MT231 - Analysis 1

Homework #1

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Problem 1

Prove that for any sets A and B, the following holds

$$A = (A \cap B) \cup (A \setminus B)$$

To prove this, we will show that any element $x \in A$ will also be a part of $(A \cap B) \cup (A \setminus B)$
For any element, it can either be in B or not be in B, therefore:

$$\begin{aligned}
&x \in A \\
&x \in A \text{ and } (x \in B \text{ or } x \notin B) \\
&(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \notin B) \\
&(x \in A \cap B) \text{ or } (x \in A \setminus B) \\
&x \in (A \cap B) \cup (A \setminus B)
\end{aligned}$$

Therefore, if $x \in A$, then $x \in (A \cap B) \cup (A \setminus B)$. This shows that $A \subseteq (A \cap B) \cup (A \setminus B)$ as every element in A has to be in $(A \cap B) \cup (A \setminus B)$.
Now we need to show that every element $x \in (A \cap B) \cup (A \setminus B)$ also has to be in A.

$$\begin{aligned}
&x \in (A \cap B) \cup (A \setminus B) \\
&x \in (A \cap B) \text{ or } x \in (A \setminus B) \\
&(x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } x \notin B) \\
&x \in A \text{ and } (x \in B \text{ or } x \notin B)
\end{aligned}$$

Therefore, if $x \in (A \cap B) \cup (A \setminus B)$, then $x \in A$ and $(x \in B$ or $x \notin B)$, which shows that regardless of wether x is or isn't in B, x will be in A. This means that every element in $(A \cap B) \cup (A \setminus B)$ is in A. Therefore, $(A \cap B) \cup (A \setminus B) \subseteq A$.
Because $(A \cap B) \cup (A \setminus B)$ is a subset of A, and A is a subset of $(A \cap B) \cup (A \setminus B)$, that means that every element in one set must be in the other, they are therefore the same.

Problem 2

Let $a, b \in \mathbb{R}$, where $a < b$, find a bijection from (a, b) to $(0, 1)$.

Fist, find a bijection from (a, b) to $(0, n)$ for some $n \in \mathbb{R}$.

$$\begin{aligned}
&g : (a, b) \to (0, n) \\
&g : (a, b) \to (a - a, b - a) \\
&g : (a, b) \to (0, b - a) \\
&g(x) = x - a
\end{aligned}$$

g is clearly a bijection as it is a linear function. Now, find a bijection from $(0, b - a)$ to $(0, 1)$.

$$\begin{aligned}
p : (0, b - a) &\to (0, 1) \\
p(0) &= 0 \\
p(b - a) &= 1 \\
p(x) &= \frac{x}{b - a}
\end{aligned}$$

If we combine the two functions above, we get:

$$\begin{aligned}
f(x) &= p \circ g \\
&= p(g(x)) \\
&= \frac{x - a}{b - a} \\
&= \frac{1}{b - a}x - \frac{a}{b - a}
\end{aligned}$$

f will have the domain of g.
The range of g is the domain of p, therefore f will have the range of p. This means that $f : (a, b) \to (0, 1)$
Because $\frac{1}{b-a}$ is a constant, the function $f(x)$ is linear, and therefore a bijection.

Problem 3

Prove that a function $f : A \to B$ which possesses an inverse must be a bijection.

For a function to be a bijection, it must be one-to-one, and onto. We will prove that if f has an inverse, then it must be both one-to-one and onto by contradiction.

f must be one-to-one

We know that f has an inverse, assuming that f isn't one-to-one, then $\exists a_1, a_2 \in A$ where $a_1 \neq a_2$ and $b \in B$ such that:

$$f(a_1) = f(a_2) = b$$

This would mean that:

$$f^{-1}(b) = a_1 \text{ and } f^{-1}(b) = a_2$$

But a function cannot return two different values when given one input.
This means that if f isn't one to one, it cannot have an inverse. This is a contradiction, as we know that f does have an inverse, meaning that f has to be one to one.

f must be a onto

We know that f has an inverse, assuming that f isn't onto, then $\exists b \in B$ such that $f(a) \neq b$ for any $a \in A$.
If f had an inverse $f^{-1}(b)$ would be undefined, meaning that $f^{-1} : B \to A$ wouldn't exist. Therefore, for f to have an inverse, it must be onto.

Because we know that f must be one-to-one, and f must be onto, f must be a bijection if it has an inverse.

Problem 4

Part a

Consider the function $f : A \to B$. Show that setting $a_1 \sim a_2$ if $f(a_1) = f(a_2)$ defines an equivalence relation on A.

1. $a \sim a$ since $f(a) = f(a)$
2. if $a \sim b$, then $b \sim a$ since $f(a) = f(b)$, then $f(b) = f(a)$
3. if $a \sim b$ and $b \sim c$, then $a \sim c$ since if $f(a) = f(b)$ and $f(b) = f(c)$, then $f(a) = f(c)$

Part b

Identify the equivalence classes under this equivalence relation if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = x^2$

$$\begin{aligned} f(a) &= f(b) \\ a^2 &= b^2 \\ \pm a &= b \end{aligned}$$

Therefore the equivalence class of any x is:

$$[x] = \{x, -x\}$$

Part c

Identify the equivalence classes under this equivalence relation if $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = \lfloor x \rfloor$

There are an infinite amount of equivalence classes, all in the form $[n, n+1)$ for some integer n

$$[x] = [\lfloor x \rfloor, \lfloor x \rfloor + 1)$$

Problem 5

Let C be the set of counties in Ireland.

Part a

Give an example of an equivalence relation on C. What are the equivalence classes of this relation?

An example of an equivalence relation on C would be $a \sim b$ if the first letter of a is the same as the first letter of b.
This is an equivalence relation as it fulfils all 3 rules:

1. $a \sim a$ since a will always have the same name as a, a will always have the same first letter as a.
2. if $a \sim b$, then $b \sim a$ since if a and b have the same first letter, then b and a will also have the same first letter.
3. if $a \sim b$, and $b \sim c$, then $a \sim c$. If a and b have the same first letter, and b and c have the same first letter, then c and a must have the same first letter, and therefore, $a \sim c$.

The equivalence relation classes would be all counties starting with a, all counties starting with b, all counties starting with c...

Part b

Give another example of an equivalence relation on C.

Another example of an equivalence relation on C would be $a \sim b$ if a and b have the same number of houses.
This is an equivalence relation as it fulfils all 3 rules:

1. $a \sim a$ since a will always have the same houses as a.
2. if $a \sim b$, then $b \sim a$ since if a and b have the same number of houses, then b and a will also have the same number of houses.
3. if $a \sim b$, and $b \sim c$, then $a \sim c$. If a and b have the same number of houses, and b and c have the same number of houses, then c and a must have the same number of houses, and therefore, $a \sim c$.

Problem 5 continued on next page. . .

Part c

Give an example of a relation on C which is not an equivalence relation.

An example of a relation on C which it not an equivalence relation would be $a \sim b$ if a and b border each other.
This is not an equivalence relation as Waterford and Cork border each other, therefore Waterford $\sim$ Cork. Cork also borders Kerry therefore Cork $\sim$ Kerry. However Waterford $\nsim$ Kerry.
This breaks the transitive property of equivalence relations.