Abstract Algebra

Homework #2

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Problem 1

Find the basis of the following subspace of $\mathbb{R}^4$:

$$S_1 := \left\{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 - 2x_2 + x_4 = 0, x_3 + x_4 = 0\right\}$$

$$S_2 := span(\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}5\\6\\5\\-6\end{pmatrix}, \begin{pmatrix}1\\2\\1\\-2\end{pmatrix}, \begin{pmatrix}3\\-4\\3\\4\end{pmatrix}\right\})$$

Also, find the dimension of $S_1 \cap S_2$

Part 1

$$x_1 - 2x_2 + x_4 = 0 \implies x_4 = 2x_2 - x_1$$
$$x_3 + x_4 = 0 \implies x_3 = -x_4$$

And so, given (x_1, x_2, x_3, x_4) can be written as $(x_1, x_2, -(2x_2 - x_1), 2x_2 - x_1)$

Therefore

$$S_1 = span(\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right\})$$

meaning that a basis for S_1 is

$$\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\-1\end{pmatrix}\right\}$$

Part 2

To find the basis for S_2, we can put the given vectors into a matrix, and simplify it to Reduced Row Echelon Form:

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 5 & 6 & 5 & -6 \\ 1 & 2 & 1 & -2 \\ 3 & -4 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - 5r_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 6 & 0 & -6 \\ 1 & 2 & 1 & -2 \\ 3 & -4 & 3 & 4 \end{bmatrix}$$
$$\xrightarrow{r_3 - r_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 6 & 0 & -6 \\ 0 & 2 & 0 & -2 \\ 3 & -4 & 3 & 4 \end{bmatrix}$$
$$\xrightarrow{r_4 - 3r_1} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 6 & 0 & -6 \\ 0 & 2 & 0 & -2 \\ 0 & -4 & 0 & 4 \end{bmatrix}$$
$$\xrightarrow{1/6r_2,\ 1/2r_3,\ -1/4r_4} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$
$$\xrightarrow{r_3 - r_2,\ r_4 - r_2} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore:

$$S_2 = span(\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right\})$$

meaning that a basis for S_2 is

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$$\left\{\begin{pmatrix}1\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\-1\end{pmatrix}\right\}$$

Part 3

Any element in $S_1 \cap S_2$ must be in S_1 and in S_2, therefore, $x \in S_2$ can be written as $(a, b, a, -b)$, also, $x \in S_1 \therefore x = (n, m, c, -c)$ This means that $a = -(-b) = b$, therefore, $x = (b, b, b, -b)$.

Therefore

$$S_1 \cap S_2 = span(\left\{\begin{pmatrix}1\\1\\1\\-1\end{pmatrix}\right\})$$

And so, the dimension of $S_1 \cap S_2$ is 1.

Problem 2

$$S := \{P \in \mathbb{R}_n[X] \mid P(1) = 0\}$$

Show that S is a subspace of $\mathbb{R}_n[X]$, find a basis for S, and determine its dimension.

Given $A, B \in S$, and $\lambda \in \mathbb{R}$, to show that S is a subspace of $\mathbb{R}_n[X]$, we need to prove:

1. $(A + B)(x) = A(x) + B(x)$
2. $(\lambda A)(x) = \lambda A(x)$

By the definition of the polynomials, $\forall P, Q \in \mathbb{R}_n[X]$, $(P + Q)(x) = P(x) + Q(x)$, and $\forall \alpha \in \mathbb{R}$, $(\alpha P)(x) = \alpha P(x)$

$$(A + B)(1) = A(1) + B(1) = 0 + 0 = 0$$

$$(\lambda A)(x) = \lambda A(x) = \lambda 0 = 0$$

And so this shows that for any $A, B \in S$, $A + B \in S$, and for any $\lambda \in \mathbb{R}$, $\lambda A \in S$, therefore, S is a subspace of $\mathbb{R}_n[X]$.

The basis of $\mathbb{R}_n[X]$ is given by $\left\{1, x, x^2, ..., x^n\right\}$. However, since $A(1) = 0$, we know that:

$$A = a_0 + a_1 x^1 + a_2 x^2 + \ldots + a_n x^n$$

$$A(1) = a_0 + a_2 + \ldots + a_n = 0 \implies a_0 = -(a_1 + \ldots + a_n)$$

Therefore, A can be written as $a_1(x-1) + a_2(x^2-1) + \ldots + a_n(x^n-1) = (a_1 x - a_1) + \ldots + (a_n x^n - a_n)$

This shows that $A = span(\left\{x - 1, x^2 - 1, ..., x^n - 1\right\})$, therefore, $dim(S) = n$

Problem 3

Show that

$$M_{n\times n}(\mathbb{R}) = Sym_n(\mathbb{R}) \oplus Skew_n(\mathbb{R})$$

To prove the above, we need to show that every $A \in M_{n\times n}(\mathbb{R})$ can be written as a unique sum of $B + C$, $B \in Sym_n(\mathbb{R})$, $C \in Skew_n(\mathbb{R})$

Note: By the definition, $b_{ij} = b_{ji}$, and $c_{ij} = -c_{ji}$ this also shows that $c_{ii} = 0$ since $c_{ii} = -c_{ii}$.

We can find the matrices A and B algebraically

$$\begin{aligned} a_{ij} &= b_{ij} + c_{ij} \\ a_{ji} &= b_{ji} + c_{ji} = b_{ij} - c_{ij} \end{aligned}$$

Since we need to solve for 2 variables (b_{ij}, and c_{ij}) with two given values (a_{ij}, and a_{ji}), there is only one unique solution.

$$\begin{aligned} b_{ij} &= a_{ji} + c_{ij} \\ a_{ij} &= a_{ji} + c_{ij} + c_{ij} = a_{ji} + 2c_{ij} \\ c_{ij} &= \frac{a_{ij} - a_{ji}}{2}, \quad c_{ji} = -\frac{a_{ij} - a_{ji}}{2} \\ b_{ij} &= b_{ji} = a_{ji} + c_{ij} = a_{ji} + \frac{a_{ij} - a_{ji}}{2} \end{aligned}$$

Since every element in B and every element in C can be written in one way in terms of the elements of A, we know that the sum is unique.

Problem 4

Which of the following is a linear functional

1. on vector space $\mathbb{R}[X]$ over $\mathbb{R}$: $f(P) = 3P(0) + 5P(1)$
2. on vector space $\mathbb{R}^3$ over $\mathbb{R}$: $f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$
3. on vector space $\mathbb{R}^3$ over $\mathbb{R}$: $f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|$
4. on vector space $\mathbb{Z}_5^3$ over $\mathbb{Z}_5$: $f(x_1, x_2, x_3) = x_1^5 + x_2^5 + x_3^5$

Definition: A function f is linear functional iff:

$$f(a+b) = f(a) + f(b)$$

$$f(\lambda a) = \lambda f(a)$$

Part 1

Take $P, Q \in \mathbb{R}[X]$, and $\lambda \in \mathbb{R}$, then:

$$\begin{aligned}
f(P+Q) &= 3(P+Q)(0) + 5(P+Q)(1) \\
&= 3(P(0) + Q(0)) + 5(P(1) + Q(1)) \\
&= 3P(0) + 3Q(0) + 5P(1) + 5Q(1) \\
&= 3P(0) + 5P(1) + 3Q(0) + 5Q(1) \\
&= f(P) + f(Q)
\end{aligned}$$

$$\begin{aligned}
f(\lambda P) &= 3(\lambda P)(0) + 5(\lambda Q)(1) \\
&= 3\lambda P(0) + 5\lambda Q(1) \\
&= \lambda(3P(0) + 5Q(1)) \\
&= \lambda f(P)
\end{aligned}$$

This shows that the function $f(P) = 3P(0) + 5P(1)$ is linear functional

Part 2

We can prove that $f(x_1, x_2, x_3) = x_1x_2 + x_2x_3 + x_3x_1$ is not linear functional by contradiction:

If f were linear functional, then $-f(x_1, x_2, x_3) = f(-x_1, -x_2, -x_3)$, however:

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$$f(-x_1, -x_2, -x_3) = (-x_1)(-x_2) + (-x_2)(-x_3) + (-x_3)(-x_1) = x_1x_2 + x_2x_3 + x_3x_1 = f(x_1, x_2, x_3)$$

And since $f(-x_1, -x_2, -x_3) = f(x_1, x_2, x_3)$, then $f(-x_1, -x_2, -x_3) \neq -f(x_1, x_2, x_3)$, meaning that f is not linear functional.

Part 3

We can show $f(x_1, x_2, x_3) = |x_1| + |x_2| + |x_3|$ is not linear functional by example.

$f(1, 1, 1) = 3$, if f were linear functional, then $-f(1, 1, 1) = f(-1, -1, -1) = -3$, however, $f(-1, -1, -1) = 3 \neq -3$.

Therefore,$f(\lambda x) \neq \lambda f(x)$ for all x, which means that f is not linear functional.

Part 4

$f(x_1, x_2, x_3) = x_1^5 + x_2^5 + x_3^5$ is linear functional.

$$\begin{aligned}
f(\lambda x_1, \lambda x_2, \lambda x_3) &= (\lambda x_1)^5 + (\lambda x_2)^5 + (\lambda x_3)^5 && \mod 5 \\
&= \lambda^5 x_1^5 + \lambda^5 x_2^5 + \lambda^5 x_3^5 && \mod 5 \\
&= \lambda^5 (x_1^5 + x_2^5 + x_3^5) && \mod 5 \\
&= \lambda (x_1^5 + x_2^5 + x_3^5) && \mod 5
\end{aligned}$$

Note: $(\lambda^5 = \lambda) \mod 5$ we can prove this by showing all possible values for $\lambda \in \mathbb{Z}_5$

$$\begin{aligned}
&1^5 = 1 \mod 5 && 2^5 = 32 = 2 \mod 5 \\
&3^5 = 243 = 3 \mod 5 && 4^5 = 1024 = 4 \mod 5 \\
&5^5 = 3125 = 0 \mod 5
\end{aligned}$$

Note: $(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 = a^5 + b^5 \mod 5$

$$\begin{aligned}
f(x_1 + y_1, x_2 + y_2, x_3 + y_3) &= (x_1 + y_1)^5 + (x_2 + y_2)^5 + (x_3 + y_3)^5 && \mod 5 \\
&= x_1^5 + y_1^5 + x_2^5 + y_2^5 + x_3^5 + y_3^5 && \mod 5 \\
&= x_1^5 + x_2^5 + x_3^5 + y_1^5 + y_2^5 + y_3^5 && \mod 5 \\
&= f(x_1, x_2, x_3) + f(y_1, y_2, y_3)
\end{aligned}$$

Problem 5

Part 1

Poof by contradiction:

S is linearly dependent if $\exists \lambda$'s $\in \mathbb{Q}$ (not all 0), st:

$$\sum_{i=1}^{\infty} \lambda_i log(p_i) = 0$$

(assume that those lambdas exist)

Note: $log(x) = 0 \implies x = 1$

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_i log(p_i) &= \sum_{i=1}^{\infty} log(p_i^{\lambda_i}) \\
&= log(\sum_{i=1}^{\infty} p_i^{\lambda_i}) \\
&= 0 \\
\sum_{i=1}^{\infty} p_i^{\lambda_i} &= 1
\end{aligned}$$

But for the sum to be equal to one, $\lambda_i = 0$ for all values of i, this shows that S is not linearly dependent, meaning that it is linearly independent.

Part 2

We showed that S is linearly independent, therefore, if we assume that $\mathbb{R}_{\mathbb{Q}}$ is finite dimensional, with a dimension of n, then the first n elements of S would be a basis for $\mathbb{R}_{\mathbb{Q}}$.

However, the $n + 1$ element of S is also an element of $\mathbb{R}_{\mathbb{Q}}$, and it cannot be written in terms of the firs n elements that we chose (because S is linearly independent), therefore, those elements cannot be a basis.

This is a contradiction, showing that $\mathbb{R}_{\mathbb{Q}}$ is not finite dimensional.

Problem 6

Let V be a vector space over a field $\mathbb{F}$, and suppose that $f, g \in V^*$ are such that

$$g(x) = 0 \iff f(x) = 0$$

Take $x_0 \in V$ such that $g(x_0) \neq 0$ (if no such x_0 exists, then $g(x) = 0$ for all x, and therefore, $f = g$, so $\lambda = 1$). Define $n := g(x_0)$, and $m := f(x_0)$. We can rewrite m in terms of n: $m = (mn^{-1})n$, let $k = mn^{-1} \in \mathbb{F}$. Therefore, $n = g(x_0)$, $kn = f(x_0)$.

First, we need to show that $\forall x \in V$, $\exists \alpha$ such that $g(x + \alpha x_0) = 0$ we can do so algebraically:

Since g is linear, $g(0) = 0$ (*), therefore, find α such that $x - \alpha x_0 = 0 \implies \alpha = xx_0^{-1}$

$$\begin{aligned} g(x - \alpha x_0) &= 0 \\ g(x) - \alpha g(x_0) &= 0 \\ g(x) &= \alpha g(x_0) \\ &= \alpha n \end{aligned}$$

$$\begin{aligned} f(x - \alpha x_0) &= 0 \\ f(x) - \alpha f(x_0) &= 0 \\ f(x) &= \alpha f(x_0) \\ &= \alpha kn \\ &= k\alpha n \\ &= kg(x) \end{aligned}$$

Let $\lambda = k$, and we have proven that there exists λ such that $f = \lambda g$

(*) Take t to be a linear functional, then $f(0) = 0$. This can be shown as follows: $f(x+0) = n = f(x)$, $f(x) + f(0) = n$