Finite Element Method Derivation For InkFEM

Ángel Sola Orbaiceta

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Abstract

InkFEM is a software package written in the Go programming language to compute the displacements and stresses over linear structures using the *Finite Element Method*. This document discusses the formulation used in the software.

1 Axial Elements

An *axial element* is a linear resistant element whose deformation happens in the direction or its directrix only. The linear elements used by *InkFEM* are a combination of axial and beam elements (see Section 2).

1.1 Energy Formulation

The total energy of an axial element...

1.2 Displacement Field And Interpolation Functions

Let $\vec{u}(x)$ be the displacements field in the axial finite element of length L. u_1 is the displacement of the element's node 1 in the X direction, and u_2 the displacement of node 2 in the X direction. The displacements field can be chosen to vary linearly inside the finite element, so it can be interpolated from the values of u_1 and u_2 like so:

$$\vec{u}(x) = N_1 u_1 + N_2 u_2 \tag{1}$$

where:

$$\begin{aligned} N_1(x) &= 1 - \frac{x}{L} \\ N_2(x) &= \frac{x}{L} \end{aligned} \tag{2}$$

1.3 Distributed Loads

We consider distributed axial loads that vary linearly with respect to the X direction:

$$q_x(x) = a + bx \tag{3}$$

The work done by these forces can be obtained by the following integration:

$$W_{q_x} = \int_0^L q_x \vec{u}(x) dx = \int_0^L (a + bx) \left[\left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2 \right] dx$$

Which yields a result of:

$$W_{q_x} = \begin{bmatrix} \frac{aL}{2} + \frac{bL^2}{6} & \frac{aL}{2} + \frac{bL^2}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Therefore, a linear axial load $q_x(x)$ can be distributed over the two nodes in the finite element adding the forces:

$$\begin{aligned} F_x^1 &= \frac{aL}{2} + \frac{bL^2}{6} \\ F_x^2 &= \frac{aL}{2} + \frac{bL^2}{3} \end{aligned} \tag{4}$$

2 Beam Elements