Gaussian elimination

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \cdot X + (-1) \cdot Y + 2 \cdot Z \\ (-2)X + 2Y + (-3)Z \\ (-3)X + (-1)Y + 2Z \end{bmatrix} = \begin{bmatrix} X - Y + 2Z \\ -2X + 2Y - 3Z \\ -3X - Y + 2Z \end{bmatrix}$$

Solve the system:

$$\begin{cases} X - Y + 2Z = 1 \\ -2X + 2Y - 3Z = -1 \\ -3X - Y + 2Z = -3 \end{cases} \iff \underbrace{\begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{b}$$

Goal:
Solve for v

Augmented matrix = $[A \mid b]$ = $\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ \boxed{-2} & 2 & -3 & -1 \\ \boxed{-3} & -1 & 2 & -3 \end{array}\right]$

pivot

Pivot = the leftmost non-zero entry on each row (not allowed to be in b)

Gaussian elimination:

using row operations, goal is for the pivot on row i should be to the right of the pivot on row j, for all $i > j$

- row eliminations (add a multiple of row j to row i) we will do $i > j$
- row exchange
- multiplying a whole row by some non-zero const

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ \boxed{-2} & 2 & -3 & -1 \\ \boxed{-3} & -1 & 2 & -3 \end{array}\right] \xrightarrow{2\cdot r_1 + r_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ \boxed{-3} & -1 & 2 & -3 \end{array}\right] \xrightarrow{3r_1 + r_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & \boxed{-4} & 8 & 0 \end{array}\right]$$

should be made 0

should be made 0

add a multiple of row 1 to row 2 so as to replace the -2 by 0

$$a \cdot row_1 + row_2 = \underbrace{\left[\begin{array}{ccc|c} \boxed{-2 + a \cdot 1} & 2 + a\cdot(-1) & -3 + a \cdot 2 & -1 + a \cdot 1 \end{array}\right]}$$

need $-2 + a \cdot 1 = 0$

need $a = 2$

$$\left[\begin{array}{ccc|c} 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

swap rows 2 and 3

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{-4} & 8 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

ROW ECHELON FORM

Gauss-Jordan elimination : after Gaussian elimination, you can also make all pivots equal to 1 and ensure that all entries above a pivot are 0

REDUCED ROW ECHELON FORM

$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{-4} & 8 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow{r_2 \cdot \left(-\frac{1}{4}\right)} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 2 & 1 \\ 0 & \boxed{1} & -2 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow{2 \cdot r_3 + r_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{(-2) r_3 + r_1}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{1 \cdot r_2 + r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \text{RREF} \checkmark$$

$$Av = b \qquad [A|b] \overset{\text{Gaussian}}{\rightsquigarrow} [U|c]$$

$$[U|c] = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & -4 & 8 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Principle: any solution v of $\boxed{Av=b}$ is a solution of $\boxed{Uv=c}$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \iff \begin{cases} X - Y + 2Z = 1 \\ -4Y + 8Z = 0 \\ 1 \cdot Z = 1 \end{cases}$$

BACK SUBSITUTION

$$\boxed{Z = 1}$$

plug it into eqn 2 : $-4Y + 8 \cdot 1 = 0 \Rightarrow \boxed{Y = 2}$

plug into eqn 1

$$X - 2 + 2 \cdot 1 = 1 \Rightarrow \boxed{X = 1}$$

SOLUTION: $\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Principle: performing a row operation on a matrix A amounts to the same thing as multiplying A on the left by a very specific matrix

- add λ times row j to row i : $A \rightsquigarrow E_{ij}^{(\lambda)} A$

elimination matrix (the λ is located on row i and column j)

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \lambda & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

- swap row i and row j : $A \rightsquigarrow P_{ij} A$

permutation matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{0} & \boxed{1} & 0 \\ 0 & 0 & 0 & \boxed{1} & \boxed{0} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$i = 4, j = 5$

- multiply row i by λ

$A \rightsquigarrow D_i^{(\lambda)} A$

diagonal matrix (the λ is on the i-th row)

$$\begin{bmatrix} 1 & & & & \\ & 1 & & \text{\Large 0} & \\ & & \lambda & & \\ & \text{\Large 0} & & 1 & \\ & & & & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -1 & 2 \\ -2 & 2 & -3 \\ -3 & -1 & 2 \end{bmatrix} \xrightarrow{2r_1 + r_2} E_{21}^{(2)} A \xrightarrow{3r_1 + r_3} E_{31}^{(3)} E_{21}^{(2)} A \xrightarrow{\text{swap } r_2 \text{ and } r_3} P_{23} E_{31}^{(3)} E_{21}^{(2)} A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 8 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$E_{ij}^{(\lambda)}$ = elimination matrix

$$\Downarrow$$

$$P_{23} E_{31}^{(3)} E_{21}^{(2)} A = U$$

if you want to solve $Av = b$

$$\underbrace{P_{23} E_{31}^{(3)} E_{21}^{(2)} A}_{U} v = \underbrace{P_{23} E_{31}^{(3)} E_{21}^{(2)} b}_{c} \implies \boxed{Uv = c}$$

Next week: we will use this matrix multiplication stuff to write

$$\boxed{A = \underbrace{L}_{\text{lower triangular matrix}} \cdot \underbrace{U}_{\text{REF}}}$$

More detail on elimination matrices & permutation matrices

3 × 3

$$E_{21}^{(a)} = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_{31}^{(b)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ b & 0 & 1 \end{bmatrix}$$

$$E_{32}^{(c)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & c & 1 \end{bmatrix}$$

$$P_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$P_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5 × 5

$$P_{24} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$