random variables $X_1, \ldots, X_n$ $\rightsquigarrow$ put them in a **random vector** $\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$

takes vector values $v_1, v_2, \ldots$ with probabilities $p_1, p_2, \ldots$

expected value: $\mu = \sum_i p_i \cdot v_i$ is a vector of numbers

(co)variance: $K = \sum_i p_i \underbrace{(v_i - \mu)}_{n\times 1} \underbrace{(v_i - \mu)^T}_{1\times n}$ is an $n \times n$ matrix → symmetric, pos. semi-def

$K = Q D Q^T$

set $\mathbf{Y} = Q^T \cdot \mathbf{X}$, then the (co)variance of $\underbrace{\mathbf{Y}}_{= \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}}$ is $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

upshot: $Y_1, \ldots, Y_n$ are uncorrelated random variable

Probabilistic $\underbrace{\text{Prisoner's Dillema}}$

A and B have a choice: either stay silent (S)
or they can confess (C)

How many years do A and B get in prison

B \ A	S	C
S	1, 1	3, 0
C	0, 3	2, 2

let's say A and B randomly / independently chooses S or C with prob $\frac{1}{2}$

random variables are X_A and X_B

years in prison for A and B

$$\mathbf{X} = \begin{bmatrix} X_A \\ X_B \end{bmatrix}$$

possible values are $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

SS   CS   SC   CC

probability 1/4

mean $\mu = \frac{1}{4}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{4}\begin{bmatrix}0\\3\end{bmatrix} + \frac{1}{4}\begin{bmatrix}3\\0\end{bmatrix} + \frac{1}{4}\begin{bmatrix}2\\2\end{bmatrix} = \begin{bmatrix}3/2\\3/2\end{bmatrix}$

covariance: $K = \frac{1}{4}\begin{bmatrix}1-3/2\\1-3/2\end{bmatrix}\begin{bmatrix}1-3/2 & 1-3/2\end{bmatrix} + \frac{1}{4}\begin{bmatrix}0-3/2\\3-3/2\end{bmatrix}\begin{bmatrix}0-3/2 & 3-3/2\end{bmatrix} +$

$+ \frac{1}{4}\begin{bmatrix}3-3/2\\0-3/2\end{bmatrix}\begin{bmatrix}3-3/2 & 0-3/2\end{bmatrix} + \frac{1}{4}\begin{bmatrix}2-3/2\\2-3/2\end{bmatrix}\begin{bmatrix}2-\frac{3}{2} & 2-\frac{3}{2}\end{bmatrix}$

$= \frac{1}{4}\begin{bmatrix}5 & -4\\-4 & 5\end{bmatrix}$ eigenvectors are $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix} = v_1$, $\lambda_1 = 9/4$

and $\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = v_2$, $\lambda_2 = 1/4$

$Q = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\-1 & 1\end{bmatrix}$, $D = \begin{bmatrix}9/4 & 0\\0 & 1/4\end{bmatrix} \Rightarrow \boxed{K = QDQ^T}$

$$\text{PCA}: \quad \mathbf{Y} = Q^T \mathbf{X} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} X_A \\ X_B \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

define
$$Y_1 = \frac{1}{\sqrt{2}}(X_A - X_B)$$
$$Y_2 = \frac{1}{\sqrt{2}}(X_A + X_B)$$

B \ A	S	C
S	1 / 1	3 / 0
C	0 / 3	2 / 2

uncorrelated

$X_A + X_B - 3$

mean = 3

B \ A	S	C
S	−1	0
C	0	1

sum of # years in prison

$X_A - X_B$

mean = 0

B \ A	S	C
S	0	−3
C	3	0

difference of # years in prison

Continuous random vectors $\mathbf{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}$ takes values in $\mathbb{R}^n$ according to probability distribution is $p : \mathbb{R}^n \longrightarrow \mathbb{R}_{\geq 0}$

expected value:

$$\boldsymbol{\mu} = \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} p(x_1, \dots, x_n) \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} dx_1 \dots dx_n \rightsquigarrow \text{a vector of numbers} \quad \boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}$$

$$K = \int_{-\infty}^{\infty} \cdots\cdots \int_{-\infty}^{\infty} p(x_1, \dots, x_n) \begin{bmatrix} x_1 - \mu_1 \\ \vdots \\ x_n - \mu_n \end{bmatrix} \begin{bmatrix} x_1 - \mu_1 & \dots\dots & x_n - \mu_n \end{bmatrix} dx_1 \dots dx_n \rightsquigarrow \text{an } n \times n \text{ matrix of numbers.}$$

example: n normally distributed random variables: $p(y_1, \dots, y_n) = \dfrac{e^{-\frac{(y_1 - \nu_1)^2}{2\Sigma_1} - \dots - \frac{(y_n - \nu_n)^2}{2\Sigma_n}}}{\sqrt{(2\pi)^n \Sigma_1 \dots \Sigma_n}}$

independent

with means $\nu_1, \dots, \nu_n$ and variances $\Sigma_1, \dots, \Sigma_n$

$$-\frac{(y_1-\nu_1)^2}{2\Sigma_1} - \cdots - \frac{(y_n-\nu_n)^2}{2\Sigma_n} = -\frac{1}{2} \overbrace{\begin{bmatrix} y_1-\nu_1 & \cdots & y_n-\nu_n \end{bmatrix}}^{(y-\nu)^T} D^{-1} \underbrace{\begin{bmatrix} y_1-\nu_1 \\ \vdots \\ y_n-\nu_n \end{bmatrix}}_{y-\nu}$$

$$D = \begin{bmatrix} \Sigma_1 & & 0 \\ & \ddots & \\ 0 & & \Sigma_n \end{bmatrix}$$

the covariance matrix of these n $\boxed{\text{independent}}$ normally distributed random variables is :

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{\text{this thing}}}{\sqrt{(2\pi)^n \Sigma_1 \cdots \Sigma_n}} \begin{bmatrix} y_1-\nu_1 \\ \vdots \\ y_n-\nu_n \end{bmatrix} \begin{bmatrix} y_1-\nu_1 & \cdots & y_n-\nu_n \end{bmatrix} dy_1 \cdots y_n$$

let $y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$ and $\nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_n \end{bmatrix}$

$$D = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(y-\nu)^T D^{-1} (y-\nu)}}{\sqrt{(2\pi)^n \det D}} (y-\nu)\cdot(y-\nu)^T \, dy_1 \cdots y_n$$

a identity in analysis

Example: dependent normally distributed random variables

fix a symmetric matrix S and a vector μ

$$p(x_1, \dots, x_n) = \frac{e^{-\frac{1}{2}(x-\mu)^T S^{-1}(x-\mu)}}{\sqrt{(2\pi)^n \det S}}$$

covariance matrix $\displaystyle\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}(x-\mu)^T S^{-1}(x-\mu)}}{\sqrt{(2\pi)^n \det S}} (x-\mu)(x-\mu)^T \, dx_1 \dots dx_n .$

$\|$

$Q \cdot$ (the covariance matrix from the independent case) Q^T

PCA: $S = QDQ^T$

$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \; y = Q^T x \Rightarrow \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} := Q^T \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$