

$$\begin{aligned}
 (a) \ E[\hat{a}] &= E[\bar{x}^2] = \text{var}(\bar{x}) + E[\bar{x}]^2 \\
 &= \text{var}\left\{\frac{1}{N} \sum_{n=0}^{N-1} x[n]\right\} + E\left[\frac{1}{N} \sum_{n=0}^{N-1} x[n]\right]^2 \\
 &= \frac{1}{N^2} (N\sigma^2) + A^2 \\
 &= \frac{1}{N} \sigma^2 + A^2 \\
 &\neq A^2 \Rightarrow \text{unbiased}
 \end{aligned}$$

$$\begin{aligned}
 (b) \ \text{var}(\hat{a}) &= \text{var}(\bar{x}^2) = E[\bar{x}^4] - E[\bar{x}^2]^2 \\
 &= \left(\cancel{A^4} + 6A^2 \frac{\sigma^2}{N} + 3 \frac{\sigma^4}{N^2} \right) - \left(\frac{\sigma^4}{N^2} + 2A^2 \frac{\sigma^2}{N} + \cancel{A^4} \right) \\
 &= 4A^2 \frac{\sigma^2}{N} + 2 \frac{\sigma^4}{N^2}
 \end{aligned}$$

$$I(\theta) = -E\left[\frac{\partial^2 \ln P(\mathbf{x}; \theta)}{\partial \theta^2}\right] = \frac{1}{\sigma^2} \cdot N$$

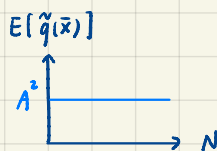
$$\text{Let } q(A) = A^2 \Rightarrow \left(\frac{\partial q(A)}{\partial A}\right)^2 = 4A^2$$

$$\Rightarrow \text{var}\{\hat{a}\} \geq \frac{4A^2}{\frac{N}{\sigma^2}} = \frac{4A^2 \sigma^2}{N}$$

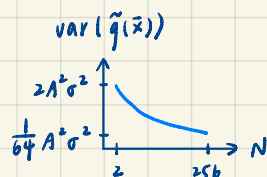
does not achieve CRLB
If $N \rightarrow \infty$, $\text{var}(\hat{a}) \rightarrow \text{CRLB}$

$$\begin{aligned}
 (c) \ \tilde{q}(\bar{x}) &= q(A) + \frac{dq(A)}{dA} (\bar{x} - A) \\
 &= A^2 + 2A(\bar{x} - A)
 \end{aligned}$$

$$\begin{aligned}
 (d) \ E[\tilde{q}(\bar{x})] &= E[A^2 + 2A(\bar{x} - A)] \\
 &= A^2 + 2A(E[\bar{x}] - A) \\
 &= A^2
 \end{aligned}$$



$$\begin{aligned}
 \text{var}(\tilde{q}(\bar{x})) &= 4A^2 \text{var}(\bar{x}) \\
 &= 4A^2 \frac{\sigma^2}{N}
 \end{aligned}$$



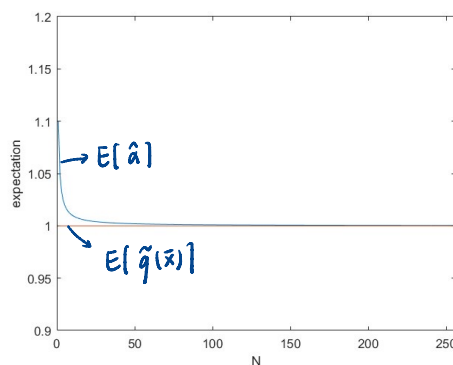
$$(e) \ \text{Let } A=1, \sigma^2=0.1$$

$$(a) \ E[\hat{a}] = \frac{0.1}{N} + 1$$

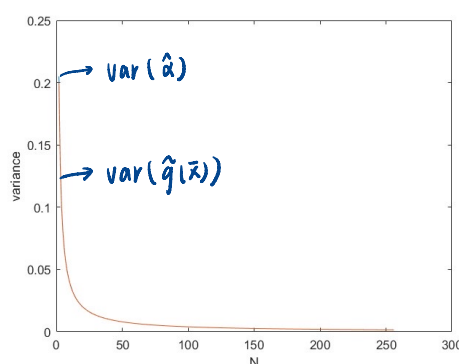
$$\begin{aligned}
 (b) \ \text{var}(\hat{a}) &= \frac{0.4}{N} + \frac{0.02}{N^2} \\
 \text{CRLB} &= \frac{0.4}{N}
 \end{aligned}$$

$$(c) \ \tilde{q}(\bar{x}) = 1 + 2(\bar{x} - 1) = 2\bar{x} - 1$$

$$\begin{aligned}
 (d) \ E[\tilde{q}(\bar{x})] &= 1 \\
 \text{var}(\tilde{q}(\bar{x})) &= \frac{0.4}{N}
 \end{aligned}$$

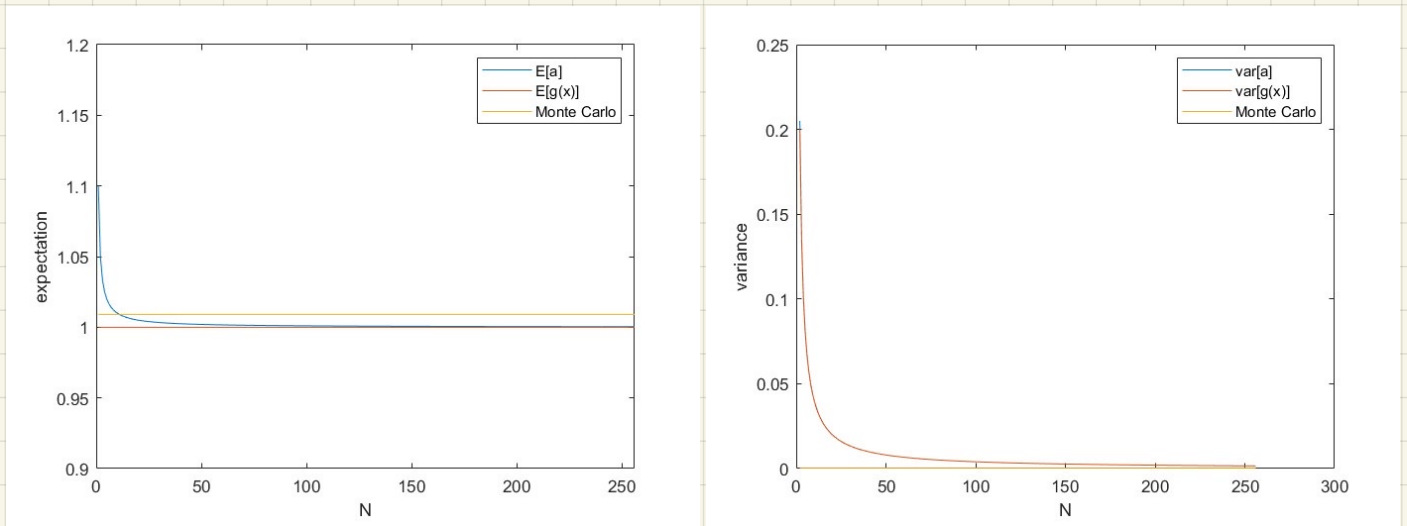


$$\begin{aligned}
 E[\hat{a}] &= E[\tilde{q}(\bar{x})] + \frac{0.1}{N} \\
 \text{When } N \rightarrow \infty, E[\hat{a}] &\rightarrow E[\tilde{q}(\bar{x})]
 \end{aligned}$$



$$\begin{aligned}
 \text{var}(\hat{a}) &= \text{var}(\tilde{q}(\bar{x})) + \frac{0.02}{N^2} \\
 \text{When } N \rightarrow \infty, \text{var}(\hat{a}) &\rightarrow \text{var}(\tilde{q}(\bar{x}))
 \end{aligned}$$

(f)

(g) Let $A=1$, $\sigma^2=1$

$$(a) E[\hat{a}] = \frac{1}{N} + 1$$

$$(b) \text{var}(\hat{a}) = \frac{4}{N} + \frac{2}{N^2}$$

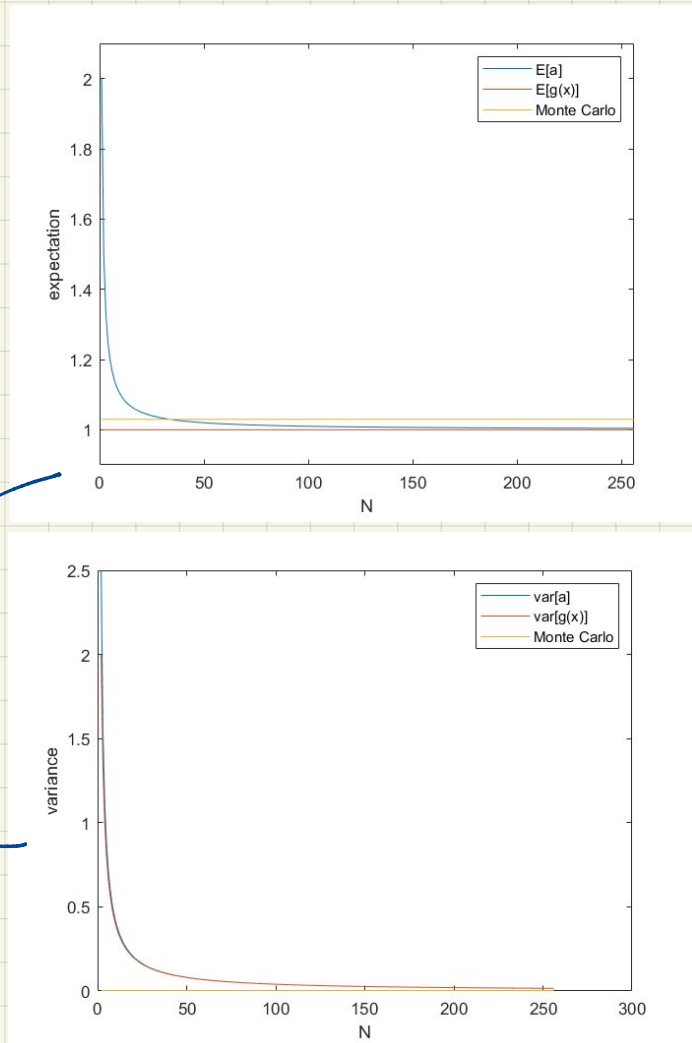
$$\text{CRLB} = \frac{4}{N}$$

$$(c) \tilde{q}(\bar{x}) = 1 + 2(\bar{x} - 1) = 2\bar{x} - 1$$

$$(d) E[\tilde{q}(\bar{x})] = 1$$

$$\text{var}(\tilde{q}(\bar{x})) = \frac{4}{N}$$

$\uparrow \sigma^2$
 $\therefore E[\hat{a}], E[\tilde{q}(\bar{x})] \uparrow$
 $\text{var}(\hat{a}), \text{var}(\tilde{q}(\bar{x})), \text{var}(\tilde{q}(\bar{x})) \uparrow$



$$(h) P(a; x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\sqrt{a})^2} + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x+\sqrt{a})^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2+a)} \left[e^{\frac{\sqrt{a}}{\sigma^2}x} + e^{-\frac{\sqrt{a}}{\sigma^2}x} \right]$$

$$\ln P(a; x) = -\frac{1}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x^2+a) + \ln \left[e^{\frac{\sqrt{a}}{\sigma^2}x} + e^{-\frac{\sqrt{a}}{\sigma^2}x} \right]$$

$$\ln P(a; \mathbf{x}) = -\frac{N}{2} \ln(2\pi\sigma^2) - \frac{N}{2\sigma^2} a - \frac{1}{2\sigma^2} \sum_i x_i^2 + \sum_i \ln \left[e^{\frac{\sqrt{a}}{\sigma^2}x_i} + e^{-\frac{\sqrt{a}}{\sigma^2}x_i} \right]$$

$$\frac{\partial \ln P(a; \mathbf{x})}{\partial a} = -\frac{N}{2\sigma^2} + \sum_i \frac{e^{\frac{\sqrt{a}}{\sigma^2}x_i} \cdot \frac{x_i}{\sigma^2} \frac{1}{2} a^{-\frac{1}{2}} + e^{-\frac{\sqrt{a}}{\sigma^2}x_i} \cdot \left(-\frac{x_i}{\sigma^2}\right) \frac{1}{2} a^{-\frac{1}{2}}}{e^{\frac{\sqrt{a}}{\sigma^2}x_i} + e^{-\frac{\sqrt{a}}{\sigma^2}x_i}}$$

$$= -\frac{N}{2\sigma^2} + \frac{1}{2} \alpha^{-\frac{1}{2}} \sum_i \frac{x_i}{\sigma^2} \frac{e^{\frac{\sqrt{\alpha}}{\sigma^2} x_i} - e^{-\frac{\sqrt{\alpha}}{\sigma^2} x_i}}{e^{\frac{\sqrt{\alpha}}{\sigma^2} x_i} + e^{-\frac{\sqrt{\alpha}}{\sigma^2} x_i}}$$

$$= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^2 \sqrt{\alpha}} \sum_i x_i \tanh\left(\frac{\sqrt{\alpha}}{\sigma^2} x_i\right)$$

$$\text{Let } \frac{\partial \ln P(\alpha; \underline{x})}{\partial \alpha} = 0$$

$$\Rightarrow \alpha^{-\frac{1}{2}} \sum_i x_i \tanh\left(\frac{\sqrt{\alpha}}{\sigma^2} x_i\right) = N$$

$$\Rightarrow \alpha = \hat{A}_{ML}^2$$