Gaussian
Gauss-Jordan } elimination : algorithms using row operations that seek to put a matrix in a prescribed form

pivot on row i+1 is to the right of pivot on row i

row echelon form

condition in green +
+ all pivots are =1 +
+ all entries above a pivot are 0

reduced row echelon form

MATRIX

$$\begin{bmatrix} 0 & \boxed{3} & 2 \\ \boxed{1} & 5 & 0 \\ 0 & 0 & \boxed{3} \end{bmatrix}$$

pivots

AUGMENTED MATRIX

$$\left[\begin{array}{ccc|c} 0 & 0 & \boxed{7} & 3 \\ \boxed{1} & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

no pivot on 3rd row

THIS MATRIX IS SINGULAR

Matrix multiplication

$A \quad B \rightsquigarrow AB$

$m \times n$ matrix $\quad n \times p$ matrix $\quad m \times p$ matrix

$$A = \begin{array}{cc} & \\ \begin{bmatrix} & \vdots & \\ \cdots & a_{ij} & \cdots \\ & \vdots & \end{bmatrix} & i \\ j & \end{array} \qquad B = \begin{array}{cc} \begin{bmatrix} & \vdots & \\ \cdots & b_{jk} & \cdots \\ & \vdots & \end{bmatrix} & j \\ k & \end{array}$$

then $$AB = \begin{array}{cc} \begin{bmatrix} & \vdots & \\ \cdots & c_{ik} & \cdots \\ & \vdots & \end{bmatrix} & i \\ k & \end{array}$$

defined by

$$c_{ik} = a_{i1} b_{1k} + \cdots + a_{in} b_{nk} = \sum_{j=1}^{n} a_{ij} \cdot b_{jk}$$

$$A = \begin{bmatrix} a_{i1} & a_{i2} & \cdots\cdots & a_{in} \end{bmatrix} \qquad B = \begin{bmatrix} b_{1k} \\ b_{2k} \\ \vdots \\ b_{nk} \end{bmatrix}$$

c_{ik} = dot product of the row vector here and the column vector here

---

$$A = \begin{bmatrix} a_{i1} & \cdots\cdots & a_{in} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{n1} & \cdots & b_{np} \end{bmatrix} ; \quad AB = \begin{bmatrix} c_{i1} & \cdots & c_{ip} \end{bmatrix}$$ is given by

$$\begin{bmatrix} c_{i1} & \cdots & c_{ip} \end{bmatrix} = a_{i1} \cdot \begin{bmatrix} b_{11} & \cdots & b_{1p} \end{bmatrix} + \cdots + a_{in} \begin{bmatrix} b_{n1} & \cdots & b_{np} \end{bmatrix}$$

UPSHOT: the rows of AB are linear combinations of the rows of B

-"- columns ——— -"- ——— columns of A

Block multiplication

$$A = \left[\begin{array}{cc|c} 1 & -1 & 2 \\ \hline -4 & 1 & 0 \end{array}\right] \qquad B = \left[\begin{array}{ccc|c} 1 & -1 & 2 & 3 \\ -4 & 0 & 3 & -6 \\ \hline 1 & 4 & 0 & 2 \end{array}\right]$$

A_{11}, A_{12}, A_{21}, A_{22}; B_{11}, B_{12}, B_{21}, B_{22}

A_{11} is 1×2 $\qquad$ B_{11} is 2×3

A_{12} is 1×1 $\qquad$ B_{12} is 2×1

A_{21} is 1×2 $\qquad$ B_{21} is 1×3

A_{22} is 1×1 $\qquad$ B_{22} is 1×1

$$AB = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$$

all 4 corners are computed by multiplication of smaller matrices

$$\left[\begin{array}{ccc|c} 7 & 7 & -1 & 7 \\ \hline -8 & 4 & -5 & 6 \end{array}\right]$$

Properties of multiplication:

addition of matrices of the same size is componentwise addition

- $A(BC) = (AB)C$
- $A(B+C) = AB + AC$ and $(A+B)C = AC + BC$
- $AI = IA = A$ where $I = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$ identity (unit) matrix

a square matrix

(we write I_n if we want to emphasize the fact that I is $n \times n$)

- if A is square,

$$\underbrace{A \cdot A \cdot A \cdots\cdots A}_{p \text{ times}} = A^p$$

$$I = A^0$$

$$\boxed{A^p \cdot A^q = A^{p+q}}$$

$$\boxed{AB \neq BA}$$

Definition: if A is square, its inverse is a matrix

$$A^{-1}$$

same size as A

such that the following equalities hold $\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I}$

Def: if A $\boxed{\text{does not have an inverse}}$, it's called $\boxed{\text{singular}}$

if A has an inverse, it's called non-singular

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$$

does not have an inverse

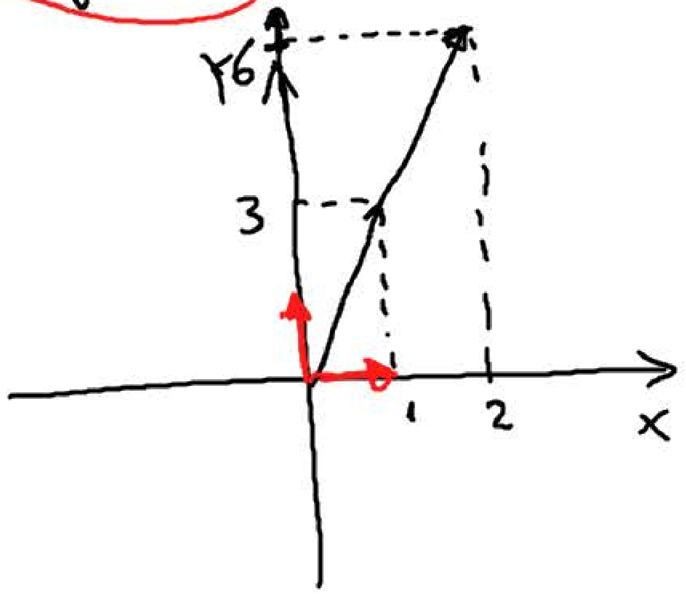
↓

singular

$$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

||

linear combinations of columns of A, i.e. $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 6 \end{bmatrix}$ $=$ $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$



• How to compute inverse of a matrix

$$\left[\underset{n\times n}{A} \;\middle|\; \underset{n\times n}{I_n}\right] \xrightarrow[\text{elimination}]{\text{Gauss-Jordan}} \begin{cases} \text{RREF is of the form } [I_n \mid X] : \text{ then } X = A^{-1} \\ \text{RREF is not of the form } [I_n \mid X] : \text{ then } A \text{ is singular} \end{cases}$$

• $$Av = b \Rightarrow \underbrace{A^{-1}A}_{I}\, v = A^{-1}b$$

$$\Rightarrow Iv = A^{-1}b \Rightarrow v = A^{-1}b$$

A must be square and invertible

Properties: $$(AB)^{-1} = B^{-1}A^{-1}$$

A & B have to be $n\times n$ invertible

A^p even if $p<0$

$$A^p = (A^{-1})^{-p} \qquad p = (-1)(-p)$$

$$A^p \cdot A^q = A^{p+q} \rightsquigarrow (A^p)^q = A^{p\cdot q}$$

Can you raise A to non-integer powers?

In general, no.

for example, $A^{0.5} = ?$

$\searrow$ should be a matrix B such that $B^2 = A$

In general, there are many such B's, so the operation $A^{0.5}$ is not uniquely defined.