$A \rightsquigarrow$ column space $C(A)$ = the subspace of $\boxed{\mathbb{R}^m}$ spanned by the columns of A

$m \times n$

$\rightsquigarrow$ nullspace $N(A)$ = a particular subspace of $\boxed{\mathbb{R}^n}$

$S \subset \mathbb{R}^3$, S is spanned by vectors $v_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix}$

plane; what is the equation $ax + by + cz = 0$ that cuts out this plane? (for suitably chosen a, b, c)

$\longleftrightarrow$ what must a, b, c be so that v_1, v_2 satisfy the equation?

i.e. $\begin{cases} a \cdot 1 + b \cdot 1 + c \cdot (-2) = 0 \\ a \cdot 5 + b \cdot (-2) + c \cdot (-3) = 0 \end{cases}$

$\longleftrightarrow$ $a = 1$

$b = 1$

$c = 1$

$S = \{x + y + z = 0\}$

$A = [1 \quad 1 \quad 1]$; by construction,

a b c

$$[1 \quad 1 \quad 1] \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = [0] \rightsquigarrow \boxed{A v_1 = 0}$$

$$[1 \quad 1 \quad 1] \cdot \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} = [0] \rightsquigarrow \boxed{A v_2 = 0}$$

A "kills"

"annihilates"

$S = \{ \alpha v_1 + \beta v_2, \text{ for various } \alpha, \beta \}$

$A v = 0$ for all $v \in S$

Key Notion

$$A(\alpha v_1 + \beta v_2) = $$
$$= \alpha \cdot A v_1 + \beta \cdot A v_2 = 0$$

"zero vector"

<u>Def</u>: for A an $m \times n$ matrix, its nullspace $N(A) \subset \mathbb{R}^n$ is the set of $v \in \mathbb{R}^n$ such that $Av = 0$

$v_1, v_2 \in N(A) \Rightarrow v_1 + v_2 \in N(A)$ | $A(v_1+v_2) = Av_1 + Av_2$

$v \in N(A), c \in \mathbb{R} \Rightarrow c \cdot v \in N(A)$ | $A(cv) = c \cdot Av$

given A, computing its nullspace $\Longleftrightarrow$ solving $\boxed{Av=0}$

Fact: if $A = L\,R$ for some invertible matrix L

A: $m\times n$, L: $m\times m$, R: $m\times n$

then $\boxed{N(A)=N(R)}$

Proof: $v\in N(A) \Longleftrightarrow LRv=0$

$\Updownarrow \qquad\qquad \Updownarrow$

$v\in N(R) \Longleftrightarrow R\cdot v=0$

Application: we will apply Gauss-Jordan elimination to A $\Rightarrow$ R will be the reduced row echelon form of A

it will turn out that $N(R)$ will be easy to compute

$\Downarrow$

we will have computed $N(A)$

Gauss elim:
- the pivots go to the right as you read the matrix top-to-bottom

extra steps to G-J elim:
- pivots are all $=1$
- all entries directly above a pivot are 0

$$A = \begin{bmatrix} \boxed{1} & 3 & -2 & 3 \\ \boxed{1} & 3 & 0 & 7 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} \boxed{1} & 3 & -2 & 3 \\ 0 & 0 & \boxed{2} & 4 \end{bmatrix}$$

Gaussian elimination

$$\xrightarrow{r_2 \cdot \frac{1}{2}} \begin{bmatrix} \boxed{1} & 3 & -2 & 3 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix} \xrightarrow{r_1 + 2 \cdot r_2} \begin{bmatrix} \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix} = R$$

$$N(A) = N(R)$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in N(R) \iff \begin{bmatrix} \boxed{1} & 3 & 0 & 7 \\ 0 & 0 & \boxed{1} & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0 \iff \begin{cases} \boxed{a} + 3b + 7d = 0 \\ \boxed{c} + 2d = 0 \end{cases} \iff \begin{cases} a = -3b - 7d \\ c = -2d \end{cases}$$

pivot vars | free vars

call a and c the pivot variables | call b and d the free variables

Principle: For any choice of values for the free variables (b and d)

you can uniquely solve for the pivot variables (a and c)

such that $v = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$ is a solution to $Av = 0$

$a = -3b - 7d$

$c = -2d$

$$\Rightarrow N(A) = \left\{ \begin{bmatrix} -3b-7d \\ b \\ -2d \\ d \end{bmatrix} \text{ for all choices of } b \text{ and } d \right\}$$

specialize the free vars to $(b,d) = (1,0)$ or to $(b,d) = (0,1)$

$$\begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in N(A) \ni \begin{bmatrix} -7 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

= the subspace spanned by these two vectors

Algorithm for computing $N(A)$:

- compute the RREF of $A \rightsquigarrow R$
- identify the pivot columns of R .... all other columns of R are called free
  (columns which have pivots)
- $N(A) =$ the set of $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where x_i's are either pivot variables or free variables

  where $\{x_i$ pivot variables$\}$ can be expressed from $\{x_i$ free variables$\}$ via the equation $\boxed{R \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0}$

- $N(A)$ is spanned by the vectors $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ of the following form $\begin{cases} \cdot \text{ one free variable} = 1 \\ \cdot \text{ other free variables} = 0 \\ \cdot \text{ pivot vars are solved for in terms of free vars by} \end{cases}$

$$S = \left\{ \alpha \cdot \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \beta \cdot \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} \right\}$$

a, b, c s.t. $\begin{cases} a \cdot 1 + b \cdot 1 + c \cdot (-2) = 0 \\ a \cdot 5 + b \cdot (-2) + c \cdot (-3) = 0 \end{cases} \iff \begin{bmatrix} 1 & 1 & -2 \\ 5 & -2 & -3 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$

$\Updownarrow$

compute $N\left(\begin{bmatrix} 1 & 1 & -2 \\ 5 & -2 & -3 \end{bmatrix} \right)$

$$A \rightsquigarrow E_{ij}^{(\lambda)} A$$

is the same as adding $\lambda \cdot$ row j to row i

true for $i > j$ as well as for $i < j$

Gaussian elimination

extra step to get Gauss-Jordan elimination