# MT251P - Euclidean Geometry

# Homework #1

**Angel Cervera Roldan**
21319203

# Problem 1

## a)

Prove that $\sqrt[3]{2}$ is irrational, for this we will use contradiction. Assume that $\sqrt[3]{2} = \frac{a}{b}$, where the fraction is in its simplest form, meaning that a and b are coprimes.

$$\begin{aligned} \sqrt[3]{2} &= \frac{a}{b} \\ 2 &= \frac{a^3}{b^3} \\ 2b^3 &= a^3 \end{aligned}$$

This shows that $a^3$ is an even number, and therefore, a is an even number. This can be proven:
An odd number $2n + 1$ times an odd number will always be an odd number: $(2n + 1)(2n + 1) = 4n^2 + 4n + 1 = 2(2n^2 + 2n) + 1$. Therefore an odd times an odd is an odd, and that times an odd is also an odd.
We can now re-write a as $2n$

$$\begin{aligned} \sqrt[3]{2} &= \frac{2n}{b} \\ 2 &= \frac{(2n)^3}{b^3} \\ 2b^3 &= (2n)^3 \\ 2b^3 &= (2n)(4n^2) \\ 2b^3 &= 8n^3 \\ b^3 &= 4n^3 = 2(2n^3) \end{aligned}$$

Therefore $b^3$ is an even number, and therefore $b$ is an even number.
But b and a cannot have 2 as a common factor, as the fraction was in the simplest form.

## b)

$\frac{x}{y}$ is irrational for all irrational numbers $x$, $y$ with $x \neq y$
This is not true, we can disprove it with a counter example: $x = \sqrt{5}$, $y = 2\sqrt{5}$. Because $\sqrt{5} \neq 2\sqrt{5}$, $x \neq y$, but $\frac{\sqrt{5}}{2\sqrt{5}} = \frac{1}{2}$, which is a rational number.
We know that $x$ is irrational (as $\sqrt{5}$ is proven to be irrational in the next question), and $y$ is also irrational as $\sqrt{5}$ is irrational, and an integer multiple of an irrational is an irrational.

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## c)

There exist rational numbers $m, t, w$ such that $m - 2w + t\sqrt{2} = 0$.
A rational number $n$, where $n \neq 0$ times an irrational number will return an irrational, therefore, the only possible value for $t$ is 0.
Therefore, the equation above can be rewritten as $m - 2w + 0 = 0$

$$m - 2w = 0$$
$$m = 2w$$

Therefore there are an infinite number of rational solutions for the above equation. For any rational $x$, the equation will hold for $m = x$, $w = \frac{x}{2}$.
Example: $m = 1$, $w = \frac{1}{2}$, $t = 0$, then $1 - 2\frac{1}{2} + 0\sqrt{2} = 1 - 1 + 0 = 0$.

# Problem 2

Prove that $\sqrt{5}$ is irrational
Proof by contradiction:
Assume that $\sqrt{5}$ is a rational number, then $\sqrt{5} = \frac{a}{b}$ for some two integers a and b, where the fraction is in its simplest form (a and b are co-primes).

$$\begin{aligned}\sqrt{5} &= \frac{a}{b}\\ 5 &= \frac{a^2}{b^2}\\ 5b^2 &= a^2\end{aligned}$$

Since $b^2 \in \mathbb{Z}$, then a is a multiple of 5. Because $a^2$ is a multiple of 5, then $a$ must be a multiple of 5. Therefore $\exists n \in \mathbb{Z}$ such that $a = 5n$.
We can therefore rewrite the above equation as:

$$\begin{aligned}\sqrt{5} &= \frac{5n}{b}\\ 5 &= \frac{(5n)^2}{b^2}\\ 5b^2 &= (5n)^2\\ 5b^2 &= 25n^2\\ b^2 &= 5n^2\end{aligned}$$

Therefore, both a and b are multiples of 5. This contradicts the beginning assumption that $\frac{a}{b}$ was in its simplest form.

# Problem 3

Prove that $\neg(\neg P \vee Q) \vee (P \wedge Q) \sim P$

# Problem 4

Is $(\neg P \wedge Q) \Rightarrow (Q \vee \neg P)$ a tautology?

# Problem 5

Do two integers $x, y$ exist such that $9y - x^2 + 3 = 0$?
To solve this, we will use proof by contradiction. Assume that there exist two integers such that the above holds.

$$\begin{aligned} 9y - x^2 + 3 &= 0 \\ 9y - x^2 &= -3 \\ \frac{9y - x^2}{3} &= 3y - \frac{x^2}{3} \\ 3y - \frac{x^2}{3} &= -1 \in \mathbb{Z} \end{aligned}$$

Because -1 is an integer, then $3y - \frac{x^2}{3}$ must also be an integer.
Given we have assumed that $y$ is an integer, then we know that $3y$ is also an integer. Because $3y$ is an integer, and $3y - \frac{x^2}{3}$ is also an integer, then $\frac{x^2}{3}$ must also be an integer.
Because of this, we know that $x^2$ must be a multiple of 3. This implies that $x$ is also a multiple of 3. Therefore $\exists n \in \mathbb{Z}$ such that $x = 3m$.
Now we can rewrite the original equation as:

$$\begin{aligned} 9y - (3n)^2 + 3 &= 0 \\ 9y - 9n^2 &= -3 \\ 9(y - n^2) &= -3 \\ y - n^2 &= -\frac{3}{9} \\ y - n^2 &= -\frac{1}{3} \end{aligned}$$

But because we have assumed that $y$ and $n$ are both integers, then we know that $n^2$ is also an integer, since an integer times an integer is an integer. We also know that $y - n^2$ is an integer because an integer minus another integer is an integer. Since $-\frac{1}{3}$ is equal to $y - n^2$, this wrongly implies that $-\frac{1}{3}$ is an integer.
This is a contradiction, meaning that there exists no two integers that satisfy the above equation.