10.10.2022

Curs 2

– P&S –

Câmp de probabilitate. Operații cu ~~evenimente~~
Formule de calcul

~~Ω~~,

experiment aleator $\rightarrow (\Omega, \mathcal{F})$

Ω – sp. stărilor, mulțimea ev. elementare

$\mathcal{F} \subseteq \mathcal{P}(\Omega)$ mulțimea ev. posibile

a) $\Omega \in \mathcal{F}$
ev. sigur

b) dacă $A \in \mathcal{F} \Rightarrow$
$A^c \in \mathcal{F}$
↳ ev. contrar lui A

c) $A, B \in \mathcal{F} \Rightarrow$
$A \cup B \in \mathcal{F}$

(a, b, c) – algebră

c') $(A_n)_n \in \mathcal{F} \Rightarrow$
$\bigcup_n A_n \in \mathcal{F}$

a), b) + c') $\rightarrow$ σ-algebră
(sigma algebră)

$$\mathbb{P} : \mathcal{F} \longrightarrow [0,1]$$
$$A \longrightarrow p$$
($A \in \mathcal{F}$, $p \in [0,1]$)

Pp. că avem un experiment aleator și un eveniment A de interes. Repetăm experimentul (în condiții similare) de un nr mare de ori ($= N$).

Notăm $N(A)$ nr de realizări ale lui A.

$\frac{N(A)}{N}$ -> frecvența relativă de realizare a lui A.

$$\mathbb{P}(A) \simeq \lim_{N \to \infty} \frac{N(A)}{N}$$

$N(A) \in \{0, \dots, N\}$

$\frac{N(A)}{N} \in [0,1]$.

$\mathbb{P}(A) \in [0,1]$

Dacă $A = \Omega$ (ev. sigur) $\Rightarrow N(A) = N$

$\cancel{\Rightarrow \frac{N(A)}{N} = 1 \Rightarrow \mathbb{P}(A) = 1}$

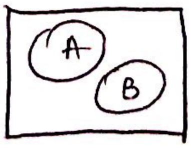
$\Rightarrow \frac{N(\Omega)}{N} = 1 \Rightarrow \mathbb{P}(\Omega) = 1$

$\mathbb{P}(A) \in [0,1]$

$\mathbb{P}(\Omega) = 1$

Pp. că avem 2 ev. $A, B \in \mathcal{F}$, $A \cap B = \emptyset$ (disjuncte)

$A \cup B \in \mathcal{F}$ (cel puțin 1 se realizează)


Ω

$N(A \cup B) = N(A) + N(B) \quad | : N$

$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ (finit aditivitate)

Def.: O funcție $\mathbb{P}: \mathcal{F} \to [0,1]$ care verifică:

a) $\mathbb{P}(\Omega) = 1$

b) $\forall (A_n)_n \subseteq \mathcal{F}$ disjuncte 2 câte 2

(σ-aditivitate)

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n)$$

($\hookrightarrow$ serie)

Se numește măsură de probabilitate pe $(\Omega, \mathcal{F})$ (probabilitate)

Experiment aleator

$\downarrow$

$(\Omega, \mathcal{F}, \mathbb{P})$ câmp de probabilitate

Experiment:

a) Aruncatul cu banul

$\Omega = \{H, T\}$

$\mathcal{F} = \mathcal{P}(\Omega) = \{\emptyset, \{H\}, \{T\}, \underbrace{\{H,T\}}_{\Omega}\}$

$\mathbb{P}: \mathcal{F} \to [0,1]$

$\mathbb{P}(\Omega) = 1, \ \mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\{H\}) = p \in [0,1) \Rightarrow \mathbb{P}(\{T\}) = 1 - p.$

$p = 1/2 \Leftarrow$ monedă echilibrată

Experiment

(2) Aruncatul cu zarul

$\Omega = \{1,2,3,4,5,6\}$

$\mathcal{F} = \mathcal{P}(\Omega)$ $\qquad 2^6$ elemente (= nr. de submulțimi)

$$\{0,1\}^\Omega = \{f : \Omega \to \{0,1\}\}$$

$A^B = \{f : B \to A\}$, A, B mulțimi

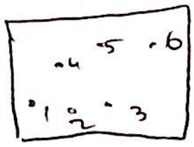
$\mathbb{P} : \mathcal{F} \to [0,1]$

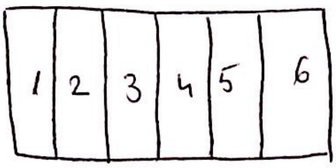
$\mathbb{P}(\Omega) = 1$

$\mathbb{P}(\emptyset) = 0$

$\mathbb{P}(\{i\}) = p_i \in [0,1)$, $i \in \{1,\dots,6\}$

$i \in \overline{0,6}$

Ω 

Ω 

$\Omega = \{1\} \cup \{2\} \cup \dots \cup \{6\}$

$p_1 + p_2 + p_3 + p_4 + p_5 + p_6 = 1$

Știm:

a) $\mathbb{P}(\Omega) = 1$

b) $(A_n)_n \in \mathcal{F}$ disjuncte 2 câte 2

$\mathbb{P}(\bigcup_n A_n) = \sum_{n \geq 1} \mathbb{P}(A_n)$

Proprietăți:

a) $\mathbb{P}(\emptyset) = 0$

$\begin{cases} \Omega \cup \emptyset = \Omega \Rightarrow \mathbb{P}(\Omega \cup \emptyset) = \mathbb{P}(\Omega) = 1 \\ \Omega \cap \emptyset = \emptyset \end{cases}$ $\quad \underbrace{\mathbb{P}(\Omega)}_{1} + \mathbb{P}(\emptyset) = 1 \Rightarrow \mathbb{P}(\emptyset) = 0$

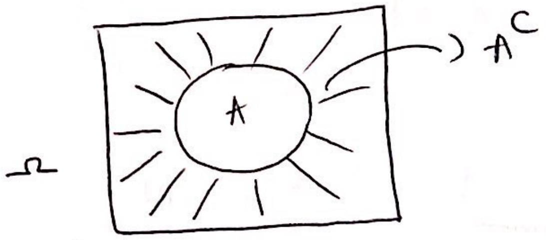
Șirul $A_n = \emptyset$

$\bigcup_n A_n = \emptyset$ $\quad$ Pp. $\mathbb{P}(\emptyset) > 0$

din b) $\underbrace{\mathbb{P}(\emptyset)}_{0 / 1} = \underbrace{\sum \mathbb{P}(\emptyset)}_{\infty}$ $\Big\}$ contradicție

b) $\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i)$, $A_1, A_2, \ldots, A_n$ disjuncte 2 câte 2

c) $A \in \mathcal{F} \Rightarrow \mathbb{P}(A^c) = 1 - \mathbb{P}(A)$



$A \cap A^c = \emptyset$

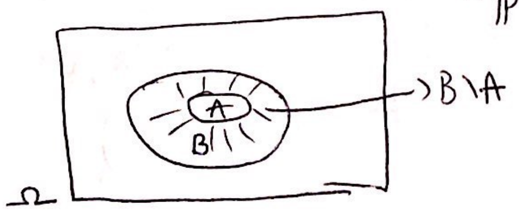
$A \cup A^c = \Omega \Rightarrow \mathbb{P}(A \cup A^c) = \mathbb{P}(\Omega) = 1$

$\qquad \mathbb{P}(A \cup A^c) = \mathbb{P}(A) + \mathbb{P}(A^c)$

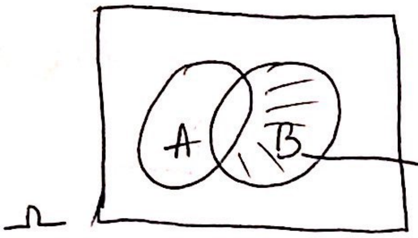
d) Proprietatea de monotonie

$A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$

$\qquad \mathbb{P}(B) = \mathbb{P}(A) + \mathbb{P}(B \setminus A)$



e) $A, B \in \mathcal{F}$, $\mathbb{P}(A\cup B) = ? = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A\cap B)$



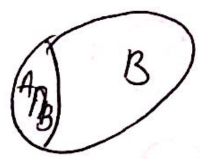
$\to B\setminus(A\cap B) = B\setminus A$

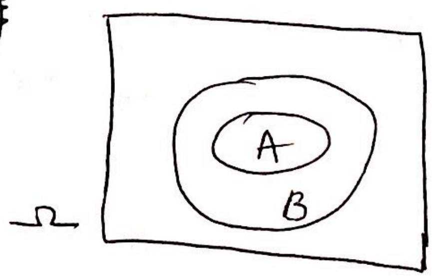
$A\cup B = A\cup(B\setminus A)$

$A\cap(B\setminus A) = \emptyset$ (sunt disjuncte)

$\mathbb{P}(A\cup B) = \mathbb{P}(A) + \mathbb{P}(B\setminus A)$

$= \mathbb{P}(A) + \underbrace{\mathbb{P}(B\setminus(A\cap B))}_{\mathbb{P}(B)-\mathbb{P}(A\cap B)}$



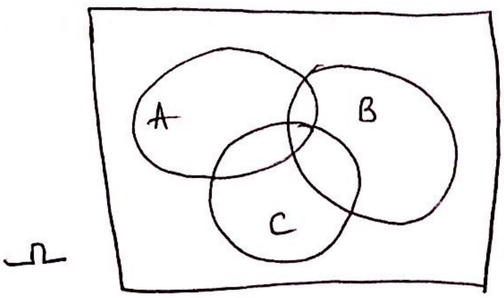


$\underline{A\subseteq B} \Rightarrow \mathbb{P}(A) \le \mathbb{P}(B)$

$\mathbb{P}(B\setminus A) = \mathbb{P}(B) - \mathbb{P}(A)$

e') A, B, C evenimente

$\mathbb{P}(A\cup B\cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A\cap B) - \mathbb{P}(A\cap C) - \mathbb{P}(B\cap C) + \mathbb{P}(A\cap B\cap C)$



f) Formula lui Poincaré

$$\mathbb{P}(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i<j} \mathbb{P}(A_i \cap A_j) + \sum_{i<j<k} \mathbb{P}(A_i \cap A_j \cap A_k) + \ldots + (-1)^{n+1} \cdot \mathbb{P}(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Temă

$\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$ → pt. probabilități mici (cutremur & fulger)

$\mathbb{P}(A \cap B) \ge \mathbb{P}(A) + \mathbb{P}(B) - 1$ → pt. probabilități mari (laptop & smartphone)

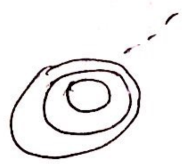
$\mathbb{P}(\{H\}) = p \in (0,1)$

$A = \{$ va pica H mai devreme sau mai târziu $\}$

$\mathbb{P}(A) = 1$

$A_n = \{$ vom obține H în n aruncări $\}$

$A = \bigcup_n A_n$



$A_1 \subseteq A_2 \subseteq A_3$

$H\ldots, \_H\ldots,$

$\mathbb{P}(A \cup B) \le \mathbb{P}(A) + \mathbb{P}(B)$

$\mathbb{P}(A \cap B) \ge \mathbb{P}(A) + \mathbb{P}(B) - 1$

$\bigcup_n A_n = \lim_n A_n$

$$\mathbb{P}(\lim_n A_n) = \lim_n \underbrace{\mathbb{P}(A_n)}_{1-(1-p)^n}$$

Modelul clasic de probabilitate
(Modelul lui Laplace)

Fie $N \geq 1$, $N \in \mathbb{N}$ și considerăm un experiment aleator cu N rezultate posibile.

$$\Omega = \{\omega_1, \omega_2, \dots, \omega_N\}$$

$$\mathcal{F} = \mathcal{P}(\Omega) \quad (2^N \text{ elemente})$$

$$\mathbb{P} : \mathcal{F} \longrightarrow [0,1] \qquad \mathbb{P}(\{\omega_i\}) = \frac{1}{N}, \ i \in \{1, \dots, N\}$$

s.m. echi~~ve~~partiție

Fie $A \in \mathcal{F}$

reuniune finită de evenimente elementare

$$\mathbb{P}(A) = \mathbb{P}\Big(\bigcup_{\omega_i \in A} \{\omega_i\}\Big) = \sum_{\omega_i \in A} \mathbb{P}(\{\omega_i\}) = \frac{1}{N} \cdot \sum_{\omega_i \in A} 1 = \frac{|A|}{N} =$$

$A = \{\omega_1, \omega_7, \omega_9\}$.

$$\cancel{= \frac{|A|}{|\omega|}} = \frac{|A|}{|\Omega|}$$

$$= \frac{\text{nr. caz fav}}{\text{nr caz posibile}}$$

(a) Formula sumei

A, B mulțimi finite și disjuncte $\Rightarrow |A \cup B| = |A| + |B|$

mulțimi oarecare $\Rightarrow |A \cup B| = |A| + |B| - |A \cap B|$

Principiul includerii - excluderii

$A_1, A_2, \ldots, A_m$ finite

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = \sum_{i=1}^{m} |A_i| - \sum_{i<j} |A_i \cap A_j| + \sum_{i<j<k} |A_i \cap A_j \cap A_k| + \ldots + (-1)^{m+1} |A_1 \cap A_2 \cap \ldots \cap A_m|$$

Aplicație (laborator)

$\varphi(m)$ - nr. de nr prime cu m $\leq m$.

fct. Euler

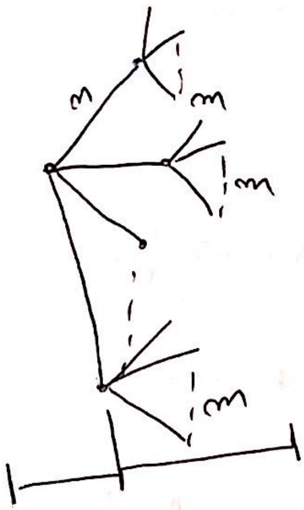
$$\varphi(m) = m \prod_{p/m} \left(1 - \frac{1}{p}\right)$$

b) Formula produs

A, B finite, $A \times B = \{(a,b) \mid a \in A$ şi $b \in B\}$

$|A \times B| = |A| \cdot |B|$

(a,b)



$n \times m$

$A^n = \{(a_1, \dots a_n) \mid a_i \in A\}$

$|A|^n$