Stochastic integration by parts

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Consider a process $X(t)$ whose increments satisfy the stochastic differential equation $dX(t) = f(t, W(t))\, dW(t)$. Putting this in the integral form:

$$X(t) = X(0) + \int_0^t f(s, W(s))\, dW(s) \tag{1}$$

where $W(t)$ is a Wiener process, whose initial value $W(0)$ is zero with probability one: its mean value and covariance function are

$$\begin{cases} \langle W(t) \rangle = 0 \\ \langle W(t_1) W(t_2) = \min(t_1, t_2) \end{cases}$$

If we consider the partition $\Pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$, then the right-hand side of Equation 1 can be written (Itô stochastic integral):

$$\int_0^t f(s, W(s))\, dW(s) = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(t_i, W(t_i))(W(t_{i+1}) - W(t_i))$$

where the limit is taken when the size of the partition, namely $|\Pi_n| = \max(t_{i+1} - t_i)$ goes to zero. Henceforth, we assume that the partition is uniform, namely $t_{i+1} - t_i = \Delta t, i = 1, 2, \cdots, n$. The convergence of the limit is assumed in the mean-square sense.

Itô lemma

One important result can be obtained by considering the Itô integral:

$$I(T) = \int_0^T (dW(t))^2 = \lim_{n\to\infty} \sum_{i=0}^{n-1} (\Delta W(t_i))^2$$

$I(T)$ is easily seen to be a Gaussian random variable, with mean value T (because of the property of independent increments in a Wiener process):

$$\langle I(T) \rangle = \lim_{n\to\infty} \sum_{i=0}^{n-1} \langle (\Delta W(t_i))^2 \rangle = \lim_{n\to\infty} \sum_{i=0}^{n-1} \Delta t = \lim_{n\to\infty} n(T/n) = T$$

and zero variance:

$$\mathrm{Var}(I(T)) = \lim_{n\to\infty} \sum_{i=0}^{n-1} \langle (\Delta W(t_i))^4 \rangle = \lim_{n\to\infty} 2n(T/n)^2 = 0$$

Therefore, the integral of all the $(dW(t))^2$ is deterministic. Formally we can write this as follows:

$$\boxed{(dW)^2 = dt} \tag{2}$$

The weird property of the Wiener process to have increments ΔW that scale down by $\sqrt{\Delta t}$, forces to get rid of the ordinary chain rule of differential calculus. This has implications also in the way integration by parts can be carried out.

Consider the process $F(t) = f(t)g(W(t))$, and assume that $f(\cdot)$ and $g(\cdot)$ are differentiable. Using the product rule:

$$\begin{aligned} dF(t) &= df(t)g(W(t)) + f(t)dg(W(t)) \\ &= f'(t)g(W(t))dt + f(t)\left[g'(W(t))dW(t) + \frac{1}{2}g''(W(t))(dW(t))^2\right] \\ &= \left[f'(t)g(W(t)) + \frac{1}{2}f(t)g''(W(t))\right]dt + f(t)g'(W(t))dW(t) \end{aligned}$$

Writing the relation in the integral form, we obtain a formula for **stochastic integration by parts**:

$$\boxed{\int_0^t f(s)g'(W(s))dW(s) = f(t)g(W(t))\Big|_0^t - \int_0^t f'(s)g(W(s))ds - \frac{1}{2}\int_0^t f(s)g''(W(s))ds} \tag{3}$$

Let us consider the case when $f(t) = 1$. Equation 3 becomes:

$$\int_0^t g'(W(s))dW(s) = g(W(t))\Big|_0^t - \frac{1}{2}\int_0^t g''(W(s))ds \tag{4}$$

Example 0.1. Calculate the Itô integral:

$$\int_0^t W(s)dW(s)$$

In this case, $f(t) = 1$ and $g(W(t)) = W^2(t)/2$. Equation 4 simplifies to:

$$\int_0^t W(s)dW(s) = \frac{1}{2}W^2(s)\Big|_0^t - \frac{1}{2}\int_0^t ds = \frac{W^2(t) - t}{2}$$

It is noted that the result differs from the one expected by applying the ordinary rule of calculus:

$$\int_0^t WdW = \frac{1}{2}W^2\Big|_0^t = \frac{W^2(t)}{2}$$

Another interesting case is when $g(W(t)) = W(t)$. Equation 3 becomes:

$$\int_0^t f(s)dW(s) = f(t)W(t)\Big|_0^t - \int_0^t f'(s)W(s)ds \tag{5}$$

Example 0.2. Calculate the Itô integral $I(t)$ using the definition in terms of Riemann-Stieltjes sums:

$$I(t) = \int_0^t sdW(s) = \lim_{n\to\infty}\sum_{i=0}^{n-1} t_i \Delta W(t_i)$$

This is a Gaussian random variable, whose mean value and variance are:

$$\begin{cases} \langle I(t)\rangle = \lim_{n\to\infty}\sum_{i=0}^{n-1} t_i\langle\Delta W(t_i)\rangle = 0 \\ \mathrm{Var}(I(t)) = \lim_{n\to\infty}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} t_i t_j\langle\Delta W(t_i)\Delta W(t_j)\rangle = \int_0^t s^2 ds = \frac{t^3}{3} \end{cases} \tag{6}$$

Example 0.3. Calculate the Itô integral $I(t)$ using the formula for stochastic integration by parts:

$$I(t) = \int_0^t s dW(s)$$

I.e., $f(t) = t$ in Equation 5, which simplifies to:

$$I(t) = \int_0^t s dW(s) = sW(s)\Big|_0^t - \int_0^t W(s) ds = tW(t) - \underbrace{\int_0^t W(s) ds}_{\substack{\text{integrated} \\ \text{Wiener process}}}$$

Denote the integrated Wiener process with $Z(t)$:

$$I(t) + Z(t) = tW(t) \tag{7}$$

$Z(t)$ is a Gaussian random variable, whose mean value and variance are:

$$\begin{cases} \langle Z(t) \rangle = \lim_{n\to\infty} \sum_{i=0}^{n-1} \langle W(t_i) \rangle \Delta t = 0 \\ \mathrm{Var}(Z(t)) = \int_0^t du \int_0^t \langle W(u)W(v) \rangle dv = 2\int_0^t du \int_0^u v dv = \int_0^t u^2 du = \frac{t^3}{3} \end{cases} \tag{8}$$

Example 0.4. Calculate the covariance between $I(t) = \int_0^t s dW(s)$ and $Z(t) = \int_0^t W(s) ds$. We know that $I(t) + Z(t) = tW(t)$, see Equation 7:

$$\mathrm{Cov}(I(t) + Z(t), I(t) + Z(t)) = \mathrm{Var}(tW(t)) = t^3$$

On the other hand:

$$\mathrm{Var}(I(t)) + \mathrm{Var}(Z(t)) + 2\mathrm{Cov}(I(t), Z(t)) = t^3$$

Using Equation 6 and Equation 8 we conclude:

$$\mathrm{Cov}(I(t), Z(t)) = \frac{t^3}{6}$$

The processes $I(t)$ and $Z(t)$ are thus not independent.