Vector space

modeled after $\mathbb{R}^n = \{ (x_1, \ldots, x_n) \} = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \right\}$

n-tuples of real numbers

- vectors can be added (component-wise) and multiplied with scalars

scalars → a number $\in \mathbb{R}$

<u>Definition</u>: a vector space V is a set endowed with two operations:

(1) given $v, w \in V$ there is a notion of addition, i.e. $v + w \in V$ is defined

(2) given $v \in V$ and $\alpha \in \mathbb{R}$ there is a notion of scalar multiplication, i.e. $\alpha \cdot v \in V$ is defined

(compatibilities)

- $v + w = w + v$
- $(v+w)+y = v + (w+y)$
- $\alpha(v+w) = \alpha \cdot v + \alpha \cdot w$
- $\vdots$

$\underset{\in \mathbb{R}}{0} \cdot \underset{\in V}{v} = \text{zero vector} = \underset{\in V}{0}$

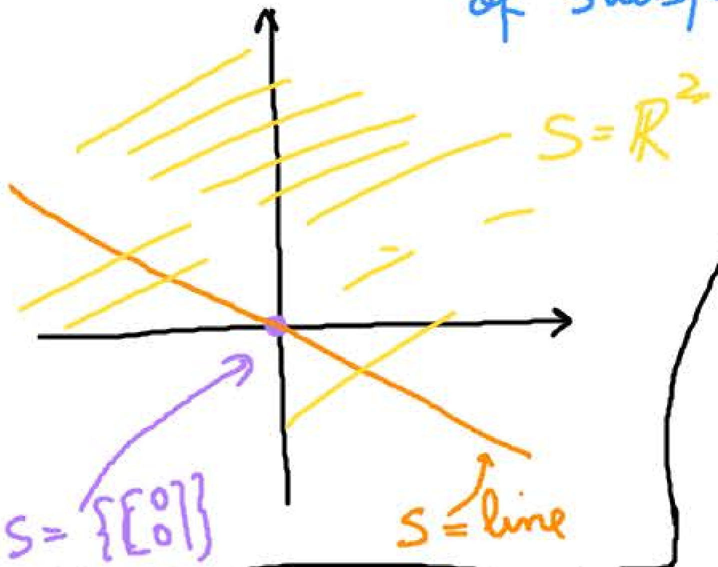
Examples

- $\mathbb{R}^n$ is a vector space
- subspaces of $\mathbb{R}^n$

Ex: $n=1$, $\mathbb{R}^1 = \mathbb{R}$ = line

2 subspaces $\begin{cases} S = \{0\} \\ S = \mathbb{R} \end{cases}$

$n=2$, $\mathbb{R}^2$ = plane, has 3 types of subspaces



"exotic"

$V = \left\{\begin{matrix}\text{all functions} \\ f:\mathbb{R}\to\mathbb{R}\end{matrix}\right\}$

$f, g : \mathbb{R} \to \mathbb{R}$

$f+g : \mathbb{R} \to \mathbb{R}$

define $(f+g)(x) = f(x) + g(x)$

Def (subspace): if V is a vector space, then a subset $S \subset V$ is called a **subspace** if S is preserved by addition & scalar multiplication

- for all $v, w \in S$, $v + w \in S$
- for all $v \in S$, $\alpha \in \mathbb{R}$, $\alpha \cdot v \in S$

a subspace of a vector space is a vector space in its own right

Linear combination : $v_1, \ldots, v_k \in V$
$\alpha_1, \ldots, \alpha_k \in \mathbb{R}$ $\rightsquigarrow$ $\boxed{\begin{array}{c}\alpha_1 v_1 + \ldots + \alpha_k v_k \\ \cap \\ V\end{array}}$

if $S \subset V$ is a subspace
if $v_1, \ldots, v_k \in S$, then $\boxed{\alpha_1 v_1 + \ldots + \alpha_k v_k \in S}$

How does this help us solve $\underbrace{A}_{m \times n} \underbrace{v}_{n \times 1} = \underbrace{b}_{m \times 1}$

<u>Definition</u> the column space of A is the subspace $C(A) \subset \mathbb{R}^m$ spanned by the n columns of A

i.e. elements of $C(A)$ are all possible linear combinations of columns of A

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

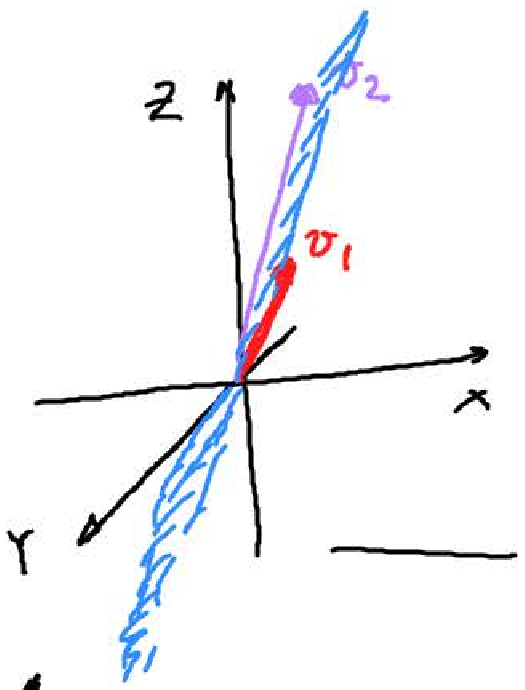
$$A = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} \quad ; \quad C(A) \subset \mathbb{R}^3 \text{ spanned by } v_1, v_2$$

v_1 v_2

$v_1, v_2 \in \mathbb{R}^3$

i.e. $C(A) = \left\{ \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ where } \alpha, \beta \in \mathbb{R} \right\}$

plane



$$B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$$

w_1 w_2

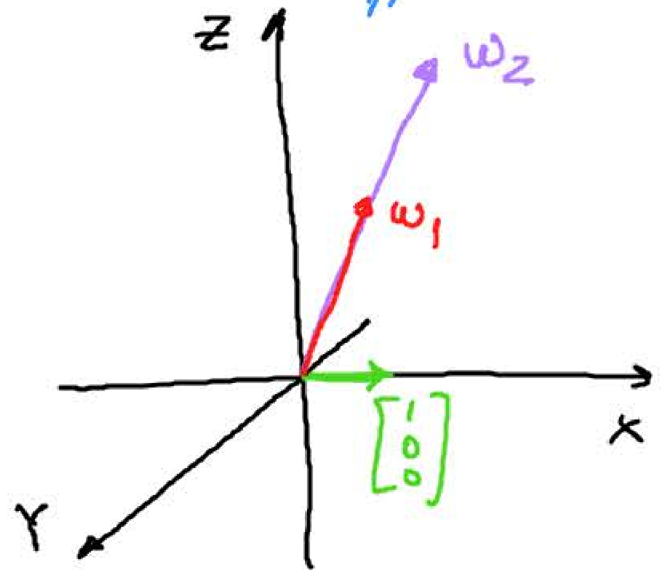
$w_2 = 2 w_1$

$$C(B) = \left\{ \alpha \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \beta \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} \text{ where } \alpha, \beta \in \mathbb{R} \right\}$$

line

$$= \left\{ (\alpha + 2\beta) \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ where } \alpha, \beta \in \mathbb{R} \right\}$$

$$= \left\{ \gamma \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \text{ where } \gamma \in \mathbb{R} \right\}$$



Theorem: system $Av = b$ has solutions if and only if $b \in C(A)$

$$b = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \cdot \left(v_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = v_1 \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= v_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + v_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

= a linear combination of the columns of the matrix $\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}$

$$b \in C\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \right)$$

does there exist a solution to

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; ?$$

$\Updownarrow$

is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in C\left(\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 6 \end{bmatrix} \right)$?

No, because this is the line in $\mathbb{R}^3$ that passes through $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$