Teorema de schimbare de variabilă

Fie $D, G \subset \mathbb{R}^n$ mulțime deschisă, $\varphi : D \to G$ un difeomorfism de clasă C^1 (φ este bijectivă și φ și φ^{-1} sunt de clasă C^1), $A \subset D$ a.î. $A \in J(\mathbb{R}^n)$ și $f : \varphi(A) \to \mathbb{R}$ mărginită.

Varianta 1 Dacă $\overline{A} \subset D \Rightarrow \varphi(A) \in J(\mathbb{R}^n)$.

Dacă f este integrabilă Riemann $\Rightarrow$

$f \circ \varphi \cdot |\det \varphi'| : A \to \mathbb{R}$ este integrabilă Riemann și

$$\int_{\varphi(A)} f(x)\,dx = \int_A f\circ\varphi(y)\,|\det \varphi'(y)|\,dy.$$

Varianta 2 Dacă $\varphi(A)$ este mărginită, $F_r G$ este neglijabilă Lebesgue $\Rightarrow \varphi(A) \in J(\mathbb{R}^n)$.

Dacă f este integrabilă Riemann $\Rightarrow f\circ\varphi \cdot |\det \varphi'| : A \to \mathbb{R}$ este integrabilă Riemann și

$$\int_{\varphi(A)} f(x)\,dx = \int_A f\circ\varphi(y)\,|\det \varphi'(y)|\,dy$$

Exemplu (coordonate polare)

$$\varphi: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}^2 \quad \varphi(r,\theta) = (\underbrace{r\cos\theta}_{x},\ \underbrace{r\sin\theta}_{y})$$

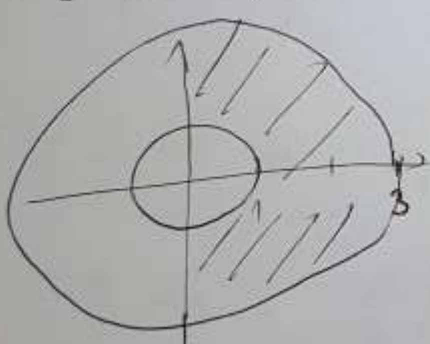
$\Rightarrow r = \sqrt{x^2+y^2}$

$\varphi: (0,\infty)\times(-\pi,\pi) \to \mathbb{R}^2 \setminus \{y=0,\ x\le 0\}$ bijectivă

φ și $\varphi^{-1} \in C^1$

Să se calculeze $I = \iint_B \ln(x^2+y^2)\,dx\,dy$

$B = \{1 \le x^2+y^2 \le 9 \quad x \ge 0\}$



Vrem A a.î. $\varphi(A) = B \Rightarrow 1 \le r^2 \le 9$ $r\cos\theta \ge 0$

$\Rightarrow r \in [1,3]$ $\quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}] \Rightarrow$

$$A = [1,3]\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right] = \bar{A} \subset (0,\infty)\times(-\pi,\pi)$$

$$\varphi'(r,\theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad \det \varphi' = r > 0$$

$$I = \iint_A \ln(r^2)\, r\, dr\, d\theta =$$

$$= \int_1^3 \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r \ln r \, d\theta \right) dr =$$

$$= 2\pi \int_1^3 r \ln r \, dr = 2\pi \int_1^3 \left(\frac{r^2}{2}\right)' \ln r \, dr =$$

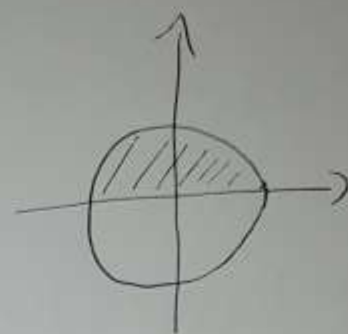
$$= 2\pi \cancel{\int_1^3} \left(\frac{r^2}{2} \ln r \Big|_1^3 - \int_1^3 \frac{r^2}{2} \cdot \frac{1}{r} \, dr \right) =$$

$$= 2\pi \left(\frac{9}{2} \ln 3 - 0 - \int_1^3 \frac{r}{2} \, dr \right) =$$

$$= 2\pi \left(\frac{9}{2} \ln 3 - \frac{r^2}{4} \Big|_1^9 \right) = 2\pi \left(\frac{9}{2} \ln 3 - 2 \right)$$

Să se calculeze $\iint_A (x^2+y^2)^2\,dx\,dy$ ④

$A = \{(x,y) \mid x^2+y^2 \le 1, y \ge 0\}$



$\mathring{A} = \{x^2+y^2 < 1, y > 0\}$

$\varphi : (0,\infty) \times \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right) \to \mathbb{R}^2 \setminus \{x=0, y\le 0\}$.

$r^2 < 1 \quad r\sin\theta > 0 \Rightarrow r \in (0,1) \quad \theta \in (0,\pi)$

$B = (0,1) \times (0,\pi) \Rightarrow \varphi(B) = A \quad (\varphi(\bar{B}) = \bar{A})$.

$I = \int_A (x^2+y^2)^2\,dx\,dy = \int_{\mathring{A}} (x^2+y^2)^2\,dx\,dy$

$\mu(\mathrm{Fr}\,A) = 0 \qquad |\det \varphi'|$

$I = \int_{\varphi(B)} (x^2+y^2)^2\,dx\,dy = \int_B r^4 \cdot \underbrace{r}_{|\det\varphi'|}\,dr\,d\theta =$

$= \int_{\bar{B}} r^5\,dr\,d\theta = \int_0^1 \left(\int_0^\pi r^5\,d\theta\right)dr$

$= \pi \int_0^1 r^5\,dr = \frac{\pi}{6}$

Să se calculeze

1) $\int_A \frac{xy}{x^2+y^2}\,dx\,dy \qquad A=\{1\le x^2+y^2\le 9,\ x\ge 0,\ y\ge 0\}$

2) $\int_A x^2y^2\,dx\,dy \qquad A=\{1\le x^2+y^2\le 4 \quad x\ge y\}$

3) $\int_A e^{x^2+y^2}\,dx\,dy \qquad A=\{x^2+y^2\le 1\}$

4) $\int_A \sqrt{4-(x^2+y^2)}\,dx\,dy \qquad A=\{1\le x^2+y^2\le 4\}$

5) $\int_A \frac{x^2}{4}+\frac{y^2}{9}\,dx\,dy \qquad A=\{\frac{x^2}{4}+\frac{y^2}{9}\le 1\}$

6) $\int_A x^2+y^2\,dx\,dy \qquad A=\{x^2+y^2\ge 1 \quad x^2+y^2\le 2x\}$

7) $\int_A xy\,dx\,dy \qquad A=\{x^2+\frac{y^2}{4}\le 3,\ x\ge 0,\ y\ge 0\}$

8) $\int_A xy\,dx\,dy \qquad A=\{x\ge 0,\ y\ge 0,\ x^{\frac{2}{3}}+y^{\frac{2}{3}}\le 1\}$

$\mu(A)=?$

(8)

Să se calculeze:

$I = \iint_A xy\,dx\,dy \qquad A = \{0 < y \leq x^2 \leq 2y,\ x < y^2 \leq 2x\}$

$x > 0 \quad y > 0 \qquad$ pr $x > 0 \quad y > 0$

Avem: $1 \leq \dfrac{x^2}{y} \leq 2 \qquad 1 \leq \dfrac{y^2}{x} \leq 2$

Notăm $u = \dfrac{x^2}{y} \qquad v = \dfrac{y^2}{x}$

$\Rightarrow \quad y = \dfrac{x^2}{u} \qquad v\,x = \dfrac{x^4}{u^2} \Rightarrow x^3 = u^2 v$

$\Rightarrow \quad x = u^{\frac{2}{3}} v^{\frac{1}{3}} \quad y = u^{\frac{1}{3}} v^{\frac{2}{3}}$

$\varphi : (0,\infty)^2 \to (0,\infty)^2 \quad \varphi(u,v) = \left(u^{\frac{2}{3}} v^{\frac{1}{3}},\ u^{\frac{1}{3}} v^{\frac{2}{3}}\right)$

$\varphi \in C^\infty$, φ bijectivă

$B = [1,2]^2$

$\varphi(B) = A$

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$$\varphi'(u,v) = \begin{pmatrix} \frac{2}{3} u^{-\frac{1}{3}} v^{\frac{1}{3}} & \frac{1}{3} u^{\frac{2}{3}} v^{-\frac{2}{3}} \\ \frac{1}{3} u^{-\frac{2}{3}} v^{\frac{2}{3}} & \frac{2}{3} u^{\frac{1}{3}} v^{-\frac{1}{3}} \end{pmatrix}$$

$$\det \varphi' = \frac{4}{9} - \frac{1}{9} = \frac{1}{3}$$

$$I = \iint_{A=\varphi(B)} xy\,dx\,dy = \iint_B \underbrace{u^{\frac{2}{3}} v^{\frac{1}{3}}}_{x} \cdot \underbrace{u^{\frac{1}{3}} v^{\frac{2}{3}}}_{y} \cdot \underbrace{\frac{1}{3}}_{\det\varphi'} \,du\,dv =$$

$$= \frac{1}{3} \int_{[1,2]^2} uv\,du\,dv = \frac{1}{3} \left(\int_1^2 u\,du \right)^2 =$$

$$= \frac{1}{3} \left(\frac{u^2}{2} \Big|_1^2 \right)^2 = \frac{1}{3} \cdot \frac{9}{4} = \frac{3}{4}$$

Ex $\iint_A x^2 y^2 \,dx\,dy \qquad A = \{ 0 < y \leq x^2 \leq 3y, \, 0 < x < y^2 \leq 2x$