In the DC level in WGN problem, let $A \sim u[-A_0, A_0]$ and the noise $w[n] \sim N(0, \sigma^2)$. We would like to obtain Bayesian estimators using different risk functions.

1. (5 points)

   Derive the posterior probability $p(A|\mathrm{x})$.

   Now, let $A_0 = 2$ and $\sigma^2 = 0.1$.

(i) $$P(x[n]|A) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2\sigma^2}(x[n]-A)^2}$$

$$\Rightarrow P(A|\underline{x}) = \frac{P(\underline{x}|A)P(A)}{\int P(\underline{x}|A)P(A)\,dA}$$

$$= \frac{\frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2} \cdot \frac{1}{2A_0}}{\underbrace{\int_{-A_0}^{A_0} \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{n=0}^{N-1}(x[n]-A)^2} \cdot \frac{1}{2A_0}\,dA}_{c''}}, \quad |A| \le A_0$$

$$= \frac{1}{c'} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2} \cdot \frac{1}{c''}$$

$$= \frac{1}{c} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2}, \quad c = \int_{-A_0}^{A_0} \frac{1}{\sqrt{2\pi\frac{\sigma^2}{N}}} e^{-\frac{1}{2\frac{\sigma^2}{N}}(A-\bar{x})^2}\,dA$$

2. (10 points)

   Let the number of observations x at each realization of A be $N = 10$. Randomly generate one realization of A. Plot $p(A|\mathrm{x})$, and $\hat{A}_{MMSE}$ (with bold dot in the figure) for 15 different realizations of x

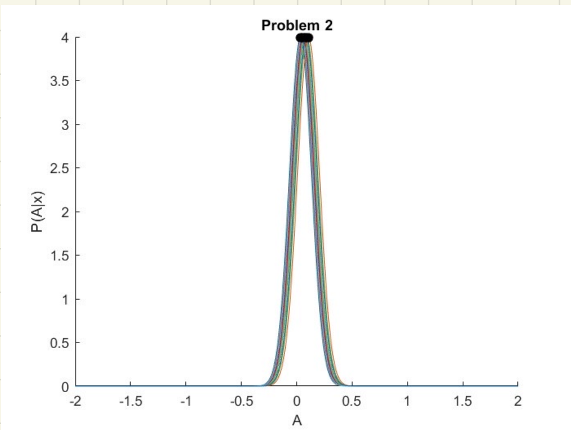
```
clc; clear all; close all;
seed = 101;
rng(seed);

A0 = 2;
A = (2*A0) * rand - A0;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

x = A + sigma2 * randn(N, 15);
x_ = mean(x, 1);

figure;
xlabel('A');
ylabel('P(A|x)');
title('Problem 2');
hold on;

for i = 1:15
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_(i)).^2) );

    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    P_ax_mean = @(a) P_ax_normalized(a) .* a;
    A_mmse = integral(P_ax_mean, -A0, A0);

    fplot(P_ax_normalized, [-A0, A0]);
    plot(A_mmse, P_ax_normalized(A_mmse), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
end
hold off;
```

→ 生成 A

} 生成 15組 $\underline{x}$，求 mean

求 $P(A|\underline{x})$

} 求 $E[A|\underline{x}]$

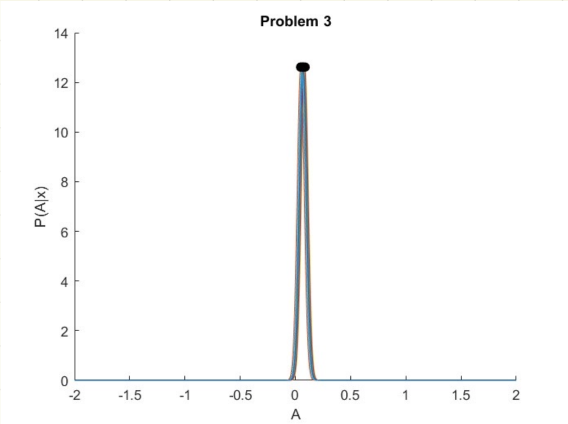


最高點的 A 會是 $\bar{x}$，
理論上 $E[A|\underline{x}] \fallingdotseq \bar{x}$

3. (5 points)

Now let $N = 100$. Repeat (2).

```
N = 100;
sigma2N = sigma2/N;

x = A + sigma2 * randn(N, 15);
x_ = mean(x, 1);

figure;
xlabel('A');
ylabel('P(A|x)');
title('Problem 3');
hold on;

for i = 1:15
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_(i)).^2) );

    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    P_ax_mean = @(a) P_ax_normalized(a) .* a;
    A_mmse = integral(P_ax_mean, -A0, A0);

    fplot(P_ax_normalized, [-A0, A0]);
    plot(A_mmse, P_ax_normalized(A_mmse), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
end
hold off;
```



$N\uparrow$, $\frac{\sigma^2}{N}\downarrow$, shape truncated Gaussian

$\Rightarrow$ Narrow Gaussian

$\Rightarrow$ Truncation effect $\downarrow$

$\Rightarrow$ The $P(A|\underline{x})$ more like the original Gaussian pdf

$\Rightarrow E_A\{A|\underline{x}\} \triangleq \hat{A}_{MMSE} \approx \bar{x}$

4. (12 points)

Following (2), now let the realizations of A be $M = 1000$. Obtain the Bayesian MSE $B_{mse}(\hat{A}_{MMSE})$ using Monte Carlo simulations.

```
M = 1000;
A0 = 2;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

A = (2*A0) * rand(1, M) - A0;

A_mmse = zeros(1,M);
for i = 1:M
    x = A(i) + sigma2 * randn(N, 1);
    x_ = mean(x);
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_).^2) );
    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    P_ax_mean = @(a) P_ax_normalized(a) .* a;
    A_mmse(i) = integral(P_ax_mean, -A0, A0);
end
A_mmse_monte = mean(A_mmse);
BMSE = mean((A-A_mmse_monte).^2);
disp(['Problem(4) A_mmse using Monte Carlo: ', num2str(A_mmse_monte)]);
disp(['Problem(4) BMSE using Monte Carlo: ', num2str(BMSE)]);
```

$\rightarrow$ 生成M個A

$P(A|\underline{x})$

$\rightarrow A_{MMSE} = E[P|\underline{x}]$

$\rightarrow \widehat{E[\hat{A}]} = \frac{1}{M}\sum_{m=1}^{M}\hat{A}_m$

$\rightarrow \widehat{var[\hat{A}]} = \frac{1}{M}\sum_{m=1}^{M}(\hat{A}_m - \widehat{E[\hat{A}]})^2$

```
Problem(4) A_mmse using Monte Carlo: -0.021435
Problem(4) BMSE using Monte Carlo: 1.3199
```

5. (5 points)

Following (2), and let $N = 100$, repeat (4).

```
Problem(5) A_mmse using Monte Carlo: -0.019988
Problem(5) BMSE using Monte Carlo: 1.3199
```

6. (32 points)

Now, the risk function is absolute error. Repeat (2)-(5). Your answer should be as detailed as possible to obtain full credit. Especially "HOW" you obtain $\hat{A}$ either in mathematical representation or via Matlab program.

$$C(\epsilon) = |\epsilon| = |\theta - \hat{\theta}|$$

$$\Rightarrow \int_\theta |\theta - \hat{\theta}|\, p(\theta|\underline{x})\, d\theta$$

$$= \int_{-\infty}^{\hat{\theta}} \underbrace{(\hat{\theta} - \theta)\, p(\theta|\underline{x})}_{h(\hat{\theta},\theta)}\, d\theta + \int_{\hat{\theta}}^{\infty} \underline{(\theta - \hat{\theta})\, p(\theta|\underline{x})}\, d\theta$$

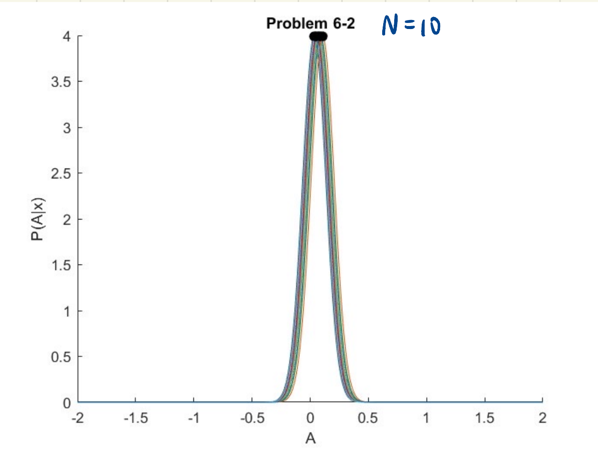
$$\frac{\partial}{\partial \hat{\theta}} = 0$$

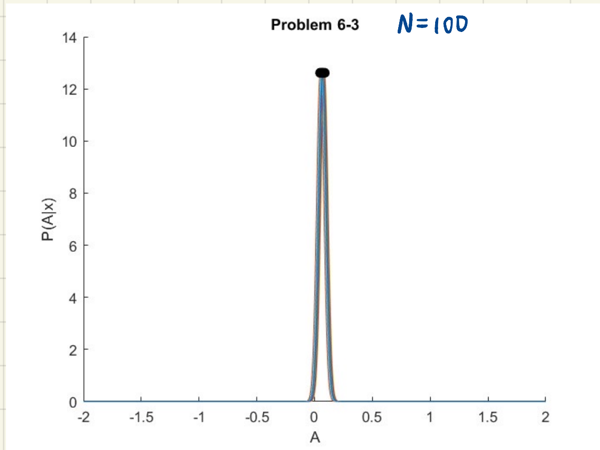
$$\Rightarrow \int_{-\infty}^{\hat{\theta}} p(\theta|\underline{x})\, d\theta - \int_{\hat{\theta}}^{\infty} p(\theta|\underline{x})\, d\theta = 0$$

$$\Rightarrow P(\theta \le \hat{\theta} \mid \underline{x}) = \frac{1}{2} \Rightarrow \hat{\theta} \text{ 是 } \theta|\underline{x} \text{ 的中位數}$$

```matlab
%%
seed = 101;
rng(seed);

A0 = 2;
A = (2*A0) * rand - A0;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

x = A + sigma2 * randn(N, 15);
x_ = mean(x, 1);

figure;
xlabel('A');
ylabel('P(A|x)');
title('Problem 6-2');
hold on;

for i = 1:15
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_(i)).^2) );

    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    CDF = @(a) integral(P_ax_normalized, -A0, a);
    median_A = fzero(@(a) CDF(a) - 0.5, 0);

    fplot(P_ax_normalized, [-A0, A0]);
    plot(median_A, P_ax_normalized(median_A), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
end
hold off;
```

選中位數、cdf = 0.5 的位置






```matlab
M = 1000;
A0 = 2;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

A = (2*A0) * rand(1, M) - A0;

median_A = zeros(1,M);
for i = 1:M
    x = A(i) + sigma2 * randn(N, 1);
    x_ = mean(x);
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_).^2) );
    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    CDF = @(a) integral(P_ax_normalized, -A0, a);
    median_A(i) = fzero(@(a) CDF(a) - 0.5, 0);
end
A_hat = mean(median_A);
disp(['Problem(6-4) A_hat: ', num2str(A_hat)]);
```

$$E[\hat{A}] = \frac{1}{M}\sum_{m=1}^{M} \hat{A}_m$$

```
Problem(6-4) A_hat: -0.021537
Problem(6-5) A_hat: -0.020008
```

7. (32 points)

Now, the risk function is Hit-or-risk. Repeat (2)-(5). Your answer should be as detailed as possible to obtain full credit. Especially "HOW" you obtain $\hat{A}$ either in mathematical representation or via Matlab program.

$$C(\epsilon) = \begin{cases} 0, & |\epsilon| < \delta \\ 1, & |\epsilon| > \delta \end{cases}$$

$$\Rightarrow |\theta - \hat{\theta}| > \delta \Rightarrow \theta - \hat{\theta} > \delta \text{ or } \hat{\theta} - \theta > \delta$$

$$\Rightarrow \theta > \hat{\theta} + \delta \text{ or } \theta < \hat{\theta} - \delta$$

$$\Rightarrow g(\hat{\theta}) = \int_\theta C(\epsilon) P(\theta|\underline{x})\, d\theta$$
$$= \int_{\hat{\theta}+\delta}^{\infty} 1 \cdot P(\theta|\underline{x})\, d\theta + \int_{-\infty}^{\hat{\theta}-\delta} 1 \cdot P(\theta|\underline{x})\, d\theta$$
$$= 1 - \int_{\hat{\theta}-\delta}^{\hat{\theta}+\delta} P(\theta|\underline{x})\, d\theta$$
$$\Rightarrow \min.\, g(\hat{\theta}) = \underline{\max.\, \int_{\hat{\theta}-\delta}^{\hat{\theta}+\delta} P(\theta|\underline{x})\, d\theta}$$

If $P(\theta|\underline{x})$ is Gaussian

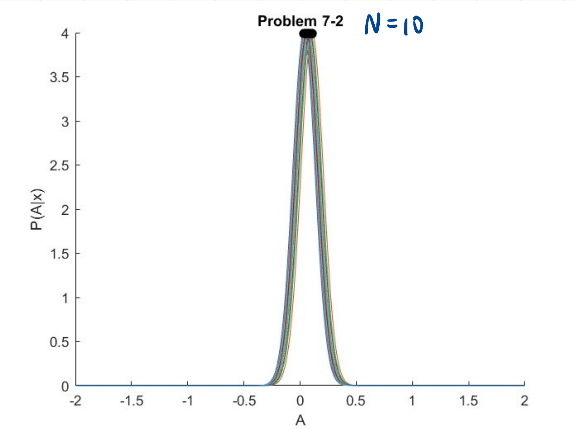
For general $P(\theta|\underline{x})$, if $\delta \to 0$

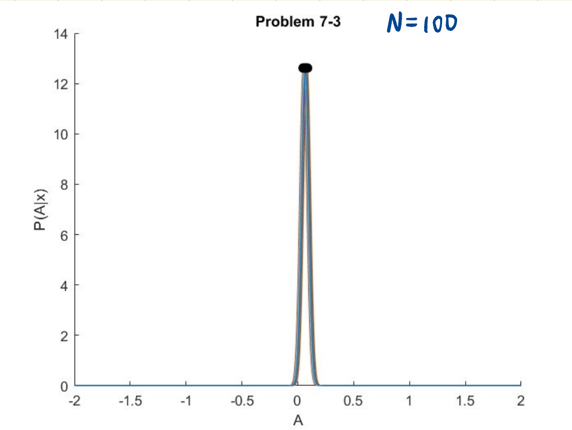
$\Rightarrow$ 取 $P(\theta|\underline{x})$ 最大時的 θ $\Rightarrow$ MAP

$\hat{\theta}$ is the peak of $P(\theta|\underline{x})$

```
%%
seed = 101;
rng(seed);

A0 = 2;
A = (2*A0) * rand - A0;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

x = A + sigma2 * randn(N, 15);
x_ = mean(x, 1);

figure;
xlabel('A');
ylabel('P(A|x)');
title('Problem 7-2');
hold on;

for i = 1:15
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_(i)).^2) );

    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;

    [max_a, max_val] = fminbnd(@(a) -P_ax_normalized(a), -A0, A0);

    fplot(P_ax_normalized, [-A0, A0]);
    plot(max_a, P_ax_normalized(max_a), 'ko', 'MarkerFaceColor', 'k', 'MarkerSize', 5, 'LineWidth', 2);
end
hold off;
```

選 max. $P(A|\underline{x})$






```
M = 1000;
A0 = 2;
sigma2 = 0.1;
N = 10;
sigma2N = sigma2/N;

A = (2*A0) * rand(1, M) - A0;

MAP_A = zeros(1,M);
for i = 1:M
    x = A(i) + sigma2 * randn(N, 1);
    x_ = mean(x);
    P_ax = @(a) (a >= -A0 & a <= A0) .* ( 1 / sqrt(2*pi*sigma2N) ) .* exp( (-0.5/sigma2N) * ((a-x_).^2) );
    c = integral(P_ax, -A0, A0);
    P_ax_normalized = @(a) P_ax(a) / c;
    [max_a, max_val] = fminbnd(@(a) -P_ax_normalized(a), -A0, A0);
    MAP_A(i) = max_a;
end
A_hat = mean(MAP_A);
disp(['Problem(7-4) A_hat: ', num2str(A_hat)]);
```

$$\widehat{E[\hat{A}]} = \frac{1}{M} \sum_{m=1}^{M} \hat{A}_m$$

```
Problem(7-4) A_hat: -0.021852
Problem(7-5) A_hat: 0.28762
```