NUMERICAL METHODS OF PARTIAL DIFFERENTIAL EQUATIONS
PROFESSOR: CRISTHIAN MONTOYA (cmontoya@yachaytech.edu.ec)
LABORATORY 4–CFL CONDITION AND ADVECTION–DIFFUSION EQUATION

The propose in this session is to study a transport and diffusion equation using FDM and the CFL condition.
Consider the equation

$$\frac{\partial u}{\partial t}(x,t) + \frac{\partial}{\partial x}(c(x)u(x,t)) - \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}(x,t)\right) = 0, \quad \text{in } \Omega = [0,L], \tag{1}$$

where $u(x,t)$ is a scalar function that represents the concentration of a chemical species on a one–dimensional atmosphere, $c(x) > 0$ is the wind velocity and $k(x) > 0$ is a turbulence coefficient, which by simplicity are constants.

Consider the space step $h = L/(N+1)$, $n \in \mathbb{N}$ and the points $x_j = jh,\ j \in \{1,\dots,N+1\}$, as well as a time step Δt, which discretizes the time as $t_n = n\Delta t$. Let u_j^n be an approximation of $u(x_j.t_n)$.
Let Ω be a spatial domain $100[Km]$, periodic boundary conditions ($u(0,t) = u(L,t)$) and wind velocity $c = 50[m/s]$. Assume an initial time as a Gaussian model given by

$$u_0(x) = \overline{u}e^{-\frac{(x-50)^2}{8}},$$

where x is the space variable (in $[Km]$) and amplitude $\overline{u} = 2$.

1. a) Prove that the following explicit centred scheme is consistent with the equation (1):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^n - u_{j-1}^n}{2h} - k\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} = 0, \quad \forall\ j = 1,\dots,N. \tag{2}$$

   b) Fix $k = 0$. Prove that (2) can be rewritten as follows:

$$\begin{pmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1 & -\alpha & 0 & \cdots & 0 \\ \alpha & 1 & -\alpha & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha & 1 \end{pmatrix} \begin{pmatrix} u_1^n \\ \vdots \\ \vdots \\ \vdots \\ u_N^n \end{pmatrix} + \begin{pmatrix} \alpha u_N^n \\ 0 \\ \vdots \\ 0 \\ -\alpha u_1^n \end{pmatrix},$$

   where $\alpha = \dfrac{c\Delta t}{2h}$.

   c) Implement the explicit centred scheme and study the CFL condition for different values of Δt ($h = 0,5$ fixed). Analyse from a theoretical point of view the stability of the scheme in the L^2 norm. It does not work well!. Comment results.

   d) Consider the initial condition $u_0(x) = 2 \times 1_{[100/3,200/3)}$, where 1_A is the indicator function on A. Repeat the previous items with this condition and show the numerical problems in its implementation.

   e) Now, for every initial condition, to use $k > 0$ and experiment.

2. a) Repeat 1.(a) for the explicit scheme (upstream scheme):

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_j^n - u_{j-1}^n}{h} - k\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2} = 0, \quad \forall\ j = 1,\dots,N. \tag{3}$$

b) Fix $k=0$. Show that (3) is equivalent to the system

$$\begin{pmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} 1-\beta & 0 & 0 & \cdots & 0 \\ \beta & 1-\beta & 0 & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \beta & 1-\beta \end{pmatrix} \begin{pmatrix} u_1^{n} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n} \end{pmatrix} + \begin{pmatrix} \beta u_N^n \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix},$$

where $\beta = \dfrac{c\Delta t}{2h}$.

c) Implement the upstream scheme. Study the stability in the L^2 norm y numerically verify the CFL condition and the diffusion for different values Δt, with $h = 0,5[Km]$ fixed. Why the upstream scheme has a " good behaviour" for $\delta t = 10$ and $h = 0,5$?

d) Repite the previous steps with the discontinuous condition given in 1.(d), namely, $u_0(x) = 2 \times 1_{[100/3,200/3)}$. What are the numerical problems in its implementation.

e) Repeat 1.(e) for the upstream scheme (3).

3. a) Repeat 1.(a) for the implicit centred scheme:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + c\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2h} - k\frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{h^2} = 0, \quad \forall\, j = 1, \dots, N. \tag{4}$$

b) Fix $k=0$. Prove that (4) can be rewritten as follows:

$$\begin{pmatrix} 1 & \alpha & 0 & \cdots & -\alpha \\ -\alpha & 1 & \alpha & \cdots & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \alpha \\ \alpha & \cdots & 0 & -\alpha & 1 \end{pmatrix} \begin{pmatrix} u_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n+1} \end{pmatrix} = \begin{pmatrix} u_1^{n} \\ \vdots \\ \vdots \\ \vdots \\ u_N^{n} \end{pmatrix},$$

where $\alpha = \dfrac{c\Delta t}{2h}$.

c) Implement the implicit scheme for $\Delta t = 10[s]$. Compare the implicit and upstream schemes through stability analysis and numerical results.

d) Finally, repite the structure given in 2.(d) and 2.(e) for the implicit scheme (4). Compare results.