if A is a diagonal / triangular, then its eigenvalues are its diagonal entries

$$A = \begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$$

$$p(\lambda) = \det(A - \lambda I) = \det\left(\begin{bmatrix} d_1-\lambda & & * \\ & \ddots & \\ 0 & & d_n-\lambda \end{bmatrix}\right) = (d_1-\lambda)\dots(d_n-\lambda)$$

roots = eigenvalues

Similar matrices have the same eigenvalues

A and B are called similar if

$$A = XBX^{-1}$$

for some X

Diagonalization:

$$A = V\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}V^{-1}$$

eigenvalues λ_i eigenvalues λ_i

if $A = XBX^{-1}$ and v is an eigenvector of A (corresponding to λ) then $X^{-1}\cdot v$ is an eigenvector of B (corresponding of λ)

When diagonalization fails?

$\underset{n\times n}{A}$, assume $\underbrace{p(\lambda)}_{\text{degree } n} = (d_1-\lambda)^{r_1} \cdots\cdots (d_s-\lambda)^{r_s}$ where $d_1, \ldots, d_s$ are distinct

The number r_i is called the $\boxed{\text{algebraic multiplicity}}$ of d_i as an eigenvalues

<u>Fact</u> : $\boxed{r_1 + \cdots + r_s = n}$

Def : the $\boxed{\text{geometric multiplicity}}$ of d_i as an eigenvalue of A is

$$\dim \left\{ \begin{array}{l} \text{eigenvectors of } A \\ \text{corresponding to } d_i \end{array} \right\} = \dim N(A - d_i \cdot I)$$

<u>Thm</u> : for any eigenvalue λ of A, its geometric multiplicity $\leq$ its algebraic multiplicity

<u>Consequence of Fact</u>

the sum of the geometric multiplicities is $\leq n$

Thm: a matrix A can be diagonalized if and only if $\underbrace{\text{the sum of geom. mult. of its eigenvalues is } = n}_{\text{means that there is a basis of } \mathbb{R}^n \text{ consisting of the eigenvectors of } A}$

Ex: $A = \begin{bmatrix} d & 0 \\ 0 & d \end{bmatrix}$ $\qquad$ $B = \begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$ $\qquad$ $p(\lambda) = (d-\lambda)^2$

A is diagonal

B can never be diagonalized

eigenvalues d, d

$\Updownarrow$

d has alg. mult. 2

eigenvalues d, d

$\Updownarrow$

d has alg. mult. 2

eigenspace of A $= N(A - dI) = N\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) = \mathbb{R}^2$

$\Updownarrow$

d has geom. mult. 2

eigenspace of B $= N(B - dI) = N\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ = a line, i.e. the x-axis

$0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ 0 \end{bmatrix} \Rightarrow y = 0$

$\rightsquigarrow$ d has geom. multiplicity 1

Def (Jordan normal form): for $\boxed{\text{any}}$ square matrix A, you can write its as:

$$A = V \underbrace{\begin{bmatrix} \boxed{J_1} & 0 & 0 \\ 0 & \boxed{J_2} & 0 \\ 0 & 0 & \ddots\ \boxed{J_t} \end{bmatrix}}_{\text{the Jordan normal form of } A} V^{-1}$$

where each J_i is a Jordan block, i.e.

$$J_i = \begin{bmatrix} \lambda & 1 & & 0 \\ & \ddots & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$$

- the total number of λ_i's which appear on the diagonal of the Jordan blocks is equal to n_i, i.e. the algebraic multiplicity of λ_i as an eigenvalue of A
- the pattern of Jordan block and the 1's above the diagonal can be read off from A
- the columns of V are bases of $N\left((A-\lambda_i I)^{n_i}\right)$

Examples of Jordan normal form

(Jordan blocks in red boxes)

2×2: $\begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$ and $\begin{bmatrix} d & 1 \\ 0 & d \end{bmatrix}$

3×3: $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$ and $\begin{bmatrix} d_1 & 1 & 0 \\ 0 & d_1 & 0 \\ 0 & 0 & d_2 \end{bmatrix}$ and $\begin{bmatrix} d & 1 & 0 \\ 0 & d & 1 \\ 0 & 0 & d \end{bmatrix}$

(also $\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 1 \\ 0 & 0 & d_2 \end{bmatrix}$, although it's similar to this one)

Problem: find the Jordan normal form of

$$A = \begin{bmatrix} 10 & -3 & 7 \\ 27 & -10 & 18 \\ 5 & -3 & 3 \end{bmatrix} = V\,?\,V^{-1}$$

$$p(\lambda) = \det(A - \lambda \cdot I) = -\lambda^3 + 3\lambda^2 - 4$$

observe that $p(-1) = 0 \Rightarrow p(\lambda) \xlongequal{\text{polynomial long division}} -(\lambda+1)^1(\lambda-2)^2$

$\Rightarrow$ eigenvalues are $d_1 = -1$ alg. mult. 1

$d_2 = 2$ alg. mult 2

$$\text{eigenspace of } d_1 = N(A - d_1 I) = N(A + I) = N\left(\begin{bmatrix} 11 & -3 & 7 \\ 27 & -9 & 18 \\ 5 & -3 & 4 \end{bmatrix}\right) \overset{\text{RREF}}{=} N\left(\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\| $$

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \iff \begin{cases} x + \frac{z}{2} = 0 \\ y - \frac{z}{2} = 0 \end{cases} \right\}$$

pivot

free

; an eigenvector for d_1 is $v_1 = \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \left(\text{or } \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right)$

line =

⇓

geom. mult of d_1 is 1

$$\text{eigenspace of } d_2 = N(A - d_2 I) = N(A - 2I) \overset{\text{RREF}}{\rightsquigarrow} N\left(\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$\| $$

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \iff \begin{cases} x + 2z = 0 \\ y + 3z = 0 \end{cases} \right\}$$

pivot

free

; an eigenvector for d_2 is $v_2 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$

=

line

⇓

geom. mult of d_2 is 1

b/c $\underbrace{1 + 1}_{\text{sum of geom. mult}} < 3$, we cannot diagonalize A

$$N\left((A-2I)^2\right) \xrightarrow{\text{RREF}} N\left(\begin{bmatrix} 1 & -1/2 & \frac{1}{2} \\ & 0 & \end{bmatrix}\right)$$

$$\parallel$$

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } \quad x - \frac{y}{2} + \frac{z}{2} \right\} = 0$$

a "plane", which contains v_2

__choose__ $v_3 = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix}$ is in this plane

$$V = [v_1 | v_2 | v_3], \text{ then } \quad V \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & 0 & 2 \end{array}\right] V^{-1} = A$$