# Cofactor expansion for determinants :

$\det A = \det A^T$

- along row $i$ :

$$\det A = a_{i1} C_{i1} + \dots + a_{in} C_{in}$$

cofactors

- along column $j$ :

$$\det A = a_{1j} C_{1j} + \dots + a_{nj} C_{nj}$$

if $A = \begin{bmatrix} * & a_{j1} & * \\ & \vdots & \\ a_{i1} \cdots & a_{ij} & \cdots a_{in} \\ & \vdots & \\ * & a_{jn} & * \end{bmatrix}$

then its $(i,j)$ - minor is

$$M_{ij} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

and its $(i,j)$ - cofactor is

$$C_{ij} = (-1)^{i+j} \det M_{ij}$$

Ex: $A = \begin{bmatrix} 7 & 0 & 3 & -1 \\ 0 & 4 & 3 & 0 \\ 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$

cofactor expansion along row 2

$$\det A = 0 \cdot \cancel{C_{21}} + 4 \cdot C_{22} + 3 \cdot C_{23} + 0 \cdot \cancel{C_{24}} = 4 \cdot (-6) + 3 \cdot 18$$

$$\boxed{= 30}$$

$M_{22} = \begin{bmatrix} 7 & 3 & -1 \\ 2 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

$$C_{22} = (-1)^{2+2} \cdot \det M_{22} = 3 \cdot (-1)^{1+2} \det \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = -3 \cdot (2 \cdot 1 - 0 \cdot 1)$$

$$\boxed{C_{22} = -6}$$

let's compute this by cofactor exp along column 2

$M_{23} = \begin{bmatrix} 7 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 2 & 1 \end{bmatrix}$

$$C_{23} = (-1)^{2+3} \cdot \det M_{23}$$

$$= -\left( 2 (-1)^{3+2} \det \begin{bmatrix} 7 & -1 \\ 2 & 1 \end{bmatrix} \right)$$

$$= 2 \cdot (7 \cdot 1 - 2 \cdot (-1))$$

$$\boxed{C_{23} = 18}$$

$$\underbrace{A}_{n\times n}\ \underbrace{v}_{n\times 1} = \underbrace{b}_{n\times 1}$$

$\rightsquigarrow$ if $A$ is non-singular $\iff \det A \neq 0$

then $\boxed{v = A^{-1}\cdot b}$

The determinant of $A$ features in the formula for $A^{-1}$

$$\det A = a_{i1} C_{i1} + \dots + a_{in} C_{in} = a_{i1} x_{1i} + \dots + a_{in} x_{ni}$$

also $$0 = a_{i1} C_{j1} + \dots + a_{in} C_{jn} = a_{i1} x_{1j} + \dots + a_{in} x_{nj} \quad \text{for all } j \neq i$$

$\Downarrow$

consider the matrix

$X$ with entries $x_{ij} = C_{ji}$

$\shortparallel$

"transposed cofactor matrix"

$$A\cdot X = \begin{bmatrix} \det A & & 0 \\ & \ddots & \\ 0 & & \det A \end{bmatrix} = \det A \cdot I$$

$(i,i)$ $(i,j)$

$$A \cdot \left(\frac{X}{\det A}\right) = I \implies A^{-1} = \frac{X}{\det A}$$

divide every entry of X by det A

$$\left(A^{-1}\right)_{ij} = \frac{X_{ij}}{\det A} = \frac{C_{ji}}{\det A}$$

obvious that you need $\det A \neq 0$ for A to be invertible

Ex: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\underbrace{\det A}_{ad-bc}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$C_{11} = (-1)^{1+1} \cdot \det [d] = d$$

$$C_{12} = (-1)^{1+2} \det [c] = -c$$

$$C_{21} = (-1)^{2+1} \det [b] = -b$$

$$C_{22} = (-1)^{2+2} \det [a] = a$$

$\rightsquigarrow$ matrix of cofactors is $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix} \rightsquigarrow$ its transpose is $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A v = b \implies v = A^{-1} b$$

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} (A^{-1})_{11} & \cdots & \vdots \\ \vdots & & \\ \cdots & & (A^{-1})_{nn} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \implies v_i = (A^{-1})_{i1} \cdot b_1 + \cdots + (A^{-1})_{in} \cdot b_n$$

$$\boxed{v_i = \frac{C_{1i}}{\det A} \cdot b_1 + \cdots\cdots + \frac{C_{ni}}{\det A} \cdot b_n}$$

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for all $i$ from 1 to $n$, consider the matrix $B_i = A$ with the $i$-th column replaced by $b$

i.e. $A = [a_1 | \cdots | a_n] \rightsquigarrow B_i = [a_1 | \cdots | a_{i-1} | b | a_{i+1} | \cdots | a_n]$

$\|$ the RHS of these two formula are equal by cofactor exp on $i$-th column of $B_i$

$$\boxed{v_i = \frac{\det B_i}{\det A}}$$

Cramer's rule

Upshot: the solution is $v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ where

$$v_1 = \frac{\det B_1}{\det A}, \ \ldots\ldots, \ v_n = \frac{\det B_n}{\det A}$$

Ex: $\begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$ ; the solution is:

$$v_1 = \frac{\det \begin{bmatrix} 0 & -2 & 0 \\ 4 & 0 & -3 \\ 1 & 0 & -5 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}} = -17 \qquad v_2 = \frac{\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 4 & -3 \\ 2 & 1 & -5 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}} = \frac{-17}{2} \qquad v_3 = \frac{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 1 \end{bmatrix}}{\det \begin{bmatrix} 1 & -2 & 0 \\ 1 & 0 & -3 \\ 2 & 0 & -5 \end{bmatrix}} = -7$$

numerator of $v_1 = (-2) \cdot (-1)^{1+2} \cdot \det \begin{bmatrix} 4 & -3 \\ 1 & -5 \end{bmatrix} = -34$ | denominator of $v_1 = (-2) \cdot (-1)^{1+2} \det \underbrace{\begin{bmatrix} 1 & -3 \\ 2 & -5 \end{bmatrix}}_{1 \cdot (-5) - 2 \cdot (-3) = 1} = 2$

$3 \times 3$ determinants give you a formula for the cross-product

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \quad w = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \rightsquigarrow v \times w = \begin{bmatrix} ?_1 \\ ?_2 \\ ?_3 \end{bmatrix}$$

$$i = e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$j = e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$K = e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$v \times w = \det \begin{bmatrix} i & v_1 & w_1 \\ j & v_2 & w_2 \\ K & v_3 & w_3 \end{bmatrix} =$$ cofactor expansion along first colum

$$= i \cdot (-1)^{1+1} \det \begin{bmatrix} v_2 & w_2 \\ v_3 & w_3 \end{bmatrix} + j \cdot (-1)^{2+1} \cdot \det \begin{bmatrix} v_1 & w_1 \\ v_3 & w_3 \end{bmatrix} + K (-1)^{3+1} \det \begin{bmatrix} v_1 & w_1 \\ v_2 & w_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix} = i \left(v_2 w_3 - v_3 w_2\right) + j \cdot \left(v_3 w_1 - v_1 w_3\right) + K \left(v_1 w_2 - v_2 w_1\right)$$