Linear transformations = functions $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

given by formula $\boxed{\phi(v) = Av}$ LAST TIME

for some particular $m \times n$ matrix THIS TIME

Properties (easy computations)

$$\boxed{\begin{array}{ll} (1) & \phi(v+v') = \phi(v) + \phi(v') \\ (2) & \phi(c \cdot v) = c \cdot \phi(v) \\ & \text{for all } v, v' \in \mathbb{R}^n \\ & \text{— " — scalars } c \end{array}}$$

(1) $\phi(v+v') = A(v+v') = Av + Av' = \phi(v) + \phi(v')$

(2) $\phi(c \cdot v) = Ac \cdot v = c \cdot A \cdot v = c \cdot \phi(v)$

Note on projections: if projecting onto a line, i.e. $A = \begin{bmatrix} \text{COLUMN} \end{bmatrix}$

then the formula

$$P_V = A \underbrace{(A^T A)}_{\text{this is just a } 1\times 1 \text{ matrix}}^{-1} A^T$$

is really easy

Linear transformations

$\phi : \mathbb{R}^n \to \mathbb{R}^m$

$\longleftrightarrow$

$m \times n$ matrices

A

Moral: $\underbrace{A \text{ represents } \phi}$ in the standard basis $e_1, \dots, e_n$ of $\mathbb{R}^n$
$e_1, \dots, e_m$ of $\mathbb{R}^m$

$$\phi(x_1 e_1 + \dots + x_n e_n) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) e_i$$

LAST TIME

entries of A give a formula for ϕ in the standard basis

TODAY: we'll generalize this setup to arbitrary bases of $\mathbb{R}^n / \mathbb{R}^m$ instead of the standard basis $\longrightarrow$ CHANGE OF BASIS

$\phi \rightsquigarrow A$ means that $\phi(v) = Av$

$\psi \rightsquigarrow B$ means that $\psi(v) = Bv$

$$\boxed{\begin{array}{l} \phi \circ \psi \rightsquigarrow A \cdot B \\ \phi^{-1} \rightsquigarrow A^{-1} \end{array}}$$

means that $\phi \circ \psi(v) = \phi(\psi(v)) = \phi(Bv) = A(Bv)$
$= (AB)v$

means that $\phi^{-1}(v) = w$

such that $v = \phi(w)$
$= Aw$

$\Updownarrow$

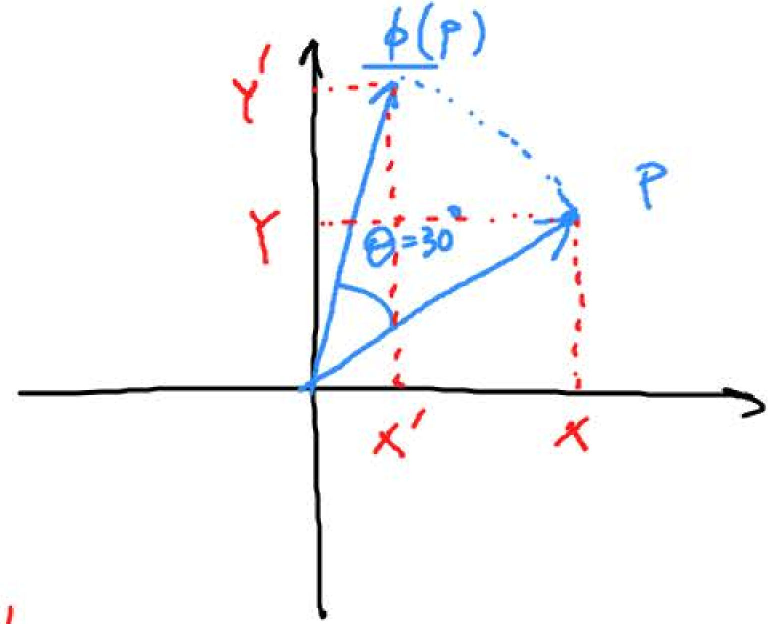
$A^{-1}v = w$

$\Longrightarrow \phi^{-1}(v) = A^{-1}v$

the part about ϕ^{-1} & A^{-1} only makes sense if ϕ, A are invertible (in particular, only if $m = n$)

Change of basis : $\phi : \mathbb{R}^2 \to \mathbb{R}^2$

$$\phi(p) = p \ \underbrace{\text{rotated by } 30^\circ}_{\theta = \frac{\pi}{6}} \text{ counter-clockwise around origin}$$



$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$$

$$\phi\left(\begin{bmatrix} X \\ Y \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \dfrac{\sqrt{3}\cdot X - Y}{2} \\ \dfrac{X + \sqrt{3}\cdot Y}{2} \end{bmatrix} \begin{matrix} \to X' \\ \\ \to Y' \end{matrix}$$

$$\Downarrow$$

$$\boxed{\phi(X\cdot e_1 + Y\cdot e_2) = \left(\frac{\sqrt{3}\cdot X - Y}{2}\right) e_1 + \left(\frac{X + \sqrt{3}\, Y}{2}\right)\cdot e_2}$$

what if we wanted an analogous formula for e_1, e_2 replaced by another basis v_1, v_2 of $\mathbb{R}^2$?

$v_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ form a basis of $\mathbb{R}^2$

can you find a formula for ϕ in terms of the basis v_1, v_2?

i.e. $\phi(x \cdot v_1 + y \cdot v_2) = \# \cdot v_1 + * \cdot v_2$

goal: find formulas for # and *

Approach:

(1) convert v_1, v_2 into e_1, e_2 $\longrightarrow$ $v_1 = 2e_1$, $v_2 = e_1 - e_2$

(2) apply formula for ϕ on the bottom of the previous slide

solve for e's in terms of the v's

(3) convert e_1, e_2 back into v_1, v_2 $\longrightarrow$ $e_1 = \frac{v_1}{2}$, $e_2 = e_1 - v_2 = \frac{v_1}{2} - v_2$

$$\phi(x\cdot v_1 + y\cdot v_2) = \phi\big(x\cdot 2e_1 + y\cdot(e_1-e_2)\big) =$$

here x, y can be anything

$$= \phi\big(\underbrace{(2x+y)}_{x'}e_1 + \underbrace{(-y)}_{y'}e_2\big)$$

by formula in purple box from 2 slides ago

$$= \frac{\sqrt{3}\,x' - y'}{2}\,e_1 + \frac{x' + \sqrt{3}\cdot y'}{2}\,e_2 = \frac{\sqrt{3}(2x+y)+y}{2}\cdot e_1 + \frac{2x+y-y\sqrt{3}}{2}\cdot e_2$$

$$= \frac{\sqrt{3}(2x+y)+y}{2}\cdot\frac{v_1}{2} + \frac{2x+y-y\sqrt{3}}{2}\left(\frac{v_1}{2} - v_2\right)$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{1}{2} \\ -1 & \frac{\sqrt{3}-1}{2} \end{bmatrix}$$

$$= \left(\frac{\sqrt{3}(2x+y)+y}{4} + \frac{2x+y-y\sqrt{3}}{4}\right)v_1 + \frac{y\sqrt{3}-2x-y}{2}\cdot v_2$$

$$= \left[\frac{x(2\sqrt{3}+2)}{4} + \frac{2y}{4}\right]v_1 + \frac{y(\sqrt{3}-1)-2x}{2}\cdot v_2 = \left[x\cdot\overset{b_{11}}{\frac{\sqrt{3}+1}{2}} + y\cdot\overset{b_{12}}{\frac{1}{2}}\right]v_1 + \left[x\cdot\overset{b_{21}}{(-1)} + y\cdot\overset{b_{22}}{\frac{\sqrt{3}-1}{2}}\right]v_2$$

Upshot: just like $A = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$ controlled the formula for ϕ in the basis e_1, e_2

the matrix $B = \begin{bmatrix} \frac{\sqrt{3}+1}{2} & \frac{1}{2} \\ -1 & \frac{\sqrt{3}-1}{2} \end{bmatrix}$ controls the formula for ϕ in the basis v_1, v_2

$$\phi(x v_1 + y v_2) = $$
$$= (x b_{11} + y b_{12}) v_1 + (x b_{21} + y b_{22}) v_2$$

what is the connection between A & B? If you know A, how to get B

$\Downarrow$

CHANGE OF BASIS FORMULA

$$\boxed{B = V^{-1} A V}, \text{ where } V = \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix} = [v_1 | v_2]$$

$$= \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}^{-1} \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}$$

the entries of V are the basis vectors v_1, v_2 in the standard basis

V is called the CHANGE OF BASIS matrix from the basis v_1, v_2 to the basis e_1, e_2

$$\Updownarrow$$

$V = [v_1 | v_2]$ means $V \cdot e_1 = v_1$ and $V \cdot e_2 = v_2$

what is the CHANGE OF BASIS MATRIX from the basis e_1, e_2 to the basis v_1, v_2 $\longleftrightarrow$ V^{-1} because

$V^{-1} \cdot v_1 = e_1$

$V^{-1} \cdot v_2 = e_2$

what is the CHANGE OF BASIS MATRIX from the basis v_1, v_2 to the basis w_1, w_2 where $V = [v_1 | v_2]$ and $W = [w_1 | w_2]$ $\longleftrightarrow$ $W^{-1} V$

all change of bases matrices are invertible

Summary: if

$$\phi(x_1 e_1 + \cdots + x_n e_n) = (a_{11} x_1 + \cdots + a_{1n} x_n) e_1 + \cdots + (a_{n1} x_1 + \cdots + a_{nn} x_n) e_n$$

then

$$\phi(y_1 v_1 + \cdots + y_n v_n) = (b_{11} y_1 + \cdots + b_{1n} y_n) v_1 + \cdots + (b_{n1} y_1 + \cdots + b_{nn} y_n) v_n$$

$\forall$ numbers

$x_1 \ldots x_n$

$y_1 \ldots y_n$

where

$$B = V^{-1} A V \qquad V = [v_1 | \cdots | v_n]$$

$$B = [\cdots b_{ij} \cdots] \qquad A = [\cdots a_{ij} \cdots]$$