Brief Applied Calculus

JAMES STEWART
McMaster University and University of Toronto

DANIEL CLEGG
Palomar College

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After learning a core group of basic functions, we will be armed with the tools to create formulas that describe scenarios as diverse as trends in the stock market, world population, historic Olympic wins, the growth of computing power, and the popularity of a new product. © Michael Nagle/Bloomberg via Getty Images

Functions and Models

The fundamental objects that we deal with in calculus are functions. This chapter prepares the way for calculus by discussing the basic ideas concerning functions, their graphs, and ways of transforming and combining them. We stress that a function can be represented in different ways: by an equation, in a table, by a graph, or in words. We look at the main types of functions that will be needed in our study of calculus and describe the process of using these functions as mathematical models of real-world phenomena.

1.1 Functions and Their Representations

Introduction to Functions

Mathematical relationships can be observed in virtually every aspect of our environment and daily lives. Populations, financial markets, the spread of diseases, setting the price of a new product, and the effects of pollution on an ecosystem can all be analyzed using mathematics.

Many mathematical relationships can be considered as *functions*. A function is a correspondence in which one quantity is determined by another. For instance, each day that the US stock market is open corresponds to a closing price of Google stock. We say that the daily closing price of the stock is a function of the date.

For additional illustrations, consider the following four situations.

A. The area A of a square plot of land depends on the length s of one side of the plot. The rule that connects s and A is given by the equation $A = s^2$. With each positive number s there is associated one value of A, and we can say that A is a *function* of s.

B. The human population of the world P depends on the time t. The table gives estimates of the world population for certain years. For instance, when $t = 1950$, $P \approx 2{,}560{,}000{,}000$. But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

Year	Population (millions)
1900	1650
1910	1750
1920	1860
1930	2070
1940	2300
1950	2560
1960	3040
1970	3710
1980	4450
1990	5280
2000	6080
2010	6870

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

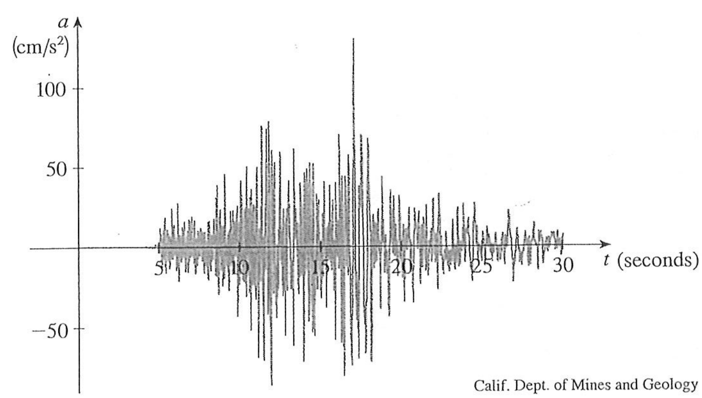


FIGURE 1
Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number (s, t, w, or t), another number (A, P, C, or a) is assigned. In each case we say that the second number is a function of the first number. You can think of a function in terms of an input/output relationship, where the function assigns an output value to each input value it accepts.

■ A **function** is a rule that assigns to each input exactly one output.

Notice that while a function can assign only one output to each input, it is perfectly acceptable for two different inputs to share the same output. Although a function can be defined for any sort of input or output, we usually consider functions for which the inputs and outputs are real numbers.

Notation and Terminology for Functions

We typically refer to a function by a single letter such as f. If x represents an input to the function f, the corresponding output is $f(x)$, read "f of x."

The set of all allowable inputs is called the **domain** of the function.

The **range** of f is the set of all possible output values, $f(x)$, as x varies throughout the domain.

A symbol that represents an arbitrary number in the *domain* of a function f is called an **independent variable**.

A symbol that represents a number in the *range* of f is called a **dependent variable**.

In Example A, for instance, s is the independent variable and A is the dependent variable. (We can choose the value of s independently, but A *depends* on the value of s.) Using function notation we can write $A = f(s)$, where f represents the area function.

It's helpful to think of a function as a **machine** (see Figure 2). If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function.

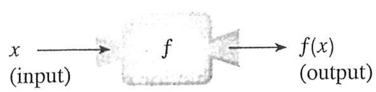


FIGURE 2
Machine diagram for a function f

For example, many cash registers used in retail stores have a button that, when pressed, automatically computes the sales tax to be added to the total. This button can be thought of as a function: An amount of money is entered as an input, and the machine outputs an amount of tax. Both the domain and range of this function are sets of positive numbers that represent amounts of money.

■ EXAMPLE 1 A Price Function

A cafe sells its basic coffee in three different cup sizes: 8, 10, and 14 ounces. They charge \$0.22 per ounce for the drinks.

(a) If the function p is defined so that $p(v)$ is the price of v ounces of coffee, find and interpret the value of $p(10)$.

(b) What are the domain and range of p?

SOLUTION

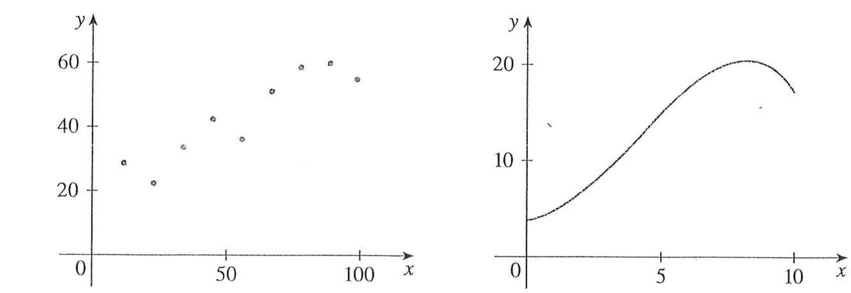
(a) The function value $p(10)$ represents the output (price) of the function when the input is 10 ounces of coffee. Thus $p(10) = \$0.22 \times 10 = \$2.20$.

(b) If we assume that the cafe sells only 8-, 10-, and 14-ounce coffee drinks, then the only allowable inputs to the price function are the three numbers 8, 10, and 14, so the domain of p is the set $\{8, 10, 14\}$. The range is the set of outputs that correspond to the inputs in the domain: $\{1.76, 2.20, 3.08\}$. ■

We use braces { } to list the elements of a set.

Although the rule defining a function may be clear, or you may have a list of inputs and outputs for a function, it is often easiest to analyze a function if you can visualize the relationship between the inputs and outputs. The most common method for visualizing a function is to view its graph. If f is a function, then its **graph** is the set of input-output pairs $(x, f(x))$ plotted as points for all x in the domain of f. In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.

If the domain consists of isolated values, as in Example 1, the data are *discrete* and the graph is a collection of individual points, called a *scatter plot*. On the other hand, if the input variable represents a quantity that can vary *continuously* through an interval of values, the graph is a curve or line (see Figure 3). We will define a continuous function more formally in Chapter 2. For now, you can think of a continuous function as one for which you can sketch its graph without lifting your pencil from the paper.



(a) Scatter plot

(b) Continuous function

FIGURE 3
Graphs of functions

The graph of a function f gives us a useful picture of the behavior or "life history" of a function. Since the y-coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x. (See Figure 4.) The graph of f also allows us to picture the domain of f on the x-axis and its range on the y-axis as in Figure 5.

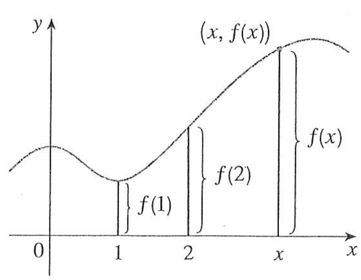


FIGURE 4

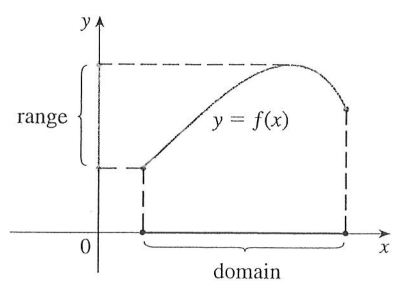


FIGURE 5

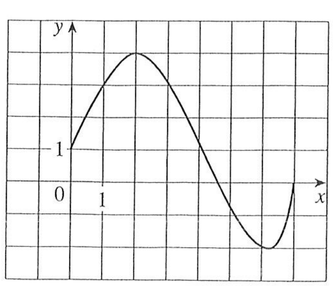


FIGURE 6

■ EXAMPLE 2 Reading Information from a Graph

The graph of a function f is shown in Figure 6.

(a) Find the values of $f(1)$ and $f(5)$.

(b) What are the domain and range of f?

Recall that a closed bracket is used with interval notation to indicate an included value, while an open parenthesis indicates that the endpoint of the interval is not included. For instance, the interval $[2, 5)$ is equivalent to $\{x \mid 2 \leqslant x < 5\}$. An interval is called *closed* if it includes both endpoints; an *open* interval includes neither endpoint. A detailed review is included in Appendix A.

SOLUTION

(a) We see from Figure 6 that the point $(1, 3)$ lies on the graph of f, so the value of f at 1 is $f(1) = 3$. (In other words, the point on the graph that lies above $x = 1$ is 3 units above the x-axis.)

When $x = 5$, the graph lies about 0.7 unit below the x-axis, so we estimate that $f(5) \approx -0.7$.

(b) We see that $f(x)$ is defined when $0 \leqslant x \leqslant 7$, so the domain of f is the closed interval $[0, 7]$. Notice that f takes on all values from -2 to 4, so the range of f is

$$\{y \mid -2 \leqslant y \leqslant 4\} = [-2, 4]$$

Representations of Functions

We have seen four possible ways to represent a function:

- verbally (by a description in words)
- numerically (by a table of values)
- visually (by a graph)
- algebraically (by an explicit formula)

If a single function can be represented in several ways, it is often useful to go from one representation to another to gain additional insight into the function. But certain functions are described more naturally by one method than by another. With this in mind, let's reexamine the four situations that we considered at the beginning of this section.

A. The most useful representation of the area of a square plot of land as a function of its side length is probably the algebraic formula $A(s) = s^2$, though it is possible to compile a table of values or sketch a graph (half a parabola). Because a square has to have a positive side length, the domain is $\{s \mid s > 0\} = (0, \infty)$, and the range is also $(0, \infty)$.

B. We are given a description of the function in words: $P(t)$ is the human population of the world at time t. For convenience, we can measure $P(t)$ in millions and let $t = 0$ represent the year 1900. Then the table of values of world population at the left provides a convenient representation of this function. If we plot these values, we get the scatter plot in Figure 7.

t	$P(t)$ (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

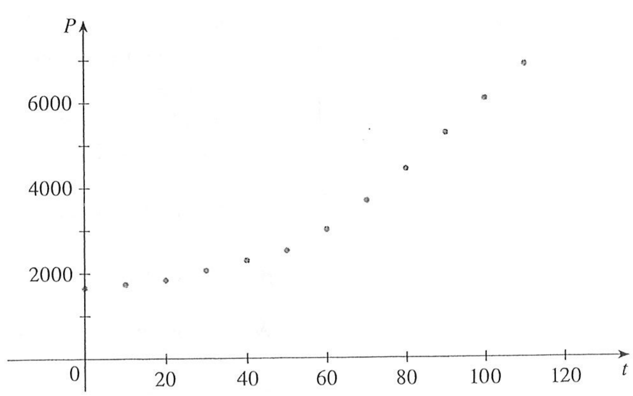


FIGURE 7

This scatter plot is a useful representation; the graph allows us to absorb all the data at once. What about a formula? Of course, it's impossible to devise an explicit formula that gives the exact human population $P(t)$ at any time t. But it is possible to find an expression for a function that *approximates* $P(t)$. In fact, using methods explained in Section 1.5, we obtain the approximation

$$P(t) \approx (1436.53) \cdot (1.01395)^t$$

Figure 8 shows that this function is a reasonably good "fit." Notice that here we have graphed a continuous curve as an approximation to discrete data. We will soon see that the ideas of calculus can be applied to discrete data as well as explicit formulas.

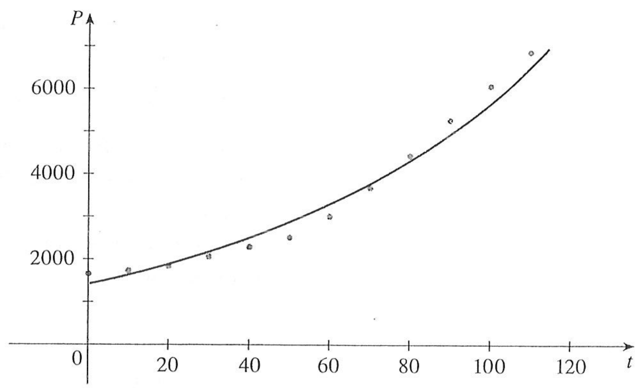


FIGURE 8

A function defined by a table of values is called a *tabular* function.

w (ounces)	$C(w)$ (dollars)
$0 < w \leq 1$	0.88
$1 < w \leq 2$	1.05
$2 < w \leq 3$	1.22
$3 < w \leq 4$	1.39
$4 < w \leq 5$	1.56
⋮	⋮

C. Again the function is described in words: $C(w)$ is the cost of mailing a large envelope with weight w. The rule that the US Postal Service used as of 2011 is as follows: The cost is 88 cents for up to 1 oz, plus 17 cents for each additional ounce (or less), up to 13 oz. The table of values at the left is the most convenient representation for this function, though it is possible to sketch a graph (see Example 10).

D. The graph shown in Figure 1 is the most natural representation of the vertical acceleration function $a(t)$. It's true that a table of values could be compiled, and it is even possible to devise an approximate formula. But everything a geologist needs to know—amplitudes and patterns—can be seen easily from the graph. (The same is true for the patterns seen in electrocardiograms of heart patients and polygraphs for lie-detection.)

In the next example we sketch the graph of a function that is defined verbally.

EXAMPLE 3 Drawing a Graph from a Verbal Description

When you turn on a hot-water faucet, the temperature T of the water depends on how long the water has been running. Draw a rough graph of T as a function of the time t that has elapsed since the faucet was turned on.

SOLUTION

The initial temperature of the running water is close to room temperature because the water has been sitting in the pipes. When the water from the hot-water tank starts flowing from the faucet, T increases quickly. In the next phase, T is constant at the temperature of the heated water in the tank. When the tank is drained, T

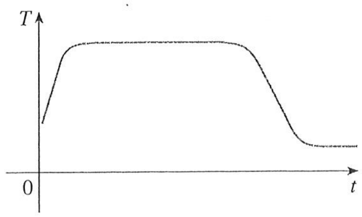


FIGURE 9

decreases to the temperature of the water supply. This enables us to make the rough sketch of T as a function of t in Figure 9. ■

A more accurate graph of the function in Example 3 could be obtained by using a thermometer to measure the temperature of the water at 10-second intervals. In general, researchers collect experimental data and use them to sketch the graphs of functions, as the next example illustrates.

■ EXAMPLE 4 A Numerically Defined Function

t	$N(t)$
1	41.4
3	25.1
5	15.5
7	10.2
9	6.0

The data shown in the margin give weekly sales figures for a video game shortly after its release. Let $N(t)$ be the number of copies sold, in thousands, during the week ending t weeks after the game's release. Sketch a scatter plot of these data, and use the scatter plot to draw a continuous approximation to the graph of $N(t)$. Then use the graph to estimate the number of copies sold during the sixth week.

SOLUTION

We plot the five points corresponding to the data from the table in Figure 10. The data points in Figure 10 look quite well behaved, so we simply draw a smooth curve through them by hand as in Figure 11. (Later in this chapter you will see how to find an algebraic formula that approximates the data.)

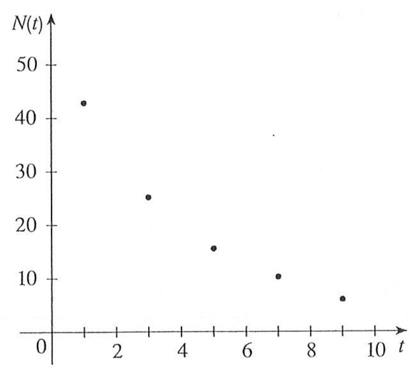


FIGURE 10

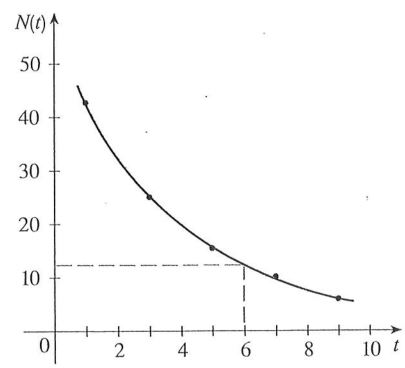


FIGURE 11

From the graph, it appears that $N(6) \approx 12.5$, so we estimate that 12,500 units were sold during the sixth week. ■

In the following example we start with a verbal description of a function in a physical situation and obtain an explicit algebraic formula. The ability to do this is a useful skill in solving optimization problems such as maximizing the profit of a company.

■ EXAMPLE 5 Expressing a Cost as a Function

A rectangular storage container with an open top has a volume of 10 m^3. The length of its base is twice its width. Material for the base costs \$10 per square meter; material for the sides costs \$6 per square meter. Express the cost of materials as a function of the width of the base.

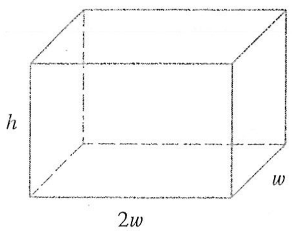


FIGURE 12

SOLUTION

We draw a diagram as in Figure 12 and introduce notation by letting w and $2w$ be the width and length of the base, respectively, and h be the height.

The area of the base is $(2w)w = 2w^2$, so the cost, in dollars, of the material for the base is $10(2w^2)$. Two of the sides have area wh and the other two have area $2wh$, so the cost of the material for the sides is $6[2(wh) + 2(2wh)]$. The total cost is therefore

$$C = 10(2w^2) + 6[2(wh) + 2(2wh)] = 20w^2 + 36wh$$

To express C as a function of w alone, we need to eliminate h and we do so by using the fact that the volume is $10\ \text{m}^3$. Thus

$$\text{volume} = \text{width} \cdot \text{length} \cdot \text{height} = w(2w)h = 10$$

which gives

$$h = \frac{10}{2w^2} = \frac{5}{w^2}$$

Substituting this into the expression for C, we have

$$C = 20w^2 + 36w\left(\frac{5}{w^2}\right) = 20w^2 + \frac{180}{w}$$

Therefore the equation

$$C(w) = 20w^2 + \frac{180}{w} \qquad w > 0$$

expresses C as a function of w. ■

In the next two examples we look at functions given by algebraic formulas.

■ EXAMPLE 6 A Function Defined by a Formula

If $f(x) = 2x^2 - 5x + 1$, evaluate

(a) $f(-3)$ **(b)** $f(4) - f(2)$ **(c)** $\dfrac{f(1 + h) - f(1)}{h}$ $(h \neq 0)$

SOLUTION

(a) Replace x by -3 in the expression for $f(x)$:

$$f(-3) = 2(-3)^2 - 5(-3) + 1 = 2 \cdot 9 + 15 + 1 = 18 + 15 + 1 = 34$$

(b) $f(4) - f(2) = [2(4)^2 - 5(4) + 1] - [2(2)^2 - 5(2) + 1] = 13 - (-1) = 14$

(c) We first evaluate $f(1 + h)$ by replacing x by $1 + h$ in the expression for $f(x)$:

$$\begin{aligned} f(1 + h) &= 2(1 + h)^2 - 5(1 + h) + 1 \\ &= 2(1 + 2h + h^2) - 5(1 + h) + 1 \\ &= 2 + 4h + 2h^2 - 5 - 5h + 1 = 2h^2 - h - 2 \end{aligned}$$

The expression

$$\frac{f(1+h)-f(1)}{h}$$

in Example 6 is called a **difference quotient** and occurs frequently in calculus. We will will begin making use of it in Chapter 2.

Then we substitute into the given expression and simplify:

$$\begin{aligned}\frac{f(1+h)-f(1)}{h} &= \frac{(2h^2-h-2)-(2-5+1)}{h}\\ &= \frac{2h^2-h-2-(-2)}{h}\\ &= \frac{2h^2-h}{h} = \frac{h(2h-1)}{h} = 2h-1\end{aligned}$$ ■

■ EXAMPLE 7

Determining the Domain of a Function Defined by a Formula

Find the domain of each function.

(a) $B(r) = \sqrt{r+2}$ **(b)** $g(x) = \dfrac{1}{x^2 - x}$

SOLUTION

If a function is given by a formula and the domain is not stated explicitly, the convention is that the domain is the set of all numbers for which the formula makes sense and defines a real number.

(a) Because the square root of a negative number is not defined (as a real number), the domain of B consists of all values of r such that $r + 2 \geqslant 0$. This is equivalent to $r \geqslant -2$, so the domain is the interval $[-2, \infty)$.

(b) Since

$$g(x) = \frac{1}{x^2 - x} = \frac{1}{x(x-1)}$$

and division by 0 is not allowed, we see that $g(x)$ is not defined when $x = 0$ or $x = 1$. Thus the domain of g is $\{x \mid x \neq 0,\ x \neq 1\}$. ■

The graph of a function is a curve or scatter plot in the xy-plane. But the question arises: Which graphs in the xy-plane represent functions and which do not? This is answered by the following test.

> ■ **The Vertical Line Test** A curve or scatter plot in the xy-plane is the graph of a function of x if and only if no vertical line intersects the graph more than once.

The reason for the truth of the Vertical Line Test can be seen in Figure 13.

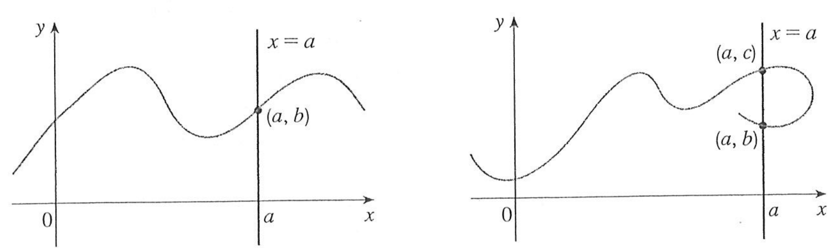


FIGURE 13

If each vertical line $x = a$ intersects a curve only once, at (a, b), then exactly one functional value is defined by $f(a) = b$. But if a line $x = a$ intersects the curve twice, at (a, b) and (a, c), then the curve can't represent a function because a function can't assign two different output values to an input a.

EXAMPLE 8 Using the Vertical Line Test

Determine whether the graph represents a function.

(a)

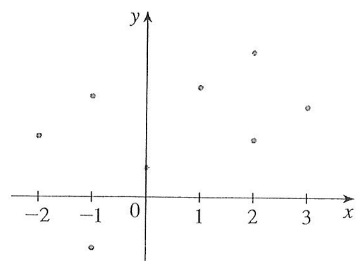


FIGURE 14

(b)

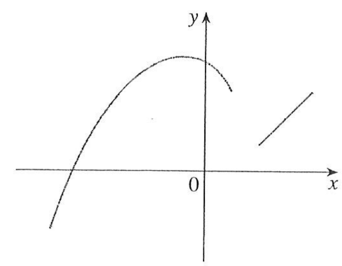


FIGURE 15

SOLUTION

(a) Notice that if we draw a vertical line on the scatter plot in Figure 14 at $x = -1$ or at $x = 2$, the line will intersect two of the points. Therefore the scatter plot does not represent a function.

(b) No matter where we draw a vertical line on the graph in Figure 15, the line will intersect the graph at most once, so this is the graph of a function. Notice that the "gap" in the graph does not pose any trouble; it is acceptable for a vertical line not to intersect the graph at all. ■

Mathematical Modeling

In Example B on page 5, we drew a scatter plot of the world population data and then found an explicit equation that approximated the behavior of the population data. The function P we used is called a *mathematical model* for the population. A **mathematical model** is a mathematical description (usually by means of a function or an equation) of a real-world scenario, such as the demand for a company's product or the life expectancy of a person at birth. Although a function used as a model may not exactly match observed data, it should be a close enough approximation to allow us to understand and analyze the situation, and perhaps to make predictions about future behavior.

Figure 16 illustrates the process of mathematical modeling. Given a real-world problem, our first task is to formulate a mathematical model by identifying and

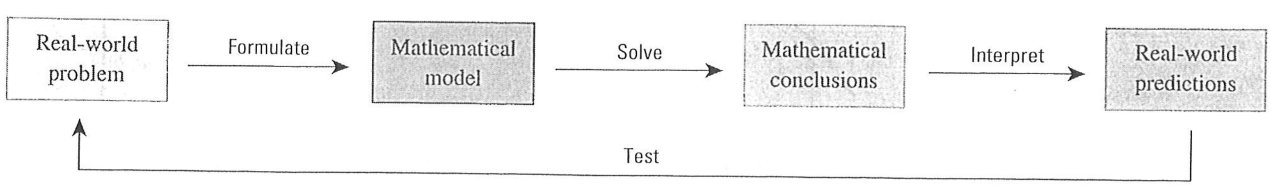


FIGURE 16 The modeling process

naming the independent and dependent variables and making assumptions that simplify the situation enough to make it mathematically tractable. We use our knowledge of the physical situation and our mathematical skills to develop equations that relate the variables. In situations where there is no physical law to guide us, we may need to collect data (either from the Internet or a library or by conducting our own experiments) and examine the data in the form of a table or a graph. In the next few sections, we will see a variety of different types of algebraic equations that are often used as mathematical models.

The second stage is to apply the mathematics that we know (such as the calculus that will be developed throughout this book) to the mathematical model that we have formulated in order to derive mathematical conclusions. Then, in the third stage, we take those mathematical conclusions and interpret them as information about the original real-world situation by way of offering explanations or making predictions. The final step is to test our predictions by checking against new real data. If the predictions don't compare well with reality, we need to refine our model or formulate a new model and start the cycle again.

Keep in mind that a mathematical model is rarely a completely accurate representation of a physical situation—it is an *idealization*. A good model simplifies reality enough to permit mathematical calculations but is accurate enough to provide valuable conclusions. It is important to realize the limitations of the model. In the end, Mother Nature and financial markets have not always been predictable!

Piecewise Defined Functions

In some instances, no single formula adequately describes the behavior of a quantity. A population may exhibit one growth pattern for 20 years but then change to a different trend. In such cases we can use a function with different formulas in different parts of the domain. We call such functions *piecewise defined functions*, and the next two examples illustrate the concept.

EXAMPLE 9 Graphing a Piecewise Defined Function

A function f is defined by

$$f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1 \end{cases}$$

Evaluate $f(-2)$, $f(-1)$, and $f(1)$ and sketch the graph.

SOLUTION

Remember that a function is a rule. For this particular function the rule is the following: First look at the value of the input x. If it happens that $x \leq -1$, then the value of $f(x)$ is $1 - x$. On the other hand, if $x > -1$, then the value of $f(x)$ is x^2.

Since $-2 \leq -1$, we have $f(-2) = 1 - (-2) = 3$.

Since $-1 \leq -1$, we have $f(-1) = 1 - (-1) = 2$.

Since $1 > -1$, we have $f(1) = 1^2 = 1$.

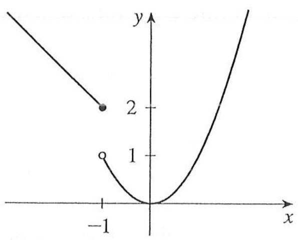


FIGURE 17

How do we draw the graph of f? We observe that if $x \leq -1$, then $f(x) = 1 - x$, so the part of the graph of f that lies to the left of $x = -1$ must coincide with the line $y = 1 - x$, which has slope -1 and y-intercept 1. (Linear equations are reviewed in Section 1.3.) If $x > -1$, then $f(x) = x^2$, so the part of the graph of f that lies to the right of the line $x = -1$ must coincide with the graph of $y = x^2$, which is a parabola. This enables us to sketch the graph in Figure 17. The solid dot indicates that the point $(-1, 2)$ is included on the graph; the open dot indicates that the point $(-1, 1)$ is excluded from the graph.

EXAMPLE 10 A Step Function

In Example C at the beginning of this section we considered the cost $C(w)$ of mailing a large envelope with weight w. In effect, this is a piecewise defined function because, from the table of values on page 6, we have

$$C(w) = \begin{cases} 0.88 & \text{if } 0 < w \leq 1 \\ 1.05 & \text{if } 1 < w \leq 2 \\ 1.22 & \text{if } 2 < w \leq 3 \\ 1.39 & \text{if } 3 < w \leq 4 \\ \vdots & \end{cases}$$

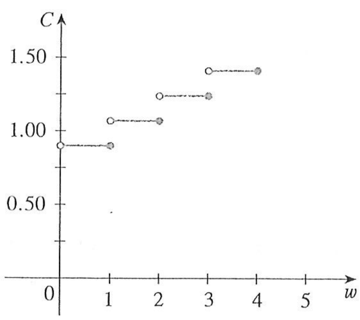


FIGURE 18

The graph is shown in Figure 18. You can see why functions similar to this one are called *step functions*—they jump from one value to the next.

Symmetry

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an **even function**. For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x)$$

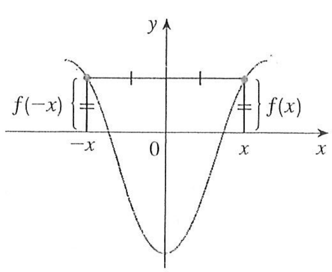


FIGURE 19 An even function

The geometric significance of an even function is that its graph is symmetric with respect to the y-axis (see Figure 19). This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.

If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an **odd function**. For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

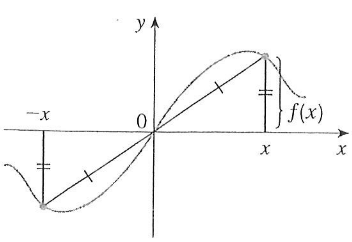


FIGURE 20 An odd function

The graph of an odd function is symmetric about the origin (see Figure 20). If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180° about the origin. Note that a function does not have to be either even or odd; many are neither.

■ EXAMPLE 11 Testing for Symmetry

Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $f(x) = x^5 + x$ **(b)** $g(x) = 1 - x^4$ **(c)** $h(x) = 2x - x^2$

SOLUTION

(a)
$$\begin{aligned} f(-x) &= (-x)^5 + (-x) = (-1)^5x^5 + (-x) \\ &= -x^5 - x = -(x^5 + x) \\ &= -f(x) \end{aligned}$$

Therefore f is an odd function.

(b)
$$g(-x) = 1 - (-x)^4 = 1 - x^4 = g(x)$$

So g is even.

(c)
$$h(-x) = 2(-x) - (-x)^2 = -2x - x^2$$

Since $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$, we conclude that h is neither even nor odd. ■

The graphs of the functions in Example 11 are shown in Figure 21. Notice that the graph of h is symmetric neither about the y-axis nor about the origin.

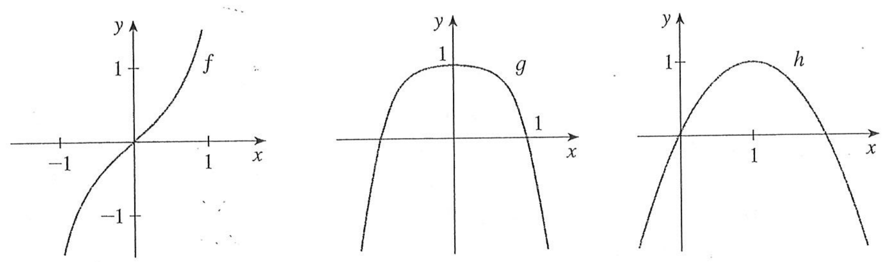


FIGURE 21 (a) Odd function (b) Even function (c) Neither even nor odd

■ Exercises 1.1

1. **Price function** A nursery sells potting soil for \$0.40 per pound, and the soil is available in 4-lb, 10-lb, and 50-lb bags. If $f(x)$ is the price of a bag of potting soil that weighs x pounds,
   (a) find and interpret the value of $f(10)$.
   (b) determine the domain and range of f.

2. **Price function** An Internet retailer charges \$4.99 to ship an order that totals less than \$25 and \$5.99 for an order up to \$75, and offers free shipping for an order over \$75. If $g(p)$ is the shipping cost for an order totaling p dollars, state the domain and range of g.

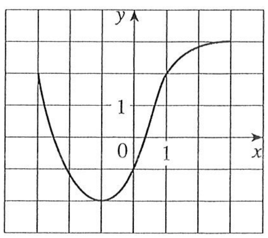
3. **Population function** Let $P(t)$ be the population, in thousands, of a city t years after January 1, 2000. Interpret the equation $P(8) = 64.3$. What does $P(4.5)$ represent?

4. **Blood alcohol content** Let $B(t)$ be the blood alcohol content (measured as the percentage by volume of alcohol in the blood) of a dinner guest t hours after her arrival. Interpret the equation $B(1.25) = 0.06$ in this context.

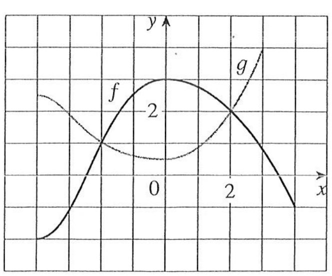
5. **Fuel economy** Let $F(s)$ be the average fuel economy of a particular car, measured in miles per gallon, when the car is being driven at s mi/h. What does the equation $F(65) = 24.7$ say in this context?

6. **Loan payments** Let $N(r)$ be the number of \$300 monthly payments required to repay an \$18,000 auto loan when the interest rate is r percent. What does the equation $N(6.5) = 73$ say in this context?

7. The graph of a function f is given.
   (a) State the value of $f(-1)$.
   (b) Estimate the value of $f(2)$.
   (c) For what values of x is $f(x) = 2$?
   (d) Estimate the values of x such that $f(x) = 0$.
   (e) State the domain and range of f.



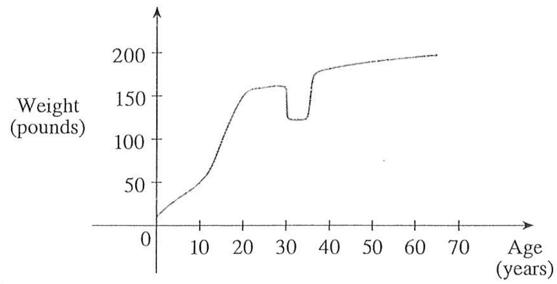
8. The graphs of f and g are given.
   (a) State the values of $f(-4)$ and $g(3)$.
   (b) For what values of x is $f(x) = g(x)$?
   (c) Estimate the solutions of the equation $f(x) = -1$.
   (d) State the domain and range of f.
   (e) State the domain and range of g.



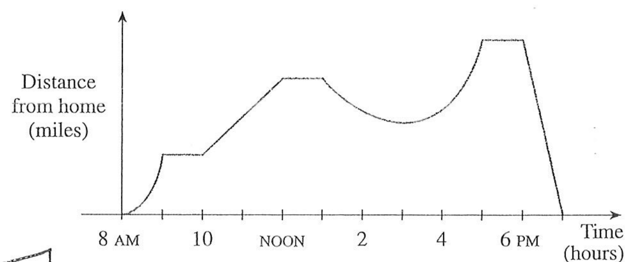
9. **Earthquakes** Figure 1 was recorded by an instrument operated by the California Department of Mines and Geology at the University Hospital of the University of Southern California in Los Angeles. Use it to estimate the range of the vertical ground acceleration function at USC during the Northridge earthquake.

10. In this section we discussed examples of ordinary, everyday functions: Population is a function of time, postage cost is a function of weight, water temperature is a function of time. Give three other examples of functions from everyday life that are described verbally. What can you say about the domain and range of each of your functions? If possible, sketch a rough graph of each function.

11. **Weight function** The graph gives the weight of a certain person as a function of age. Describe in words how this person's weight varies over time. What do you think happened when this person was 30 years old?



12. **Distance function** The graph gives a salesman's distance from his home as a function of time on a certain day. Describe in words what the graph indicates about his travels on this day.



13. **Temperature function** You put some ice cubes in a glass, fill the glass with cold water, and then let the glass sit on a table. Describe how the temperature of the water changes as time passes. Then sketch a rough graph of the temperature of the water as a function of the elapsed time.

14. **Hours of daylight** Sketch a rough graph of the number of hours of daylight as a function of the time of year.

15. **Temperature function** Sketch a rough graph of the outdoor temperature as a function of time during a typical spring day.

16. **Market value** Sketch a rough graph of the market value of a new car as a function of time for a period of 20 years. Assume the car is well maintained.

17. **Retail sales** Sketch a rough graph of the average daily amount of a particular type of coffee bean (measured in pounds) sold by a store as a function of the price of the beans.

18. **Temperature function** You place a frozen pie in an oven and bake it for an hour. Then you take it out and let it cool before eating it. Describe how the temperature of the pie changes as time passes. Then sketch a rough graph of the temperature of the pie as a function of time.

19. **Lawn height** A homeowner mows the lawn every Wednesday afternoon. Sketch a rough graph of the height of the grass as a function of time over the course of a four-week period.

20. Air travel An airplane flies from an airport and lands an hour later at another airport, 400 miles away. If t represents the time in minutes since the plane has left the terminal building, let $x(t)$ be the horizontal distance traveled and $y(t)$ be the altitude of the plane.

(a) Sketch a possible graph of $x(t)$.

(b) Sketch a possible graph of $y(t)$.

(c) Sketch a possible graph of the ground speed.

21. Phone subscribers The number N (in millions) of US cellular phone subscribers is shown in the table. (End of year estimates are given.)

t	1996	1998	2000	2002	2004	2006
N	44	69	109	141	182	233

(a) Use the data to sketch a rough graph of N as a function of t.

(b) Use your graph to estimate the number of cell-phone subscribers at the end of 2001 and 2005.

22. Temperature Temperature readings T (in °F) were recorded every two hours from midnight to 2:00 PM in Baltimore on September 26, 2007. The time t was measured in hours from midnight.

t	0	2	4	6	8	10	12	14
T	68	65	63	63	65	76	85	91

(a) Use the readings to sketch a rough graph of T as a function of t.

(b) Use your graph to estimate the temperature at 11:00 AM.

23. If $f(x) = 3x^2 - x + 2$, find $f(2)$, $f(-2)$, $f(a)$, $f(-a)$, $f(a+1)$, $2f(a)$, $f(2a)$, $f(a^2)$, $[f(a)]^2$, and $f(a+h)$.

24. If $g(t) = 4t - t^2$, find $g(3)$, $g(-1)$, $g(x)$, $g(x-2)$, and $g(x+h)$.

25–30 ■ Evaluate the difference quotient for the given function. Simplify your answer.

25. $f(x) = x^2 + 1$, $\quad \dfrac{f(4+h) - f(4)}{h}$

26. $f(x) = 2x^2 - x$, $\quad \dfrac{f(t+h) - f(t)}{h}$ $\quad$ $t = 1$ $\quad$ $h = .5$

27. $f(x) = 4 + 3x - x^2$, $\quad \dfrac{f(3+h) - f(3)}{h}$

28. $f(x) = x^3$, $\quad \dfrac{f(a+h) - f(a)}{h}$

29. $f(x) = \dfrac{1}{x}$, $\quad \dfrac{f(x) - f(a)}{x - a}$

30. $f(x) = \dfrac{x+3}{x+1}$, $\quad \dfrac{f(x) - f(1)}{x - 1}$

31–34 ■ Find the domain of the function.

31. $f(x) = \dfrac{x}{3x - 1}$

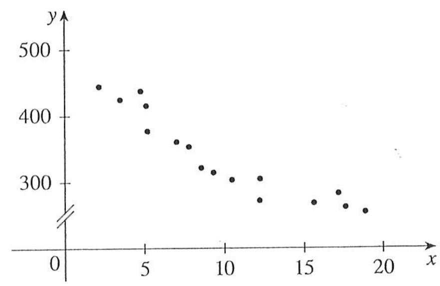
32. $f(x) = \dfrac{3x + 4}{x^2 - x}$

33. $f(t) = \sqrt{2t + 6}$

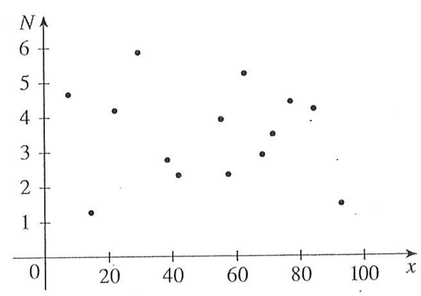
34. $g(u) = \sqrt{u - 4} + 1.5u$

35–36 ■ Determine whether the scatter plot is the graph of a function of x. Explain how you reached your conclusion.

35.

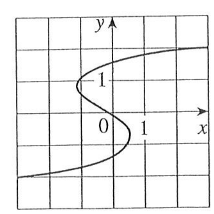


36.



37–40 ■ Determine whether the curve is the graph of a function of x. If it is, state the domain and range of the function.

37.



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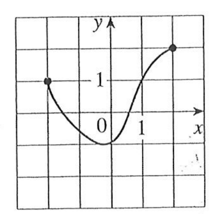

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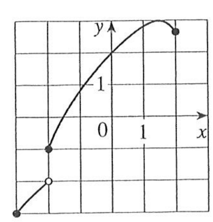

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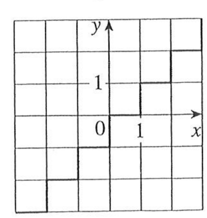


41–44 ■ Evaluate $f(-3)$, $f(0)$, and $f(2)$ for the piecewise defined function. Then sketch the graph of the function.

41. $f(x) = \begin{cases} x + 2 & \text{if } x < 0 \\ 1 - x & \text{if } x \geqslant 0 \end{cases}$

42. $f(x) = \begin{cases} 3 - \frac{1}{2}x & \text{if } x < 2 \\ 2x - 5 & \text{if } x \geqslant 2 \end{cases}$

43. $f(x) = \begin{cases} x + 1 & \text{if } x \leqslant -1 \\ x^2 & \text{if } x > -1 \end{cases}$

44. $f(x) = \begin{cases} -1 & \text{if } x \leqslant 1 \\ 7 - 2x & \text{if } x > 1 \end{cases}$

45–48 ■ Find a formula for the described function and state its domain.

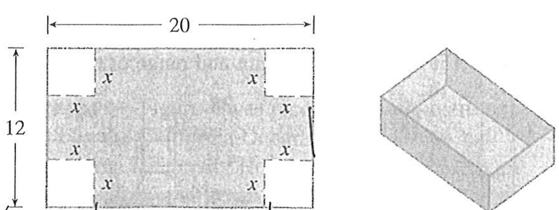
45. A rectangle has perimeter 20 m. Express the area of the rectangle as a function of the length of one of its sides.

46. A rectangle has area 16 m^2. Express the perimeter of the rectangle as a function of the length of one of its sides.

47. Surface area An open rectangular box with volume 2 m^3 has a square base. Express the surface area of the box as a function of the length of a side of the base.

48. Height and width A closed rectangular box with volume 8 ft^3 has length twice the width. Express the height of the box as a function of the width.

49. Box design A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions 12 in. by 20 in. by cutting out equal squares of side x at each corner and then folding up the sides as in the figure. Express the volume V of the box as a function of x.



50. Taxi fares A taxi company charges two dollars for the first mile (or part of a mile) and 20 cents for each succeeding tenth of a mile (or part). Express the cost C, in dollars, of a ride as a function of the distance x traveled, in miles, for $0 < x < 2$, and sketch the graph of this function.

51. Income tax In a certain country, income tax is assessed as follows. There is no tax on income up to \$10,000. Any income beyond \$10,000 is taxed at a rate of 10%, up to an income of \$20,000. Any income over \$20,000 is taxed at 15%.

(a) Sketch the graph of the tax rate R as a function of the income I.

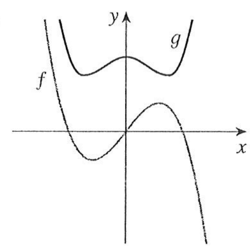
(b) How much tax is assessed on an income of \$14,000? On \$26,000?

(c) Sketch the graph of the total assessed tax T as a function of the income I.

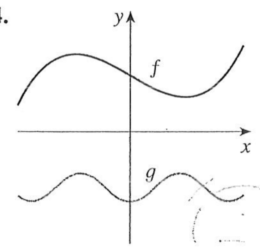
52. The functions in Example 10 and Exercises 50 and 51(a) are called *step functions* because their graphs look like stairs. Give two other examples of step functions that arise in everyday life.

53–54 ■ Graphs of f and g are shown. Decide whether each function is even, odd, or neither. Explain your reasoning.

53.



54.



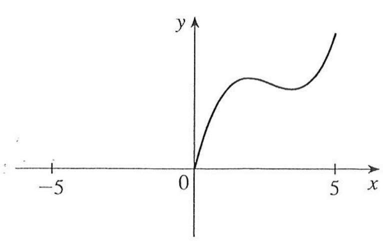
55. (a) If the point (5, 3) is on the graph of an even function, what other point must also be on the graph?

(b) If the point (5, 3) is on the graph of an odd function, what other point must also be on the graph?

56. A function f has domain $[-5, 5]$ and a portion of its graph is shown.

(a) Complete the graph of f if it is known that f is even.

(b) Complete the graph of f if it is known that f is odd.



57–62 ■ Determine whether f is even, odd, or neither. If you have a graphing calculator, use it to check your answer visually.

57. $f(x) = \dfrac{x}{x^2 + 1}$

58. $f(x) = \dfrac{x^2}{x^4 + 1}$

59. $f(x) = \dfrac{x}{x + 1}$

60. $f(x) = x|x|$

61. $f(x) = 1 + 3x^2 - x^4$

62. $f(x) = 1 + 3x^3 - x^5$

Challenge Yourself

63. If f and g are both even functions and $h(x) = f(x) + g(x)$, is h even? If f and g are both odd functions, is h odd? What if f is even and g is odd? Justify your answers.

64. Window area A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 ft, express the area A of the window as a function of the width x of the window.

1.2 Combining and Transforming Functions

In this section we form new functions by combining existing functions in various ways. We also learn how to transform functions by shifting, stretching, or reflecting their graphs. These skills will enable you to use a basic set of functions, studied in the sections ahead, to design specific functions that model a wide variety of applications.

Combinations of Functions

Two functions f and g can be combined to form new functions using the operations of addition, subtraction, multiplication, and division in a manner similar to the way we add, subtract, multiply, and divide real numbers. For instance, we can define a new function h that is the sum of f and g by the equation $h(x) = f(x) + g(x)$. This means that the output of the new function h is the sum of the outputs of the individual functions f and g. This definition makes sense if both $f(x)$ and $g(x)$ are defined. Thus the domain of the function h consists of only those values that belong to both the domain of f and the domain of g.

Suppose a company has two different shipping centers, one on the West Coast and the other on the East Coast. If $W(t)$ is the number of packages shipped from the western facility t weeks after the start of the year, and $E(t)$ is the number of packages shipped from the eastern facility t weeks after the start of the year, then we can define a new function $N(t)$ by

$$N(t) = W(t) + E(t)$$

Thus $N(t)$ measures the combined number of packages sent from both shipping centers t weeks after the start of the year. Notice that the input for each function is the same; if the inputs of two functions are not measuring the same quantities, the sum of the functions is not meaningful.

We can subtract, multiply, or divide functions in a similar way. For instance, $k(x) = f(x)\,g(x)$ means that the output of the function k is the product of the outputs of the functions f and g. The domain of each of these new functions consists of all the numbers that appear in both the domain of f and the domain of g, with the exception that if we divide f by g, we must ensure that no division by 0 will occur. So the domain of $q(x) = f(x)/g(x)$ is all values shared by the domains of f and g where $g(x) \neq 0$.

EXAMPLE 1 Combining Two Functions

If $N(v) = \sqrt{v}$ and $T(v) = 3 - v$, find equations and the domains for the functions $A(v) = N(v)\,T(v)$ and $B(v) = N(v)/T(v)$.

SOLUTION

The domain of $N(v) = \sqrt{v}$ is $[0, \infty)$, all the real numbers greater than or equal to 0. The domain of $T(v) = 3 - v$ is $\mathbb{R}$, all real numbers. The domain of $A(v) = N(v)\,T(v)$ consists of those values that are shared by both these domains, namely $[0, \infty)$. The formula for the product function is

$$A(v) = N(v)\,T(v) = \sqrt{v}\,(3 - v)$$

Similarly,

$$B(v) = \frac{N(v)}{T(v)} = \frac{\sqrt{v}}{3 - v}$$

Notice that $T(v) = 0$ when $v = 3$, so 3 must be excluded from the domain of B. Thus the domain of B is all real numbers greater than or equal to 0, except 3. In set-builder notation, we write $\{v \mid v \geqslant 0, v \neq 3\}$. ■

EXAMPLE 2 Combining Revenue and Cost Functions

Suppose the annual revenue, in millions of dollars, of a company is $R(t) = 0.2t^2 + 3t + 5$, where t is measured in years and $t = 0$ corresponds to the year 2000. The annual cost, in millions of dollars, for the company is $C(t) = 4t + 9$.

(a) Find a formula for the function $P(t) = R(t) - C(t)$.

(b) Compute and interpret $P(7)$.

SOLUTION

(a)

$$\begin{aligned} P(t) &= R(t) - C(t) = (0.2t^2 + 3t + 5) - (4t + 9) \\ &= 0.2t^2 + 3t + 5 - 4t - 9 \\ &= 0.2t^2 - t - 4 \end{aligned}$$

(b) We can find $P(7)$ by subtracting the output values of the functions R and C, or we can use the formula from part (a) directly:

$$P(7) = 0.2(7^2) - 7 - 4 = -1.2$$

Notice that $P(t)$ is the annual revenue minus the annual cost, so it represents the annual profit for the company. Since $t = 7$ corresponds to 2007, and the output is negative, we know that during 2007 the company lost 1.2 million dollars. ▪

Composition of Functions

There is another way of combining two functions to form a new function. As a simple illustration, suppose that a company's annual profit for year t is given by $P(t)$ and the total amount of tax the company pays, $f(P)$, is determined by its profit P. Since the tax paid is a function of profit and profit is, in turn, a function of t, it follows that the amount of tax paid is ultimately a function of t. In effect, the output of the profit function P can be used as the input for the tax function f, and $f(P(t))$ is the amount of tax the company paid during year t. This new function is called the *composition* of the functions P and f.

If we have equations for two functions, we can write a formula for their composition. For example, suppose $y = f(t) = \sqrt{t}$ and $t = g(x) = x^2 + 1$. Now y is a function of t and t is a function of x, so y can be considered as a function of x. We compute this by substitution:

$$y = f(t) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

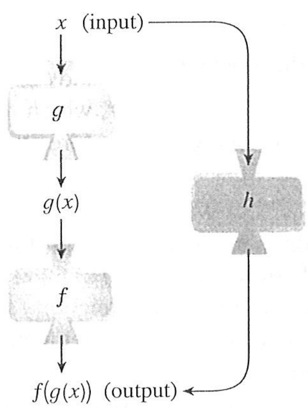


FIGURE 1
The h machine is composed of the g machine (first) and then the f machine.

> ■ **Definition** Given two functions f and g, the **composition** of f and g is defined by
>
> $$h(x) = f(g(x))$$

The domain of $h(x) = f(g(x))$ is the set of all values x in the domain of g such that $g(x)$ is in the domain of f. In other words, $f(g(x))$ is defined whenever both $g(x)$ and $f(g(x))$ are defined. It is probably easier to picture the composition of f and g with a machine diagram (see Figure 1).

EXAMPLE 3 Composing Two Functions

Let $f(x) = x^2$ and $g(x) = x - 3$. If $h(x) = f(g(x))$ and $k(x) = g(f(x))$, compute $h(5)$ and $k(5)$.

SOLUTION

First let's trace the path the input 5 takes under the function h. Since $h(5) = f(g(5))$, we first input 5 into the inner function g, where $g(5) = 2$. The output 2 is then used as an input into the outer function f, which gives an output of $f(2) = 2^2 = 4$. Thus $h(5) = f(g(5)) = f(2) = 4$. Similarly, $k(5) = g(f(5)) = g(25) = 22$. Notice that the original input always goes through the inner function first, and the resulting output is used as an input into the outer function.

We can also write formulas for h and k:

$$h(x) = f(g(x)) = f(x - 3) = (x - 3)^2$$
$$k(x) = g(f(x)) = g(x^2) = x^2 - 3$$

Then it is easy to compute

$$h(5) = (5 - 3)^2 = 2^2 = 4 \quad \text{and} \quad k(5) = 5^2 - 3 = 25 - 3 = 22$$ ▪

NOTE: You can see from Example 3 that, in general, $f(g(x)) \neq g(f(x))$. Remember, the notation $f(g(x))$ means that the function g is applied first and then f is applied second. In Example 3, $f(g(x))$ is the function that *first* subtracts 3 and *then* squares; $g(f(x))$ is the function that *first* squares and *then* subtracts 3.

EXAMPLE 4 Interpreting a Composition of Functions

The altitude of a small airplane t hours after taking off is given by $A(t) = -2.8t^2 + 6.7t$ thousand feet, where $0 \leq t \leq 2$. The air temperature in the area at an altitude of x thousand feet is $f(x) = 68 - 3.5x$ degrees Fahrenheit.

(a) What does the composition $h(t) = f(A(t))$ measure?

(b) Compute $h(1)$ and interpret your result in this context.

(c) Find a formula for $h(t)$.

(d) Does $A(f(x))$ give a meaningful result in this context?

SOLUTION

(a) The hours t that the airplane has been flying is first used as an input into the inner function A, which outputs the altitude of the plane $A(t)$ in thousands of feet. This altitude in turn is used as an input into the outer function f, which outputs a temperature in degrees Fahrenheit. Thus h is the air temperature at the airplane's location t hours after take-off.

(b) The input 1 first enters the function A, giving $A(1) = 3.9$. We then input 3.9 into the function f, which gives $f(3.9) = 54.35$. This means that 1 hour after take-off, the air temperature at the plane's location is 54.35°F.

(c)
$$\begin{aligned} h(t) &= f(A(t)) = f(-2.8t^2 + 6.7t) = 68 - 3.5(-2.8t^2 + 6.7t) \\ &= 9.8t^2 - 23.45t + 68 \end{aligned}$$

Using this direct formula, you can verify that $h(1) = 54.35$ as we found in part (b).

(d) Although we could compute a formula for $A(f(x))$, it wouldn't be a meaningful quantity here. The inner function f outputs a temperature in °F, but this is not an appropriate value to pass to the outer function A as an input, because A is a function of t, a number of hours.

So far we have used composition to build complicated functions from simpler ones. But we will see in later chapters that in calculus, it is often useful to be able to *decompose* a complicated function into simpler ones, as in the following example.

EXAMPLE 5 Decomposing a Function

If $L(t) = (2t - 1)^3$, find functions f and g such that $L(t) = f(g(t))$.

It is not important what letter we use to represent the variable in the outer function f. The function $f(x) = x^3$ is the same function as $f(a) = a^3$ or $f(q) = q^3$.

SOLUTION

The formula for L says: First double t and subtract 1, then cube the result. One option is to think of $2t - 1$ as the inner function and call it g. Then

$g(t) = 2t - 1$ and $L(t) = (g(t))^3$. The outer function is the cubing function, so if we let $f(x) = x^3$, then

$$L(t) = f(g(t)) = f(2t - 1) = (2t - 1)^3$$

Note that there are other choices we could have made, such as $g(t) = 2t$ and $f(x) = (x - 1)^3$, but the first solution is probably the most useful one. ■

Transformations of Functions

Next we discuss how to modify a function to change the shape or location of its graph. Armed with these techniques, we can use familiar graphs to design functions that will fit a wide variety of applications. The first of these *transformations* we will consider are called **translations**. If you compare the graphs of $y = f(x)$ and $y = f(x) + 3$ in Figure 2, you will notice that the shapes are identical, but the second graph is located 3 units higher on the coordinate plane. The second function increases each output of the first function by 3, so each point on its graph moves 3 units higher. In effect, we have shifted the entire graph upward 3 units. Similarly, $y = f(x) - 3$ shifts the graph of $y = f(x)$ downward 3 units.

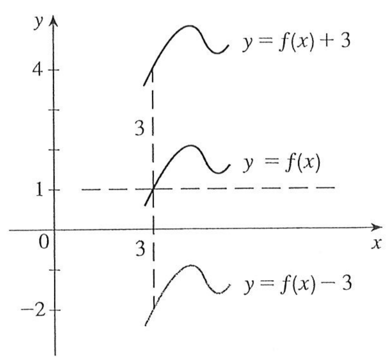


FIGURE 2
Vertical translations of the graph of f

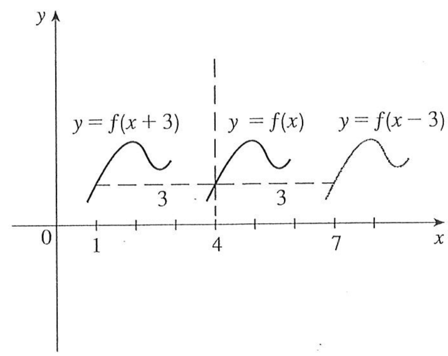


FIGURE 3
Horizontal translations of the graph of f

Next compare the graph of $y = f(x)$ with the graph of $y = f(x + 3)$ in Figure 3. The graph of $y = f(x + 3)$ is the same as the graph of $y = f(x)$ but shifted 3 units to the *left*. To see why this is the case, note that if $g(x) = f(x + 3)$, then $g(1) = f(1 + 3) = f(4)$, so the output corresponding to $x = 4$ in the graph of f is plotted with $x = 1$ in the graph of g, 3 units to the left. Similarly, $y = f(x - 3)$ shifts the graph of f to the right 3 units.

■ **Vertical and Horizontal Shifts** Suppose c is a positive number.

translation of the graph of $y = f(x)$	equation
shift c units upward	$y = f(x) + c$
shift c units downward	$y = f(x) - c$
shift c units to the right	$y = f(x - c)$
shift c units to the left	$y = f(x + c)$

We can also **stretch** (or compress) graphs. For instance, compare the graphs of $y = f(x)$ and $y = 2f(x)$ in Figure 4. The second graph has a shape similar to the first, but it has been stretched vertically by a factor of 2. Each output of the original function is doubled, so the vertical distance between each point of the graph and the x-axis is doubled. If we graph $y = \frac{1}{2}f(x)$, each output is halved, so the graph appears to be compressed vertically (toward the x-axis).

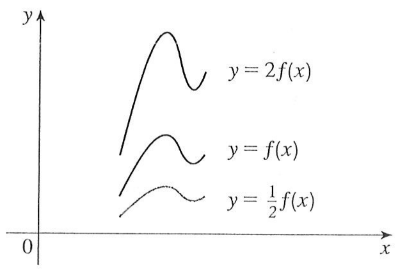


FIGURE 4
Stretching the graph of f vertically

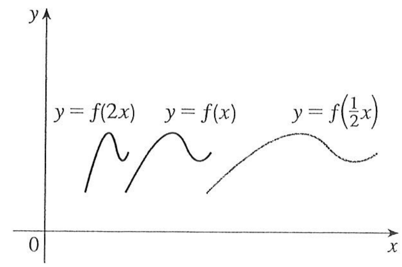


FIGURE 5
Stretching the graph of f horizontally

Now compare the graphs of $y = f(x)$ and $y = f(2x)$ in Figure 5. This time we have compressed the graph horizontally (toward the y-axis) by a factor of 2. To see why this occurs, observe that if $g(x) = f(2x)$, then $g(1) = f(2 \cdot 1) = f(2)$, so the output corresponding to $x = 2$ in the graph of f is plotted with $x = 1$ in the graph of g, half the distance from the y-axis. Similarly, the graph of $y = f\left(\frac{1}{2}x\right)$ is the graph of $y = f(x)$ stretched horizontally by a factor of 2.

Vertical and Horizontal Stretching Suppose $c > 1$.

transformation of the graph of $y = f(x)$	equation
stretch vertically by a factor of c	$y = cf(x)$
compress vertically by a factor of c	$y = \frac{1}{c}f(x)$
compress horizontally by a factor of c	$y = f(cx)$
stretch horizontally by a factor of c	$y = f\left(\frac{1}{c}x\right)$

Finally, we can **reflect** graphs in either a vertical or horizontal direction. If we compare the graphs of $y = f(x)$ and $y = -f(x)$ in Figure 6, the graph of $y = -f(x)$ is the graph of $y = f(x)$ but flipped upside down. Each point (x, y) on the original graph is replaced by the point $(x, -y)$, so the graph appears to be reflected about the

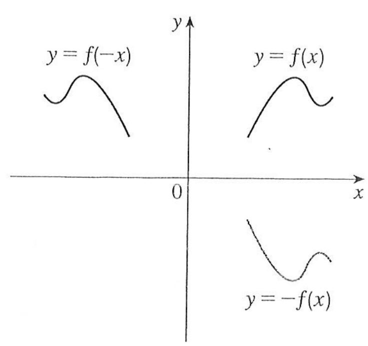


FIGURE 6
Reflecting the graph of f

x-axis. If you compare the graph of $y = f(x)$ with the graph of $y = f(-x)$ in Figure 6, you'll notice that this time the x-values are made opposite, so the graph appears reflected about the y-axis.

■ Vertical and Horizontal Reflections

reflection of the graph of $y = f(x)$	equation
reflect about the x-axis	$y = -f(x)$
reflect about the y-axis	$y = f(-x)$

Figure 7 illustrates several combinations of various transformations.

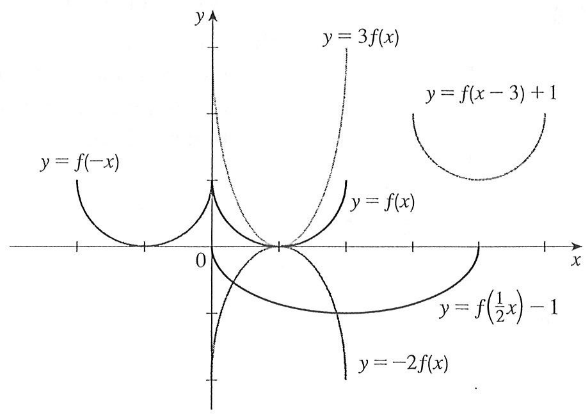


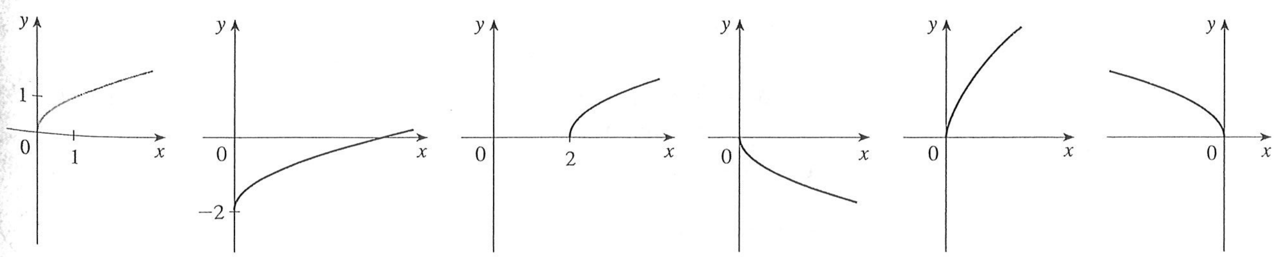
FIGURE 7

■ EXAMPLE 6 Sketching Transformations of a Function

Given the graph of $y = \sqrt{x}$, use transformations to graph $y = \sqrt{x} - 2$, $y = \sqrt{x-2}$, $y = -\sqrt{x}$, $y = 2\sqrt{x}$, and $y = \sqrt{-x}$.

SOLUTION

The graph of the square root function $y = \sqrt{x}$ is shown in Figure 8(a). If we let $f(x) = \sqrt{x}$, then $y = \sqrt{x} - 2 = f(x) - 2$, so the graph is shifted 2 units downward. Similarly, $y = \sqrt{x-2} = f(x-2)$ shifts the graph 2 units to the right, $y = -\sqrt{x} = -f(x)$ reflects the graph about the x-axis, $y = 2\sqrt{x} = 2f(x)$ stretches the graph vertically by a factor of 2, and $y = \sqrt{-x} = f(-x)$ reflects the graph about the y-axis.



(a) $y = \sqrt{x}$ (b) $y = \sqrt{x} - 2$ (c) $y = \sqrt{x-2}$ (d) $y = -\sqrt{x}$ (e) $y = 2\sqrt{x}$ (f) $y = \sqrt{-x}$

FIGURE 8

■

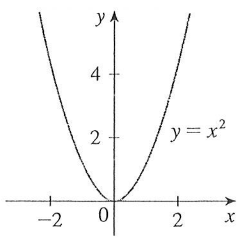


FIGURE 9

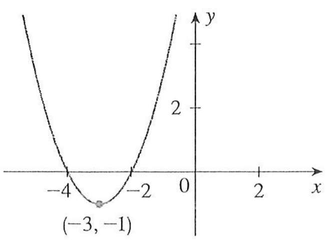


FIGURE 10

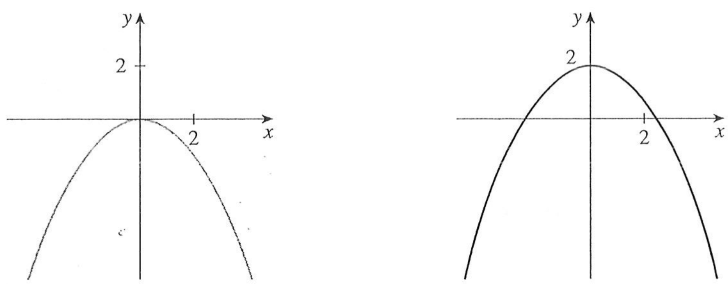
■ EXAMPLE 7 Sketching Multiple Transformations

Given the graph of the function $y = x^2$ shown in Figure 9, sketch the graphs of **(a)** $f(x) = (x + 3)^2 - 1$ and **(b)** $g(x) = -\frac{1}{3}x^2 + 2$.

SOLUTION

(a) The graph of $y = (x + 3)^2$ is the graph of $y = x^2$ shifted 3 units to the left. If we then shift the graph down 1 unit, we have the graph of $f(x) = (x + 3)^2 - 1$ shown in Figure 10.

(b) The graph of $y = -\frac{1}{3}x^2$ is the graph of $y = x^2$ compressed vertically by a factor of 3 and reflected across the x-axis. [See Figure 11(a).] Shift the resulting graph up 2 units to arrive at the graph of $g(x) = -\frac{1}{3}x^2 + 2$ as shown in Figure 11(b).



(a) $y = -\frac{1}{3}x^2$ (b) $y = -\frac{1}{3}x^2 + 2$

FIGURE 11 ■

■ EXAMPLE 8 Interpreting Transformations of Functions

Let $C(x)$ be the amount, in thousands of dollars, that a manufacturer charges for an order of x thousand computer memory chips.

(a) The price (in thousands of dollars) that a rival manufacturer charges to provide x thousand chips is given by $f(x) = C(x) + 12$. How does the rival company's price compare to that of the first company?

(b) What if the amount that the rival company charges for an order of x thousand chips is given by $g(x) = 1.4C(x)$?

(c) What if the rival charges $h(x) = C(x - 2)$ to provide x thousand chips?

SOLUTION

(a) The output of f is always 12 greater than the output of C (for the same input), so the rival supplier charges \$12,000 more for each order than the first manufacturer.

(b) The output of g is 1.4 times the output of C, so the price that the rival company charges is 1.4 times greater, or 40% more, than the first manufacturer's price.

$P = R - C$

$C = R - P$

(c) The graph of $h(x)$ is the graph of $C(x)$ shifted two units to the right. This means that for the same price, the rival manufacturer will supply 2000 more chips than the first manufacturer. For instance, if $x = 10$ then $h(10) = C(8)$. ■

Exercises 1.2

1. **Class attendance** Let $M(t)$ be the number of male students and $F(t)$ the number of female students that attended a math class at a local university on day t of this year. If we define a function g where $g(t) = M(t) + F(t)$, describe what g measures.

2. **Price of gas** Let $A(x)$ be the total amount charged to a consumer for x gallons of premium gasoline at a particular gas station, and let $T(x)$ be the total amount of tax the station pays for x gallons of the gasoline. What does the function $f(x) = A(x) - T(x)$ measure?

3. **Bank holdings** Let $g(n)$ be the amount of gold, in ounces, that a bank has in its vault at the end of the nth day of this year, and let $v(n)$ be the value, in dollars, of one ounce of gold at the end of the nth day of this year. What does the function $f(n) = g(n)\,v(n)$ measure?

4. **Investments** Let $P(t)$ be the daily closing price of one share of General Electric stock t days after January 1, 2010, and let $Q(t)$ be the number of shares owned by a pension fund at the end of that same day. What does the function $g(t) = P(t)\,Q(t)$ measure?

5. **Crops** A farm devotes $A(x)$ acres of its land to growing corn during year x. If $B(x)$ is the number of bushels of corn the farm yielded during year x, what does the function $C(x) = B(x)/A(x)$ represent?

6. **Phone usage** Let $M(n)$ be the total number of minutes Kathi talked on her cellular phone during the nth month of last year, and let $C(n)$ be the amount she paid for her phone service during that month. What does the function $h(n) = C(n)/M(n)$ represent?

7. **Salary** An employee's annual salary, in thousands of dollars, is given by $S(t) = 42 + 1.8t$, where t is the year with $t = 0$ corresponding to 2000, and $C(t) = 16.4 + 0.6t$ is the total amount of commissions, in thousands of dollars, the employee earned that year.
   (a) Find a formula for the function $f(t) = S(t) + C(t)$.
   (b) Compute $f(4)$ and interpret your result in this context.

8. **Revenue and profit** The annual revenue of a small store, in thousands of dollars, is given by $R(t) = 645 + 21t$, where t is the year, with $t = 0$ corresponding to 2000. Similarly, the store's annual profit is given by $P(t) = 175 + 16t - 0.3t^2$.
   (a) Write a formula for the annual cost function $C(t)$ for the store.
   (b) Compute $C(3)$ and interpret your result in this context.

9. If $f(x) = x^2 - 5x$ and $g(x) = 3x + 12$, write a formula for each of the following functions.
   (a) $A(x) = f(x) + g(x)$ (b) $B(x) = f(x) - g(x)$
   (c) $C(x) = f(x)\,g(x)$ (d) $D(x) = f(x)/g(x)$

10. If $p(x) = \sqrt{x + 1}$ and $q(x) = 2x - 4$, write a formula for each of the following functions. What is the domain?
   (a) $A(x) = p(x) + q(x)$ (b) $B(x) = p(x) - q(x)$
   (c) $C(x) = p(x)\,q(x)$ (d) $D(x) = p(x)/q(x)$

11. If $f(x) = x^2 + 1$, $g(t) = 4t - 2$, $A(t) = f(g(t))$, and $B(x) = g(f(x))$, compute $A(3)$ and $B(3)$.

12. If $h(n) = 2 - 5n$, $p(n) = n^2 - 3$, $u(n) = h(p(n))$, and $v(n) = p(h(n))$, compute $u(2)$ and $v(2)$.

13. If $M(t) = t + \sqrt{t}$, $N(t) = 3t + 7$, $C(t) = M(N(t))$, and $D(t) = N(M(t))$, compute $C(3)$ and $D(4)$.

14. If $f(t) = t^3 + 2$, $g(x) = 2x + 3$, $p(x) = f(g(x))$, and $r(t) = g(f(t))$, compute $p(-1)$ and $r(-2)$.

15–20 ■ Find the functions $p(x) = f(g(x))$ and $q(x) = g(f(x))$.

15. $f(x) = x^2 - 1$, $g(x) = 2x + 1$

16. $f(x) = 1 - x^3$, $g(x) = 1/x$

17. $f(x) = x^3 + 2x$, $g(x) = 1 - \sqrt{x}$

18. $f(x) = 1 - 3x$, $g(x) = 5x^2 + 3x + 2$

19. $f(x) = x + \dfrac{1}{x}$, $g(x) = x + 2$

20. $f(x) = \sqrt{2x + 3}$, $g(x) = x^2 + 1$

21. **Surfboard production** Let $N(t)$ be the number of surfboards a manufacturer produces during year t. If $P(x)$ is the profit, in thousands of dollars, the manufacturer earns by selling x surfboards, what does the function $f(t) = P(N(t))$ represent?

22. Car maintenance If $C(m)$ is the average annual cost for maintaining a Honda Civic that has been driven m thousand miles and $f(t)$ is the number of miles on Sean's Honda Civic t years after he purchased it, what does the function $g(t) = C(f(t))$ represent?

23. Carpooling As fuel prices increase, more drivers carpool. The function $f(p)$ gives the average percentage of commuters who carpool when the cost of gasoline is p dollars per gallon. If $g(t)$ is the average monthly price (per gallon) of gasoline, where t is the time in months beginning January 1, 2011, which composition gives a meaningful result, $f(g(t))$ or $g(f(p))$? Describe what the resulting function measures.

24. Home prices People are moving into a small community and driving the home prices higher. Suppose $p(t)$ is the population of the community t years after January 1, 2000, and $f(n)$ is the median home price when the population of the area is n people. Which function gives a meaningful result, $p(f(n))$ or $f(p(t))$? What does it represent in this context?

25. Scuba diving The pressure a scuba diver experiences at a depth of d feet is approximately $P(d) = 14.7 + 0.433d$ PSI (pounds per square inch). Suppose that for the first portion of Paul's dive, his depth after m minutes is $f(m) = 0.5m + 3\sqrt{m}$ feet.

(a) Write a formula for the function $A(m) = P(f(m))$. What does A measure?

(b) Compute $A(25)$ and interpret your result in this context.

26. Electric power A town produces a portion of its electricity using windmills. Suppose that with winds that average s mi/h, the windmills generate $p(s) = \sqrt{1400s}$ kilowatts of power. The town estimates that $f(x) = 0.34x$ is the number of people that can be supported by a power level of x kilowatts.

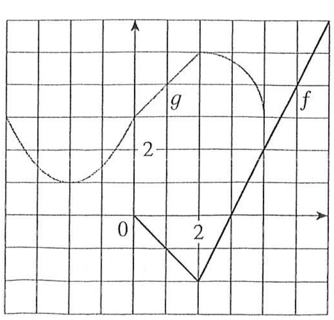
(a) Write a formula for the function $r(s) = f(p(s))$. What does r measure?

(b) Compute $r(18)$ and interpret your result in this context.

27. Use the given graphs of f and g to evaluate each expression.

(a) $f(g(2))$ (b) $g(f(0))$

(c) $f(g(0))$ (d) $f(f(4))$



28. Use the table to evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$

(c) $g(f(3))$ (d) $f(g(6))$

x	1	2	3	4	5	6
$f(x)$	3	1	4	2	2	5
$g(x)$	6	3	2	1	2	3

29–32 ■ Find functions f and g so that $h(x) = f(g(x))$.

29. $h(x) = (x^2 + 1)^{10}$ **30.** $h(x) = \sqrt{x^3 - 1}$

31. $h(x) = \sqrt{2x^2 + 5}$ **32.** $h(x) = \dfrac{1}{x^2 - 5}$

---

33. Suppose the graph of f is given. Write equations (in terms of f) for the graphs that are obtained from the graph of f as follows.

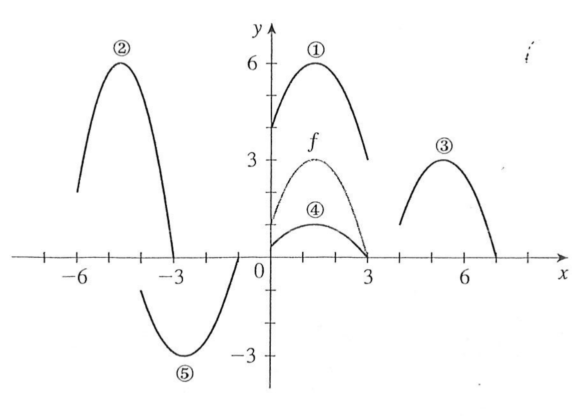
(a) Shift 4 units upward.
(b) Shift 4 units downward.
(c) Shift 4 units to the right.
(d) Shift 4 units to the left.
(e) Reflect about the x-axis.
(f) Reflect about the y-axis.
(g) Stretch vertically by a factor of 3.
(h) Shrink vertically by a factor of 3.

34. Explain how the following graphs are obtained from the graph of $y = f(x)$.

(a) $y = 5f(x)$ (b) $y = f(x - 5)$
(c) $y = -f(x)$ (d) $y = -5f(x)$
(e) $y = f(5x)$ (f) $y = 5f(x) - 3$

35. The graph of $y = f(x)$ is given. Match each equation with its graph and give reasons for your choices.

(a) $y = f(x - 4)$ (b) $y = f(x) + 3$
(c) $y = \frac{1}{3}f(x)$ (d) $y = -f(x + 4)$
(e) $y = 2f(x + 6)$

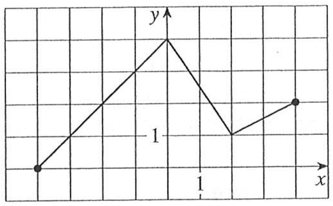


$\boxed{x}$ $y = 2x + 5$ / $\cdot\ 2(x-1)+5$
$\cdot\ 4x + 10$
$\cdot\ -2x - 5$

36. The graph of f is given. Draw the graph of each of the following functions.

(a) $y = f(x + 4)$ **(b)** $y = f(x) + 4$

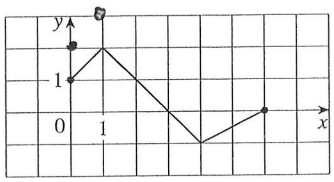
(c) $y = 2f(x)$ **(d)** $y = -\frac{1}{2}f(x) + 3$



37. The graph of f is given. Use it to graph the following functions.

(a) $y = f(2x)$ **(b)** $y = f(\frac{1}{2}x)$

(c) $y = f(-x)$ **(d)** $y = -f(-x)$



38–42 ■ The graph of $y = \sqrt{x}$ is shown in Figure 8(a). Use transformations to graph each of the following functions.

38. $y = \sqrt{x} + 3$

39. $y = \sqrt{x + 3}$ **40.** $y = -\frac{1}{2}\sqrt{x}$

41. $y = -\sqrt{x - 1}$ **42.** $y = \sqrt{-x} + 2$

43–46 ■ The graph of $y = x^2$ is shown in Figure 9. Use transformations to graph each of the following functions.

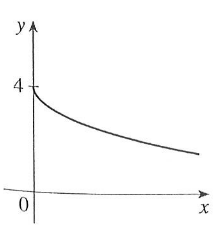
43. $y = -x^2 + 2$ **44.** $y = (x - 1)^2 - 4$

45. $f(x) = \frac{1}{4}x^2 - 3$

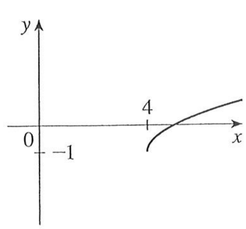
46. $g(x) = -(x + 5)^2 + 3$

47. Given the graph of $y = \sqrt{x}$ as shown in Figure 8(a), use transformations to create a function whose graph is as shown.

(a)

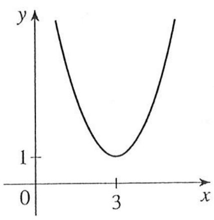


(b)

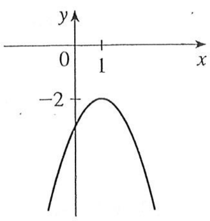


48. Given the graph of $y = x^2$ as shown in Figure 9, use transformations to create a function whose graph is as shown.

(a)



(b)



49. Water depth The depth, in feet, of water in a reservoir is given by $f(t)$, where t is the time in months beginning January 1, 2000.

(a) If a second reservoir's water depth is given by $g(t) = f(t) - 15$, how do the water levels of the two reservoirs compare?

(b) What if the second reservoir's depth is $g(t) = f(t - 2)$?

(c) What if the second reservoir's depth is $g(t) = f(t + 2)$?

(d) What if the second reservoir's depth is $g(t) = 0.8f(t)$?

50. Temperature The temperature, in degrees Fahrenheit, at Bob Hope Airport in California x days after the start of the year is given by $T(x)$.

(a) If the temperature x days after the start of the year at Los Angeles International Airport (LAX) is given by $h(x) = T(x) - 8$, how does the temperature at LAX compare to the temperature at Bob Hope Airport?

(b) What if the temperature at LAX is $h(x) = 0.9T(x)$?

51. Music sales The number of songs sold, in thousands, during the nth month of last year by an Internet music service is $A(n)$.

(a) If a rival service sold $B(n) = 1.3A(n)$ songs, how does the number of songs sold by the rival service compare to that of the first service?

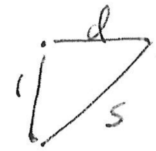
(b) What if the rival service sold $B(n) = A(n) + 23$ songs?

(c) What if the rival service sold $B(n) = A(n - 1) + 5$ songs?

52. Bear population An ecologist has been observing the populations of brown bears and black bears in a region of Alaska. Let $R(t)$ represent the estimated number of brown bears, and $L(t)$ the estimated number of black bears, t years after January 1, 1990.

(a) If there are always 500 more black bears than brown bears, write a formula [in terms of $R(t)$] for $L(t)$.

(b) If there are always 15% fewer black bears than brown bears, write a formula [in terms of $R(t)$] for $L(t)$.



(c) If the number of black bears at any point in time matches the number of brown bears two years prior, write a formula [in terms of $R(t)$] for $L(t)$.

53. Motion A ship is moving at a speed of 30 km/h parallel to a straight shoreline. The ship is 6 km from shore and it passes a lighthouse at noon.

(a) Express the distance s between the lighthouse and the ship as a function of d, the distance the ship has traveled since noon; that is, find f so that $s = f(d)$.

(b) Express d as a function of t, the time elapsed since noon; that is, find g so that $d = g(t)$.

(c) Find $f(g(t))$. What does this function represent?

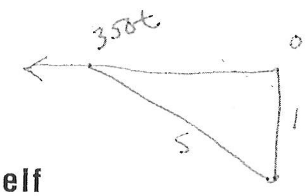
54. Motion An airplane is flying at a speed of 350 mi/h at an altitude of one mile and passes directly over a radar station at time $t = 0$.

(a) Express the horizontal distance d (in miles) that the plane has flown as a function of t. $d = 350t$

(b) Express the distance s between the ~~plane and~~ the radar station as a function of d. Q1 $s = \sqrt{d^2+1}$

(c) Use composition to express s as a function of t. Q2

55. Water ripple A stone is dropped into a lake, creating a circular ripple that travels outward at a speed of 60 cm/s.



(a) Express the radius r of this circle as a function of the time t in seconds.

(b) If A is the area of this circle as a function of the radius, find $A(r(t))$ and interpret it.

56. Electric current The **Heaviside function** H is defined by

$$H(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geqslant 0 \end{cases}$$

It is used in the study of electric circuits to represent the sudden surge of electric current, or voltage, when a switch is instantaneously turned on.

(a) Sketch the graph of the Heaviside function.

(b) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 0$ and 120 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$.

(c) Sketch the graph of the voltage $V(t)$ in a circuit if the switch is turned on at time $t = 5$ seconds and 240 volts are applied instantaneously to the circuit. Write a formula for $V(t)$ in terms of $H(t)$. (Note that starting at $t = 5$ corresponds to a translation.)

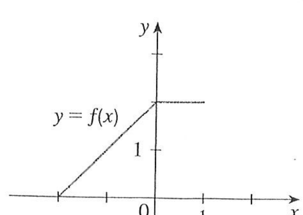
Challenge Yourself

57–58 ■ Find a formula for $p(x) = f(g(h(x)))$.

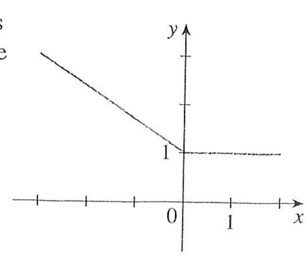
57. $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 2$, $h(x) = x + 3$

58. $f(x) = 2x - 1$, $g(x) = x^2$, $h(x) = 1 - x$

59. The graph of a function $y = f(x)$ is given.

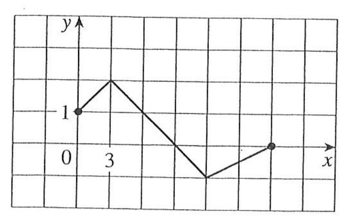


Write an equation (in terms of f) for the function whose graph is as shown.



60–61 ■ If f is the graph given in Exercise 37, write a formula (in terms of f) for the function whose graph is shown.

60.



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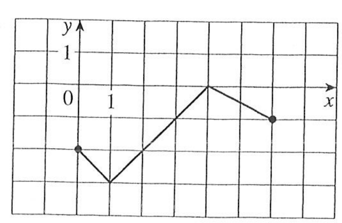
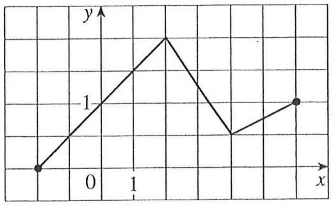
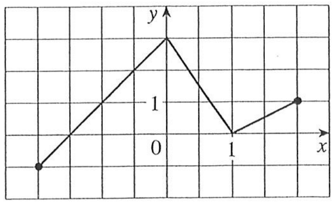


62–63 ■ If f is the graph given in Exercise 36, write a formula (in terms of f) for the function whose graph is shown.

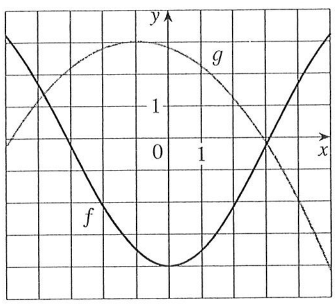
62.



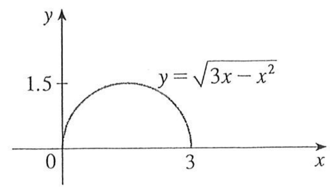
63.



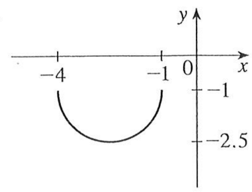
64. Use the given graphs of f and g to estimate the value of $h(x) = f(g(x))$ for $x = -5, -4, -3, \ldots, 5$. Use these estimates to sketch a rough graph of h.



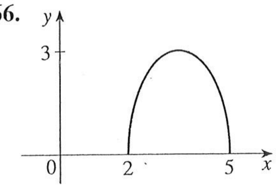
65–66 ■ The graph of $y = \sqrt{3x - x^2}$ is given. Use transformations to create a function whose graph is as shown.



65.



66.



67. Let f and g be linear functions with equations $f(x) = m_1x + b_1$ and $g(x) = m_2x + b_2$. If $h(x) = f(g(x))$, is h also a linear function? If so, what is the slope of its graph?

68. If you invest x dollars at 4% interest compounded annually, then the amount $A(x)$ of the investment after one year is $A(x) = 1.04x$. Find formulas for $A(A(x))$, $A(A(A(x)))$, and $A(A(A(A(x))))$. What do these compositions represent? Find a formula for the composition of n copies of A.

1.3 Linear Models and Rates of Change

Of the many different types of functions that can be used to model relationships observed in the real world, one of the most common is the *linear function.* When we say that one quantity is a linear function of another, we mean that the graph of the function is a line.

Review of Lines

Recall that the *slope* of a line is a measure of its steepness. We measure the slope by computing the "rise over run" between any two points on the line:

$$\text{slope} = \frac{\text{rise}}{\text{run}}$$

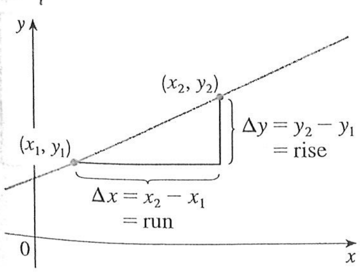


FIGURE 1

As we can see in Figure 1, the rise is simply the difference or change in y-values between the two points and the run is the difference in x-values. Thus we can think of the slope as the "change in y over the change in x."