real valued → periodic $f: \mathbb{R} \to \mathbb{R}$

even part

$$f(x) \overset{*}{=} a_0 + a_1 \cdot \cos x + a_2 \cos 2x + a_3 \cos 3x + \dots$$
$$+ b_1 \cdot \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots$$

odd part

a_k, b_k's are called (real) Fourier coeffs

$$(f(x), g(x))_{\mathbb{R}} = \int_{-\pi}^{\pi} f(x) g(x) \, dx$$

complex valued → periodic $f: \mathbb{R} \to \mathbb{C}$

($\mathbb{C}$ = complex #)

$$f(x) \overset{*}{=} \dots + c_{-2} e^{-2ix} + c_{-1} e^{-ix} + c_0 + c_1 e^{ix} + c_2 e^{2ix} + \dots$$

c_k's are called (complex) Fourier coeffs

$$(f(x), g(x))_{\mathbb{C}} = \int_{-\pi}^{\pi} f(x) \overline{g(x)} \, dx$$

$$(f(x), f(x))_{\mathbb{C}} = \int_{-\pi}^{\pi} f(x) \overline{f(x)} \, dx = \int_{-\pi}^{\pi} |f(x)|^2 dx \in \mathbb{R}_{>0}$$

periodic

$$f(x + 2\pi) = f(x)$$

(*) is a particular case of decomposing a vector as a linear combination of an orthogonal basis

Ex: $f(x) = \cos 7x$

real Fourier series $= 0 + 0 \cdot \cos x + 0 \cdot \cos 2x + \dots\dots + 1 \cdot \cos 7x + \dots$
$+ 0 \cdot \sin x + 0 \cdot \sin 2x + \dots\dots$

complex Fourier series $= \dots + 0 \cdot e^{-8ix} + \frac{e^{-7ix}}{2} + 0 \cdot e^{-6ix} + \dots\dots + 0 \cdot e^{6ix} + \frac{e^{7ix}}{2} + 0 \cdot e^{8ix} + \dots$

$\boxed{\text{any real Fourier series can be converted to a complex Fourier series by } \cos kx = \frac{e^{kix} + e^{-kix}}{2} \quad \sin kx = \frac{e^{kix} - e^{-kix}}{2i}}$

$\overline{e^{ix}} = e^{-ix}$

when is $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{kix}$ real?

$\Updownarrow$

$f(x) = \overline{f(x)}$

equal iff $c_{-k} = \overline{c_k}$ for all k

$\overline{f(x)} = \sum_{k=-\infty}^{\infty} \overline{c_k}\, e^{-kix}$

$e^{ix} = \cos x + i \cdot \sin x$

$e^{-ix} = \cos x - i \cdot \sin x$

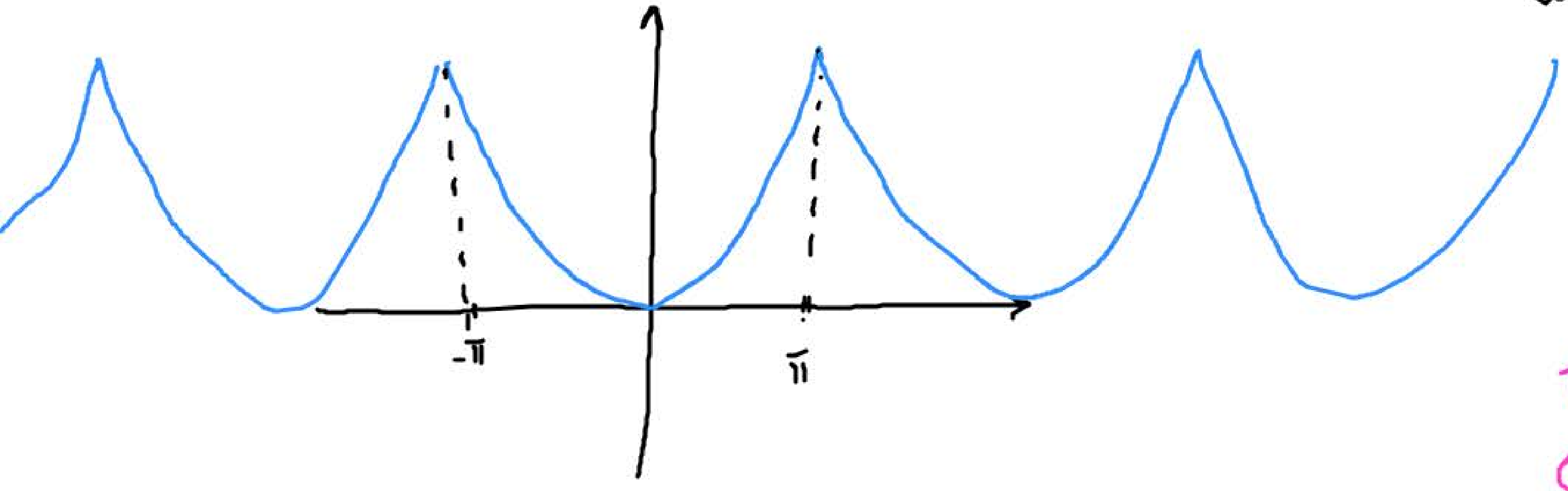
complex conjugates

$e^{7ix} = \cos 7x + i \cdot \sin 7x$

$e^{-7ix} = \cos 7x - i \sin 7x \quad (+)$

$e^{7ix} + e^{-7ix} = 2 \cdot \cos 7x$

Ex: complex Fourier series of $f(x) = x^2 \quad \forall x \in [-\pi, \pi]$
and then extended periodically



$$f(x) \overset{\circledast}{=} \sum_{k=-\infty}^{\infty} c_k e^{kix}$$

to find c_k's, take the inner product of $\circledast$ with a given e^{Lix} for any integer L

$$(f(x), e^{Lix})_{\mathbb{C}} = \sum_{k=-\infty}^{\infty} c_k \underbrace{(e^{kix}, e^{Lix})} = c_L \cdot 2\pi$$

this is $=0$ unless $k = L$, in which case it's 2π

$$\Downarrow$$

$$c_L = \frac{1}{2\pi} (f(x), e^{Lix})_{\mathbb{C}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \underbrace{e^{-Lix}} dx$$

this is $\overline{e^{Lix}}$

$$C_L = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-Lix}\,dx$$

$f(x)$ $\quad g'(x)$; we need $g(x) = \dfrac{e^{-Lix}}{-Li}$

Integration by parts

$$\int_a^b f(x)g'(x)\,dx = -\int_a^b f'(x)g(x)\,dx$$

$$+ f(b)g(b) - f(a)g(a)$$

I.B.P.

$$= -\frac{1}{2\pi}\int_{-\pi}^{\pi} 2x\cdot\frac{e^{-Lix}}{-Li}\,dx + \frac{1}{2\pi}\frac{x^2 e^{-Lix}}{-Li}\Bigg|_{-\pi}^{\pi}$$

$f(x)$ $\quad g'(x)$; we need $g(x) = \dfrac{e^{-Lix}}{(-Li)^2} = \dfrac{e^{-Lix}}{-L^2}$

$$e^{2\pi i} = 1$$
$$e^{\pi i} = -1$$
$$e^{Li\pi} = (-1)^L$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} 2\cdot\frac{e^{-Lix}}{(-Li)^2}\,dx - \frac{1}{2\pi}\,2x\cdot\frac{e^{-Lix}}{(-Li)^2}\Bigg|_{-\pi}^{\pi} + \frac{1}{2\pi}\frac{x^2 e^{-Lix}}{-Li}\Bigg|_{-\pi}^{\pi}$$

$$\frac{1}{2\pi}\frac{e^{-Lix}}{(-Li)^3}\Bigg|_{-\pi}^{\pi} = \frac{1}{2\pi}\left(\frac{e^{-Li\pi}}{(-Li)^3} - \frac{e^{Li\pi}}{(-Li)^3} - 2\pi\cdot\frac{e^{-Li\pi}}{(-Li)^2} + 2\pi\frac{e^{Li\pi}}{(-Li)^2} + \frac{(2\pi)^2 e^{-Li\pi}}{-Li} - \frac{(2\pi)^2 e^{Li\pi}}{-Li}\right)$$

Bonus problem (a.k.a. mathematical chocolate):

Show that any $\underbrace{\text{rotation}}_{}$ of a sphere leaves $\underbrace{\text{two points fixed}}_{}$.

a transformation which preserves lines, lengths and angles and handedness

the transformation does not change the points in question

(1) $rot\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = Q\cdot\begin{bmatrix}x\\y\\z\end{bmatrix}$ where Q is an orthagonal matrix with $\det Q = 1$

$$\boxed{\begin{gathered} Q^TQ = I \\ \Downarrow \\ \det Q^T\cdot\det Q = 1 \Rightarrow \det Q = \pm 1 \end{gathered}}$$

$rot(v) = Av$, $v\cdot w = |v|\cdot|w|\cdot\cos\alpha$

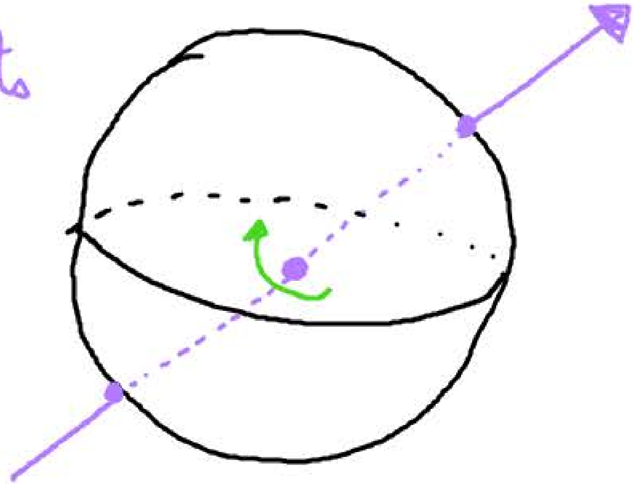
$$v^TA^TAw = rot(v)\cdot rot(w) = v\cdot w = v^Tw \quad \text{for all } v,w$$

$$\Downarrow$$

$$A^TA = I$$

• = fixed points

• = direction of rotation



(2) : any orthagonal matrix with det 1 has $\lambda = 1$ as an eigenvalue

$$\Downarrow$$

$\exists\, v$ s.t. $Qv = v \Rightarrow rot(v) = v$