Today: let A be $n \times n$, i.e. square

Gaussian elimination

LU factorization

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$$

Assume Gaussian elimination for A does not require row exchanges

step 1: add multiples of row 1 to other rows

$\rightarrow$ add λ_{21} times row 1 to row 2

$\rightarrow$ add λ_{31} times row 1 to row 3

$\vdots$

$\rightarrow$ add λ_{n1} times row 1 to row n

step 2: add multiples of row 2 to lower rows

$\rightarrow$ add λ_{32} times row 2 to row 3

$\vdots$

$\rightarrow$ add λ_{n2} times row 2 to row n

$\vdots$

step $n-1$: add multiples of row $n-1$ to lower row

$\rightarrow$ add $\lambda_{n,n-1}$ times row $n-1$ to row n

$$\underbrace{E_{n,n-1}^{(\lambda_{n,n-1})}}_{\text{step } n-1} \cdots\cdots \underbrace{E_{n2}^{(\lambda_{n2})} \cdots\cdots E_{32}^{(\lambda_{32})}}_{\text{step } 2} \underbrace{E_{n1}^{(\lambda_{n1})} \cdots E_{31}^{(\lambda_{31})} E_{21}^{(\lambda_{21})}}_{\text{step } 1} A = \overset{\text{REF}}{U}$$

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} & u_{14} \\ 0 & u_{22} & u_{23} & u_{24} \\ 0 & 0 & u_{33} & u_{34} \\ 0 & 0 & 0 & u_{44} \end{bmatrix}$$

diagonal

to isolate A in the left-hand side of this formula, we need to multiply by inverses of all the E matrices

$$\left(E_{ij}^{(\lambda)}\right)^{-1} = E_{ij}^{(-\lambda)}$$

show for 2×2 matrices

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix}$$

$$\Updownarrow$$

$$\begin{bmatrix} 1\cdot 1 + 0\cdot(-\lambda) & \cancel{1\cdot 0 + 0\cdot 1} \\ \lambda\cdot 1 \cancel{+1}(-\lambda) & \lambda\cdot 0 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -\lambda & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

A square matrix in REF is upper triangular, i.e. all entries underneath diagonal are 0's

$$E_{n,n-1}^{(\lambda_{n,n-1})} \cdots\cdots E_{n1}^{(\lambda_{n1})} \cdots E_{21}^{(\lambda_{21})} A = U$$

multiply on left with $\left(E_{n,n-1}^{(\lambda_{n,n-1})}\right)^{-1} = E_{n,n-1}^{(-\lambda_{n,n-1})}$

$$I \cdots\cdots E_{n1}^{(\lambda_{n1})} \cdots E_{21}^{(\lambda_{21})} A = E_{n,n-1}^{(-\lambda_{n,n-1})} U$$

$\vdots$

multiply on left with $\left(E_{n1}^{(\lambda_{n1})}\right)^{-1} \cdots\cdots$

then with $\left(E_{21}^{(\lambda_{21})}\right)^{-1}$

$$\boxed{A = \underbrace{E_{21}^{(-\lambda_{21})} \cdots E_{n1}^{(-\lambda_{n1})} \cdots\cdots E_{n,n-1}^{(-\lambda_{n,n-1})}}_{\text{call this } L} \cdot U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -\lambda_{21} & 1 & 0 & 0 & 0 \\ -\lambda_{31} & -\lambda_{32} & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & 1 & 0 \\ -\lambda_{n1} & -\lambda_{n2} & \cdots & -\lambda_{n,n-1} & 1 \end{bmatrix}$$

Important: L is lower triangular $\Longrightarrow$ Factorization: (unique if A is non-singular)

$$A = \overset{\text{lower}}{L} \cdot \overset{\text{upper}}{U}$$

$$A = \begin{bmatrix} \boxed{2} & 4 & 1 \\ \boxed{-4} & -5 & 0 \\ \boxed{-2} & 5 & 6 \end{bmatrix} \xrightarrow{2\cdot r_1 + r_2} \begin{bmatrix} \boxed{2} & 4 & 1 \\ 0 & \boxed{3} & 2 \\ \boxed{-2} & 5 & 6 \end{bmatrix} \xrightarrow{r_1 + r_3} \begin{bmatrix} \boxed{2} & 4 & 1 \\ 0 & \boxed{3} & 2 \\ 0 & \boxed{9} & 7 \end{bmatrix}$$

$$\xrightarrow{(-3)r_2 + r_3} \begin{bmatrix} \boxed{2} & 4 & 1 \\ 0 & \boxed{3} & 2 \\ 0 & 0 & \boxed{1} \end{bmatrix} = U$$

upper triangular

$$E_{32}^{(-3)} E_{31}^{(1)} E_{21}^{(2)} A = U \quad \Big| \cdot \left(E_{32}^{(-3)}\right)^{-1}$$

$$E_{31}^{(1)} E_{21}^{(2)} A = E_{32}^{(3)} U \quad \Big| \cdot \left(E_{31}^{(1)}\right)^{-1}$$

$$E_{21}^{(2)} A = E_{31}^{(-1)} E_{32}^{(3)} U \quad \Big| \cdot \left(E_{21}^{(2)}\right)^{-1}$$

$$\rightsquigarrow \quad A = \underbrace{E_{21}^{(-2)} E_{31}^{(-1)} E_{32}^{(3)}}_{L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}} U$$

Note: this formula only holds because of the specific order that we do our row eliminations

Diagonal matrices :

$$D = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

$\Longleftarrow$

$$D^{-1} = \begin{bmatrix} \frac{1}{d_1} & & & 0 \\ & \frac{1}{d_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{d_n} \end{bmatrix}$$

$$A = L \cdot U$$

L: lower triangular with 1's on diagonal

U: upper triangular

$\downarrow$

you can force a diagonal matrix out of U so that what remains will be an upper triangular matrix with 1's on diagonal

$$U = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \underbrace{D}_{\text{diagonal}} \cdot \underbrace{U'}_{\text{upper triangular matrix with 1's on diagonal}}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U = DU' \iff D^{-1}U = U'$$

$$D^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1/2 \\ 0 & 1 & 2/3 \\ 0 & 0 & 1 \end{bmatrix}$$

Upshot: any* square matrix A can be written as

$$A = LDU$$

this U is actually U' from previous slide

lower triangular with 1's on diag

diagonal

upper triangular with 1's on diag

* = assuming Gaussian elimination for A does not require row exchanges

Why?

- product of upper triangular matrices is upper triangular
- product of lower triangular matrices is lower triangular
- product of diagonal matrices is diagonal

○ : your lower triangular matrix can have non 1's on the diagonal

○ : your upper triangular matrix can have non 1's on the diagonal