Estimation and Detection (Fall - 2024)

*** Total points: 100**

*** Due 10/21/2024**

For all simulations, let the seed be 101, and the noise is WGN with variance σ^2 and zero mean.

1. (100 points)

   In the DC level in WGN problem:

   $$x[n] = A + \omega[n],\ n = 0, 1, ..., N-1.$$

   We mentioned that $\hat{\alpha} = \bar{x}^2$ is asymptotically optimal for estimating A^2.

   (a) (5 points) Obtain $\mathbb{E}\{\hat{\alpha}\}$ in terms of A. Is it unbiased?

   (b) (10 points) Obtain $\text{var}\{\hat{\alpha}\}$ in terms of A. Does it achieve the CRLB?

   (c) (5 points) Let $\bar{x}$ be a variable. $\text{g}(\bar{x}) = \bar{x}^2$ is a nonlinear function of $\bar{x}$. Let $\widetilde{g}(\bar{x})$ be the first order Taylor approximation at the Center A. Write down $\widetilde{g}(\bar{x})$.

   (d) (15 points) Following (c), when $\text{N} \to \infty$, $\bar{x}$ has very high probability in the linear approximation region. That is $\widetilde{g}(\bar{x}) \approx \text{g}(\bar{x})$. Obtain $\mathbb{E}\{\widetilde{g}(\bar{x})\}$ and $\text{var}\{\widetilde{g}(\bar{x})\}$ as functions of N for $N = 2:1:256$.

   (e) (10 points) Repeat (a)-(d), let $A = 1$ and $\sigma^2 = 0.1$. Plot $\mathbb{E}\{\hat{\alpha}\}$ and $\mathbb{E}\{\widetilde{g}(\bar{x})\}$ as functions of N for $N = 1:1:256$. Also, plot $\text{var}\{\hat{\alpha}\}$ and $\text{var}\{\widetilde{g}(\bar{x})\}$ as functions of N for $N = 2:1:256$.

(f) (30 points) Now, we would like to verify the results via Monte Carlo simulation. Let the number M of the realizations be 4096, and $\widehat{\mathbb{E}\{\widetilde{g}(\bar{x})\}} = \frac{1}{M}\sum_{m=1}^{M} \bar{x_m}^2$, $\widehat{\mathrm{var}\{\widetilde{g}(\bar{x})\}} = \frac{1}{M}\sum_{m=1}^{M} (\bar{x_m}^2 - A^2)^2$. Plot $\widehat{\mathbb{E}\{\widetilde{g}(\bar{x})\}}$ and $\widehat{\mathrm{var}\{\widetilde{g}(\bar{x})\}}$ in the same figures for (e).

(g) (10 points) Let $A = 1$, $\sigma^2 = 1$. Repeat (e)-(f).

(h) (15 points) We would like to obtain an estimator that maximize the likelihood function, i.e. $\widehat{A^2_{ML}}$. Using transformation of variable, write down the likelihood function of A^2, i.e. $\mathrm{P}(\alpha; \underline{x})$. Then, obtain $A^{\hat{2}}_{ML}$ by maximizing $\mathrm{P}(\alpha; \underline{x})$.