Finite Element Method Derivation For InkFEM

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Abstract

InkFEM is a software package written in the Go programming language to compute the displacements and stresses over linear structures using the *Finite Element Method*. This document discusses the formulation used in the software.

1 Axial Elements

An *axial element* is a linear resistant element whose deformation happens in the direction or its directrix only. The linear elements used by *InkFEM* are a combination of axial and beam elements (see Section 2).

1.1 Energy Formulation

The total energy of an axial element...

1.2 Displacement Field And Interpolation Functions

Let $\vec{u}(x)$ be the displacements field in the axial finite element of length L. u_1 is the displacement of the element's node 1 in the X direction, and u_2 the displacement of node 2 in the X direction. The displacements field can be chosen to vary linearly inside the finite element, so it can be interpolated from the values of u_1 and u_2 like so:

$$\vec{u}(x) = N_1 u_1 + N_2 u_2 \tag{1}$$

where:

$$\begin{aligned} N_1(x) &= 1 - \frac{x}{L} \\ N_2(x) &= \frac{x}{L} \end{aligned} \tag{2}$$

1.3 Distributed Loads

We consider distributed axial loads that vary linearly with respect to the X direction:

$$q_x(x) = a + bx \tag{3}$$

The work done by these forces can be obtained by the following integration:

$$W_{q_x} = \int_0^L q_x \vec{u}(x) dx = \int_0^L (a + bx) \left[\left(1 - \frac{x}{L}\right) u_1 + \left(\frac{x}{L}\right) u_2 \right] dx$$

Which yields a result of:

$$W_{q_x} = \begin{bmatrix} \frac{aL}{2} + \frac{bL^2}{6} & \frac{aL}{2} + \frac{bL^2}{3} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Therefore, a linear axial load $q_x(x)$ can be distributed over the two nodes in the finite element adding the forces:

$$\begin{aligned} F_x^1 &= \frac{aL}{2} + \frac{bL^2}{6} \\ F_x^2 &= \frac{aL}{2} + \frac{bL^2}{3} \end{aligned} \tag{4}$$

1.4 Axial Stress

Each element's axial stress...

1.5 Example

This example is implemented as an integration test in the *axial_member_test.go* to ensure the correct working of the FEM procedure.

Consider an axial member with one of its ends fixed and an axial distributed load that goes from 400 $\frac{\text{N}}{\text{cm}}$ in the fixed end to 0 $\frac{\text{N}}{\text{cm}}$ in the free end. The bar has a length of 100 cm, a cross section area of 14 cm^2 and is made of a material whose Young modulus is $20 \times 10^6 \frac{\text{N}}{\text{cm}^2}$. This axial bar is depicted in Figure 1.5.

The load can be formulated as:

$$q_x(x) = a + bx = 400 - 4x$$

From mechanics of materials, we know that in an axial member, the normal stress σ is:

$$\sigma = E\epsilon$$

where E is the material's Young modulus and ϵ is the strain. Given the displacement field $u(x)$ inside the bar subject to external loads, the strain can be expressed as:

$$\epsilon = \frac{du}{dx}$$

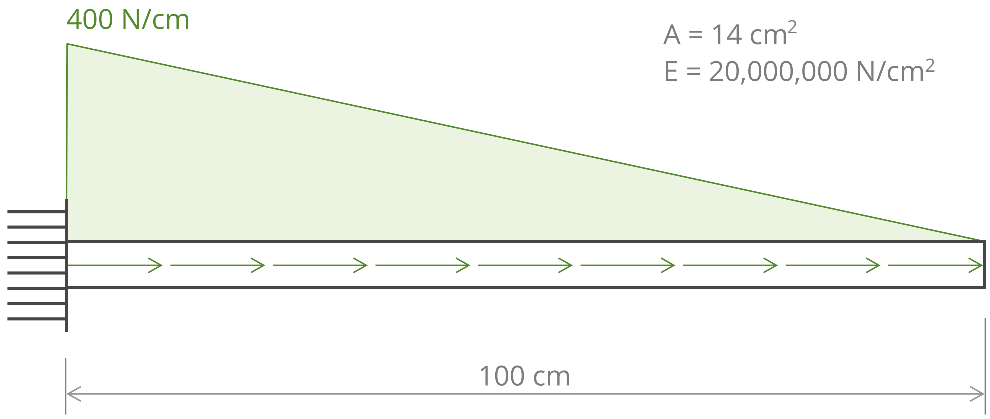

Figure 1: Axial bar subject to a distributed load

and therefore:

$$\sigma = E\frac{du}{dx}$$

At the same time, we know that the equilibrium of a differential slice in the bar yields the following relation:

$$\frac{\partial\sigma}{\partial x} = -\frac{q_x(x)}{A}$$

We can obtain the analytical value for the stress function of the position x integrating:

$$\sigma(x) = -\int \frac{q_x(x)}{A}\,dx = -\int \left(\frac{a+bx}{A}\right)\,dx = \sigma_r - \frac{1}{A}\left(ax + \frac{bx^2}{2}\right)$$

Where σ_r is the normal stress in the fixed end, which can be obtained computing the reaction force first:

$$N_r = \frac{400\frac{\mathrm{N}}{\mathrm{cm}} \times 100\mathrm{cm}}{2} = 20,000\mathrm{N}$$

And then dividing the reaction force N_r by the bar's cross section area A:

$$\sigma_r = \frac{20,000\mathrm{N}}{14\mathrm{cm}^2} = 1428.57\frac{\mathrm{N}}{\mathrm{cm}^2}$$

2 Beam Elements

From the mechanics of materials beam theory, the shear stress V in a beam is related to the vertical distributed load $f_y(x)$ as shown by Equation 5:

$$\frac{dV}{dx} = f_y(x) \tag{5}$$

Let $v(x)$ be the vertical displacement of the beam's centroidal axis. Then, the bending moment $M(x)$ can be expressed in terms of this displacement (Equation 6):

$$M(x) = EI\frac{d^2v(x)}{dx^2} \tag{6}$$

The relationship between the shear stress $V(x)$ and the bending moment $M(x)$ is given by Equation 7:

$$V(x) = \frac{dM(x)}{dx} = \frac{d}{dx}\left(EI\frac{d^2v(x)}{dx^2}\right) \tag{7}$$

And hence, the beam's governing differential equation, provided EI is constant, is given by Equation 8:

$$EI\frac{d^4v(x)}{dx^4} = f_y(x) \tag{8}$$

2.1 Energy Formulation

Energy...

2.2 Displacement Field And Interpolation Functions

The vector $\{q^e\}$ contains the beam finite element start and end nodes vertical displacements and rotations:

$$\{q^e\} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} v_1 \\ \left(\frac{dv}{dx}\right)_1 \\ v_2 \\ \left(\frac{dv}{dx}\right)_2 \end{Bmatrix}$$

Let $v(x)$ be the vertical displacements field in the beam finite element of length L. We can assume a cubic displacement field inside the element, as described by Equation 9:

$$v(x) = N_1 v_1 + N_2 \theta_1 + N_3 v_2 + N_4 \theta_2 \tag{9}$$

where:

$$\begin{aligned} N_1(x) &= \frac{1}{L^3}\left(2x^3 - 3x^2 L + L^3\right) \\ N_2(x) &= \frac{1}{L^2}\left(x^3 - 2x^2 L + xL^2\right) \\ N_3(x) &= \frac{1}{L^3}\left(-2x^3 - 3x^2 L\right) \\ N_4(x) &= \frac{1}{L^2}\left(x^3 - x^2 L\right) \end{aligned} \tag{10}$$

These N interpolation functions fulfill the following conditions:

$$\begin{array}{llll} N_1(0) = 1 & N_1'(0) = 0 & N_1(L) = 0 & N_1'(L) = 0 \\ N_2(0) = 0 & N_2'(0) = 1 & N_2(L) = 0 & N_2'(L) = 0 \\ N_3(0) = 0 & N_3'(0) = 0 & N_3(L) = 1 & N_3'(L) = 0 \\ N_4(0) = 0 & N_4'(0) = 0 & N_4(L) = 0 & N_4'(L) = 1 \end{array}$$

The displacement field can be rewritten as:

$$v(x) = [N]\{q^e\} = \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

2.3 Distributed Loads

loads...