

MT231 - Analysis 1

Homework #1

Angel Cervera Roldan
21319203

Problem 1

Prove that for any sets A and B , the following holds

$$A = (A \cap B) \cup (A \setminus B)$$

To prove this, we will show that any element $x \in A$ will also be a part of $(A \cap B) \cup (A \setminus B)$. For any element, it can either be in B or not be in B , therefore:

$$\begin{aligned} x &\in A \\ x &\in A \text{ and } (x \in B \text{ or } x \notin B) \\ (x \in A \text{ and } x \in B) &\text{ or } (x \in A \text{ and } x \notin B) \\ (x \in A \cap B) &\text{ or } (x \in A \setminus B) \\ x &\in (A \cap B) \cup (A \setminus B) \end{aligned}$$

Therefore, if $x \in A$, then $x \in (A \cap B) \cup (A \setminus B)$. This shows that $A \subseteq (A \cap B) \cup (A \setminus B)$ as every element in A has to be in $(A \cap B) \cup (A \setminus B)$.

Now we need to show that every element $x \in (A \cap B) \cup (A \setminus B)$ also has to be in A .

$$\begin{aligned} x &\in (A \cap B) \cup (A \setminus B) \\ x &\in (A \cap B) \text{ or } x \in (A \setminus B) \\ (x \in A \text{ and } x \in B) &\text{ or } (x \in A \text{ and } x \notin B) \\ x &\in A \text{ and } (x \in B \text{ or } x \notin B) \end{aligned}$$

Therefore, if $x \in (A \cap B) \cup (A \setminus B)$, then $x \in A$ and $(x \in B \text{ or } x \notin B)$, which shows that regardless of whether x is or isn't in B , x will be in A . This means that every element in $(A \cap B) \cup (A \setminus B)$ is in A . Therefore, $(A \cap B) \cup (A \setminus B) \subseteq A$.

Because $(A \cap B) \cup (A \setminus B)$ is a subset of A , and A is a subset of $(A \cap B) \cup (A \setminus B)$, that means that every element in one set must be in the other, they are therefore the same.

Problem 2

Let $a, b \in \mathbb{R}$, where $a < b$, find a bijection from (a, b) to $(0, 1)$.

Fist, find a bijection from (a, b) to $(0, n)$ for some $n \in \mathbb{R}$.

$$g : (a, b) \rightarrow (0, n)$$

$$g : (a, b) \rightarrow (a - a, b - a)$$

$$g : (a, b) \rightarrow (0, b - a)$$

$$g(x) = x - a$$

g is clearly a bijection as it is a linear function. Now, find a bijection from $(0, b - a)$ to $(0, 1)$.

$$p : (0, b - a) \rightarrow (0, 1)$$

$$p(0) = 0$$

$$p(b - a) = 1$$

$$p(x) = \frac{x}{b - a}$$

If we combine the two functions above, we get:

$$f(x) = p \circ g$$

$$= p(g(x))$$

$$= \frac{x - a}{b - a}$$

$$= \frac{1}{b - a}x - \frac{a}{b - a}$$

f will have the domain of g .

The range of g is the domain of p , therefore f will have the range of p . This means that

$$f : (a, b) \rightarrow (0, 1)$$

Because $\frac{1}{b-a}$ is a constant, the function $f(x)$ is linear, and therefore a bijection.

Problem 3

Prove that a function $f : A \rightarrow B$ which possesses an inverse must be a bijection.

For a function to be a bijection, it must be one-to-one, and onto. We will prove that if f has an inverse, then it must be both one-to-one and onto by contradiction.

f must be one-to-one

We know that f has an inverse, assuming that f isn't one-to-one, then $\exists a_1, a_2 \in A$ where $a_1 \neq a_2$ and $b \in B$ such that:

$$f(a_1) = f(a_2) = b$$

This would mean that:

$$f^{-1}(b) = a_1 \text{ and } f^{-1}(b) = a_2$$

But a function cannot return two different values when given one input.

This means that if f isn't one to one, it cannot have an inverse. This is a contradiction, as we know that f does have an inverse, meaning that f has to be one to one.

f must be a onto

We know that f has an inverse, assuming that f isn't onto, then $\exists b \in B$ such that $f(a) \neq b$ for any $a \in A$.

If f had an inverse $f^{-1}(b)$ would be undefined, meaning that $f^{-1} : B \rightarrow A$ wouldn't exist. Therefore, for f to have an inverse, it must be onto.

Because we know that f must be one-to-one, and f must be onto, f must be a bijection if it has an inverse.

Problem 4

Part a

Consider the function $f : A \rightarrow B$. Show that setting $a_1 \sim a_2$ if $f(a_1) = f(a_2)$ defines an equivalence relation on A .

1. $a \sim a$ since $f(a) = f(a)$
2. if $a \sim b$, then $b \sim a$ since $f(a) = f(b)$, then $f(b) = f(a)$
3. if $a \sim b$ and $b \sim c$, then $a \sim c$ since if $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$

Part b

Identify the equivalence classes under this equivalence relation if $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^2$

$$f(a) = f(b)$$

$$a^2 = b^2$$

$$\pm a = b$$

Therefore the equivalence class of any x is:

$$[x] = \{x, -x\}$$

Part c

Identify the equivalence classes under this equivalence relation if $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = \lfloor x \rfloor$

There are an infinite amount of equivalence classes, all in the form $[n, n + 1)$ for some integer n

$$[x] = [\lfloor x \rfloor, \lfloor x \rfloor + 1)$$

Problem 5

Let C be the set of counties in Ireland.

Part a

Give an example of an equivalence relation on C . What are the equivalence classes of this relation?

An example of an equivalence relation on C would be $a \sim b$ if the first letter of a is the same as the first letter of b .

This is an equivalence relation as it fulfils all 3 rules:

1. $a \sim a$ since a will always have the same name as a , a will always have the same first letter as a .
2. if $a \sim b$, then $b \sim a$ since if a and b have the same first letter, then b and a will also have the same first letter.
3. if $a \sim b$, and $b \sim c$, then $a \sim c$. If a and b have the same first letter, and b and c have the same first letter, then c and a must have the same first letter, and therefore, $a \sim c$.

The equivalence relation classes would be all counties starting with a , all counties starting with b , all counties starting with c ...

Part b

Give another example of an equivalence relation on C .

Another example of an equivalence relation on C would be $a \sim b$ if a and b have the same number of houses.

This is an equivalence relation as it fulfils all 3 rules:

1. $a \sim a$ since a will always have the same houses as a .
2. if $a \sim b$, then $b \sim a$ since if a and b have the same number of houses, then b and a will also have the same number of houses.
3. if $a \sim b$, and $b \sim c$, then $a \sim c$. If a and b have the same number of houses, and b and c have the same number of houses, then c and a must have the same number of houses, and therefore, $a \sim c$.

Part c

Give an example of a relation on C which is not an equivalence relation.

An example of a relation on C which is not an equivalence relation would be $a \sim b$ if a and b border each other.

This is not an equivalence relation as Waterford and Cork border each other, therefore Waterford \sim Cork. Cork also borders Kerry therefore Cork \sim Kerry. However Waterford $\not\sim$ Kerry.

This breaks the transitive property of equivalence relations.