let A be an $m \times n$ matrix

$C(A) \subset \mathbb{R}^m$ subspace spanned by the columns of A

$N(A) \subset \mathbb{R}^n$ the s pace of vectors annihilated by A,

$$C(A) = \{ b \in \mathbb{R}^m \text{ such that } \underbrace{\text{exists some } v \text{ such that } Av = b}\}$$

$\Longleftrightarrow$ some linear combination of the columns of A produces the vector b

$$N(A) = \{ v \in \mathbb{R}^n \text{ such that } Av = 0 \}$$

$\boxed{Av = b}$ $\longrightarrow$ solve for any given A, b

STEP 1: is $b \in C(A)$? If NO, there are no solutions
If YES, there are solutions

Pick a particular solution; call it $v_{particular}$; it satisfies $\boxed{A v_{particular} = b}$

STEP 2: consider an arbitrary solution; call it $v_{general}$; it satisfies $\boxed{A v_{general} = b}$

subtract purple boxes:

$$A(\underbrace{v_{general} - v_{particular}}_{w}) = b - b = 0 \implies \underset{\substack{\| \\ v_{general} - v_{particular}}}{w} \in N(A)$$

UPSHOT : $V_{general} = V_{particular} + W_{general}$

an arbitrary / general element of $N(A)$

How to do this in practice?

$$\underbrace{\begin{bmatrix} 2 & 6 & 2 & 2 & -1 \\ 2 & 6 & 1 & -1 & 2 \\ 3 & 9 & -1 & -9 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} -7 \\ 6 \\ -6 \end{bmatrix}}_{b}$$

Put $[A|b]$ RREF.

$$\left[\begin{array}{ccccc|c} \boxed{2} & 6 & 2 & 2 & -1 & -7 \\ \boxed{2} & 6 & 1 & -1 & 2 & 6 \\ \boxed{3} & 9 & -1 & -9 & 1 & -6 \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccccc|c} \boxed{2} & 6 & 2 & 2 & -1 & -7 \\ 0 & 0 & \boxed{-1} & -3 & 3 & 13 \\ \boxed{3} & 9 & -1 & -9 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{r_3 - \frac{3}{2} r_1} \left[\begin{array}{ccccc|c} \boxed{2} & 6 & 2 & 2 & -1 & -7 \\ 0 & 0 & \boxed{-1} & -3 & 3 & 13 \\ 0 & 0 & \boxed{-4} & -12 & 5/2 & 9/2 \end{array}\right] \xrightarrow{r_3 - 4 \cdot r_2} \left[\begin{array}{ccccc|c} \boxed{2} & 6 & 2 & 2 & -1 & -7 \\ 0 & 0 & \boxed{-1} & -3 & 3 & 13 \\ 0 & 0 & 0 & 0 & \boxed{-\frac{19}{2}} & -\frac{95}{2} \end{array}\right]$$

$$\xrightarrow[r_3 \cdot \left(-\frac{2}{19}\right)]{r_1 \cdot \frac{1}{2} \quad r_2 \cdot (-1)} \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 1 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \boxed{1} & 3 & -3 & -13 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right] \xrightarrow{r_2 + 3 \cdot r_3} \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 1 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 3 & 1 & 1 & -\frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right] \xrightarrow{r_1 + \frac{1}{2}\cdot r_3}$$

$$\left[\begin{array}{ccccc|c} \boxed{1} & 3 & 1 & 1 & 0 & -1 \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right] \xrightarrow{r_1 - r_2}$$

$$[R|c] = \left[\begin{array}{ccccc|c} \boxed{1} & 3 & 0 & -2 & 0 & -3 \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & \boxed{1} & 5 \end{array}\right]$$

Instead of solving original system, solve:

$$\underbrace{\begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}}_{R} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{\begin{bmatrix} -3 \\ 2 \\ 5 \end{bmatrix}}_{c}$$

pivot columns: 1, 3, 5

pivot variables: x_1, x_3, x_5

free variables: x_2, x_4

$Rv = c$

$$\begin{cases} \boxed{x_1} + 3x_2 - 2x_4 = -3 \\ \boxed{x_3} + 3x_4 = 2 \\ \boxed{x_5} = 5 \end{cases} \iff \begin{cases} x_1 = -3x_2 + 2x_4 - 3 \\ x_3 = -3x_4 + 2 \\ x_5 = 5 \end{cases}$$

for any choice of the free variables x_2, x_4, the system allows you to solve uniquely for x_1, x_3, x_5

$$V_{general} = \begin{bmatrix} -3a+2b-3 \\ a \\ -3b+2 \\ b \\ 5 \end{bmatrix} \qquad V_{particular} = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 5 \end{bmatrix} \qquad \underset{N(A)}{\overset{\displaystyle W_{general}}{\cap}} = \begin{bmatrix} -3a+2b \\ a \\ -3b \\ b \\ 0 \end{bmatrix}$$

just set free variables $=0$

for any a, b

An interesting thing is if one or more rows of R do not have pivots, e.g.

$$\left[\begin{array}{ccccc|c} \boxed{1} & 3 & 0 & -2 & 0 & -3 \\ 0 & 0 & \boxed{1} & 3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

entries in the "extended" column do not count as pivots

$$\begin{cases} x_1 + 3x_2 - 2x_4 = -3 \\ x_3 + 3x_4 = 2 \\ 0 = 5 \end{cases}$$

NO SOLUTIONS

If R has a full row of 0's, the system only has solutions if the corresponding number on the "extended" column is also a 0

Def : The rank of a matrix A is the number of its pivot columns

the columns where the pivots of (R)REF of A lie

$$\begin{bmatrix} \boxed{2} & 6 & 2 & 2 & -1 \\ \boxed{2} & 6 & 1 & -1 & 2 \\ \boxed{3} & 9 & -1 & -9 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \boxed{1} & 3 & 0 & -2 & 0 \\ 0 & 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

Property : $\boxed{r \leq \min(m,n)}$

Proof : pivots are all on different rows & columns

$r \leq \min(m,n)$

- if $r = m$, then A is called **full row rank**
- if $r = n$, then A is called **full column rank**

full column rank $\Rightarrow$ $N(A)$ is as small as possible

i.e. $N(A) = 0$

$\Downarrow$

$Av = b$ has at most one solution for all b's

full row rank $\Rightarrow$ $C(A)$ is as big as possible

i.e. $C(A) = \mathbb{R}^m$

$\Downarrow$

$Av = b$ has at least one solution for all b's if solving $A\,w = 0$, the solution is a general element of $N(A)$

if solving $A\,v = b$, the solution is $V_{general} = V_{particular} + W$

what if v, b vectors are replaced by V and B matrices?

$$\underset{m\times n}{A}\ \underset{n\times p}{V} = \underset{m\times p}{B} \rightsquigarrow \text{break this up into systems} \quad \begin{array}{l} A v_1 = b_1 \\ A v_2 = b_2 \\ \vdots \\ A v_p = b_p \end{array}$$

where $V = [v_1 | \dots | v_p]$

$B = [b_1 | \dots | b_p]$