

In the DC level in WGN problem, let  $A \sim u[-A_0, A_0]$  and the noise  $w[n] \sim N(0, \sigma^2)$ . We would like to obtain Bayesian estimators using different risk functions.

### 1. (5 points)

Derive the posterior probability  $p(A|x)$ .

Now, let  $A_0 = 2$  and  $\sigma^2 = 0.1$ .

$$\begin{aligned}
 (i) \quad P(x[n]|A) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x[n]-A)^2} \\
 \Rightarrow P(A|\bar{x}) &= \frac{P(\bar{x}|A)p(A)}{\int P(\bar{x}|A)p(A)dA} \\
 &= \frac{\frac{1}{(2\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n]-A)^2} \cdot \frac{1}{2A_0}}{\int_{-A_0}^{A_0} \frac{1}{(2\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2} \sum_{n=0}^{N-1} (x[n]-A)^2} \cdot \frac{1}{2A_0} dA} , |A| \leq A_0 \\
 &= \frac{1}{c'} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2} \cdot \frac{1}{c''} \\
 &= \frac{1}{c} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2}, c = \int_{-A_0}^{A_0} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2} dA
 \end{aligned}$$

### 2. (10 points)

Let the number of observations  $x$  at each realization of  $A$  be  $N = 10$ .

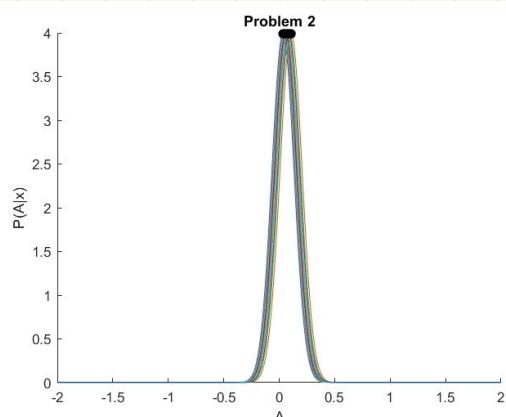
Randomly generate one realization of  $A$ . Plot  $p(A|x)$ , and  $\hat{A}_{MMSE}$  (with bold dot in the figure) for 15 different realizations of  $x$

```

1 clc; clear all; close all;
2 seed = 101;
3 rng(seed);
4
5 A0 = 2;
6 A = (2*A0) * rand - A0;  $\rightarrow$  生成 A
7 sigma2 = 0.1;
8 N = 10;
9 sigma2N = sigma2/N;
10
11 x = A + sigma2 * randn(N, 15);  $\left\{ \text{生成 } 15 \text{ 组 } x, \text{ 不 mean} \right.$ 
12 x_ = mean(x, 1);
13
14 figure;
15 xlabel('A');
16 ylabel('P(A|x)');
17 title('Problem 2');
18 hold on;
19
20 for i = 1:15
21 P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_(i)).^2));
22
23 c = integral(P_ax, -A0, A0);
24 P_ax_normalized = @(a) P_ax(a) / c;
25 P_ax_mean = @(a) P_ax_normalized(a) .* a;  $\left\{ \text{求 } E[A|\bar{x}] \right.$ 
26 A_mmse = integral(P_ax_mean, -A0, A0);
27
28 fplot(P_ax_normalized, [-A0, A0]);
29 plot(A_mmse, P_ax_normalized(A_mmse), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
30 end
31 hold off;

```

求  $P(A|\bar{x})$



最高點的  $A$  等於  $\bar{x}$   
理論上  $E[A|\bar{x}] \neq \bar{x}$

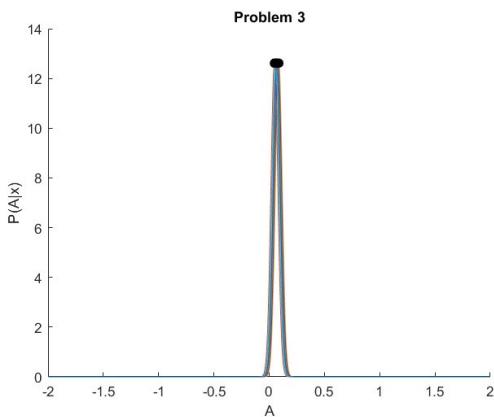
### 3. (5 points)

Now let  $N = 100$ . Repeat (2).

```

33 N = 100;
34 sigma2N = sigma2/N;
35
36 x = A + sigma2 * randn(N, 1);
37 x_ = mean(x, 1);
38
39 figure;
40 xlabel('A');
41 ylabel('P(A|x)');
42 title('Problem 3');
43 hold on;
44
45 for i = 1:15
46 P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_).^2));
47
48 c = integral(P_ax, -A0, A0);
49 P_ax_normalized = @(a) P_ax(a) / c;
50 P_ax_mean = @(a) P_ax_normalized(a) .* a;
51 A_mmse = integral(P_ax_mean, -A0, A0);
52
53 fplot(P_ax_normalized, [-A0, A0]);
54 plot(A_mmse, P_ax_normalized(A_mmse), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
55 end
hold off;

```



$N \uparrow, \frac{\sigma^2}{N} \downarrow$ , shape truncated Gaussian  
 $\Rightarrow$  Narrow Gaussian  
 $\Rightarrow$  Truncation effect  
 $\Rightarrow$  The  $P(A|z)$  more like the original Gaussian pdf  
 $\Rightarrow E_A\{A|z\} \approx \hat{A}_{MMSE} \approx \bar{x}$

### 4. (12 points)

Following (2), now let the realizations of  $A$  be  $M = 1000$ . Obtain the

Bayesian MSE  $B_{mse}(\hat{A}_{MMSE})$  using Monte Carlo simulations.

```

58 M = 1000;
59 A0 = 2;
60 sigma2 = 0.1;
61 N = 10;
62 sigma2N = sigma2/N;
63
64 A = (2*A0) * rand(1, M) - A0;  $\rightarrow$  生成 M 个 A
65
66 A_mmse = zeros(1,M);
67 for i = 1:M
68 x = A(i) + sigma2 * randn(N, 1);
69 x_ = mean(x);
70 P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_).^2));
71 P_ax_normalized = @(a) P_ax(a) / c;
72 P_ax_mean = @(a) P_ax_normalized(a) .* a;
73 A_mmse(i) = integral(P_ax_mean, -A0, A0);  $\rightarrow \hat{A}_{MMSE} = E[\hat{A}]$ 
74
75 end
76 A_mmse_monte = mean(A_mmse);  $\rightarrow E[\hat{A}] = \frac{1}{M} \sum_{m=1}^M \hat{A}_m$ 
77 BMSE = mean((A-A_mmse_monte).^2);  $\rightarrow \text{var}[\hat{A}] = \frac{1}{M} \sum_{m=1}^M (\hat{A}_m - E[\hat{A}])^2$ 
78 disp(['Problem(4) A_mmse using Monte Carlo: ', num2str(A_mmse_monte)]);
79 disp(['Problem(4) BMSE using Monte Carlo: ', num2str(BMSE)]);

```

Problem(4) A\_mmse using Monte Carlo: -0.021435

Problem(4) BMSE using Monte Carlo: 1.3199

### 5. (5 points)

Following (2), and let  $N = 100$ , repeat (4).

Problem(5) A\_mmse using Monte Carlo: -0.019988

Problem(5) BMSE using Monte Carlo: 1.3199

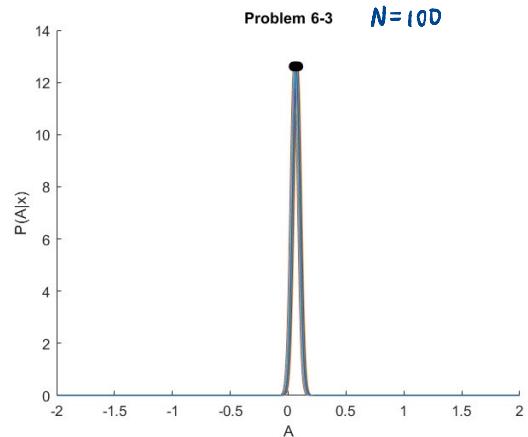
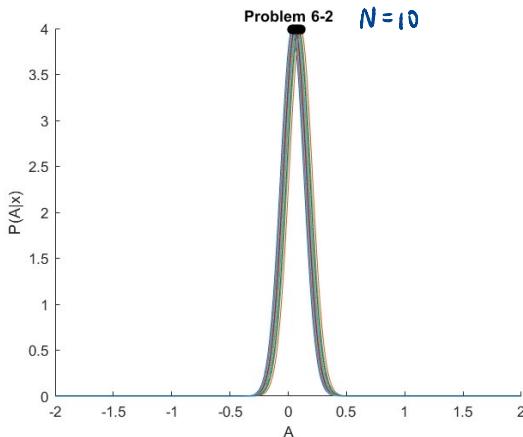
## 6. (32 points)

Now, the risk function is absolute error. Repeat (2)-(5). Your answer should be as detailed as possible to obtain full credit. Especially "HOW" you obtain  $\hat{A}$  either in mathematical representation or via Matlab program.

```

100    %%%
101    seed = 101;
102    rng(seed);
103
104    A0 = 2;
105    A = (2*A0) * rand - A0;
106    sigma2 = 0.1;
107    N = 10;
108    sigma2N = sigma2/N;
109
110    x = A + sigma2 * randn(N, 15);
111    x_ = mean(x, 1);
112
113    figure;
114    xlabel('A');
115    ylabel('P(A|x)');
116    title('Problem 6-2');
117    hold on;
118
119    for i = 1:15
120        P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_(i)).^2));
121
122        c = integral(P_ax, -A0, A0);
123        P_ax_normalized = @(a) P_ax(a) / c;
124        CDF = @(a) integral(P_ax_normalized, -A0, a);
125        median_A = fzero(@(a) CDF(a) - 0.5, 0);
126
127        fplot(P_ax_normalized, [-A0, A0]);
128        plot(median_A, P_ax_normalized(median_A), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
129    end
130    hold off;

```



$$C(\theta) = |\epsilon| = |\theta - \hat{\theta}|$$

$$\Rightarrow \int_{-\infty}^{\hat{\theta}} |\theta - \hat{\theta}| P(\theta | \xi) d\theta$$

$$h(\hat{\theta}, \theta)$$

$$= \int_{-\infty}^{\hat{\theta}} (\hat{\theta} - \theta) P(\theta | \xi) d\theta + \int_{\hat{\theta}}^{\infty} (\theta - \hat{\theta}) P(\theta | \xi) d\theta$$

$$\frac{\partial}{\partial \theta} = 0$$

$$\Rightarrow \int_{-\infty}^{\hat{\theta}} P(\theta | \xi) d\theta - \int_{\hat{\theta}}^{\infty} P(\theta | \xi) d\theta = 0$$

$$\Rightarrow P(\theta \leq \hat{\theta} | \xi) = \frac{1}{2} \Rightarrow \hat{\theta} \text{ 是 } \theta | \xi \text{ 的中位数}$$



```

157    M = 1000;
158    A0 = 2;
159    sigma2 = 0.1;
160    N = 10;
161    sigma2N = sigma2/N;
162
163    A = (2*A0) * rand(1, M) - A0;
164
165    median_A = zeros(1, M);
166    for i = 1:M
167        x = A(i) + sigma2 * randn(N, 1);
168        x_ = mean(x);
169        P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_.)^2));
170        c = integral(P_ax, -A0, A0);
171        P_ax_normalized = @(a) P_ax(a) / c;
172        CDF = @(a) integral(P_ax_normalized, -A0, a);
173        median_A(i) = fzero(@(a) CDF(a) - 0.5, 0);
174    end
175    A_hat = mean(median_A);
176    disp(['Problem(6-4) A_hat: ', num2str(A_hat)]);

```

$$E[\hat{A}] = \frac{1}{M} \sum_{m=1}^M \hat{A}_m$$

Problem(6-4) A\_hat: -0.021537

Problem(6-5) A\_hat: -0.020008

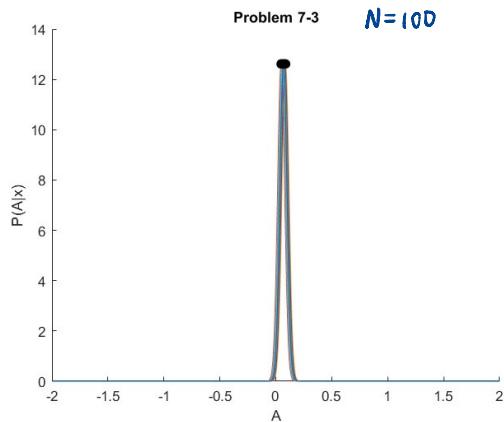
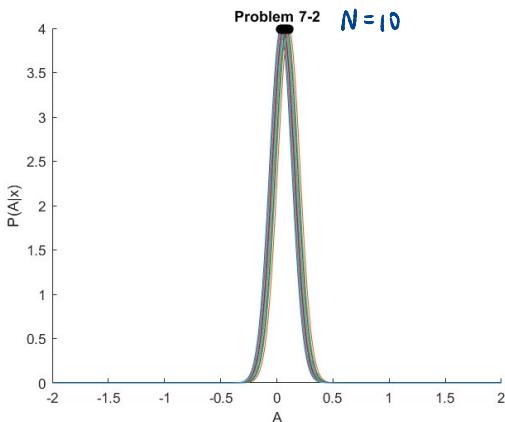
7. (32 points)

Now, the risk function is Hit-or-risk. Repeat (2)-(5). Your answer should be as detailed as possible to obtain full credit. Especially "HOW" you obtain  $\hat{A}$  either in mathematical representation or via Matlab program.

```

195   %% seed = 101;
196   rng(seed);
197
198   A0 = 2;
199   A = (2*A0) * rand - A0;
200   sigma2 = 0.1;
201   N = 10;
202   sigma2N = sigma2/N;
203
204   x = A + sigma2 * randn(N, 15);
205   x_ = mean(x, 1);
206
207   figure;
208   xlabel('A');
209   ylabel('P(A|x)');
210   title('Problem 7-2');
211   hold on;
212
213   for i = 1:15
214       P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_(i)).^2));
215
216       c = integral(P_ax, -A0, A0);
217       P_ax_normalized = @(a) P_ax(a) / c;
218
219       [max_a, max_val] = fminbnd(@(a) -P_ax_normalized(a), -A0, A0); → max. P(A|z)
220
221       fplot(P_ax_normalized, [-A0, A0]);
222       plot(max_a, P_ax_normalized(max_a), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
223   end
224   hold off;

```



$$C(\theta) = \begin{cases} 0, & |\theta| < \delta \\ 1, & |\theta| \geq \delta \end{cases}$$

$$\Rightarrow |\theta - \hat{\theta}| > \delta \Rightarrow \theta - \hat{\theta} > \delta \text{ or } \hat{\theta} - \theta > \delta$$

$$\Rightarrow \theta > \hat{\theta} + \delta \text{ or } \theta < \hat{\theta} - \delta$$

$$\Rightarrow g(\hat{\theta}) = \int_{-\infty}^{\hat{\theta}} C(\theta) P(\theta|x) d\theta$$

$$= \int_{\hat{\theta}-\delta}^{\hat{\theta}} 1 \cdot P(\theta|x) d\theta + \int_{\hat{\theta}+\delta}^{\infty} 1 \cdot P(\theta|x) d\theta$$

$$\Rightarrow \min. g(\hat{\theta}) = \max. \int_{\hat{\theta}-\delta}^{\hat{\theta}+\delta} P(\theta|x) d\theta$$

If  $P(\theta|x)$  is Gaussian

For general  $P(\theta|x)$ , if  $\delta \rightarrow 0$   
 $\Rightarrow \max. P(\theta|x) \xrightarrow{\text{极大化}} \theta \rightarrow \text{MAP}$

$\hat{\theta}$  is the peak of  $P(\theta|x)$

```

252   M = 1000;
253   A0 = 2;
254   sigma2 = 0.1;
255   N = 10;
256   sigma2N = sigma2/N;
257
258   A = (2*A0) * rand(1, M) - A0;
259
260   MAP_A = zeros(1, M);
261   for i = 1:M
262       x = A(i) + sigma2 * randn(N, 1);
263       x_ = mean(x);
264       P_ax = @(a) (a >= -A0 & a <= A0) .* (1 / sqrt(2*pi*sigma2N)) .* exp((-0.5/sigma2N) * ((a-x_.)^2));
265
266       c = integral(P_ax, -A0, A0);
267       P_ax_normalized = @(a) P_ax(a) / c;
268       [max_a, max_val] = fminbnd(@(a) -P_ax_normalized(a), -A0, A0);
269       MAP_A(i) = max_a;
270   end
271   A_hat = mean(MAP_A); →  $E[\hat{A}] = \frac{1}{M} \sum_{m=1}^M \hat{A}_m$ 
272   disp(['Problem(7-4) A_hat: ', num2str(A_hat)]);

```

Problem(7-4)  $A\_hat: -0.021852$

Problem(7-5)  $A\_hat: 0.28762$