

Frequency resolution of spectral analysis

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Frequency resolution is the size of the smallest frequency for which details in the frequency response and the spectrum can be resolved by the estimate. For example, a resolution of 0.1 Hz means that the frequency response variations at frequency intervals at or below 0.1 Hz cannot be resolved.

Consider an analog band-limited signal $x(t)$ with bandwidth B Hz; $x(t)$ is observed over a sample period of T_r s and sampled at a sampling frequency of f_s Hz ($T_s = 1/f_s$ denotes the sampling interval). The total number of samples of $x(t)$ is then $N = \lfloor T_r/T_s \rfloor$, where $\lfloor \cdot \rfloor$ denotes the floor function (or greatest integer function), namely the function that takes as input a real number r and gives as output the greatest integer less than or equal to r .

According to the Shannon-Nyquist sampling theorem, if the sampling frequency f_s is chosen to be $2B$, the maximum resolvable frequency is:

$$f_{\max} = \frac{f_s}{2}$$

On the other hand, the minimum resolvable frequency is inversely related to the sample period:

$$f_{\min} = \frac{1}{T_r} = \frac{1}{NT_s} \tag{1}$$

and hence the number of frequencies that can be resolved from f_{\min} to f_{\max} is

$$N_f = \frac{f_{\max} - f_{\min}}{\Delta f}$$

where Δf is the frequency resolution. Since $\Delta f = f_{\min}$, we also have:

$$N_f = \frac{\frac{f_s}{2} - \frac{f_s}{N}}{\frac{f_s}{N}} = \frac{N}{2} - 1$$

This implies that there will be $N/2$ discrete frequencies from 0 to f_{\max} .

The accurate detection of frequency components in the spectrum $X(f)$ of the signal $x(t)$ is challenged by the phenomenon known as **amplitude ambiguity**; it consists of ambiguous and false amplitudes occurring in $X(f)$ whenever the sample period T_r is not an integer multiple of *all* of the contributory periods in $x(t)$. That is, false or ambiguous amplitudes will occur at frequencies that are immediately adjacent to the actual frequency.

For aperiodic signals, T_r theoretically must be infinite. For periodic signals, T_r must be equal to the least common integer multiple of all the periods contained in the signal. When this condition cannot be fulfilled, windowing is the elective method to minimize the effect of amplitude ambiguity (not used herein). Application of the DFT (Discrete Fourier Transform) or FFT (Fast Fourier Transform) to an aperiodic signal implicitly assumes that the signal is infinite in length and formed by repeating the signal of length T_r an infinite number of times. This leads to discontinuities in the amplitude that occur at each integer multiple of T_r . These discontinuities are step-like, which introduce false amplitudes that decrease around the main frequencies.

💡 Can discrete-time sinusoids be non-periodic?

The normalized frequency f in cycles per sample of a **discrete-time sinusoid**:

$$x[n] = \cos(2\pi f n)$$

is restricted to values in the interval $-1/2 \leq f \leq 1/2$. This is because, for any discrete-time sinusoid with frequency $|f| > 1/2$, an integer m exists such that $f = f_0 + m$ and $|f_0| \leq 1/2$:

$$\cos 2\pi f n = \cos[2\pi(f_0 + m)n] = \cos(2\pi f_0 n) + \cos(2\pi m n) = \cos(2\pi f_0 n)$$

It is worth noting that the highest rate of oscillation of discrete-time sinusoids is when, at every time instant, the output sample flips with respect to the previous one:

$$f_0 = \frac{1}{2} \rightarrow \cos(2\pi f_0 n) = \cos(\pi n) = (-1)^n$$

Moreover, a discrete-time sinusoid is periodic if its frequency can be expressed in terms of a rational number; an additive mixture of periodical discrete-time sinusoids (called here **sinusoidal mixture**) is periodic with period equal to the least common integer multiple of their periods.

As outlined above, a spectrum line at frequency f will be accurately represented by the DFT when $f_s > 2f_{\max}$ and $T_r = mT$, where $m = 1, 2, \dots$, and $T = 1/f$. This implies that $N = mf_s/f$. If T_r is not an integer multiple of T , leakage will occur in the DFT. This appears as amplitudes at f spilling onto adjacent frequencies.

As an example, consider a sinusoidal mixture with three unit-amplitude components at 2.875 Hz, 3 Hz, 3.125 Hz. The sampling frequency is set to $f_s = 16$ Hz and the sample period to $T = 8$ s ($N = 128$): the component at 2.875 Hz is observed for 23 periods, the component at 3 Hz for 24 periods, and the component at 3.125 Hz for 25 periods. The resolution calculated according to Equation 1 is 0.125 Hz.

A chunk of code written in MATLAB (Natick, Massachusetts: The MathWorks, Inc., <https://www.mathworks.com>) shows how to simulate the sinusoidal mixture and perform spectral analysis.

```
% *****
% sinusoidal mixture generation
% *****

fs = 16;    % sampling frequency, Hz
Ts = 1/fs;  % sampling interval, s
T = 8;      % sample period, s

t = (0:Ts:T-Ts); % time domain, s
N = length(t);   % number of samples

f1 = 2.9375;      % frequency component 1, Hz
f2 = 3;           % frequency component 2, Hz
f3 = 3.0625;      % frequency component 3, Hz

x1 = cos(t.*(2*pi*f1)); % sinusoid 1
x2 = cos(t.*(2*pi*f2)); % sinusoid 2
x3 = cos(t.*(2*pi*f3)); % sinusoid 3
X = x1 + x2 + x3;      % sinusoidal mixture

% *****
% calculation of single-side spectrum (see comments below)
% *****

P          = fft(X, N);          % FFT calculation
P2         = abs(P/N);          % double-side spectrum, rescaled by N
P1         = P2(1:N/2+1);
P1(:, 2:end-1) = 2*P1(2:end-1);
```

```

Y          = P1(1:end-1);          % single-side spectrum
f          = (0:N/2-1).*(fs/N); % frequency domain, Hz

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i Comments

1. Convert the FFT spectrum P of the signal X to the single-sided amplitude spectrum. Because the FFT function includes a scaling factor N between the original and the transformed signals, rescale P by dividing by N .
2. Take the complex magnitude of the FFT spectrum. The two-sided amplitude spectrum $P2$, where the spectrum in the positive frequencies is the complex conjugate of the spectrum in the negative frequencies, has half the peak amplitudes of the time-domain signal.
3. To convert to the single-sided spectrum, take the first half of the two-sided spectrum $P2$. Multiply the spectrum in the positive frequencies by 2. $P1(1)$ and $P1(\text{end})$ are not multiplied by 2 because these amplitudes correspond to the zero and Nyquist frequencies, respectively, and they do not have the complex conjugate pairs in the negative frequencies.
4. Define the frequency domain f for the single-side spectrum Y .

The single-sided spectrum of the sinusoidal mixture is shown in Figure 1.

Consider now the case that the three components of the sinusoidal mixture have frequencies 2.9375 Hz, 3 Hz, 3.0625 Hz. To resolve them, we would need to double the frequency resolution of the FFT, with respect to the scenario of Figure 1. Erroneously, I may try to double the sampling frequency, Figure 2: this expedient leaves unaltered the frequency resolution.

To resolve the components of the sinusoidal mixture and avoid amplitude ambiguity, the frequency resolution of 0.0625 Hz can be achieved by extending the time period to $T_r = 16$ s, which corresponds to $N = 256$ samples at the original sampling frequency $f_s = 16$ Hz, see Figure 3.

To conclude: whereas the sampling frequency sets the time resolution, the sample period occupies a central place in setting the frequency resolution when spectral analysis is performed. The sample period being the same, just increasing the sampling frequency simply increases the number of samples by the same ratio, leaving their ratio unaltered. The only way to increase the frequency resolution is to increase the sample period.

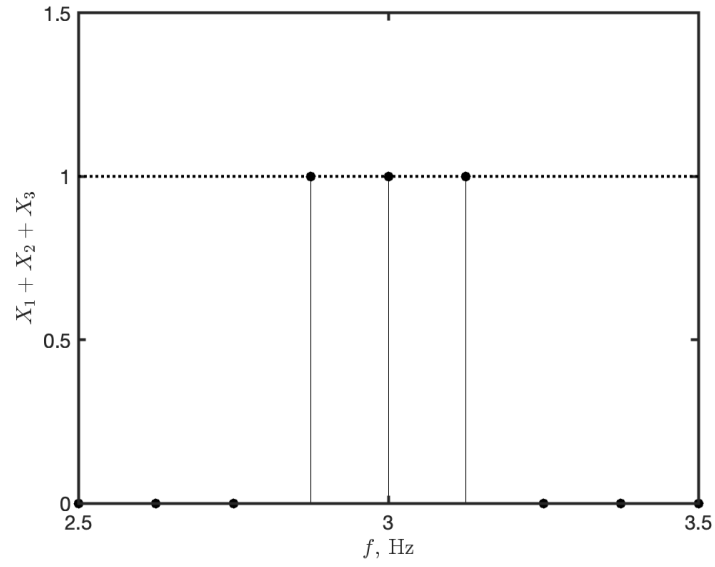


Figure 1: Single-sided spectrum of the sinusoidal mixture composed of three components at 2.875 Hz, 3 Hz, 3.125 Hz; sampling frequency: 16 Hz; sample period: 8 s. For the sake of visualization the frequency interval shown is limited to 2.5-3.5 Hz.

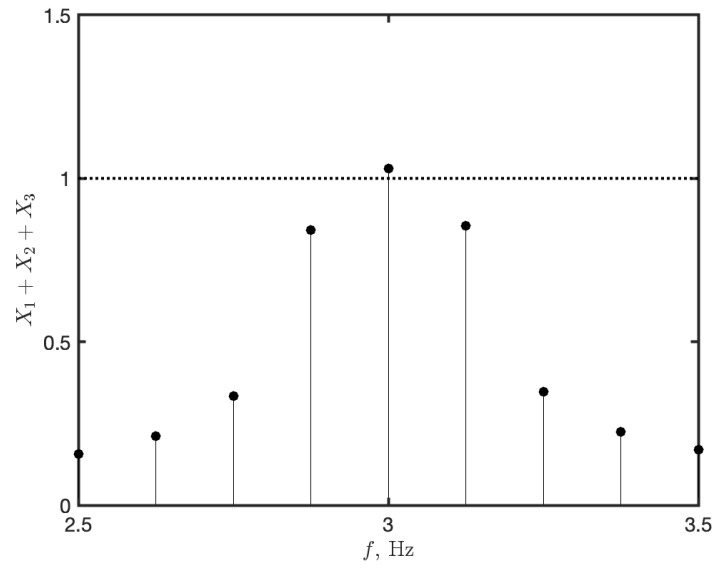


Figure 2: Single-sided spectrum of the sinusoidal mixture composed of three components at 2.9375 Hz, 3 Hz, 3.0625 Hz; sampling frequency: 32 Hz; sample period: 8 s.

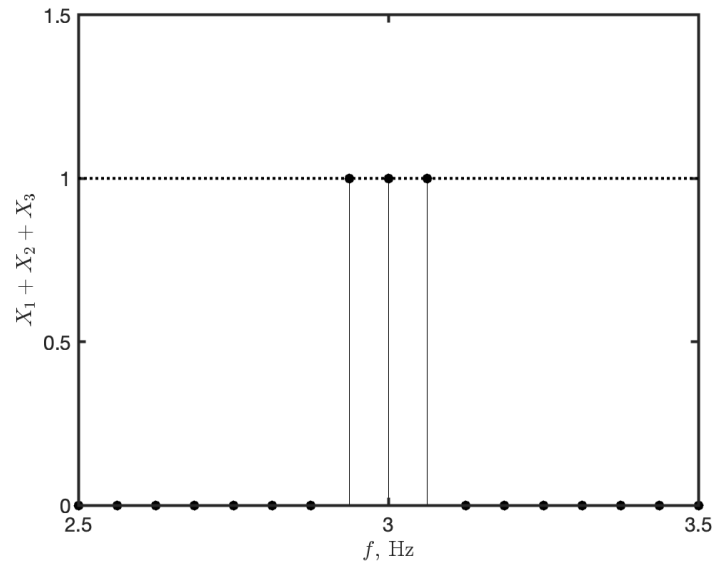


Figure 3: Single-sided spectrum of the sinusoidal mixture composed of three components at 2.9375 Hz, 3 Hz, 3.0625 Hz; sampling frequency: 16 Hz; sample period: 16 s.