Common mistakes on midterm

- REF $\rightsquigarrow$

$$\begin{pmatrix} &&&& * &&&&& \\ &&&& * &&&&& \\ &&&& * &&&&& \\ 0&0&0&0&\boxed{a}&*&*&*&*&* \\ &&&& 0 &&&&& \\ &&&& 0 &&&&& \\ &&&& 0 &&&&& \end{pmatrix}$$

any non-zero pivot

- RREF $\rightsquigarrow$

$$\begin{pmatrix} &&&&& 0 & * &&&& \\ &&&&& 0 & * &&&& \\ &&&&& 0 & * &&&& \\ 0&0&0&0&0&\boxed{1}&*&*&*&*&* \\ &&&&& 0 &&&&& \\ &&&&& 0 &&&&& \\ &&&&& 0 &&&&& \end{pmatrix}$$

say this is not a pivot column

$\rightsquigarrow$ when applying Gaussian elim to $[A \mid b]$, $\boxed{\text{perform the same row operations to the } b \text{ column as to the } A \text{ matrix}}$

· Projections : $P_V = A(A^TA)^{-1}A^T$

always invertible as long as the columns of A are independent

· vector spaces can be perpendicular to each other (also vectors)

But it doesn't make sense to say that matrices are perpendicular to each other

- vectors & vector spaces are geometric objects
- matrices are just representations of geometric objects, but they do not have intrinsic geometry

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \text{ s.t. } \boxed{3x + 4y = 0} \right\}$$

write $V = C(A)$; what is A?

$\to$ 2×1 matrix

$V \neq C\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$ doesn't make mathematical sense

$V = C\left(\begin{bmatrix} 4 \\ -3 \end{bmatrix}\right)$ b/c $(x,y) = (4,-3)$ satisfies equation

Today: linear transformations

$$\mathbb{R}^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ where } x_1, \dots, x_n \in \mathbb{R} \right\}$$

Def: a linear transformation is a function

$$\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

such that
- $\phi(v + v') = \phi(v) + \phi(v')$
- $\phi(c \cdot v) = c \cdot \phi(v)$

} reminiscent of the defining properties of vector spaces

for any $v, v' \in \mathbb{R}^n$
for any scalar c

Linear transformations send objects (lines, triangles, ellipses...) to objects of the same nature

Examples:

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

- reflections (mirror images)
- dilation/contractions
- rotations
- shearing

Matrices represent $\underbrace{\text{linear transformations}}_{\phi : \mathbb{R}^n \to \mathbb{R}^m}$

take the basis $e_1, \dots, e_n$ of $\mathbb{R}^n$
$e_1, \dots e_m$ of $\mathbb{R}^m$

$$e_i = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad 1 \to \text{i-th spot}$$

· $\phi(e_j)$ = some vector in $\mathbb{R}^m$ $= \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$ ← j-th column of A

$= a_{1j} e_1 + a_{2j} e_2 + \dots + a_{mj} e_m$

for all $j \in \{1, \dots, n\}$

$\rightsquigarrow$ determines a matrix

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

linear transformation

$\to$ $m\times n$ matrix

what is the connection between ϕ and A?

$$\boxed{\phi(e_j) = a_{1j}e_1 + \dots + a_{mj}e_j}$$

what about $\phi(v)$ for any $v \in \mathbb{R}^n$

$$v = v_1e_1 + v_2e_2 + \dots + v_ne_n = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$\phi(v)$

$\Vert$ b/c ϕ is linear

$\phi(v_1e_1) + \phi(v_2e_2) + \dots + \phi(v_ne_n)$

$\Vert$ b/c ϕ is linear

$v_1\phi(e_1) + v_2\phi(e_2) + \dots + v_n\phi(e_n)$

$\Vert$

$$\sum_{j=1}^{n} v_j\phi(e_j) = \sum_{j=1}^{n} v_j \sum_{i=1}^{m} a_{ij}e_i = \sum_{i=1}^{m}\left(\sum_{j=1}^{n} a_{ij}v_j\right)e_i = \begin{bmatrix} a_{11}v_1 + \dots + a_{1n}v_n \\ a_{21}v_1 + \dots + a_{2n}v_n \\ \vdots \\ a_{m1}v_1 + \dots + a_{mn}v_n \end{bmatrix} = Av$$

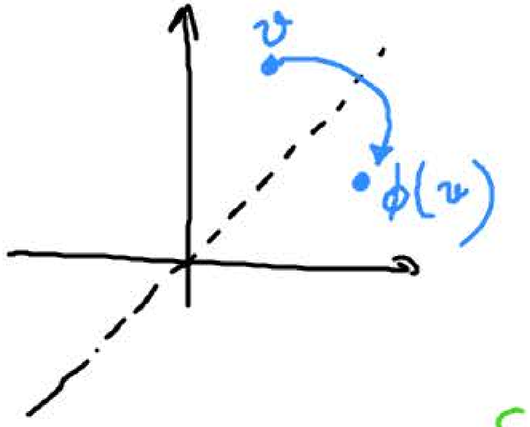
Upshot: any linear transformation $\phi : \mathbb{R}^n \to \mathbb{R}^m$

is of the form $\phi(v) = Av$

for a uniquely determined $\boxed{m \times n \text{ matrix } A}$ $\longrightarrow$ the matrix associated to ϕ

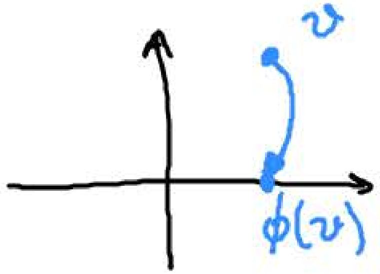
Examples: $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ is

- reflection in the line $\{x = y\} \subset \mathbb{R}^2$



$\longleftrightarrow A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

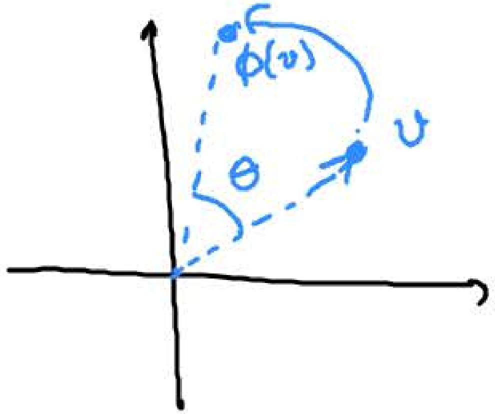
- projection onto x axis



$\longleftrightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

- dilation by λ in the x direction and by μ in the y direction $\longleftrightarrow A = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$

- rotation by angle θ around the origin of $\mathbb{R}^2$



$$\longleftrightarrow A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

- projection onto a subspace $V \subset \mathbb{R}^n$ is a linear transformation $\longleftrightarrow A(A^TA)^{-1}A^T$ where $V = C(A)$

$$\boxed{\begin{aligned} \phi(v) &= \mathrm{proj}_V v \\ &\;\Vert \\ A&(A^TA)^{-1}A^T v \end{aligned}}$$

Theorem: composition of linear transformations corresponds to matrix multiplication

$$\phi : \mathbb{R}^n \to \mathbb{R}^m \rightsquigarrow A^{m\times n}$$
$$\psi : \mathbb{R}^p \to \mathbb{R}^n \rightsquigarrow B^{n\times p}$$
$$\Rightarrow \phi\circ\psi : \mathbb{R}^p \longrightarrow \mathbb{R}^m \rightsquigarrow AB^{m\times p}$$

$$\mathbb{R}^p \xrightarrow{\psi} \mathbb{R}^n \xrightarrow{\phi} \mathbb{R}^m$$

This is why we only multiply an $m\times \underline{n}$ matrix with an $\underline{n}\times p$ matrix