Diagonalizability $\longleftrightarrow$ eigenvalues, eigenvectors

$n \times n$

A is diagonalizable if and only if A is similar to a diagonal matrix

$$A = V \cdot \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} V^{-1}$$

scales by a factor d_i in the e_i-direction

scales by a factor d_i in the $v_i = V e_i$ direction

$V = [v_1 | \dots | v_n]$, $v_1, \dots, v_n$ form a basis of $\mathbb{R}^n$

$$A(c_1 v_1 + c_2 v_2 + \dots + c_n v_n) = c_1 d_1 v_1 + c_2 d_2 v_2 + \dots + c_n d_n v_n$$

Almost all $n \times n$ matrices are diagonalizable

(even those not diagonalizable have the next best thing, i.e. a Jordan normal form) given A, what are $d_1, \dots, d_n$ and $v_1, \dots, v_n$

columns of V

diagonal entries in

$$\boxed{A = V \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} V^{-1}}$$

eigenvalues of A

eigenvectors of A

<u>Def</u>: given a $n \times n$ matrix A, an eigenvector of A is some non-zero vector $v \in \mathbb{R}^n$ s.t. $\boxed{Av = \lambda v}$ for some number λ called an eigenvalue of A

<u>Theorem</u>: if λ is an eigenvalue of A, the eigenvectors corresponding to λ are

$$\{v \text{ s.t. } Av = \lambda v\} = \{v \text{ s.t. } (A - \lambda I)v = 0\} = N(A - \lambda I)$$

you know how to compute this vector space

How about the eigenvalues of a given matrix A?

λ is an eigenvalue $\Longleftrightarrow$ $N(A-\lambda\cdot I)\neq 0 \iff \det(A-\lambda I)=0$

EIGENVALUES are the roots of the CHAR. POLY.

Def: The characteristic polynomial of a matrix A is:

$$p(\lambda)=\det(A-\lambda I)$$

$$A=\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix} \Rightarrow A-\lambda I=\begin{bmatrix} a_{11}-\lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda\end{bmatrix} \Rightarrow \det(A-\lambda I) \overset{\text{big formula}}{=\!=}$$

$=$ a sum of products of a_{ij}'s and $(a_{ii}-\lambda)$'s

DEGREE (largest power of λ) of $P(\lambda)$ is n

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightsquigarrow A - \lambda \cdot I = \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} \rightsquigarrow$$

$$p(\lambda) = \det \begin{bmatrix} a-\lambda & b \\ c & d-\lambda \end{bmatrix} = (a-\lambda)(d-\lambda) - bc$$

$$= \lambda^2 - \lambda \underbrace{(a+d)}_{\operatorname{Tr} A} + \underbrace{ad - bc}_{\det A \,=\, p(0)}$$

Thm: for an $n \times n$ matrix A, its char poly will be:

$$p(\lambda) = (-1)^n \cdot \lambda^n + (-1)^{n-1} \lambda^{n-1} \cdot \boxed{\operatorname{Tr} A} + \dots\dots + \underbrace{\boxed{\det A}}_{\text{the coefficient of } \lambda^0 \text{ in } p(\lambda)}$$

$$p(\lambda) = (d_1 - \lambda)(d_2 - \lambda) \dots (d_n - \lambda)$$

b/c eigenvalues are roots of char poly

eigenvalues

the trace of a square matrix is the sum of its diagonal entries, i.e.

$$\operatorname{Tr} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \dots + a_{nn}$$

Vieta: $\operatorname{Tr} A$ = sum of its eigenvalues
$\det A$ = product of its eigenvalues

$A = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$; find eigenvalues & eigenvectors.

char poly of A is $p(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 5 & 2-\lambda \end{bmatrix}$

$= (2-\lambda)^2 - 5 = \lambda^2 - 4\lambda - 1$

quad. formula says roots of this poly are

$d_1 = \frac{4 + \sqrt{16+4}}{2} = 2 + \sqrt{5}$

$d_2 = \frac{4 - \sqrt{16+4}}{2} = 2 - \sqrt{5}$

$= (2 - \lambda - \sqrt{5})(2 - \lambda + \sqrt{5})$

roots are $2 - \sqrt{5}$ & $2 + \sqrt{5}$

Expect: $d_1 + d_2 = 2 + \sqrt{5} + 2 - \sqrt{5} = 4 = \operatorname{Tr} A$

$d_1 \cdot d_2 = (2+\sqrt{5})(2-\sqrt{5}) = 2^2 - \sqrt{5}^2 = 4 - 5 = -1 = \det A$

what about eigenvectors corresponding to $d_1 = 2 + \sqrt{5}$, i.e. we want to solve

$A v_1 = d_1 v_1$

$\Updownarrow$

$N(A - d_1 I) = \begin{bmatrix} 2 - d_1 & 1 \\ 5 & 2 - d_1 \end{bmatrix} =$

$= \begin{bmatrix} -\sqrt{5} & 1 \\ 5 & -\sqrt{5} \end{bmatrix}$

RREF: $\begin{bmatrix} -\sqrt{5} & 1 \\ 5 & -\sqrt{5} \end{bmatrix} \xrightarrow{r_2 + r_1 \cdot \sqrt{5}} \begin{bmatrix} -\sqrt{5} & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_1 \cdot \left(-\frac{1}{\sqrt{5}}\right)} \begin{bmatrix} \boxed{1} & -\frac{1}{\sqrt{5}} \\ 0 & 0 \end{bmatrix}$

pivot

$\begin{bmatrix} 1 & -\frac{1}{\sqrt{5}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \iff X - \frac{Y}{\sqrt{5}} = 0$

pivot

free

$\rightsquigarrow$ a basis of $N(A - d_1 I)$ consists of the vector $\begin{bmatrix} +\frac{1}{\sqrt{5}} \\ 1 \end{bmatrix} = v_1$

a line

pivot

free

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eigenvalues

$d_1 = 2 + \sqrt{5}$

$d_2 = 2 - \sqrt{5}$

eigenvectors

$v_1 = \begin{bmatrix} +\frac{1}{\sqrt{5}} \\ 1 \end{bmatrix}$

$v_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 1 \end{bmatrix}$

- $\begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} = V \begin{bmatrix} 2+\sqrt{5} & 0 \\ 0 & 2-\sqrt{5} \end{bmatrix} V^{-1}$

where $V = [v_1 | v_2] = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ 1 & 1 \end{bmatrix}$

a basis of $N(A - d_2 I)$ consists of the vector $\begin{bmatrix} -\frac{1}{\sqrt{5}} \\ 1 \end{bmatrix} = v_2$

Steps to diagonalizing $\underbrace{A}_{n\times n}$

(1) compute char poly of A : $p(\lambda) = \det(A - \lambda \cdot I)$

(2) solve for the roots of $p(\lambda)$, call them $\underbrace{\lambda_1, \ldots, \lambda_n}_{\text{these will be eigenvalues}}$

(3) for each λ_i, find a basis of the corresponding vector subspace of eigenvectors, a.k.a. $N(A - \lambda_i \cdot I)$, call this basis $\underbrace{v_1, \ldots, v_n}_{\text{these vectors in } \mathbb{R}^n \text{ will be the eigenvectors}}$

Upshot :

$$A = V \cdot \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} V^{-1}, \text{ where } V = [v_1 | \ldots | v_n]$$