MT241P-solution1

MT241P assignment 1
from 5 October to 14 October 2022

Question 1

Part 1

Prove that for any $n \geq 1$

$$\sum_{i=1}^{n} i^2 = \frac{n \cdot (n+1) \cdot (2n+1)}{6}$$

First we show that the formula holds up for $n = n_0 = 1$

$$\sum_{i=1}^{1} i^2 = 1^2 = \frac{1 \cdot (1+1) \cdot (2+1)}{6} = 1$$

Therefore it holds up for $n = n_0 = 1$, now we assume that the formula holds up for $n = k$, where $k \geq n_0$, and we check if it will also hold up for $n = k + 1$.

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= \frac{(k+1) \cdot (k+1+1) \cdot (2(k+1)+1)}{6} \\ &= \frac{(k+1) \cdot (k+2) \cdot (2k+3)}{6} \end{aligned}$$

Also,

$$\begin{aligned} \sum_{i=1}^{k+1} i^2 &= (k+1)^2 + \frac{k \cdot (k+1) \cdot (2k+1)}{6} \\ &= \frac{6(k+1)^2}{6} + \frac{k \cdot (k+1) \cdot (2k+1)}{6} \\ &= \frac{(k+1)[6(k+1) + k \cdot (2k+1)]}{6} \\ &= \frac{(k+1) \cdot (6k + 6 + 2k^2 + k)}{6} \\ &= \frac{(k+1) \cdot (2k^2 + 7k + 6)}{6} \\ &= \frac{(k+1) \cdot (k+2) \cdot (2k+3)}{6} \end{aligned}$$

Therefore, if it is true for $n = k$, it then also is true for $n = k + 1$, and because it is true for $n = n_0$, it is also true for all the successors of n_0, therefore it must be true for every $n \in \{1, 2, 3, \dots\}$

Part 2

Prove that for any $n \geq 5$

$$n^2 < 2^n$$

We start off y checking if if holds up for the base case, when $n = 5$=

$$\begin{aligned} 5^2 &< 2^5 \\ 25 &< 32 \end{aligned}$$

Therefore we know that it holds up for the base case, now we assume it works for some n, and we show that if it holds for n, then it must hold for $n + 1$.

$$\begin{aligned} (n+1)^2 &< 2^{n+1} \\ n^2 + 2n + 1 &< 2^n \cdot 2 \\ n^2 + 2n + 1 &< 2^n + 2^n \end{aligned}$$

Because we have assumed that $n^2 < 2^n$, then we know that the above is true if $2n + 1 \leq 2^n$, and again, because $n^2 < 2^n$, we know that if $2n + 1 \leq n^2$, then $2n + 1 \leq 2^n$.

$$
\begin{aligned}
2n + 1 &\leq n^2 \\
-n^2 + 2n + 1 &\leq 0
\end{aligned}
$$

Using the minus b formula, we can see that $2n + 1 \leq n^2$ for any n where $1 + \sqrt{2} < n$, and $1 + \sqrt{2} < 5$, therefore it is true for any n later or equal to 5.

Question 2

Given the following function:

$$
\begin{aligned}
f(1,1) &= 2 \\
f(m+1,n) &= f(m,n) + 2 \cdot (m+n) \\
f(m,n+1) &= f(m,n) + 2 \cdot (m+n-1)
\end{aligned}
$$

Prove that $f(m,n) = (m+n)^2 - (m+n) - 2n + 2$.

In the base case, $m = 1$, and $n = 1$.

$$
f(1,1) = 2 = (1+1)^2 - (1+1) - 2 + 2 = 4 - 2 = 2
$$

Therefore, it holds up for the base case.

Now we will assume that it holds up for $f(m,1)$, and if that's the case, that it also holds up for $f(m+1,1)$.

From the definition of the function, we know that $f(m+1,1) = f(m,1) + 2 \cdot (m+1)$, and because we have assumed that the formula holds up for $f(m,1)$, we know that

$$
\begin{aligned}
f(m+1,1) &= f(m,1) + 2 \cdot (m+1) \\
&= (m+1)^2 - (m+1) - 2 + 2 + 2 \cdot (m+1) \\
&= (m+1)^2 - (m+1) + 2 \cdot (m+1) \\
&= (m+1)^2 + (m+1) \\
&= m^2 + 2m + 1 + m + 1 \\
&= m^2 + 3m + 2
\end{aligned}
$$

$$
\begin{aligned}
f(m+1,1) &= (m+1+1)^2 - (m+1+1) - 2 + 2 \\
&= (m+2)^2 - (m+2) \\
&= m^2 + 2m + 2m + 4 - m - 2 \\
&= m^2 + 3m + 2
\end{aligned}
$$

Therefore, if $f(m,1)$ holds for some $m \in \mathbb{N}$, then it will hold for every value of m.

Now we will assume that it holds for $f(m,n)$, and show by induction that if that holds, then $f(m,n+1)$ will also hold.

$$
\begin{aligned}
f(m,n+1) &= f(m,n) + 2 \cdot (m+n-1) \\
&= (m+n)^2 - (m+n) - 2n + 2 + 2 \cdot (m+n-1) \\
&= (m+n)^2 + m - n
\end{aligned}
$$

$$
\begin{aligned}
f(m,n+1) &= (m+n+1)^2 - (m+n+1) - 2(n+1) + 2 \\
&= m^2 + n^2 + 2mn + 2m + 2n + 1 - m - n - 1 - 2n - 2 + 2 \\
&= m^2 + n^2 + 2mn + m - n \\
&= (m+m)^2 + m - n
\end{aligned}
$$

Therefore we have shown that if it holds for some $f(m,n)$, it will also hold for $f(m,n+1)$ where m is constant, and we have also shown that if $f(m,1)$ holds for some m, then it will hold for any $f(m,1)$. Because our base test showed that it holds for $f(1,1)$, then we know it woks for $f(m,1)$, and because it woks for $f(m,1)$, it will work for $f(m,n)$.

Question 3

Given the "Lucas Sequence":
$T_1 = 1$, $T_2 = 3$, $T_n = T_{n-1} + T_{n-2}$, find a rational x, where $x < \frac{7}{4}$, such that

$$
T_n < x^n
$$

To solve this, we will assume that there exist a non empty set A such that $T_n < a^n$, $\forall a \in A$, where every element in A is smaller than $\frac{7}{4}$.

We need to use strong induction, therefore there are two base cases, $n = 1$, and $n = 2$

$$
\begin{aligned}
T_1 &< a^1 \\
1 &< 1 \\
T_2 &< a^2 \\
\sqrt{3} &< a
\end{aligned}
$$

We will set the restriction of every element $a \in A$ to be larger than the positive root of 3. With this restriction, the two base cases will pass for every element in A.

$$
\begin{aligned}
T_{n+1} &= T_n + T_{n-1} \\
a^{n+1} &> a^n + a^{n-1} \\
&> a^{n-1}(a+1) \\
a^{n-1} \cdot a^2 &> a^{n-1}(a+1) \\
a^2 &> a+1 \\
0 &> a^2 - a - 1
\end{aligned}
$$

Using the minus b formula, we can see that the above is true for all numbers larger than $\frac{1+\sqrt{5}}{2}$. Because $\sqrt{3} > \frac{1+\sqrt{5}}{2}$, we can see that the set A can be defined as $A := \{x | x \in \mathbb{Q} \wedge \sqrt{3} < x < \frac{7}{4}\}$

We have by induction proven that $\forall a \in A$, $T_n < a^n$. A sample value for a would be $\frac{174}{100}$.

Question 4

Part 1

Show that if $a|b$, and $b|c$, then $a|c$

If $a|b$, then $\exists p \in \mathbb{Z}$ such that $\frac{b}{a} = p$, $\therefore pa = b$, and if $b|c$, then $\exists q$ such that $qb = c$.

If $a|c$, then there must exist a k such that $ka = c$.

Because we know that $pa = b$ and $qb = c$, $qpa = c$, and that an integer times an integer will always be an integer, we know that $k = qp$ will be an integer. Therefore we have prove that if $a|b$, and $b|a$, there must exist an integer k such that $ka = c$, therefore $a|c$.

Part 2

Show that if $a|b$, and $a|c$, then $a|(ub + vc)$.

First i will show that if $a|n$ and $a|m$, then, $a|(n+m)$:

If $a|n$, then there exist an integer p such that $ap = n$, and if $a|m$, then there exists an integer q such that $aq = m$.

To show that $a|(n+m)$, we need to demonstrate that there exists an integer k such that $ka = n + m$

$$
\begin{aligned}
ap &= n \\
aq &= m \\
ap + aq &= n + m \\
a(p+q) &= n + m \\
\text{Define } k: &= p + q \\
ak &= n + m
\end{aligned}
$$

Where we know that k is an integer since p and q are both integers, and adding two integers will result in an integer.

Therefore, if $a|n$ and $a|m$, then, $a|(n+m)$, now that we know this, we can show that $a|b$, and $a|c$, then $a|(ub+vc)$ if $a|ub$ and $a|vc$.

Now we need to show that for any integer u, if $a|n$, then $a|un$. To show that $a|un$, we need to prove that $un = la$ for some integer l.
Because we know that $a|n$, that means that n is a multiple of a, therefore $n = ka$ for some integer k. Therefore $un = uka$, because u and k are both integers, their product l will also be an integer. That that means that $un = la$ for some integer l. Therefore, if $a|n$, then $a|un$ for any integer n.

Now that we know that if $a|n$ and $a|m$, then, $a|(n+m)$, and that if $a|n$, then $a|un$, we prove that if $a|b$, and $a|c$, then $a|(ub+vc)$.

If $a|(ub+vc)$, then $a|vc$ $a|ub$, and if $a|c$, then $a|vc$ is true for any integer v, and $a|ub$ is true for any integer u. Therefore if $a|b$, and $a|c$, then $a|(ub+vc)$.

Question 5

Part 1

Show that 2003 divides $4007^n - 1$, for all integers $n \geq 0$.

We say that a number a divides by b, if b is a multiple of a. Therefore, if a divides b there exist a p such that $\frac{b}{a} = p$, where $p \in \mathbb{N}_0$

Base case, when $n = 0$

$$\frac{4007^0 - 1}{2003} = \frac{0}{2003} = 0$$

Therefore it holds for $n = 0$. Now we assume for some n, where $n \geq 0$. We now need to show that if it holds for n, it will also hold for $n + 1$.

$$\begin{aligned}
\frac{4007^{n+1} - 1}{2003} &= \frac{4007^n \cdot 4007 - 1}{2003} \\
&= 4007 \cdot \frac{4007^n - \frac{1}{4007}}{2003} \\
&= 4007 \cdot \frac{4007^n + 1 - 1 - \frac{1}{4007}}{2003} \\
&= 4007 \cdot (\frac{4007^n - 1}{2003} + \frac{1 - \frac{1}{4007}}{2003}) \\
&= 4007(\frac{4007^n - 1}{2003}) + \frac{4007 - 1}{2003} \\
&= 4007(\frac{4007^n - 1}{2003}) + 2
\end{aligned}$$

And because $\frac{4007^k - 1}{2003}$ is, by assumption an integer, and, an integer times an integer is an integer, then $4007(\frac{4007^k - 1}{2003})$ is an integer. Adding two to an integer, will also return an integer.

Question 6

Show that 5 divides $n^5 - n$, for any $n \in \mathbb{Z}$

If 5 divides $n^5 - n$, for any value of n, $\exists p$, where $p \in \mathbb{Z}$ such that $\frac{n^5 - n}{5} = p$.

For the base case we can use $n = 0$

$$\frac{0^2 - 0}{5} = 0$$

Therefore it hold for $n = 0$

Now we assume that it works for some n, and we try to prove that if it holds for n, then it also holds for $n + 1$

$$\begin{aligned}
\frac{(n+1)^5 - n - 1}{5} &= \frac{1 + 5n + 10n^2 + 10n^3 + 5n^4 + n^5 - n - 1}{5} \\
&= \frac{n^5 + 5n^4 + 10n^3 + 10n^2 + 4n}{5} \\
&= \frac{n^5 - n}{5} + \frac{5n^4 + 10n^3 + 10n^2 + 5n}{5} \\
&= \frac{n^5 - n}{5} + 5 \cdot \frac{n^4 + 2n^3 + 2n^2 + n}{5} \\
&= \frac{n^5 - n}{5} + n^4 + 2n^3 + 2n^2 + n
\end{aligned}$$

Because we know by assumption that $\frac{n^5 - n}{5}$ is an integer, and that any series of multiplication on integers, and addition on integers will result in an integer, then we know that $\frac{n^5 - n}{5} + n^4 + 2n^3 + 2n^2 + n$ must be an integer for any $n \in \mathbb{Z}$.