Change of basis =

= the relationship between matrices A and B

$m \times n$

$$\boxed{A = WBV^{-1}}$$

A and B represent one and the same linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^m$, albeit in different bases

columns of V on $\mathbb{R}^n$
columns of W on $\mathbb{R}^m$

Similar matrices : we call A and B similar if

$(m = n, V = W)$

$$\boxed{A = VBV^{-1}}$$

for some matrix V

A and B represent the same $\phi : \mathbb{R}^n \to \mathbb{R}^n$ but the bases are modified in the same way on the domain of ϕ as on the codomain ϕ

Diagonalization: almost all square matrices A are similar to a diagonal matrix

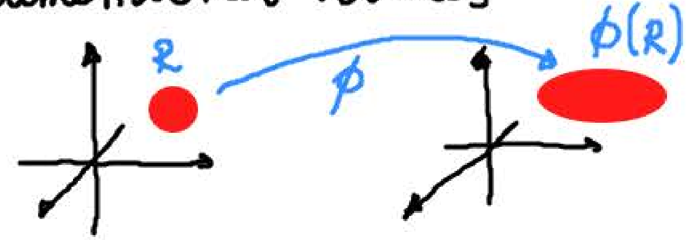
$$A = V \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} V^{-1}$$

Determinants

Definition: given an $n \times n$ matrix A, its determinant is that number

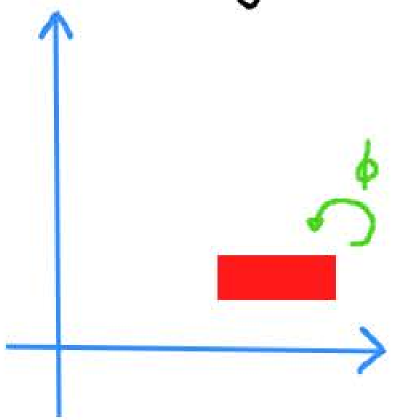
$\det A = $ the factor/ratio by which A rescales n-dimensional volumes $= \dfrac{\operatorname{vol} \phi(R)}{\operatorname{vol} R}$ → can be < 0

$\phi : \mathbb{R}^n \to \mathbb{R}^n$



1-dim volume = length
2-dim volume = area
3-dim volume = volume
. . . . .

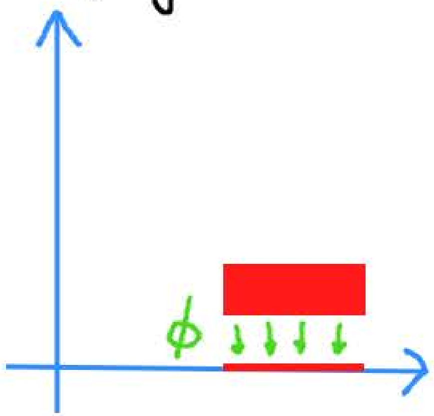
identity



$\det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

projection

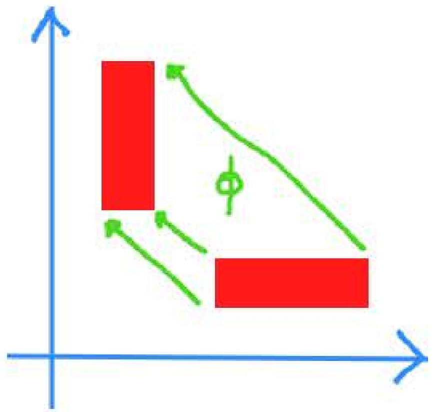


$\det \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \end{bmatrix}$

TYPO: this should be 0

reflection across x=y

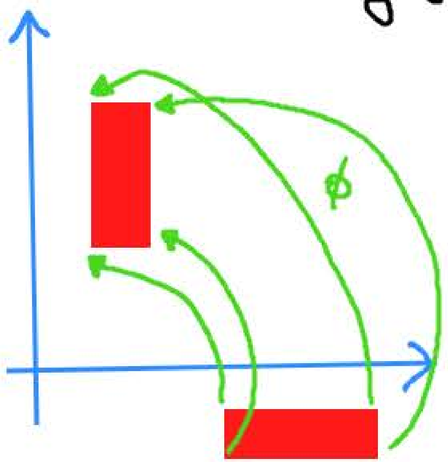


$\det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1$

b/c switches handedness

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$

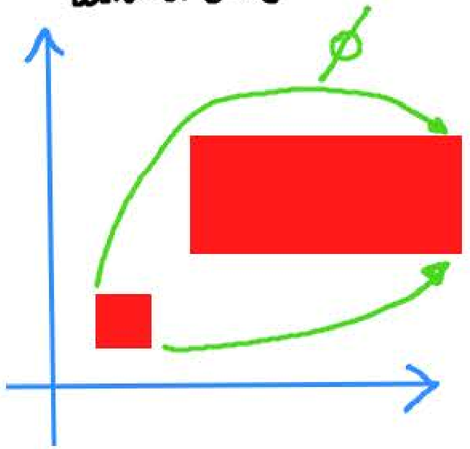
rotation by 90°



$\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 1$

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}$

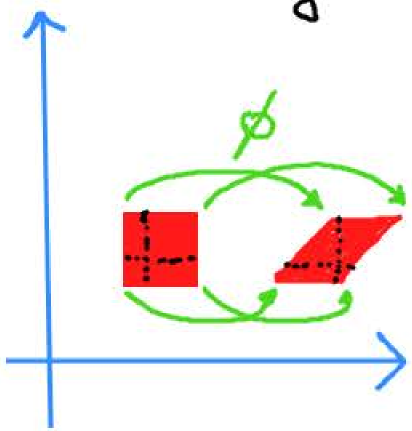
dilation



$\det \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} = 5 \cdot 3$

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 5x \\ 3y \end{bmatrix}$

shearing



$\det \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} = 1$

$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + cy \\ y \end{bmatrix}$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \cdot d - b \cdot c$$

Inverse matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

det of matrix

For any two $n \times n$ matrices A and B

$$\boxed{\det(AB) = \det A \cdot \det B}$$

(explanation in lecture notes)

$$R \xrightarrow{\psi} \psi(R) \xrightarrow{\phi} \phi(\psi(R))$$

$A \rightsquigarrow \phi$

$B \rightsquigarrow \psi$

$AB \rightsquigarrow \phi \circ \psi$

$$\det(AB) = \frac{\operatorname{vol} \phi(\psi(R))}{\operatorname{vol} R} = \underbrace{\frac{\operatorname{vol} \phi(\psi(R))}{\operatorname{vol} \psi(R)}}_{\det A} \cdot \underbrace{\frac{\operatorname{vol} \psi(R)}{\operatorname{vol} R}}_{\det B}$$

Caution:

$$\det(A+B) \neq \det(A) + \det(B)$$

$\det(AB) = \det A \cdot \det B$

$\Downarrow$

How does $\det A$ behave under row operations?

- adding a multiple of row i to row j does not change determinant
- exchanging two rows multiplies the determinant by -1
- multiplying a row by λ multiplies the determinant by λ

$\det E_{ij}^{(c)} = 1$

$\det P_{ij} = -1$

$\det D_i^{(\lambda)} = \lambda$

$A \rightsquigarrow E_{ij}^{(c)} \cdot A$

$\det(E_{ij}^{(c)} \cdot A) = \det A$

$A \rightsquigarrow P_{ij} A$

$\det(P_{ij} A) = \det A \cdot (-1)$

$A \rightsquigarrow D_i^{(\lambda)} A$

$\det(D_i^{(\lambda)} A) = \det A \cdot \lambda$

$$\det \begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ & & \ddots \\ 0 & & & d_n \end{bmatrix} = d_1 \cdots\cdots d_n$$

$$\det \begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix} = d_1 \cdots\cdots d_n$$

page 3 of Lecture 19

$$\det \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ * & & d_n \end{bmatrix} = d_1 \cdots\cdots d_n$$

$\det A = \pm \cdot$ product of pivots in REF(A)

$(-1)^{\#\text{ of row exchanges}}$

$$A \xrightarrow{\text{elimination}} E_{ij}^{(c)} A \xrightarrow{\text{exchange}} U$$

$\det A = \pm \cdot \det U$

$$U = \begin{bmatrix} \boxed{d_1} & & * \\ & \ddots & \\ 0 & & \boxed{d_n} \end{bmatrix}$$

pivots of U

$$PA = LDU$$

$$L = \begin{bmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ * & & & 1 \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \qquad U = \begin{bmatrix} 1 & & & * \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{bmatrix}$$

$$\underbrace{\det P}_{\substack{\pm 1 \\ \| \\ (-1)^{\#\text{ row exchanges}}}} \cdot \det A = \underbrace{\det L}_{1} \cdot \underbrace{\det D}_{\substack{d_1 \cdots d_n \\ \| \\ \text{product of the} \\ \text{pivots of REF}(A)}} \cdot \underbrace{\det U}_{1}$$

what you're trying to compute