SCHOOL OF
MATHEMATICS SCIENCES AND
INFORMATION TECHNOLOGY

LABORATORY 4-CFL CONDITION AND ADVECTION -DIFFUSION EQUATION

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March 21, 2018

1) a) Consistency of the explicit centered scheme(explicit with respect time and centered with respect space).

Let us fix x, therefore by Taylor's series.

$u(x, t + \Delta t) = u(x, t) + u_t(x, t)\Delta t + \mathcal{O}(\Delta t^2)$

So, $u_t(x, t) = \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} + \mathcal{O}(\Delta t)$

Let us fix t ,then by Taylor's series

i. $u(x + \Delta x, t) = u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{\Delta x^2}{2} + u_{xxx}(x, t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$
ii. $u(x - \Delta x, t) = u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{\Delta x^2}{2} - u_{xxx}(x, t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$

In order to get an approximation of the first-order derivative of u with respect x, we subtract i from ii. Then,

iii. $u(x + \Delta x, t) - u(x - \Delta x, t) = 2u_x(x, t)\Delta x + \mathcal{O}(\Delta x^3)$

In order to get the second-order derivative of u with respect to x, we add i and ii. Then,

iv. $u(x + \Delta x, t) + u(x - \Delta x, t) = 2u(x, t) + u_{xx}(x, t)\Delta x^2 + \mathcal{O}(\Delta x^4)$

We define

$$u(x, t) = u_j^n$$
$$u(x + \Delta x, t) = u_{j+1}^n$$
$$u(x - \Delta x, t) = u_{j-1}^n$$

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From iii.

$$u_x(x,t) = \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$u_{xx}(x,t) = \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Therefore the error between the numerical approximation and the exact one is $\mathcal{O}(\Delta t + (\Delta x)^2)$. Which by definition the scheme used is consistent

b) If k=0, the system is equivalent to

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial c(x)(x,t)}{\partial x} = 0 \quad in \quad \Omega = [0, L]$$

and the explicit centered scheme is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2h} = 0 \quad \forall j = 1, ..., N$$

Regrouping and multiplying by Δt,

We define $\alpha = c\frac{\Delta t}{2h}$

$$u_j^{n+1} = (\alpha)u_{j-1}^n + u_j^n + u_{j+1}^n(-\alpha)$$

Analyzing the boundary conditions in the boundary solutionu_0 and u_{n+1} Then in

j=1

$$u_1^{n+1} = (\alpha)u_0^n + u_1^n + u_2^n(-\alpha) = u_1^n + u_2^n(-\alpha) + \alpha u_{N+1}^n$$

with u_{N+1}^n a known value.

j=2

$$u_2^{n+1} = (\alpha)u_1^n + u_2^n + u_3^n(-\alpha)$$

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.
.

j=N-1

$$u_{N-1}^{n+1} = (\alpha)u_{N-2}^n + u_{N-1}^n + u_N^n(-\alpha)$$

j=N

$$u_N^{n+1} = (\alpha)u_{N-1}^n + u_N^n + u_{N+1}^n(-\alpha)$$

with $u^n_{N+1}(-\alpha)$ as known value.

In matrix form

$$\begin{bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{bmatrix} = \begin{bmatrix} 1 & -\alpha & 0 & \cdots & 0 \\ \alpha & 1 & -\alpha & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & 1 \end{bmatrix} \begin{bmatrix} u_1^n \\ \vdots \\ \vdots \\ \vdots \\ u_N^n \end{bmatrix} + \begin{bmatrix} \alpha u_0^n \\ 0 \\ \vdots \\ 0 \\ -\alpha u_{N+1}^n \end{bmatrix}$$

c) By Von Neumann Stability analysis, we have the ansatz :

$$\xi_j^n \sim e^{jkx_j}$$

Then,
$\xi_j^{n+1} = e^{jkx_j} - \frac{c\Delta t}{2\Delta x}(e^{jk(x_j+\Delta x)} - e^{jk(x_j-\Delta x)} = [1 - \frac{c\Delta t}{2\Delta x}(e^{jk\Delta x} - e^{-jk\Delta x})]\xi_j^n$

$\xi_j^{n+1} = 1 - \frac{c\Delta t}{\Delta x}(jsin(k\Delta x)) = g(k)$

$|g(k)| = \sqrt{1 + (\frac{c\Delta t}{\Delta x} sin(k\Delta x))^2}$

$|g(k)| > 1$
The Von Neumann Stability condition for the amplification factor g(k) say.

$$|g(k)| \leq 1 \quad \forall k$$

Therefore, $|g(k)| > 1$ implies that the instability occurs for all given c,Δt and Δx i.e.
Te scheme is unconditionally unstable in the L^2 norm.

d)

e)

2) a) Consistency of the implicit centered scheme(implicit with respect time and centered with respect space).

Let us fix x, therefore by Taylor's series.

$u(x, t + \Delta t) = u(x, t) + u_t(x, t)\Delta t + \mathcal{O}(\Delta t^2)$

So, $u_t(x, t) = \frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} + \mathcal{O}(\Delta t)$

Let us fix t ,then by Taylor's series

i. $u(x + \Delta x, t) = u(x, t) + u_x(x, t)\Delta x + u_{xx}(x, t)\frac{\Delta x^2}{2} + u_{xxx}(x, t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$
ii. $u(x - \Delta x, t) = u(x, t) - u_x(x, t)\Delta x + u_{xx}(x, t)\frac{\Delta x^2}{2} - u_{xxx}(x, t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$

We define

$$u(x, t) = u_j^n$$
$$u(x + \Delta x, t) = u_{j+1}^n$$
$$u(x - \Delta x, t) = u_{j-1}^n$$

In order to get an approximation of the first-order derivative of u with respect to x we use ii.

$$u_x(x,t) = \frac{u_j^n - u_{j-1}^n}{\Delta x} + \mathcal{O}(\Delta x^2)$$

In order to get the second-order derivative of u with respect to x, we add i and ii. Then,

iii. $u(x+\Delta x,t) + u(x-\Delta x,t) = 2u(X,t) + u_{xx}(x,t)\Delta x^2 + \mathcal{O}(\Delta x^4)$

From iii.

$$u_{xx}(x,t) = \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Therefore the error between the numerical approximation and the exact one is $\mathcal{O}(\Delta t + (\Delta x)^2)$. Which by definition the scheme used is consistent

b) If k=0, the system is equivalent to

$$\frac{\partial u(x,t)}{\partial t} + \frac{\partial c(x)(x,t)}{\partial x} = 0 \quad in \quad \Omega = [0,L]$$

and the explicit upstream scheme is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{h} = 0 \quad \forall j = 1,...,N$$

Regrouping and multiplying by Δt,

We define $\beta = c\frac{\Delta t}{h}$

$$u_j^{n+1} = (\beta)u_{j-1}^n + u_j^n - u_j^n(\beta)$$

Analyzing the boundary conditions in the boundary solutionu_0 and u_{n+1} Then in

j=1

$$u_1^{n+1} = (\beta)u_0^n + u_1^n - u_1^n(\beta)$$

.

j=2

$$u_2^{n+1} = (\beta)u_1^n + u_2^n - u_2^n(\beta)$$

.
.
.

j=N-1

$$u_{N-1}^{n+1} = (\beta)u_{N-2}^n + u_{N-1}^n - u_{N-1}^n(\beta)$$

j=N

$$u_N^{n+1} = (\beta)u_{N-1}^n + u_N^n - u_N^n(\beta)$$

In matrix form

$$\begin{bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{bmatrix} = \begin{bmatrix} 1-\beta & 0 & 0 & \cdots & 0 \\ \beta & 1-\beta & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta & 1-\beta \end{bmatrix} \begin{bmatrix} u_1^n \\ \vdots \\ \vdots \\ \vdots \\ u_N^n \end{bmatrix} + \begin{bmatrix} \beta u_0^n \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

c) By Von Neumann Stability Analysis, we have that

$$u(x,t) = A(t)e^{jkx}$$

with $j = \sqrt{-1}, k$ is the wave number, and A is the amplitude of the wave. Therefore, the solution at time step n and at x=xj written as:

$$A^n e^{jkx_j}$$

we have :

$$A^{n+1}e^{jkx_j} = A^n e^{jkx_j} - \frac{c\Delta t}{\Delta x}(A^n e^{jk(x_j+\Delta x)} - A^n e^{jk(x_j-\Delta x)}$$

Divide by $A^{n+1}e^{jkx_j}$, and $\xi = \frac{A^{n+1}}{A^n}$

$\xi = 1 - \frac{c\Delta t}{\Delta x}(1 - e^{jkx\Delta x)})$

$\xi = 1 - \frac{c\Delta t}{\Delta x}(1 - cosk\Delta x) - j\frac{c\Delta t}{\Delta x}sin(k\Delta x)$

Let $\frac{c\Delta t}{\Delta x} = \lambda$
Hence

$$\xi = 1 - \lambda(1 - cos(k\Delta x) - j\lambda sin(k\Delta x)$$

Then,
$|\xi^2| = (1 - \lambda(1 - cos(k\Delta x))^2 + (\lambda sin(k\Delta x)^2$

$|\xi^2| = 1 - 2\lambda(1 - cos(k\Delta x)) + \lambda^2(1 - cos(k\Delta x)^2\lambda^2 sin^2(k\Delta x)$

$|\xi^2| = 1 - 2\lambda + 2\lambda cos(k\Delta x) + \lambda^(2) + \lambda^2 cos^2(k\Delta x) - 2\lambda^2 cos(k\Delta x) + \lambda^2 sin^2(k\Delta x)$

$|\xi^2| = 1 - 2\lambda + 2\lambda cos(k\Delta x) + 2\lambda^2 - 2\lambda^2 cos(k\Delta x)$

$\left|\xi^2\right| = 1 - 2\lambda(1-\lambda)(1 - cos(k\Delta x)$

Hence for stability it is required that

$$1 - 2\lambda(1-\lambda)(1 - cos(k\Delta x) \leq 1$$

or

$$-2\lambda(1-\lambda)(1 - cos(k\Delta x) \leq 0$$

Which will be true only if $(1-\lambda) \geq 0$ or $\lambda \leq 1$ hence this implies

$$\frac{c\Delta t}{\Delta x} \leq 1$$

Hence the upstream scheme is stable if the CFL conditions is satisfied.

d)

e)

3) a) Consistency of the implicit centered scheme(implicit with respect time and centered with respect space).

Let us fix x, therefore by Taylor's series.

$u(x, t+\Delta t) = u(x,t) + u_t(x,t)\Delta t + \mathcal{O}(\Delta t^2)$

So, $u_t(x,t) = \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + \mathcal{O}(\Delta t)$

Let us fix t ,then by Taylor's series

i. $u(x+\Delta x, t) = u(x,t) + u_x(x,t)\Delta x + u_{xx}(x,t)\frac{\Delta x^2}{2} + u_{xxx}(x,t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$

ii. $u(x-\Delta x, t) = u(x,t) - u_x(x,t)\Delta x + u_{xx}(x,t)\frac{\Delta x^2}{2} - u_{xxx}(x,t)\frac{\Delta x^3}{6} + \mathcal{O}(\Delta x^4)$

In order to get an approximation of the first-order derivative of u with respect x, we subtract $u(x+\Delta x, t+\Delta t)$ with $u(x-\Delta x, t-\Delta t)$. Then,

iii. $u(x+\Delta x, t+\Delta t) - u(x-\Delta x, t-\Delta t) = 2u_x(x,t)\Delta x + \mathcal{O}(\Delta x^3)$

In order to get the second-order derivative of u with respect to x, we add $u(x+\Delta x, t+\Delta t)$ with $u(x-\Delta x, t-\Delta t)$. Then,

iv. $u(x+\Delta x, t+\Delta t) + u(x-\Delta x, t-\Delta t) = 2u(x, t+\Delta t) + u_{xx}(x,t)\Delta x^2 + \mathcal{O}(\Delta x^4)$

We define

$$u(x,t) = u_j^n$$
$$u(x+\Delta x, t) = u_{j+1}^n$$
$$u(x-\Delta x, t) = u_{j-1}^n$$
$$u(x+\Delta x, t+\Delta t) = u_{j+1}^{n+1}$$
$$u(x-\Delta x, t-\Delta t) = u_{j-1}^{n-1}$$

From iii.

$$u_x(x,t) = \frac{u_{j+1}^{n+1} - u_{j-1}^{n-1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$$u_{xx}(x,t) = \frac{u_{j+1}^{n+1} + u_{j-1}^{n-1} - 2u_j^{n+1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

Therefore the error between the numerical approximation and the exact one is $\mathcal{O}(\Delta t + (\Delta x)^2)$. Which by definition the scheme used is consistent

b) If k=0, the system is equivalent to

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and the implicit centered scheme is

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} = 0 \quad \forall j = 1, ..., N$$

Regrouping and multiplying by Δt,

We define $\alpha = c\frac{\Delta t}{2h}$

$$u_j^{n+1} = (\alpha)u_{j-1}^{n+1} + u_j^n - u_{j+1}^{n+1}(\alpha)$$

Analyzing the boundary conditions in the boundary solutionu_0 and u_{n+1} Then in

j=1

$$u_1^{n+1} + u_2^{n+1}(\alpha) - (\alpha)u_0^{n+1} = u_1^n$$

.

j=2

$$u_2^{n+1} + u_3^{n+1}(\alpha) - (\alpha)u_1^{n+1} = u_2^n$$

.
.
.

j=N-1

$$u_{N-1}^{n+1} + u_N^{n+1}(\alpha) - (\alpha)u_{N-2}^{n+1} = u_{N-1}^n$$

j=N

$$u_N^{n+1} + u_{N+1}^{n+1}(\alpha) - (\alpha)u_{N-1}^{n+1} = u_N^n$$

with $u_{N+1}^n(\alpha)$ as known value.
In matrix form

$$\begin{bmatrix} 1 & \alpha & 0 & \cdots & -\alpha \\ -\alpha & 1 & \alpha & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ \alpha & \cdots & 0 & -\alpha & 1 \end{bmatrix} \begin{bmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{bmatrix} = \begin{bmatrix} u_1^n \\ \vdots \\ \vdots \\ \vdots \\ u_N^n \end{bmatrix}$$

c) By Von Neumann Stability Analysis, we have that

$$u(x,t) = A(t)e^{jkx}$$

with $j = \sqrt{-1}$, k is the wave number, and A is the amplitude of the wave. Therefore, the solution at time step n and at x=xj written as:

$$A^n e^{jkx_j}$$

We have:

$$A^n e^{jkx_j} = A^{n+1}e^{jkx_j} + \frac{c\Delta t}{2\Delta x}(A^{n+1}e^{jk(x_j+\Delta x)} - A^{n+1}e^{jk(x_j-\Delta x)}$$

Divide by $A^n e^{jkx_j}$, and $\xi = \frac{A^{n+1}}{A^n}$

$1 = \xi + \frac{c\Delta t}{2\Delta x}(\xi e^{jk\Delta x} - \xi e^{-jk\Delta x})$

$1 = \xi[1 + \frac{c\Delta t}{2\Delta x}(e^{jk\Delta x}e^{-jk\Delta x})]$

$1 = \xi(1 + j\frac{c\Delta t}{\Delta x}sin(k\Delta x))$

$g(k) = \frac{1}{1+j\frac{c\Delta t}{\Delta x}sin(k\Delta x)} = \frac{1-j\frac{c\Delta t}{\Delta x}sin(k\Delta x)}{1+\frac{c\Delta t}{\Delta x}sin(k\Delta x)}$

$|g(k)| = \frac{\sqrt{1+(\frac{c\Delta t}{\Delta x})^2 sin^2(k\Delta x)}}{1+\frac{c\Delta t}{\Delta x}sin(k\Delta x)} < 1$

Therefore, for the amplification factor the implicit centered scheme is unconditionally stable.

d)

e)