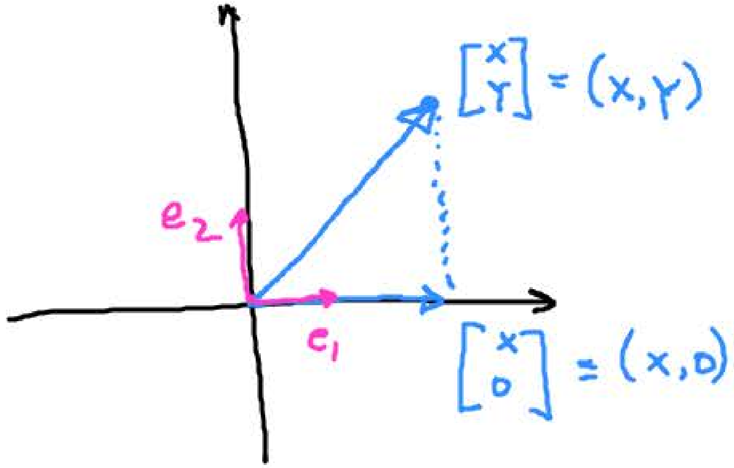
Change of basis (continued): expressing a linear transformation ϕ in terms of various bases

$\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ projection onto $a = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$\phi(v) = A v$ where $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

very simple matrix

$$\phi\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\Updownarrow$$

$$\phi(x \cdot \underbrace{e_1}_{} + y \cdot \underbrace{e_2}_{}) = x \cdot e_1$$

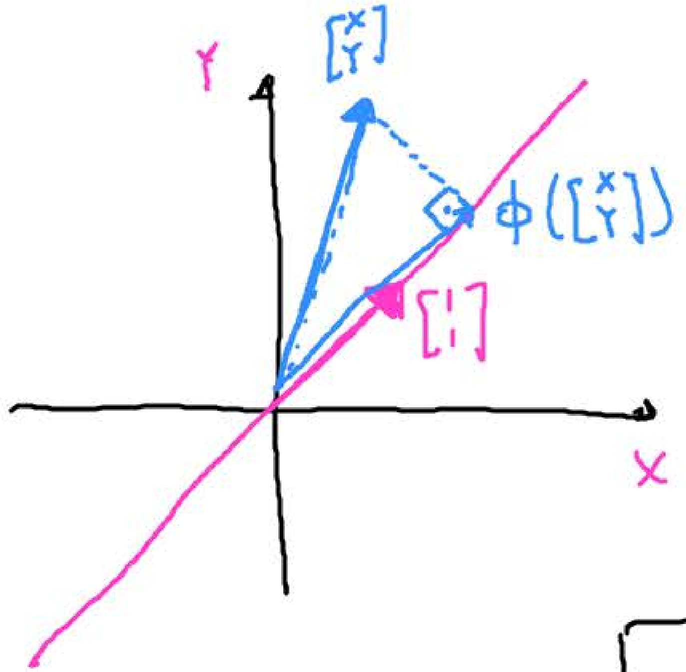
standard basis

ϕ preserves the e_1 component and annihilates the e_2 component

What about projections onto the line spanned by $\begin{bmatrix}1\\1\end{bmatrix}$?

What is a formula for ϕ?

$$\phi(v) = P_{\text{line}}\, v = \underbrace{\begin{bmatrix}1\\1\end{bmatrix}\left(\begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right)^{-1}\begin{bmatrix}1 & 1\end{bmatrix}}_{\text{projection matrix onto the line spanned by } a=\begin{bmatrix}1\\1\end{bmatrix}} v$$



$$= \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}2\end{bmatrix}^{-1}\begin{bmatrix}1 & 1\end{bmatrix} v$$

$$= \begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}\tfrac{1}{2}\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} v$$

$$= \frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}\begin{bmatrix}1 & 1\end{bmatrix} v$$

$$= \frac{1}{2}\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} v$$

$$= \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix} v$$

What is the connection between?

$$A = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} \rightsquigarrow \text{proj matrix onto } \begin{bmatrix}1\\0\end{bmatrix}$$

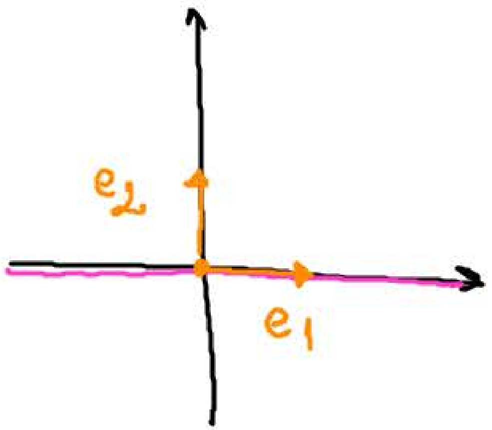
$$B = \begin{bmatrix}1/2 & 1/2\\1/2 & 1/2\end{bmatrix} \rightsquigarrow \text{proj matrix onto } \begin{bmatrix}1\\1\end{bmatrix}$$

$\rightarrow$

CHANGE OF BASIS FORMULA

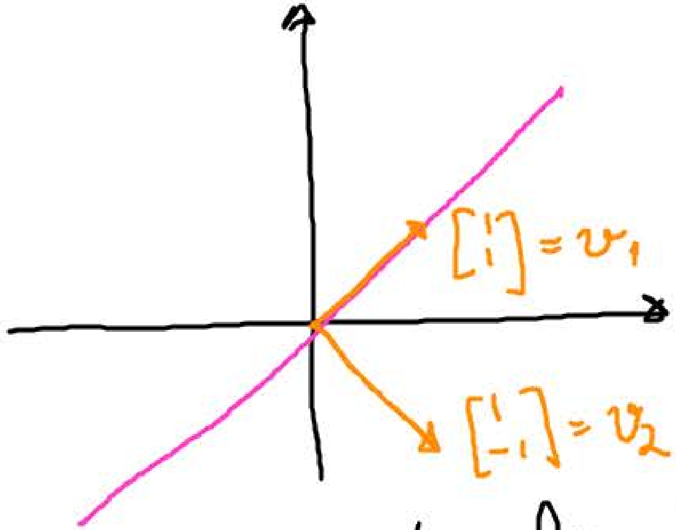
V an appropriate matrix

$$B = V^{-1}AV$$



ϕ = projection onto x axis

$\phi(x \cdot e_1 + y \cdot e_2) = x \cdot e_1$



what plays the role of e_1 & e_2

ϕ' = projection onto line $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\phi'(x \cdot v_1 + y \cdot v_2) = x \cdot v_1$

Erratum: while VB = AV is correct, the explicit 2 x 2 matrices corresponding to A and B should be switched

we need the matrix that changes from the basis v_1, v_2 to the basis e_1, e_2

$$V = [v_1 | v_2] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Change of basis says:

$$B = V^{-1} A V \iff VB = AV$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

General principle: take $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

- if ϕ is represented by a matrix A in bases $e_1, \dots, e_n$ of $\mathbb{R}^n$, $e_1, \dots, e_m$ of $\mathbb{R}^m$

$$\phi(v) = Av \Rightarrow \phi(x_1 e_1 + \dots + x_n e_n) = (a_{11} x_1 + \dots + a_{1n} x_n) e_1 + \dots + (a_{m1} x_1 + \dots + a_{mn} x_n) e_n$$

- then ϕ is represented by a matrix B in basis $v_1, \dots, v_n$ of $\mathbb{R}^n$, $w_1, \dots, w_m$ of $\mathbb{R}^m$

$$\phi(x_1 \cdot v_1 + \dots + x_n v_n) = (b_{11} x_1 + \dots + b_{1n} x_n) w_1 + \dots + (b_{m1} x_1 + \dots + b_{mn} x_n) w_n$$

- where B and A are connected by the change of basis formula

$$\boxed{\underbrace{B}_{m \times n} = \underbrace{W^{-1}}_{m \times m} \underbrace{A}_{m \times n} \underbrace{V}_{n \times n}}$$

where $V = [v_1 | \dots | v_n]$

$W = [w_1 | \dots | w_m]$

last time, we were studying the particular case when $V = W$

$\Downarrow$

$m = n$

Application (details in a few weeks):

- dilation linear transformations: $\phi\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} d_1 x_1 \\ \vdots \\ d_n x_n \end{bmatrix} = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

diagonal matrices correspond to dilations in the directions of the standard basis $e_1, \dots, e_n$

Thm: (almost) any $n \times n$ $\boxed{\text{complex}}$ matrix A corresponds to dilation in the direction of some basis $v_1, \dots, v_n$ of $\mathbb{R}^n$

do the same thing, but in different bases

change of basis

$V = [v_1 | \dots | v_n]$

$$V^{-1} A V = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} \iff A = V \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix} V^{-1} \iff A \text{ is similar to a diagonal matrix}$$

Def: A & B are called similar if $A = VBV^{-1}$ for some matrix B.

Precise example: given $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix}$ $\quad (\Leftarrow \; \phi : \mathbb{R}^3 \to \mathbb{R}^2)$

$$\phi(x_1 e_1 + x_2 e_2 + x_3 e_3) = (1\cdot x_1 + 1\cdot x_2 + 2\cdot x_3)\, e_1 + (2\cdot x_1 + 3\cdot x_2 + 4\cdot x_3)\, e_2$$

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$\downarrow$

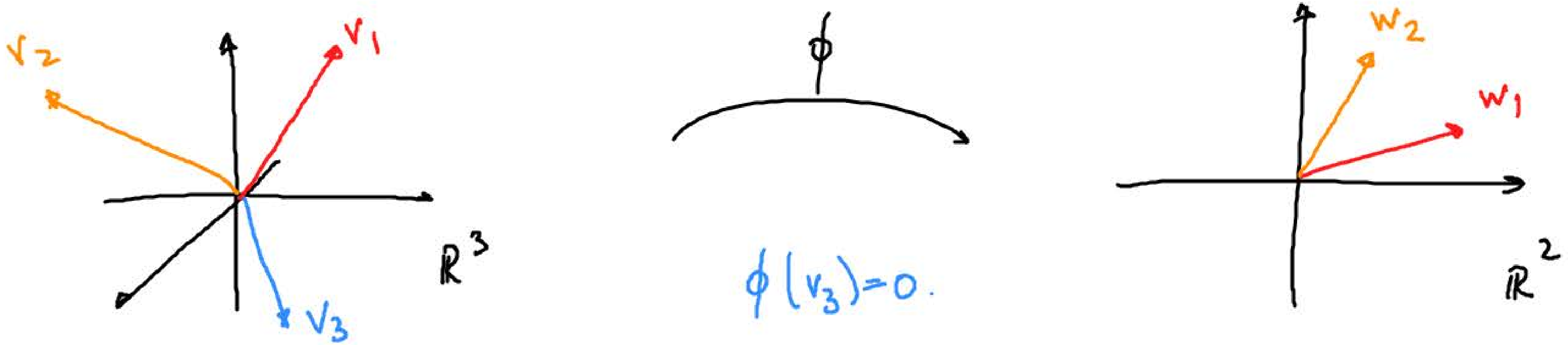
find bases v_1, v_2, v_3 of $\mathbb{R}^3$

and w_1, w_2 of $\mathbb{R}^2$

in which ϕ corresponds to the simple matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

$$\phi(x_1 v_1 + x_2 v_2 + x_3 v_3) = (1\cdot x_1 + 0\cdot x_2 + 0\cdot x_3)\, w_1 + (0\cdot x_1 + 1\cdot x_2 + 0\cdot x_3)\, w_2 = x_1 w_1 + x_2 w_2$$

simpler formula



Change of basis: find matrices $\overset{3\times 3}{V}$ & $\overset{2\times 2}{W}$ → matrices whose columns are the sought-for bases

such that $B = W^{-1} A V$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = W^{-1} \cdot \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \cdot V$$

Gauss-Jordan elimination is all about multiplying a complicated matrix on the left by stuff and getting a simpler matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} = R$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix}}_{\text{almost as simple as } B} = E_{12}^{(-1)} E_{21}^{(-2)} \cdot A$$

is of the form we want, i.e. $W^{-1} A \cdot V$

multiplying on the right by a matrix corresponds to column operations instead of row operations

$$R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{c_3 - 2c_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = B$$

$$R \cdot E_{13}^{(-2)} = B$$

$$R = E_{12}^{(-1)} E_{21}^{(-2)} A$$

$$\Rightarrow \quad \underbrace{E_{12}^{(-1)} E_{21}^{(-2)}}_{\text{let this be } W^{-1}} A \cdot \underbrace{E_{13}^{(-2)}}_{\text{let this be } V} = B$$

let this be W^{-1}

let this be $V = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

the columns of V & W are the sought-for basis

$$W^{-1} = E_{12}^{(-1)} E_{21}^{(-2)}$$

$$\Updownarrow$$

$$W = E_{21}^{(2)} E_{12}^{(1)} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$$