$n \times n$ matrix A can be diagonalized if

- it has n eigenvalues $\lambda_1, \ldots, \lambda_n$; and it always does so in the world of complex numbers
- they have n linearly independent eigenvectors $v_1, \ldots, v_n$ } if this fails, use Jordan normal forms

$\Downarrow$

$$A = V \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V^{-1}$$

where $V = [v_1 | \ldots | v_n]$

$\lambda_1, \ldots, \lambda_n$ are the roots of the degree n characteristic polynomial $p(\lambda) = \det(A - I \cdot \lambda)$

e.g. $n = 2$: $p(\lambda) = \lambda^2 - t \cdot \lambda + d$

$t \rightarrow \mathrm{Tr}\, A$

$d \rightarrow \det A$

$$\lambda_1 = \frac{t + \sqrt{t^2 - 4d}}{2}$$

$$\lambda_2 = \frac{t - \sqrt{t^2 - 4d}}{2}$$

λ_1, λ_2 are not real if $t^2 - 4d < 0$; but, they are complex numbers

Def: define the symbol i such that

$$\boxed{i^2 = -1}$$

imaginary numbers

a complex number is any expression

$$z = \underbrace{a}_{\text{real part}} + \underbrace{b}_{\text{imaginary part}} \cdot i \quad \text{where } a, b \in \mathbb{R}$$

$a = \operatorname{Re} z$

$b = \operatorname{Im} z$

Algebra with complex numbers:

- $(a+bi) \pm (c+d\cdot i) = (a \pm c) + (b \pm d)\cdot i$
- $(a+bi)(c+di) = ac + bc\cdot i + ad\cdot i + bd\underbrace{i^2}_{-1}$
  $= \underbrace{ac - bd}_{\text{real part}} + \underbrace{(bc+ad)}_{\text{imaginary}}\cdot i$

Complex conjugate of

$z = a + bi$ is

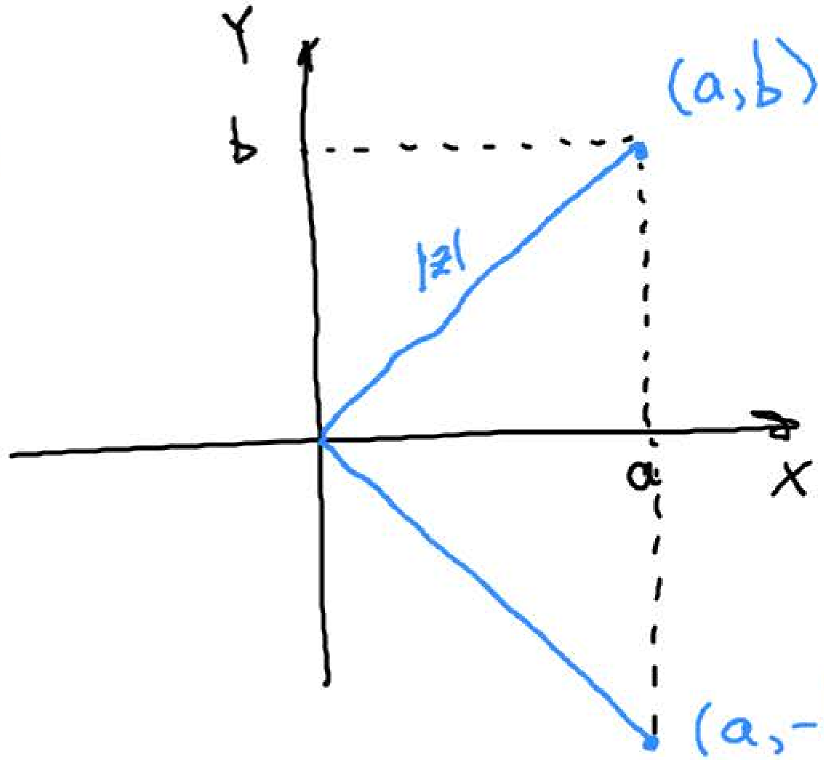
$\bar{z} = a - bi$

Absolute value of z is

$$|z| = \sqrt{a^2 + b^2}$$

Geometric interpretation of complex numbers :

the complex plane



(a,b) is identified with $z = a+bi$

$(a,-b)$ is identified with $\overline{z}$

Theorem :

$$z \cdot \overline{z} = |z|^2$$

positive real numbers

$$(a+bi)(a-bi)$$
$$\|$$
$$a^2 + \cancel{bai} - \cancel{abi} \underbrace{-b^2 i^2}_{b^2}$$
$$\|$$
$$a^2+b^2$$

$\mathbb{R} \subset \mathbb{C}$

real numbers　　complex number

$a \rightsquigarrow a + 0 \cdot i$

$$\frac{1}{z} = \frac{1 \cdot \overline{z}}{z \cdot \overline{z}} = \frac{\overline{z}}{|z|^2} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2} \cdot i$$

if $z = a+bi$

Division of complex numbers:

$$\frac{2+i}{4+3i} = \frac{(2+i)(4-3i)}{(4+3i)(4-3i)} = \frac{8+4i-6i-3i^2}{4^2+3^2} = \frac{11-2i}{25} = \frac{11}{25} - \frac{2i}{25}$$

$p(x) = ax^2+bx+c$ ⇝ its roots are $\boxed{\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}}$

- if $b^2-4ac>0$, the roots are real and distinct
- if $b^2-4ac=0$, there is a single root with multiplicity 2
- if $b^2-4ac<0$, its roots are: $\dfrac{-b \pm \sqrt{(-1)\cdot(4ac-b^2)}}{2a}$

$$b^2-4ac<0 \iff 4ac-b^2>0$$

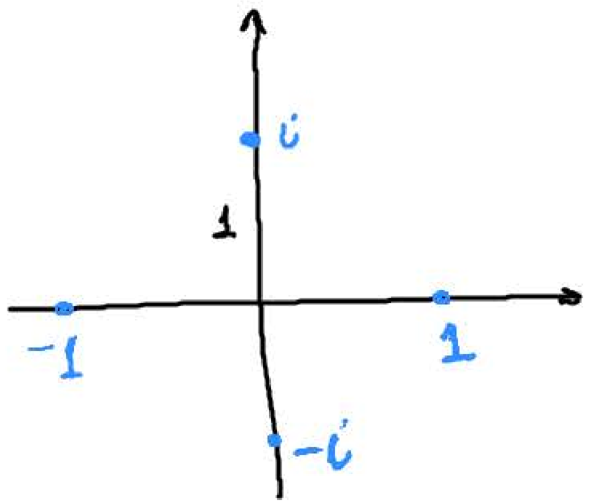
namely $\dfrac{\overset{\text{real}}{-b} \pm i \cdot \overset{\text{real}}{\sqrt{4ac-b^2}}}{\underset{\text{real}}{2a}}$ → the two roots are $\boxed{\text{conjugate}}$ complex numbers, if a, b, c are real

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rightsquigarrow \text{diagonalize}$$

$$p(\lambda) = \det(A - \lambda I) = \boxed{\lambda^2 - \lambda \cdot \mathrm{Tr}\, A + \det A} = \overset{\overset{a}{\parallel}}{1} \cdot \lambda^2 - \overset{\overset{b}{\parallel}}{0} \cdot \lambda + \overset{\overset{c}{\parallel}}{1}$$

$$= \lambda^2 + 1$$

roots are $\dfrac{0 \pm i \cdot \sqrt{4 \cdot 1 \cdot 1 - 0^2}}{2} = \pm i \implies \lambda_1 = i, \quad \lambda_2 = -i$



eigenvectors are $v_1 \in N(A - \lambda_1 I)$

$v_2 \in N(A - \lambda_2 I)$

$$N(A - i \cdot I) = N\left(\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix}\right) \xrightarrow{r_2 - i \cdot r_1} N\left(\begin{bmatrix} -i & -1 \\ \underbrace{1 - i(-i)}_{1 + i^2 = 0} & \underbrace{-i + i}_{0} \end{bmatrix}\right) = N\left(\begin{bmatrix} \boxed{-i} & -1 \\ 0 & 0 \end{bmatrix}\right)$$

$$\xrightarrow{R_1 \cdot \frac{1}{-i}} N\left(\begin{bmatrix} 1 & \frac{-1}{-i} \\ 0 & 0 \end{bmatrix}\right) = N\left(\begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}\right)$$

$$\frac{1}{i} = \frac{(-i)}{i\cdot(-i)} = \frac{-i}{1} = -i$$

$$v_1 = \begin{bmatrix} X \\ Y \end{bmatrix} \text{ s.t. } \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = 0 \iff X - Y\cdot i = 0$$

pivot

free

set $Y = 1$, solve for $X = i$

$$\Downarrow$$

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

similarly $v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$

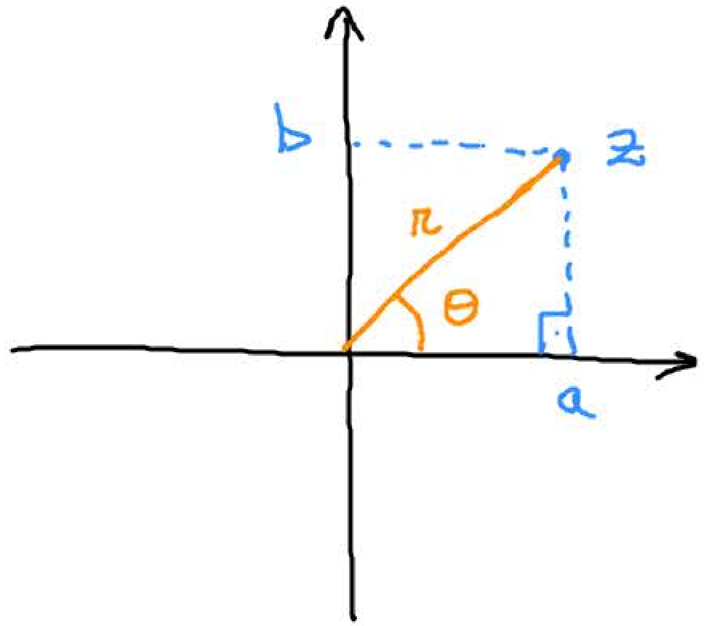
upshot :

inverses of 2×2 matrices

$$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{\text{rotation by } 90^\circ} = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\cdot\begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}^{-1}$$

Cartesian form is $\boxed{z = a + b \cdot i}$

Polar form of z is $\boxed{z = r e^{i\theta}}$



$$r = \sqrt{a^2 + b^2} = |z|$$

$$\theta = \arccos \frac{a}{\sqrt{a^2+b^2}} = \arg z$$

$$\cos \theta = \frac{a}{\sqrt{a^2+b^2}}$$

$$\sin \theta = \frac{b}{\sqrt{a^2+b^2}}$$

$$\Rightarrow \cos \theta + i \cdot \sin \theta = \frac{a + bi}{r}$$

$$\Updownarrow$$

$$r(\cos \theta + i \cdot \sin \theta) = a + bi$$

Thm:

$$\cos \theta + i \cdot \sin \theta = e^{i \cdot \theta}$$

$$z = r \cdot e^{i\theta}$$
$$z' = r' \cdot e^{i\theta'}$$
$$\Rightarrow z \cdot z' = rr' \cdot e^{i(\theta+\theta')}$$
$$\Rightarrow z^n = r^n \cdot e^{in\theta}$$

Roots of unity are $z^n = 1$ $\rightsquigarrow$ $e^{i\cdot\frac{2\pi}{n}}, e^{2i\cdot\frac{2\pi}{n}}, \ldots\ldots, e^{n\cdot i\cdot\frac{2\pi}{n}}$

if $z = r \cdot e^{i\theta}$, we need $r^n \cdot e^{in\theta} = 1$

$$\underbrace{e^{(n+1)i\cdot\frac{2\pi}{n}}} = \underbrace{e^{i\cdot\frac{2\pi}{n}}}$$

$$\Downarrow$$

$r^n = 1 \Longrightarrow r = 1$

$e^{in\theta} = 1 \Longrightarrow n\theta$ is an integer multiple of 2π

$$\Updownarrow$$

$\theta = \frac{2\pi k}{n}$ where $k \in \mathbb{Z}$

$$e^{i\cdot 2\pi} = 1$$