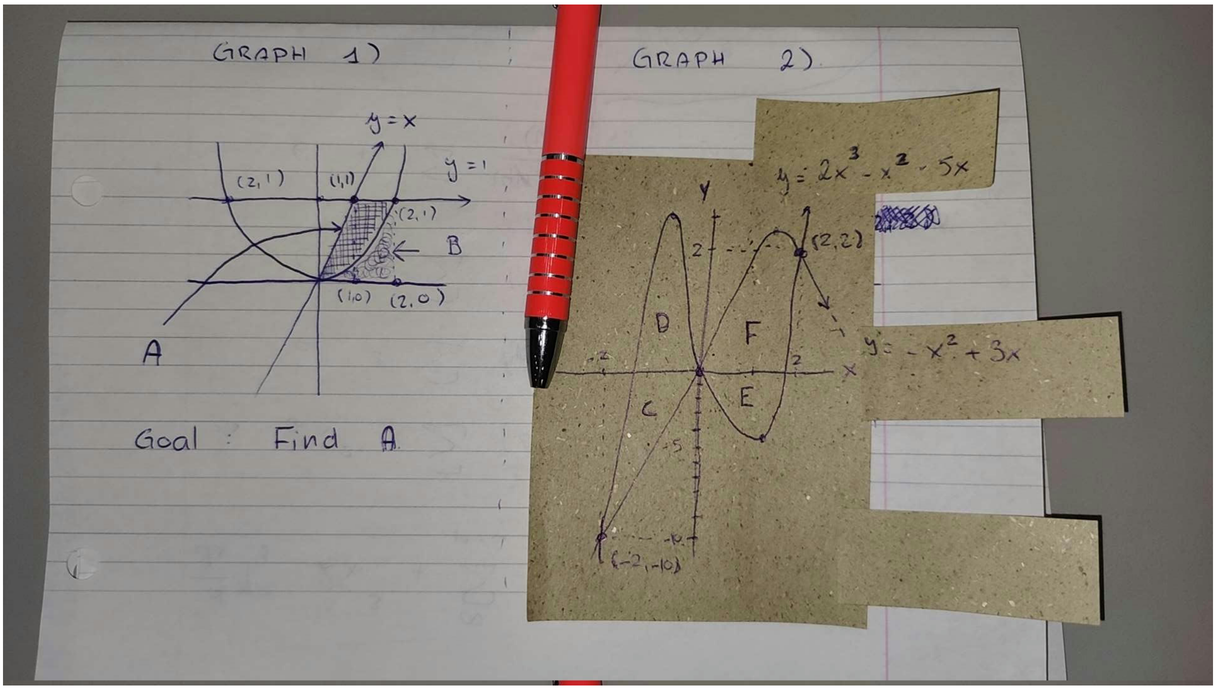

Figure 1: Sketch of Graph 1, and of Graph 2

Part 1

We need to find A. First, let $C = A + B$. We can solve for C as it is the area of the triangle with vertices at $(0,0), (1,0), (1,1)$ plus the area of the rectangle with vertices $(1,0), (2,0), (1,1), (2,1)$.

$$C = \left(\frac{1}{2} \times 1 \times 1\right) + (1 \times 1) = \frac{3}{2}$$

Also,

$$\begin{aligned} B &= \int_0^2 \frac{1}{4}x^2 \\ &= \left[\frac{1}{12}x^3\right]_0^2 \\ &= \frac{8}{12} = \frac{4}{6} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} C = A + B &\implies A = C - B \\ &\implies A = \frac{3}{2} - \frac{2}{3} \\ &= \frac{5}{6} \end{aligned}$$

Part 2

We can shift both of the functions by adding 10. This will make it easier to integrate, and will have the same area. Call the area between the two curves F.

$$\begin{aligned}
F &= \left(\int_{-2}^{0} 2x^3 - x^2 - 5x + 10 \,\mathrm{d}x - \int_{-2}^{0} -x^2 + 3x + 10 \,\mathrm{d}x\right) \\
&+ \left(\int_{0}^{2} -x^2 + 3x + 10 \,\mathrm{d}x - \int_{0}^{2} 2x^3 - x^2 - 5x + 10 \,\mathrm{d}x\right) \\
&= \left(\left[\frac{x^4}{2} - \frac{x^3}{3} - 5\frac{x^2}{2} + 10x\right]_{-2}^{0} - \left[-\frac{1}{3}x^3 + 3\frac{x^2}{2} + 10x\right]_{-2}^{0}\right) \\
&+ \left(\left[-\frac{1}{3}x^3 + 3\frac{x^2}{2} + 10x\right]_{0}^{2} - \left[\frac{x^4}{2} - \frac{x^3}{3} - 5\frac{x^2}{2} + 10x\right]_{0}^{2}\right) \\
&= \left(\frac{58}{3} - \frac{34}{3}\right) + \left(\frac{70}{3} - \frac{46}{3}\right) = 16
\end{aligned}$$