if $V, W \subset \mathbb{R}^n$ are orthogonal complements, then $V \perp W$ and $\dim V + \dim W = n$

$\Updownarrow$

$V = W^\perp = \{ w \in \mathbb{R}^n \text{ s.t. } w \perp W \}$

$\Updownarrow$

$W = V^\perp = \{ v \in \mathbb{R}^n \text{ s.t. } v \perp V \}$

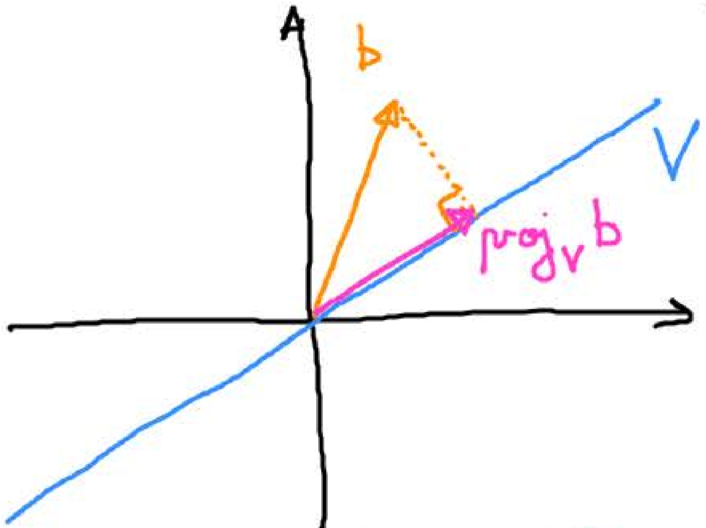
$$a = \text{proj}_V a + \text{proj}_W a$$

any vector in $\mathbb{R}^n$ $\quad$ $\in V$ $\quad$ $\in W$

$\boxed{\text{Today}}$ projections

<ins>Def</ins>: given a subspace $V \subset \mathbb{R}^n$ and a vector $b \in \mathbb{R}^n$, the projection $\boxed{\text{proj}_V b}$ is defined as the $\underbrace{\text{closest}}$ vector $\in V$ to b

size of error $= |v - b|$ is minimal



$\Longleftrightarrow$ $\text{proj}_V b$ is that vector $v \in V$ such that $b - v \perp V$

call p in lecture notes

"error"

Case 1: V = line, say spanned by $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \neq 0$

We want a formula for $\text{proj}_a b$ for any $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

means "projection of b onto the line spanned by a"

let $p = \text{proj}_a b$

- $p = \lambda \cdot a$ for some $\lambda \in \mathbb{R}$
- $b - p \perp a \iff a \cdot (b-p) = 0$

determine p

$$a \cdot b - \lambda\, a \cdot a = 0$$

$$a \cdot b = \lambda\, a \cdot a$$

$$\lambda = \frac{a \cdot b}{a \cdot a} = \frac{a^T b}{a^T a}$$

1x1 matrices

$\neq 0$ because $a \cdot a = \|a\|^2 \neq 0$ b/c $a \neq 0$

In general,

$\dfrac{A}{B}$ makes no sense

(exception: if B is 1x1, we identify B with its entry

Upshot:

$$\text{proj}_a b = a \cdot \frac{a^T b}{a^T a}$$

this is λ

Ex: $a = e_i$ = i-th standard basis vector of $\mathbb{R}^n$ $= \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ → i-th spot

$$\text{proj}_{e_i} b = e_i \frac{e_i^T b}{e_i^T e_i} = \underbrace{e_i}_{\text{vector}} \underbrace{\frac{b_i}{1}}_{\text{number}} = b_i e_i$$

if $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

$$b = b_1 e_1 + \ldots + b_i e_i + \ldots + b_n e_n$$

$$\left(\text{because } \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_n \end{bmatrix} \right)$$

$$\downarrow$$

$$\text{proj}_{i\text{-th axis}} b = b_i e_i$$

YOU CAN WRITE PROJECTIONS AS MATRIX MULTI-PLICATION

"projection formula"

$$\text{proj}_a b = a \frac{a^T b}{a^T a} = \frac{\overset{n\times 1}{a}\,\overset{1\times n}{a^T}}{a^T a} b = P_a \cdot b$$

"best" projection formula

$$X \cdot (YZ) = (XY) Z$$

$\frac{a a^T}{a^T a} = P_a$ is an $n \times n$ projection matrix

Theorem: for any subspace $V \subset \mathbb{R}^n$, there is a projection matrix $\underbrace{P_V}_{n\times n}$

such that $\text{proj}_V b = P_V \cdot b$ → projection formula

Goal: find a formula for P_V

pick a basis for V $\Longleftrightarrow$ picking a matrix A s.t. $V = C(A)$

$v_1, \ldots, v_k$

A: $n \times k$, where $k = \dim V$

$A = [v_1 | \ldots | v_k]$

$p = \text{proj}_V b$ is determined by $\begin{cases} p \in V \Leftrightarrow p \in C(A) \Leftrightarrow \text{exists } v \text{ s.t. } Av = p \\ (b-p) \perp V \Leftrightarrow (b-p) \perp C(A) \Leftrightarrow A^T(b-p) = 0 \end{cases}$

$(b-p) \perp C(A) \Leftrightarrow b - p \in N(A^T) \Leftrightarrow A^T(b-p) = 0$

$A^T b = A^T p$

$A^T b = A^T A v$

Claim: $\underbrace{A^T A}_{k\times k}$ is invertible b/c.

A has linearly independent column

$\Downarrow$

$(A^T A)^{-1} A^T b = v$

$$p = A v = A \cdot (A^T A)^{-1} \cdot A^T b$$

$$\Downarrow$$

Projection formula: $\mathrm{proj}_V b = A (A^T A)^{-1} A^T b$

$$A (A^T A)^{-1} A^T = P_V$$

Projection matrix $\boxed{P_V = A(A^T A)^{-1} A^T}$

where $V = C(A)$

NB: the formula for P_V does not depend on the choice of A, as long as $V = C(A)$

but hey? isn't

$$(A^T A)^{-1} \neq A^{-1} \cdot (A^T)^{-1}$$

No!

b/c A is $n \times k$

only makes sense for A square

if so, then isn't

$$A \cdot (A^T A)^{-1} A^T = A \cdot A^{-1} (A^T)^{-1} \cdot A^T$$

$$= I \cdot I = I ?$$

Ex: compute P_V where $V \subset \mathbb{R}^3$ spanned by $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$V = C(A)$, where $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}$; $P_V = A \cdot (A^T A)^{-1} \cdot A^T$

$$A^T A = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}}} \implies (A^T A)^{-1} = \underline{\underline{\frac{1}{6} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}}}$$

always symmetric

$$P_V = \underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}}_{3\times 2} \underbrace{\frac{1}{6} \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}}_{2\times 2} \underbrace{\begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}}_{2\times 3} = \frac{1}{6} \begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}$$

3×3

square, of size = dim $\mathbb{R}^n$

$$\forall \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad \mathrm{proj}_V b = P_V \cdot b = \frac{1}{6}\begin{bmatrix} 5 & -1 & -2 \\ -1 & 5 & -2 \\ -2 & -2 & 2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 5b_1 - b_2 - 2b_3 \\ -b_1 + 5b_2 - 2b_3 \\ -2b_1 - 2b_2 + 2b_3 \end{bmatrix}$$

2 remarks :

- P_V are symmetric
- $\boxed{P_V P_V = P_V}$ → idempotent