Finite Element Method Derivation For InkFEM

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Abstract

InkFEM is a software package written in the Go programming language to compute the displacements and stresses over linear structures using the *Finite Element Method*. This document discusses the formulation used in the software.

1 Axial Elements

An *axial element* is a linear resistant element whose deformation happens in the direction or its directrix only. The linear elements used by *InkFEM* are a combination of axial and beam elements (see Section 2).

From mechanics of materials, we know that in an axial member, the normal stress σ_x is:

$$\sigma_x = E\epsilon_x \tag{1}$$

where E is the material's Young modulus and ϵ_x is the strain. Given the displacement field $u(x)$ inside the bar subject to external loads, the strain can be expressed as:

$$\epsilon_x = \frac{du}{dx} \tag{2}$$

and therefore:

$$\sigma_x = E\frac{du}{dx}$$

The equilibrium of a differential slice in the bar yields the following relation:

$$\frac{\partial \sigma_x}{\partial x} = -\frac{q_x(x)}{A} \tag{3}$$

1.1 Energy Formulation

The total potential energy in an axial element is:

$$\Pi = \frac{1}{2}\int_L \sigma_x \epsilon_x A dx - \int_L u(x) f_v(x) A dx - \int_L u(x) q_x(x) dx - \sum_i u_i P_i \tag{4}$$

where:

- $f_v(x)$: distributed force by unit of volume in the x direction
- $q_x(x)$: distributed force by unit of area in the x direction
- P_i: punctual axial force in point i

1.2 Displacement Field And Interpolation Functions

Let $u(x)$ be the displacements field in the axial finite element of length L. u_1 is the displacement of the element's node 1 in the X direction, and u_2 the displacement of node 2 in the X direction. The displacements field can be chosen to vary linearly inside the finite element, so it can be interpolated from the values of u_1 and u_2 like so:

$$u(x) = N_1 u_1 + N_2 u_2 \tag{5}$$

where:

$$\begin{aligned} N_1(x) &= 1 - \frac{x}{L} \\ N_2(x) &= \frac{x}{L} \end{aligned} \tag{6}$$

1.3 Distributed Loads

We consider distributed axial loads that vary linearly with respect to the X direction:

$$q_x(x) = a + bx \tag{7}$$

The work done by these forces can be obtained by the following integration:

$$W_{q_x} = \int_0^L q_x(x)\vec{u}(x)dx = \int_0^L (a+bx)\left[\left(1 - \frac{x}{L}\right)u_1 + \left(\frac{x}{L}\right)u_2\right]dx$$

Which yields a result of:

$$W_{q_x} = \begin{bmatrix} \frac{aL}{2} + \frac{bL^2}{6} & \frac{aL}{2} + \frac{bL^2}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Therefore, a linear axial load $q_x(x)$ can be distributed over the two nodes in the finite element adding the forces:

$$\begin{aligned} F_x^1 &= \frac{aL}{2} + \frac{bL^2}{6} \\ F_x^2 &= \frac{aL}{2} + \frac{bL^2}{3} \end{aligned} \tag{8}$$

Figure 1.3 depicts this distibution of a linear axial load inside a finite element of length L.

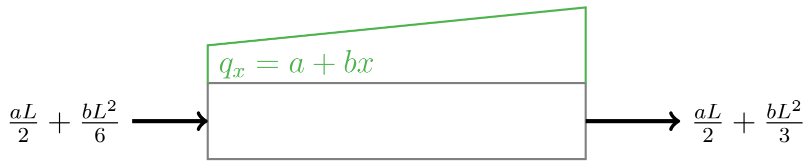

Figure 1: Distribution of a distributed axial load

1.4 Axial Stress

Once the finite element's node displacements in the X direction u_1 and u_2 are obtained, the axial stress σ_e can be computed as follows:

$$\sigma_e = E\epsilon_e = E\left(\frac{u_2 - u_1}{L}\right)$$

Given the stress inside the element, σ_e, we can now compute the values of the axial stress in the nodes:

$$\sigma_e^1 = \sigma_e + \frac{F_x^1}{A} = \sigma_e + \frac{1}{A}\left(\frac{aL}{2} + \frac{bL^2}{6}\right)$$
$$\sigma_e^2 = \sigma_e - \frac{F_x^2}{A} = \sigma_e + \frac{1}{A}\left(\frac{aL}{2} + \frac{bL^2}{3}\right)$$

In the finite element's left node, the axial stress value equals the element's stress plus the left load over the cross section area. The axial stress at the right node is the element's stress plus the right load over the cross section area.

1.5 Example

This example is implemented as an integration test in the *axial_member_test.go* to ensure the correct working of the FEM procedure.

Consider an axial member with one of its ends fixed and an axial distributed load that goes from 400 $\frac{\text{N}}{\text{cm}}$ in the fixed end to 0 $\frac{\text{N}}{\text{cm}}$ in the free end. The bar has a length of 100 cm, a cross section area of 14 cm^2 and is made of a material whose Young modulus is $20 \times 10^6 \frac{\text{N}}{\text{cm}^2}$. This axial bar is depicted in Figure 1.5.

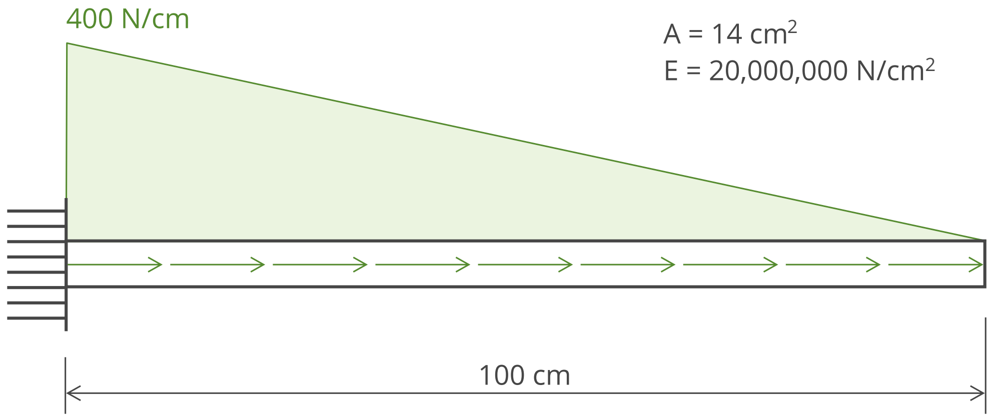

Figure 2: Axial bar subject to a distributed load

1.5.1 The Axial Stress

Let's start by finding the expression in terms of the position x of the axial stress inside the bar. The distributed load can be formulated as:

$$q_x(x) = a + bx = 400 - 4x$$

We can obtain the analytical value for the stress function of the position x integrating:

$$\sigma(x) = -\int \frac{q_x(x)}{A}\, dx = -\int \left(\frac{a+bx}{A}\right) dx = \sigma_r - \frac{1}{A}\left(ax + \frac{bx^2}{2}\right)$$

Where σ_r is the normal stress in the fixed end, which can be obtained computing the reaction force first:

$$N_r = \frac{400\frac{\text{N}}{\text{cm}} \times 100\text{cm}}{2} = 20,000\text{N}$$

And then dividing the reaction force N_r by the bar's cross section area A:

$$\sigma_r = \frac{20,000\text{N}}{14\text{cm}^2} = 1428.57\frac{\text{N}}{\text{cm}^2}$$

1.5.2 The Axial Displacements

Given the stress-strain equation, to compute the displacement in the x direction:

$$du = \frac{\sigma}{E} dx$$

Integrating:

$$u(x) = \cancel{u_r} + \frac{1}{E} \int \sigma(x) dx = \frac{1}{E} \int \left[\sigma_r - \frac{1}{A} \left(ax + \frac{bx^2}{2} \right) \right] dx$$

where u_r is the displacement in the fixed end, which we know must equal zero. This integration yields:

$$u(x) = \frac{\sigma_r x}{E} - \frac{1}{EA} \left(\frac{ax^2}{2} + \frac{bx^3}{6} \right)$$

2 Beam Elements

From the mechanics of materials beam theory, the shear stress V in a beam is related to the vertical distributed load $f_y(x)$ as shown by Equation 9:

$$\frac{dV}{dx} = f_y(x) \tag{9}$$

Let $v(x)$ be the vertical displacement of the beam's centroidal axis. Then, the bending moment $M(x)$ can be expressed in terms of this displacement (Equation 10):

$$M(x) = EI\frac{d^2v(x)}{dx^2} \tag{10}$$

The relationship between the shear stress $V(x)$ and the bending moment $M(x)$ is given by Equation 11:

$$V(x) = \frac{dM(x)}{dx} = \frac{d}{dx}\left(EI\frac{d^2v(x)}{dx^2}\right) \tag{11}$$

And hence, the beam's governing differential equation, provided EI is constant, is given by Equation 12:

$$EI\frac{d^4v(x)}{dx^4} = f_y(x) \tag{12}$$

2.1 Energy Formulation

Energy...

2.2 Displacement Field And Interpolation Functions

The vector $\{q^e\}$ contains the beam finite element start and end nodes vertical displacements and rotations:

$$\{q^e\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \left(\frac{dv}{dx}\right)_1 \\ v_2 \\ \left(\frac{dv}{dx}\right)_2 \end{Bmatrix}$$

Let $v(x)$ be the vertical displacements field in the beam finite element of length L. We can assume a cubic displacement field inside the element, as described by Equation 13:

$$v(x) = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2 \tag{13}$$

where:

$$\begin{aligned}
N_1(x) &= \frac{1}{L^3}\left(2x^3 - 3x^2 L + L^3\right) \\
N_2(x) &= \frac{1}{L^2}\left(x^3 - 2x^2 L + xL^2\right) \\
N_3(x) &= \frac{1}{L^3}\left(-2x^3 - 3x^2 L\right) \\
N_4(x) &= \frac{1}{L^2}\left(x^3 - x^2 L\right)
\end{aligned} \tag{14}$$

These N interpolation functions fulfill the following conditions:

$$\begin{array}{llll}
N_1(0) = 1 & N_1'(0) = 0 & N_1(L) = 0 & N_1'(L) = 0 \\
N_2(0) = 0 & N_2'(0) = 1 & N_2(L) = 0 & N_2'(L) = 0 \\
N_3(0) = 0 & N_3'(0) = 0 & N_3(L) = 1 & N_3'(L) = 0 \\
N_4(0) = 0 & N_4'(0) = 0 & N_4(L) = 0 & N_4'(L) = 1
\end{array}$$

The displacement field can be rewritten as:

$$v(x) = [N]\{q^e\} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

2.3 Distributed Loads

Considering the distributed load formulation as in Equation 7, but in the y direction $q_y(x)$, the work done by the load is:

$$W_{q_y} = \int_0^L q_y(x)\vec{v}(x)dx$$

which yields a result of:

$$W_{q_y} = \begin{bmatrix} \frac{aL}{2} + \frac{3L^2b}{20} & \frac{aL^2}{12} + \frac{bL^3}{30} & \frac{aL}{2} + \frac{7bL^2}{20} & -\frac{aL^2}{12} - \frac{bL^3}{20} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

Therefore, a linear load $q_y(x)$ can be distributed over the two nodes in the finite element adding the forces and moments:

$$\begin{aligned} F_y^1 &= \frac{aL}{2} + \frac{3L^2b}{20} \\ M_z^1 &= \frac{aL^2}{12} + \frac{bL^3}{30} \\ F_y^2 &= \frac{aL}{2} + \frac{7bL^2}{20} \\ M_z^2 &= -\frac{aL^2}{12} - \frac{bL^3}{20} \end{aligned} \tag{15}$$

Figure 2.3 depicts this distibution of a linear shear load inside a finite element of length L.

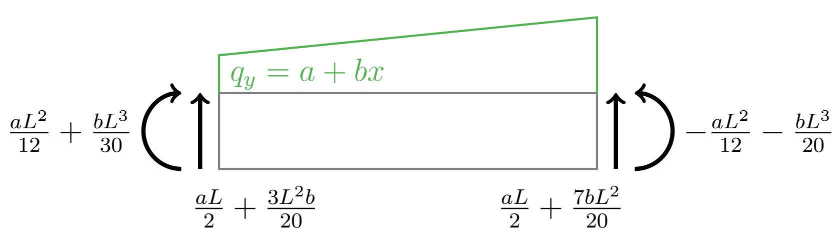

Figure 3: Distribution of a distributed shear load

2.4 Shear Force

Given the expression for the shear force $V(x)$ function of the displacement field $v(x)$:

$$V(x) = \frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) \tag{16}$$

and knowing that $v(x)$ is:

$$v(x) = \left[\frac{2}{L^3}(v_1 - v_2) + \frac{1}{L^2}(\theta_1 + \theta_2)\right] x^3 + \left[\frac{-3}{L^2}(v_1 - v_2) - \frac{1}{L}(2\theta_1 + \theta_2)\right] x^2 + \theta_1 x + v_2 \tag{17}$$

The shear force inside the element V_e is then:

$$V_e = \frac{12EI}{L^3}(v_1 - v_2) + \frac{6EI}{L^2}(\theta_1 + \theta_2)$$

Given the shear force inside the element, V_e, we can compute the values for the shear force in the element nodes:

$$\begin{aligned} V_e^1 &= V_e - F_y^1 \\ V_e^2 &= V_e + F_y^2 \end{aligned}$$

2.5 Bending Moment

Given the expression for the bending moment $M(x)$ function of the displacement field $v(x)$:

$$M(x) = EI\frac{d^2v}{dx^2} \tag{18}$$

and the vertical displacement field expression in Formula 17, the bending moment can be computed as follows:

$$M(x) = EI\left(\frac{12x - 6L}{L^3}v_1 + \frac{6x - 4L}{L^2}\theta_1 + \frac{-12x + 6L}{L^3}v_2 + \frac{6x - 2L}{L^2}\theta_2\right)$$

Evaluating this equation for $x = 0$ and $x = L$, we obtain:

$$\begin{aligned} M(0) &= \frac{6EI}{L^2}(v_2 - v_1) - \frac{2EI}{L}(\theta_2 + 2\theta_1) \\ M(L) &= \frac{6EI}{L^2}(v_1 - v_2) + \frac{2EI}{L}(\theta_1 + 2\theta_2) \end{aligned}$$

And finally, accounting for the external bending moments in the nodes, we can compute the bending moment values for the nodes of the finete element as follows:

$$M_e^1 = M(0) + M_z^0$$
$$M_e^2 = M(L) - M_z^1$$