Oxford Physics

M. Phys. Major Option C1

Astrophysics

Problem Sets 2022-2023

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NB: For some problems you might find it useful to refer to courses from previous years including the 1st and 3rd year astrophysics short options and the 3rd year cosmology course (end of the General Relativity course). However, you do ***not*** *require knowledge of General Relativity: the cosmological concepts discussed can be understood in a Newtonian gravitational framework.*

Chapter 1

Hot Big Bang Model

For Tutorial 1

1.1 Cosmological models

State the cosmological principle and define the terms *homogeneity* and *isotropy.* Give an example showing that a homogeneous universe need not be isotropic.

The Friedmann-Lemaître-Robertson-Walker metric for a homogeneous and isotropic universe is given by

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + a(t)^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right]$$

where ds is the proper time interval between two events, t is the cosmic time, k measures the spatial curvature, and r, θ and ϕ are the radial, polar and azimuthal co-ordinates respectively. Discuss the physical significance of $a(t)$, the scale factor, sketching its form for the three cases of a matter-only universe with positive, zero, and negative spatial curvature.

Define the term *luminosity distance.* By considering the amount of radiation which is received in a unit area located a co-moving distance away from a source, show that the luminosity distance d_{lum} is given by the formula

$$d_{\mathrm{lum}} = a_0 r_0 (1+z)$$

where r_0 is the co-moving distance and z is the red-shift.

Describe how the luminosity distance of Type Ia supernova might be used to constrain cosmological parameters, and discuss in detail the observations which are required, carefully outlining any key assumptions of the method.

1.2 Early Universe

Give an account of the observational evidence for the hot Big Bang model of the Universe.
The Friedmann-Lemaître and fluid equations respectively are given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}$$

and

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$$

where a is the scale factor, ρ is the density and p is the pressure ($\dot{a}$ and $\dot{\rho}$ are the derivatives of these quantities with respect to time.) Use these equations to derive the acceleration of the Universe.
Hence demonstrate that if the Universe is homogeneous and the strong energy condition $\rho c^2 + 3p > 0$ holds, the Universe must have undergone a Big Bang.

1.3 Units

It is often useful to change between different units when looking at a problem. A common practice is to convert a quantity into completely different units by multiplying by the fundamental constants such as c, $\hbar$, k_B and unit conversions.
For example, take the Hubble constant

$$H_0 = 100h\mathrm{km\,s^{-1}Mpc^{-1}}.$$

We can re-express it as a length by dividing by c to get

$$\frac{H_0}{c} = 3.334 \times 10^{-4} Mpc^{-1}.$$

To re-express as an energy (in eV) we must first multiply by $\hbar$ to get

$$\hbar H_0 = 1.05h \times 10^{-32}\mathrm{K\,m\,J\,Mpc^{-1}}.$$

We then need to convert Mpc to Km and J to eV to get

$$\hbar H_0 1.01h \times 10^{-34}.$$

Finally to convert it into an inverse timescale (in years) we can simply convert Mpc to Mm and s into yr

$$H_0 = 1.02h \times 10^{-10}\mathrm{yr^{-1}}.$$

These correspond to an upper limit on the mass of the dark energy particle, the inverse Hubble length, inverse approximate age.
Now do the same for the following quantities. Note that you will have to look up the values of the fundamental constants and the conversions between units.

- $\rho_c = 3H_0^2/8\pi G$ into (a) g cm^{-3}, (b) GeV^4, (c) eV cm^{-3}, (d) protons cm^{-3}, (e) $M_\odot$Mpc^{-3}. If the cosmological constant has $\rho_V = 2\rho_c/3$, what is its energy scale in eV (i.e. $\rho_V^{1/4}$). Compare to the Planck mass, $M_{Pl} = (8\pi G)^{-1/2}$

- The photon temperature, T_{CMB}=2.728K to (a) eV^4. Assuming a black body distribution, convert this to a number density, n_γ in photons cm^{-3} and energy density, ρ_γ in (a) eV, (b) g cm^{-3} and $\Omega_\gamma = \rho_\gamma/\rho_c$.

- The neutrino temperature, $T_\nu = (4/11)^{1/3}T_{CMB}$. Use this to express n_ν, ρ_ν and Ω_ν in the above units assuming that the neutrinos are relativistic (and have three species).

- With the above relic *number* density, now consider the case where one out of three neutrino species has a mass of 1 eV and the rest are massless. What is the energy density of relic neutrinos in units of the critical density, $\Omega_{\nu,massive}$. For what mass is the energy density at the critical value?

1.4 Expansion of the Universe

In the lectures you have been given an expression for the evolution of the Hubble constant, $H = \frac{\dot{a}}{a}$ as a function of the scale factor. It can depend on a number of constants that encode the fractional energy density in the various energy species. These are the fractional energy density in non-relativistic matter, $\Omega_{M,0}$, in relativistic matter, $\Omega_{\gamma,0}$, in the cosmological constant, $\Omega_{V,0}$ and in curvature, $\Omega_{K,0}$. You will use the evolution equation for $H(a)$ to solve the following problems.

- Assume the universe today with no curvature or relativistic matter, but with both non-relativistic matter and a cosmological constant. Write an expression for the age of the universe.

- Assume instead that the universe is open (hyperbolic) with non-relativistic matter but no cosmological constant or relativistic matter.

- Assume that there is only relativistic (radiation) and non-relativistic matter in the universe (no cosmological constant) and that the universe is flat. Integrate the age equation to determine the time at which the cosmic temperature was 10^9 K (remember that the temperature of the Universe today is 2.7 K) and when it was 1/3 eV (1 eV is approximately 11600 K). In this case you can solve the integral exactly. Assume that the ratio of radiation energy density to non-relativistic matter energy density *today*, $\Omega_{\gamma,0}/\Omega_{M,0} \simeq 2.5h^{-2} \times 10^{-5}$ where the Hubble constant today is $H_0 = 100h\text{km s}^{-1}\text{Mpc}^{-1}$.

Chapter 2

Large Scale Structure / Lensing

For Tutorial 2

2.1 Growth of inhomogeneities

The equation for the evolutions of small perturbations in an expanding background is:

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{c_s^2}{a^2}\nabla^2\delta - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2\Omega\delta = 0$$

where a is the scale factor, c_s^2 is the speed of sound, and Ω is the fractional energy density in non-relativistic matter.

a) In a Universe dominated by non-relativistic matter (or dust), we have that $a \propto t^{2/3}$ and $c_S^2 \simeq 0$. Find the solutions to δ. Which one dominates?

b) The Newton-Poisson equation in an inhomogeneous universe can be rewritten as

$$\nabla^2\Phi = 4\pi G a^2 \bar{\rho}\delta$$

where $\bar{\rho}$ is the mean energy density, and (as above) the gradient is taken in terms of *conformal coordinates* (i.e. coordinates which are fixed on the expanding space time). From what you know about the evolution of the energy-density of non-relativistic matter, find the time evolution of Φ, the Newtonian potential.

c) Assume that the universe is a mixture of dust and cosmological constant and that the cosmological constant is constant over all of space. Furthermore, assume that the cosmological constant dominates, so that the fractional energy density in dust is effectively zero. Explain why the the equation for the evolution of perturbations in the dust is given by

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}=0$$

Find the solution of δ as a function of time.

d) Compare a Universe in which the cosmological constant dominates with one in which it is absent. Consider a structure with a density contrast $\delta(t_0)$ today. It can be a galaxy or a cluster of galaxies. Will it have been around for longer in the Universe with or without the cosmological constant? Think of a way for testing for the presence of delayed growth of structure by looking for galaxies at large redshifts.

2.2 A baryon dominated Universe

In the very early Universe, before recombination, baryons and photons interact very strongly to form a tightly coupled fluid. Let us assume that the transition between radiation and matter domination occurs at the same time as recombination. Before recombination, the evolution equation for the perturbations in the baryon/photon fluid is

$$\delta''+\frac{a'}{a}\delta'-\frac{1}{3}\nabla^2\delta-4\left(\frac{a'}{a}\right)^2\delta=0 \tag{2.1}$$

with $a\propto\eta$. After recombination the evolution equation is

$$\delta''+\frac{a'}{a}\delta'-\frac{3}{2}\left(\frac{a'}{a}\right)^2\delta=0 \tag{2.2}$$

with $a\propto\eta^2$.

- Rewrite these equations in Fourier space by replacing ∇ by $-i\vec{k}$.
- Solve both of these equations in the long wavelength limit.
- Before recombination, there is a scale (let us call it $k_J=2\pi/\lambda_J$ where λ_J is the Jeans length) that separates the large k behaviour from the

small k behaviour. What is it? How does it evolve with time? (Please show the time evolution for the *physical* and *conformal* Jeans wave number).

A very rough approximation to the solution of the evolution equations before recombination on small wavelengths is

$$\delta = C_1 \cos\left(\frac{1}{\sqrt{3}}k\eta\right) + C_2 \sin\left(\frac{1}{\sqrt{3}}k\eta\right)$$

where C_1 and C_2 are constants.

- In the pre-recombination period, how does a perturbation that starts off with a $k < k_J$ evolve (answer this question qualitatively)? What type of behaviour will it have at sufficiently late times?

- (This is hard). Start off with a perturbation with an amplitude A on scales much larger than Jeans scale at some very early time. Follow its evolution until today. You should find two types of behaviour: modes whose wavenumber k are *smaller* than k_J at recombination and modes whose wave numbers are *greater* than k_J. By matching the large wavelength solutions to short wavelength solutions at the time when $k_J = k$ you can get a solution that covers all times. What form will it have at late times? (No need to do the full calculation, just figure out what kind of solution will be picked out at late times.)

- (This is hard). Define the power spectrum of perturbations to be

  $$P(k) \equiv |\delta(t_0, \vec{k})|^2$$

  Sketch what you expect the power spectrum of perturbations to be today if we assume that initially all modes had the same amplitude at some early time.

2.3 Neutrinos and free streaming

Consider a universe with critical density full of nonrelativistic matter. The evolution equation for density perturbations, is

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \delta = 0.$$

with $\delta = \delta\rho/\rho_0$, where we have expanded the total density $\rho = \rho_0 + \delta\rho$ into a homogeneous (ρ_0) and imhomogeneous ($\delta\rho$) part. Find the solution for the evolution of δ as a function of time.

Now consider a critical density universe which has a mixture of massive neutrinos (with fractional density Ω_ν) and non-relativistic matter (with fractional density Ω_M) so that $\Omega_\nu + \Omega_M = 1$. It will expand at the same rate as in the case considered above. On large scales δ will evolve as above but on scales smaller than about $40\mathrm{h}^{-1}$Mpc, the evolution of perturbations in the nonrelativistic matter is set by

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} - \frac{3}{2}\left(\frac{\dot{a}}{a}\right)^2 \Omega_{M,0}\,\delta = 0.$$

Find the solution for the evolution of δ as a function of time.

What is the difference in the growth rate of density perturbations on small scales for these two perturbations? Explain the reason for the difference in terms of the behaviour of the neutrinos.

We are now able to measure the angular positions and redshifts, z of millions of galaxies in the universe. In some cases we are even able to measure their distances from us. At low redshifts, a measurement of the distance, d and the redshift of a galaxy can be used to determine its *peculiar velocity* away from us, v, through

$$cz = H_0 d + v$$

where H_0 is the Hubble constant today and c is the speed of light. From conservation of energy we have that the peculiar velocity and the density contrast at a given point are related through:

$$\vec{\nabla}\cdot\vec{v} = -\dot{\delta}$$

Show that, in regions of equal δ, galaxies will have on average lower peculiar velocities in a universe with massive neutrinos as compared to a universe with only non-relativistic matter.

2.4 Gravitational Lensing

For a galaxy cluster potential defined by

$$4\pi G\rho_H(r) = \frac{V_H^2}{r^2 + a_H^2} \tag{2.3}$$

where a_H and V_H are constants, obtain the surface density $\Sigma(R)$ projected in a circle of radius R on the sky and derive the corresponding projected mass

$M(< b)$ within radius b. The angle bending of a ray of light passing at radius b from a gravitational lens writes:

$$\alpha(b) = \frac{4G}{c^2} \frac{M(< b)}{b}. \tag{2.4}$$

Show that when both the distance to the source and the lens-source distance are much larger than the distance to the lens, a light ray passing far enough from the center of the halo, so that b is much larger than a_H is bent by an angle:

$$\alpha \approx \frac{2\pi V_H^2}{c^2} \text{ radians.} \tag{2.5}$$

A distant source exactly aligned with the center of the "dark halo" or cluster, will appear as a ring of radius $\theta_E \approx \alpha$ on the sky. Abell 383 shows a large tangential arc 16 arcsec from its center, corresponding to $V_H \approx 1800$ km/s. What is the kinetic energy of a particle in circular orbit in this potential? To what virial temperature does this correspond? Show that this is similar to the observed X-ray temperature $T_X \approx 6 \times 10^7$ K.

Chapter 3

Advanced Stellar Astrophysics and Galactic Dynamics

For Tutorial 3

3.1 Stars

3.1.1 The Sun

Considering the effective temperature of the Sun, $T_{\text{eff}} = 5800$ K and its average radius, $R_\odot = 6.96 \times 10^{10}$ cm, (a) estimate the solar luminosity, assuming that the Sun is a spherical star. (b) Compare your result with the brightness obtained from the flux on the surface of the Sun, $F_\odot = 6.33 \times 10^{10}$ erg cm^{-2} s^{-1}.

3.1.2 Cold Stars

A cold star has an effective temperature $T_{\text{eff}} = 3200$ K, apparent bolometric magnitude $m_{\text{bol}} = 6.0$ and absolute bolometric magnitude $M_{\text{bol}} = -2.9$. Calculate its distance, luminosity and radius.

3.1.3 Stellar Structure

Explain what is meant by the term *hydrostatic equilibrium* in stellar structure and discuss its importance. Derive an expression for the timescale on which changes occur if equilibrium conditions are disturbed.
A star is completely supported by radiation pressure, and transport of energy is by radiation only. Use the equation of state $p = aT^4/3$ and the radiative

transport equation

$$L = \frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{\mathrm{d}T}{\mathrm{d}r}$$

(where L is the luminosity, κ the opacity, ρ the density, r the radius, c the speed of light and a the radiation constant) to show that in hydrostatic equilibrium the luminosity is given by

$$L = \frac{4\pi GMc}{\kappa}$$

where G is the gravitational constant and M is the total mass of the star. What would happen if L were suddenly increased beyond this value?

3.1.4 Stellar Cores

By Taylor expanding as a function of radius, determine the behaviour of the interior mass, M(r) and the pressure P(r) in the neighbourhood of a star centre, to the lowest non-zero and non-constant order.

3.2 Star Formation

(a) Consider a spherical cloud with mass M, radius R, density ρ, temperature T and molecular weight μ. Show that the minimum mass for this cloud to condense to form stars can be written in the form

$$M > M_J \approx \frac{KT^{3/2}}{\mu^{3/2}\sqrt{\rho}}$$

which is the mass of Jeans. Determine the value of the constant K.

(b) Assume that the pre-collapse cloud is in hydrostatic equilibrium and can be treated as an isothermal sphere, i.e. a sphere of gas at constant temperature T where the supporting thermal pressure is given by $P = \rho\, c_\mathrm{s}^2$ and where $c_\mathrm{s} = \sqrt{kT/\mu m_\mathrm{H}}$ is the isothermal sound speed of the gas. Show that the density as a function of radius r from the centre of the sphere is approximately given by

$$\rho = \frac{c_\mathrm{s}^2}{2\pi G}\frac{1}{r^2},$$

and the mass enclosed within a radius r by

$$M(r) = \frac{2c_\mathrm{s}^2}{G}\, r.$$

3.3 Stars in Galaxies

In a galaxy located at a distance d Mpc from our own Milky Way, what would be the apparent $B-$magnitude of a star like our Sun? Show that in such a galaxy, an angle of 1 arcsec on the sky corresponds to a length of $5d$ pc. If its surface brightness is $I_B = 27$ mag arcsec^{-2}, how much $B-$band light does a patch of one square arcsecond of this galaxy emit? Show that although this is equivalent to ~ 1 L$_\odot$pc^{-2} in the $B-$band, the same surface brightness corresponds to ~ 0.3 L$_\odot$pc^{-2} in the $I-$band.

Data: $M_V^\odot = 4.83$; $B - V = 0.65$; $V - I = 0.72$; $m_B =$ -2.5 $\log_{10}$ (F_B) + 8.29 if the $B-$band flux F_B is in 10^{-12} erg/s/cm^2/Å units.

3.4 Principles of Galactic Dynamics - Spheroids

The *Plummer sphere* of total mass M_P and scale radius a_P is a simple if crude model for star clusters and round galaxies. Its gravitational potential:

$$\Phi_P(r) = -\frac{GM_P}{\sqrt{r^2 + a_P^2}}, \tag{3.1}$$

approaches that of a point mass when $r \gg a_P$. Show that its density is given:

$$\rho_P(r) = \frac{3a_P^2 M_P}{4\pi(r^2 + a_P^2)^{5/2}}. \tag{3.2}$$

and calculate U_P, its potential energy. What is the mass $M(< R)$ enclosed within a Plummer sphere of radius R? When viewed from a great distance along the z-axis, what is its surface density $\Sigma_P(R)$ at a distance R from the center? Check that the core radius r_c, where $\Sigma_P(R)$ drops to half its central value, is $r_c \approx 0.644\, a_P$.

3.5 Principles of Galactic Dynamics - Disks

A simple disk model potential is that of the *Kuzmin disk* of total mass M_K and scale length a_K, which reads, in cylindrical polar coordinates:

$$\Phi_K(R, z) = -\frac{GM_K}{\sqrt{R^2 + (a_K + |z|)^2}}. \tag{3.3}$$

Irrespective of whether z is positive or negative, this is the potential of a point mass M_K at $R = 0$, displaced by a distance a_K along the z-axis, on the opposite side of the $z = 0$ plane (i.e. negative if z is positive and positive if z is negative). Show that $\nabla^2\Phi_K = 0$ everywhere except in the plane $z = 0$ and use the divergence theorem to obtain the surface density $\Sigma_K(R)$ there.

3.6 Dark Matter Halos

The *Navarro-Frenk-White* (NFW) model is used to describe the density profile of cold dark matter halos with characteristic scale lengths a_N and densities ρ_N that form in cosmological N-body simulations:

$$\rho_{NFW}(r) = \frac{\rho_N}{(r/a_N)(1 + (r/a_N))^2}. \tag{3.4}$$

Note the cusp in the center where the density diverges like $1/r$ and the rapid fall off in r^{-3} at large radii. Calculate the associated gravitational potential, as a function of a_N and the characteristic velocity dispersion squared, $\sigma_N^2 = 4\pi G\rho_N a_N^2$. Use this result to obtain the speed $V(r)$ of a test particle on a circular orbit at radius r in this potential.

Chapter 4

Galaxies

For Tutorial 4

4.1 The Milky Way: Our Very Own Super Massive Black Hole

Using an 8-metre telescope to observe the Galactic center regularly over two decades, you notice that one star moves back and forth across the sky in a straight line: its orbit is edge on. You take one spectra to measure its radial velocity V_r, and find that this repeats exactly each time the star is at the same point in the sky. You are in luck: the furthest points of the star's motion on the sky are also when it is closest to the black hole (pericenter) and furthest from it (apocenter). You measure $s = 0.248$ arcsec, the separation of these two points on the sky, and the orbital period $P = 15.24$ yr. Assuming that the black hole provides all the gravitational force, determine the properties of the elliptical orbit of the star around it (eccentricity e and semi-major axis a), to find both the mass M_{BH} of the black hole, and its distance d_{BH} from us.

Data: $V_r^{apocenter} = 473$ km/s; $V_r^{pericenter} = 7326$ km/s.

4.2 The Milky Way and the Local Group: Collision with Andromeda

Collision in the local group: the case of the Milky Way and the Andromeda galaxy. The distance r of separation between two point masses m_1 and m_2 moving under their mutual gravitational attraction obeys the same equation as a body of much smaller mass attracted by a mass $M = m_1 + m_2$. Now the trajectory of a body orbiting in the plane $z = 0$ around this much larger mass M changes according to:

$$\frac{d^2r}{dt^2} - \frac{L_z^2}{r^3} = -\frac{GM}{r^2}. \tag{4.1}$$

where L_z is the conserved z angular momentum. Show that the solution to this equation is an ellipse of eccentricity e and semi-major axis a, which can be written in the parametric form

$$r = a(1 - e\cos\eta)\ ;\quad t = \sqrt{\frac{a^3}{GM}}(\eta - e\sin\eta) \tag{4.2}$$

with $a = L_z^2/(GM(1-e^2))$ and time t is measured from one of the pericenter passages where $\eta = 0$. Taking $e = 1$ and giving r and dr/dt their current measured values of 770 kpc and -120 km/s respectively, show that $\eta = 4.2$ corresponds to $t_0 \approx 12.5$ Gyr and $a \approx 517$ kpc for the Milky Way/Andromeda galaxy system. What is the combined mass M then and why is it the smallest possible? Repeat the calculation for $\eta = 4.25$: what conclusion can you draw from this exercise? Show that the Milky Way and M31 will again come close to each other in about 3 Gyr.

4.3 The Main Types of Galaxies: Tully-Fisher

Ignoring the presence of a bulge, explain why we might expect the mass M of a spiral galaxy to follow approximately

$$M \propto V_{max}^2 h_R \tag{4.3}$$

where V_{max} is the maximum of the rotation curve of the disk and h_R its scale length (see formula for surface brightness profile $I(R)$ below). Show that if the surface brightness (averaged over features like spiral arms) in a spiral galaxy follows the exponential profile:

$$I(R) = I(0)\exp(-R/h_R) \tag{4.4}$$

then its luminosity is given by $L = 2\pi I(0)h_R^2$ and hence that if the ratio M/L and central surface brightness are constants then $L \propto V_{max}^4$. In fact, $I(0)$ is lower in low-surface-brightness galaxies: show that if these objects are to follow the same Tully-Fisher relation they must have higher mass-to-light ratios, with approximately $M/L \propto 1/\sqrt{I(0)}$

4.4 The Main Types of Galaxies: Spheroids in the Sky

Looking from a random direction, the fraction of galaxies that we see at an angle between i and $i + \Delta i$ to the polar axis is just $\sin i \Delta i$, i.e. the fraction of a sphere around each galaxy corresponding to this viewing directions. If they are all oblate with axis ratio B/A, then the fraction of galaxies $f_{obl}\Delta q$ with apparent axis ratios between q and $q + \Delta q$ is given by:

$$f_{obl}(q)\Delta q = \frac{\sin i \Delta q}{|dq/di|} = \frac{q\Delta q}{\sqrt{1-(B/A)^2}\sqrt{q^2-(B/A)^2}} \tag{4.5}$$

What is the fraction of oblate elliptical galaxies with true axis ratio B/A that appear more flattened than axis ratio q? If these galaxies have $B/A = 0.8$, show that the number seen in the range $0.95 < q < 1$ should be about one third that of those with $0.8 < q < 0.85$. Finally, show that for very oblate galaxies ($B/A \leq 0.1$), an even higher proportion of the images will be nearly circular, i.e. with $0.95 < q < 1$, in comparison to images with $B/A < q < B/A + 0.05$.

4.5 Optics and spectroscopy

Explain what is meant by the angular resolution and magnification of a telescope. Show that the limit of resolution is given approximately by λ/D where λ is the wavelength of the incident radiation and D is the diameter of the telescope aperture. What is the typical angular resolution (in seconds of arc) achieved with a large ground-based telescope operating at visible wavelengths from a good site?

Give expressions for the resolving power and angular dispersion of a diffraction grating, defining all quantities.

In order to study the internal motion of galaxies astronomers often need to measure velocity differences differences of order $10\,\mathrm{km\,s^{-1}}$. Consider a grating spectrograph equipped with a camera of focal length 2 m and a CCD detector of pixel size $20\,\mu$m. The spectrograph is used to study galaxies in the [O III] emission line whose wavelength is 500.7 nm. Given that the light is incident normally on the grating and the first-order spectrum falls on the detector, estimate the minimum grating ruling, in lines per mm, required to give the desired velocity resolution.

4.6 High-redshift galaxies

Discuss the significance of the observation that special lines of distant galaxies are redshifted compared to their rest wavelengths.
For what range of redshifts would the hydrogen Lyman-α line (rest wavelength 121.6 nm) be redshifted in to the visible part (400–700 nm) of the spectrum?
What particular problems are encountered in attempting to detect Lyman-α from galaxies with redshifts greater than about 7?
Assuming the Hubble constant H_0 to be $70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, determine the range of distances (in Mpc) corresponding to the range of redshifts you have calculated. You may take the relationship between distance d and redshift z to be

$$d = \frac{cz}{H_0}\frac{(1+\frac{z}{2})}{(1+z)^2}.$$

What sort of celestial objects are observed with redshifts $z > 2$? Outline further evidence which favours these objects being located at cosmological distances.

Chapter 5

Radiative Processes I

For Tutorial 5

5.1 Blackbody spectra and magnitude systems

Explain what is meant by a black-body and describe applications of this concept in astrophysics.

The intensity of radiation emitted per unit wavelength interval by a black-body of temperature T is given as a function of wavelength by

$$B(\lambda) = \frac{2hc^2}{\lambda^5} \frac{1}{\exp(hc/\lambda kT) - 1}.$$

Obtain an approximation for this formula appropriate for wavelengths such that $hc/\lambda kT << 1$. In what region of the electromagnetic spectrum would you expect this approximation to apply to the radiant emission from stars?

The apparent magnitude m of a hot star of surface temperature 40 000 K is measured at the two wavelengths 440 nm and 550 nm, corresponding to the colour filters known as B and V. The colour index $m_\mathrm{B} - m_\mathrm{V}$ is found to be -0.35. Compare this with the theoretical colour index expected for a black-body at this temperature.

[The apparent magnitude is related to the observed flux f by $m = -2.5 \log_{10} f + C$ where C is a constant for a given wavelength. The colour index for a star of surface temperature 11 700 K is zero.]

5.2 Specific intensity, flux and brightness temperature

(a) An accreting neutron star in the galaxy M82 (distance $d = 3.5$ Mpc) radiates a total luminosity of $L_s = 10^{38}$ erg/s from its spherical surface. The radiated intensity is a strong function of θ, which is the zenith angle with respect to the north magnetic pole of the star:

$$I(\theta) \propto 200 \cos^2 \theta + 1.$$

Given that we view the neutron star directly down the pole ($\theta = 0$), calculate the flux we measure from it in units of erg/s/cm^2. What luminosity would we infer the star to have from this measured flux by assuming isotropic radiation?

(b) In the lectures we discussed the core of the quasar 3C273, an extremely bright radio source. Consider now a less-extreme example, where a source has a specific flux of 10 Jy at 178 MHz and subtends a solid angle of one arcmin^2. What is the brightness temperature of the radio source at 178 MHz?

(c) You may be surprised that the temperature is not dramatically higher than that of very hot stars. Why then is it so difficult to observe stars at radio wavelengths?

5.3 Emission lines in planetary nebulae

a) Explain why most of the bright emission lines in the spectrum of a planetary nebula arise from elements whose abundances are small compared to hydrogen.

b) What determines the relative brightness of lines from different elements and from different stages of ionization of a given element, and what quantities need to be estimated or measured in order to estimate the abundance of the elements that produce the emission lines?

c) How do observations made in different spectral wavebands contribute to our knowledge of the processes that produce the spectrum and the conditions in the nebula?

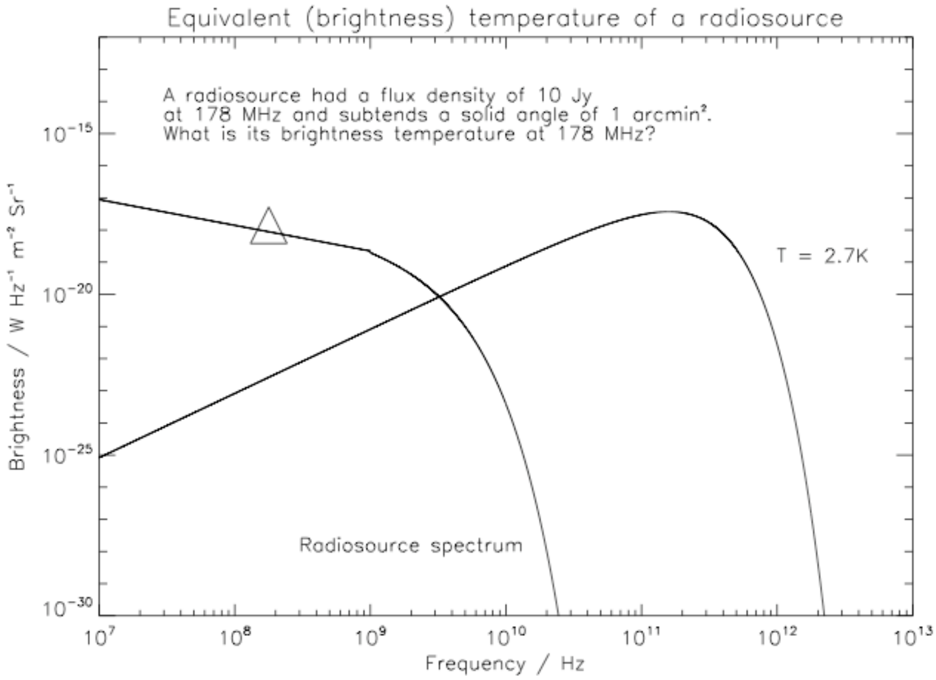


Figure 5.1: Brightness temperature.

5.4 Critical density of a planetary nebula

a) An element of total number density n_{E} exists mainly in two stages of ionization, i and $i+1$, with number densities n_i and n_{i+1}. Express n_i/n_{E} and n_{i+1}/n_{E} in terms of the ionization rate β_i (from i) and the recombination rate γ_i (to i). In a planetary nebula what processes are the main contributors to β_i and γ_i? How do n_i/n_{E} and n_{i+1}/n_{E} vary if the electron number density n_{e} is increased?

b) Consider a forbidden transition in the optical spectrum of a planetary nebula. The transition occurs between an excited level (2), with number density n_2, and the ground level (1), with number density n_1. Write down the appropriate form of n_2/n_1. Hence explain what is meant by the *critical density.*

c) Two such lines occur in an ion, from levels 3 and 2 to level 1. Assuming that the wavelengths $\lambda_{21} \simeq \lambda_{31}$, show that their flux ratio is given by

$$\frac{F_{31}}{F_{21}} \simeq \frac{g_3}{g_2}\frac{A_{31}}{A_{21}}\frac{(n_{\mathrm{e}}^*(2)+n_{\mathrm{e}})}{(n_{\mathrm{e}}^*(3)+n_{\mathrm{e}})},$$

where A_{ji} is the spontaneous transition probability, g_j is the statistical weight of level j and $n_{\mathrm{e}}^*(j)$ is the critical density for a transition from level j.

Using the data for singly ionized sulphur (S II) given in the table below, find:

d) the values of F_{31}/F_{21} when (i) $n_{\mathrm{e}} \gg n_{\mathrm{e}}^*(2)$ and (ii) when $n_{\mathrm{e}} \ll n_{\mathrm{e}}^*(3)$

e) the value of n_{e} at which F_{31}/F_{21} has its greatest dependence on $\ln(n_{\mathrm{e}})$, and the corresponding value of F_{31}/F_{21}. Show that if this value of F_{31}/F_{21} can be measured to within $\pm 10\%$ then $\log_{10}(n_{\mathrm{e}})$ can be determined to within ± 0.18.

Table: Data for S II lines.

λ_{ji}/nm	Transition (ji)	A_{ji}/s^{-1}	Critical density $n_{\mathrm{e}}^*(j)/\mathrm{m}^{-3}$
673.1	${}^2\mathrm{D}_{3/2} - {}^4\mathrm{S}_{3/2}$ (21)	1.7×10^{-3}	3×10^{10}
671.6	${}^2\mathrm{D}_{5/2} - {}^4\mathrm{S}_{3/2}$ (31)	6.3×10^{-4}	1×10^{10}

[At a fixed electron temperature T_{e} the collisional excitation rate is

$$C_{ij} \propto \frac{\exp(-hc/\lambda_{ij}k_{\mathrm{B}}T_{\mathrm{e}})}{g_i}\ ]$$

5.5 Detailed balance and coronal approximation

a) Discuss what is meant by *detailed balance* and *the coronal approximation* in the context of processes that determine the number density n_2 of an excited state in a two level atom. In each case give the expression for n_2/n_1, where n_1 is the ground state number density.

b) Emission lines arising from permitted and spin-forbidden electric dipole transitions to a common atomic energy level are observed in stellar transition regions. Explain how their relative intensities can be used to determine the electron number density n_e. Explain why there is an electron density below which this method cannot be used.

c) The diagram shows some transitions observed in Fe XIV.

Transition	λ /nm	C_{ij} (relative)	A_{ji} (relative)
${}^2D_{5/2} \to {}^2P_{3/2}$	21.9	9	6
${}^2D_{3/2} \to {}^2P_{1/2}$	21.1	10	5
${}^2D_{3/2} \to {}^2P_{3/2}$	22.0	1	1

Using the relative collisional excitation rate coefficients C_{ij} and the relative transition probabilities A_{ji} given in the table, derive an expression for the relative intensities of the lines at wavelengths $\lambda = 21.9\,\mathrm{nm}$ and $\lambda = 21.1\,\mathrm{nm}$ in terms of the population ratio $n({}^2P_{3/2})/n({}^2P_{1/2})$. What relative intensity would be expected if the ${}^2P_{3/2}$ and ${}^2P_{1/2}$ level number densities were determined by detailed balance at a temperature $T_\mathrm{e} = 2 \times 10^6\,\mathrm{K}$?

d) In a solar active region the observed intensity ratio of the above lines is 0.15. Use the collisional excitation rate coefficient $C_{ij} = 7\times10^{-15}\,\mathrm{m^3\,s^{-1}}$ and the transition probability $A_{ji} = 60\,\mathrm{s^{-1}}$ for the ${}^2P_{3/2} \to {}^2P_{1/2}$ transition to show that $n({}^2P_{3/2})/n({}^2P_{1/2})$ depends on n_e. Find the value of n_e.

[The relation between the collisional de-excitation and excitation rate coefficients is $C_{ji} = (g_i/g_j)C_{ij}\exp\left(W_{ij}/k_\mathrm{B}T_\mathrm{e}\right)$ where g_i and g_j are the statistical weights of the lower and upper levels respectively and W_{ij} is the excitation energy of level j above level i.]

5.6 Intersystem lines of cool stars

a) Using a model for a two-level ion (ground state plus one excited state), discuss the processes which should be included when considering the formation of an intersystem (semi-forbidden) line in a cool-star transition region. Hence show that the rate at which energy is emitted in an intersystem line of wavelength λ_{21} is given by

$$L_{21} = \frac{hc}{\lambda_{21}} A_{21} \frac{n_{\rm E}}{n_{\rm H}} \int \left(\frac{n_{\rm ion}}{n_{\rm E}}\right) \frac{f_1(n_{\rm e}, T_{\rm e}) n_{\rm H}}{A_{21} + f_1(n_{\rm e}, T_{\rm e}) + f_2(n_{\rm e}, T_{\rm e})} {\rm d}V \, ,$$

where A_{21} is the spontaneous transition probability, $T_{\rm e}$ is the electron temperature and $n_{\rm E}$, $n_{\rm H}$, $n_{\rm ion}$ and $n_{\rm e}$ are the number densities of the element under consideration, hydrogen, ions and electrons, respectively.

Show that

$$f_1(n_{\rm e}, T_{\rm e}) = C_{12} n_{\rm e} \quad \text{and} \quad f_2(n_{\rm e}, T_{\rm e}) = C_{21} n_{\rm e},$$

where C_{12} and C_{21} are rate coefficients for collisional excitation and de-excitation respectively.

b) Intersystem lines of Si III and C III are observed in spectra of cool stars with a range of surface gravities. Assuming that both lines are formed at $T_{\rm e} = 4.5 \times 10^4$K, use the data in the table below to calculate the maximum and minimum values of the ratio L(Si III)/L(C III).

c) In the spectrum of a planetary nebula, the ratio L(Si III)/L(C III) is observed to be less than 0.1. Discuss the differences between the physical conditions in planetary nebulae and cool star transition regions and suggest the main cause of this small energy ratio.

Data for Si III and C III

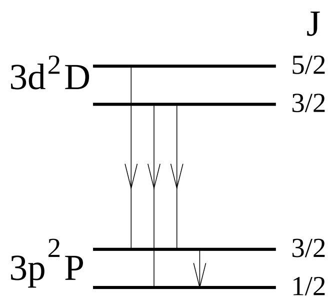
Ion	Transition	λ (nm)	Ω	$A_{21}({\rm s}^{-1})$	$N_{\rm E}/N_{\rm H}$	$N_{\rm ion}/N_{\rm E}$
Si III	$3{\rm s}^2\,{}^1{\rm S}_0{-}3{\rm s}3{\rm p}\,{}^3{\rm P}^\circ_1$	189.2	2.8	1.5×10^4	3.5×10^{-5}	0.79
C III	$2{\rm s}^2\,{}^1{\rm S}_0{-}2{\rm s}2{\rm p}\,{}^3{\rm P}^\circ_1$	190.9	0.32	1.0×10^2	3.5×10^{-4}	0.46

The ionization potentials of Si III and C III are 33.5eV and 47.9eV, respectively.

[The rate coefficient for collisional excitation is

$$C_{12} = \frac{8.63 \times 10^{-12} \, \Omega \, 10^{-\left[\frac{6.25 \times 10^6}{\lambda_{21} T_e}\right]}}{g_1 T_e^{1/2}} \mathrm{m}^3 \, \mathrm{s}^{-1} \; ,$$

where Ω is the collision strength given in the table, g_1 is the statistical weight of the lower level, λ_{21} is in nm and T_e is in K.]


J
3d²D
5/2
3/2
3p²P
3/2
1/2

Chapter 6

Radiative Processes II

For Tutorial 6

6.1 Dust

A hot star is embedded in a dusty region. Silicate dust grains have a radius of 0.1μm and have an efficiency $Q_\nu \propto \nu$, such that the efficiency ratio $<Q_{IR}>$ / $<Q_{UV}> = 2\times10^{-2}$a T_{gr}, while carbon grains have a radius of 0.01μm and have an efficiency $Q_\nu \propto \nu^2$, such that the efficiency ratio $<Q_{IR}>$ / $<Q_{UV}>$ $= 4\times10^{-4} a^2 T_{gr}^2$, where a is in μm.

a) Calculate the temperatures that these grains will attain at a distance of 20 AU from the central star of a planetary nebulae with a temperature of 40 000 K and a luminosity of 10^4 $L_\odot$. Comment on these temperatures and the possible behaviour as a function of distance from the star in this object.

b) Observations in the mid-infrared at $\lambda = 10\mu$m show emission extending to distances of 0.01pc from the central star. What is the likely explanation for this emission?

6.2 Stellar Opacity

a) Describe the main sources of opacity at visible wavelengths in the photospheres of hot (B-type) stars, solar-type stars and cool stars. Explain the physical conditions that give rise to the different opacity sources.

b) Estimate the temperature at which the number of hydrogen atoms in the first excited state is equal to the number in the ground state for a thermal distribution. Is this level population likely to be realised in practice?

c) Without detailed derivation, justify the following expression for a grey atmosphere, defining all terms used and stating any assumptions made:

$$\mathrm{S}(\tau) = \tfrac{3}{4\pi}\left(\tau + \tfrac{2}{3}\right)\mathrm{F}$$

d) Adopting a grey atmosphere approximation, estimate the temperature ranges which may be investigated by measurements at the limb and at the centre of the solar disk.

6.3 HII Region

Observations of a HII region in a galaxy at a distance of 3Mpc give the following values for the fluxes of Hydrogen recombination lines:

Line	Wavelength (μm)	Observed Flux F (10^{-18} Wm^{-2})	Relative Intrinsic Flux
H(Gamma)	0.434	0.0119	47
H(Beta)	0.486	0.0455	100
H(alpha)	0.656	0.702	285

Theoretical line intensities relative to H_β = 100 calculated by Hummer & Storey are also listed.
The extinction expressed in magnitudes relative to an extinction of A(V)= 1 magnitude in the optical can be approximated by:

$$\mathrm{A(x)} = 1.0 + 0.826(\mathrm{x}-1.83) - 0.320(\mathrm{x}-1.83)^2$$

where x is the inverse wavelength $1/\lambda$ in units of μm^{-1}.

a) What is the extinction, expressed as magnitudes of visual extinction A(V) towards the emitting region?

b) What information can be obtained from the extinction-corrected hydrogen line fluxes, and how might the calculated value of the extinction be checked?

c) Give an estimate of the total number of ionizing photons emitted by the stars.

d) Use this to estimate the size of the resulting HII region, assuming an electron density $n_e \sim 10^{10}\mathrm{m}^{-3}$.

The hydrogen recombination coefficient $\alpha = 2.6 \times 10^{-19}\mathrm{m}^3\mathrm{s}^{-1}$

6.4 Absorption lines from a quasar

Rest-frame ultraviolet absorption lines from the ground state of C II have been detected towards the quasar 0347 - 3819 at a redshift of 3.025. The ground state of the C^+ ion consists of two levels $2s^2 2p\ 2P^0_{1/2,3/2}$ with an energy separation of $\Delta E = 63.42\ \mathrm{cm}^{-1}$.
Assuming thermal equilibrium, obtain an expression for the level population of the ground and excited fine structure levels within the ground state, and estimate the excitation temperature using the column densities in the two levels and the oth er information below. Comment on the value of T_{ex} found.

The column densities estimated from the transitions from the ground and excited levels within the ground state are $5.05 \pm 0.28 \times 10^{15}\ \mathrm{cm}^{-2}$ and $1.92 \pm 0.1 \times 10^{13}\ \mathrm{cm}^{-2}$ respectively. For the ground level J=1/2 and for the excited level J=3/2.

Appendix: Useful Constants and Unit Conversions

- $G = 6.673\ 10^{-8}\ \mathrm{cm}^3\ \mathrm{g}^{-1}\ \mathrm{s}^{-2}$
- $m_H = 1.673\ 10^{-24}$ g
- 1 pc = $3.086\ 10^{16}$ m
- 1 yr = $3.160\ 10^7$ s

Chapter 7

High-Energy Astrophysics I

For Tutorial 7

7.1 Shocks

a) Using the ideal gas law and the strong shock jump conditions, show that the temperature of gas downstream of a shock is given by

$$T_d = \frac{3}{16} \frac{m v_u^2}{k_\mathrm{B}}$$

where m is the mean particle mass in the gas and v_u^2 is the bulk speed of the gas upstream of the shock.

b) Why do the temperature and density of the upstream gas not appear in this relation?

c) The blast wave of supernova SN1993J had an initital expansion speed of 20 000 km s^{-1}. Assuming the interstellar medium around the supernova to be hydrogen plasma, estimate the initial temperature behind the blast wave.

d) Hydrogen plasma falls radially onto a white dwarf, passing through a shock very close to the surface. Show that the temperature behind the shock is given by

$$\frac{k_\mathrm{B} T}{m_\mathrm{e} c^2} = \frac{3}{32} \frac{m_\mathrm{p}}{m_\mathrm{e}} \frac{R_\mathrm{S}}{R_*}$$

where R_* is the radius of the star and R_S is its Schwarzschild radius. What is $k_\mathrm{B} T$ if $R_* = 6000$ km and $M = M_\odot$?

7.2 Particle acceleration

a) Outline the physical mechanism by which it is thought that electrons are accelerated to ultra-relativistic energies in strong non-relativistic shocks such as are found in supernova remnants.

b) Suppose that an electron is involved in a collision which increases the electron's total energy by a factor β. Suppose further that there is a probability p that this electron remains within the region where further collisions may occur. Show that the expected distribution of electron energies is

$$N(E)dE \propto E^k dE$$

Where $k = -1 + (\ln p / \ln \beta)$.

c) Assuming that $k = -2$ and that the electrons are accelerated to energies at which they emit synchrotron radiation, show that the power-law region of the synchrotron spectrum will have a form

$$I_\nu \propto \nu^{-0.5}$$

You may assume that a synchrotron electron emits almost all its radiation at a characteristic frequency $\nu_{\rm crit} \sim \frac{\gamma^2 eB}{2\pi m_{\rm e}}$ and that the power radiated is $P = \frac{4\gamma^2\beta^2 c\sigma_{\rm T} B^2}{3\mu_0}$

7.3 Synchrotron spectrum

a) Describe, with the aid of an annotated sketch, the shape of the spectrum of continuum radio emission from the lobes of a powerful extragalactic radio source. Include in your discussion the *optically thick* and *optically thin* regions of the spectrum, along with an explanation of these terms, and describe how the shape of the spectrum is modified by radiation losses as the radio source ages.

b) Show that a relativistic electron with Lorentz factor γ, passing through a region containing a uniform magnetic flux density B, has a gyrofrequency

$$\nu_g = \frac{eB}{\gamma m_{\rm e}}$$

c) Hence show that such an electron emits radiation whose spectrum is strongly peaked at a characteristic frequency ν_{crit} which is given by

$$\nu_{\text{crit}} \sim \frac{\gamma^2 eB}{m_{\text{e}}}$$

d) The power radiated by the electron is given by

$$P = \frac{4\gamma^2\beta^2 c\sigma_{\text{T}} B^2}{6\mu_0}$$

where β is the speed of the electron relative to the speed of light and σ_{T} is the Thomson cross section. Using these formulae for P and ν_{crit}, calculate a characteristic timescale for the synchrotron lifetime of the electron.

e) The powerful giant radiosource 3C236 is 6 Mpc across, with the host galaxy at the centre. The radio spectrum of the lobes close to the host galaxy shows a cut-off in emission, assumed to be due to radiative losses, at frequencies above about 1 GHz. Taking the magnetic flux density in this region to be 0.3 nT, estimate the age of the synchrotron plasma near the host galaxy.

On the assumption that this plasma was left behind near the host galaxy by the radiosource jets just as they began to expand into intergalactic space, estimate the expansion speed of the radiosource. What factors may cause these estimates of age and expansion speed to be unreliable?

7.4 Equipartition/Minimum Energy

a) Describe the evidence that the radio emission from powerful extragalactic radio sources is produced by the synchrotron mechanism.

b) A radio source contains a population of relativistic electrons with Lorentz factors $\gamma \gg 1$ and a uniform magnetic flux density of magnitude B. The number density of electrons with Lorentz factors in the range γ to $\gamma + \mathrm{d}\gamma$ is given by

$$n(\gamma)\mathrm{d}\gamma = n_1 \left(\frac{\gamma}{\gamma_1}\right)^{-k} \mathrm{d}\gamma$$

for γ greater than some limit γ_1, where n_1 is a constant and k is a constant greater than 2. Show that J_ν, the power emitted per unit volume per unit frequency interval, is given by

$$J_\nu \propto n_1 \gamma_1^k \gamma^{1-k} B.$$

You may assume that each electron emits all its synchrotron radiation at a frequency $\nu = \gamma^2 eB/m_\mathrm{e}$ and that the power radiated by each electron is $P = \frac{4}{3} c\sigma_\mathrm{T} u_\mathrm{m} \gamma^2$ where u_m is the energy density of the magnetic field and σ_T is the Thomson scattering cross-section.

c) If the energy density stored in relativistic electrons, u_e, is

$$u_\mathrm{e} = \frac{n_1 \gamma_1^2 m_\mathrm{e} c^2}{k-2}$$

show that, for a given observed value of J_ν, the total energy density in the source in the form of relativistic electrons and magnetic field has a minimum value which occurs when

$$u_\mathrm{e} = \frac{4}{3} u_\mathrm{m}.$$

d) The radio source Cygnus A is about 100 kpc across. The flux density of the magnetic field in the radio source is thought to be about 6 nT. Estimate a lower limit to the total energy content of the radio source and explain what implications this has for how Cygnus A is powered.

7.5 The Innermost Stable Circular Orbit

In General Relativity, the equation for the radial coordinate r of a test particle orbiting a non-rotating black hole of mass M can be written as

$$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2GM}{c^2 r}\right)\left(\frac{L^2}{r^2} + c^2\right) = \frac{1}{2}\frac{E^2}{c^2}, \tag{7.1}$$

where $\dot{r} = \mathrm{d}r/\mathrm{d}t$ and L and E are the angular momentum per unit rest mass and the energy per unit rest mass of the particle, respectively (the particle is assumed to have non-zero rest mass). This equation resembles the energy conservation equation in Newtonian dynamics, $E_\mathrm{N} = 1/2\,\dot{r}^2 + V_\mathrm{eff}(r)$, except for the additional term $-GML^2/c^2\,r^3$ in the effective potential V_eff that becomes dominant at small radii.

a) Treating the problem like a Newtonian one, sketch the effective potential for a particle near a black hole as a function of radius, both for a small and a large value of L. Characterize the possible types of trajectories/orbits in both cases.

b) Show that for each value of L there are two possible circular orbits

$$r_\pm = \frac{L^2 \pm [L^4 - 12G^2M^2L^2/c^2]^{1/2}}{2GM}, \tag{7.2}$$

provided that $L^2 > 12G^2M^2/c^2$.

c) Show that the r_+ solution has a minimum value of $r_+^{\rm min} = 6GM/c^2$ and argue that this is a stable orbit (i.e. corresponds to a minimum of the effective potential). What does this imply for the r_- solution?

d) Calculate the energy E of a particle at this innermost stable circular orbit and show that it's binding energy per unit rest mass $E_{\rm B}$ is

$$E_{\rm B} = (1 - (8/9)^{1/2})\, c^2 \simeq 0.06\, c^2.$$

e) Discuss briefly what happens as matter orbiting a black hole in an accretion disc approaches the innermost stable orbit. Compare this case to accretion onto a non-magnetic neutron star.

Chapter 8

High-Energy Astrophysics II

For Tutorial 8

8.1 Accretion Basics

Explain what is meant by *escape velocity* and device an expression for the escape velocity of a particle at distance R from a compact object of mass M. By assuming that the maximum escape velocity of a black hole of mass M is the speed of light, c, obtain an expression for the Schwarzschild radius R_{S} of this body. Discuss the validity of the approximations used to derive the Schwarzschild radius in this fashion and evaluate it for the cases $M = 10M_{\odot}$ and $M = 10^6 M_{\odot}$.

What rate of matter infall (in units of $M_{\odot}$ per year) would be needed to power a quasar of luminosity 10^{39} W if a black hole of mass $10^6 M_{\odot}$ lay at the quasar core? Assume conversion of gravitational potential energy into luminosity with 100% efficiency. Does your answer support accretion of matter by a black hole as a likely model for quasar energy generation?

8.2 Accretion discs

a) What is the physical significance of the Eddington luminosity, L_{Edd}? Derive the expression

$$L_{\mathrm{Edd}} = \frac{4\pi G m_{\mathrm{p}} M c}{\sigma_{\mathrm{T}}}$$

for pure hydrogen plasma accreting onto an object of mass M, where σ_{T} is the Thomson cross-section, clearly stating any assumptions made.

What is the physical significance of your answer to question 5.2a regarding the Eddington luminosity?

b) Assume that the emission from a geometrically-thin, optically-thick accretion disc is dominated by a component from its innermost part. Take this to be a uniform circular region of radius $r \sim 6GM/c^2$. By equating the luminosity emitted from this region to the Eddington luminosity, derive an approximate expression for the black-body temperature of radiation from the inner disc. Estimate the temperatures for black holes of $10\,\mathrm{M}_\odot$ and $10^8\,\mathrm{M}_\odot$ and comment on your results in the context of the observed spectra of X-ray binaries and active galaxies.

c) The galaxy NGC 4258 contains a black hole of mass $3.5 \times 10^7\,\mathrm{M}_\odot$. Calculate the Eddington luminosity and compare it with the observed X-ray luminosity of $4 \times 10^{33}\,\mathrm{W}$.

d) Contrast the properties of accretion discs with $\dot{M} \sim \dot{M}_{\mathrm{Edd}}$ and $\dot{M} \gg \dot{M}_{\mathrm{Edd}}$.

[$\sigma_{\mathrm{T}} = 6.6 \times 10^{-29}\,\mathrm{m}^2$.]

8.3 Doppler beaming and boosting

In the lectures we derived the formula for the apparent projected velocity $\beta_{\mathrm{app}}c$ of material in a jet with velocity βc at an angle θ to the line of sight,

$$\beta_{\mathrm{app}} = \frac{\beta \sin\theta}{1 - \beta\cos\theta}.$$

a) Use this formula to show that for any particular jet, the maximum apparent speed is

$$\beta_{\mathrm{app}}^{\mathrm{max}} = \gamma\beta,$$

which is seen when the jet is viewed from a direction such that

$$\beta = \cos\,\theta.$$

b) A source moving along the $\theta = 0$ axis emits photons isotropically in its rest frame (the S' frame). Use the relativistic abberation formulae to transform θ' to the observer's (S) frame θ and hence show that the distribution of photons in the S frame is given by

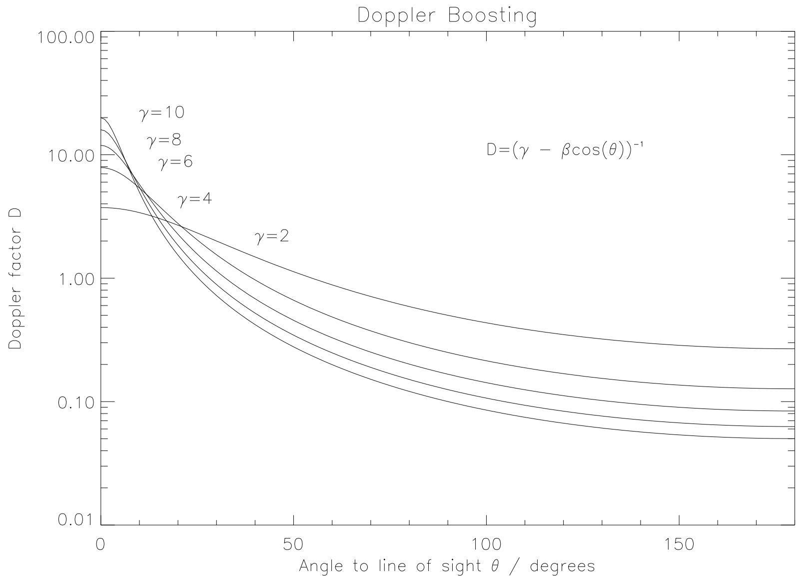


Figure 8.1: Doppler boosting of a relativistic jet.

$$P(\theta)\ \mathrm{d}\theta = D^2 \frac{1}{2} \sin\theta\ \mathrm{d}\theta,$$

where D is the relativistic Doppler factor, $[\gamma(1-\beta\cos\theta)]^{-1}$.

c) Detailed study of the single-sided jet in the quasar 3C273 suggests that the jet is pointing towards us, at about 10° from the line-of-sight, has a bulk Lorentz factor $\gamma = 11$, and a synchrotron spectral index of 0.5. The deepest radio maps of 3C273 have a dynamic range of about 5000 and show no sign whatsoever of a counterjet. Is this surprising?

Figure 8.1 shows how the Doppler factor varies as a function of γ and θ. There are couple of points to note.

First, D varies with $\cos(\theta)$, so for a receding source you can calculate it as $[\gamma(1-\beta\cos\theta)]^{-1}$ with $90 < \theta < 180$, or you can say $[\gamma(1+\beta\cos\theta)]^{-1}$ with $0 < \theta < 90$. Be alert!

Second, D becomes less than one *while the jet is still pointing towards you!* The peak of the boosting is occuring off to the side; you're looking at the fainter "shoulder" of the boosted beam.

8.4 Inverse-Compton scattering

a) Explain what is meant by *Thomson scattering* and *inverse-Compton scattering*. Give **three** examples where inverse-Compton scattering is important in astrophysics.

b) The power P in scattered radiation due to Thomson scattering is given by

$$P = c\sigma_{\mathrm{T}}U$$

where σ_{T} is the Thomson scattering cross-section and U is the energy density of incident radiation. Use this relation to derive an expression for the average power of radiation inverse Compton scattered by a population of relativistic electrons with Lorentz factors $\gamma \sim 1000$. Assume that the distribution of electrons is isotropic and that the radiation being scattered is at radio frequencies. Justify each step in your derivation.

c) Ultra-relativistic electrons of energy 10^{11} eV are observed at the Earth. By considering the effect on the electrons of inverse Compton scattering off the microwave background radiation, calculate the maximum length of time for which the electrons could have had energies larger than the observed value. Given that the age of the Galaxy is $\sim 10^{10}$ years, what does this imply? Suggest some possible sources of such high-energy electrons.

[The energy density of the microwave background is $2.6 \times 10^5\,\mathrm{eV\,m^{-3}}$; $\sigma_{\mathrm{T}} = 6.6 \times 10^{-29}\,\mathrm{m}^2$.]

8.5 Self-absorbed Synchrotron

By considering the intensity of radiation emitted by a black body, show how an estimate of the magnetic flux density in a compact radio source can be obtained from a measurement of its specific intensity at a frequency at which the source is optically thick. You may use the Rayleigh-Jeans approximation of the Planck function.

8.6 Thermal Bremsstrahlung

a) What is the mechanism that produces bremsstrahlung radiation? Beginning with the result that an interaction between a stationary proton and an electron with speed v at impact parameter b emits a total energy $\propto 1/(b^3 v)$, show that the spectral energy distribution of bremsstrahlung radiation in a hydrogen plasma as a function of frequency ν is given by

$$F_\nu \propto \frac{n_\mathrm{e}^2}{v} \ln\left(\frac{mv^2}{2h\nu}\right),$$

where n_e is the electron density.

b) The integrated bremsstrahlung emissivity ϵ (the power per unit volume) for a hydrogen plasma is given by

$$\epsilon = 1.7 \times 10^{-39} T^{1/2} n_\mathrm{e}^2 \ \mathrm{W\,m^{-3}},$$

where T is the temperature of the plasma in units of K and n_e is in units of $\mathrm{m^{-3}}$. Considering the core of a virialized cluster of galaxies with a gas mass of $10^{13}\, M_\odot$ (i.e. about 10% of the total core mass) and a size (0.1 $\mathrm{R_{500}}$) of 200 kpc, estimate the time it takes for the gas to cool, explaining quantitatively any assumptions you need to make.

c) Why is this estimate an upper limit to the cooling time? What are the implications for the existence of such clusters in the local Universe, given that the time since the Big Bang is approximately 13.7 Gyr?

d) The central galaxy in such a cluster is found to harbour a black hole of mass $10^9 M_\odot$. Describe, with a quantitative calculation, how this may reconcile any difficulties implied by your previous calculations.

8.7 Sunyaev-Zel'dovich Effect and H_0

a) What is meant by the *Sunyaev–Zel'dovich effect*? How is it measured? Under certain conditions, the magnitude of the Sunyaev–Zel'dovich effect is

$$\frac{\delta T_b}{T_b} = -8 \int \frac{\sigma_\mathrm{T} n_e k_\mathrm{B} T_e}{m_e c^2}\, \mathrm{d}l.$$

Give an account of the physical origin of this effect that qualitatively explains why $\delta T_b/T_b$ is given by this formula. Estimate the frequency at which this formula becomes invalid.

b) The galaxies of the Coma cluster have a mean recession velocity with respect to Earth of $7000\,\mathrm{km\,s^{-1}}$. Observations of the Coma cluster with X-ray telescopes show that the intracluster plasma has an angular diameter of 25 arcminutes, a temperature of $5 \times 10^7\,\mathrm{K}$ and an electron density of $\sim 3000\,m^{-3}$. Observations of the Cosmic Microwave Background toward the cluster at 30 GHz show a diminution in brightness temperature of 0.75 mK. Use these data to make an estimate of the Hubble parameter, describing any assumptions made.

c) Identify two difficulties involved in the measurement of the Hubble parameter via the Sunyaev–Zel'dovich effect, and describe how their influence may be minimized.

[$\sigma_{\mathrm{T}} = 6.6 \times 10^{-29}\,\mathrm{m}^2$; the mean temperature of the Cosmic Microwave Background is 2.73 K.]