EE 364B Project Proposal:
Fast Eigenvalue Optimization for Spectral-Element PDEs

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1 Goal

Eigenvalue optimization is a common task in many control problems [BEGFB94] and, recently, in inference problems on continuous spaces. In this project, we'll attempt to write a fast solver for eigenvalue problems arising from spectral element discretizations of partial differential equations (PDEs).

2 Introduction

A *spectral element method* (SEM)[1] is a finite-element method for solving PDEs which makes use of high-degree polynomials to approximate functions on a compact domain. The idea is to then construct a quadrature rule such that all polynomials of lesser degree have exact integrals when sampled at a particular set of points.

In general, these SEM-discretized problems—while sparse—are usually slow to solve in modern optimization packages for even a modest number of points in the discretization domain as they lack the tridiagonal structure that many PDE problems have (e.g., when discretizing by assuming $h^2 f''(x) \approx f(x+h) - 2f(x) + f(x-h)$ for fixed $h > 0$). In particular, the matrix generated by the spectral elements method is essentially a block-diagonal matrix with small blocks, each of which overlap with the previous block by exactly one element, so no completely trivial decomposition can be applied.[2]

Our project will take advantage of the banded structure of these approximately-block-diagonal matrices for fast solutions of eigenvalue problems.

[1]Note that this is a class of methods, each generated by different polynomials or quadrature rules.

[2]Further, it's important to note that this matrix sparsity pattern is a special case of chordal sparsity [VA$^+$15], which, while fast to solve in general, does not take advantage of the further structure which is obvious from this problem.

3 Approaches

Let $\mathscr{L} : \mathbf{R}^n \to \mathbf{S}^m$ be affine and have the sparsity structure described above and let $x \in \mathbf{R}^n$, then the problem of interest can be written as the dual of the standard form of SDPs,

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & -b^T x + t \\ \text{subject to} & \mathscr{L}(x) \preceq tI, \\ & Cx = d, \\ & x \succeq 0. \end{array} \tag{1}$$

The first inequality is with respect to the semidefinite cone, while the latter is with respect to the positive orthant.

This general optimization problem can be solved in several ways. We will attempt to explore two possibilities: interior point methods for solving these large-scale SDPs and first-order cone-splitting methods for solving these problems. In the former case, consider a rewriting of the problem using a barrier method is straightforward

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & -b^T x + t - \mu \log\det(tI - \mathscr{L}(x)) \\ \text{subject to} & Cx = d, \\ & x \succeq 0. \end{array}$$

If the Cholesky factorization of $X \equiv tI - \mathscr{L}(x)$ can be easily computed, say, $X = L^T L$, then we're done as $\log\det X = 2\sum_i \log L_{ii}$, since the determinant of a triangular matrix is the product of its diagonal elements. Solving this barrier-method minimization via most methods is then straightforward.

As in [ZFP+17] define $E_i \in \mathbf{R}^{d\times m}$ to be the 'selection matrix' defined by, where b is the number of sub-blocks of $\mathscr{L}$ and d is their dimension,

$$E_i = \underbrace{\begin{bmatrix} \underbrace{0_{d\times(d-1)} \quad \dots \quad 0_{d\times(d-1)}}_{i-1} & I_d & 0_{d\times(d-1)} & \dots & 0_{d\times(d-1)} \end{bmatrix}}_{b}.$$

This allows us to rewrite (1) above to

$$\begin{array}{ll} \underset{x,t}{\text{minimize}} & -b^T x + t \\ \text{subject to} & \mathscr{L}(x) - tI = \sum_i E_i Z_i E_i^T, \\ & Cx = d, \\ & x \succeq 0, \\ & Z_i \succeq 0,\ i \in \{1, 2, \dots, b\}, \end{array} \tag{2}$$

where projecting the much smaller Z_i matrices (if $d \ll m$, as is the case with our problem) into the PSD cone is much faster than projecting $\mathscr{L}(x) - tI$. Both approaches will be compared with each other and then to available solvers.

References

[BEGFB94] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. Siam, 1994.

[VA+15] Lieven Vandenberghe, Martin S Andersen, et al. Chordal graphs and semidefinite optimization. *Foundations and Trends® in Optimization*, 1(4):241–433, 2015.

[ZFP+17] Yang Zheng, Giovanni Fantuzzi, Antonis Papachristodoulou, Paul Goulart, and Andrew Wynn. Fast admm for semidefinite programs with chordal sparsity. In *American Control Conference (ACC), 2017*, pages 3335–3340. IEEE, 2017.