Soluciones de onda viajera de un modelo continuo de transmisión de energía en medios no homogéneos
Traveling-wave solutions of a continuous model of energy transmission in non-homogeneous media

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Abstract   En este trabajo, se investigará la existencia de soluciones de un model continuo de transmisión de ondas en medios no homogéneos. Una extensión del método clásico de integrales directas (también conocido como el método de la ecuación de prueba) será empleada en este trabajo para derivar soluciones analíticas del modelo bajo estudio. Todas las soluciones que se obtendrán son ondas viajeras descritas de forma exacta.

Abstract   In this work, we investigate the existence of exact solutions of a continuous model of wave transmission in nonhomogeneous media. An extension of the classical method of direct integrals (also known as the trial equation method) is used in this work in order to derive some analytical solutions of the model under study. All the solutions derived in this work are actually traveling waves that are presented in exact form.

Ecuaciones hiperbólicas alineales — Coeficientes temporalmente variables — Soluciones de onda viajera
Nonlinear hyperbolic equations — Time-dependent coefficients — Traveling-wave solutions

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Introduction

The classical Klein–Gordon equation and its (continuous and discrete) nonlinear generalizations are models that describe a wide range of phenomena in modern physics. Among many other recent reports, equations of the Klein–Gordon and sine-Gordon types have been employed to obtain a vast family of localized solutions in exact form [1], to study the behavior of a massive scalar field in a charged rotating hairy black hole [2], to explore the dynamics of genuine continuum breathers and the conditions under which it persists in a $\mathscr{PT}$-symmetric medium [3], as a model to construct non-equilibrium steady states in arbitrary spatial dimensions under a local quench [4], as an example of a discrete model for which phase-shift breathers cannot be supported [5] and in the investigation of the phenomenon of nonlinear supratransmission in nonlinear regimes [6] among other applications [7, 8].

Most of the Klein–Gordon models investigated in the literature consider the presence of constant coefficients. Indeed, most of the relevant models are particular cases or (continuous or discrete) extensions of the nonlinear hyperbolic partial differential equation

$$u_{tt} + du_t - a^2 u_{xx} + bu + cu^3 = 0, \tag{1}$$

where u is a function of the real variables x and t. Meanwhile, the symbols u_t, u_{tt} and u_{xx} represent the partial derivatives

of u with respect to t, t^2 and x^2, respectively, and a, b, c and d are usually fixed real numbers. This last requirement physically implies that the media described by those Klein–Gordon models are homogeneous in both space and time.

Motivated by these facts, we will consider here an inhomogeneous generalization of the damped Klein–Gordon equation in which the coefficients are functions of time. The goal of this work is to propose expressions for traveling-wave solutions of that model, as well as conditions on the (nonconstant) coefficients and the wave velocities that guarantee the existence of such solutions. To that end, we will employ the extension of the method of direct integrals (also known as the trial equation method) proposed in [1]. Such approach is a generalization of the method designed by W.-X. Ma and B. Fuchssteiner for partial differential equations with constant coefficients [9]. It is important to mention that different approaches have been employed in the literature to determine traveling-wave solutions of nonlinear models in physics [10]. However, the generalized trial equation technique is particularly interesting in view of its simplicity and the fact that various families of solutions are fully characterized through the values of discriminant systems (for instance, see [11, 12] and references therein).

This note is sectioned as follows. In Section 1, we present the damped Klein–Gordon equation with time-dependent co-

efficients that motivates this work. The method of the direct integral is used then to determine traveling-wave solutions of the model. The expressions of the constant parameters required by the direct integral method are derived, along with necessary conditions for the existence of the solutions. In Section 2, we consider various particular cases and, in each of them, we derive traveling-wave solutions of the parametric Klein-Gordon equation of interest. We will note in that section that some of the traveling-wave solutions that we will obtain are complex-valued functions. Finally, we close this work with a section of concluding remarks.

1. Klein–Gordon equation

In this work, we will use the symbol $\overline{\mathbb{R}^+}$ to represent the set of nonnegative real numbers, and we will suppose that a, b, c and d are real-valued and Riemann-integrable functions defined on $\overline{\mathbb{R}^+}$. Throughout we will assume that u is a nonnegative real function on the vector $(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}$ with continuous partial derivatives up to the second order, satisfying the partial differential equation

$$u_{tt} + d(t)u_t - a^2(t)u_{xx} + b(t)u + c(t)u^3 = 0. \quad (2)$$

Note that this model is a generalization of the Klein–Gordon equation (1), which considers now non-constant coefficients that depend on the temporal variable.

In the present work, we will use an extension of the method of direct integrals (or trial equation method) proposed by W.-X. Ma and B. Fuchssteiner in [9] in order to account for the presence of time-dependent coefficients. Such parametric approach has been used recently for simpler diffusive models [1, 13]. Following the approach described in [1], we assume the existence of traveling-wave solutions of the model (2) which take the form

$$u(x,t) = u(\xi), \quad (3)$$

where

$$\xi = \xi(x,t) = \kappa(t)x + \omega(t), \quad (4)$$

for suitable functions $\kappa, \omega : \overline{\mathbb{R}^+} \to \mathbb{R}$ with continuous derivatives up to the second order.

Using these assumptions and the chain rule, the following identities readily follow:

$$u_t = (\kappa'(t)x + \omega'(t))\dot{u}, \quad (5)$$
$$u_{tt} = (\kappa'(t)x + \omega'(t))^2\ddot{u} + (\kappa''(t)x + \omega''(t))\dot{u}, \quad (6)$$
$$u_{xx} = \kappa^2(t)\ddot{u}. \quad (7)$$

Here, $\dot{u}$ represents the derivative of $u = u(\xi)$ with respect to ξ, and $\ddot{u}$ denotes the respective second derivative of u. Suppose that there exist a nonnegative integer m and real constants α_i for each $i = 0, 1, \ldots, m$, such that

$$(\dot{u})^2 = \sum_{k=0}^{m} \alpha_k u^k. \quad (8)$$

As a consequence,

$$\ddot{u} = \frac{1}{2}\sum_{k=1}^{m} k\alpha_k u^{k-1}. \quad (9)$$

Substituting the formulas (5)–(7) as well as the expansion (8) into our model, we obtain the ordinary differential equation

$$\Theta(x,t)\dot{u} + \Lambda(x,t)\sum_{k=1}^{k} k\alpha_k u^{k-1} + b(t)u + c(t)u^3 = 0, \quad (10)$$

where

$$\Theta(x,t) = [k''(t) + d(t)\kappa'(t)]x + \omega''(t) + d(t)\omega'(t), \quad (11)$$
$$\Lambda(x,t) = \frac{1}{2}\left[(\kappa'(t)x + \omega'(t))^2 - a^2(t)\kappa^2(t)\right]. \quad (12)$$

Note that the left-hand side of (10) is a polynomial in the variables u and $\dot{u}$ which must be equal to zero. Applying he balancing principle, it follows that $m = 4$. Moreover, as additional consequences of the balancing principle we obtain that

$$\Theta(x,t) = 0, \quad (13)$$
$$2\alpha_2\Lambda(x,t) + b(t) = 0, \quad (14)$$
$$4\alpha_4\Lambda(x,t) + c(t) = 0, \quad (15)$$

for every $(x,t) \in \mathbb{R} \times \overline{\mathbb{R}^+}$. Note also that $\alpha_1 = \alpha_3 = 0$ must be satisfied.

Several simplifications readily follow from these last identities. Indeed, the facts that both α_2 and α_4 are constants together with Equations (14) and (15) imply that Λ depends only on the temporal variable. This and the expression of Λ yield that $\kappa'(t) = 0$. Equivalently, $\kappa(t)$ must be equal to a constant $\kappa \in \mathbb{R}$. Therefore both Θ and Λ are functions exclusively of t, and the conditions (13)–(14) can be rewritten as

$$\omega''(t) + d(t)\omega'(t) = 0, \quad (16)$$
$$\alpha_2 = \frac{b(t)}{\kappa^2a^2(t) - (\omega'(t))^2}, \quad (17)$$
$$\alpha_4 = \frac{c(t)}{2\left[\kappa^2a^2(t) - (\omega'(t))^2\right]}, \quad (18)$$

for each $t \in \overline{\mathbb{R}^+}$. Observe that κ must be a real number such that the right-hand sides of the last two equations are constants.

Before we close this section, it is important to note that the solution (16) can be readily expressed in analytic form. Moreover, constant functions $\omega(t) = \omega$ satisfy this equation. Also, in the absence of the damping term the function ω adopts the form $\omega(t) = \omega_1 t + \omega_2$, where ω_1 and ω_2 are real constants.

2. Exact solutions

Following the results in Section 1, let $m = 4$, let $\alpha_1 = \alpha_3 = 0$, and suppose that $\alpha_0 \in \mathbb{R}$ and that the expressions in (17) and (18) are constant real numbers. Using the change of variable $u = v^{1/2}$ and some algebraic simplifications, Equation (8) becomes

$$(\dot{v})^2 = 4\alpha_4 v^3 + 4\alpha_2 v^2 + 4\alpha_0 v. \qquad (19)$$

To simplify this expression we let $w = (4\alpha_4)^{1/3} v$. Expressing the identity (19) in terms of w, separating variables in the resulting equation and integrating both sides, we obtain

$$\int \frac{dw}{\sqrt{w(w^2 + b_1 w + b_0)}} = \pm (4\alpha_4)^{1/3}(\xi - \xi_0), \qquad (20)$$

where $b_1 = 4\alpha_2(4\alpha_4)^{-2/3}$ and $b_0 = 4\alpha_0(4\alpha_4)^{-1/3}$, and ξ_0 is an arbitrary real number.

Let F be the polynomial inside the radical in the integrand of the last equation as a function of w. Note that the discriminant of $F(w)$ is given by $\Delta = b_1^2 - 4b_0$. Accordingly, various cases can be considered now depending on the discriminant and the constants in the expansion (8). Following the approach used in [1], we will consider those cases next.

Case 1. $\Delta = 0$, $w > 0$, $\alpha_2 < 0$ and $a_4 > 0$
In this case, the following functions are solutions of (2):

$$u = \pm\sqrt{\frac{|\alpha_2|}{2\alpha_4}} \tanh\left(\sqrt{\frac{|\alpha_2|}{2}}(\xi - \xi_0)\right), \qquad (21)$$

$$u = \pm\sqrt{\frac{|\alpha_2|}{2\alpha_4}} \coth\left(\sqrt{\frac{|\alpha_2|}{2}}(\xi - \xi_0)\right). \qquad (22)$$

Case 2. $\Delta = 0$, $w > 0$, $\alpha_2 > 0$ and $a_4 > 0$
Under these conditions, the solution is given by

$$u = \pm\sqrt{\frac{\alpha_2}{2\alpha_4}} \tan\left(\sqrt{\frac{\alpha_2}{2}}(\xi - \xi_0)\right). \qquad (23)$$

Case 3. $\Delta = 0$, $w > 0$ and $b_1 = 0$
In this case, the corresponding solution is

$$u = \pm\frac{\sqrt[3]{2}}{\sqrt[3]{\alpha_4}(\xi - \xi_0)}. \qquad (24)$$

Case 4. $\Delta > 0$, $b_0 = 0$, $w > -b_1$, $\alpha_2 > 0$ and $\alpha_4 > 0$
In this case, the integrations yield the solutions

$$u = \pm\sqrt{\frac{\alpha_2}{\alpha_4}} \left[\frac{1}{2}\tanh^2\left(\sqrt{\frac{\alpha_2}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}, \qquad (25)$$

$$u = \pm\sqrt{\frac{\alpha_2}{\alpha_4}} \left[\frac{1}{2}\coth^2\left(\sqrt{\frac{\alpha_2}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}. \qquad (26)$$

It is worth mentioning here that, unfortunately, these functions take on values in the system of the complex numbers

and they do not provide meaningful real solutions of (2). However, more relevant solutions are given by (see [13])

$$u = \pm\sqrt{\frac{\alpha_2}{\alpha_4}} \left(\frac{\operatorname{sech}(\sqrt{\alpha_2}(\xi - \xi_0))}{1 - \sqrt{2}\tanh(\sqrt{\alpha_2}(\xi - \xi_0))}\right). \qquad (27)$$

Case 5. $\Delta > 0$, $b_0 = 0$, $w > -b_1$, $\alpha_2 < 0$ and $\alpha_4 > 0$
The following function is a solution of the damped Klein-Gordon equation:

$$u = \pm\sqrt{\frac{|\alpha_2|}{\alpha_4}} \left[\frac{1}{2}\tan^2\left(\sqrt{\frac{|\alpha_2|}{2}}(\xi - \xi_0)\right) - 1\right]^{1/2}. \qquad (28)$$

Again, it is easy to realize that this solution exists only for values of ξ for which the term in the brackets is a nonnegative number.

Case 6. $\Delta > 0$, $b_0 = 0$, $w > -b_1$, $\alpha_2 > 0$ and $\alpha_4 < 0$
In this case, we have the following exact solutions of (see [13]):

$$u = \pm\sqrt{\frac{2\alpha_2}{|\alpha_4|}} (1 + \cosh(2\sqrt{\alpha_2}(\xi - \xi_0))^{-1/2}, \quad (29)$$

$$u = \pm\sqrt{\frac{\alpha_2}{|\alpha_4|}} \operatorname{sech}(\sqrt{\alpha_2}(\xi - \xi_0)). \qquad (30)$$

Case 7. $\Delta > 0$, $b_0 \neq 0$ and $\alpha < \beta < \gamma$
In a first stage, we suppose that $\Delta > 0$, $b_0 \neq 0$, $\alpha < \beta < \gamma$ are real numbers with one of them equal to zero, the others are solutions of $w^2 + b_1 + b_0 = 0$, and $\alpha < w < \beta$. Under these circumstances,

$$u = \pm\sqrt[6]{4\alpha_4} \left[\alpha + (\beta - \alpha)\operatorname{sn}^2\left(\sqrt[3]{\alpha_4/2}\sqrt{\gamma - \alpha}(\xi - \xi_9), m\right)\right]^{1/2}. \qquad (31)$$

Here, sn represents the Jacobi elliptic sine function and

$$m^2 = \frac{\beta - \alpha}{\gamma - \alpha}. \qquad (32)$$

Secondly, suppose that $\Delta > 0$, $b_0 \neq 0$, $\alpha < \beta < \gamma$ are as before and $w > \gamma$. The solution of this case is provided by the formula

$$u = \pm\sqrt[6]{4\alpha_4} \frac{\left[\gamma + \beta\operatorname{sn}^2\left(\sqrt[3]{\alpha_4/2}\sqrt{\gamma - \alpha}(\xi - \xi_9), m\right)\right]^{1/2}}{\operatorname{cn}^2\left(\sqrt[3]{\alpha_4/2}\sqrt{\gamma - \alpha}(\xi - \xi_9), m\right)}, \qquad (33)$$

where m is defined as in Case 6 and cn represents the Jacobi elliptic cosine function.

3. Conclusions

In this letter, we used an extension of the direct integral method or trial equation method to determine exact solutions of a

damped Klein-Gordon equation with time-dependent coefficients. The extension of the direct integral method used here is based on a paper by Liu [1], and it does not require for the coefficients of the trial equation to be constant. Instead, the approach followed here assumes that those coefficients are functions of the temporal variable. This technique is applied to a partial differential equation with non-constant coefficients that generalizes the well-known damped Klein–Gordon model with nonlinear reaction. As a result of our derivations, some traveling-wave solutions of our model have been calculated in exact form, and pertinent conditions on the coefficients of the model and the wave velocity are established in order to guarantee the existence of such solutions.

References

[1] Yang Liu. Exact solutions to nonlinear Schrödinger equation with variable coefficients. *Applied Mathematics and Computation*, 217(12):5866–5869, 2011.

[2] B Pourhassan. The Klein–Gordon equation of a rotating charged hairy black hole in (2+ 1) dimensions. *Modern Physics Letters A*, 31(09):1650057, 2016.

[3] Nan Lu, Panayotis G Kevrekidis, and Jesús Cuevas-Maraver. $\mathscr{PT}$-symmetric sine-Gordon breathers. *Journal of Physics A: Mathematical and Theoretical*, 47(45):455101, 2014.

[4] Benjamin Doyon, Andrew Lucas, Koenraad Schalm, and MJ Bhaseen. Non-equilibrium steady states in the Klein–Gordon theory. *Journal of Physics A: Mathematical and Theoretical*, 48(9):095002, 2015.

[5] Vassilis Koukouloyannis. Non-existence of phase-shift breathers in one-dimensional Klein–Gordon lattices with nearest-neighbor interactions. *Physics Letters A*, 377(34):2022–2026, 2013.

[6] J E Macías-Díaz and A Puri. An application of nonlinear supratransmission to the propagation of binary signals in weakly damped, mechanical systems of coupled oscillators. *Physics Letters A*, 366(4):447–450, 2007.

[7] Abdul-Majid Wazwaz and SA El-Tantawy. Gaussian soliton solutions to a variety of nonlinear logarithmic Schrödinger equation. *Journal of Electromagnetic Waves and Applications*, 30(14):1909–1917, 2016.

[8] AR Prasanna. On photon trajectories and electromagnetics near strongly gravitating cosmic sources. *Journal of Electromagnetic Waves and Applications*, 29(3):283–330, 2015.

[9] Wen-Xiu Ma and Benno Fuchssteiner. Explicit and exact solutions to a Kolmogorov-Petrovskii-Piskunov equation. *International Journal of Non-Linear Mechanics*, 31(3):329–338, 1996.

[10] Abdul-Majid Wazwaz. Solitary waves solutions for extended forms of quantum Zakharov–Kuznetsov equations. *Physica Scripta*, 85(2):025006, 2012.

[11] Cheng-Shi Liu. Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations. *Computer Physics Communications*, 181(2):317–324, 2010.

[12] Cheng-Shi Liu. A new trial equation method and its applications. *Communications in Theoretical Physics*, 45(3):395, 2006.

[13] Houria Triki and Abdul-Majid Wazwaz. Trial equation method for solving the generalized Fisher equation with variable coefficients. *Physics Letters A*, 380(13):1260–1262, 2016.