University of Minnesota
ECON 8702
Problem Set 1
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Extended version of Bianchi (2011)

This file presents an extended version of Bianchi (2011) allowing for production. Households solve the following problem:

$$\max_{c_t, n_t, b_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[U(c_t) - v(n_t)\right]$$

$$st: \quad b_{t+1} + c_t^T + p_t^N c_t^N = y_t^T + b_t(1+r) + n_t w_t + \pi_t \qquad (\lambda_t)$$

$$b_{t+1} \geq -\kappa\left(y_t^T + w_t n_t + \pi_t\right) \qquad (\mu_t)$$

where

$$c_t = \left[\omega\left(c_t^T\right)^{-\eta} + (1-\omega)\left(c_t^N\right)^{-\eta}\right]^{-\frac{1}{\eta}}, \quad \eta > -1, \quad \omega \in (0,1)$$

Firms solve:

$$\pi = \max_h h^{\alpha} - wh, \qquad \alpha \in (0,1)$$

and the market clearing conditions are simply: $h_t = n_t$, and $y_t^N = h_t^{\alpha}$.

Theoretical Results

Taking the FOC for the household's problem we have:

$$(c_t^T): \quad \underbrace{U'(c_t) \times \frac{-1}{\eta}\left[\omega(c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}-1} \times -\eta\omega(c_t^T)^{-\eta-1}}_{U_T(t)} = \lambda_t \tag{1}$$

$$(c_t^N): \quad U'(c_t) \times \frac{-1}{\eta}\left[\omega(c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}-1} \times -\eta(1-\omega)(c_t^N)^{-\eta-1} = \lambda_t p_t^N \tag{2}$$

Dividing equations (1) and (2):

$$p_t^N = \frac{1-\omega}{\omega}\left(\frac{c_t^T}{c_t^N}\right)^{\eta+1} \tag{3}$$

The other FOCs from the household:

$$(n_t): \quad v'(n_t) = w_t(\lambda_t + \mu_t \kappa) \tag{4}$$

$$(b_{t+1}): \quad \lambda_t = \mathbb{E}_t \lambda_{t+1} \beta(1+r) + \mu_t \tag{5}$$

The Euler equation in the decentralized equilibrium is thus given by:

$$U_T(t) = \beta(1+r)\mathbb{E}_t U_T(t+1) \tag{6}$$

We now turn to the firm's problem:

$$(h_t): \quad \alpha p_t^N h_t^{\alpha-1} = w_t \tag{7}$$

We can thus write the firm's profits such as:

$$\pi_t = (1-\alpha) p_t^N h_t^\alpha \tag{8}$$

$$w_t h_t + \pi_t = p_t^N h_t^\alpha$$

Using the equation (3):

$$w_t h_t + \pi_t = \frac{1-\omega}{\omega}\left(\frac{c_t^T}{h_t^\alpha}\right)^{\eta+1} h_t^\alpha$$

We rewrite the household's budget constraint:

$$b_{t+1} + c_t^T = y_t^T + (1+r)b_t \tag{9}$$

We can also rewrite the credit constraint such as:

$$b_{t+1} \geq -\kappa\left[y_t^T + \frac{1-\omega}{\omega}(c_t^T)^{\eta+1} h_t^{-\alpha\eta}\right] \tag{10}$$

Note that in Bianchi (2011) the total endowment of the economy $y_t = (y_t^T, y_t^N)$ is stochastic. Here, the total amount of non-tradable goods available is determined by the labor supply/demand. Since the firm's problem is deterministic this economy is not facing *domestic* uncertainty directly. All the uncertainty in this version of the model comes from the shocks on the tradable endowment. By the assumption of the small open economy, the tradable goods (as the interest rates) are determined from abroad.

The recursive formulation of the planner's problem will be thus given by:

$$V(b,y) = \max_{b',h} U(c) - v(h) + \beta \mathbb{E}_{y'|y}[V(b',y')]$$

$$st: \qquad (9) \text{ and } (10)$$

Taking the planner's FOCs:

$$(c^T): \quad \lambda_t^{sp} = U_T(t) + \mu_t^{sp} \underbrace{\kappa \frac{1-\omega}{\omega}(\eta+1)\left(\frac{c^T}{h^\alpha}\right)^\eta}_{\Psi}$$

$$(h): \quad v'(h) = U'(c)\frac{dc}{dc^N}\alpha h^{\alpha-1} + \underbrace{\mu_t^{sp}\kappa\frac{1-\omega}{\omega}(-\alpha\eta)\left(c^T\right)^{\eta+1} h^{-\alpha\eta-1}}_{=0(\text{ since the borrowing constraint does not bind at } t)}$$

Using the functional form for the consumption aggregator and computing its derivative with respect to c^N:

$$v'(h) = U'(c_t) \times \frac{-1}{\eta}\left[\omega(c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}-1} \times (-\eta)(1-\omega)(c_t^N)^{-\eta-1} \times \alpha h^{\alpha-1} \tag{11}$$

$$(b'): \quad \lambda_t^{sp} = \beta(1+r)\mathbb{E}_{y'|y}[\lambda_{t+1}^{sp}] + \mu_t^{sp}$$

From (c^T):

$$\lambda_t^{sp} = U_T(t) + \mu_t^{sp}\Psi_t \quad \text{and} \quad \lambda_{t+1}^{sp} = U_T(t+1) + \mu_{t+1}^{sp}\Psi_{t+1}$$

Plug the above equation on (b') to get the following Euler equation for the planner:

$$U_T(t) = \beta(1+r) \times \mathbb{E}_t[U_T(t+1) + \mu_{t+1}^{sp}\Psi_{t+1}] + \mu_t^{sp}(1-\Psi_t) \tag{12}$$

Just as in the paper when the credit constraint binds, agents undervalue wealth since they don't internalize the effects of their consumption of tradable goods in the price of non-tradable and how this relaxes their borrowing constraints. Computing the Euler equation in a TDCE we get that:

$$U_T(t) = \beta(1+r)(1+\tau_t)\mathbb{E}_t[U_T(t+1)] + \mu_t^{sp}$$

Since the tax τ_t induces the same allocation between the two solutions:

$$\tau_t = \frac{\mathbb{E}_t[\mu_{t+1}^{sp}\Psi_{t+1}]}{\mathbb{E}_t[U_T(t+1)]} - \underbrace{\frac{\mu_t^{sp}\Psi_t}{\beta(1+r)\mathbb{E}_t[U_T(t+1)]}}_{=0}$$

For the labor supply decision, in the decentralized equilibrium, the household understands that increments in the labor supply relax the budget constraint and the borrowing constraint according to the wages. When solving the planner's problem we impose that the non-tradable consumption clears the market. Market clearing together with firm optimality implies that the household budget constraint remains unchanged by labor supply decisions. On the other hand, by changing equilibrium wages, labor supply decisions change the borrowing constraint, which is not internalized by the households in the decentralized equilibrium.

In a TDCE:

$$v'(h) = \lambda_t(1-\tilde{\tau}_t)w_t + \underbrace{\kappa\mu_t(1-\tilde{\tau}_t)w_t}_{=0 \text{ (since } \mu_t = 0)} = \lambda_t(1-\tilde{\tau}_t)\alpha p_t^N h_t^{\alpha-1}$$

Using the FOC with respect to non-tradable goods:

$$v'(h) = U'(c_t) \times \frac{-1}{\eta}\left[\omega(c_t^T)^{-\eta} + (1-\omega)(c_t^N)^{-\eta}\right]^{-\frac{1}{\eta}-1} \times (-\eta)(1-\omega)(c_t^N)^{-\eta-1} \times \alpha h^{\alpha-1}(1-\tilde{\tau}_t) \quad (13)$$

Let $\tilde{\tau}_t$ be such that it equates the allocation chosen by the planner and the one obtained in a TDCE. We can thus compare (13) and (11) to get:

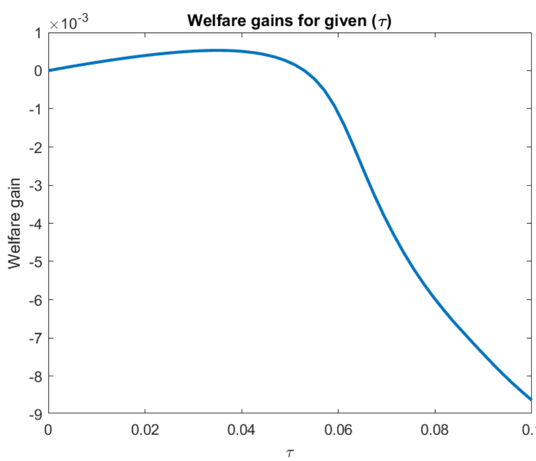
$$\tilde{\tau}_t = 0$$

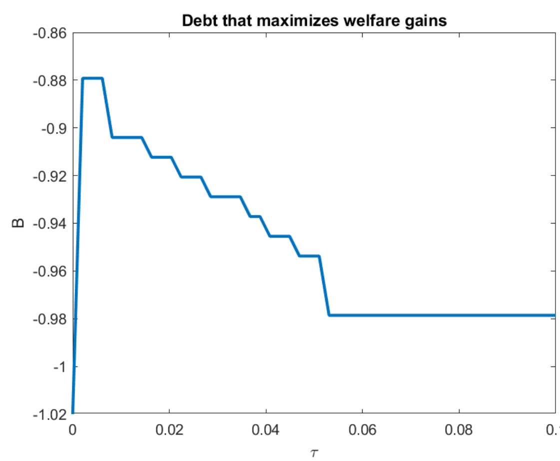
Fixed tax experiment in Bianchi (2011)

This section evaluates the effects of a constant tax rate policy[1]. We started by setting a grid for τ values: $\mathcal{T} = \{\tau_1, \tau_2, ..., \tau_M\}$. This will modify the computation of the decentralized equilibrium (section 3 in the code). We added the taxes and the rebate to compute the TDCE. We computed the welfare gains of $\tau = \overline{\tau} > 0$ when compared to the $\tau = 0$ case, in the same way, Bianchi (2011) compares the DE and the SP welfare.

$$[1 + \gamma(b, \mathbf{y})]^{1-\sigma} V(b, \mathbf{y}; \tau = 0) = V(b, \mathbf{y}; \tau = \overline{\tau})$$

Using the same approach as the replication code, we fixed $\mathbf{y} = (y^T, y^N)$ at a specific gridpoint and found which value of bond holdings maximizes the welfare gains[2]. The results exhibit some welfare gains from introducing a fixed tax on borrowing.





The above figures show a small increase in welfare for small values of τ when compared to the decentralized equilibrium without any taxes. Even though the welfare gains are not as high as the case with state-contingent taxes, the fixed tax can still prevent agents from overborrowing. The fixed tax $\overline{\tau} = 3.47\%$ maximizes welfare gains over the values in our grid[3]. In the right panel, we see that the value of debt holdings for which the welfare gains are maximized is decreasing on taxes. The idea is that the closer we are to the maximum debt value (where the borrowing constraint binds) the higher has to be the tax in order to avoid the borrowing constraint to bind. Using the simulated series for the optimal state-contingent and averaging it we get that $\tau^* = \int \tau(b, \mathbf{y}) d\Pi = 5.15\%$, which provides a worse

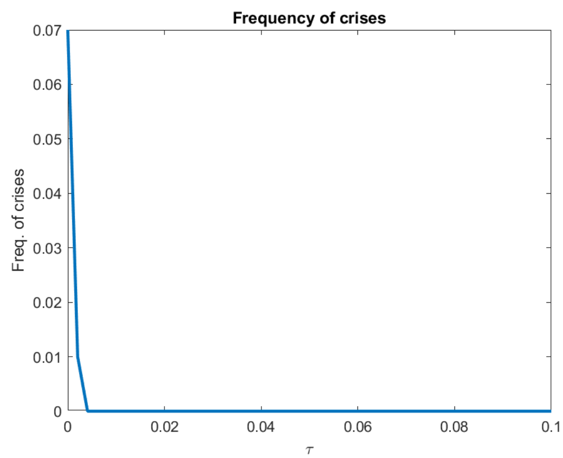
[1] The MATLAB code used can be found here.

[2] Basically, we generated Figure (5) in Bianchi (2011) for each $\tau \in \mathcal{T}$ and report the maximum values and its correspondent bond holdings.

[3] We tried $M = 50$ gridpoints from $[0, 0.1]$.

result than the one that maximizes welfare gains. The idea is that $\overline{\tau}$ is already enough to reduce drastically the frequency of crises (as we show later). Therefore, increasing taxes from that level would just distort equilibrium allocations instead of correcting the externality.

Next, we compute the crisis events as suggested and plot the frequency of crises for each value of tax.



Note that low values of tax are not enough to prevent overborrowing, and thus the frequency of crises for low taxes is higher. However, a pretty small tax is already enough considerably reduce the frequency of crises.

	Freq. of crises	Avg. CA in a crisis event
Decentralized Equilibrium ($\tau = 0.00\%$)	0.07%	10.26%
Decentralized Equilibrium ($\tau = 3.47\%$)	0.00%	-
Social Planner Problem	0.00%	-

In our result, the fixed tax is already enough to avoid crises. The problem is that it ends up being too restrictive to borrow in periods for which the level of debt is not that high.

Stochastic interest rate model

The household's problem is to choose stochastic processes $\left\{c_t^T, c_t^N, b_{t+1}\right\}_{t\geq 0}$ taking b_0 and $\left\{p_t^N, R_t\right\}_{t\geq 0}$ as given to maximize:

$$\max_{c_t, b_{t+1}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$st: \quad \frac{b_{t+1}}{R_t} + c_t^T + p_t^N c_t^N = y_t^T + b_t + p_t^N y_t^N \qquad (\lambda_t)$$

$$\frac{b_{t+1}}{R_t} \geq -\kappa\left(y_t^T + p_t^N y_t^N\right) \qquad (\mu_t)$$

where

$$c_t = \left[\omega\left(c_t^T\right)^{-\eta} + (1-\omega)\left(c_t^N\right)^{-\eta}\right]^{-\frac{1}{\eta}}, \quad \eta > -1, \quad \omega \in (0,1)$$

The household's first-order conditions require the following:

$$\lambda_t = u_T(t)$$

$$p_t^N = \left(\frac{1-\omega}{\omega}\right)\left(\frac{c_t^T}{c_t^N}\right)^{\eta+1}$$

$$\lambda_t = \beta(1+R_t)\mathbb{E}_t\lambda_{t+1} + \mu_t$$

$$\frac{b_{t+1}}{R_t} + \left(\kappa^N p_t^N y_t^N + \kappa^T y_t^T\right) \geq 0, \quad \text{with equality if } \mu_t > 0$$

The results seem similar to those derived in Exercise 1, but now the state variables are $\left(b, y^T, R_t\right)$. When making a decision on b_{t+1}, the agent not only takes into account the interest rate to compute the marginal benefit of an additional unit of savings but also computes the expected value of future utility considering the transition matrix for the interest rate. Ie, $\mathbb{E}_t\lambda_{t+1} = \sum_{y_{t+1}^T}\sum_{R_{t+1}'}\pi(R_{t+1})\pi(y^t)\lambda_{t+1}(b', y_{t+1}, R_{t+1})$.
Taking into account this, the FOC characterizing the social planner problem are given by:

$$\lambda_t^{sp} = u_T(t) + \mu_t^{sp}\Psi_t$$

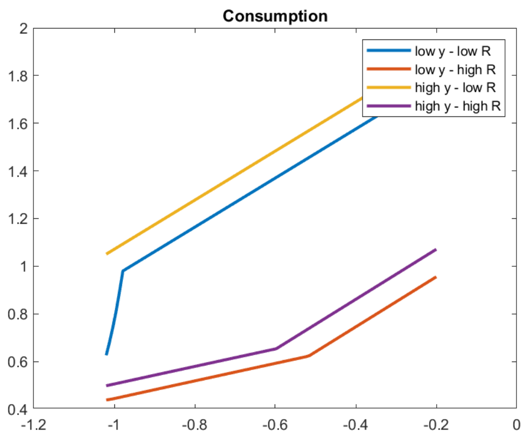
$$\lambda_t^{sp} = \beta(1+R_t)\mathbb{E}_t\lambda_{t+1}^{sp} + \mu_t^{sp}$$

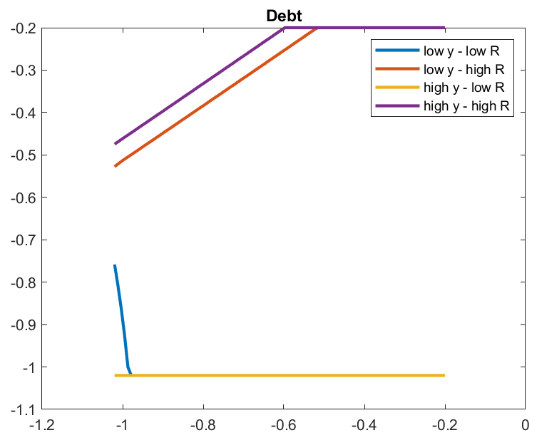
$$\frac{b_{t+1}}{R_t} + \left(\kappa^N \frac{1-\omega}{\omega}\left(\frac{c^T}{y_t^N}\right)^{\eta+1} y_t^N + \kappa^T y_t^T\right) \geq 0, \quad \text{with equality if } \mu_t^{sp} > 0$$

Where $\Psi_t = \kappa\frac{1-\omega}{\omega}(\eta+1)\left(\frac{c^T}{y_N^\alpha}\right)^{\eta}$. The $\tau(b, y_t, R_t)$ taxes on debt that would decentralize the planner's problem when the borrowing constraint is not currently binding are given by:

$$\tau_t = \frac{\mathbb{E}_t[\mu_{t+1}^{sp}\Psi_{t+1}]}{\mathbb{E}_t[U_T(t+1)]} - \underbrace{\frac{\mu_t^{sp}\Psi_t}{\beta(1+r)\mathbb{E}_t[U_T(t+1)]}}_{=0}$$

To solve the model we looped over interest rates and endowments, with the constraints as stated above. The results for consumption and debt in the DE are the following:





References

J. Bianchi. Overborrowing and systemic externalities in the business cycle. *American Economic Review*, 101(7):3400–3426, 2011.