Deterministic Finite Automata

COMP2600 — Formal Methods for Software Engineering

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Pop quiz

What is

$$x - y\left(\frac{x}{y}\right)$$

where

- $x = 4195835$;
- $y = 3145727$?

Pop quiz - answer

If you answered $\mathbf{256}$, congratulations! You are a **Pentium microprocessor** released by **Intel** in 1993.

This floating point error evaded Intel's testing, only to be discovered by computational number theorist **Thomas R. Nicely** while generating large prime numbers.

A full recall of the chip ended up costing Intel ***US$475 million***.

But how can Intel ever have confidence that their hardware is error-free and so won't cost them money and reputation?

Sounds like a job for **formal methods**!

Formal modelling

Intel are now world leaders in the formal verification of hardware.

A full description of the formal methods approaches they use is outside the bounds of this course.

But here's a specific problem - how do you reason **mathematically** about a **physical** object like a computer chip?

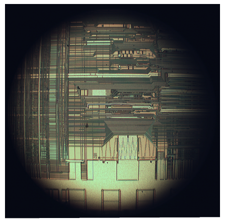


Photo by Matt Britt & Matt Gibbs

Abstract Machines

The solution is to model the physical object via a mathematically defined **abstract machine**.

This approach is also useful for software, and is particularly well suited to certain applications such as parsing.

Abstract machines come in various flavours, and will be the focus of the rest of this course.

The technical highlight will be **Turing machines**, which are powerful enough to perform any computation that can be imagined!

However less powerful abstract machines are often easier to reason about and hence more useful for formal methods applications.

Finite State Automata

The first flavour of abstract machine we will look at will be the *least* powerful - **Finite State Automata (FSA)**.

They are still powerful enough to encode the actions of microprocessors, and so are a fundamental formal methods concept applied by Intel and others.

In fact they were invented, not by computer scientists interested in hardware, but by two **neuropsychologists**, Warren S. McCullough and Walter Pitts, who back in 1943 were trying to model neurons in the human brain!

Such unlikely connections are a common theme in the history of science.

Input and Output

Finite state automata in this course will

- Accept as **input** a string of symbols.
- Produce as **output** success or failure only.

Therefore one way to think about abstract machines is to ask which strings are accepted and which rejected by them.

Conversely, given a set of strings (called a **language**) we might try to design an abstract machine that recognises all and only those strings.

The study of abstract machines and the study of (formal) languages are therefore closely linked.

Language Examples

Formal Language Theory gets applied to natural and artificial languages and to programming languages, but in this course we will look at simpler examples over small alphabets, e.g.:

1. finite sets.

$$\{a, aa, ab, aaa, aab, aba, abb\}$$

2. Palindromes consisting of bits (0,1):

$$\{0, 1, 00, 11, 010, 101, 000, 111, 0110, ...\}$$

In each case the language is the set of strings. The first is a finite language and the second is an infinite language.

Language Terminology

- The **alphabet** (or **vocabulary**) of a formal language is a set of **tokens** (or **letters**). It is usually denoted Σ.
- A **string** (or **word**) over Σ is a *sequence* of tokens, or the null-string ε.
  For example, if $\Sigma = \{a, b, c\}$, then $ababc$ is a string over Σ.
- A **language** with alphabet Σ is some set of strings over Σ.
- The strings of a language are called the **sentences** of the language.

Notation:

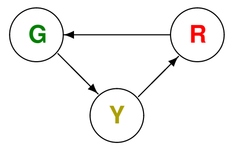
- Σ^* is the set of all strings over Σ.
- Therefore, every language with alphabet Σ is some subset of Σ^*.

Finite State Automata

The simplest useful abstraction of a "computing machine" consists of:

- A fixed, finite set of **states**
- A **transition relation** over the states

Simplified Example: a traffic light machine has 3 states:

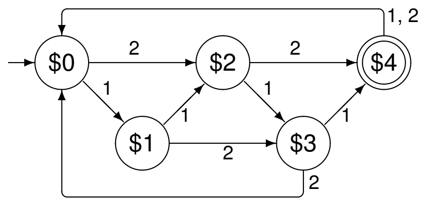


G names state in which light is green.
Y names state in which light is yellow.
R names state in which light is red.

Some information is missing – where does the machine ***start*** and ***stop***, and what ***triggers*** transitions between states?

Example: Vending Machine

Imagine a vending machine which (1) accepts \$1 and \$2 coins;
(2) refunds all money if more than \$4 is added;
(3) is ready to deliver if exactly \$4 has been added.



Note that the ***transitions*** are labelled with important information, and the ***start state*** and ***end state(s)*** are specially labelled.

The Vending Machine ctd

- You start at the state "\$0", indicated by the $\rightarrow$ at the left
- the alphabet $\Sigma = \{1, 2\}$
- at the "\$4" state (circled) you have credit for a purchase
- what strings of tokens (starting at the "\$0") leave you in the circled state?
  - the empty string ε ✗
  - 22 ✓
  - 1222 ✗
  - 1222221111 ✓
- the accepted strings are the language defined by the automaton.

Terminology

We will study **deterministic** finite automata (**DFA**) first.

- The **alphabet** of a DFA is a finite set of *input tokens* that an automaton acts on.
- a DFA has a finite set of **states**.
- One of the states is the **initial** state — where the automaton starts.
- At least one of the states is a **final** state.
- A **transition function** *(next state function)*:

$$State \times Token \ \rightarrow \ State$$

Formal Definition of DFA

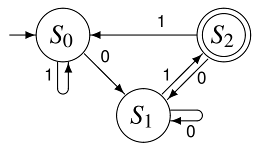
A Deterministic Finite State Automaton (DFA) is completely characterised by five items:

$$(\Sigma, S, s_0, F, N)$$

- an input alphabet Σ, the set of tokens
- a finite set of states S
- an “initial” state $s_0 \in S$ (we start here)
- a set of “final” states $F \subseteq S$ (we hope to finish in one of these)
- a transition function $N : S \times \Sigma \ \rightarrow \ S$

(We will see later that N being a ***function*** is the reason the automaton is deterministic.)

Example 1



- Alphabet - $\{0, 1\}$
- States - $\{S_0, S_1, S_2\}$
- Initial state - S_0
- Final states - $\{S_2\}$
- Transition function (as a table) -

	0	1
S_0	S_1	S_0
S_1	S_1	S_2
S_2	S_1	S_0

(Note that the actual state names are irrelevant.)

Example 1, ctd

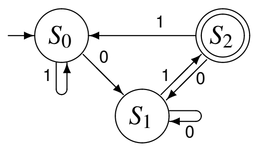
If N is the transition function, then we write $N(S_0, 0)$ when we want to refer to the state that the above automaton goes to when it starts in S_0 and absorbs a 0.

From the definition, $N(S_0, 0) \ = \ S_1$.

Similarly, $N(S_1, 1) \ = \ S_2$

Eventual State Function

Revisit example 1:



- Input 0101 takes the DFA from S_0 to S_2,
  Input 1011 takes the DFA from S_1 to S_0, etc.
- A complete list of such possibilities is a function from a given state and a string to an 'eventual state.'

This is the automaton's ***Eventual State Function.***

Eventual State Function — Definition

Suppose A is a DFA with states S, alphabet Σ, and transition function N. The eventual state function for A is

$$N^* : S \times \Sigma^* \to S$$

$N^*(s,w)$ is the state A reaches, starting in state s and reading string w.

N^* can be defined inductively:

$$N^*(s,\varepsilon) \;=\; s \qquad \text{(N1)}$$

$$N^*(s,x\alpha) \;=\; N^*(N(s,x),\alpha) \qquad \text{(N2)}$$

It is the eventual state function acting on the **start state** that we are really interested in.

An Important (but Unsurprising) Theorem about N^*

The "Append" Theorem

For all states $s \in S$ and for all strings $\alpha,\beta \in \Sigma^*$

$$N^*(s,\alpha\beta) \;=\; N^*(N^*(s,\alpha),\beta)$$

Proof by induction on the length of α.

Base case: $\alpha = \varepsilon$

$$\begin{aligned} \mathrm{LHS} &= N^*(s,\varepsilon\beta) = N^*(s,\beta) \\ \mathrm{RHS} &= N^*(N^*(s,\varepsilon),\beta) \\ &= N^*(s,\beta) = \mathrm{LHS} && \text{(by (N1))} \end{aligned}$$

Proof ctd: Step case:

Suppose $N^*(s,\alpha\beta) \;=\; N^*(N^*(s,\alpha),\beta)$ (IH)

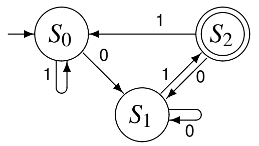
$$\begin{aligned} \mathrm{LHS} &= N^*(s,(x\alpha)\beta) \\ &= N^*(s,x(\alpha\beta)) \\ &= N^*(N(s,x),\alpha\beta) && \text{(by (N2))} \\ &= N^*(N^*(N(s,x),\alpha),\beta) && \text{(by IH)} \end{aligned}$$

$$\begin{aligned} \mathrm{RHS} &= N^*(N^*(s,x\alpha),\beta) \\ &= N^*(N^*(N(s,x),\alpha),\beta) && \text{(by (N2))} \end{aligned}$$

Corollary — when β is a single token

$$N^*(s,\alpha y) \;=\; N(N^*(s,\alpha),y)$$

Example



$$
\begin{aligned}
N^*(S_1, 1011) &= N^*(N(S_1,1), 011)\\
&= N^*(S_2, 011)\\
&= N^*(S_1, 11)\\
&= N^*(S_2, 1)\\
&= N^*(S_0, \varepsilon)\\
&= S_0
\end{aligned}
$$

Language of an Automaton

We say a DFA accepts a string if, starting from the start state, it terminates in one of the final states. More precisely, let $A = (\Sigma, S, s_0, F, N)$ be a DFA and w be a string in Σ^*.

We say w is ***accepted*** by A if

$$N^*(s_0, w) \in F$$

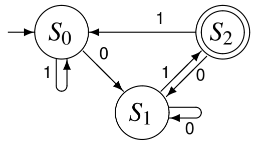
The ***language*** accepted by A is the set of all strings accepted by A:

$$L(A) = \{w \in \Sigma^* \mid N^*(s_0, w) \in F\}$$

That is, $w \in L(A)$ iff $N^*(s_0, w) \in F$.

Example 1 again

A_1:



It is easy to see whether a given string is accepted or not.

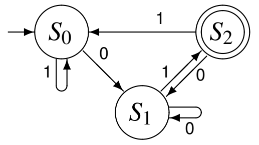
For example, 0011101 takes the machine from state S_0 through states S_1, S_1, S_2, S_0, S_0, S_1 to S_2 (a final state).

$$N^*(S_0, 0011101) = N^*(S_1, 011101) = N^*(S_1, 11101) = \ldots = N^*(S_1, 1) = S_2$$

Other strings that are accepted: 01, 001, 101, 0001, 0101, 00101101 . . .

Example 1 (ctd.)

A_1:



Accepted strings: 01, 001, 101, 0001, 0101, 00101101 . . .

Not accepted: ε, 0, 1, 00, 10, 11, 100 . . .

How is the first list different from the second list...?

How do we *justify* our guess at this answer?

Proving an Acceptance Predicate — in General

If we are to claim that a machine M accepts the language that is characterised by a predicate P, then we must prove the following:

Proof obligation 1:

Show that any string satisfying P is accepted by M.

Proof obligation 2:

Show any string accepted by M satisfies P.

Proving an Acceptance Predicate for A_1

Proof obligation 1:

If a string ends in 01, then it is accepted by A_1. That is:

For all $\alpha \in \Sigma^*$, $N^*(S_0, \alpha 01) \in F$

Proof obligation 2:

If a string is accepted by A_1, then it ends in 01. That is:

For all $w \in \Sigma^*$, if $N^*(S_0, w) \in F$ then $\exists \alpha \in \Sigma^*.\ w = \alpha 01$

Part 1: $\forall \alpha \in \Sigma^*,\ N^*(S_0, \alpha 01) \in F$

Lemma:

$$\forall s \in S.\ N^*(s, 01) = S_2$$

Proof: Prove that $N^*(S_i, 01) = S_2$ by cases:

$$N^*(S_0, 01) = N^*(S_1, 1) = S_2$$
$$N^*(S_1, 01) = N^*(S_1, 1) = S_2$$
$$N^*(S_2, 01) = N^*(S_1, 1) = S_2$$

So, by the "append" theorem above, whether $N^*(S_0, \alpha)$ is S_0, S_1 or S_2,

$$N^*(S_0, \alpha 01) = N^*(N^*(S_0, \alpha), 01) = S_2 \qquad \square$$

Part 2: $N^*(S_0, w) = S_2 \implies \exists \alpha.\ w = \alpha 01$

Proof:

- If w were the empty string ε, then $N^*(S_0, \varepsilon) = S_0 \neq S_2$, so the antecedent must be false, so by the rules of propositional logic, the claim is true!
- If w were one token long then again the antecedent is false - there is no one step path from S_0 to S_2.
- There clearly ***are*** paths from S_0 to S_2 of two or more steps, so suppose we are in that position with $N^*(S_0, \alpha xy) = S_2$.

  Then by the corollary to the "append" theorem,

  $$N(N^*(S_0, \alpha x), y) = S_2$$

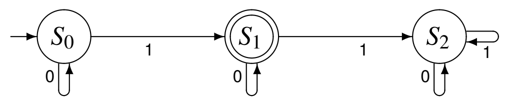
  By the definition of N, y must be 1 and $N^*(S_0, \alpha x)$ must be S_1.

  Similarly, $N(N^*(S_0, \alpha), x) = S_1$, so x is 0, again by the definition of N.

Another Example

What language does this DFA accept?

$O1B$:



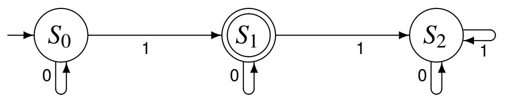
Answer for $O1B$

$O1B$ accepts the language of bit-strings containing *exactly one 1-bit.*

Proof obligations:

- Show that if a bit-string contains exactly one 1-bit then it is accepted by $O1B$.
- Show that if a string is accepted by $O1B$ it contains exactly one 1-bit.

$O1B$:



Mapping to Mathematics

Expressed mathematically, the main conclusion is

$$L(O1B) = \{w \in \Sigma^* \mid w = 0^m 1 0^n\}$$

where m, n are any natural numbers.

The two subgoals are

1. If $w = 0^m 1 0^n$ then $N^*(S_0, w) = S_1$
2. If $N^*(S_0, w) = S_1$ then $w = 0^m 1 0^n$.

For this DFA the phrase "w is accepted by $O1B$" is captured by the expression $N^*(S_0, w) = S_1$.

Proving these subgoals

The first subgoal follows immediately from the following two lemmas, which are easily proved by induction:

$$\begin{aligned}\forall n \geq 0.\ N^*(S_0, 0^n) &= S_0 \\ \forall n \geq 0.\ N^*(S_1, 0^n) &= S_1\end{aligned}$$

Therefore

$$\begin{aligned}N^*(S_0, 0^n 1 0^m) &= N^*(N^*(S_0, 0^n), 10^m) = N^*(S_0, 10^m) \\ &= N^*(N(S_0, 1), 0^m) = N^*(S_1, 0^m) = S_1\end{aligned}$$

The second subgoal, stated more formally as

$$\forall w:\ N^*(S_0, w) = S_1 \implies \exists n, m \geq 0.\ w = 0^n 1 0^m$$

is proved on the next slide.

Proving these subgoals ctd.

Lemma: $N^*(S_0, \beta) = S_0 \implies \beta = 0^n$ for some $n \geq 0$ (easy induction)

We now prove $N^*(S_0, w) = S_1 \implies w = 0^n10^m$ for some $n, m \geq 0$, *by induction on the length of w*.

The base case holds because the antecedent is false – $N^*(S_0, \varepsilon) \neq S_1$.

Suppose for induction that the theorem holds for all words of length k, and say that $N^*(S_0, \alpha) = S_1$ for α with length $k+1$.

- **Case One:** $\alpha = \beta 1$ and $N^*(S_0, \beta) = S_0$. Then by the **lemma** $\beta = 0^k$, so $\alpha = 0^k 10^0$.
- **Case Two:** $\alpha = \beta 0$ and $N^*(S_0, \beta) = S_1$. Then by the **induction hypothesis** $\beta = 0^n10^m$, so $\alpha = 0^n10^{m+1}$.

Limitations of DFAs

General Question: How powerful are DFAs ?

Specifically, what class of languages can be recognised by DFAs ?

A very important example:

Consider this language: $L = \{ a^n b^n \mid n \in \mathbb{N} \}$

That is, $L = \{\varepsilon, ab, aabb, aaabbb, a^4b^4, a^5b^5, \ldots\}$

This language cannot be recognised by any finite state automata

This is because the memory of a DFA is limited, but the machine would have to remember how many 'a's it has seen.

Proof by contradiction

Suppose A is a DFA that accepts L. That is $L = L(A)$.

Each of the following expressions denotes a state of A

$$N^*(S_0, a),\ N^*(S_0, aa),\ N^*(S_0, a^3)\ \ldots$$

Since this list is infinite and the number of states in A is finite, some of these expressions must denote the same state.

Choose distinct i and j such that $N^*(S_0, a^i) = N^*(S_0, a^j)$.

(What we have done here is pick on two initial string fragments that the automaton will not be able to distinguish, in terms of what is allowed for the rest of the string)

Proof by contradiction (ctd)

Since a^ib^i is accepted, we know

$$N^*(S_0, a^ib^i) \in F$$

By the *append* theorem

$$N^*(N^*(S_0, a^i), b^i) = N^*(S_0, a^ib^i) \in F$$

Now, since $N^*(S_0, a^i) = N^*(S_0, a^j)$

$$N^*(S_0, a^jb^i) = N^*(N^*(S_0, a^j), b^i) \in F$$

So a^jb^i is accepted by A, but a^jb^i is not in L, contradicting the initial assumption.

Pigeon-Hole Principle

The proof used the *pigeon-hole principle*:

If we have more pigeons than pigeon-holes, then at least two pigeons must be sharing a hole.

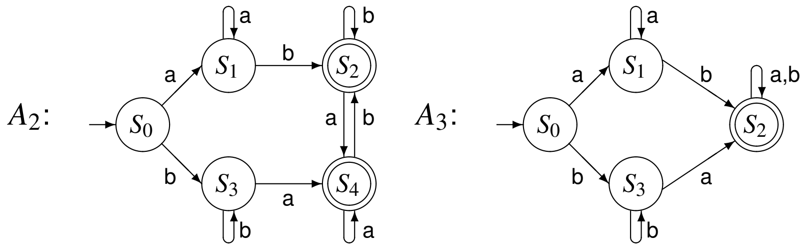
This was obviously true in our example as we had infinitely many pigeons crammed into finitely many holes.

This is a useful generic way to prove that no *finite* state automata can recognise certain *infinite* languages.

(but there are many infinite languages that ***can*** be recognised by DFAs, as we've seen, so be careful!)

Equivalence of Automata

Two automata are said to be **equivalent** if they accept the same language.



Interesting question:

Can we *simplify* a DFA? Is there an equivalent DFA with *fewer states?*

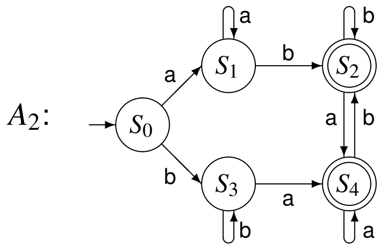
Equivalence of States

Two states S_j and S_k of an DFA are **equivalent** if, for all input strings w

$N^*(S_j, w) \in F$ if and only if $N^*(S_k, w) \in F$

Note that $N^*(S_j, w)$ and $N^*(S_k, w)$ may be different states - we only care that both are in, or not in, F.

In the following example, S_2 is equivalent to S_4.



An Algorithm for Finding Equivalent States

There is an iterative algorithm to compute a list of equivalence classes of states.

- The working data structure for the algorithm is a list of lists ("groups") of states
- Each group contains states that appear to be equivalent, *given the tests we have done so far*
- On each iteration, we test one of the groups with a symbol from the alphabet.
- If we notice differing behaviour, we *split the group.*

The Algorithm Details

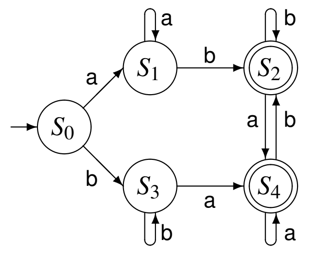
- **Input:** A list containing two “groups” (a group is represented as a list of states). One group consists of the Final states and the other consists of the non-final states.
- **Data:** The working data structure, $WDS : [[State]]$, is a list of groups of states. When two states are in different groups, we *know* they are not equivalent.
- **Loop:** Pick a group, $\{s_1,...s_j\}$ and a symbol, x. If the states $\{N(s_i,x)\}$ are all in the same group, then the group $\{s_1,...s_j\}$ is not split.

  If the states $\{N(s_i,x)\}$ belong to different groups of WDS, then the group $\{s_1,...s_j\}$ should be split accordingly.
- **Continue until** we cannot, by *any* choice of letter, split *any* group.

Our Previous Example

We split the non-final states, but leave the final states together:

A_2:



$$[[s_0,s_1,s_3],[s_2,s_4]]$$
$$\downarrow \text{a}$$
$$[[s_0,s_1],[s_3],[s_2,s_4]]$$
$$\downarrow \text{b}$$
$$[[s_0],[s_1],[s_3],[s_2,s_4]]$$
$$\downarrow \text{a}$$
$$[[s_0],[s_1],[s_3],[s_2,s_4]]$$
$$\downarrow \text{b}$$
$$[[s_0],[s_1],[s_3],[s_2,s_4]]$$

Elimination of States

Suppose $A=(\Sigma,S,S_0,F,N)$ is a DFA with state S_k equivalent to S_j.
(and S_k is not S_0.)

We can eliminate S_k from this automaton by defining a new automaton $A'=(\Sigma,S',S_0,F',N')$ as follows:

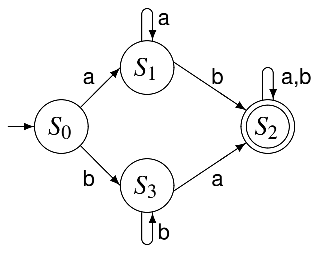
- S' is S without S_k
- F' is F without S_k
- $N'(s,w)=\begin{cases} S_j & \text{if } N(s,w)=S_k \\ N(s,w) & \text{otherwise.} \end{cases}$

Example

Since $S_2 \equiv S_4$ in A_2, let’s eliminate S_4.

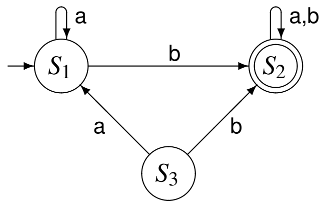
- New set of states is $\{S_0,S_1,S_2,S_3\}$
- New set of final states is $\{S_2\}$
- New transition function is:

A_3:



DFA Minimisation

Consider the DFA below:



None of $S_1,\ S_2,\ S_3$ are equivalent...

But S_3 is *inaccessible* from the start state so can safely be deleted (along with the transitions emerging from it).

Deleting equivalent and inaccessible states will give a ***minimal*** DFA.