(a) $E[\hat{\alpha}] = E[\bar{x}^2] = var(\bar{x}) + E[\bar{x}]^2$

$= var\{\frac{1}{N}\sum_{n=0}^{N-1} x[n]\} + E[\frac{1}{N}\sum_{n=0}^{N-1} x[n]]^2$

$= \frac{1}{N^2}(N\sigma^2) + A^2$

$= \frac{1}{N}\sigma^2 + A^2$

$\neq A^2 \ni$ unbiased

(b) $var(\hat{\alpha}) = var(\bar{x}^2) = E[\bar{x}^4] - E[\bar{x}^2]^2$

$= (\cancel{A^4} + 6A^2\frac{\sigma^2}{N} + 3\frac{\sigma^4}{N^2}) - (\frac{\sigma^4}{N^2} + 2A^2\frac{\sigma^2}{N} + \cancel{A^4})$

$= 4A^2\frac{\sigma^2}{N} + 2\frac{\sigma^4}{N^2}$

does not achieve CRLB

If $N \to \infty$, $var(\hat{\alpha}) \to$ CRLB

$I(\theta) = -E[\frac{\partial^2 \ln P(x;\theta)}{\partial\theta^2}] = \frac{1}{\sigma^2}\cdot N$

Let $g(A) = A^2 \ni (\frac{\partial g(A)}{\partial A})^2 = 4A^2$

$\ni var\{\hat{\alpha}\} \geq \frac{4A^2}{\frac{N}{\sigma^2}} = \frac{4A^2\sigma^2}{N}$

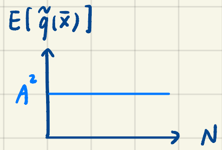
(c) $\tilde{g}(\bar{x}) = g(A) + \frac{dg(A)}{dA}(\bar{x} - A)$

$= A^2 + 2A(\bar{x} - A)$

(d) $E[\tilde{g}(\bar{x})] = E[A^2 + 2A(\bar{x} - A)]$

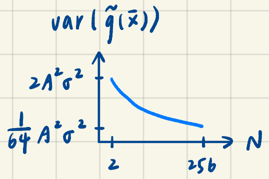
$= A^2 + 2A(\cancel{E[\bar{x}]}^{A} - A)$

$= A^2$



$var(\tilde{g}(\bar{x})) = 4A^2 var(\bar{x})$

$= 4A^2\frac{\sigma^2}{N}$



(e) Let $A = 1$, $\sigma^2 = 0.1$

(a) $E[\hat{\alpha}] = \frac{0.1}{N} + 1$

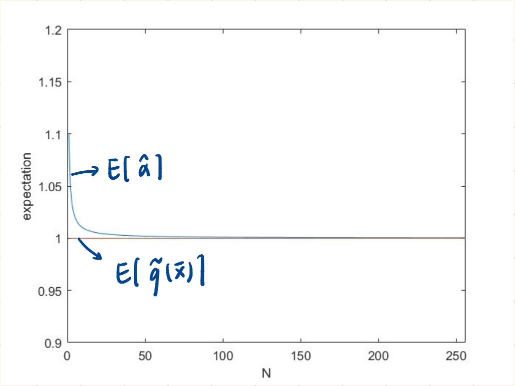
(b) $var(\hat{\alpha}) = \frac{0.4}{N} + \frac{0.02}{N^2}$

CRLB $= \frac{0.4}{N}$

(c) $\tilde{g}(\bar{x}) = 1 + 2(\bar{x} - 1) = 2\bar{x} - 1$

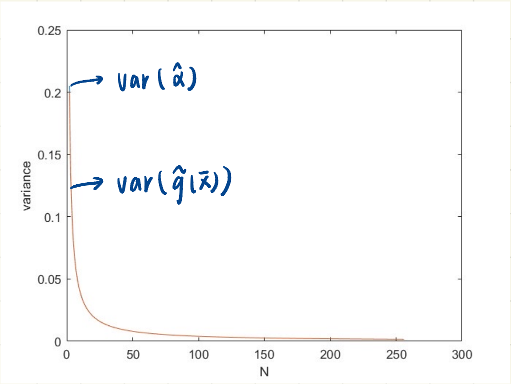
(d) $E[\tilde{g}(\bar{x})] = 1$

$var(\tilde{g}(\bar{x})) = \frac{0.4}{N}$



$E[\hat{\alpha}] = E[\tilde{g}(\bar{x})] + \frac{0.1}{N}$

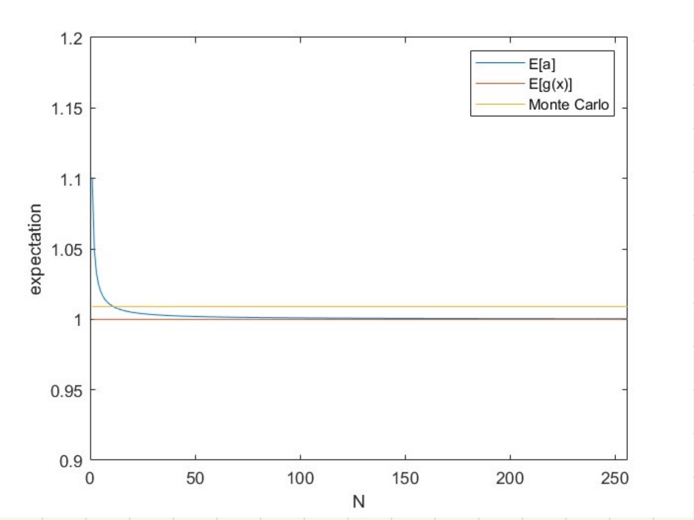
when $N \to \infty$, $E[\hat{\alpha}] \to E[\tilde{g}(\bar{x})]$

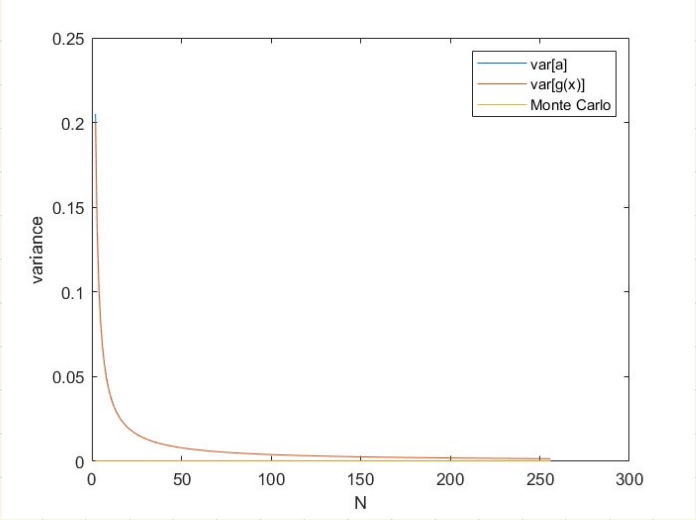


$var(\hat{\alpha}) = var(\tilde{g}(\bar{x})) + \frac{0.02}{N^2}$

when $N \to \infty$, $var(\hat{\alpha}) \to var(\tilde{g}(\bar{x}))$

(f)





(g) Let $A=1$, $\sigma^2=1$

(a) $E[\hat{\alpha}] = \frac{1}{N} + 1$

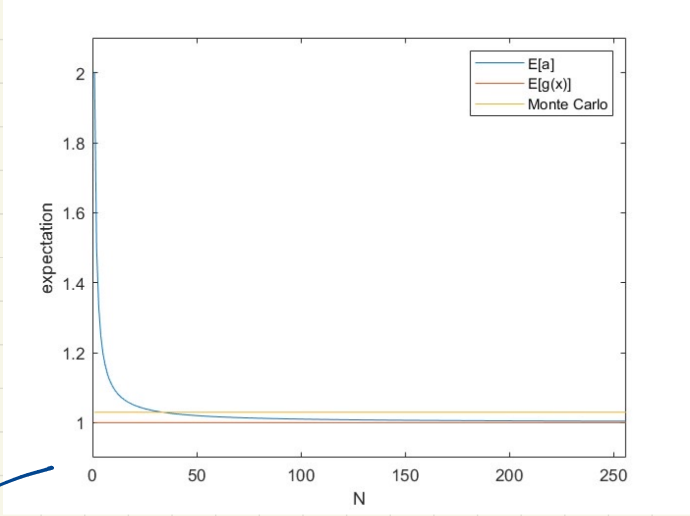
(b) $\mathrm{var}(\hat{\alpha}) = \frac{4}{N} + \frac{2}{N^2}$

$\mathrm{CRLB} = \frac{4}{N}$

(c) $\tilde{g}(\bar{x}) = 1 + 2(\bar{x}-1) = 2\bar{x} - 1$

(d) $E[\tilde{g}(\bar{x})] = 1$

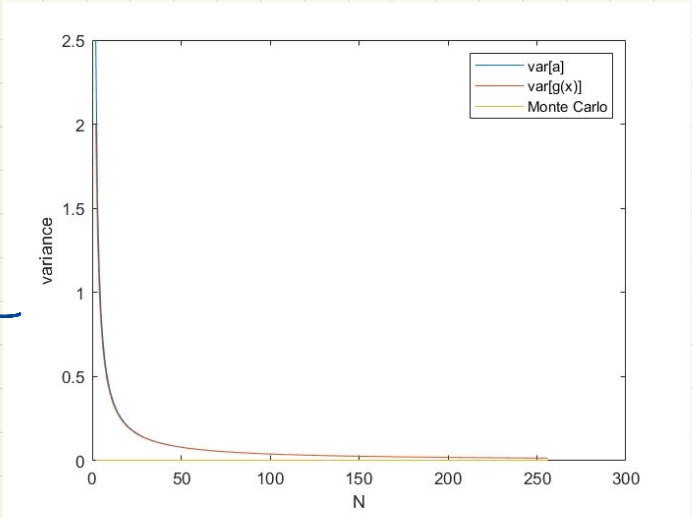
$\mathrm{var}(\tilde{g}(\bar{x})) = \frac{4}{N}$



$\because \sigma^2 \uparrow$

$\therefore E[\hat{\alpha}],\ \widehat{E[\tilde{g}(\bar{x})]} \uparrow$

$\mathrm{var}(\hat{\alpha}),\ \mathrm{var}(\tilde{g}(\bar{x})),\ \widehat{\mathrm{var}(\tilde{g}(\bar{x}))} \uparrow$



(h) $$P(\alpha; x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x-\sqrt{\alpha})^2} + \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x+\sqrt{\alpha})^2}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x^2+\alpha)} \left[e^{\frac{\sqrt{\alpha}}{\sigma^2}x} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x} \right]$$

$$\ln P(\alpha; x) = -\frac{1}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}(x^2+\alpha) + \ln\left[e^{\frac{\sqrt{\alpha}}{\sigma^2}x} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x} \right]$$

$$\ln P(\alpha; \underline{x}) = -\frac{N}{2}\ln(2\pi\sigma^2) - \frac{N}{2\sigma^2}\alpha - \frac{1}{2\sigma^2}\sum_i x_i^2 + \sum_i \ln\left[e^{\frac{\sqrt{\alpha}}{\sigma^2}x_i} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x_i} \right]$$

$$\frac{\partial \ln P(\alpha; \underline{x})}{\partial \alpha} = -\frac{N}{2\sigma^2} + \sum_i \frac{e^{\frac{\sqrt{\alpha}}{\sigma^2}x_i} \cdot \frac{x_i}{\sigma^2} \frac{1}{2} \alpha^{-\frac{1}{2}} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x_i} \cdot \left(-\frac{x_i}{\sigma^2}\right) \frac{1}{2} \alpha^{-\frac{1}{2}}}{e^{\frac{\sqrt{\alpha}}{\sigma^2}x_i} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x_i}}$$

$$= -\frac{N}{2\sigma^2} + \frac{1}{2}\alpha^{-\frac{1}{2}} \sum_i \frac{x_i}{\sigma^2} \frac{e^{\frac{\sqrt{\alpha}}{\sigma^2}x_i} - e^{-\frac{\sqrt{\alpha}}{\sigma^2}x_i}}{e^{\frac{\sqrt{\alpha}}{\sigma^2}x_i} + e^{-\frac{\sqrt{\alpha}}{\sigma^2}x_i}}$$

$$= -\frac{N}{2\sigma^2} + \frac{1}{2\sigma^2\sqrt{\alpha}} \sum_i x_i \tanh\left(\frac{\sqrt{\alpha}}{\sigma^2} x_i\right)$$

Let $\frac{\partial \ln P(\alpha; \underline{x})}{\partial \alpha} = 0$

$$\Rightarrow \alpha^{-\frac{1}{2}} \sum_i x_i \tanh\left(\frac{\sqrt{\alpha}}{\sigma^2} x_i\right) = N$$

$$\Rightarrow \alpha = \hat{A}^2_{ML}$$