SCHOOL OF
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STOKES EQUATION (FINITE ELEMENT METHOD): The entrained cavity flow of an incompressible fluid

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May 24, 2018

In this project we tackle the unsteady Stokes system using finite elements in two dimensions.More precisely, we consider two interesting cases: given an open bounded subset convex and connected $\Omega \subset \mathbb{R}^2$ with normal exterior derivative n, find $u \in H^1(\Omega)^2$ and $p \in L^2(\Omega)/\mathbb{R}$ such that

$$\begin{cases} \nu\Delta u + \nabla p = f & \text{in } \Omega \\ \nabla.u = 0, & \text{in } \Omega \\ u = g & \text{on } \Gamma_d \\ \sigma(u,p).n = 0 & \text{on } \Gamma_n \\ u = 0, & \text{on } \partial\Omega s.t(\Gamma_d \cup \Gamma_n) \end{cases} \tag{1}$$

where u is the velocity, p is the pressure, $\nu > 0$,f $\in L^2(\Omega)^2, g \in H^{1/2}$, $\int g.nd\sigma = 0$.Moreover,

$$\sigma(u,p) = -Ip + 2\nu(e(u)), \qquad e(u) := \frac{1}{2}(\nabla u + \nabla^t u).$$

1 Theoretical Framework

1.1 Compressible and incompressible fluid

Incompressible fluid: are the fluids with constant density, they could be liquids and gases. Although there is no such thing in reality as an incompressible fluid, we use this term where the change in density with pressure is so small as to be negligible. This is usually the case with liquids. We may also consider gases to be incompressible when the pressure variation is small compared with the absolute pressure. In our case we have $\nabla.u = 0$. Therefore the fluid is incompressible.

Compressible fluid is a flow in which the density changes are significant so that pressure changes normally occur along a fluid flow, and these pressure changes, in general, induce a change in the density of the fluid. Therefore, in a compressible flow, the density changes that result from these pressure changes have a significant influence on the flow. Then, changes in the flow that result from changes in density are often called compressibility effects. All fluids are compressible. However, compressibility effects are found more frequently in gas flows than in liquid flows. In our case if we have $\nabla.u = h$ the fluid is compressible.

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1.2 Reynolds number

The Reynolds number is the ratio of inertial forces to viscous forces and is a convenient parameter for predicting if a flow condition will be laminar or turbulent. It can be interpreted that when the viscous forces are dominant (slow flow, low Re) they are sufficient enough to keep all the fluid particles in line, then the flow is laminar. Even very low Re indicates viscous creeping motion, where inertia effects are negligible. When the inertial forces dominate over the viscous forces (when the fluid is flowing faster and Re is larger) then the flow is turbulent. A measure of the non-linearity of Navier-Stokes equation is given by the Reynolds number

$$R_e = \frac{UL}{\nu}$$

where **L** and **U** are respectively the typical length scale and velocity of the fluid and ν is the kinematic viscosity. The Reynolds number plays a fundamental role in turbulence, since it gives a dimensional estimate of the relative wight between the inertial term $u.\nabla u$ and the viscous term $\nu \Delta u$:

$$\frac{[u.\nabla u]}{[\nu \Delta u]} \sim \frac{UL}{\nu}$$

Reynolds Number Regimes

Laminar flow: for practical purposes, if the Reynolds number is less than 2000, the flow is laminar.

Transitional flow: At Reynolds numbers between about 2000 and 4000 the flow is unstable as a result of the onset of turbulence. These flows are sometimes referred to as transitional flows.

Turbulent flow: If the Reynolds number is greater than 3500, the flow is turbulent. Most fluid systems in nuclear facilities operate with turbulent flow.

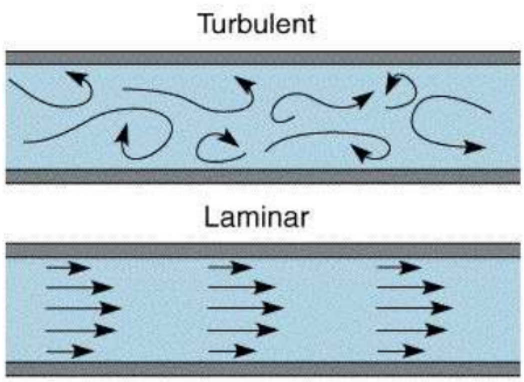

Figure 1: Laminar flow vs turbulent flow

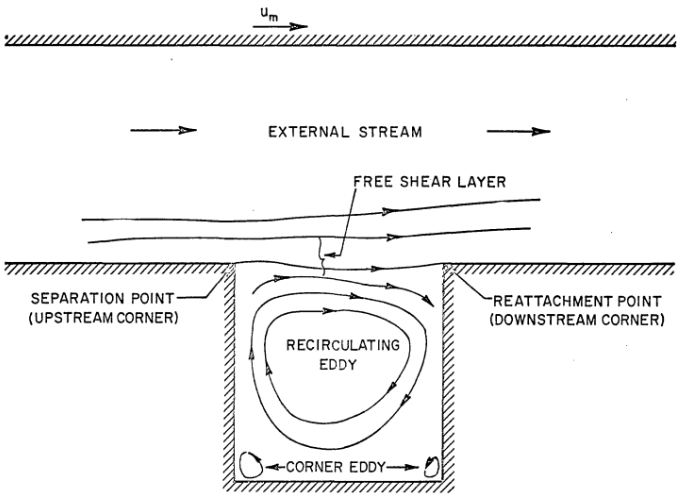

Figure 2: Problem definition of the flow over a cavity

Laminar flow:

* $R_e < 2000$
* low velocity
* Fluid particles move in straight lines
* Layers of water flow over one another at different speeds with virtually no mixing between layers.
* The flow velocity profile for laminar flow in circular pipes is parabolic in shape,with a maximum flow in the center of the pipe and a minimum flow at the pipe walls.
* The average flow velocity is approximately one half of the maximum velocity.
* Simple mathematical analysis is possible.

Turbulent flow

* $R_e > 4000$
* High velocity
* The flow is characterized by the irregular movement of particles of the fluid.
* Average motion is in the direction of the flow
* The flow velocity profile for turbulent flow is fairly flat across the center section of a pipe and drops rapidly extremely close to the walls.
* The average flow velocity is approximately equal to the velocity at the center of the pipe.
* Mathematical analysis is very difficult

The dynamics of an incompressible Newtonian fluid is determined by Navier -Stokes equations (1823), supplemented by the incompressibility condition

$$\frac{\partial u}{\partial t} + u.\nabla u = \nu \Delta u - \frac{\nabla p}{\rho} = f \qquad \nabla .u = 0$$

Where P is the pressure , ρ is the density of the fluid, $\nu = \frac{\mu}{\rho}$ is its kinematic viscosity, and f represents the sum of the external forces for unit mass which sustain the motion.

* $u.\nabla u$**:** is the inertial, or non-linear term which characterizes Navier- Stokes equation,and is responsible for the transfer of kinetic energy in the turbulent cascade.

* $-\frac{\nabla P}{\rho}$**:** the pressure gradients which guarantee the incompressibility of the flow. In absence of external forces they are determined by the Poisson equation

$$\Delta_p = -\rho\partial_i\partial_j u_i u_j$$

which is obtained taking the divergence of the equation.

* $\nu\Delta u$: the dissipative viscous term. It is originated by the Reynolds stresses of the Newtonian fluid, and it is proportional to the viscosity.

The Reynolds number R_e is the only dimensionless parameter in the equation of motion.Now, we shall study the fluid dynamics resulting from that the Reynolds number is very small compared to unity, $R_e \ll 1$.Since $R_e = \frac{UL}{\nu}$, the smallness of R_e can be achieved by considering extremely small lengths scales, or by dealing with a very viscous liquid, or by treating flows of very small velocity.

The choice $R_e \ll 1$ is an interesting and important assumption, for it is relevant to many practical problems, especially in a world where many products of technology, including those manipulating fluids, are reduction in size.

Since R_e is indicative of the ratio of inertial to viscous forces, the assumption of small R_e will mean that viscous forces dominate the dynamics. That suggests that we may be able to drop entirely the term $\frac{Du}{Dt}$ from the Navier-Stokes equations, rendering the system linear. The linearity of the problem will be a major simplification.

$$R_e(\frac{\partial u}{\partial t} + u.\nabla u + \nabla p) = \nabla^2 u, \qquad \nabla.u = 0,$$

It is tempting to say that the smallness of Re means that we can neglect the left-hand side of the first equation, leading to the reduced (linear) system

$$\nabla^2 u = 0, \qquad \nabla.u = 0$$

1.3 Existence and uniqueness

1.3.1 The Stokes Equations

Let us consider the Navier-Stokes equations describing de N-dimensional motion of an incompressible viscous fluid:

$$\rho(\frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j}) - \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} = \rho f_i, 1 \leq i \leq N \tag{2}$$

$$\nabla.\mathbf{u} = \sum_{i=1}^{N} D_{ii}(\mathbf{u}) = 0 \qquad \text{incompressibility condition} \tag{3}$$

where

$$\rho_{ij} = -P\delta_{ij} + 2\mu D_{ij}(\mathbf{u}), 1 \leq i, j \leq N, \tag{4}$$

and

$$D_{ij}(\mathbf{u}) = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i) \tag{5}$$

In these equations, $\mathbf{u} = (u_1, ..., u_N)$ is the velocity of the fluid , ρ is its density (assumed to be constant), $\mu > 0$ is its viscosity (also assumed to be constant) an P is its pressure, (σ_{ij}) is the stress tensor and $\mathbf{f} = (f_1, ..., f_N)$ represents a density of body forces per unit mass (gravity for instance).
As usual, we set

$$p = P/\rho, \qquad v = \mu/\rho \tag{6}$$

Here, p is the kinematic pressure and v is the kinematic viscosity for the sake of simplicity, they will be called in the sequel pressure and viscosity. With these notations, the Navier-Stokes equations become

$$\frac{\partial u_i}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial u_i}{\partial x_j} - 2v \sum_{j=1}^{N} u_j \frac{\partial D_{ij}\mathbf{u}}{\partial x_j} + \frac{\partial p}{\partial x_i}, 1 \leq i, j \leq N, \tag{7}$$

$$\nabla.\mathbf{u} = 0. \tag{8}$$

Note that , when $div\mathbf{u} = 0$, the following identity holds

$$\sum_{j=1}^{N} \frac{\partial D_{ij}(\mathbf{u})}{\partial x_j} = (1/2) \sum_{j=1}^{N} (\frac{\partial^2 u_i}{\partial x_j^2} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}) = (1/2)\Delta u_i \tag{9}$$

So that 7 can be written more conveniently

$$\begin{cases} \frac{\partial u}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial u}{\partial x_j} - v\Delta u + \nabla p = \mathbf{f} \\ \nabla.\mathbf{u} = 0 \end{cases} \tag{10}$$

Another canonical way of writing the Navier-Stokes equations consists in introducing a parameter (the Reynolds number) which measures the effect of viscosity on the flow. for a given problem, let L be a characteristic length and U be a characteristic velocity. This determines a characteristic time $T = L/U$. Then, we introduce the dimensionless quantities

$$x' = x/L, \qquad \mathbf{u}' = \mathbf{u}/U, \qquad t' - t/T \tag{11}$$

Using this change of variables , it is easy to check that the Navier-Stokes equations become:

$$\begin{cases} \frac{\partial \mathbf{u}'}{\partial t'} + \sum_{j=1}^{N} u'_j \frac{\partial \mathbf{u}'}{\partial x'_j} - \frac{v}{LU}\Delta' \mathbf{u}' + \nabla' p' = \mathbf{f}' \\ \nabla_{x'}.\mathbf{u}' = 0, \end{cases} \tag{12}$$

where

$$p' = P/(\rho U^2), \qquad \mathbf{f}' = L\mathbf{f}/U^2 \tag{13}$$

Now, if we define the Reynolds number R_e to be the dimensionless number

$$R_e = LU/v, \tag{14}$$

we find that the Navier-Stokes equations may be written in dimensionless variables

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{N} u_j \frac{\partial \mathbf{u}}{\partial x_j} - \frac{1}{R_e}\Delta \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla.\mathbf{u} = 0. \end{cases} \tag{15}$$

We obtain again the equations 10 with v replaced by $1/R_e$ For the time being, we introduce two simplifications in the equations 7 or 10. We only consider the steady-state (or stationary) case, that is $\partial \mathbf{u}/\partial t = 0$, and furthermore we assume that the velocity $\mathbf{u}$ is sufficiently small for ignoring the nonlinear convection terms $u_j(\partial u_i/\partial x_j)$. This we are led to the Stokes equations

$$\begin{cases} -2v\sum_{j=1}^{N} \frac{\partial D_{ij}(\mathbf{u})}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i, 1 \leq i \leq N, \\ \nabla.\mathbf{u} = 0, \end{cases} \tag{16}$$

which can be written more conveniently

$$\begin{cases} -v\Delta \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla.\mathbf{u} = 0. \end{cases} \tag{17}$$

The Stokes equations are linear but nevertheless they deserve special attention because of the incompressibility condition $\nabla.\mathbf{u} = 0$. In this paragraph we shall establish the existence and uniqueness of the solution of the Stokes equations and we shall derive several variational formulations that will be used later on for approximation purposes.

1.3.2 The Dirichlet Problem in the Velocity-Pressure Formulation

In order to get a well-posed problem for the Stokes equations 17, we have to complete them with appropriate boundary conditions. We being with the Dirichlet boundary conditions.
Lemma 2.2 Let Ω be connected . For each $\mathbf{g} \in H_{1/2}(\Gamma)^N$ satisfying $\int_\Gamma \mathbf{g}\cdot\mathbf{n}ds = 0$ there exists a function $\mathbf{u} \in H^1(\omega)^N$, unique up to an additive function of V, such that

$$\nabla.u = 0 \quad \text{in } \Omega, \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma \tag{18}$$

Moreover, we have

$$\inf_{\mathbf{v}\in V} \|\mathbf{u}+\mathbf{v}\|_{1,\Omega} \leq C \|\mathbf{g}\|_{1/2,\Gamma}, \tag{19}$$

where the constant $C > 0$ is independent of $\mathbf{u}$ and $\mathbf{g}$.

Theorem 5.1. Let Ω be a bounded and connected open subset of $\mathbb{R}^N$ with a Lipschits-continuous boundary Γ. Given $\mathbf{f} \in H^{-1}(\Omega)^N$ and $\mathbf{g} \in H^{1/2}(\Gamma)^N$ such that

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n}ds = 0, \tag{20}$$

there exists a unique pair $(\mathbf{u}, p) \in H^1(\Omega)^N \times L^2_O(\Omega)$ solution of the equations

$$\begin{cases} -v\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega \\ \nabla.\mathbf{u} = 0 \quad \text{in } \Omega \\ \mathbf{u} = \mathbf{g} \quad \text{on } \Gamma \end{cases} \tag{21}$$

Proof. By virtue of 20 and Lemma 2.2, there exists a function $\mathbf{u}_0 \in H^1(\Omega)^N$ such that

$$\nabla.\mathbf{u}_0 = 0 \quad \text{in } \Omega, \qquad \mathbf{u}_0 = \mathbf{g} \quad \text{on } \Gamma. \tag{22}$$

Now, let us put problem 21 into the following framework:
Let X and M denote two real Hilbert spaces with the norms $||\cdot||_X$ and $||\cdot||_M$, respectively. Let X' and M' be their corresponding dual spaces and let $||\cdot||_{X'}$ and $||\cdot||_{M'}$, by $<\cdot,\cdot>$,
so we introduce two continuous bilinear form:

$$a(\cdot,\cdot) : X \times X \to \mathbb{R}, b(\cdot,\cdot) : X \times M \to \mathbb{R}$$

with norm,

$$||a|| = \sup_{\mathbf{u},\mathbf{v} \in X, u\neq 0, v\neq 0} \frac{a(u,v)}{||u||_X ||v||_X}, ||b|| = \sup_{\mathbf{v}\in X, \mathbf{u}\in M u\neq 0, v\neq 0} \frac{b(v,\mu)}{||v||_X ||\mu||_M}$$

and consider the variation problem:
For ℓ given in X' and ψ in M', find a pair (u, λ) in $X \times M$ such that:

$$\begin{cases} a(u,v) + b(v,l) =< \ell, v > & \forall v \in X \\ b(u,\mu) =< u, \mu >, & \forall \mu \in M \end{cases} \tag{23}$$

Then, we set:

$$X = H^1_0(\Omega)^N, M = L^2_0(\Omega) \tag{24}$$

with norms $\|\cdot\|_X = \|\cdot\|_{1,\Omega}, \qquad \|\cdot\|_M = \|\cdot\|_{0,\Omega},$

$$a(\mathbf{u}, \mathbf{v}) = v \sum_{i,j=1}^{N} (\frac{\partial u_i}{\partial x_j}, \frac{\partial v_i}{\partial x_j}) = v(\nabla u, \nabla v), \tag{25}$$

$$b(\mathbf{v}, q) = -(q, \nabla.v),$$
$$< \mathbf{l}, \mathbf{v} >=< \mathbf{f}, \mathbf{v} > -a(\mathbf{u}_0, \mathbf{v}), \qquad \chi = 0.$$

Then

$$V = \mathbf{v} \in H^1_0(\Omega)^N ; \nabla.\mathbf{v} = 0.$$

We must check that the form $a(.,.)$ is V-elliptic and the form $b(.,.)$ satisfies the inf-sup condition. On the one hand, the ellipticity property is obvious since

$$a(\mathbf{v}, \mathbf{v}) = v|\mathbf{v}|^2_{1,\omega},$$

On the other hand, the inf-sup condition says that

$$\sup_{\mathbf{v} \ inH^1_0(\Omega)^N} \frac{(q, \nabla.\mathbf{v})}{|\mathbf{v}|_{1,\Omega}} \geq \beta \|q\|_{0,\Omega} \qquad \forall q, L^2_0(\Omega). \tag{26}$$

Let $q \in L^2_0(\Omega)$; by virtue of corollary 2.4, there exists a unique function $\mathbf{v} \in V$ such that

$$\nabla.\mathbf{v} = q, \qquad |\mathbf{v}|_{1,\Omega} \leq C \|q\|_{0,\Omega} \tag{27}$$

Hence

$$\frac{(q, \nabla.v)}{|v|_{1,\Omega}} = \frac{\|q\|^2_{0,\Omega}}{|V|_{1,\Omega}} \geq (1/C) \|q\|_{0,\Omega}$$

from which 26 follows with $\beta = 1/C$.
Now there exist a unique pair of functions (w,p)$\in H^1_0(\Omega)^N x L^2_0(\Omega)$ such that

$$\begin{cases} a(w,v)+b(v,p)=<\ell,v> & \forall v\in H_0^1(\Omega)^N \\ b(w,q)=0, & \forall q\in L_0^2(\Omega)\end{cases} \tag{28}$$

Equivalently, $(u=u_0+w,p)\in[u_0+H_0^1(\Omega)^N]\times L_0^2(\Omega)$ is the solution of the equations

$$\begin{cases} \nu(\nabla u,\nabla p)-(p,\nabla\cdot v)=<f,v> & \forall v\in H_0^1(\Omega)^N \\ (q,\nabla.u)=0, & \forall q\in L_0^2(\Omega)\end{cases} \tag{29}$$

Applying Corollary 2.4
Corollary 2.4 Let Ω be connected. Then

1) the operator ∇ is an isomorphism of $L_0^2(\Omega)$ onto V^o

2) the operator $\nabla.u$ is an isomorphism of $V^{\perp}$ onto $L_0^2(\Omega)$

this last equatiton is equivalent to $\nabla\cdot u=0$. Moreover, $u\in u_0+H_0^1(\Omega)^N$ if and only if

$$u\in H^1(\Omega)^N,\quad u_\Gamma=g$$

Hence there exists a unique pair $(u,p)\in H_0^1(\Omega)^N\times L_0^2(\Omega)$ such that

$$\begin{cases} \nu(\nabla u,\nabla p)-(p,\nabla\cdot v)=<f,v> & \forall v\in H_0^1(\Omega)^N \\ \nabla.u=0, \\ u_\Gamma=g\end{cases} \tag{30}$$

Implicitly the theorem 5.1 is proved by the de Rham's Theorem in order to well-defined the existence of the scalar function p(presure) in the variational formulation:
Theorem de Rham.- Let ℓ belong to $H_0^{-1}(\Omega)^N$ and satisfy

$$<\ell,v>=0\quad\forall v\in V \tag{31}$$

Then, there exists exactly one function ϕ in $L_0^2(\Omega)$ such that

$$<\ell,v>=\int_\Omega\phi\nabla\cdot v=<\Delta\phi,v>\quad\forall v\in H_0^{-1}(\Omega)^N \tag{32}$$

Important Remark: We use the same indices for theorems, corollaries from the book[1] in order that the reader can reference quickly them and learn other proposed Stokes equation from this book.

1.4 Implementation in Freefem++

First, we remember the variation approach developed before in our problem
In the domain $\Omega\in\mathbb{R}^2$, we denote by u the vector field, p is the scalar function and $\mu thedynamicviscosity$. The classical mechanical stress σ_* of the fluid is:

$$\sigma^*(u,p)=2\mu e(u)-pI_2\quad e(u)=\frac{1}{2}(\nabla u+(\nabla u)^t$$

which is equivalent to:

$$\nabla\cdot\sigma(u,p)=\mu\nabla u-pI_2$$

because $\nabla\cdot 2\mu e(u)=\mu\nabla\cdot\nabla u+\mu\nabla\cdot(\nabla u)^t=2\mu\nabla\cdot\nabla u+2\mu\nabla\nabla\cdot u=2\mu\Delta u$
since $\nabla\cdot u=0$ It follows that the Stokes problem is to find the velocity field u and the pressure p, satisfying:

$$\begin{cases} -\nabla\cdot\sigma(u,p)=f & \in\Omega \\ \nabla\cdot u=0\Omega\end{cases} \tag{33}$$

or equivalent to

$$\begin{cases} \nu\Delta u+\nabla p=f & \text{in }\Omega \\ \nabla.u=0, & \text{in }\Omega\end{cases} \tag{34}$$

where f is the density of external forces plus boundary condition defined through the variational form:

$$\forall v,q:\int_\Omega 2\mu\nabla u:\nabla v-q\nabla\cdot u-p\nabla\cdot v=\int_\Omega f\dot{v}+\int_\Gamma(\nabla\cdot\sigma(u,p)=\mu\nabla u-pI_2n)^tv$$

where $a : b = \Sigma_{i,j} a_{ij} b_{ij}$
The pressure p is defined up to an additive constant and the weak formulation can use a regularization (to remove the problem of the additive constant and impose a zero mean value for the pressure).

$$\forall v \in (H_0^1)^2, q \in L^2 : \int_\Omega 2\mu\nabla u : \nabla v - q\nabla \cdot u - p\nabla \cdot v - \epsilon pq = \int_\Omega f \cdot v$$

With the following code, we translate the mathematical variational formulation into a symbolic language, specifically, in FreeFem++ software:

```
int nn=10;
mesh Th=square(nn,nn);
macro grad(u) [dx(u),dy(u)] //
macro Grad(u1,u2) [grad(u1),grad(u2)]  //
macro D(u1,u2) [ [dx(u1),(dy(u1)+dx(u2))*.5] , [(dy(u1)+dx(u2))*.5,dy(u2)] ]   //
macro div(u1,u2) (dx(u1)+dy(u2))//
real epsp =1e-8, mu = 1;
fespace Vh(Th,P2); fespace Ph(Th,P1); // Taylor Hood Finite element

Vh u1,u2, v1,v2;    Ph p,q ;
solve Stokes([u1,u2,p],[v1,v2,q]) =
int2d(Th) ( mu*(Grad(u1,u2):Grad(v1,v2))
           - div(u1,u2)*q   - div(v1,v2)*p  -epsp*p*q )
 + on(1,2,u1=0,u2=0) + on(3,u1=1,u2=0) ;
 plot([u1,u2],p,wait=1);
 cout << " mean value pressure= " << int2d(Th)(p)/Th.area<<endl;
```

In this code, we generate

1) a domain $\Omega^2 = [0,1] \times [0,1]$ with the command mesh

2) Sobolev space for $\mathbb{P}_{\not{k}}, \mathbb{P}_{\not{k}}-$finite element with command fespace in order to use Taylor Hood Finite element

3) the variational formulation with $g(x, y) = (1, 0) \in \Gamma_d$ throung the command on(3,u1=1,u2=0); and for on(1,2,4,u1=0,u2=0) we choose $u = 0 \in \delta\Omega/(\Gamma_d \cup \Gamma_n)$

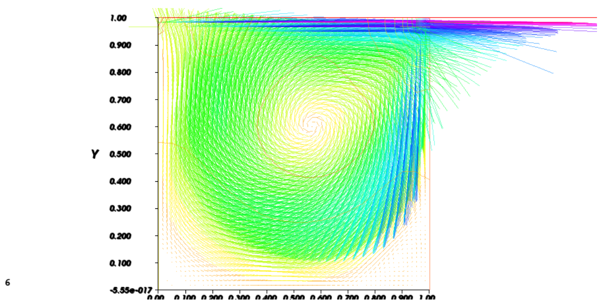

Figure 3: Fluid dynamics of a entrained cavity

1.5 References

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