To apply the formula for P_V, the projection matrix onto a subspace V, you need to pick a basis

$$v_1, \ldots\ldots, v_n \quad \text{of } V$$

take the matrix $A = [v_1 | \ldots\ldots | v_n]$

then $P_V = A(A^TA)^{-1}A^T$

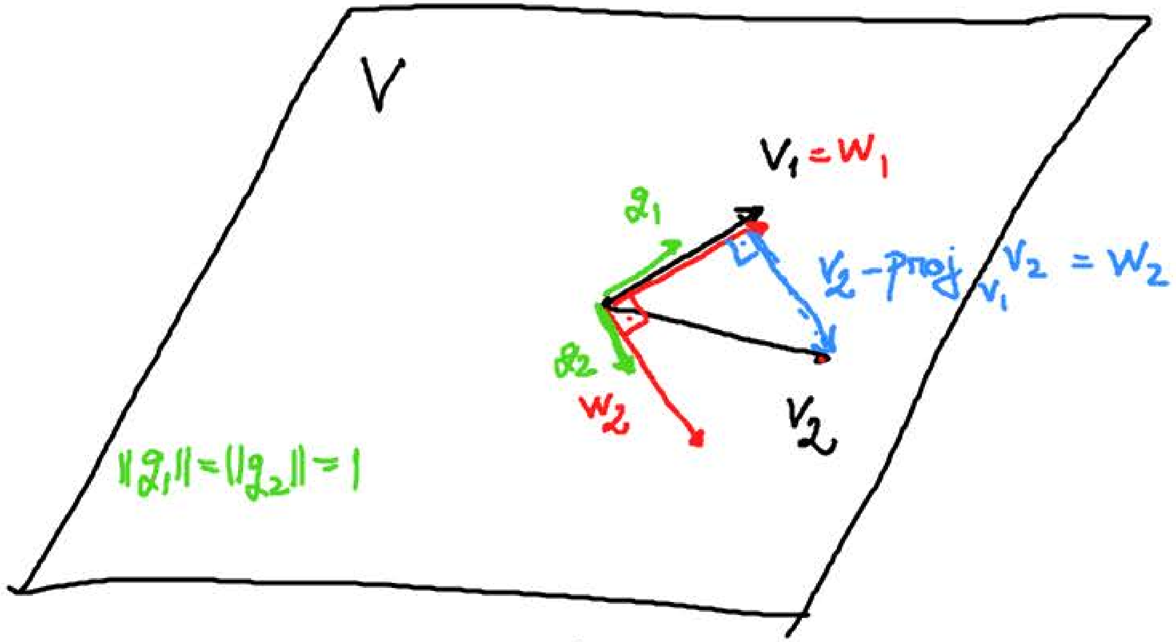
if instead you pick an orthonormal basis

$$\boxed{q_1, \ldots\ldots, q_n} \quad \text{of } V$$

orthonormality of columns of Q

take the matrix $Q = [q_1 | \ldots\ldots | q_n]$

then $P_V = QQ^T$, because $\boxed{Q^TQ = I}$



$v_1, v_2 \rightsquigarrow$ basis

$w_1, w_2 \rightsquigarrow$ orthogonal basis

$q_1, q_2 \rightsquigarrow$ orthonormal basis

The Gram-Schmidt process:

convert any basis $v_1, \ldots, v_n$ into an orthonormal basis $q_1, \ldots, q_n$

of whatever vector space you work with

basis of the same vector space

allow operations $v_i \rightsquigarrow \alpha_i v_i + \boxed{\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_{i-1} v_{i-1}}$

step 1 (modify v_1): define $q_1 = \dfrac{v_1}{\|v_1\|}$

step 2 (modify v_2): define $w_2 = v_2 - \text{proj}_{q_1} v_2$

(now $w_2 \perp v_1$)

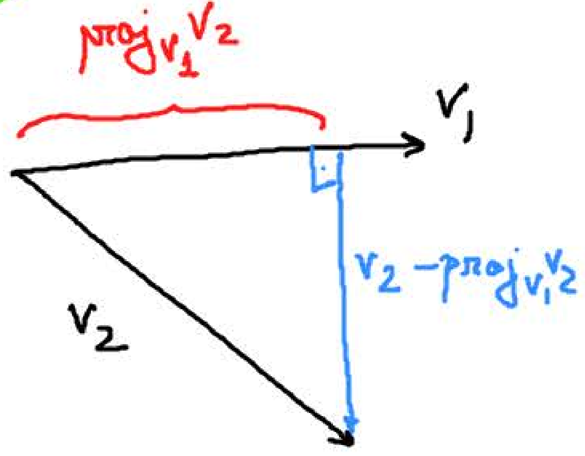
$$q_2 = \frac{w_2}{\|w_2\|}$$

step 3 (modify v_3): define $w_3 = v_3 - \text{proj}_{q_1} v_3 - \text{proj}_{q_2} v_3$

$$q_3 = \frac{w_3}{\|w_3\|}$$

⋮

step n (modify v_n)



Obs: $v_2 - \text{proj}_{v_1} v_2$ is $\perp v_1$

Ex: $V \subset \mathbb{R}^3$, $V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } x+y+z=0 \right\}$

dim 2

we want a orthonormal basis of V; last time, we eyeballed

$$\boxed{q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}}$$

Let's do the same thing via Gram-Schmidt

↳ pick an arbitrary basis of V, say $v_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

step 1 (modify v_1): $q_1 = \frac{v_1}{\|v_1\|} = \frac{v_1}{\sqrt{1^2+2^2+3^2}} = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

dot product

step 2 (modify v_2): $w_2 = v_2 - \text{proj}_{q_1} v_2 = v_2 - \frac{q_1 q_1^T v_2}{q_1^T q_1} = v_2 - q_1 (q_1 \cdot v_2)$

$q_1^T q_1 = \|q_1\|^2 = 1$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \left(\frac{1 \cdot 0}{\sqrt{14}} + \frac{2 \cdot 1}{\sqrt{14}} + \frac{(-3)(-1)}{\sqrt{14}} \right)$$

$$= \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} - \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \cdot \frac{5}{\sqrt{14}} = \frac{1}{14} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{w_2}{\|w_2\|} = \frac{w_2}{\frac{\sqrt{1^2+4^2+5^2}}{14}} = \frac{14}{\sqrt{42}} w_2 = \frac{1}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{42}} \begin{bmatrix} -5 \\ 4 \\ 1 \end{bmatrix}$$

$v_1, \dots, v_n \rightsquigarrow q_1, \dots q_n$

$A = [v_1 | \dots | v_n] \rightsquigarrow Q = [q_1 | \dots | q_n]$

- replace v_i by $v_i - \mathrm{proj}_{q_{i-1}} v_i - \dots - \mathrm{proj}_{q_1} v_1 = v_i - q_{i-1}\alpha_{i-1} - \dots - q_1 \alpha_1$

calculated from the projections of v_i onto $q_1, \dots q_{i-1}$

- replace v_i by $\frac{v_i}{\|v_i\|}$

subtract a linear combination of first $i-1$ columns from the i-th column

divide the i-th column by a number

replacing the i-th column by the i-th column $+ \lambda \cdot$ the j-th column is achieved by $A \rightsquigarrow A\, E_{ji}^{(\lambda)}$

multiplying the i-th column by λ is achieved by the operation

$A \rightsquigarrow A \cdot D_i^{(\lambda)}$

Gram - Schmidt

$$A = [v_1 | \dots | v_n]$$

$$\mu_1 = \frac{1}{\|v_1\|}$$

$$A \rightsquigarrow \underbrace{A \cdot D_1^{(\mu_1)} \, E_{12}^{(\lambda_{12})} \, D_2^{(\mu_2)} \, E_{13}^{(\lambda_{13})} \, E_{23}^{(\lambda_{23})} \cdot D_3^{(\mu_3)} \cdots}_{Q}$$

step 1: $v_1 \rightsquigarrow q_1 = \dfrac{v_1}{\|v_1\|}$

step 2: $v_2 \rightsquigarrow w_2 = v_2 - \mathrm{proj}_{q_1} v_2 = v_2 + q_1 \cdot \lambda_{12}$

a number that is prescribed by the equality $-\mathrm{proj}_{q_1} v_2 = q_1 \lambda_{12}$

$$-\mathrm{proj}_{q_1} v_2 = -q_1 \cdot \frac{q_1^T v_2}{q_1^T q_1}$$

$w_2 \rightsquigarrow q_2 = \dfrac{w_2}{\|w_2\|}$

step 3: $v_3 \rightsquigarrow w_3 = v_3 + \lambda_{13} \cdot q_1 + \lambda_{23} q_2$

$w_3 \rightsquigarrow q_3 = \dfrac{w_3}{\|w_3\|} = \mu_3 w_3$

$$Q = A\, D_1^{(\mu_1)}\, E_{12}^{(\lambda_{12})}\, D_2^{(\mu_2)} \cdots\cdots$$

$$Q \cdot \underbrace{\cdots\cdots \underbrace{D_3^{\left(\frac{1}{\mu_1}\right)} E_{13}^{(-\lambda_3)} E_{23}^{(-\lambda_{23})}}_{\text{step 3}} \underbrace{D_2^{\left(\frac{1}{\mu_2}\right)} E_{12}^{(-\lambda_{12})}}_{\text{step 2}} \underbrace{D_1^{\left(\frac{1}{\mu_1}\right)}}_{\text{step 1}}}_{\text{upper triangular square matrix}} = A$$

$$R = \begin{bmatrix} * & * & * & * \\ & * & * & * \\ 0 & & * & * \\ & & & * \end{bmatrix}$$

$E_{ij}^{(\lambda)}$, $D_i^{(\mu)}$ are upper triangular if $i<j$

Thm ($\Longleftrightarrow$ Gram-Schmidt)

any matrix A can be factored as

$$A = \underbrace{Q}_{\text{orthonormal columns}} \underbrace{R}_{\text{upper triangular square}}$$

$$v_2 - \text{proj}_{g_1} v_2 = v_2 - g_1 \cdot \underbrace{\frac{g_1^T v_2}{\cancel{g_1^T g_1}}}_{\text{projection formula}}$$

$$g_1^T g_1 = 1$$
$$\|$$
$$\|g_1\|^2$$

could also multiply the red matrices first, but we choose the order of operations on the right to ensure answer is of the form $g_1 \cdot$ times constant

$$\underset{g_1}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} \left(\underset{g_1^T}{\begin{bmatrix} & & & \end{bmatrix}} \underset{v_2}{\begin{bmatrix} \\ \\ \\ \end{bmatrix}} \right)$$

$$\boxed{g_1 \cdot \text{ a constant}}$$