# Lecture 4: Proof of CFMM Construction

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## 1 Note

We'll be using the same notation from class: $V : \mathbf{R}^n_+ \to \mathbf{R}$ is the portfolio value function we wish to replicate.

**Consistent portfolio values.** A portfolio value function $V$ is *consistent* if it is concave, nondecreasing, and 1-homogeneous, *i.e.*, if for any $t \geq 0$,

$$V(tc) = tV(c).$$

For the note we will assume that the function $V$ is differentiable, though there is a simple generalization using subgradient calculus. This means that, since $V$ is concave, we have, for any $c, q \in \mathbf{R}^n_+$,

$$V(q) \leq \nabla V(c)^T(q - c) + V(c). \tag{1}$$

(This is one definition of concavity for differentiable functions, which we will assume here.)

## 2 Replicating trading function

We want to show that the trading function defined in the following way:

$$\tilde{\varphi}(R) = \inf_c (c^T R - V(c)) \tag{2}$$

is a trading function that 'replicates' $V$; *i.e.*, its portfolio value function is equal to $V$.

**Proof strategy.** Let $\tilde{V}(c)$ denote the value of the no-arbitrage problem for this trading function $\tilde{\varphi}$:

$$\begin{array}{ll} \text{minimize} & c^T R \\ \text{subject to} & \tilde{\varphi}(R) \geq 0, \end{array} \tag{3}$$

with variable $R' \in \mathbf{R}^n$. We need to show that $\tilde{V} = V$, which we will do this in two steps. First, we will show that $\tilde{V} \geq V$ (this is the easy part of the proof) and then we will show that, given any $c$, there is always a feasible point $R$ for problem (3) with objective value equal to $V(c)$.

**Upper bound.** The fact that $\tilde{V} \geq V$ is a single line: let $R$ be feasible for problem (3), then

$$c^T R - V(c) \geq \tilde{\varphi}(R) \geq 0,$$

so $c^T R \geq V(c)$. The first inequality follows from the definition of $\tilde{\varphi}$ in (2), while the second follows from the fact that $R$ is feasible. Since this is true for any feasible $R$, and the objective value for this $R$ is $c^T R$, then necessarily $\tilde{V}(c) \geq V(c)$.

**Lower bound.** To construct the lower bound, we will show that, for any choice of $c$, there exists a feasible $R$ for problem (3) with objective value equal to $V(c)$, this will imply that the optimal value of (3), which is no larger than $c^T R$ by definition, is therefore no larger than $V(c)$. For this point, we will choose $R = \nabla V(c)$ (which is nonnegative since $V$ is nondecreasing!) and first show that $c^T R = V(c)$.

Using the definition of concavity (1) and our choice of $R$, we have that, for any $q \in \mathbf{R}^n_+$,

$$V(q) \geq R^T(q - c) + V(c).$$

Setting $q = 0$, we find

$$0 = V(0) \geq -c^T R + V(c),$$

or, that $c^T R \geq V(c)$, where the first equality follows from the 1-homogeneity of $V$. On the other hand, setting $q = 2c$, we find

$$2V(c) = V(2c) \geq c^T R + V(c),$$

which, after some rearrangement, gives $V(c) \geq c^T R$. Putting these two statements together, we get $V(c) = c^T R$, so this choice of $R$, if feasible, has objective value $c^T R$.

Let's now show the last part: that $R$ is indeed feasible for (3). For any $q \in \mathbf{R}^n_+$, we know, from (1) that

$$V(q) \geq R^T(q - c) + V(c).$$

Rearranging slightly, we have

$$V(q) - q^T R \geq V(c) - c^T R = 0.$$

Taking the infimum over $q$ on the left hand side and using the definition of $\tilde{\varphi}$ in (2), gives that $\tilde{\varphi}(R) \geq 0$ as required, so $\tilde{V}(c) \leq V(c)$.

**Equality.** Putting both statements together gives the final claim that $V = \tilde{V}$. To summarize: we were first given some consistent payoff $V$. Based on this $V$, we constructed a trading function $\tilde{\varphi}$, with some payoff $\tilde{V}$. We then showed that this $\tilde{V}$ was *actually equal* to $V$, which means that the function $\tilde{\varphi}$ we constructed has the desired payoff we wanted all along!

**Two-value property.** Why do we use zero for the constraint in problem (3)? This is because $\tilde{\varphi}(R)$ takes on exactly two values, either $\tilde{\varphi}(R) = 0$ or $\tilde{\varphi}(R) = -\infty$. To see this, note that $c = 0$ is always feasible for (2) so $\tilde{\varphi}(R) \leq 0$. On the other hand, if for fixed $R$ there exists some $c'$ such that $c'^T R - V(c') < 0$, then

$$\inf_c (c^T R - V(c)) \leq tc'^T R - V(tc') = t(c'^T R - V(c')) \to -\infty$$

as $t \to \infty$. In other words, if $R$ is feasible, then necessarily $\tilde{\varphi}(R) = 0$.