Common Tools in Succinct Proofs

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Session 2, SPLA Study Club

Outline

Quick recap

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- This one, we will start doing real(-ish!) stuff
- Let's put some of the previous tools to the test!

High level overview

- We're going to start by discussing *models*
- Then progress onto specific tools
- Will mostly deal with 'exact' checks
- (Leave 'sparse' checks for homework/paper)

In other words...

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Let's build up the toolbox!

Outline

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- Want to certify that some given vector $x \in \mathbf{F}^n$ is *sparse*
- Say check that $\leq 10\%$ of entries are nonzero
- Can start checking until we see 90% of entries are nonzero
- If the vector is large, this is not great

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- See homework :)

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- Along with cryptographic tools needed
- We will discuss two models of interaction

Direct access model

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Direct access model

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- There exists a vector $x \in \mathbf{F}^n$
- We would like to verify some claim Q about x
- Can query entries of x (*e.g.*, can query x_i)

Coding model

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- We have a (known) code matrix $G \in \mathbf{F}^{m \times n}$
- There exists some message $x \in \mathbf{F}^n$
- Would like to verify some claim Q about x
- Can query individual *symbols* of the encoded message Gx

Truthfulness and commitments

- By assumption, the models require truthfulness
- In general, this is achieved via cryptographic techniques
- Ex.: the direct model can be achieved via a Merkle tree
- We elide this here! (But it is fascinating and worth studying)

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Exact check model

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- Very hard in the direct access model
- See homework exercises!

Set up

- For the remainder of section: fix a code matrix $G \in \mathbf{F}^{m \times n}$
- Code has distance $d > 0$
- And r will be uniformly drawn from $\{1, \ldots, m\}$
- (Think of r as drawing a uniformly random row of G)

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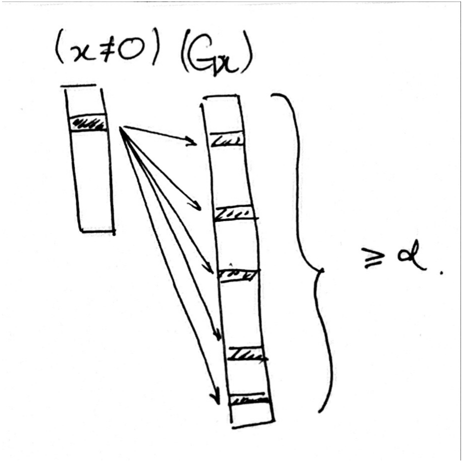
- We can write this as

$$(Gx)_r = 0 \quad \underset{p}{\Longrightarrow} \quad x = 0$$

- Here, $p \leq 1 - d/m$

Intuition



Discussion

- Equivalent to 1D Schwartz–Zippel lemma
- But for general linear codes!
- (Schwartz–Zippel is the special case of Reed–Solomon)

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- Let $\tilde{x}_i$ denote ith row, then

$$(G\tilde{x}_1)_r = 0, \ \ldots, \ (G\tilde{x}_n)_r = 0 \quad \underset{p}{\Longrightarrow} \quad X = 0$$

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- What should p be here? (See homework 1, exercise 5!)

Matrix zero check (equiv.)

- Another (identical) way of writing this:
  1. Pick random row r from G
  2. Use row to take linear combination of cols $x_1, \ldots, x_n$ of X
  3. Check if this linear combination is zero

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$$G_{r1}x_1 + \cdots + G_{rn}x_n = 0 \quad \underset{p}{\Longrightarrow} \quad X = 0$$

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- Can we 'put them together'? (Yes! See homework)

Aside: reduced matrix zero check

- This check is a Schwartz–Zippel generalization
- Indeed, we get the same bounds for Reed–Solomon codes
- (But can get better ones! See §3.1.3 of the paper)
- Shows Schwartz–Zippel is not tight (by a very small factor...)

Vector subspace check

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Vector subspace check

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- Let's start with some matrix $X \in \mathbf{F}^{k \times n}$
- Want to check if all columns of X are in a subspace $V \subseteq \mathbf{F}^k$
- Can we do this cheaply?

Vector subspace check

- Let's do something similar to matrix zero check:
  1. Pick random row r
  2. Use row for linear combination of columns of X
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$$G_{r1}x_1 + \cdots + G_{rn}x_n \in V \quad \underset{p}{\Longrightarrow} \quad x_i \in V, \quad i = 1, \ldots, n$$

Proof via probabilistic implications

- Let C be parity check matrix for V
- Then

$$G_{r1}x_1 + \cdots + G_{rn}x_n \in V \;\; \text{implies} \;\; C(G_{r1}x_1 + \cdots + G_{rn}x_n) = 0$$

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- But, by linearity (see homework 1, problem 1)

$$0 = C(G_{r1}x_1 + \cdots + G_{rn}x_n) = G_{r1}(Cx_1) + \cdots + G_{rn}(Cx_n)$$

- Note the last quantity!

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- By definition of parity check matrix then

$$Cx_1 = \cdots = Cx_n = 0 \quad \text{implies} \quad x_1, \ldots, x_n \in V$$

Putting it all together

- We only used one implication with error p (matrix zero check)
- Rest were all ‘standard’ implications (*i.e.*, zero error)
- This means that we can write:

$$G_{r1}x_1 + \cdots + G_{rn}x_n \in V \quad \underset{p}{\Longrightarrow} \quad x_i \in V, \quad i = 1, \ldots, n$$

- Where p is the same as the matrix zero check from before

Outline

Analogues of exact checks

- There is a sparse analogue of each exact check
- We will not explore proofs here (as they are more complicated)
- But check the paper (and the homework!) for some of these

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- It's good to be comfortable with notation before that :)

Outline

Summary

- Rewrote a bunch of tools used in many papers!
- Some maybe felt familiar
- But did all of them in very short order

Next lecture

- We will see how to apply these tools to explain FRI
- (As a quasi-'capstone' project)
- But many more applications exist :)