TOWER OF HANOI

Algorithm Design and Analysis

AUTHOR

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1. Problem description

The **Tower of Hanoi or Tower of Brahma** is a mathematical puzzle or game, invented by the French mathematician Édouard Lucas in 1883.

The puzzle consists of **three pegs and a number of disks of different sizes**, which can be introduced into any peg. Besides, the initial position of the puzzle is a stack in ascending order of size on one peg, the smallest at the top.

The aim of the puzzle is to move the stack to another rod, following the next 3 rules:

- Only one disk can be moved at the same time.
- Each move consists on taking the disk at the top of one peg and pushing into the top of another.
- No disk can be placed above a smaller disk.

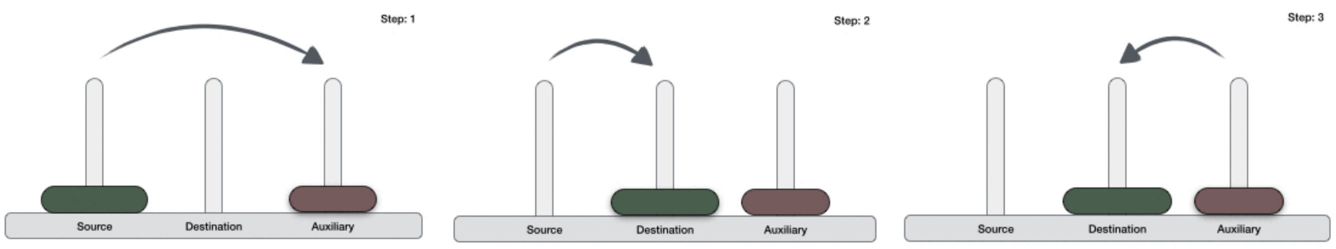
One interesting fact is that there is a story about an Indian temple which contains a large room with 3 pegs in it, surrounded by 64 golden disks. According to the legend, when the last move of the puzzle is completed, the world will end. We will make the calculations later!

2. Recursive Solution

To write the algorithm for Tower of Hanoi, first we need to understand the problem and solve it with 1 or 2 amount of disks. For a better comprehension we will call the pegs source, destination and auxiliary (this last one is only for helping moving the disks).

If we have only one disk, the base case, it can be moved directly from source to destination. However, if we have 2 disks we have to follow this 3 steps:

1. Move the smaller disk to auxiliary peg.
2. Move the bottom disk to the destination peg.
3. Move the smaller disk from auxiliary to destination peg.



If we generalize this steps, we can design a recursive solution for Tower of Hanoi problem. First, we divide the stack of disks in two parts, **the first one contains the nth disk and the other one n-1 disks.** The steps are:

1. Move the n-1 disks from source to auxiliary peg.
2. Move the nth disk from source to destination peg.

3. Move n-1 disks from auxiliary peg to destination peg.

If we follow those steps in each call, we will divide the problem until we reach the base case.

2.1. Pseudocode

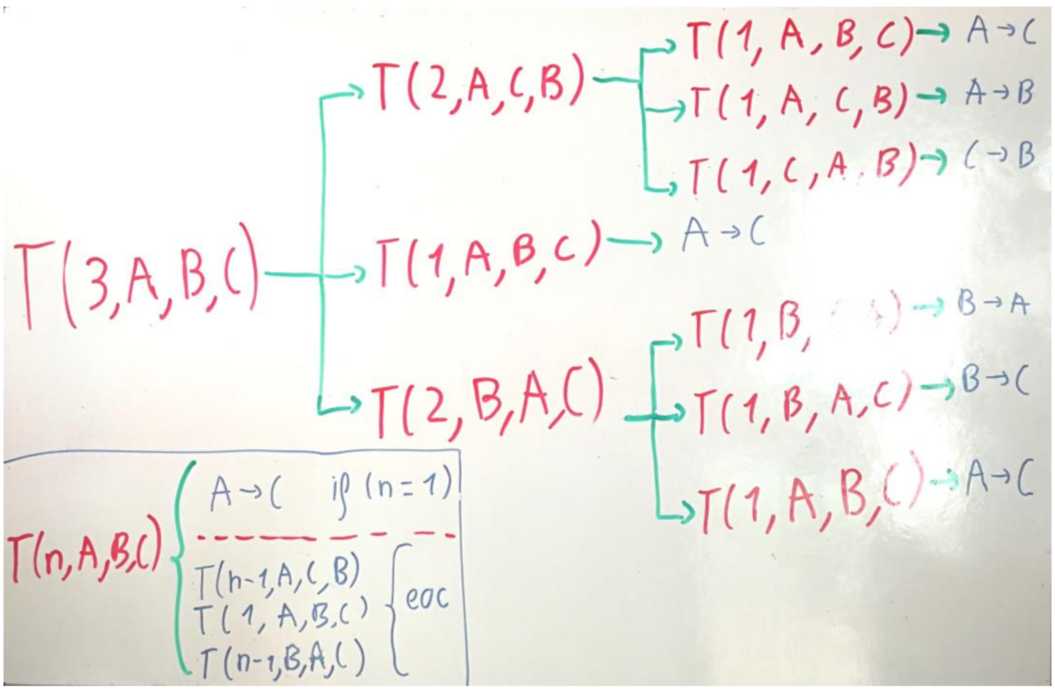
```
void moveDisks(int numberOfDisks, Peg sourcePeg, Peg auxiliaryPeg, Peg destinationPeg) {

    if (numberOfDisks == 1) { // Base case
        // Move directly the disk from sourcePeg to destinationPeg.
        destinationPeg.push(sourcePeg.pop());
    }

    else {
        // 1. Move the n-1 disks from source to auxiliary peg.
        moveDisks(numberOfDisks - 1, sourcePeg, destinationPeg, auxiliaryPeg);
        // 2. Move the nth disk from source to destination peg.
        moveDisks(1, sourcePeg, auxiliaryPeg, destinationPeg);
        // 3. Move n-1 disks from auxiliary peg to destination peg.
        moveDisks(numberOfDisks - 1, auxiliaryPeg, sourcePeg, destinationPeg);
    }

}
```

The initial call would be with the origin values, as:

```
        moveDisks(this.numberOfDisks, this.sourcePeg, this.auxiliaryPeg, this.destinationPeg);
```

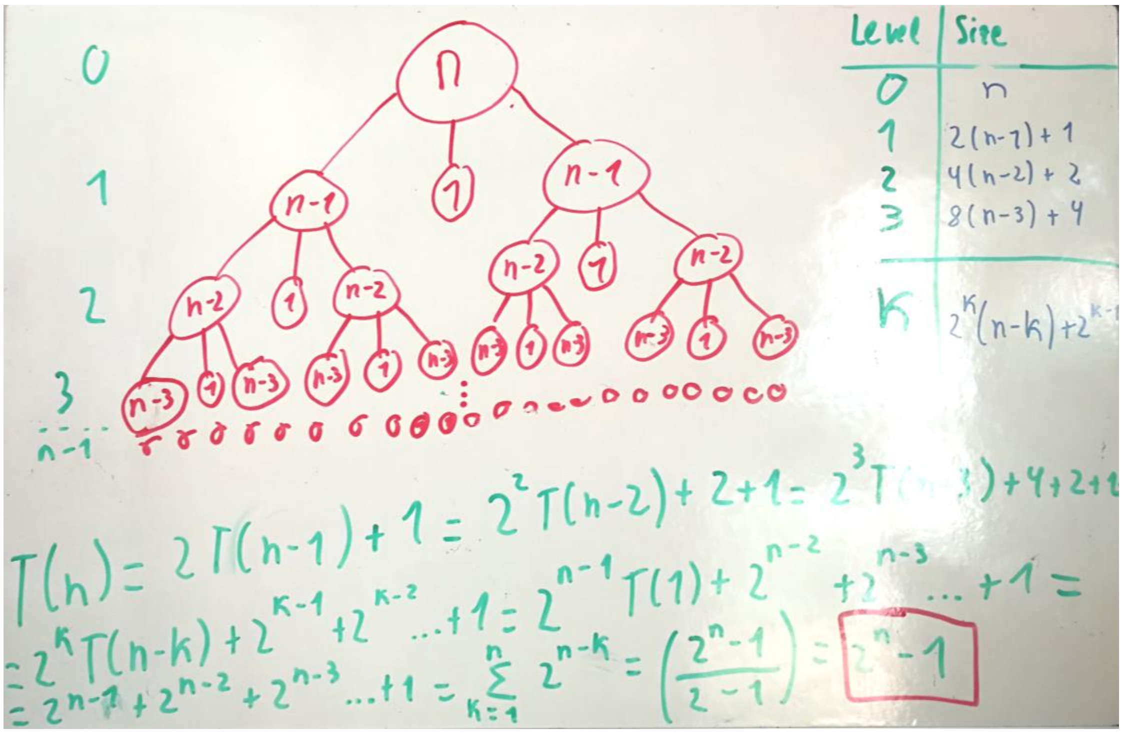
3. Algorithm complexity

To analyze the algorithm complexity first we will write the **recursive function**. For finding it we will represent graphically the recursive calls that the algorithm does. We will use the example of 3 as number of disks in the Tower of Hanoi. In red there are the recursive calls and in blue the final movement.



At the left-bottom corner we can see the recursive call. The base case is the direct movement, when the number of disks is 1. In any other case the algorithm makes 3 recursive calls, or 2 if we deduce that the second call is constant.

For analyzing the complexity of the algorithm I will represent graphically the recursion tree:


Level
Size
0
n
1
2(n-1) + 1
2
4(n-2) + 2
3
8(n-3) + 4
k
2^k(n-k) + 2^(k-1)
n
n-1
1
n-1
n-2
1
n-2
n-2
1
n-2
n-3
1
2
3
n-1
T(n) = 2T(n-1) + 1 = 2^2 T(n-2) + 2 + 1 = 2^3 T(n-3) + 4 + 2 + 1
= 2^k T(n-k) + 2^(k-1) + 2^(k-2) ... + 1 = 2^(n-1) T(1) + 2^(n-2) + 2^(n-3) ... + 1 =
= 2^(n-1) + 2^(n-2) + 2^(n-3) ... + 1 = Σ_(k=1)^n 2^(n-k) = (2^n - 1)/(2 - 1) = 2^n - 1

As you can see in the picture there are n-1 levels of the recursion tree, starting on level 0. In every level the size is the sum of all its sub problems. At the right-top corner you can see a table where the level of the tree is associated with the size of the sub problem. If we follow the pattern we can deduce that in the k-level the size is $2^k(n-k) + 2^{k-1}$.

Following that pattern, we can find the complexity of the problem, which is represented in the bottom part of the picture. We start from the complexity of the level 1 (2T(n-1) + 1), then the level 2 (2^2 T(n-2) + 2 + 1), and we find the pattern. After that we can make the final summation and, after applying its formula, we end up with a complexity of **2^n – 1.**

4. Algorithm test

For testing proposals, we will make a table and graphics with the execution time for different input times in the recursive algorithm that is represented in the point **2.1**. Furthermore, to test which structure type is more efficient between stack or Array List we will execute the algorithm using that 2 different types.

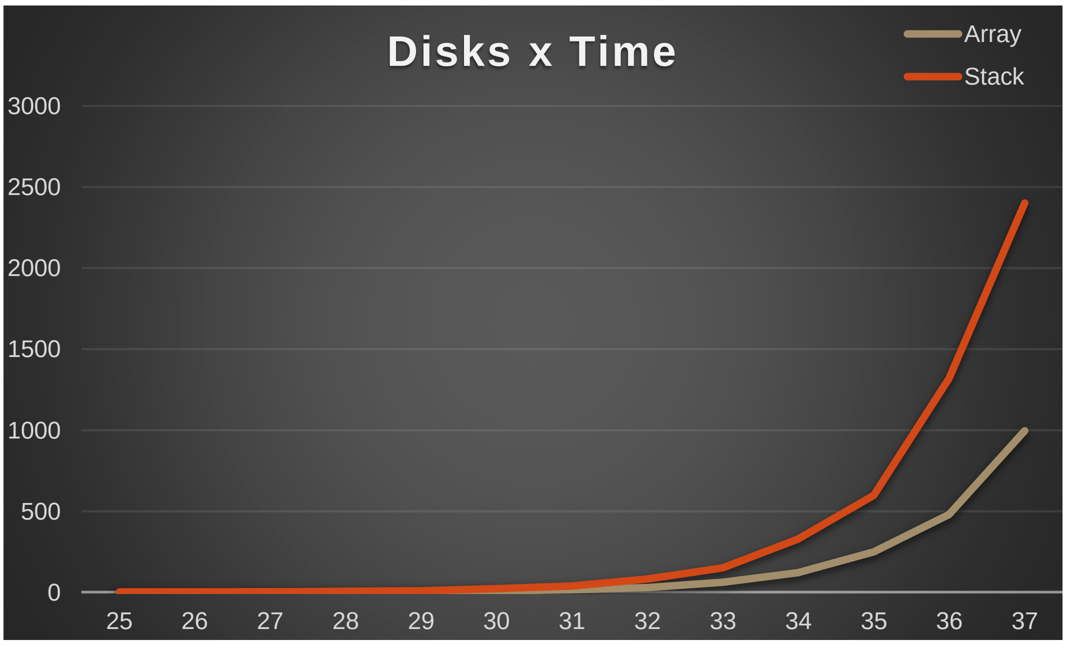
Note: All the tests are made with the Java programming language.

For showing the mean elapsed time for tests we have chosen the range between 25 and 37 because it's where we can see a big difference in time. The result table is:

Disks	Movements	Time Array	Time Stack	1 Instruction Array	Approximated Time
25	33554431	0.243228258	1.070206778	7.24877E-09	0.238980811
26	67108863	0.462992979	1.285602947	6.89913E-09	0.477961629
27	134217727	0.967381949	2.344186338	7.20756E-09	0.955923264
28	268435455	1.863706762	5.107864988	6.94285E-09	1.911846536
29	536870911	3.928877135	9.365830825	7.3181E-09	3.823693079
30	1073741823	7.550789245	20.39955628	7.03222E-09	7.647386165
31	2147483647	15.52857167	37.4117279	7.23105E-09	15.29477234
32	4294967295	30.15800081	82.11768546	7.02171E-09	30.58954468
33	8589934591	62.22456867	149.0334849	7.24389E-09	61.17908937
34	17179869183	119.8980741	326.9518977	6.97899E-09	122.3581787
35	34359738367	248.4288268	598.2588013	7.23023E-09	244.7163575
36	68719476735	479.5327992	1320.087659	6.97812E-09	489.432715
37	1.37439E+11	997.2242081	2402.569439	7.25576E-09	978.86543

As we can see **the elapsed time using the peg as array is lot faster** than the stack elapsed time, so we conclude that the Array Set of Disk is more efficient than the Stack one (always that the elements are entered intelligently). **We will use it for our calculations.**

If we look the time array difference between one disk and its predecessor, we can see a difference of approximately its double. In fact, if we make a graphic representing in the y axis the elapsed time and in the y axis the disks for that elapsed time. **We can see that the function is very similar to 2^n**, as expected.



If we divide the array elapsed time and the number of movements for all the number of disks between 25 and 37, and finally we make the mean we get:

```
1 Instruction -> 7.25576E-09
```

So, if the Hanoi Tower legend is true and we calculate the elapsed time for 64 disks the result in years would be 4166. However, it's impossible to move the disks manually at that speed, so we are safe!

Bibliography

- Recursion Algorithm - Tower of Hanoi: https://www.youtube.com/watch?v=5_6nsViVM00
- TutorialsPoint: https://www.tutorialspoint.com/data_structures_algorithms/tower_of_hanoi.htm
- Wikipedia: https://en.wikipedia.org/wiki/Tower_of_Hanoi