SEMINAR INTEGRABILITATE.

RECAPITULARE

$f:[a,b]\to\mathbb{R}$ mărginită

$$P=\Delta=(a=x_0<x_1<\dots<x_n=b)$$

$$S_P(f)=\sum_{i=0}^{n-1}M_i(x_i-x_{i-1}) \qquad M_i=\sup f|[x_{i-1},x_i]$$

$\vee$|

$$\sigma_P(f,\xi=(\xi_i)_{i=\overline{0,n-1}})=\sum_{i=0}^{n-1}f(\xi_i)(x_i-x_{i-1}) \quad (\xi_i\in[x_{i-1},x_i])$$

$\vee$|

$$s_\Delta(f)=\sum_{i=0}^{n-1}m_i(x_i-x_{i-1}) \qquad m_i=\inf_{x\in[x_{i-1},x_i]}f(x)$$

$$\int_a^b f=\lim_{\|P\|\to 0}\sigma_\Delta(f,\xi)$$

$$\overline{\int_a^b} f=\inf S_\Delta(f)=\lim_{\|\Delta\|\to 0}S_\Delta(f)$$

$\vee$|

$$\underline{\int_a^b} f=\sup s_\Delta(f)=\lim_{\|\Delta\|\to 0}s_\Delta(f)$$

<u>T.D</u> f este int R. $\Leftrightarrow$ $\overline{\int_a^b} f=\underline{\int_a^b} f$

<u>T.L</u> f este int. R. $\Leftrightarrow$ D_f este neglijabilă Lebesgue.

Să se calculeze $\overline{S}_a^b f$, $\underline{S}_a^b f$ pentru funcțiile

1). $f:[-1,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}.\end{cases}$

2). $f:[0,2]\to\mathbb{R}$ $\quad f(x)=\begin{cases}x, & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$

3). $f:[0,\frac{\pi}{2}]\to\mathbb{R}$ $\quad f(x)=\begin{cases}\sin x & x\in\mathbb{Q}\\ \cos x & x\notin\mathbb{Q}\end{cases}$

Rezolvări.

1) $s_\Delta(f)=\sum\limits_{i=0}^{n-1} m_i(x_i-x_{i-1})$ $\quad (\Delta = 0 = x_0 < x_1 < \dots < x_n = 1)$

$m_i = \inf\limits_{x\in[x_{i-1},x_i]} f(x) = 0 \Rightarrow s_\Delta(f)=0$

$\Rightarrow \underline{S}_0^1 f = 0.$

$S_\Delta(f)=\sum\limits_{i=0}^{n-1} M_i(x_i-x_{i-1})$

$M_i = \sup\limits_{x\in[x_{i-1},x_i]} f(x)=1$ $\Bigg| \Rightarrow$

$S_\Delta(f) = \sum\limits_{i=0}^{n-1}(x_i-x_{i-1}) = x_n - x_0 = 1 \Rightarrow \overline{S}_0^1 f = 1$

$\overline{S}_0^1 f \neq \underline{S}_0^1 f \Rightarrow f$ nu este int. R.

sau $\quad D f = [0,1]$

2) $\Delta = (0 = x_0 < x_1 < \dots < x_n = 2)$

$m_i = \inf f([x_{i-1}, x_i]) = 0$

$M_i = \sup f([x_{i-1}, x_i]) = x_i$

$s_\Delta(f) = \sum_{i=0}^{n-1} m_i (x_i - x_{i-1}) = 0 \Rightarrow \underline{\int_0^2} f = 0$

$S_\Delta(f) = \sum_{i=0}^{n-1} x_i (x_i - x_{i-1})$

Dacă $\|\Delta_n\| \to 0 \Rightarrow S_{\Delta_n}(f) \to \overline{\int_0^2} f$

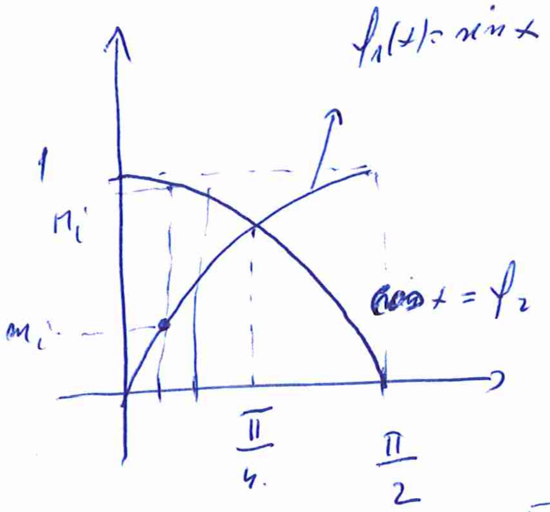
Alegem $\Delta_n = x_0 = 0 < x_1 = \frac{1}{n} < \dots < x_k = \frac{k}{n} < \dots < x_{2n} = 2$

$S_{\Delta_n}(f) = \sum_{i=0}^{2n-1} \frac{i}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{(2n-1) \cdot 2n}{2} \to \frac{1}{2} = \overline{\int_0^2} f.$

$\overline{\int_0^2} f \neq \underline{\int_0^2} f \Big/ \Rightarrow$ f nu este integrabilă Riemann.

$D_f = (0, 2]$

3) $f_1, f_2 : [0, 2] \to \mathbb{R}$ $\quad f_1(x) = \sin x \quad f_2(x) = \cos x$



$$\Delta_n = \underbrace{0 = x_0 < \dots < x_n = \tfrac{\pi}{4}}_{\Delta_n^1} \overbrace{< \dots < x_{2n} = \tfrac{\pi}{2}}^{\Delta_n^2}$$

$$m_i = \begin{cases} \sin x_{i-1} & i \leq n \\ \cos x_i & i > n \end{cases} \qquad M_i = \begin{cases} \cos x_{i-1} & i \leq n \\ \sin x_i & i > n \end{cases}$$

$$s_{\Delta_n}(f) = \sum_{i=0}^{n-1} (\sin x_{i-1})(x_i - x_{i-1}) + \sum_{i=n}^{2n-1} \cos x_i (x_i - x_{i-1})$$

$$\sigma_{\Delta_n^1}\left(\sin x, (x_{i-1})_{i=\overline{0,n-1}}\right) \qquad \sigma_{\Delta_n^2}\left(\cos x, (x_i)_{i=\overline{n,2n-1}}\right)$$

$$s_{\Delta_n}(f) \to \underline{\int_0^{\frac{\pi}{2}}} f \qquad \to \int_0^{\frac{\pi}{4}} \sin x \, dx \qquad \to \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

2) $$\underline{\int_0^{\frac{\pi}{2}}} f = \int_0^{\frac{\pi}{4}} \sin x \, dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x \, dx$$

$$= -\cos x \Big|_0^{\frac{\pi}{4}} + \sin x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = -\frac{\sqrt{2}}{2} + 1 + 1 - \frac{\sqrt{2}}{2} = 2 - \sqrt{2}.$$

Stabiliți integrabilitatea următoarelor funcții și calculați $\overline{\int_a^b} f$ și $\underline{\int_a^b} f$

1) $f:[-1,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases} x^3 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q}\end{cases}$

2) $f:[0,3]\to\mathbb{R}$ $\quad f(x)=\begin{cases} x^2 & x\in\mathbb{Q} \\ 3x-2 & x\notin\mathbb{Q}\end{cases}$

3) $f:[0,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases} x^2 & x\in A=\{\frac{1}{n}, n\geq 1\} \\ x^3 & \text{rest.}\end{cases}$

4) $f:[0,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases} 0 & x=0 \\ \dfrac{1}{2+[\frac{1}{x}]}, & x>0\end{cases}$

5) $f:[0,1]\to\mathbb{R}$ $\quad f(x)=\begin{cases} 0 & x=0 \\ (-1)^{[\frac{1}{x}]} & x>0\end{cases}$

6) $f:[0\ 5]\to\mathbb{R}$ $\quad f(x)=\begin{cases} x^2 & x\in\mathbb{Q} \\ 5x-6 & x\notin\mathbb{Q}.\end{cases}$