Last time: linear (in)dependence, bases and dimensions

collections of vectors — vector spaces

How to find a basis of a $\boxed{\text{given}}$ vector space?

Way 1: V is spanned by vectors $v_1, \dots, v_n$; $V = C(A)$ where $A = [v_1 | \dots | v_n]$

(your basis will be a subset of $v_1, \dots, v_n$)

Way 2: V is cut out by equations ; $V = N(B)$ where

$$V = \left\{ \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \text{ such that } \begin{matrix} a_1 x_1 + \dots + a_n x_n = 0 \\ b_1 x_1 + \dots + b_n x_n = 0 \\ \vdots \end{matrix} \right\}$$

$$\underbrace{\begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ & \vdots & \end{bmatrix}}_{B} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

Way 1: $V = C\left(\begin{bmatrix} 2 & 1 & 4 \\ 5 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}\right)$

$\begin{bmatrix} 2 & 1 & 4 \\ 5 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix} = A$

$A \overset{\text{REF}}{\rightsquigarrow} \begin{bmatrix} \boxed{2} & 1 & 4 \\ 0 & \boxed{-5.5} & -11 \\ 0 & 0 & 0 \end{bmatrix} = U$

pivot columns are #1 and #2

so a basis for $C(A)$ is given by its columns #1 and #2, i.e. $\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \\ -1 \end{bmatrix}$

Caveat: $C(A) \neq C(U)$, although $N(A) = N(U)$

dim V = 2

Way 2: $V = N\left(\begin{bmatrix} 2 & -3 & 11 \end{bmatrix}\right)$

$\begin{bmatrix} 2 & -3 & 11 \end{bmatrix} = B$

$B \overset{\text{RREF}}{\rightsquigarrow} \begin{bmatrix} \boxed{1} & -\frac{3}{2} & \frac{11}{2} \end{bmatrix} = R$

$N(B) = N(R)$; $R\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \iff x = \frac{3}{2}y - \frac{11}{2}z$

pivot var (x)

free vars (y, z)

a basis of $N(B)$ is given by $\begin{bmatrix} 3/2 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -11/2 \\ 0 \\ 1 \end{bmatrix}$

Four fundamental subspaces of an $m \times n$ matrix A

A^T is $n \times m$

- the column space $C(A) \subset \mathbb{R}^m$
- the null space $N(A) \subset \mathbb{R}^n$
- the row space $C(A^T) \subset \mathbb{R}^n$
- the left nullspace $N(A^T) \subset \mathbb{R}^m$

definitions

meaning of $C(A^T)$
$=$
set of linear combinations of columns of A^T
$=$
set of linear combinations of rows of A

meaning of $N(A^T) = \left\{ \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} \text{ s.t. } A^T \begin{bmatrix} v_1 \\ \vdots \\ v_m \end{bmatrix} = 0 \right\}$

$= \left\{ [v_1 \ldots v_m] \text{ s.t. } [v_1 \ldots v_m] \cdot A = 0 \right\}$

transpose

"left nullspaces"

Dimensions:

$C(A), N(A^T) \subset \mathbb{R}^m$
$N(A), C(A^T) \subset \mathbb{R}^n$

$$\dim C(A) \overset{\text{definition}}{=} r \quad \Longrightarrow \quad \dim C(A^T) = r$$

row operations do not change the row space, i.e. $C(A^T) = C(R^T)$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$\dim N(A) = n - r \qquad\qquad \dim N(A^T) = m - r$$

" $\Downarrow$ " a basis of $N(A)$ is given by setting one of the free vars $= 1$ and the others 0

hence there are $(\#$ of free variables$) = n - \#$pivots $= n - r$ vectors in a basis of $N(A)$

$\Downarrow$ is just $\Downarrow$ for A^T instead of A

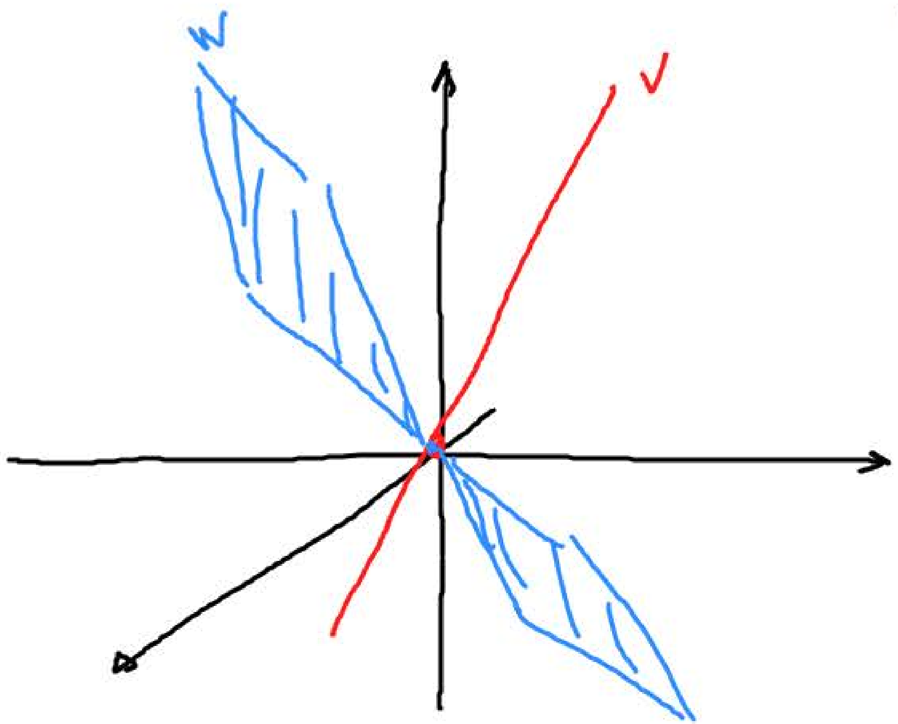
" $\Longrightarrow$ "

$$A \rightsquigarrow R = \begin{bmatrix} \boxed{1} & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

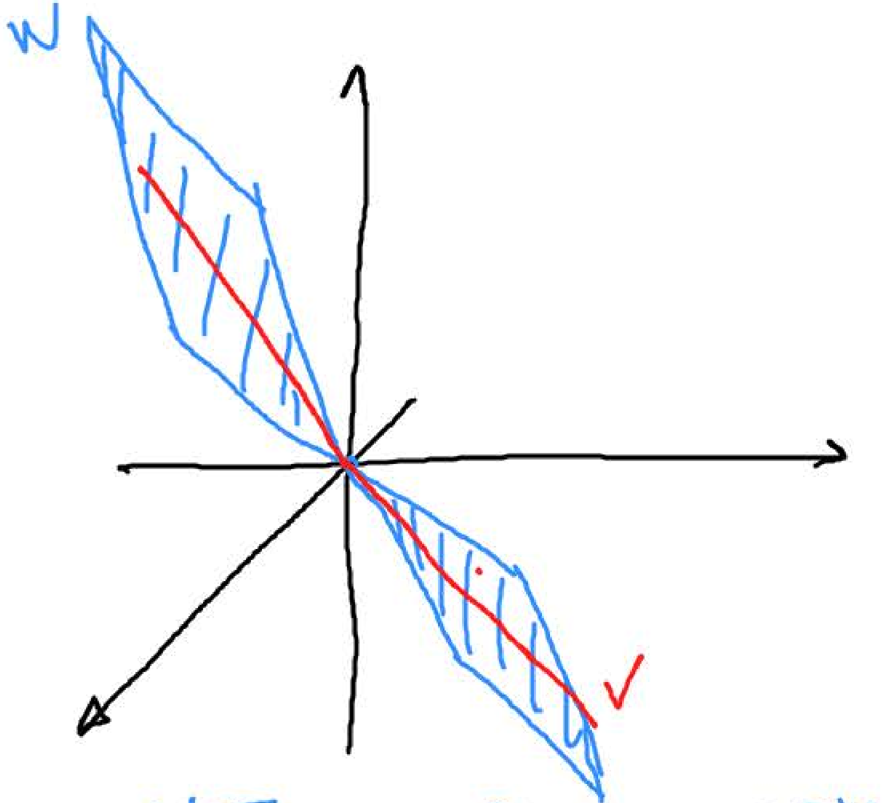
$\dim C(R^T) = r \Longleftarrow$ basis for $C(R^T)$ is given by its non-zero rows

$C(A)$ and $N(A^T)$ are complementary subspaces of $\mathbb{R}^m$

$N(A)$ and $C(A^T)$ ——— " ——— $\mathbb{R}^n$



COMPLEMENTARY



NOT COMPLEMENTARY

$V, W \subset \mathbb{R}^n$ are called complementary if

- $V \cap W = \{0\}$
- $\dim V + \dim W = n$

complementary subspaces span the entire ambient space together

How to compute a basis for $C(A^T)$ and $N(A^T)$?

"easy" way : just put A^T in (R)REF

"natural" way : how do we naturally do this in terms of (R)REF of A itself?

$$A \xrightarrow{\text{REF}} U \xrightarrow{\text{RREF}} R$$

. $C(A^T) = C(U^T) = C(R^T)$ = a basis given by pivot rows of R

. what about $N(A^T) = ?$

$N(A^T) = \{ v \text{ such that } v \cdot A = 0 \}$

1 x m row vectors

Gauss-Jordan : $K \cdot A = R$ → a product of elimination, diagonal, permutation matrices

$\Updownarrow$

$A = K^{-1}R$

$\boxed{vA = 0} \iff \underbrace{vK^{-1}}_{w}R = 0 \iff \boxed{wR = 0}$

$$[w_1\ w_2\ w_3\ w_4] \overset{\displaystyle R}{\overset{\parallel}{\begin{bmatrix} \boxed{1} & * & * & 0 & * & * & 0 \\ 0 & 0 & 0 & \boxed{1} & * & * & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}} = 0 \iff \begin{cases} w_1 = 0 \\ w_2 = 0 \\ w_3 = 0 \end{cases}$$

w_4 = anything

the subspace of W's is just the subspace $[0\ 0\ 0\ w]$

correspond to full zero rows of R

$$v = w \cdot K = [0\ 0\ 0\ *] \cdot K$$

subspace of v's is going to have a basis given by the bottom rows of K

as many of them as full zero rows of R

$$N(A) = N(R) \qquad C(A) \neq C(R)$$

$$C(A^T) = C(R^T) \qquad N(A^T) \neq N(R^T)$$

but they have the same dim

where R is RREF of A

if P is the RREF of A^T $\longrightarrow$ $N(P) = N(A^T)$

$$C(P) \neq C(A^T)$$

$$P \neq R$$

(people usually work with the RREF of A, so P does not often arise naturally in applications)