Friday's midterm: everything up to and including 4/23

heavy emphasis on Lectures 15-26

any rank 1 matrix can be written as follows: $u \cdot \sigma \cdot v^T$

where $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$, $\sigma \in \mathbb{R}_{>0}$

$|u| = |v| = 1$

SVD: any matrix can be written as

$$A = u_1 \sigma_1 v_1^T + \dots + u_r \sigma_r v_r^T$$ where $r = \mathrm{rk}\, A$

$m \times n$

$u_1, \dots, u_r, u_{r+1}, \dots, u_m$ are left singular vectors

o.n. basis

$v_1, \dots, v_r, v_{r+1}, \dots, v_n$ are right singular vectors

$\sigma_1, \dots, \sigma_r > 0$ are called singular values

$$A = U \Sigma V^T$$

$U = [u_1 | \dots | u_m]$

$V = [v_1 | \dots | v_n]$

$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 & \\ & \ddots & & 0 \\ & & \sigma_r & \\ 0 & & 0 & 0 \end{bmatrix}$$

Thm.. $A v_i = \sigma_i u_i$

$A^T u_i = \sigma_i v_i$

also $A v_{r+1} = \dots = A v_n = 0$

$A^T u_{r+1} = \dots = A^T u_m = 0$

Proof: $A = u_1 \sigma_1 v_1^T + \dots + u_r \sigma_r v_r^T$

$A v_i = u_1 \sigma_1 \underbrace{v_1^T v_i}_{=0} + \dots + u_i \sigma_i \underbrace{v_i^T v_i}_{1} + \dots + u_r \sigma_r \underbrace{v_r^T v_i}_{=0}$

$= \sigma_i u_i$

$A^T = v_1 \sigma_1 u_1^T + \dots + v_r \sigma_r u_r^T$

$A = U \Sigma V^T$

$\Updownarrow$

$A^T = V \Sigma^T U^T$

$v \in \mathbb{R}^n \rightsquigarrow Av = A(c_1 v_1 + \dots + c_n v_n)$

$\|$

$c_1 v_1 + \dots + c_n v_n$

for some $c_1, \dots, c_n$

$c_i = v \cdot v_i$

$= c_1 \underbrace{A v_1}_{\sigma_1 u_1} + \dots + c_r \underbrace{A v_r}_{\sigma_r u_r} + \dots + c_n \underbrace{A v_n}_{0}$

$= \underbrace{c_1 \sigma_1 u_1 + \dots + c_r \sigma_r u_r}$

$\Rightarrow \|Av\| = \sqrt{c_1^2 \sigma_1^2 + \dots + c_r^2 \sigma_r^2}$

$\|v\| = \sqrt{c_1^2 + \dots + c_n^2}$

How to get a bound for $\frac{\|Av\|}{\|v\|}$ as v varies?

(i.e. $c_1, \dots, c_n$ vary)

$$= \sqrt{\frac{c_1^2 \sigma_1^2 + \dots + c_r^2 \sigma_r^2}{c_1^2 + \dots + c_r^2 + \dots + c_n^2}} \leq \sigma_1$$

we have equality precisely when $c_1 = 1, c_2 = \dots = c_n = 0$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix}$$

Assume $\sigma_1 \geq \dots \geq \sigma_r > 0$

$\Downarrow$

$$\|A\| \overset{\text{def}}{:=} \max_{v \in \mathbb{R}^n \setminus 0} \left(\frac{\|Av\|}{\|v\|} \right)$$

$\|A\|$ is no greater than σ_1

also, $v = v_1$ is the vector for which $\frac{\|Av\|}{\|v\|}$ is maximal

$$A = U \Sigma V^T$$

any $m \times n$ — rotation — dilation — another inverse rotation, $V^T = V^{-1}$

symmetric $n \times n$

$$S = Q D Q^T$$

Pseudo-inverses: let A be an $m \times n$ matrix;

does A have an inverse? No

but what if it did have an inverse?

$$A v_i = \sigma_i u_i \quad \Downarrow \quad A^{-1} u_i = \frac{v_i}{\sigma_i}$$

$$A v_j = 0 \text{ if } j > r \quad \rightsquigarrow \quad v_j = \cancel{A^{-1}(0)}$$

<u>Def</u>: if A is $\boxed{m \times n}$, its pseudo-inverse is the $\boxed{n \times m}$ matrix

$A = U \Sigma V^T$

expect

$$A^{-1} = V \cdot \Sigma^{-1} \cdot U^T$$

but $\Sigma = \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ & \ddots & & & \\ 0 & & \sigma_r & & \end{bmatrix}$

is not invertible

$$A^+ = V \Sigma^+ U^T$$

where $\Sigma^+ = \begin{bmatrix} \frac{1}{\sigma_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_r} \\ & 0 & \end{bmatrix}$

compare

$$A^T = V \cdot \Sigma^T U^T$$

where $\Sigma^T = \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \\ & 0 & \end{bmatrix}$

$$A^+ A = V \Sigma^+ \underbrace{U^T U}_{I} \Sigma V^T = V \underbrace{\Sigma^+ \Sigma}_{\substack{n \times n \\ \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]}} V^T =$$ projection matrix onto subspace spanned by $v_1, \dots, v_r$ $=$ projection matrix onto $C(A^T)$

$$A A^+ = \dots\dots\dots\dots = U \underbrace{\Sigma \Sigma^+}_{\substack{m \times m \\ \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]}} U^T =$$ projection matrix onto subspace spanned by $u_1, \dots, u_r$ $=$ projection matrix onto $C(A)$

$$A = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \begin{bmatrix} 3 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}}_{u_1} \cdot \underbrace{25}_{\sigma_1} \cdot \underbrace{\begin{bmatrix} 3/5 & 4/5 \end{bmatrix}}_{v_1^T} \rightsquigarrow A^+ = v_1 \frac{1}{\sigma_1} u_1^T$$

$$= \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} \cdot \frac{1}{25} \begin{bmatrix} 3/5 & 4/5 \end{bmatrix}$$

$$= \frac{1}{625} \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$$

$$Av = b$$

least squares solution is $v = (A^T A)^{-1} A^T \cdot b$

Thm: least squares solution is

$$v = A^+ \cdot b$$

$m = n$: $A = U \Sigma \cdot V^T = \underbrace{U \underbrace{V^T \cdot V}_{I}}_{Q} \cdot \underbrace{\Sigma \cdot V^T}_{S}$

Q is orthogonal

S is symmetric (actually positive semi-def)

Thm: any square matrix has a factorization

$$A = Q S$$

orthogonal → positive semidefinite