let A be an $n \times n$ matrix:

has eigenvalues $\lambda_1, \dots, \lambda_n$ which are the roots of the characteristic polynomial of A, which is $p(\lambda) = \det(A - \lambda \cdot I)$

degree n in λ

Ex: $A = \begin{bmatrix} 5 & -3 \\ 0 & 5 \end{bmatrix} \rightsquigarrow p(\lambda) = \det \begin{bmatrix} 5-\lambda & -3 \\ 0 & 5-\lambda \end{bmatrix} = (5-\lambda)^2$

the fact that this exponent is 2 means that $\lambda = 5$ is a root of multiplicity 2 of $p(\lambda)$

$\Downarrow$

we say that the eigenvalues are $\lambda_1 = 5, \lambda_2 = 5$

($\lambda = 5$ has algebraic multiplicity 2)

Ex: $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightsquigarrow p(\lambda) = \det \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1$

this poly has 2 complex number roots

Fact: an $n \times n$ matrix has exactly n eigenvalues, but they may be complex & they must be counted with multiplicities

Still for A an $n \times n$ matrix, the space of eigenvectors corresponding to an eigenvalue λ is

eigenspace of λ $= \{ v \text{ such that } Av = \lambda v \} = \underbrace{N(A - \lambda \cdot I)}$ → is a vector subspace of $\mathbb{R}^n$

What about eigenvectors corresponding to different eigenvalues,

say $\lambda_1 \neq \lambda_2$

Thm : $N(A - \lambda_1 I) \cap N(A - \lambda_2 I) = 0$

Proof: suppose there were $0 \neq v \in N(A - \lambda_1 I) \cap N(A - \lambda_2 I)$

then $Av = \lambda_1 v$ and $Av = \lambda_2 v$

$\lambda_1 v = \lambda_2 v \Rightarrow \lambda_1 = \lambda_2$ contradiction

Upshot: eigenvectors corresp. to different eigenvalues are independent

$\Downarrow$

if the n eigenvalues are all distinct their eigenvectors give a basis of $\mathbb{R}^n$

Assume A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$

Pick n eigenvectors $v_1, \dots, v_n$; these are independent, hence $\underbrace{v_1, \dots, v_n \text{ form a basis of } \mathbb{R}^n}$

let $V = [v_1 | \dots | v_n]$; it is invertible

$$\|$$

Diagonalization theorem : $A = V \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V^{-1}$ $\iff$ $AV \overset{*}{=} V \cdot \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$

to prove $*$, it suffices to prove $A V e_i \overset{*}{=} V \cdot \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_n \end{bmatrix} e_i$ for all $i \in \{1, \dots, n\}$

$$\underset{\displaystyle A v_i \overset{*}{=} \lambda_i v_i = V \cdot \lambda_i e_i}{\|}$$

$* \Rightarrow *$ and $*$ holds by the def of eigenvectors

Application: Lucas numbers $L_n = L_{n-1} + L_{n-2}$ for all $n \geq 2$

$L_0 = 2$, $L_1 = 1$

using linear algebra

2×1 vector

$a_n = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix}$

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$
$\|$

. $a_n = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = \begin{bmatrix} L_n + L_{n-1} \\ L_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} L_n \\ L_{n-1} \end{bmatrix} = A \cdot a_{n-1}$

$a_n = A\, a_{n-1} = A^2 a_{n-2} = \dots = A^n \cdot a_0 = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\Leftarrow$

$\begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

We need an effective way to compute A^n

Diagonalize A!

$$\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

only for diagonal matrices

• char poly of A is $p(\lambda) = \det(A - \lambda \cdot I) = \lambda^2 - \lambda \cdot \mathrm{Tr}\, A + \det A$

$\overset{\|}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}$

2×2 case

$= \lambda^2 - \lambda - 1$

roots are $\dfrac{1 \pm \sqrt{5}}{2}$

• eigenvalues are $\lambda_1 = \dfrac{1+\sqrt{5}}{2}$ and $\lambda_2 = \dfrac{1-\sqrt{5}}{2}$

• compute eigenvectors v_1 and v_2

$v_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ s.t. $\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$

$\iff x \cdot \dfrac{1-\sqrt{5}}{2} + y = 0 \rightsquigarrow v_1 = \begin{bmatrix} 1 \\ \frac{\sqrt{5}-1}{2} \end{bmatrix}$

$v_1 \in N(A - \lambda_1 \cdot I)$

$$N\left(\begin{bmatrix} 1 - \frac{1+\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}\right) = N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 1 & -\frac{1+\sqrt{5}}{2} \end{bmatrix}\right) \overset{\text{REF}}{=} N\left(\begin{bmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ 0 & 0 \end{bmatrix}\right)$$

similarly $v_2 = \begin{bmatrix} 1 \\ \frac{-\sqrt{5}-1}{2} \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = V \begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} V^{-1}, \quad V = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \end{bmatrix}$$

explicit

$$V^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{bmatrix} = D$$

$$A = V \cdot D V^{-1}$$

$$A^2 = V D V^{-1} V D V^{-1} = V \cdot D^2 V^{-1}$$

$$A^3 = A^2 A = V D^2 V^{-1} V D V^{-1} = V D^3 V^{-1}$$

$$\vdots$$

$$A^n = V D^n V^{-1}$$

$$\Rightarrow A^n = V \cdot \begin{bmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^n & 0 \\ 0 & \left(\frac{1-\sqrt{5}}{2}\right)^n \end{bmatrix} V^{-1} = \text{explicit } 2\times 2 \text{ matrix whose entries are functions of } \left(\frac{1\pm\sqrt{5}}{2}\right)^n$$

$$a_n = \begin{bmatrix} L_{n+1} \\ L_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \text{an explicit } 2\times 1 \text{ vector with entries depending on } \left(\frac{1\pm\sqrt{5}}{2}\right)^n$$

$$\Downarrow$$

$$L_n = \underbrace{\left(\frac{1+\sqrt{5}}{2}\right)^n}_{>1} + \underbrace{\left(\frac{1-\sqrt{5}}{2}\right)^n}_{-1<\ldots<1} \xrightarrow{n\to\infty} \left(\frac{1+\sqrt{5}}{2}\right)^n$$

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suppose you want to compute $\underbrace{A^k}_{n\times n} \cdot \underbrace{v}_{\text{arbitrary vector in } \mathbb{R}^n}$ $\qquad A v_i = \lambda_i v_i \Rightarrow A^k v_i = \lambda_i^k v_i$

$\downarrow$

express v in terms of the eigenvectors $\underbrace{v_1, \ldots, v_n}_{\text{assume form a basis of } \mathbb{R}^n}$ of A

$$\Longrightarrow v = c_1 v_1 + \ldots + c_n v_n \Rightarrow A^k v = c_1 A^k v_1 + \ldots + c_n A^k v_n$$

$$= c_1 \lambda_1^k v_1 + \ldots + c_n \lambda_n^k v_n$$