an $n \times n$ matrix with complex entries has n complex eigenvalues

an $n \times n$ matrix with real entries has n complex eigenvalues, but the non-real ones among the eigenvalues come in complex conjugate pairs (also the respective eigenvectors will be complex conjugate

$\boxed{\text{an } n \times n \text{ symmetric matrix with real entries has } n \text{ real eigenvalues and } n \text{ orthonormal eigenvectors}}$

acceptable eigenvalues for a real 3×3

- $\lambda_1 = 2$, $\lambda_2 = \frac{7}{2}$, $\lambda_3 = \pi$ ✓
- $\lambda_1 = 7$, $\lambda_2 = 5+4i$, $\lambda_3 = 5-4i$ ✓
- $\lambda_1 = 3+4i$ $\lambda_2 = 3-4i$, $\lambda_3 = 7+5i$ ✗

e.g. $\lambda_1 = 3$ and $\lambda_2 = -7$ ✓

$$\begin{bmatrix} 3 & 0 \\ 0 & -7 \end{bmatrix}$$

e.g. $\lambda_1 = i$ and $\lambda_2 = -i$ ✓

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

e.g. $\lambda_1 = 3$ and $\lambda_2 = 7-5i$ ✗

e.g. $\lambda_1 = 3+2i$ and $\lambda_2 = 1-8i$ ✗

2×2 symmetric $\begin{bmatrix} a & b \\ b & c \end{bmatrix} \rightsquigarrow p(\lambda) = \lambda^2 - \lambda(a+c) + ac - b^2$

$$\lambda_{1,2} = \frac{a+c \pm \sqrt{(a+c)^2 - 4ac + 4b^2}}{2} = \frac{a+c \pm \sqrt{\underbrace{(a-c)^2 + 4b^2}_{>0}}}{2}$$

$\Big\}$ real

- why are the eigenvectors of symmetric matrices mutually orthogonal?

  - v is an eigenvector of S corresp. to $\lambda \Rightarrow Sv = \lambda v$

    $S \longrightarrow$ symm. matrix

  - v' is an eigenvector of S corresp to $\lambda' \Rightarrow Sv' = \lambda' v'$

assume $\boxed{\lambda \neq \lambda'}$

$$\lambda \cdot v^T v' = (Sv)^T v' \qquad v^T(Sv') = \lambda' v^T v' \implies v^T v' = 0 \Rightarrow v \perp v'$$

$$\begin{matrix} \parallel & & \parallel \\ v^T S^T v' & = & v^T S v' \end{matrix}$$

b/c S is symmetric

b/c $\lambda \neq \lambda'$

$S \rightsquigarrow$ an $n \times n$ symmetric matrix

has n mutually orthonormal eigenvectors $q_1, \dots, q_n$ } will give an orthonormal basis of $\mathbb{R}^n$

corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$

$$Q = [q_1 | \dots | q_n]$$

because the columns of Q are orthonormal, Q is an $\boxed{\text{orthogonal matrix}}$

$$\Updownarrow$$

$$\boxed{Q^{-1} = Q^T}$$

Diagonalization theorem :

$$S = Q \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^{-1} = Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^T$$

rotation scaling inverse rotation

2×2 case,

$$Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$S = Q \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} Q^T$$

Rank 1 $n \times n$ matrix

$$= [q_1 | \cdots | q_n] \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = \lambda_1 \underbrace{q_1 q_1^T}_{n \times n} + \cdots + \lambda_n \underbrace{q_n q_n^T}_{n \times n}$$

$\Downarrow$

given $v \in \mathbb{R}^n$, how do I compute $Sv \in \mathbb{R}^n$

(Singular Value Decomposition ..... in a couple of weeks)

$$v = c_1 q_1 + \cdots + c_n q_n$$

$\Downarrow$ computationally efficient

$$Sv = c_1 \lambda_1 q_1 + \cdots + c_n \lambda_n q_n$$

Def: if a symmetric real matrix S has all its eigenvalues

- positive / negative, S is called positive/negative definite ($\lambda_i > 0$)
- nonnegative / nonpositive, S — , ________ semidefinite ($\lambda_i \geq 0$)

any symmetric real matrix has as many positive eigenvalues as positive pivots

Energy (of a vector v in relation to a symmetric matrix S)

$v^T S v$

Theorem S is positive definite $\iff$ energy of any $v \neq 0$ is > 0

$\lambda_i > 0$

S is positive semidefinite $\iff$ energy of any v is ≥ 0

$\lambda_i \geq 0$

any $S = A^T A$ is positive semidefinite

$v^T S v = v^T A^T A v = (Av)^T A v$

$= \|Av\|^2 \geq 0$

$S = A^T A$ positive definite if

$Av \neq 0$ for any nonzero v

i.e. the columns of A are independent

Proof: if $v = c_1 q_1 + \cdots + c_n q_n$, then $Sv = c_1 \lambda_1 q_1 \cdots c_n \lambda_n q_n$

$v^T S v = \sum_{1 \leq i,j \leq n} c_i \lambda_j c_j \; q_i^T q_j$

$q_i^T q_j = 0$ unless $i = j$

also $q_i^T q_i = 1$

$= \sum_{i=1}^{n} \lambda_i c_i^2 > 0$ if $\lambda_1 \dots \lambda_n > 0$

energy

$$S = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

$$\lambda_{1,2} = \frac{a+c \pm \sqrt{(a-c)^2 + 4b^2}}{2}$$

easy to prove that

λ_1 and λ_2 are > 0

$\Updownarrow$

$\underbrace{\mathrm{Tr}\, S}_{a+c}$ and $\underbrace{\det S}_{ac-b^2} > 0$

$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$v^T S v = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2bxy + cy^2$$

$\Downarrow$

equation of a conic (ellipse, hyperbola) curve in $\mathbb{R}^2$