Fourier series:

$f, g \in V \rightsquigarrow f+g \in V$
$f \in V, c \in \mathbb{R} \rightsquigarrow c \cdot f \in V$

Def: let V be the vector space whose elements are periodic differentiable functions $f: \mathbb{R} \to \mathbb{R}$

f' is defined

$f(x+2\pi) = f(x)$ for all x

$\dim V = \infty$

$f(x) = 1$
$f(x) = \cos x$
$f(x) = \cos 2x$
$\vdots$
$f(x) = \sin x$
$f(x) = \sin 2x$
$\vdots$

all of these are elements of V $\rightsquigarrow$ they are all linearly independent

(but they are non-linearly dependent, e.g. $(\cos x)^2 = 1 - (\sin x)^2$)

Def: the inner product is the following analogue of the dot product for the ∞-ly dimensional vector space V

$$\underset{V}{\overset{\cap}{(f}}\,,\,\underset{V}{\overset{\cap}{g}}) = \int_{-\pi}^{\pi} f(x)\,g(x)\,dx \in \mathbb{R}$$

you would get the same value for the integral if you replaced $[-\pi,\pi]$ by any other interval of length 2π

Def: the norm is:

$$\|f\| = \sqrt{(f,f)} = \sqrt{\int_{-\pi}^{\pi} f(x)^2\,dx} \geq 0 \quad (\text{equality iff } f\equiv 0)$$

Thm: $(\sin Kx, \cos Lx) = 0$ for all integers K, L

$(\sin Kx, \sin Lx) = 0$ —— // —— $K \neq L$

$(\cos Kx, \cos Lx) = 0$ —— // —— $K \neq L$

$(1,1) = \int_{-\pi}^{\pi} 1 \cdot 1 \cdot dx = 2\pi$

$(\sin Kx, \sin Kx) = \pi$ for all $K \neq 0$

$(\cos Kx, \cos Kx) = \pi$ for all $K \neq 0$

Thm: any function $f \in V$ can be written as a $\boxed{\text{infinite}}$ linear combination of $1, \cos kx, \sin kx$ (as $k \in \mathbb{N}$)

i.e. $f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \dots$

$\underbrace{\qquad + b_1 \sin x + b_2 \sin 2x + \dots}_{\text{Fourier series expansion of } f(x)}$

Fourier coeffs of $f(x)$

makes sense in that the RHS converges as a series for all x, and the limit of this series is $f(x)$

You can get the Fourier coefficients from the inner products of $f(x)$:

$$(f(x), 1) = (a_0 + a_1 \cos x + \dots + b_1 \sin x + \dots, 1)$$

$$\int_{-\pi}^{\pi} f(x)\,dx \qquad a_0 \cdot (1,1) = a_0 \cdot 2\pi$$

$$\Longrightarrow a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

Similarly,

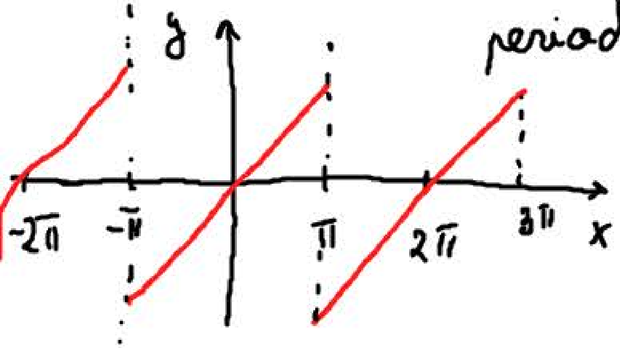
$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx$$

$(\cos kx, \cos kx)$ $\qquad$ $(f(x), \cos kx)$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx$$

$(\sin kx, \sin kx)$ $\qquad$ $(f(x), \sin kx)$

Ex: $f(x) = x$ for $x \in [-\pi, \pi)$ and extend to all x periodically



$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \cos(kx)\, dx = 0$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x \cdot \sin(kx)\, dx = (-1)^{k-1} \cdot \frac{2}{k}$$

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x\, dx = \frac{1}{2\pi} \left. \frac{x^2}{2} \right|_{-\pi}^{\pi} = 0$$

$$\Rightarrow X = \frac{2 \sin x}{1} - \frac{2 \sin 2x}{2} + \frac{2 \sin 3x}{3} - \dots$$

holds for all $x \in (-\pi, \pi)$

still $f(x) = x$

$$(f, f) = \int_{-\pi}^{\pi} x \cdot x \cdot dx = \frac{x^3}{3}\Big|_{-\pi}^{\pi} = \frac{2\pi^3}{3}$$

because

$$f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} 2}{k} \cdot \sin kx$$

$$\| \qquad \sum_{k=1}^{\infty} \frac{2^2}{k^2} (\sin kx, \sin kx) = \sum_{k=1}^{\infty} \frac{4\pi}{k^2}$$

$$\Rightarrow 4\pi \cdot \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{2\pi^3}{3}$$

$$\Downarrow$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots = \frac{\pi^2}{6}$$

There is an analogue of this for periodic functions $f : \mathbb{R} \to \mathbb{C}$

complex numbers

The role of $\sin kx, \cos kx$ will be taken by $\left\{ e^{ikx} \right\}_{k \in \text{integers}}$

$$e^{ik(x+2\pi)} = e^{ik\cdot x} \cdot e^{2\pi i \cdot k} = e^{ikx}$$

$$e^{2\pi i} = 1$$

$$(f, g) = \int_{-\pi}^{\pi} f(x)\, \overline{g(x)}\, dx$$

$$(e^{ikx}, e^{iLx}) = \int_{-\pi}^{\pi} e^{ikx} \cdot \overline{e^{iLx}}\, dx = \int_{-\pi}^{\pi} e^{ikx} \cdot e^{-iLx}\, dx =$$

$$= \int_{-\pi}^{\pi} e^{ix(k-L)}\, dx = \frac{e^{ix(k-L)}}{k-L} \Bigg|_{-\pi}^{\pi} = 0$$

$k \neq L$

$$(e^{ikx}, e^{ikx}) = 2\pi$$

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}$$

complex Fourier series coefficient

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-ikx}\, dx$$