Problem set 2: Simple games

In this (single exercise) problem set, we show that all equilibria of the 'A vs B' game discussed in lecture have zero payoff. (In fact, we'll show the result that the equilibria of any two-player symmetric, zero-sum game has zero payoff.) In slightly looser terms, this means that, if this game is only played by 'rational' agents, the best possible outcome is that all players have zero payoff, unless some agents have additional information about what other players would be playing.

1 Payoff of two-player symmetric zero-sum games

First we'll define a two-player game and its equilibria and then pose the general question.

Two-player games. A (finite) *two-player game* is defined by an *action space* for player one, denoted $i = 1, \dots, m$, and an action space for player two, denoted $j = 1, \dots, n$. The game is played when player one chooses some action i and player two chooses some action j simultaneously, yielding a *payoff* of $A_{ij} \in \mathbf{R}$ for player one and $B_{ij} \in \mathbf{R}$ for player two.

Player strategies. Because players are allowed to randomize their actions, we will let $x \in \mathbf{R}^m$ be a distribution over the possible actions of player one (*i.e.*, $x \geq 0$ and $\mathbf{1}^T x = 1$) and $y \in \mathbf{R}^n$ be a distribution of the possible actions for player two (ditto for y). We call these distributions the players' *strategies*. Assuming each player randomly draws their strategy according to the distributions denoted x and y for players one and two respectively, their *expected payoffs* are

$$x^T A y \qquad \text{and} \qquad x^T B y,$$

respectively.

Equilibria. We say that x and y are an *equilibrium* if for any other strategies $x' \in \mathbf{R}^m$ and $y' \in \mathbf{R}^n$, satisfying $x' \geq 0$ and $\mathbf{1}^T x = 1$ (similarly for y') we have

$$x^T A y \geq x'^T A y \qquad \text{and} \qquad x^T B y \geq x^T B y'.$$

In other words, x and y are at an equilibrium if no single player can change their strategy and improve their expected payoff.

Zero-sum games. A game is *zero-sum* if

$$A = -B.$$

(The term 'zero sum' comes from the fact that the sum of both players' payoffs are equal to zero for any strategies x, y.) This has the interpretation that any gain for one player is a loss for the other, and vice versa.

Symmetric games. A game is *symmetric* if $m = n$ and

$$A = B^T.$$

In an intuitive sense, this means that every player has the same payoffs up to 'relabeling.'

Problem statement. With all of that set up, we can get to the problem statement: show that any equilibrium of a two-player, symmetric zero-sum game has zero payoff.

(*Hint:* consider the contrapositive. Assume that the payoff for player one is $x^T Ay > 0$, then what is player two's payoff? What is a (simple) strategy she can use to improve her payoff?)