NUMERICAL METHODS OF PARTIAL DIFFERENTIAL EQUATIONS
PROFESSOR: CRISTHIAN MONTOYA (cmontoya@yachaytech.edu.ec)
LABORATORY 3–FINITE DIFFERENCE METHOD–2D

The propose in this session is to solve the Poisson's equation in a rectangular domain through finite differences as well as the study of convergence for a second order equation in one dimension.

1. Consider the equation $-\Delta u = 0$ in the domain $\Omega = [0,1] \times [0,1]$ with the following boundary conditions:

$$\begin{cases} u(0,y) = u(1,y) = 0 & y \in [0,1], \\ u(x,0) = 0, & x \in [0,1], \\ u(x,1) = g(x), & x \in [0,1], \end{cases} \tag{1}$$

   where $g \in C^1([0,1])$, with $g(0) = g(1) = 0$.
   Prove that the Laplace problem with boundary conditions (1) admits a unique solution.

2. Poisson's problem: Find u in an appropriate space satisfying

$$\begin{cases} -\Delta u(x,t) = f(x,y) & (x,y) \in \Omega, \\ u(x,y) = g(x,y), & (x,y) \in \partial\Omega, \end{cases} \tag{2}$$

   where f and g are known functions (in appropriate spaces).
   We define the following approximation for the square domain $\Omega = [0,1]^2$:

$$\Omega_h = \{(x_j, y_k) : 0 \leq j,k \leq N+1\},$$

   such that $h = \dfrac{1}{N+1}$, $x_j = jh,\ y_k = kh$. The discrete Laplacian operator to 5 nodes is given by:

$$\Delta_h u_{j,k} = \frac{1}{h^2}(-4u_{j,k} + u_{j-1,k} + u_{j+1,k} + u_{j,k+1} + u_{j,k+1}),$$

   where $u_{j,k} = u(x_j, y_k)$.
   Show that Δ_h is consistent of second order with respect to Δ. In other words, prove

$$\|(\Delta u) - (\Delta_h u)\|_\infty \leq Ch^2, \quad \text{for } u \in C^4(\overline{\Omega}).$$

3. Taking into account the Dirichlet boundary conditions, the discrete unknown variables are N^2, which represent the interior nodes. If those internal nodes are enumerate from left to right and bottom-up, then is possible to enumerate each node by the rule:

$$(u_h)_n = u_h(x_j, y_k), \quad n = (k-1)N + j,\ 1 \leq j,k \leq N.$$

   Thus, the discrete problem can be rewritten by the linear system

$$A_h u_h = b_h, \tag{3}$$

   where

$$A_h = \frac{1}{h^2}\begin{pmatrix} L_4 & -I & 0\cdots & & & 0 \\ -I & L_4 & -I & 0 & & \\ 0 & \ddots & \ddots & \ddots & & \\ \vdots & \ddots & & & & 0 \\ & & & -I & L_4 & -I \\ 0 & & & 0 & -I & L_4 \end{pmatrix} \qquad L_4 = \begin{pmatrix} 4 & -1 & 0 & \cdots & 0 \\ 4 & -1 & -1 & \cdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & -1 & 4 & -1 \\ 0 & \cdots & 0 & -1 & 4 \end{pmatrix} \tag{4}$$

and

$$b_h = \begin{pmatrix} f_1 + \frac{1}{h^2}(g(h,0)+g(0,h)) \\ f_2 + \frac{1}{h^2}g(2h,0) \\ \vdots \\ f_N + \frac{1}{h^2}(g(Nh,0)+g(Nh+h,h)) \\ f_{N+1} + \frac{1}{h^2}(g(0,2h)) \\ f_{N+2} \\ \vdots \\ f_{2N} + \frac{1}{h^2}(g(Nh+h,2h)) \\ f_{2N+1} + \frac{1}{h^2}(g(0,3h)) \\ f_{2N+2} \\ \vdots \end{pmatrix} \tag{5}$$

Use the boundary conditions mentioned in (1) with $g(x) = x(1-x)$ and

a) Make a function with input N and output A_h. Likewise, another function with output the vector b_h.

b) Solve the system (3) for $N = 10, 20, 30, 40, 50$ and study the condition number as well.

4. Consider the previous Laboratory (just Exercise 1–Exercise 4).
For different values of N, for example $N = 6 \times 2^j,\ j = 1, \ldots, 7$ estimate the error $\|U_h - \theta\|_1$, $\|U_h - \theta\|_2$ and $\|U_h - \theta\|_\infty$. Graphic the three curves at the same figure, in function of N and $\log - \log$–scale. To deduce the order of convergence and compare with the theoretical results.