Determinant of an $n \times n$ matrix A is $\det A = (-1)^{\#\text{row exchanges}} \cdot$ product of pivots of REF(A)

$$\begin{bmatrix} p_1 & & & * \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{bmatrix}$$

- what if REF(A) has a full row of zeroes, i.e. no pivot on some row?
- for the purpose of this formula, the "pivot" on a full 0 row is taken to be 0

A singular

$\Updownarrow$

$\det A = 0 \iff$ REF(A) has a full row of zeroes

$\iff$ rows of A are linearly dependent

$\Updownarrow$

$\det A^T = 0 \iff$ columns of A are linearly dependent (rows of A^T)

$\Updownarrow$

rank $A < n \iff A$ fails to be invertible

$$\boxed{\det A = \det A^T}$$

$\det A \neq 0 \iff A$ is invertible

$\det(AB) = \det A \cdot \det B$

$$A \cdot A^{-1} = I$$

$$\det A \cdot \det A^{-1} = \det I = 1 \implies \boxed{\det A^{-1} = \frac{1}{\det A}} \implies \det A \text{ must be non-zero for } A \text{ to be invertible}$$

$$\det(A+B) \neq \det A + \det B$$

however, something "along these lines" holds

$$\det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ a_1+b_1 & \cdots & a_n+b_n \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} = \det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,n} \\ a_1 & \cdots & a_n \\ x_{i+1,1} & \cdots & x_{i+1,n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix} + \det \begin{bmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{i-1,1} & \cdots & x_{i-1,n} \\ b_1 & \cdots & b_n \\ x_{i+1,1} & \cdots & x_{i+1,n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{bmatrix}$$

see Lecture notes for an argument

i-th row

Assume that the matrices above are equal on all rows $\neq i$

The formula above $\Longrightarrow$ "the big formula" for det A

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \det\begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\det\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} + \det\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} + \det\begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix}$$

$\det\begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} = 0$ b/c full column of 0's

$\det\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = a_{11}a_{22}$ b/c diagonal

$\det\begin{bmatrix} 0 & a_{12} \\ a_{21} & 0 \end{bmatrix} = -a_{12}a_{21}$ b/c diagonal after row exchange

$\det\begin{bmatrix} 0 & a_{12} \\ 0 & a_{22} \end{bmatrix} = 0$ b/c full column of 0's

$$\det\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

$$\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \text{sum of } 3^3 = 27 \text{ terms} = \begin{array}{l} 21 \text{ Terms are } 0 \\ \text{b/c their columns} \\ \text{are dependent} \end{array} +$$

$$= \det \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ a_{31} & 0 & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & a_{13} \\ a_{21} & 0 & 0 \\ 0 & a_{32} & 0 \end{bmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$\{1,2,3\} \to \{1,2,3\}$ $\{1,2,3\} \to \{2,3,1\}$ $\{1,2,3\} \to \{3,1,2\}$

$$+ \det \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & 0 & a_{23} \\ 0 & a_{32} & 0 \end{bmatrix} + \det \begin{bmatrix} 0 & a_{12} & 0 \\ a_{21} & 0 & 0 \\ 0 & 0 & a_{33} \end{bmatrix} + \det \begin{bmatrix} 0 & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & 0 \end{bmatrix} \quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$\{1,2,3\} \to \{1,3,2\}$ $\{1,2,3\} \to \{2,1,3\}$ $\{1,2,3\} \to \{3,2,1\}$

# of terms $= 6 = 3!$

(formula above gives the same answer as the diagonals method / Sarrus' rule for computing det (3×3))

Thm: (the big formula for det)

# of row exchanges

of $\underbrace{i<j \text{ such that } \sigma(i)>\sigma(j)}_{\text{an "inversion" of } \sigma}$

$$\det \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix} = \sum_{\substack{\text{permutations} \\ \sigma:\{1,\dots,n\}\to\{1,\dots,n\}}} (-1)\; a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

useful for sparse matrices with lots of 0's

$n\times n$ matrix A

zoom in on i-th row

$$\det\begin{bmatrix} & * & \\ a_{i1} & \cdots & a_{in} \\ & * & \end{bmatrix} = \det\begin{bmatrix} & * & \\ a_{i1} & 0 \cdots & 0 \\ & * & \end{bmatrix} + \cdots + \det\begin{bmatrix} & * & \\ 0 & \cdots 0 & a_{in} \\ & * & \end{bmatrix}$$

using big formula only pick up those permutations s.t $\sigma(i)=1$

using big formula, only pick up perms s.t. $\sigma(i)=n$

$$A = \begin{bmatrix} * & a_{1j} & * \\ & \vdots & \\ a_{i1} \cdots\cdots & a_{ij} & \cdots\cdots a_{in} \\ & \vdots & \\ * & a_{nj} & * \end{bmatrix} \rightsquigarrow M_{ij} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

is the $n-1 \times n-1$ matrix obtained by removing i-th row and j-th column of A

COFACTOR EXPANSION:

$$\det A = a_{i1} \cdot \underbrace{(-1)^{i+1} \det M_{i1}}_{C_{i1}} + a_{i2} \cdot \underbrace{(-1)^{i+2} \det M_{i2}}_{C_{i2}} + \cdots\cdots + a_{in} \cdot \underbrace{(-1)^{i+n} \det M_{in}}_{C_{in}}$$

$C_{ij} = (-1)^{i+j} \det M_{ij}$ are called the cofactors of A.