Midterm exam: in-class on March 26 (everything is included up to and including next week; the material from exam week is not included)

- NO alternative exam
- NO one can take exam before the class does
- if you have exceptional circumstances (e.g. medical) then with S^3 approval we can drop your exam altogether
- if you have another class at the same time, please ask that instructor to allow you to take exam live (if not, then you may start the 18.06 exam as soon as your other class ends, if your instructor confirms to Jeremy that you were present in that class all along)
- if it's the middle of the night for you, you may start the exam "at first light"

. contact Jeremy ASAP

. do not access Canvas / Gradescope / Piazza between the class exam start time and your exam start time ($\exists$ user access logs)

$m \times n$ matrix A

- A is full row rank if $r = m$, i.e. $C(A) = \mathbb{R}^m$ $\rightsquigarrow$ $Av = b$ has ≥ 1 solution
- A is full column rank if $r = n$, i.e. $N(A) = 0$ $\rightsquigarrow$ $Av = b$ has ≤ 1 solution

if A is square, i.e. $m = n$, when $r = n$ we say that A is full rank

$\Updownarrow$

the n columns of A are linear independent, and a basis of $\mathbb{R}^n$

$\Updownarrow$

A is invertible

$Av = b$ has exactly 1 solution for all b

That solution is $v = A^{-1} \cdot b$

if A is not full rank $\rightsquigarrow$ $< n$ pivots $\rightsquigarrow$ its (R)REF has a full row of zeroes

square A is not invertible $\Longleftrightarrow$ A is singular

Perpendicularity (Orthogonality) : $\underbrace{v \perp w}_{\text{vectors in } \mathbb{R}^n} \iff \underbrace{v \cdot w}_{\text{number}} = \underbrace{v^T w}_{1\times 1 \text{ matrix}} = 0$

Pythagoras : $\| v + w \|^2 = \|v\|^2 + \|w\|^2$ if $v \perp w$, where $\|v\|^2 = v \cdot v$

Def : two subspaces $V, W \subset \mathbb{R}^n$ are orthogonal if $v \perp w$ for any $v \in V$, $w \in W$

(1) $C(A) \perp N(A^T) \subset \mathbb{R}^m$

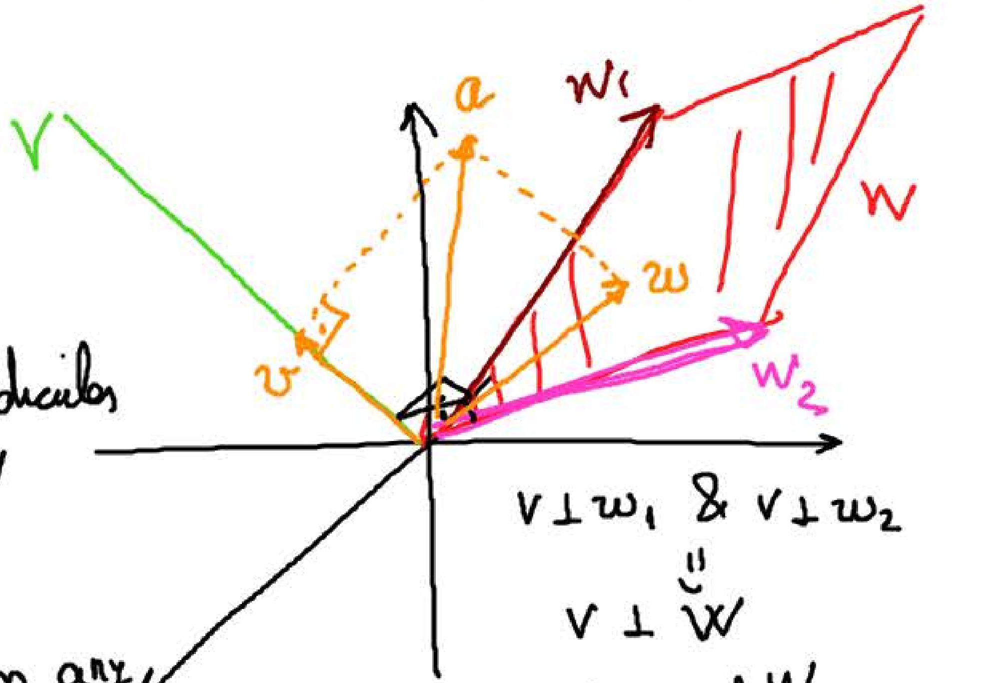
(2) $N(A) \perp C(A^T) \subset \mathbb{R}^n$

A is a $m \times n$ matrix

$V \perp W$ if any vector in V is perpendicular to any vector in W

$\Updownarrow$

$V \perp W$ if any vector in any basis of V is perpendicular to any vector in any basis of W



if A is an $m \times n$ matrix, why is $C(A) \perp N(A^T)$?

proof: $\underbrace{b \in C(A)}_{b = A \cdot v \text{ for some } v}$, $\underbrace{z \in N(A^T)}_{A^T z = 0}$; we need to show that $b \perp z$

$A^T z = 0 \iff z^T A = 0$

$b \perp z \iff z^T b = 0 \iff z^T A v = 0 \iff 0 v = 0$ true

if $V, W \subset \mathbb{R}^n$ are orthogonal, then

$$\dim(V) + \dim(W) \leq n$$

why? if $\dim(V) + \dim(W) > n$, these two subspaces would intersect in a non-zero vector $a \neq 0$

$\Downarrow$

$a \in V$ and $a \in W \implies a \perp a \implies \|a\|^2 = 0 \implies a = 0$

<u>Def</u>: if $V \perp W$ and $\dim(V) + \dim(W) = n$ ($V, W \subset \mathbb{R}^n$)

then V, W are called orthogonal complements

<u>Thm</u> : if V, W are orthogonal complements, then

$$W = V^{\perp} := \{ v \in \mathbb{R}^n \text{ such that } v \perp V \}$$

$$V = W^{\perp} := \{ w \in \mathbb{R}^n \text{ s.t. } w \perp W \}$$

(Obs: any subspace V has a unique orthogonal complement)

<u>Thm</u> : for any matrix A, $C(A)$ and $N(A^T)$ are orthogonal complements

$N(A)$ and $C(A^T)$ —————— // ——————

$$A = \begin{bmatrix} \boxed{1} & 0 & 3 & -1 \\ 0 & \boxed{1} & -2 & 1 \end{bmatrix}$$

$C(A^T)$ = a basis given by $v_1 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix}$

$$N(A) = \left\{ \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \text{ s.t. } \begin{bmatrix} \boxed{1} & 0 & 3 & -1 \\ 0 & \boxed{1} & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0 \right. \quad (*)$$

pivot vars

free vars

$$\Longleftrightarrow \begin{cases} x = -3z + t \\ y = 2z - t \end{cases}$$

a basis of $N(A)$ is given by all vectors who have 1 for one of the free variables, 0 for all the other free variables, and the pivot variables determined by $(*)$

$$w_1 = \begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad w_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

$2+2=4 \Rightarrow C(A^T)$ and $N(A)$ are complements

$C(A^T) \perp N(A)$

$\Updownarrow$

$v_1 \perp w_1 \quad v_1 \perp w_2$

$v_2 \perp w_1 \quad v_2 \perp w_2$

$$\begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 1\cdot 1 + 0\cdot(-1) + 3\cdot 0 + (-1)\cdot 1 = 0$$

Why are orthogonal complements V, W important?

- if V, W are just complementary subspaces of $\mathbb{R}^n$

  - $\dim V + \dim W = n$
  - $V \cap W = 0$

  $\Updownarrow$

  any vector $a \in \mathbb{R}^n$ can be uniquely written as $a = v + w$ for some $v \in V$, $w \in W$

- if V, W are orthogonal complements, the v and w from here are the projections of a onto the two subspaces:

$$a = v + w \quad \text{where} \quad \begin{aligned} v &= \text{proj}_V a \\ w &= \text{proj}_W a \end{aligned}$$

learn about these next time