

$$(a) x[n] = A + w[n], \quad n = 0, 1, \dots, N-1$$

$$w[n] \sim N(0, A)$$

$$P(x; A) = \frac{1}{(2\pi A)^{\frac{N}{2}}} e^{-\frac{1}{2A} \sum_{n=0}^{N-1} (x[n] - A)^2}$$

$$\Rightarrow \frac{\partial \ln P}{\partial A} = -\frac{N}{2A} + \frac{1}{2A^2} \sum_{n=0}^{N-1} (x[n] - A)^2 + \frac{1}{A} \sum_{n=0}^{N-1} (x[n] - A) = 0$$

$$\Rightarrow \hat{A}^2 + \hat{A} - \frac{1}{N} \sum_{n=0}^{N-1} x[n]^2 = 0$$

$$\Rightarrow \hat{A} = \frac{-1 \pm \sqrt{1 + \frac{4}{N} \sum_{n=0}^{N-1} x[n]^2}}{2}$$

$$= -\frac{1}{2} \pm \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} x[n]^2} \quad (\text{負不合})$$

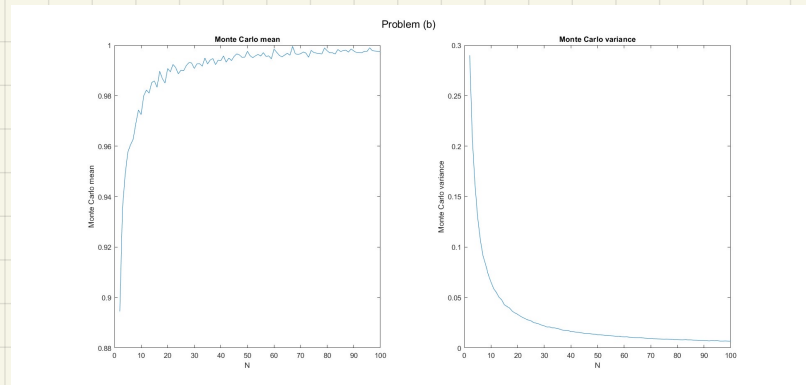
$$\Rightarrow \hat{A}_{MLE} = -\frac{1}{2} + \sqrt{\frac{1}{N} \sum_{n=0}^{N-1} x[n]^2}$$

(b)

```

1  A = 1;
2  M = 10000;
3  N = 2:1:100;
4  E_A = zeros(1,99);
5  var_A = zeros(1,99);
6
7  seed = 101;
8  rng(seed);
9
10 for j = N
11     A_MLE = zeros(1,M);
12     for i = 1:M
13         x = normrnd(A,sqrt(A),1,j);
14         sigma_x2 = sum(x.^2);
15         A_MLE(i) = -1/2+sqrt(1/4+sigma_x2/j);
16     end
17     E_A(j-1) = sum(A_MLE)/M;
18     A_MLE = A_MLE - E_A(j-1);
19     var_A(j-1) = sum(A_MLE.^2)/M;
20 end
21
22 figure;
23 sgtitle('Problem (b)');
24
25 subplot(1,2,1);
26 plot(N,E_A);
27 xlabel('N');
28 ylabel('Monte Carlo mean');
29 title('Monte Carlo mean');
30
31 subplot(1,2,2);
32 plot(N,var_A);
33 xlabel('N');
34 ylabel('Monte Carlo variance');
35 title('Monte Carlo variance');

```

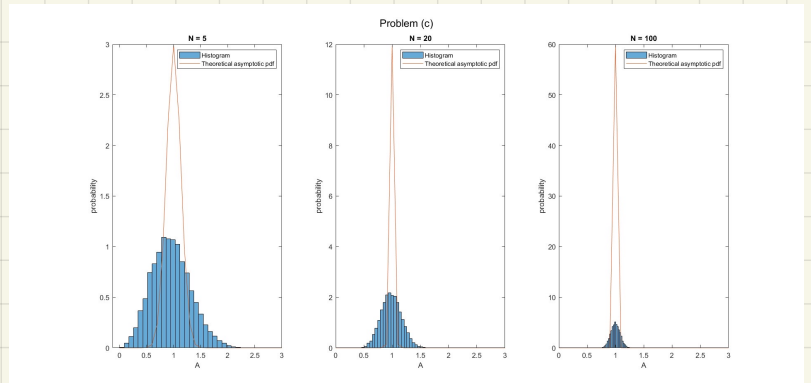


(c)

```

37 %%
38 A = 1;
39 N = [5,20,100];
40 M = 10000;
41
42 figure;
43 sgtitle('Problem (c)');
44 for n = 1:3
45     j = N(n);
46     A_MLE = zeros(1,M);
47     for i = 1:M
48         x = normrnd(A,sqrt(A),1,j);
49         sigma_x2 = sum(x.^2);
50         A_MLE(i) = -1/2+sqrt(1/4+sigma_x2/j);
51     end
52
53     x_Gaussian = 0:0.1:3;
54     sigma_the = (A^2)/(j*(A+1/2));
55     y_Gaussian = normpdf(x_Gaussian, A, sigma_the);
56
57     subplot(1,3,n);
58     histogram(A_MLE, 30, 'Normalization', 'pdf');
59     hold on;
60     plot(x_Gaussian, y_Gaussian);
61     title(['N = ', num2str(j)]);
62     legend('Histogram', 'Theoretical asymptotic pdf');
63     xlabel('A');
64     ylabel('probability');
65 end

```

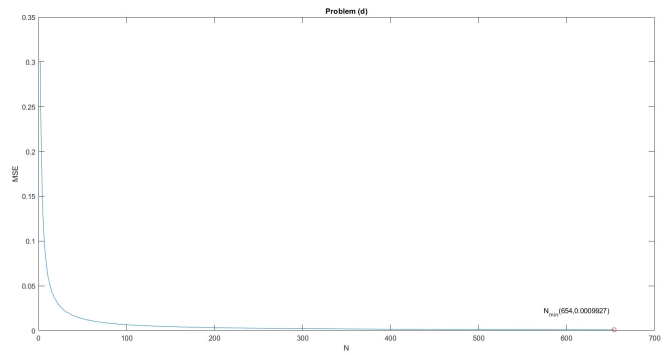


(d)

```

67 %%
68 A = 1;
69 M = 10000;
70 N = 2;
71
72 seed = 101;
73 rng(seed);
74
75 MSE = zeros(1,1000);
76
77 while 1
78     A_MLE = zeros(1,M);
79     for i = 1:M
80         x = normrnd(A,sqrt(A),1,N);
81         sigma_x2 = sum(x.^2);
82         A_MLE(i) = -1/2+sqrt(1/4+sigma_x2/N);
83     end
84
85     MSE(N-1) = mean((A_MLE-A).^2);
86
87     if MSE(N-1) < 1e-3
88         break
89     end
90
91     N = N+1;
92 end
93
94 figure;
95 plot(2:N,MSE(1:N-1));
96 hold on;
97 plot(N,MSE(N-1),'ro');
98 text(N-80,MSE(N-1)+0.02,['N_m_i_n(',num2str(N),',',num2str(MSE(N-1)),')']);
99 xlabel('N');
100 ylabel('MSE');
101 title('Problem (d)');

```



求出 MSE.
MSE < 1e-3 即跳出 while loop

(e)

```

103 %%
104 A = 1;
105 N = [5,20,100];
106 M = 10000;
107 u = zeros(1,M);
108
109 g_double_prime = @(x) (-1/4).*(x+1/4).^(-3/2);
110 u0 = A+A^2;
111 error_bound_func = @(x) (abs(g_double_prime(x))/2) * (x-u0).^2;
112
113 figure;
114 sgtitle('Problem (e)');
115 for n = 1:3
116     j = N(n);
117     error = zeros(1,M);
118     error_bound = zeros(1,M);
119     for i = 1:M
120         x = normrnd(A,sqrt(A),1,j);
121         u(i) = sum(x.^2)/j;
122         error_bound(i) = error_bound_func(u(i));
123     end
124     subplot(2,3,n);
125     histogram(u, 50, 'Normalization', 'probability');
126     title(['N = ', num2str(j), ', u']);
127     xlabel('u');
128     ylabel('u probability');
129
130     subplot(2,3,n+3);
131     histogram(error_bound, 50, 'Normalization', 'probability');
132     title(['N = ', num2str(j), ', error bound']);
133     xlabel('error bound');
134     ylabel('error bound probability');
135 end

```

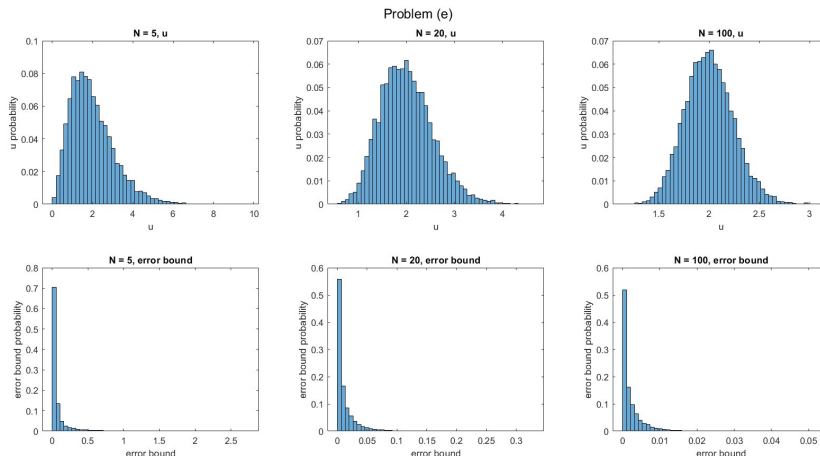
利用 2 阶微分的部分当作 error bound

∵ 使用 2 阶近似

$$\Rightarrow q(u) \approx q(u_0) + \frac{\partial q(u)}{\partial u} \Big|_{u=u_0} (u-u_0) + \frac{\partial^2 q(u)}{\partial u^2} \Big|_{u=u_0} (u-u_0)^2$$

和 1 阶近似的差

画出 error bound 的 histogram



Let $u_0 = E[u] = A + A^2 = 2$
if $N \uparrow \Rightarrow u \rightarrow u_0$

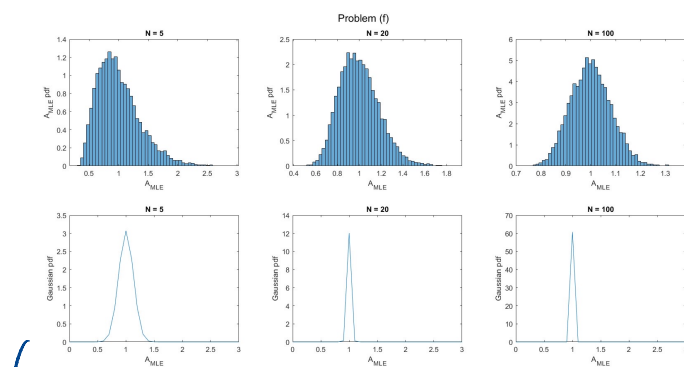
$N \uparrow \Rightarrow$ error bound \downarrow
 \Rightarrow Taylor approximation is more accurate

(f)

```

155 %%
156 A = 1;
157 N = [5, 20, 100];
158 M = 10000;
159 u = zeros(1, M);
160
161 g = @(x) (-1/2)*sqrt(1/4+x);
162 g_prime = @(x) (1/2).*(x+1/4).^(-1/2);
163 u0 = A+A^2;
164 Taylor_approx = @(x) g(u0)+g_prime(u0)*(x-u0);
165
166 figure;
167 sgtitle('Problem (f)');
168 for n = 1:3
169     j = N(n);
170     Taylor = zeros(1, M);
171     for i = 1:M
172         x = normrnd(A, sqrt(A), 1, j);
173         u(i) = sum(x.^2)/j;
174         Taylor(i) = Taylor_approx(u(i));
175     end
176
177     x_Gaussian = 0:0.1:3;
178     y_Gaussian = normpdf(x_Gaussian, mean(Taylor), var(Taylor));
179
180     subplot(2,3,n);
181     histogram(Taylor, 50, 'Normalization', 'pdf');
182     title(['N = ', num2str(j)]);
183     xlabel('A_MLE');
184     ylabel('A_MLE pdf');
185
186     subplot(2,3,n+3);
187     plot(x_Gaussian, y_Gaussian);
188     title(['N = ', num2str(j)]);
189     xlabel('A_MLE');
190     ylabel('Gaussian pdf');
191 end

```



底下三張圖是以上面三張圖求出的 \hat{A}_{MLE} 的 mean & var, 畫出的 Gaussian pdf 分布

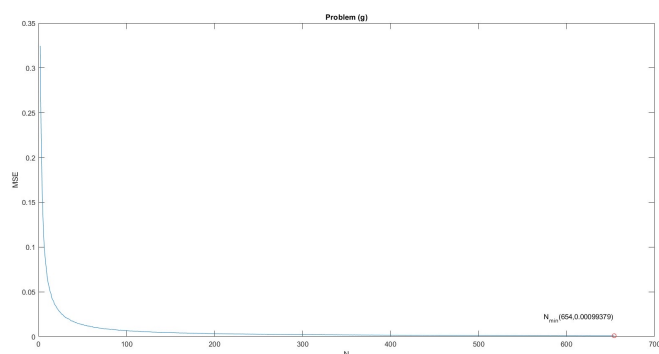
- < Discussion >
1. $N \uparrow$, 趨近似 Gaussian distribution
 2. $N \uparrow$, $\text{var}\{\hat{A}_{MLE}\} \downarrow$
 3. $N \uparrow$, $E\{\hat{A}_{MLE}\} \rightarrow A = 1$

(g)

```

193 %%
194 A = 1;
195 M = 10000;
196 N = 2;
197
198 g = @(x) (-1/2)*sqrt(1/4+x);
199 g_prime = @(x) (1/2).*(x+1/4).^(-1/2);
200 u0 = A+A^2;
201 Taylor_approx = @(x) g(u0)+g_prime(u0)*(x-u0);
202
203 seed = 101;
204 rng(seed);
205
206 MSE = zeros(1, 1000);
207
208 while 1
209     Taylor = zeros(1, M);
210     u = zeros(1, M);
211
212     for i = 1:M
213         x = normrnd(A, sqrt(A), 1, N);
214         u(i) = sum(x.^2)/N;
215         Taylor(i) = Taylor_approx(u(i));
216     end
217
218     MSE(N-1) = mean((Taylor-A).^2);
219
220     if MSE(N-1) < 1e-3
221         break
222     end
223
224     N = N+1;
225 end
226
227 figure;
228 plot(2:N, MSE(1:N-1));
229 hold on;
230 plot(N, MSE(N-1), 'ro');
231 text(N-80, MSE(N-1)+0.02, ['N_MSE(', num2str(N), ', ', num2str(MSE(N-1)), ')']);
232 xlabel('N');
233 ylabel('MSE');
234 title('Problem (g)');

```



使用 Taylor approximation 求出的 N_{mse} 和 Problem (d) 求出的相同