$$\underbrace{A}_{\text{square}} = \underbrace{L}_{\substack{\text{lower triangular}\\ \text{1's on diag}}}\ \underbrace{U}_{\text{upper triangular}}$$

$$= L\ \underbrace{D}_{\text{diagonal}}\ \underbrace{U'}_{\text{upper triangular 1's on diag}}$$

Gaussian elimination

$$\boxed{U = DU'}$$

- This only holds for A's whose Gaussian elimination does not involve row swaps
- If A does require row swaps, then we get the factorization

$$\underbrace{P}_{\text{permutation matrix}}\underbrace{A = LU}_{\text{as before}}$$

Permutation matrices : a $n \times n$ matrix with lots of 0's and a single 1 on each row & on each column

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \{\underset{\sigma(1)}{1}, \underset{\sigma(2)}{2}, \underset{\sigma(3)}{3}\}$$

let $\sigma(i)$ be the index of the column on which the 1 on row i lies, for all $i \in \{1, \dots, n\}$

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

every entry in this 2×2 is 0

$\{\sigma(1), \dots, \sigma(n)\}$ is a $\boxed{\text{permutation}}$ of $\{1, \dots, n\}$

$$\{\underset{\sigma(1)}{2}, \underset{\sigma(2)}{1}, \underset{\sigma(3)}{4}, \underset{\sigma(4)}{3}\}$$

There are $n! = 1 \cdot 2 \cdot \dots \cdot n$ such permutations

$\Downarrow$

there $n!$ permutation matrices of size $n \times n$

P_{ij} from our second class correspond to permutation $\{1, 2, \dots, i-1, j, i+1, \dots, j-1, i, j+1, \dots, n\}$

$$A = \begin{bmatrix} 0 & 0 & \boxed{3} \\ \boxed{1} & 1 & 2 \\ \boxed{1} & 2 & -1 \end{bmatrix} \xrightarrow{\text{swap } r_1 \text{ and } r_3} \begin{bmatrix} \boxed{1} & 2 & -1 \\ \boxed{1} & 1 & 2 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

$$\xrightarrow{r_2 - r_1} \begin{bmatrix} \boxed{1} & 2 & -1 \\ 0 & \boxed{-1} & 3 \\ 0 & 0 & \boxed{3} \end{bmatrix} = U$$

in general $\Rightarrow$ $PA = LU$

for a suitably chosen permutation matrix P

$$E_{21}^{(-1)} P_{13} A = U \quad \Big| \; \cdot \left(E_{21}^{(-1)}\right)^{-1}$$

$$P_{13} A = \underbrace{E_{21}^{(1)}}_{L} \cdot U = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

To get $\boxed{PA = LU}$

you have to do all row swaps **before** all the row eliminations

(if you do all your row swaps **after** row eliminations, then you get a

$A = LPU$ factorization)

how do you know which row swaps to do

- stare at the matrix and guess
- do Gaussian elimination in the order from class (add multiples of row 1 to lower rows then swap add multiples of row 2 to lower rows then swap)

and record all row swaps that you do along the way

then come back to the original matrix, do the same row swaps first (before any elimination) and then do elimination on resulting matrix

Take A rectangular $(m \times n)$

<u>Def</u>: the transpose of A, denoted by A^T, is the $n \times m$ matrix obtained by switching the roles of rows and columns

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \rightsquigarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Properties:

- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

$$(\underbrace{A}_{m\times n}\underbrace{B}_{n\times p})^T = \underbrace{B^T}_{p\times n}\underbrace{A^T}_{n\times m}$$

- $(A^{-1})^T = (A^T)^{-1}$ if A is square and invertible

$$\underbrace{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}_{=v} \cdot \underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}}_{=w} = v_1 w_1 + \cdots + v_n w_n$$

$$\underbrace{\underbrace{\begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}}_{1\times n}}_{=v^T} \underbrace{\underbrace{\begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}}_{n\times 1}}_{=w} = \begin{bmatrix} v_1 w_1 + \cdots + v_n w_n \end{bmatrix}$$

Moral: $\boxed{\underbrace{v \cdot w}_{\text{number}} = \underbrace{v^T w}_{1\times 1 \text{ matrix}}}$

compare

$$\underbrace{v}_{n\times 1}\ \underbrace{w^T}_{1\times n} = \underbrace{n\times n \text{ matrix}}_{\text{important for SVD}}$$

Definition: a square matrix S is called symmetric if $\boxed{S = S^T}$

$$\begin{bmatrix} 4 & 3 & 6 \\ 3 & 7 & -1 \\ 6 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 6 \\ 3 & 7 & -1 \\ 6 & -1 & 5 \end{bmatrix}, \quad \begin{bmatrix} 4 & \boxed{3} & \boxed{6} \\ \boxed{3} & 7 & \boxed{-1} \\ \boxed{6} & \boxed{-1} & 5 \end{bmatrix}$$

Theorem: for any matrix A (even $m \times n$), the matrix $S = \underset{n\times m}{A^T} \cdot \underset{m\times n}{A}$ is symmetric

Proof: $S^T = (A^T A)^T = A^T (A^T)^T = A^T A = S$.

Def: an antisymmetric matrix A is one such that $A = -A^T$

Theorem: The LDU factorization of a symmetric matrix S takes the form

$$S = L D \cdot L^T$$

L: lower triang, 1's on diag

D: diag

L^T: upper triang, 1's on diag

$$L = \begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} \Rightarrow L^T = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

Proof (for S is invertible $\rightsquigarrow$ unique LDU factorization)

take $S = LDU \implies S^T = (LDU)^T = U^T D^T L^T \implies L = U^T$, $U = L^T$

U^T: lower triang

$D^T = D$

L^T: upper triang

example: $O = L \cdot O \cdot U$ for any L & U

zero matrix

$\Downarrow$

LDU is not unique for singular matrices