S is a real symmetric $n \times n$

$$S = Q \cdot D \cdot Q^T \qquad \text{where} \quad \boxed{Q^T = Q^{-1}}$$

columns are eigenvectors → an o.n. basis of R^n

real eigenvalues

$$S = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \Rightarrow \lambda_1 = 9, \quad v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1, \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$p(\lambda) = \lambda^2 - \lambda \cdot 10 + 9$$

$$= (\lambda - 9)(\lambda - 1)$$

choose these s.t. $|v_1| = |v_2| = 1$

$$S = Q D Q^T$$

where $Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$D = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

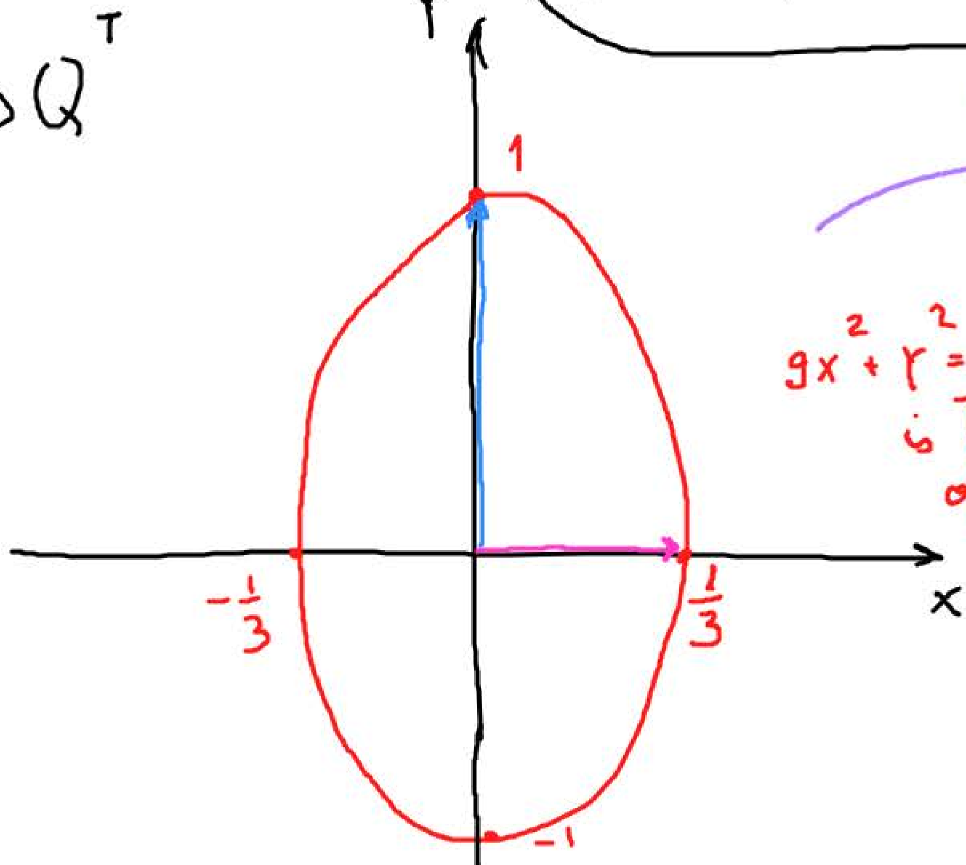
b/c $9, 1 > 0$, S is positive definite energy of $\begin{bmatrix} x \\ y \end{bmatrix}$ w.r.t. S is $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 5x^2 + 8xy + 5y^2 > 0$ for all $(x,y) = (0,0)$

equiv. with positive definiteness of S

energy of $\begin{bmatrix} x \\ y \end{bmatrix}$ w.r.t. D is $\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 9x^2 + y^2 > 0$

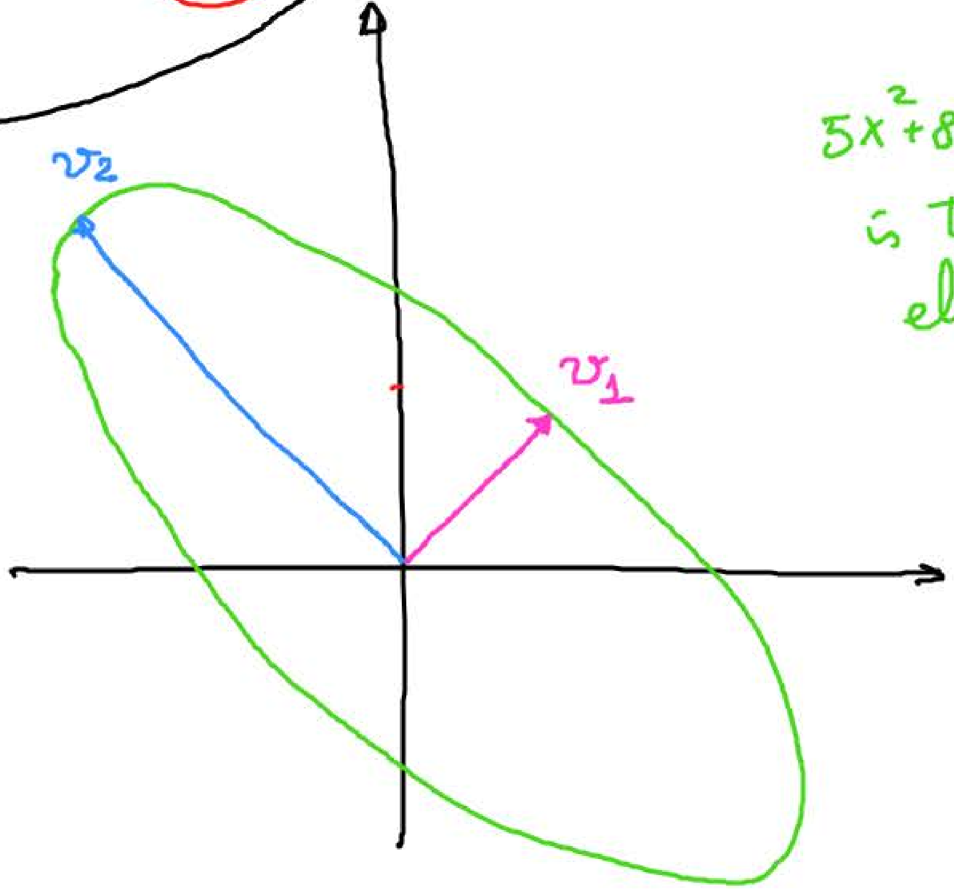
$$D = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$$

$$S = QDQ^T$$



$9x^2 + y^2 = 1$ is the equation of an ellipse

$\xrightarrow{Q}$



$5x^2 + 8xy + 5y^2 = 1$ is the equation of this ellipse.

$$S = QDQ^T \iff S = \lambda_1 q_1 q_1^T + \dots + \lambda_n q_n q_n^T$$

rank n

rank 1

$$Q = [\, q_1 | \dots | q_n \,]$$

$$D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Singular value decomposition:

any $m \times n$ matrix A can be written as

$$A = \sigma_1 \cdot u_1 v_1^T + \dots\dots + \sigma_r u_r v_r^T$$

singular values

rank 1

rank r

$m \times 1$ vectors "left singular vectors"

$n \times 1$ vectors "right singular"

white

red

Flag:
$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

How do you get: $\underset{\text{of rank } r}{\overset{m\times n}{A}} = u_1 \cdot \sigma_1 \cdot v_1^T + \dots + u_r \sigma_r v_r^T$

- singular values $\sigma_1, \dots, \sigma_r > 0$
- left singular vectors $u_1, \dots, u_r, \dots, u_m$ form a basis of $\mathbb{R}^m$
- right singular vectors $v_1, \dots, v_r, \dots, v_n$ form a basis of $\mathbb{R}^n$

positive

$$A \rightsquigarrow \underset{m\times m}{AA^T}, \text{ eig: } \sigma_1^2, \dots, \sigma_r^2, \overbrace{0, 0, \dots, 0}^{m-r}$$

$$A \rightsquigarrow \underset{n\times n}{A^TA}, \text{ eig. } \sigma_1^2, \dots, \sigma_r^2, \underbrace{0, 0, \dots, 0}_{n-r}$$

Def: the singular values $\sigma_1, \dots, \sigma_r$ of the matrix A are the $\sqrt{\text{the non-zero eigenvalues of } AA^T \text{ or } A^TA}$

Fact: AA^T and A^TA have the same non-zero eigenvalues

there are positive b/c AA^T & A^TA are positive semidefinite symm. matrices

Def: the singular vectors are eigenvectors of AA^T and A^TA:

- $\underline{AA^T} u_i = \underline{\sigma_i^2 u_i}$ (left singular vectors) $\to$ for all $1 \le i \le m$ (just set $\sigma_i^2 = 0$ if $i > r$)

- $\underline{A^TA} v_i = \underline{\sigma_i^2 v_i}$ (right singular vectors) $\to$ for all $1 \le i \le n$ (just set $\sigma_i^2 = 0$ if $i > r$)

symmetric

- the $u_1, \dots, u_m$ give you an o.n. basis of $\mathbb{R}^m$
- the $v_1, \dots, v_n$ ______ // ______ of $\mathbb{R}^n$

Thm. $A v_i = \sigma_i u_i$

$A^T u_i = \sigma_i v_i$

(these formulas also hold for $i > r$, just assume that $\sigma_i = 0$ for $i > r$).

$$A = u_1 \sigma_1 v_1^T + \cdots + u_r \sigma_r v_r^T$$

$$= U \Sigma V^T$$

A: $m \times n$, U: $m \times m$, Σ: $m \times n$, V^T: $n \times n$

, where $U = [u_1 | \cdots | u_m]$, $V = [v_1 | \cdots | v_n]$ → orthogonal matrices

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & 0 \\ & \ddots & & & \\ & & \sigma_r & & \\ 0 & & & 0 & \\ & & & & \ddots \end{bmatrix}$$

$$A^T = V \cdot \Sigma^T U^T$$

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix} \rightsquigarrow A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix} \rightsquigarrow \begin{array}{l} \lambda_1 = 25 \Rightarrow \sigma_1 = 5 \\ \lambda_2 = 9 \Rightarrow \sigma_2 = 3 \end{array}$$

rank 2

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$A v_1 = \sigma_1 u_1 \implies u_1 = \frac{A v_1}{\sigma_1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$A v_2 = \sigma_2 u_2 \qquad u_2 = \frac{A v_2}{\sigma_2} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$$

$$\Rightarrow A = u_1 \sigma_1 v_1^T + u_2 \sigma_2 v_2^T$$

to get u_3, do Gram-Schmidt $\Rightarrow u_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$