Least squares approximation

given t_1, t_2, t_3
Y_1, Y_2, Y_3 find a and b

such that $d_1^2 + d_2^2 + d_3^2$ is minimal

$$\Big\| \quad \text{lost time}$$

$$\| Av - b \|^2$$

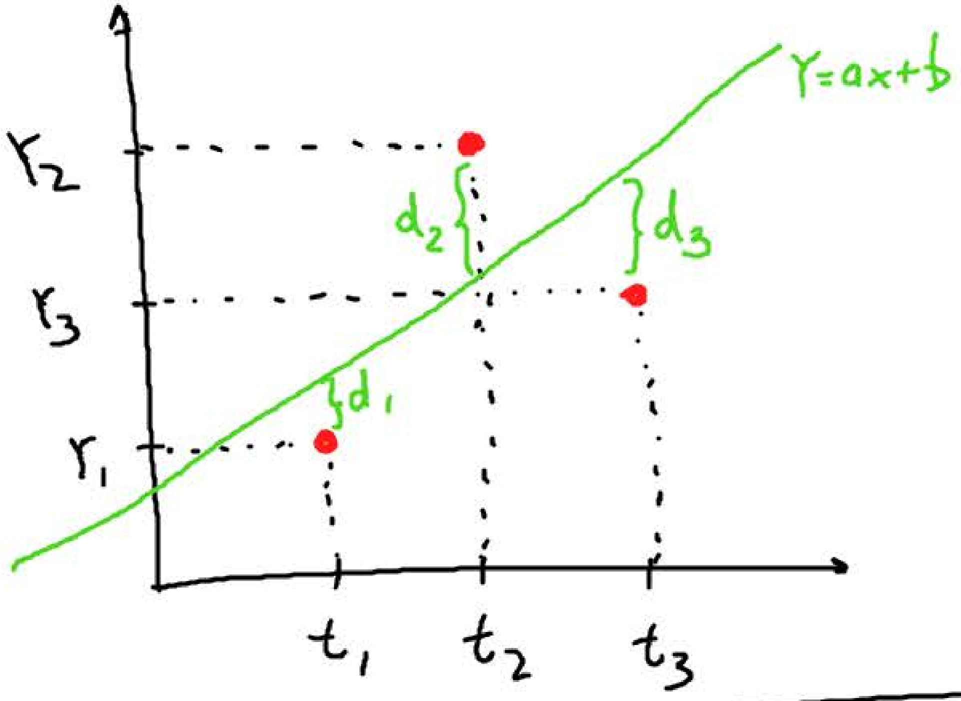
where $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix}$, $v = \begin{bmatrix} a \\ b \end{bmatrix}$, $b = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$

the sought-for v is given by $Av = \text{proj}_{C(A)}\, b$

$$\| $$

$$A(A^TA)^{-1}A^T b$$

$$v = (A^TA)^{-1}A^T b$$



$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \end{bmatrix} = \begin{bmatrix} 3 & t_1+t_2+t_3 \\ t_1+t_2+t_3 & t_1^2+t_2^2+t_3^2 \end{bmatrix}$$

$$(A^TA)^{-1} = \underbrace{\frac{1}{3(t_1^2+t_2^2+t_3^2) - (t_1+t_2+t_3)^2} \begin{bmatrix} t_1^2+t_2^2+t_3^2 & -t_1-t_2-t_3 \\ -t_1-t_2-t_3 & 3 \end{bmatrix}}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (2\times 2 \text{ matrix}) \cdot \begin{bmatrix} 1 & 1 & 1 \\ t_1 & t_2 & t_3 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix}$$

Orthogonality: suppose $V, W \subset \mathbb{R}^n$

how do we encode $\boxed{V \perp W}$?

pick a basis $v_1, \ldots, v_k$ of V $\quad A = [v_1 | \ldots | v_k]$ $n \times k$

pick a basis $w_1, \ldots, w_L$ of W $\quad B = [w_1 | \ldots | w_L]$ $n \times L$

$C(A) = V, \quad C(B) = W$

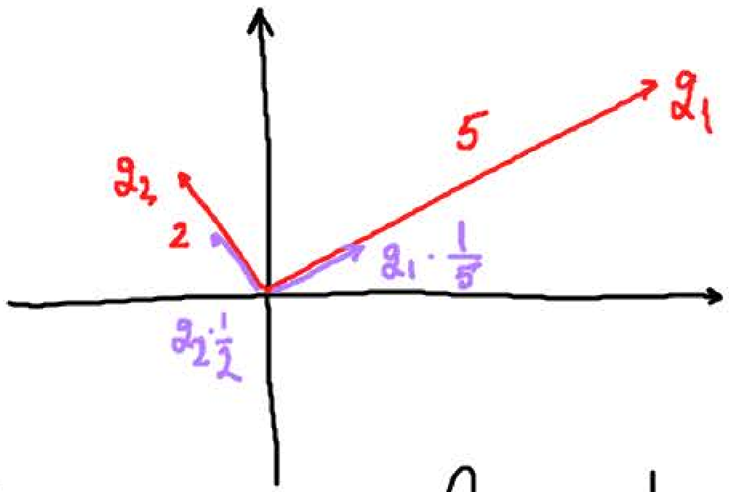
$\boxed{V \perp W} \iff \boxed{v_i \perp w_j \text{ for all } i,j} \iff \boxed{v_i^T w_j = 0 \text{ for all } i,j} \iff \boxed{A^T B = 0}$

$v_i^T w_j = 0$: $k \cdot L$ numbers are 0

$A^T B = 0$: $(k \times n)(n \times L)$ — a $k \times L$ matrix is 0

Def:

- a collection of non-zero vectors $q_1, \ldots, q_n$ are called orthogonal if: $\boxed{q_i \perp q_j} \iff \boxed{q_i^T q_j = 0}$ for all $i \neq j$
- ——— " ——— orthonormal if they are orthogonal and length 1, i.e. $\|q_i\| = 1$, i.e. $q_i^T q_i = [1]$ for all i



q_1, q_2 are orthogonal

$\frac{q_1}{5}, \frac{q_2}{2}$ are orthonormal.

$\Rightarrow$ a basis of whatever vector space they span

Fact: orthogonal vectors are <u>always</u> linearly independent

Can we encode the orthogonality/orthonormality of $q_1, \dots, q_n \in \mathbb{R}^m$ as a matrix identity?

Yes: consider the $m \times n$ matrix $Q = [q_1 | \dots | q_n]$;

the q_i's being orthogonal means that $Q^T Q = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$ $\quad d_i = \|q_i\|^2$

the q_i's —//— orthonormal means that $Q^T Q = I_n$

if $m=n$ (i.e. Q is a square matrix), we call the matrix Q orthogonal if $Q^T = Q^{-1}$ $\Longrightarrow Q^T Q = I_n$

only applies to square matrices

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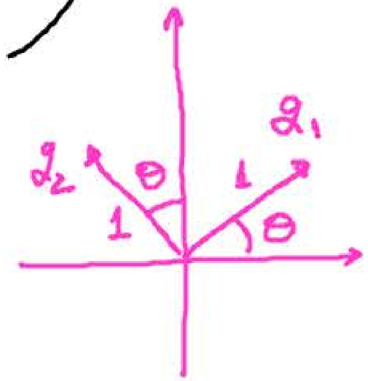
examples of orthogonal matrices:

- any permutation matrix $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

  (just multiply $P^T P$, you will get I_3)

- rotation matrix (angle θ) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with $q_1 = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $q_2 = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$



$$Q^T Q = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \left[\begin{array}{c|c} \cos\cdot\cos + \sin\cdot\sin & -\cos\cdot\sin + \sin\cdot\cos \\ \hline -\sin\cdot\cos + \cos\cdot\sin & (-\sin)(-\sin) + \cos\cdot\cos \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Orthogonal matrices preserve:

- perpendicularity : $v \perp w \iff Qv \perp Qw$

- lengths : $\|Qv\| = \|v\|$

(more generally, they preserve angles)

Proof: $\|Qv\|^2 = (Qv)^T Qv = v^T \underbrace{Q^T Q}_{I} v = v^T v = \|v\|^2$

Proof: $v \perp w \iff v^T w = 0$

$Qv \perp Qw \iff (Qv)^T Qw = 0$

$\iff v^T \underbrace{Q^T Q}_{I} w = 0$

$\iff v^T w = 0$

back to rectangular matrices; why do we love the property $Q^T Q = I_n \iff$ columns of Q are orthonormal

answer: because this makes the projection formula very simple

suppose we have $V \subset \mathbb{R}^m$ and we want to compute P_V:

- before: picked a basis $v_1, \dots, v_n$ of V put it in a matrix $A = [v_1 | \dots | v_n]$ $\implies P_V = A \, (A^T A)^{-1} A^T$

hard to invert this

- now: pick an orthonormal basis $q_1, \dots, q_n$ of V put it in a matrix $Q = [q_1 | \dots | q_n]$ $\implies P_V = Q \cdot Q^T$

because $Q^T Q = I_n \implies (Q^T Q)^{-1} = I_n$

Example: $V \subset \mathbb{R}^3$, $\underbrace{V}_{\text{2-dimensional}} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } x+y+z=0 \right\} = N\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right)$

$\boxed{\text{compute } P_V}$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = x+y+z$$

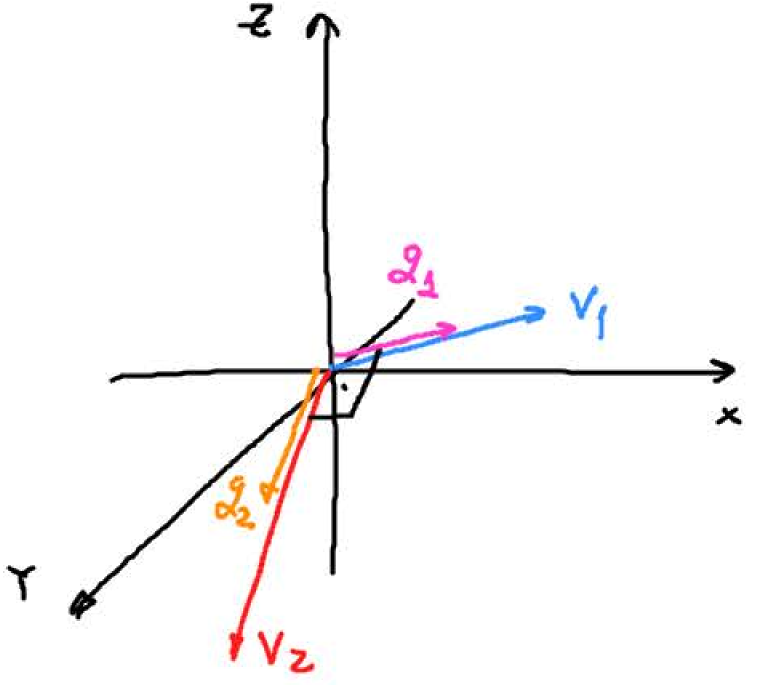
$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad ; \text{ pick } v_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ s.t. } \begin{cases} v_1 \perp v_2 & \Longleftrightarrow x = y \\ v_2 \in V & \Longleftrightarrow x+y+z=0 \end{cases} \quad , \text{ e.g. pick } v_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

to make v_1, v_2 orthonormal, rescale them by the inverses of their lengths

$$q_1 = \frac{1}{\|v_1\|} v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$q_2 = \frac{1}{\|v_2\|} v_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

$\Longrightarrow$ orthonormal basis



$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{2}{\sqrt{6}} \end{bmatrix} \quad \text{satisfies} \quad Q^T Q = I_2$$

$$P_V = Q \underbrace{(Q^T Q)}_{I}{}^{-1} Q^T = Q Q^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

- orthogonality of m vectors in n-dimensional space
- orthogonality of square matrices
  $(m=n)$