Simpson 1/3:

$$I = \int_a^b f(x)\,dx \longrightarrow f_1(x) \approx \frac{(x-b)(x-x_m)}{(a-b)(a-x_m)} f(a) + \frac{(x-a)(x-b)}{(x_m-a)(x_m-b)} f(x_m) + \frac{(x-a)(x-x_m)}{(b-a)(b-x_m)} f(b)$$

S. $$\left. h = \frac{b-a}{2} = x_m - a = b - x_m \right\} \quad f_1(x) = f(a)\frac{(x-x_m)(x-b)}{2h^2} + \frac{f(x_m)(x-a)(x-b)}{-h^2} + f(b)\frac{(x-a)(x-x_m)}{2h^2}$$

$$f_1(x) = \underbrace{\frac{f(a)}{2h^2}}_{cte}(x-x_m)(x-b) - \underbrace{\frac{f(x_m)}{h^2}}_{cte}(x-a)(x-b) + \underbrace{\frac{f(b)}{2h^2}}_{cte}(x-x_m)(x-a)$$

cada término de la forma

$$c(x-\alpha)(x-\lambda)$$

$$\int (x-\alpha)(x-\lambda)\,dx \qquad u = x-\alpha \quad du = dx \qquad dv = (x-\lambda)\,dx \quad v = \frac{(x-\lambda)^2}{2}$$

$$(x-\alpha)\frac{(x-\lambda)^2}{2} - \int \frac{(x-\lambda)^2}{2}\,dx = (x-\alpha)\frac{(x-\lambda)^2}{2} - \frac{(x-\lambda)^3}{6}$$

calculo la integral de cada término

$$\int_a^b f_1(x)\,dx = \frac{f(a)}{2h^2}\left[\int_a^b (x-x_m)(x-b)\,dx\right] - \frac{f(x_m)}{h^2}\left[\int_a^b (x-a)(x-b)\,dx\right] + \frac{f(b)}{2h^2}\left[\int_a^b (x-x_m)(x-a)\,dx\right]$$

$$= \frac{f(a)}{2h^2}\underbrace{\left[(x-x_m)\frac{(x-b)^2}{2} - \frac{(x-b)^3}{6}\right]_a^b}_{P_1} - \frac{f(x_m)}{h^2}\underbrace{\left[(x-a)\frac{(x-b)^2}{2} - \frac{(x-b)^3}{6}\right]_a^b}_{P_2} - \frac{f(b)}{2h^2}\underbrace{\left[(x-x_m)\frac{(x-a)^2}{2} - \frac{(x-a)^3}{6}\right]_a^b}_{P_3}$$

$$p_1 = (x_m - a)\frac{(a-b)^2}{2} + \frac{(a-b)^3}{6} = h\,\frac{(-2h)^2}{2} + \frac{(-2h)^3}{6} = 2h^3 - \frac{4}{3}h^3 = \frac{2}{3}h^3$$

$$p_2 = \frac{(a-b)^3}{b} = \frac{(-2h)^3}{6} = -\frac{4}{3}h^3$$

$$p_3 = (b-a)\frac{(b-x_m)^2}{2} - \frac{(b-x_m)^3}{6} + \frac{(a-x_m)^3}{6} = h^3 - \frac{h^3}{6} - \frac{h^3}{6} = h^3 - \frac{h^3}{3} = \frac{2h^3}{3}$$

Entonces:

$$\int_a^b f_1(x)\,dx = \frac{f(a)}{2h^2}\left(\frac{2}{3}h^3\right) - \frac{f(x_m)}{h^2}\left(-\frac{4}{3}h^3\right) + \frac{f(b)}{2h^2}\left(\frac{2}{3}h^3\right)$$

$$= f(a)\frac{h}{3} + f(x_m)\frac{4}{3}h + f(b)\frac{h}{3} = \frac{h}{3}\left[f(a) + 4f(x_m) + f(b)\right]$$

$$\longrightarrow \int_a^b f(x)\,dx \approx \int_a^b f_1(x)\,dx = \frac{h}{3}\left[f(a) + 4f(x_m) + f(b)\right]$$