Numerical Examples: The Karplus-Strong Algorithm (Python)

Let's explore the Python code implementing the equation

$$y[n] = \alpha y[n-M] + x[n],$$

where $x[n]$ is the input signal, M is the delay, and α is the decay. We assume that the value of the delay M is equal to the length of the input signal x. In words, it is like filling the delay buffer and then recursively going over it. Additionally, we set the length of the output to be a multiple of the delay M (i.e., of the length of the input signal). This is controlled by the argument D, so that the output length is $D * M$.

Direct implementation

A simple and intuitive approach is to use a for loop

In [1]:
```
def ks_loop(x, alpha, D) :
    import numpy as np
    '''
    Length of the output signal must be larger than the length of the in
put signal,
    that is, D must be larger than 1
    '''
    if D < 1:
        print('Duration D must be greater than 1')

    # Make sure the input is a row-vector
    if x.ndim != 1:
        print('The array entered is of the wrong size')
        return None

    # Number of input samples
    M = len(x)
```

```
    # N umber of output samples
    size_y = D*M

    # Initialize with random input x
    y = np.zeros((size_y,1))
    for i in range(M):
        y[i] = x[i]

    for index in range(M,size_y):
        y[index] = float(alpha * y[index - M])

    return y
```

We can run the algorithm with the following input data: x randomly generated,

In [2]:
```
import numpy as np
x = np.random.randn(100)
```

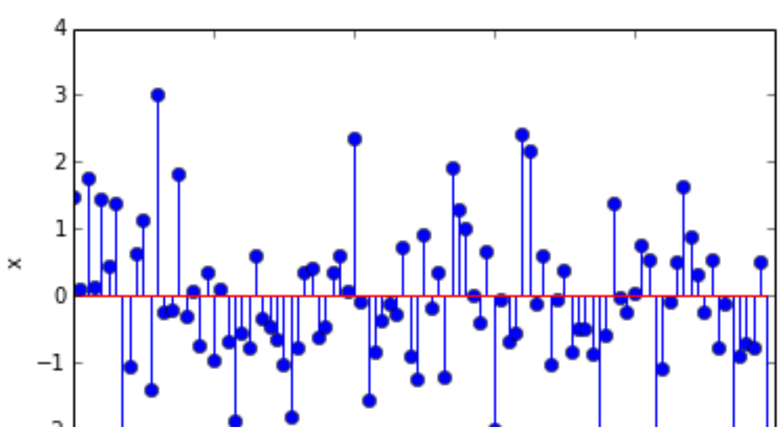
(a noise burst of 100 samples), $\alpha = 0.9$, and $D = 10$, i.e., an output signal of $10 \times 100 = 1000$ samples. Note that this is good for visualization, but it will not sound good.

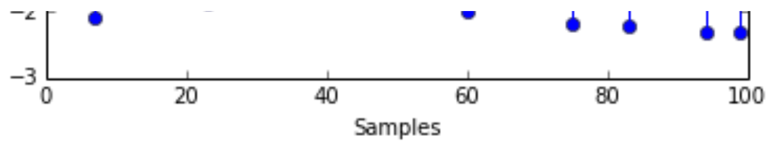
In [3]:
```
y = ks_loop(x, 0.9, 10)
```

The input looks like (plotted with `pylab.stem`)

In [4]:
```
%pylab inline
stem(np.arange(x.size),x,)
xlabel('Samples')
ylabel('x')
show()
```

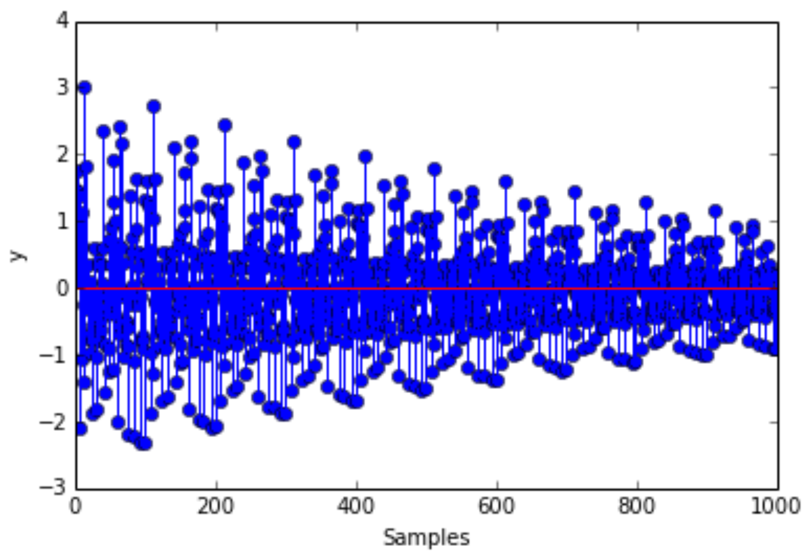
```
Populating the interactive namespace from numpy and matplotlib
```





And the output is

In [5]:
```
stem(np.arange(y.size),y)
xlabel('Samples')
ylabel('y')
show()
```



Matrix implementation

Alternatively, the equation can be implemented using only matrix operations, thus optimizing the execution time. Given the total output length $N = M \times D$, we create the following vectors and matrices. Their respective sizes are indicated in subscript.

$$alphaVector = \begin{pmatrix} 1 \\ \alpha \\ \vdots \\ \alpha^{D-1} \end{pmatrix}_{D\times 1}$$

$$alphaMatrix = \begin{pmatrix} 1 & \dots & 1 \\ \alpha & \cdots & \alpha \\ \vdots & & \vdots \\ \alpha^{D-1} & \cdots & \alpha^{D-1} \end{pmatrix}_{D\times M}$$

$$xMatrix = \begin{pmatrix} x[1] & \cdots & x[M] \\ \vdots & & \vdots \\ x[1] & \cdots & x[M] \end{pmatrix}_{D\times M}$$

$$yMatrix = alphaMatrix * xMatrix = \begin{pmatrix} x[1] & \dots & x[M] \\ \alpha x[1] & \cdots & \alpha x[m] \\ \vdots & & \vdots \\ \alpha^{D-1}x[1] & \cdots & \alpha^{D-1}x[M] \end{pmatrix}_{D\times M}$$

In [6]:

```
def ks(x, alpha, D) :
    import numpy as np
    #   Length of the output signal must be larger than the length of th
e input signal,
    #   that is, D must be larger than 1
    if D < 1:
        print('Duration D must be greater than 1')

    #   Make sure the input is a row-vector
    if x.ndim != 1:
        print('The array entered is of the wrong size')
        return None
    #   Number of input samples
    M = len(x)

    #   Create a vector of the powers of alpha, [alpha^0 alpha^1 ....]
    a = np.ones((1,D)) * alpha
    b = np.arange(D)
    alphaVector = pow(a,b)

    #Create a matrix with M columns, each being the vector of the powers
of alpha
    alphaMatrix = np.eye(D,M)
    for index in range(M):
        alphaMatrix[:,index] = alphaVector

    #Create a matrix with D rows filled by the input signal x
    xMatrix = np.tile(x,(D,1))

    #Multipliy the two, so we can read it out
    #column-by-column
    yMatrix = alphaMatrix * xMatrix

    #Read out the output column by columnn
```

```
    y = yMatrix.flatten()

    return y
```

Of course, the result is the same, but the matrix based algorithm runs much faster.

An example

Let's now play the guitar! We write a Python code that plays the opening chord of Hard day's night (http://en.wikipedia.org/wiki/A_Hard_Day%27s_Night_%28song%29#Opening_chord), a famous song by The Beatles.

In the Western system of music, A4 or middle A serves as reference with a frequency at 440Hz. The frequency of all other notes can be computed from it using the formula $F_0 = 440 \times 2^{n/12}$ where n is the number of half-tones between A4 and the desired note. The exponent n is positive if the note is above middle A, and negative otherwise.

The chord is composed of the following notes: D3, F3, G3, F4, A4, C5, G5. To give it a "wider" feeling we added another D2 below. Each note is generated using a separate Karplus-Strong algorithm. The resulting signals are added to form the chord. The original chord was played with 2 guitars (one of them 12-string and a bass guitar). We try to imitate differences in level by assigning a different gain to each note. Also, we sustain Paul's D note on the bass a bit longer by changing the corresponding decay factor.

To play a signal, we use the Python function `scipy.io.wavfile.write()`, and Audacity, which can take as arguments the signal to be played and the sampling frequency F_s.

We obtain the following implementation

In [7]:
```
# Parameters:
#
# - Fs       : sampling frequency
# - F0       : frequency of the notes forming chord
# - gain     : gains of individual notes in the chord
# - duration : duration of the chord in second
# - alpha    : attenuation in KS algorithm

Fs = 48000

import numpy as np
# D2, D3, F3, G3, F4, A4, C5, G5
F0 = 440*np.array([pow(2,(-31./12.)), pow(2,(-19./12.)), pow(2,(-16./12.
)), pow(2,(-14./12.)), pow(2,(-4./12.)), 1, pow(2,(3./12.)), pow(2,(10./
12.))])
gain = np.array([1.2, 3.0, 1.0, 2.2, 1.0, 1.0, 1.0, 3.5])
```

```
duration = 4
alpha = 0.9785

# Number of samples in the chord
nbsample_chord = Fs * duration

# This is used to correct alpha later, so that all the notes decay toget
her
# (with the same decay rate)
first_duration = np.ceil(float(nbsample_chord)/round(float(Fs)/float(F0[
0])))

# Initialization
chord = np.zeros(nbsample_chord)

for i in range(len(F0)):

    # Get M and duration parameter
    current_M = round(float(Fs)/float(F0[i]));
    current_duration = np.ceil(float(nbsample_chord)/float(current_M))

    # Correct current alpha so that all the notes decay together (with t
he
    # same decay rate)
    current_alpha = pow(alpha,(float(first_duration)/float(current_durat
ion)))

    # Let Paul's high D on the bass ring a bit longer
    if i == 1:
        current_alpha = pow(current_alpha,8)

    # Generate input and output of KS algorithm
    x = np.random.rand(current_M)
    y = ks(x, current_alpha, current_duration)
    y = y[0:nbsample_chord]

    # Construct the chord by adding the generated note (with the
    # appropriate gain)
    chord = chord + gain[i] * y

import numpy as np
from scipy.io.wavfile import write
```

```
data = chord
scaled = np.int16(data/np.max(np.abs(data)) * 32767)

write('hard_days.wav', 44100, scaled)
```

Play the chord using a 48000 Hz sampling frequency, hear how it sounds!

In [8]:
```
import Audio
Audio.Audio(data=data, rate=48000, embed=True)
```

Out[8]: 00:00 -00:04

Created Fri 11 Apr 20148:57 AM EDT

Last Modified Tue 20 Jan 2015 5:06 AM EST