let V be the subspace spanned by vectors $v_1, \ldots, v_n \in \mathbb{R}^m$

$$\| \quad \left\{ \underbrace{\alpha_1 v_1 + \cdots + \alpha_n v_n}_{\text{a linear combination}}, \text{ for all real } \alpha_1, \ldots, \alpha_n \in \mathbb{R} \right\}$$

But what if one of the vectors, say v_i, is a linear combination of the others, i.e.

$$v_i = \beta_1 v_1 + \cdots + \beta_{i-1} v_{i-1} + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n$$

for some reals $\beta_1 \ldots \beta_n$?

$$V = \left\{ \alpha_1 v_1 + \cdots + \alpha_{i-1} v_{i-1} + \alpha_i \left(\beta_1 v_1 + \cdots + \beta_{i-1} v_{i-1} + \beta_{i+1} v_{i+1} + \cdots + \beta_n v_n \right) + \alpha_{i+1} v_{i+1} + \cdots + \alpha_n v_n \right\}$$

$$= \left\{ \gamma_1 v_1 + \cdots + \gamma_{i-1} v_{i-1} + \gamma_{i+1} v_{i+1} + \cdots + \gamma_n v_n \right\}$$

$\Longrightarrow$ UPSHOT: V is actually spanned by $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$

Def: vectors $v_1, \dots, v_n$ are called (linearly) independent if none of them is a linear combination of the others

(they are called linearly dependent if one of them is a linear combination of the others)

independent: if $\alpha_1 v_1 + \dots + \alpha_n v_n \neq 0$ except if all the α's are 0

dependent: there exists a relation $\boxed{\alpha_1 v_1 + \dots + \alpha_n v_n = 0}$ without the α's being all 0

if a vector space V is spanned by a bunch of linearly dependent $v_1, \dots, v_n$

then you can remove some of the v_i's such that the remaining ones are linearly independent and span the same vector space V

$\alpha_1 v_1 + \dots + \alpha_n v_n = 0$

assume $\alpha_i \neq 0$

$$\Rightarrow \alpha_i v_i = -\alpha_1 v_1 - \dots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} - \dots - \alpha_n v_n$$

$$v_i = \frac{1}{\alpha_i}\left(-\alpha_1 v_1 - \dots - \alpha_{i-1} v_{i-1} - \alpha_{i+1} v_{i+1} - \dots - \alpha_n v_n\right)$$

$V_1 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

are they independent ? YES

if they were dependent $\Rightarrow \alpha V_1 + \beta V_2 = 0 \Rightarrow V_1 = -\frac{\beta}{\alpha} V_2$

$\Rightarrow$ one is a multiple of the other $\Rightarrow$ not true

(if $V_1 = c \cdot V_2 \Rightarrow \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = c \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} \Rightarrow$ $c = 2$, $= -\frac{5}{3}$, $= -1$ simultaneously)

what about V_1, V_2 and

$V_3 = \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix}$; NO

because there exists a linear relation

$$1 \cdot \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 4 \\ -1 \\ -1 \end{bmatrix} = 0$$

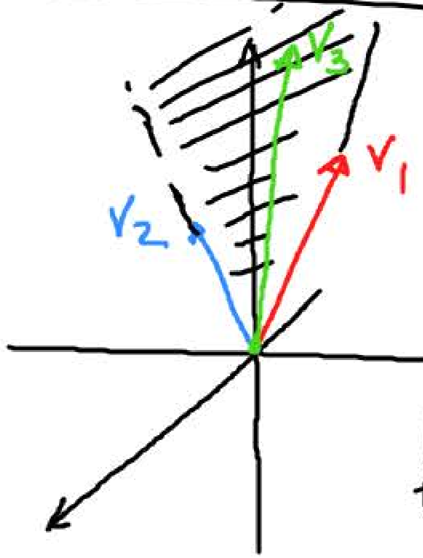
i.e. $1 \cdot V_1 + 2 \cdot V_2 + (-1) V_3 = 0$ relation

$\Rightarrow$

$V_1 = -2V_2 + V_3$

$V_2 = \frac{-V_1 + V_3}{2}$

$V_3 = V_1 + 2V_2$



DEPENDENCE

$\Updownarrow$

any one of the V_i's is in the plane spanned by the other two why $1 \cdot v_1 + 2 \cdot v_2 + (-1) \cdot v_3 = 0$?

suppose you want $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

$$\Updownarrow$$

$$[v_1 | v_2 | v_3] \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\Updownarrow$$

$$\underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 5 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}}_{= A} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0 \iff \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} \in N(A)$$

Null spaces are computed by Gauss-Jordan of A

<u>Def</u>: a basis of a vector space V is a collection of $\boxed{\text{linearly independent}}$ vectors which span V

(ex: v_1, v_2 were a basis, but also v_2, v_3 were a basis but also v_1, v_3 were a basis)

<u>Def</u>: the dimension of V is the number of vectors that comprise any basis of V

dim (line) = 1

dim (plane) = 2

dim (origin) = 0

if given $v_1, \ldots, v_n$, how can you tell if they are dependent or not?

$\Updownarrow$

exists $\alpha_1 v_1 + \ldots + \alpha_n v_n = 0$

$\Updownarrow$

$$A \cdot \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} = 0 \quad \text{where } A = [v_1 | \ldots | v_n]$$

$\Updownarrow$

dependent if $N(A) \neq 0$

independent if $N(A) = 0$

v_1, v_2, v_3 from before $\Rightarrow A = \begin{bmatrix} 2 & 1 & 4 \\ 5 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

$\downarrow$ REF

$$U = \begin{bmatrix} \boxed{2} & 1 & 4 \\ 0 & \boxed{-5.5} & -11 \\ 0 & 0 & 0 \end{bmatrix}$$

$\downarrow$ RREF

at most 2 pivots

at most 2 pivot rows

at least 1 free var

$\Leftarrow$

you can always find α, β, γ not all 0 such that $U \cdot \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = 0$

(A)

$\Updownarrow$

$\exists$ a relation $\alpha v_1 + \beta v_2 + \gamma v_3 = 0$

<u>Fact</u> : $\text{rank} = \dim C(A)$

$\parallel$

(# of pivots of A in (R)REF)

i.e. $\text{rank}(A) = \text{rank}(R)$ its reduced row echelon form

every column of A is a linear combination of its pivot columns

$$R = \begin{bmatrix} \boxed{1} & * & 0 & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & * & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}$$

all the non-pivot columns are linear combinations of the pivot columns

2 ways to present a subspace $V =$ spanned by $v_1, \dots, v_n$

$\parallel$

cut out by some equations

how do we go from one presentation to the other?

Ex: $V = \left\{ \begin{array}{l} 2x - 2y - 6z - 4t = 0 \\ 5x + y - 3z + 8t = 0 \\ -3x + 2y + 7z + 3t = 0 \end{array} \right\} \subset \mathbb{R}^4 \rightsquigarrow \begin{bmatrix} 2 & -2 & -6 & -4 \\ 5 & 1 & -3 & 3 \\ -3 & 2 & 7 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$

$\parallel$

expect V to be a line in $\mathbb{R}^4$

not so fast!

i.e. $V = N(A)$

I want a basis of $V = N(A)$

$$A \overset{RREF}{\rightsquigarrow} R = \begin{bmatrix} \boxed{1} & 0 & -1 & 1 \\ 0 & \boxed{1} & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot vars

$$\begin{cases} x = z - t \\ y = -2z - 3t \end{cases} \iff R \cdot \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$$

free vars

solve for x, y

; a basis of $N(A)$ is given by setting one of the free variables $= 1$ and all other free variables $= 0$

$$v_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

a basis of $V \Rightarrow \dim V = 2$.