

# Ligerito: A Small and Concretely Fast Polynomial Commitment Scheme

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## Abstract

In this note we present Ligerito, a small and practically fast polynomial commitment and inner product scheme. For the case of univariate and multilinear polynomial evaluations, the scheme has a proof size of  $\sim \log(N)^2 / \log \log(N)$  up to constants and for a large enough field, where  $N$  is the size of the input. Ligerito is also fast on consumer hardware: when run on an M1 MacBook Pro for a polynomial with  $2^{24}$  coefficients over a 32-bit binary field, our Julia prover implementation has a proving time of 1.3 seconds and a proof size of 255 KiB. Ligerito is also relatively flexible: any linear code for which the rows of the generator matrix can be efficiently evaluated can be used. Such codes include Reed–Solomon codes, Reed–Muller codes, among others. This, in turn, allows for a high degree of flexibility on the choice of field and can likely give further efficiency gains in specific applications.

## 1 Introduction

Polynomial commitment schemes such as KZG [KZG10], Ligerito [Ame+17], FRI [Ben+18], BaseFold [ZCF23], Blaze [Bre+24], and WHIR [Arn+24] have become workhorses for zero-knowledge proofs and other applications such as blockchain scaling [EMA25]. In many cases (with the notable exception of Ligerito), state-of-the-art schemes require the use of specific error correcting codes, such as Reed–Solomon codes, or require fields with particular properties. In this note, we present Ligerito, a new polynomial commitment scheme and inner product argument, which works over essentially any code that has the following property: any row of the code’s generator matrix can be efficiently evaluated. While this flexibility is nice, we show that, even when using basic Reed–Solomon codes over binary fields, Ligerito has small proofs and fast in practice on standard consumer hardware. This scheme also allows for the relatively ‘rich’ language of sumcheck to express more complicated constraints (such as arbitrary inner products with the coefficients of the provided polynomial) with nearly no additional overhead for the prover or the proof size. Finally, Ligerito is asymptotically small with a proof size of  $\sim \log(N)^2 / \log \log(N)$ , up to constants, for a large enough field, where  $N$

is the size of the input. The proving time is, essentially, proportional to the encoding time of the codes used.

**Structure of this note.** The rest of this note is structured as follows. In section 2, we describe the notation and conventions we will use throughout the note. In sections 3 and 4, we describe the matrix-vector product protocol and partial sumcheck protocols, which we will use as black boxes in the construction of Ligerito. In section 5, we describe the matrix-vector product protocol with partial sumcheck, which is a ‘simple’ way of merging the matrix-vector product and sumcheck protocols, and serves as a stepping stone for the final protocol. In section 6, we describe the Ligerito protocol in its entirety and provide a simple proof of its error bounds, ending with a small set of practical experiments in section 7.

## 2 Notation and conventions

We use the notation and conventions of [EA23] for simplicity. While familiarity with the Ligerito protocol [Ame+17] is not strictly necessary to read this note, we refer readers to [AER24] and [DG24] for proofs which we will rely on throughout.

### 2.1 Basic notation

Let  $\mathbf{F}$  denote some finite field. We will sometimes make use of the fact that  $\mathbf{F}' \subseteq \mathbf{F}$  is a subfield of  $\mathbf{F}$ . Throughout this note, we use uppercase letters such as  $X$  and  $G$  to refer to matrices, use lowercase letters to refer to column vectors, such that  $v \in \mathbf{F}^n$  denotes an  $n$ -vector and  $G \in \mathbf{F}^{m \times n}$  denotes an  $m \times n$  matrix, each over the field  $\mathbf{F}$ . In general  $G \in \mathbf{F}^{m \times n}$  will refer to some (linear) code with distance  $d > 0$ , where  $n$  is called the *message length* and  $m$  is called the *block length*. We write  $\|v\|$  to denote the Hamming weight of a vector  $v$  (*i.e.*, the number of nonzero entries of  $v$ ) and  $\|X\|$  to denote the number of nonzero rows of a matrix  $X$ . Note that this notation is consistent if we interpret a vector  $v$  as an  $n \times 1$  matrix, as we generally will in this note.

### 2.2 Matrix and polynomial notation

Since we constantly reshape matrices and vectors throughout the note, we will overload notation to simplify and clean up the exposition.

**Matrices to vectors (and back).** For a given matrix  $\tilde{X} \in \mathbf{F}^{m \times n}$ , we write  $\mathbf{vec}(\tilde{X})$  to the  $mn$ -vector corresponding to stacking the columns of the matrix  $\tilde{X}$  into a single column vector. Similarly, for a given vector  $v \in \mathbf{F}^{mn}$ , we write  $\mathbf{Mat}(v)$  to denote matrix corresponding to reshaping the vector  $v$  into a column-major  $m \times n$  matrix. In general, it should always be clear from context what the dimensions of this matrix are such that the expressions correctly parse. When there is any ambiguity, we will be explicit about the dimensions of the resulting matrix. We will also sometimes write  $\mathbf{Mat}(\tilde{X})$  for the reshaping of a matrix  $\tilde{X}$  to the appropriate dimensions. (This is equivalent to writing  $\mathbf{Mat}(\mathbf{vec}(\tilde{X}))$ , which is now well-defined, but this is rather cumbersome.)

**Kronecker product.** We will define the Kronecker product  $\otimes$ , defined for two matrices  $A \in \mathbf{F}^{m \times n}$  and  $B \in \mathbf{F}^{m' \times n'}$  as

$$A \otimes B = \begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ A_{21}B & A_{22}B & \dots & A_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}.$$

The resulting matrix  $A \otimes B$  is therefore of size  $mm' \times nn'$ . To avoid overly complicated expressions we will write, for a given vector  $r \in \mathbf{F}^k$ , the ‘barred’ vector

$$\bar{r} = (1 - r_1, r_1) \otimes \dots \otimes (1 - r_k, r_k), \quad (1)$$

which is a (column) vector of size  $2^k$ .

**Polynomials and vectors.** Let  $v \in \mathbf{F}^{2^{k'}}$  be a general vector with elements in  $\mathbf{F}$ . It will be very convenient in what follows to overload notation and write, for  $z \in \mathbf{F}^{k'}$ ,

$$v(z) = v^T \bar{z} = v^T ((1 - z_1, z_1) \otimes \dots \otimes (1 - z_k, z_k)). \quad (2)$$

(Note that we have used the barred notation (1) for  $z$ .) In other words, when writing  $v$  as a function, evaluated at  $z \in \mathbf{F}^{k'}$ , we interpret the vector  $v$  as the coefficients of a multilinear polynomial in  $k$  variables and evaluate it at the point  $z$ . This lets us define the *partial evaluation* of  $v$  in the first  $k \leq k'$  variables at a point  $r \in \mathbf{F}^{k'}$ , as

$$y = \mathbf{Mat}(v) \bar{r}. \quad (3)$$

Note that, from our definition (1),  $\bar{r}$  is a vector of size  $2^k$ . This means that we can infer the shape of  $\mathbf{Mat}(v)$  to be a  $2^{k-k'} \times 2^{k'}$  matrix corresponding to a column-major reshaping of the elements of  $v$ . This, in turn, means that  $y$  is a  $2^{k-k'}$ -vector, which is the partial evaluation of  $v$  (interpreted as a polynomial) when the first  $k$  variables are equal to  $r$ , with the rest free; *i.e.*,

$$y(z) = v(r, z),$$

for all  $z \in \mathbf{F}^{k-k'}$ . (As before, we are overloading notation as in (2) for both  $y$  and  $v$ .) Finally, for matrices  $\tilde{X} \in \mathbf{F}^{2^k \times 2^{k'}}$  it will also be convenient to write

$$\tilde{X}(z) = (\mathbf{vec}(\tilde{X}))(z),$$

where  $\mathbf{vec}(\tilde{X})$  corresponds to stacking the columns of the matrix  $\tilde{X}$  into a size  $2^{k+k'}$  vector.

### 3 Matrix-vector product protocol

At a high level, the main tool we will use is the Ligero protocol [Ame+17], viewed as a matrix-vector product protocol [AER24], with the logarithmic randomness of [DP24]. The Ligero protocol allows us to compute  $\tilde{X} \bar{r}$ , where  $r$  will be some randomness we specify next, for any

matrix  $\tilde{X} \in \mathbf{F}^{n \times n'}$  with communication proportional to  $n + n'$ . To simplify the exposition, we will assume that  $n' = 2^k$ , though the results hold more broadly (by, *e.g.*, padding the matrix  $\tilde{X}$ ). We will also assume there is a known code  $G \in \mathbf{F}^{m \times n}$  with distance  $d > 0$ . We break the protocol down into two parts—the algorithm performed by the prover and the algorithm performed by the verifier.

**Prover algorithm.** Here we describe the prover’s algorithm for the matrix-vector product protocol. We assume that the prover algorithm has a matrix  $\tilde{X} \in \mathbf{F}^{n \times n'}$ . The algorithm is as follows:

1. Encode the columns of  $\tilde{X}$  to get  $X = G\tilde{X}$
2. Commit to the rows of  $X$ , via, say, a Merkle commitment
3. Receive some random vector  $r \in \mathbf{F}^k$
4. Compute and send  $y_r = \tilde{X}\bar{r}$

We will call the first two steps of the prover’s algorithm the *commitment phase*.

**Verifier algorithm.** Here we describe the verifier’s algorithm for the matrix-vector product protocol.

1. Receive the Merkle commitments to the rows of some opaque matrix  $X$
2. Sample and send some random vector  $r \in \mathbf{F}^k$ , receiving some vector  $y_r \in \mathbf{F}^n$
3. Uniformly sample  $S \subseteq \{1, \dots, m\}$ , of some fixed size  $|S|$
4. Verify that  $X_S \bar{r} = G_S y_r$

Similar to the previous, we say that the first step is the *commitment phase* of the verifier’s algorithm. If any step of this algorithm fails, the verifier rejects the proof, otherwise it accepts. The randomness used for  $\bar{r}$  here can will be *logarithmic* randomness, originally discovered in [DP24] and improved in [AER24], though it can be uniform randomness, as in [Ame+17]. (In the special case that  $G$  is Reed–Solomon, the randomness can also be the powers of some uniformly random element [Ben+23].) For the remainder of this note, we will assume that  $G$  is a Reed–Solomon code and use logarithmic randomness, for which the results of [DG24] are the best-known and we present them below. Finally, we note that, with some very small modifications, the results hold in the general setting, for arbitrary linear codes  $G$ , though potentially with slightly worse constants.

**Guarantees.** It is clear that, if the prover follows the above protocol, the verifier’s algorithm will always succeed. (This property is often called *completeness*.) Additionally, from [AER24] and [DG24], we know that the verifier, if all steps pass, has the following guarantees:

1. There exists some unique matrix  $\tilde{X}$  such that  $\|X - G\tilde{X}\| < d/2$ , where  $d$  is the distance of the Reed–Solomon code  $G$
2. The vector  $y_r$  satisfies  $y_r = \tilde{X}\tilde{r}$ ; in other words, the received  $y_r$  is the partial evaluation of  $\tilde{X}$ , interpreted as a polynomial, where the first  $k$  variables are equal to  $r$

All of these are true except with error probability no more than

$$\left(\frac{m-n-1}{2m}\right)^{|S|} + \frac{mk}{|\mathbf{F}|} \quad (4)$$

in the special case that  $G$  is a Reed–Solomon code, which is the case we use throughout the note, though, again, the results hold more generally except with potentially slightly worse constants.

**Communication complexity.** The communication complexity of this protocol is easy to tally up. The verifier received an  $n$ -vector (namely,  $y_r$ ) and the  $|S|$  requested rows of  $X$ . This totals to

$$(n + |S|n') \log_2(|\mathbf{F}|). \quad (5)$$

The prover also has to send the corresponding openings of each of the  $|S|$  rows, which, if using a binary Merkle commitment, totals to

$$|S| \log_2(m). \quad (6)$$

(In a practical protocol, the concrete numbers can be made slightly better in expectation by using a Merkle tree multi-opening.) Letting  $N = nn'$  be the total number of entries in  $\tilde{X}$ , then we may set  $n = n' = \sqrt{N}$  to get a communication complexity of

$$|S| \log_2(|\mathbf{F}|) \sqrt{N}, \quad (7)$$

not including the binary Merkle openings, which are small relative to this number as  $N$  becomes large.

**Discussion.** The matrix-vector product protocol is relatively simple, and we will use it as a black box in what follows. Note that we only access  $y_r$  via the inner products with the  $|S|$  rows of  $G$ , which is an observation we will use next.

## 4 Partial sumcheck protocol

The sumcheck protocol is a well-known interactive protocol for computing an inner product between two vectors. We will show a (only small) generalization here, which we use later in this note.

## 4.1 High level protocol

Given some known vector  $w \in \mathbf{F}^{2^k}$ , which we interpret as an  $k$  variable multilinear polynomial, and some claimed inner product  $\alpha \in \mathbf{F}$ , the sumcheck protocol is an interactive protocol for reducing a claims of the form

$$\sum_{z \in \{0,1\}^k} w(z) \tilde{X}(z) = \alpha \quad (8)$$

to a claim of the form

$$\sum_{z' \in \{0,1\}^{k'}} w(r, z') \tilde{X}(r, z') = \alpha'. \quad (9)$$

For some  $\alpha'$ , randomness  $r$  which we specify next, and  $k' < k$ . (In other words, we reduce the sum over  $k$  variables to a sum over  $k'$  variables and some partial evaluations of  $w$  and  $\tilde{X}$ .) Another way to view (8) and the protocol is that we begin with an inner product between  $\tilde{X}$  and the vector  $w$ :

$$w^T \mathbf{vec}(\tilde{X}) = \alpha,$$

which is then reduced to a new inner product between the partial evaluations

$$(\mathbf{Mat}(w)\bar{r})^T (\tilde{X}\bar{r}) = \alpha',$$

where  $\mathbf{Mat}(w)$  is the reshaping of  $w$  to the appropriate dimensions. We will make use of this particular observation later in the note.

## 4.2 Partial sumcheck protocol

As before, we will describe the prover and verifier algorithms separately. In this construction, we will reduce the original sum over  $n$  variables to a sum over  $n'$  variables.

**Prover algorithm.** The prover algorithm is as follows.

1. For  $i = 1, \dots, k - k'$ 
  1. Send the polynomial  $s_i(t) = \sum_{z \in \{0,1\}^{k-i}} w(r_1, \dots, r_{i-1}, t, z) \tilde{X}(r_1, \dots, r_{i-1}, t, z)$
  2. Receive a random challenge  $r_i \in \mathbf{F}$ , to be used in the next round

**Verifier algorithm.** The verifier algorithm is as follows.

1. Set  $s_0 = \alpha$  to be a constant polynomial and  $r_0 = 0$  for convenience
2. For each of  $i = 1, \dots, k - k'$ :
  1. Send some uniformly sampled  $r_{i-1} \in \mathbf{F}$

2. Receive some degree 2 polynomial  $s_i : \mathbf{F} \rightarrow \mathbf{F}$
3. Verify that  $s_i(0) + s_i(1) = s_{i-1}(r_{i-1})$

If all verifications pass, the verifier accepts the proof. Otherwise, it rejects. Note that the verifier does not need to know neither  $w$  nor  $\tilde{X}$  to run this algorithm.

**Guarantees.** The guarantees of the partial sumcheck is as follows. If the verifier knows that, for a new uniformly randomly sampled  $r_{k-k'} \in \mathbf{F}$ ,

$$s_{k-k'}(r_{k-k'}) = \alpha' = \sum_{z' \in \{0,1\}^{k'}} w(r_1, \dots, r_{k-k'}, z') \tilde{X}(r_1, \dots, r_{k-k'}, z') \quad (10)$$

then, after running the verifier algorithm it is guaranteed that

$$\sum_{z \in \{0,1\}^k} w(z) \tilde{X}(z) = \alpha,$$

except with additional error probability no more than

$$\frac{2(k - k')}{|\mathbf{F}|}. \quad (11)$$

In particular, if  $k' = 0$ , this reduces to the standard sumcheck since the terminal condition (10) is simply to evaluate the product of  $w$  and  $\tilde{X}$  at  $(r_1, \dots, r_k)$ :

$$s_k(r_k) = w(r_1, \dots, r_k) \tilde{X}(r_1, \dots, r_k).$$

It is very easy (and relatively standard) to see that if the prover follows the prover algorithm, the verifier will always accept the proof. It is also easy to see the (partial) converse: if all steps of the verifier algorithm pass, including the final check (10), then, the verifier knows that the inner product  $w^T \mathbf{vec}(\tilde{X}) = \alpha$ , except with error probability no more than specified in (11). The proof is identical to the standard sumcheck protocol proof and we refer readers to [Tha22], chapter 4.

**Discussion.** It will be convenient to note that equation (10) can be written in matrix form as

$$s_{k-k'}(r_{k-k'}) = (\mathbf{Mat}(w)\bar{r})^T (\tilde{X}\bar{r}),$$

where, as before,  $\bar{r} = (1 - r_1, r_1) \otimes \dots \otimes (1 - r_{k-k'}, r_{k-k'})$ .

### 4.3 Observations

In this section, we make some simple observations about the partial sumcheck protocol which we use in the protocol that follows.

**Inner products.** As mentioned before, the sumcheck protocol can be viewed as a protocol for verifying that the inner product between two vectors,  $w$  and  $\mathbf{vec}(\tilde{X})$  is, say,  $\alpha$ ; written out, this means that

$$w^T \mathbf{vec}(\tilde{X}) = \sum_{z \in \{0,1\}^k} w(z) \tilde{X}(z) = \alpha.$$

**Batching evaluations.** If we are given a list of  $q$  vectors  $w_1, \dots, w_q \in \mathbf{F}^{2^k}$ , and claimed inner products  $\alpha_1, \dots, \alpha_q \in \mathbf{F}$ , we can ‘batch’ the evaluations and proofs into a single partial sumcheck protocol instance. The observation is very simple: once  $\alpha_1, \dots, \alpha_q$  are received, the verifier can draw some uniform randomness  $\beta_1, \dots, \beta_q \in \mathbf{F}$  and simply run the partial sumcheck protocol over the new vector

$$\tilde{w} = \beta_1 w_1 + \dots + \beta_q w_q, \tag{12}$$

and then new claimed inner product

$$\tilde{\alpha} = \beta_1 \alpha_1 + \dots + \beta_q \alpha_q.$$

holding  $\tilde{X}$  constant. This adds only a small probability of error, namely  $1/|\mathbf{F}|$ , to the error probability of the protocol, if we use uniform and independent randomness for each  $\beta_i$ . We will call this new vector  $w$  of (12), resulting from the batching process, the *batched vector* or *batched polynomial*.

**Gluing sumchecks.** One simple observation is that, given a partial sumcheck protocol instance, where the we are reducing a sum over  $k$  variables to a sum over  $k''$  variables, it is possible to, at any round  $k'$  with  $k'' < k' < k$ , view the partial sumcheck protocol from  $k$  to  $k''$  variables as two ‘glued’ instances of the sumcheck protocol: one from  $k$  to  $k'$  variables and one from  $k'$  to  $k''$  variables. It is then possible, via the batching observation above, to batch two inner products: one sum over  $k$  variables and one sum over  $k'$  variables (with the rest of the variables fixed by the randomness of the first partial sumcheck).

**Complete batched and glued sumcheck.** For completeness, we describe the batched and glued sumcheck below. We assume that the prover has some matrix  $\tilde{X} \in \mathbf{F}^{2^k \times 2^{k'}}$  (which itself could be a partial evaluation of some larger matrix). The prover algorithm is as follows:

1. Receive some vectors  $w_1, \dots, w_q \in \mathbf{F}^{2^{k+k'}}$
2. Send evaluations  $\alpha_i = \mathbf{vec}(\tilde{X})^T w_j$  for  $j = 1, \dots, q$
3. Receive some batching randomness  $\beta \in \mathbf{F}^q$
4. Construct a batched vector  $\tilde{w} = \sum_{j=1}^q \beta_j w_j$
5. For  $i = 1, \dots, k - k'$



1. Send the polynomial  $s_i(t) = \sum_{z \in \{0,1\}^{k-i}} \tilde{w}(r_1, \dots, r_{i-1}, t, z) \tilde{X}(r_1, \dots, r_{i-1}, t, z)$
2. Receive a random challenge  $r_i \in \mathbf{F}$ , to be used in the next round, if  $i < k - k'$

For the verifier:

1. Send some vectors  $w_1, \dots, w_q \in \mathbf{F}^{2^{k+k'}}$
2. Receive (claimed) evaluations  $\alpha_j$  for  $j = 1, \dots, q$
3. Send batching randomness  $\beta \in \mathbf{F}^q$
4. Construct a batched vector  $\tilde{w} = \sum_{j=1}^q \beta_j w_j \in \mathbf{F}^{2^{k+k'}}$
5. Set  $s_0 = \sum_{j=1}^q \beta_j \alpha_j$  to be a constant polynomial and  $r_0 = 0$  for convenience
6. For  $i = 1, \dots, k - k'$ :
  1. Send some uniformly sampled  $r_{i-1} \in \mathbf{F}$
  2. Receive some degree 2 polynomial  $s_i : \mathbf{F} \rightarrow \mathbf{F}$
  3. Verify that  $s_i(0) + s_i(1) = s_{i-1}(r_{i-1})$

If all verifier checks pass, the verifier accepts the proof. Additionally, if the verifier knows that, for a new uniformly randomly sampled  $r_{k-k'} \in \mathbf{F}$ , the following is true:

$$s_{k-k'}(r_{k-k'}) = \sum_{z' \in \{0,1\}^{k'}} \tilde{w}(r_1, \dots, r_{k-k'}, z') \tilde{X}(r_1, \dots, r_{k-k'}, z'), \quad (13)$$

then the verifier is guaranteed that

$$\sum_{z \in \{0,1\}^k} w_j(z) \tilde{X}(z) = \alpha_j,$$

with error probability no more than

$$\frac{2(k - k')}{|\mathbf{F}|} + \frac{1}{|\mathbf{F}|},$$

if the  $\beta$  are uniformly randomly chosen. It is important to note that the vectors  $w_j$  can be themselves partial evaluations of some larger vectors and that the verifier only needs access to  $\tilde{X}$  via the terminal condition (13). Note also that (from the gluing observation) the final check (13) can itself be reduced via the sumcheck protocol to a sum over even fewer variables using the above protocol.

## 5 Matrix-vector product with partial sumcheck

We will now show a simple combination of the matrix-vector product protocol and the partial sumcheck protocol. This construction allows a prover to evaluate and prove that some inner product between a matrix  $\tilde{X}$  and some public vector  $w$  is equal to  $\alpha$ . This protocol essentially uses the following observation: the vector  $y_r$  received by the verifier in the matrix-vector product protocol is, itself, a partial evaluation of  $\tilde{X}$  at some point  $g_r$ . This is essentially the right hand side of the final check of the partial sumcheck protocol in (10).

**Prover algorithm.** Given some matrix  $\tilde{X} \in \mathbf{F}^{2^k \times 2^{k'}}$ , the prover algorithm is as follows.

1. Run the commitment phase (steps 1-2) of the matrix-vector product on  $\tilde{X}$
2. Receive some vector  $w \in \mathbf{F}^{2^{k+k'}}$
3. Run the partial sumcheck prover protocol on the vector  $w$ , the matrix  $\tilde{X}$ , and the claimed inner product  $\alpha$  to reduce the sum over  $(k + k')$  to  $k$  variables. In doing so, receive randomness  $r_1, \dots, r_{k'-1}$  and send final polynomial  $s_{k'}$
4. Receive  $r_{k'} \in \mathbf{F}$  and construct  $\bar{r} = (1 - r_1, r_1) \otimes \dots \otimes (1 - r_{k'}, r_{k'})$
5. Send  $y_r = \tilde{X}\bar{r}$ , like the matrix-vector product protocol

**Verifier algorithm.** We assume that the verifier has some vector  $w \in \mathbf{F}^{2^{k+k'}}$  over which they would like to compute some inner product with the (unknown and opaque) matrix  $\tilde{X}$ . The verifier algorithm is as follows.

1. Receive the Merkle commitments to the rows of some opaque matrix  $X$
2. Send the vector  $w \in \mathbf{F}^{2^{k+k'}}$  and receive claimed evaluation  $\alpha$
3. Run the partial sumcheck verifier protocol with the claimed evaluation  $\alpha$ , receiving some degree 2 polynomial  $s_{k'}$  and some randomness  $r_1, \dots, r_{k'-1}$
4. Sample and send  $r_{k'} \in \mathbf{F}$  and construct  $\bar{r} = (1 - r_1, r_1) \otimes \dots \otimes (1 - r_{k'}, r_{k'})$
5. Receive some vector  $y_r \in \mathbf{F}^{2^k}$
6. Uniformly sample  $S \subseteq \{1, \dots, m\}$ , of some fixed size  $|S|$
7. Verify that  $X_S \bar{r} = G_S y_r$ , therefore conclude that  $y_r = \tilde{X}\bar{r}$  from the matrix-vector product protocol guarantees, where  $\tilde{X}$  is the unique matrix with  $\|X - G\tilde{X}\| < d/2$
8. Verify the partial sumcheck's final check  $s_{k'}(r_{k'}) = y_r^T \mathbf{Mat}(w)\bar{r}$

If all check pass then the verifier accepts the proof. Otherwise, it rejects. Step 8 can be easily interpreted as computing the right hand side of the final check (10):

$$y_r^T \mathbf{Mat}(w)\bar{r} = \sum_{z' \in \{0,1\}^k} y_r(z')w(r_1, \dots, r_{k'}, z') = \sum_{z' \in \{0,1\}^k} \tilde{X}(r_1, \dots, r_{k'}, z')w(r_1, \dots, r_{k'}, z'),$$

where the second equality comes from the guarantee given in step 7 of the verifier algorithm. Note that the total error probability of the protocol can be bounded by the sum of the error probabilities of the matrix-vector product protocol and the partial sumcheck protocol:

$$\left(\frac{m-n-1}{2m}\right)^{|S|} + \frac{m \log_2(n)}{|\mathbf{F}|} + \frac{2k'}{|\mathbf{F}|}. \quad (14)$$

**Guarantees.** The guarantees of the above protocol are essentially the same as those of the matrix-vector product protocol—*i.e.*, there exists a unique matrix  $\tilde{X}$  closest to the committed matrix  $X$  and that of step 7—with the additional conclusion that, indeed

$$w^T \mathbf{vec}(\tilde{X}) = \alpha;$$

in other words, that the inner product of this unique  $\tilde{X}$  with  $w$  is, indeed,  $\alpha$ .

**Discussion.** As before, we can use the observation above to batch multiple  $\{w_i\}$  vectors and so on. We note that this style of protocol is not new and has been used, in slightly different forms, in other contexts such as [DP25].

## 6 Ligerito

In this section, we present Ligerito, the main protocol, which shows how to, roughly speaking, recurse the protocol presented in the previous section.

### 6.1 High level idea

The high level idea for this protocol is very simple. In the general matrix-vector product protocol and its extension with the partial sumcheck, the verifier receives all of  $y_r$  in order to verify that  $G_S y_r$  matches the opened rows of  $X$ ; *i.e.*, that  $X_S \bar{r} = G_S y_r$ , as in step 7 of the verifier protocol. The main thing we observe is that the verifier only accesses the received partial evaluation  $y_r$  via the  $|S|$  inner products with the rows of  $G$  and the final inner product of step 8,  $y_r^T \mathbf{Mat}(w) \bar{r}$ . Note, of course, that the prover does not need to send  $y_r$ : it can, instead, commit to  $\mathbf{Mat}(y_r)$  as a new matrix, and then, again, use the previous section's protocol to prove the desired inner products with the rows of  $G$ . We can continue to recurse this, say  $\ell$ , times, where, in the last step, we simply run the matrix-vector product with the partial sumcheck of the previous section.

### 6.2 Protocol algorithms

For now, it will be useful to specify a parameter  $\ell \geq 2$  which will determine the number of rounds we will run; we will then later show an optimal choice of  $\ell$ . The protocol requires, as in the previous protocol, a sequence of dimensions  $k_1, \dots, k_\ell$  and  $k'_1, \dots, k'_\ell$  with  $k_{i+1} + k'_{i+1} = k_i$ , and a number of codes  $G_i \in \mathbf{F}^{m_i \times 2^{k_i}}$  for  $i = 1, \dots, \ell - 1$ , each with distance  $d_i > 0$ . The only requirements for these code matrices will be that it is efficient to evaluate a (generally

random) row of each  $G_i$ . This is true for a large number of code matrices such as Reed–Solomon codes, Reed–Muller codes, among many others.

**Prover algorithm.** The prover has some matrix  $\tilde{X}_1 \in \mathbf{F}^{2^{k_1} \times 2^{k'_1}}$  along with the previously mentioned public data. After committing to  $\tilde{X}_1$ , the prover will receive some  $w \in \mathbf{F}^{2^{k_1+k'_1}}$  and will show that  $w^T \mathbf{vec}(\tilde{X}_1) = \alpha$ .

1. For  $i = 1, \dots, \ell - 1$ 
  1. Run the commitment phase (steps 1-2) of the matrix-vector product on  $\tilde{X}_i$
  2. If  $i = 1$  receive  $w \in \mathbf{F}^{k_1+k'_1}$  and set  $\tilde{w}_0 = w$
  2. If  $i \geq 2$ , receive the sampled indices  $S_{i-1} \subseteq \{1, \dots, m_{i-1}\}$
  3. If  $i \geq 2$ , compute the expected symbols  $v_{i-1} = (G_{i-1} \mathbf{vec}(\tilde{X}_i))_{S_{i-1}}$ , this is  $|S_{i-1}|$  inner products with  $\tilde{X}_i$
  4. Run the batched and glued sumcheck
    1. If  $i \geq 2$ , send the  $|S|$  expected symbols  $v_{i-1}$
    2. Batch the inner product evaluations of the  $v_{i-1}$  with  $\tilde{w}_{i-1}$ , getting a new batched vector,  $\tilde{w}_i$
    3. Run the partial sumcheck protocol to reduce  $\tilde{w}_i^T \mathbf{vec}(\tilde{X}_i)$  by  $k_i$  variables, receiving randomness  $r_i \in \mathbf{F}^{k_i}$  and sending final partial sum  $s_i$
  5. Set  $\tilde{X}_{i+1} = \mathbf{Mat}(\tilde{X}_i \bar{r}_i) \in \mathbf{F}^{2^{k_{i+1}} \times 2^{k'_{i+1}}}$  to be committed in the next iteration
2. Send  $y_\ell = \mathbf{vec}(\tilde{X}_\ell)$

**Verifier algorithm.** The verifier algorithm is as follows.

1. For  $i = 1, \dots, \ell - 1$ 
  1. Receive a Merkle commitment to the rows of some opaque matrix  $X_i$
  2. If  $i = 1$ , send  $w$ , set  $\tilde{w}_0 = w$  for convenience, receive evaluation  $\alpha$ , set  $s_0 = \alpha$
  3. If  $i \geq 2$ :
    1. Uniformly sample and send  $S_{i-1} \subseteq \{1, \dots, m_{i-1}\}$  of fixed size  $|S_{i-1}|$
    2. Receive purported vector of symbols  $v_{i-1} \in \mathbf{F}^{|S_{i-1}|}$
    3. Verify that  $(X_{i-1} \bar{r}_{i-1})_{S_{i-1}} = v_{i-1}$
  4. Run the batched partial sumcheck verifier of section 4:

1. Batch the following checks into a new batched vector  $\tilde{w}_i$ :
  - The running sum  $s_{i-1}((r_{i-1})_{k_{i-1}})$  should be equal to the inner product with  $\tilde{w}_{i-1}$
  - If  $i \geq 2$ , then  $v_i$  should be equal to the inner products with each row of  $(G_{i-1})_{S_{i-1}}$
2. Run the partial sumcheck protocol over the batched vector  $\tilde{w}_i$ , sending randomness  $r_i \in \mathbf{F}^{k'_i}$  and receiving the final degree-2 polynomial  $s_i$
2. Receive vector  $y_\ell$
3. Uniformly sample  $S_{\ell-1} \subseteq \{1, \dots, m_{\ell-1}\}$  of fixed size  $|S_{\ell-1}|$
4. Verify that  $(X_{\ell-1} \bar{r}_{\ell-1})_{S_{\ell-1}} = G_{S_\ell} y_\ell$
5. Verify that  $s_{\ell-1}((r_{\ell-1})_{k_{\ell-1}}) = \tilde{w}_\ell^T y_\ell$

**Discussion.** Note that when  $\ell = 2$ , this reduces exactly to both the prover and verifier algorithms for the matrix-vector product protocol with the partial sumcheck protocol of section 5.

### 6.3 Guarantees

The guarantees of this protocol are as follows. If the prover follows the prover algorithm, the verifier will always accept the proof. (This is easy to see from the verifier algorithm.) On the other hand, if the verifier follows the verifier algorithm and accepts, then the verifier is guaranteed that (1) there exists some unique matrix  $\tilde{X}_i$  such that

$$\|X_i - G_i \tilde{X}_i\| < \frac{d_i}{2},$$

(*i.e.*, the prover ‘knows’ a unique matrix  $\tilde{X}_i$  which is close to the committed matrix  $X_i$ ) for  $i = 1, \dots, \ell - 1$ , and (2) that the inner product of the unique matrix  $\tilde{X}_1$  with the vector  $w$  is equal to the claimed evaluation  $\alpha$ :

$$w^T \mathbf{vec}(\tilde{X}_1) = \alpha.$$

Finally, the verifier guarantees that

$$y_\ell = \tilde{X}_{\ell-1} \bar{r}_{\ell-1} = \mathbf{Mat}(\tilde{X}_{\ell-2})(\bar{r}_{\ell-2} \otimes \bar{r}_{\ell-1}) = \dots = \mathbf{Mat}(\tilde{X}_1)(\bar{r}_1 \otimes \dots \otimes \bar{r}_{\ell-1}).$$

Or, in other words, that  $y_\ell$  is the partial evaluation of  $\tilde{X}_1$  over randomness  $(\bar{r}_1 \otimes \dots \otimes \bar{r}_{\ell-1})$ . Both statements are true except with error probability no more than the sum of the error probabilities of the matrix-vector product protocol and the partial sumcheck protocol:

$$\begin{aligned}
& \sum_{i=1}^{\ell-1} \left( \frac{2k_{i-1}}{|\mathbf{F}|} + \frac{|S_i|+1}{|\mathbf{F}|} + \left( \frac{m_i - 2^{k_i} - 1}{2m_i} \right)^{|S_i|} + \frac{m_i k_i}{|\mathbf{F}|} \right) \\
& + \frac{2k_{\ell-1}}{|\mathbf{F}|} + \left( \frac{m_{\ell} - 2^{k_{\ell}} - 1}{2m_{\ell}} \right)^{|S_{\ell}|} + \frac{m_{\ell} k_{\ell}}{|\mathbf{F}|}. \tag{15}
\end{aligned}$$

Here the first two terms in the sum correspond to the error from the sumcheck and the second part comes from the matrix-vector product protocol. The tail term is simply the term in the sum without the error incurred by batching the evaluations, hence the lack of the  $|S_i|$  term.

**Proof.** The proof essentially follows from the guarantees of the matrix-vector product protocol and the partial sumcheck protocol. We proceed by induction on  $\ell \geq 2$ . The base case,  $\ell = 2$ , is simply the matrix-vector product with the partial sumcheck protocol of section 5, where the guarantees follow from the guarantees provided in that section. To reiterate those guarantees, the verifier knows that there exists a unique matrix  $\tilde{X}_1$  whose encoding is closest to the committed matrix  $X_1$ , that  $\tilde{X}_1 \bar{r} = \mathbf{vec}(\tilde{X}_2) = y_2$ , and that the inner product  $w^T \mathbf{vec}(\tilde{X}_1) = \alpha$ . The error probability of this protocol is exactly that of (14), where  $n = 2^{k_1}$  and  $k' = k'_1$ :

$$\frac{2k'_1}{|\mathbf{F}|} + \left( \frac{m_1 - 2^{k_1} - 1}{2m_1} \right)^{|S|} + \frac{m_1 k'_1}{|\mathbf{F}|}.$$

Now, assume that the guarantees hold for  $\ell - 1$ : that is, for the matrix  $X_2$  (and, indeed, all ‘later’ matrices  $X_3, \dots, X_{\ell}$ ) we have that there exists a unique matrix  $\tilde{X}_2$  with

$$\|X_2 - G_2 \tilde{X}_2\| < \frac{d_2}{2},$$

that  $\tilde{X}_2 \bar{r}_2 = \mathbf{vec}(\tilde{X}_3)$ , and that the inner product  $\tilde{w}_1^T \mathbf{vec}(\tilde{X}_2) = \tilde{\alpha}_1$  holds, where  $\tilde{w}_1$  is the batched vector of the previous round and  $\tilde{\alpha}_1$  is the expected batched result. From the guarantees of the base case, we know that

$$(G_1)_{S_1} \mathbf{vec}(\tilde{X}_2) = v_1,$$

except with additional error probability no more than

$$\frac{1}{|\mathbf{F}|},$$

from the batching of the evaluations in the sumcheck protocol. The verifier checks, in step 3.3 that

$$(X_1)_{S_1} \bar{r}_1 = v_1,$$

hence

$$(X_1)_{S_1} \bar{r}_1 = (G_1)_{S_1} \mathbf{vec}(\tilde{X}_2),$$

and, from the matrix-vector product protocol guarantees of 3, we know that this implies

$$\|X_1 - G_1 \tilde{X}_1\| < \frac{d_1}{2},$$

and that  $\mathbf{vec}(\tilde{X}_1)$  satisfies

$$\mathbf{vec}(\tilde{X}_2) = \tilde{X}_1 \bar{r}_1.$$

except with (additional) error probability no more than

$$\left( \frac{m_1 - 2^{k_1} - 1}{2m_1} \right)^{|S_1|} + \frac{m_1 k_1}{|\mathbf{F}|}.$$

In other words,  $\tilde{X}_2$  is known to be the partial evaluation of the unique matrix  $\tilde{X}_1$  at  $\bar{r}_1$ . This, in turn, means that, from the guarantees of the partial sumcheck protocol, we know that

$$w^T \mathbf{vec}(\tilde{X}_1) = \tilde{w}_0^T \mathbf{vec}(\tilde{X}_1) = \alpha,$$

except with additional error probability no more than

$$\frac{2k'_1}{|\mathbf{F}|},$$

Tallying up the error gives a total error probability of

$$\frac{2k'_1}{|\mathbf{F}|} + \frac{|S_1| + 1}{|\mathbf{F}|} + \left( \frac{m_1 - 2^{k_1} - 1}{2m_1} \right)^{|S_1|} + \frac{m_1 k_1}{|\mathbf{F}|} + p_{\text{rest}},$$

where  $p_{\text{rest}}$  is the total error probability incurred by the inductive case (rounds 2 through  $\ell$ ).

## 6.4 Communication complexity

The communication complexity is also simple to total up and we will assume that the verifier randomness is derived noninteractively (via, *e.g.*, Fiat–Shamir) and hence not sent by the verifier nor counted in the total communication. At round  $i < \ell$ , the prover sends  $|S_{i-1}| + 1$  field elements for the expected symbols  $v_{i-1}$  and inner product with  $\tilde{w}_i$ . It also sends  $3k_i$  field elements, 3 elements for each degree-2 polynomial in the sumcheck protocol in step 4 of the prover algorithm. Finally, it sends  $|S_{i-1}|$  openings of the rows of  $X_{i-1}$ , which total to  $|S_{i-1}|2^{k'_{i-1}}$  field elements and  $|S_{i-1}| \log_2(m_{i-1})$  hashes for the Merkle openings of the corresponding rows. At round  $i = \ell$ , the prover sends  $\tilde{X}_\ell$  in its entirety, which totals  $2^{k_\ell + k'_\ell} =$

$2^{k_{\ell-1}}$  field elements, along with the row openings and corresponding Merkle proofs. The total communication complexity is therefore

$$\sum_{i=2}^{\ell-1} (3k'_{i-1} \log_2(|\mathbf{F}|) + |S_{i-1}| ((2^{k_{i-1}} + 1) \log_2(|\mathbf{F}|) + C \log_2(m_{i-1}))) \\ + (|S_{\ell-1}| + 1 + 2^{k_{\ell-1}}) \log_2(|\mathbf{F}|) + C |S_{\ell-1}| \log_2(m_{\ell-1}).$$

where  $C$  is the size in bits of the hash function used (in our case,  $C = 256$ , as we use SHA256).

**Asymptotics.** Let  $\lambda$  be the number of bits of security and  $\mathbf{F}$  be a field with size  $|\mathbf{F}| \gg 2^\lambda$ . Also set  $\log = \log_2$  for notational convenience. The asymptotic communication complexity is, ignoring small terms

$$\left( \sum_{i=1}^{\ell} |S_{i-1}| 2^{k'_{i-1}} \log(|\mathbf{F}|) \right) + 2^{k_{\ell-1}} \log(|\mathbf{F}|) + C \ell \log\left(\frac{N}{\rho}\right)$$

where  $N = k_1 + k'_1$  is the total number of entries in  $\tilde{X}_1$  and we use the (very coarse) bound  $m_i \leq N/\rho$ . If we assume that all  $G_i$  have the same rate  $2^{k'_i}/m_i$ , say  $\rho$ , then

$$|S_i| = \left\lceil -\frac{\lambda + \log(\ell)}{\log((1 + \rho)/2)} \right\rceil$$

suffices to ensure that the error probability is smaller than  $2^{-\lambda}$  for any  $\ell$ . (This follows from (15) and the fact that  $|\mathbf{F}| \gg \lambda$ .) Finally, set  $\ell = 2 \log(N) / \log(\log(N))$  and  $k'_i = \log(N)/\ell$  for each  $i$ , which results in, asymptotically, a communication complexity no larger than

$$\frac{2 \log(N)}{\log(\log(N))} ((\lambda + \log(N)^{1/2}) t \log_2(|\mathbf{F}|) + C \log(N)) \sim 2C \frac{\log(N)^2}{\log(\log(N))},$$

as required, where  $t = -1/\log((1 + \rho)/2)$  for convenience. (It is not hard to show that this choice of  $k'_i$  and  $\ell$  is optimal up to constants by solving a simple convex optimization problem, though we do not do so here.) We note that, even at this optimal size of the  $k_i$  and  $\ell$ , the dominant cost of communication comes from the Merkle proofs of the opening of the rows of the matrix  $X$ , rather than from the field elements in the provided rows. This likely points to a potential source for cost savings.

## 6.5 Extensions

A simple extension of the protocol is to allow the prover to commit to a matrix  $\tilde{X}_1$  which lies in a subfield  $\mathbf{F}' \subseteq \mathbf{F}$  of the main field. This makes the openings over the rows of  $X_1$  smaller, which, in turn, decreases the communication complexity of the protocol for the first round. We use this flexibility in what follows.



## 7 Numerics

In this section, we present some numerical results for the protocol. Our implementation is written in Julia, a high level language for scientific computing, and uses the `CryptoUtilities.jl` package for fast cryptographic operations in binary fields. This package includes a fast Reed–Solomon encoding library based on the algorithms described in [Lin+16]. All code for this work is available at

<https://github.com/bcc-research/Ligerito.jl>

**Numerical experiments.** In this set of numerical experiments, we choose the binary field  $|\mathbf{F}| = 2^{128}$  as the main field and  $\mathbf{F}' \subseteq \mathbf{F}$  with  $|\mathbf{F}'| = 2^{32}$  as the subfield in the first round. As in the asymptotic section, we assume that all of the matrices  $G_i$  are Reed–Solomon code matrices, each with rate  $\rho = 1/4$ . We assume  $\lambda = 100$  bits of security, making  $|S_i| = 148$  (if  $\ell \leq 8$ , which is the case here). We choose  $\ell$  by performing a grid search over the possible values of  $\ell = 1, \dots, 8$  and choose the  $\{k_i\}$  and  $\{k'_i\}$  by using a simple rounding scheme on the solution to a simple convex program, given  $\ell$ ; for more details, see the code. We run the prover on an M1 MacBook Pro with 64 GB of RAM and report our times (and proof sizes) below. Since the MacBook Pro has 8 performance cores (and 2 efficiency cores), we run the prover on 8 threads.

Coefficient count	Proof size	Proving time
$2^{20}$	145 KiB	.08 s
$2^{24}$	255 KiB	1.3 s
$2^{28}$	360 KiB	21 s
$2^{30}$	420 KiB	80 s

**Table 1:** Proof size and time for various multilinear polynomials over  $|\mathbf{F}'| = 2^{32}$  on 8 threads of an M1 MacBook Pro (64 GB).

Note that the proof sizes can differ from run to run since the Merkle batched opening size is not deterministic and depends on the exact rows opened. We also note that the scheme is relatively memory efficient, with the prover using (asymptotically) a little over twice the size of the largest encoded matrix  $X_1$  in the protocol.

Coefficient count	Polynomial size	Polynomial encoding size	Prover allocated memory
$2^{20}$	4 MiB	16 MiB	49 MiB
$2^{24}$	64 MiB	256 MiB	630 MiB
$2^{28}$	1 GiB	4 GiB	9.8 GiB
$2^{30}$	4 GiB	16 GiB	31 GiB

**Table 2:** Total allocated memory for the prover, relative to the size of the largest matrix  $X_1$  in the protocol, with rate  $\rho = 1/4$ .

We note that this is the total allocated memory for the prover during proving time, not the highest amount of memory used at any one time, which is likely much closer to the polynomial encoding size. This suggests that modern phones (which, almost universally, have at least 1 GiB of RAM) should be able to run the protocol for polynomials of size  $2^{24}$  or so, even if the expected proving time is on the order of a few seconds.

## 8 Conclusion

In this note, we presented a new polynomial commitment scheme and inner product protocol, Ligerito. We showed that it is a concretely small and fast protocol that is practical for even relatively large circuits, while also being asymptotically efficient. Given that large, practical circuits are currently on the order of around  $2^{24}$  or so constraints, this sets basic groundwork for the ability to do ‘real-time proving’ of blockchains—*i.e.*, the ability to prove that the state transition function of a blockchain is correct within the allowed blocktime—or the ability to prove important, but smaller circuits (degree  $2^{20}$  or so) on mobile devices such as phones.

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## Bibliography

- [KZG10] A. Kate, G. Zaverucha, and I. Goldberg, “Constant-size commitments to polynomials and their applications,” in *Advances in Cryptology—ASIACRYPT 2010: 16th International Conference on the Theory and Application of Cryptology and Information Security, Singapore, December 5-9, 2010. Proceedings 16*, 2010, pp. 177–194.
- [Ame+17] S. Ames, C. Hazay, Y. Ishai, and M. Venkatasubramanian, “Ligerito: Lightweight Sublinear Arguments Without a Trusted Setup,” in *Proceedings of the 2017 ACM SIGSAC Conference on Computer and Communications Security*, Dallas Texas USA: ACM, Oct. 2017, pp. 2087–2104. doi: 10.1145/3133956.3134104.
- [Ben+18] E. Ben-Sasson, I. Bentov, Y. Horesh, and M. Riabzev, “Fast Reed-Solomon Interactive Oracle Proofs of Proximity,” pp. 1–17, 2018, doi: 10.4230/LIPICS.ICALP.2018.14.
- [ZCF23] H. Zeilberger, B. Chen, and B. Fisch, “BaseFold: Efficient Field-Agnostic Polynomial Commitment Schemes from Foldable Codes.” [Online]. Available: <https://eprint.iacr.org/2023/1705>
- [Bre+24] M. Brehm, B. Chen, B. Fisch, N. Resch, R. D. Rothblum, and H. Zeilberger, “Blaze: Fast SNARKs from Interleaved RAA Codes.” [Online]. Available: <https://eprint.iacr.org/2024/1609>

- [Arn+24] G. Arnon, A. Chiesa, G. Fenzi, and E. Yogev, “WHIR: Reed–Solomon Proximity Testing with Super-Fast Verification.” [Online]. Available: <https://eprint.iacr.org/2024/1586>
- [EMA25] A. Evans, N. Mohnblatt, and G. Angeris, “ZODA: Zero-Overhead Data Availability.” [Online]. Available: <https://eprint.iacr.org/2025/034>
- [EA23] A. Evans and G. Angeris, “Succinct Proofs and Linear Algebra.” [Online]. Available: <https://eprint.iacr.org/2023/1478>
- [AER24] G. Angeris, A. Evans, and G. Roh, “A Note on Ligerio and Logarithmic Randomness.” [Online]. Available: <https://eprint.iacr.org/2024/1399>
- [DG24] B. E. Diamond and A. Gruen, “Proximity Gaps in Interleaved Codes.” [Online]. Available: <https://eprint.iacr.org/2024/1351>
- [DP24] B. E. Diamond and J. Posen, “Proximity Testing with Logarithmic Randomness,” *IACR Commun. Cryptol.*, vol. 1, no. 1, p. 2, 2024, doi: 10.62056/AKSDKP10.
- [Ben+23] E. Ben-Sasson, D. Carmon, Y. Ishai, S. Kopparty, and S. Saraf, “Proximity Gaps for Reed–Solomon Codes,” *Journal of the ACM*, p. 3614423, Aug. 2023, doi: 10.1145/3614423.
- [Tha22] J. Thaler, “Proofs, Arguments, and Zero-Knowledge,” *Foundations and Trends® in Privacy and Security*, vol. 4, no. 2–4, pp. 117–660, 2022, doi: 10.1561/33000000030.
- [DP25] B. E. Diamond and J. Posen, “Succinct Arguments over Towers of Binary Fields,” in *Advances in Cryptology – EUROCRYPT 2025*, S. Fehr and P.-A. Fouque, Eds., Cham: Springer Nature Switzerland, 2025, pp. 93–122.
- [Lin+16] S.-J. Lin, T. Y. Al-Naffouri, Y. S. Han, and W.-H. Chung, “Novel Polynomial Basis With Fast Fourier Transform and Its Application to Reed–Solomon Erasure Codes,” *IEEE Transactions on Information Theory*, vol. 62, no. 11, pp. 6284–6299, 2016, doi: 10.1109/TIT.2016.2608892.