The Lamperti transform

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Consider the stochastic differential equation (SDE):

$$dX_t = a(t, X_t)\, dt + b(X_t)\, dW_t \tag{1}$$

where the diffusion coefficient $b(X_t)$ depends only on the state variable. The **Lamperti transform** is defined as follows:

$$Y_t = F(X_t) = \int_{x'}^{X_t} \frac{1}{b(u)}\, du \tag{2}$$

where x' is any value of the state variable X in its state space. We assume that the function $F(\cdot)$ defines a one-to-one mapping from the state space of X_t to $\mathbb{R}$.

The SDE solved by the process Y_t can be formulated by applying the Itô formula to the function:

$$f(t, x) = \int_{x'}^{x} \frac{1}{b(u)} du$$

We calculate the derivatives of $f(t, x)$ that are needed in the Itô formula:

$$\begin{cases} f_t(t, x) = \dfrac{\partial}{\partial t} f(t, x) = 0 \\ f_x(t, x) = \dfrac{\partial}{\partial x} f(t, x) = \dfrac{1}{b(x)} \\ f_{xx}(t, x) = \dfrac{\partial^2}{\partial x^2} f(t, x) = -\dfrac{\dfrac{d}{dx} b(x)}{b^2(x)} = -\dfrac{b_x(x)}{b^2(x)} \end{cases} \tag{3}$$

The Itô formula tells us that:

$$
\begin{aligned}
df(t,X_t) &= f_t(t,X_t)dt + f_x(t,X_t)dX_t + \frac{1}{2}f_{xx}(t,X_t)(dX_t)^2 \\
&= \frac{a(t,X_t)dt + b(X_t)dW_t}{b(X_t)} - \frac{1}{2}\frac{b_x(X_t)}{b^2(X_t)}(a(t,X_t)dt + b(X_t)dW_t)^2 \\
&= \left[\frac{a(t,X_t)}{b(X_t)} - \frac{1}{2}b_x(X_t)\right]dt + dW_t
\end{aligned}
\tag{4}
$$

Recall that $Y_t = F(X_t) = f(t,X_t)$ and $X_t = F^{-1}(Y_t)$. Equation 4 can be written:

$$
dY_t = a_Y(t,Y_t))dt + dW_t \tag{5}
$$

where:

$$
a_Y(t,y) = \left[\frac{a(t,F^{-1}(Y_t))}{b(F^{-1}(Y_t))} - \frac{1}{2}b_x(F^{-1}(Y_t))\right]
$$

The Lamperti transform changes the generic SDE Equation 1 into another SDE with a unitary diffusion coefficient, Equation 5.

Example 0.1. Consider the SDE:

$$
dX_t = -\theta_1 X_t\, dt + \theta_2\sqrt{1+X_t^2}\, dW_t
$$

where $\theta_1 + \theta_2^2/2 > 0$. It can be shown that, setting $\nu = 1 + 2\theta_1/\theta_2^2$:

$$
X_t \propto t(\nu)/\sqrt{\nu}
$$

I.e., the stationary distribution of X_t is Student t with ν degrees of freedom. Applying the Lamperti transform:

$$
F(x) = \int_0^x \frac{1}{\theta_2\sqrt{1+u^2}}du = \frac{\text{arcsinh}}{\theta_2}
$$

Using Equation 4 we have:

$$
dF(X_t) = -\left[\frac{\theta_1}{\theta_2} + \frac{\theta_2}{2}\right]\frac{X_t}{\sqrt{1+X_t^2}}dt + dW_t
$$

In terms of the Y_t process we have:

$$dY_t = -\left[\frac{\theta_1}{\theta_2} + \frac{\theta_2}{2}\right]\tanh(\theta_2 Y_t) + dW_t$$