Să se studieze convergența următoarelor serii:

(1)

(a) Criteriul comparației

1. $\sum\limits_{n\geq 1} \frac{\sqrt{n+1}-\sqrt{n}}{n}$ 2. $\sum\limits_{n\geq 1} \frac{n\sqrt{n}}{3n^2+2}$

3. $\sum\limits_{n\geq 1} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n}$ 4. $\sum\limits_{n\geq 1} \frac{1+\sqrt{2}+\dots+\sqrt{n}}{n^3}$

5. $\sum\limits_{n\geq 1} \sqrt{n^4+3n+1}-n^2$ 6. $\sum\limits_{n\geq 1} 2^n \sin\frac{\pi}{4^n}$

(b) Criteriul raportului

7. $\sum\limits_{n\geq 1} \frac{x^n}{\sqrt[n]{C_{3n}^n}}$ $x>0$ ✓8. $\sum\limits_{n\geq 1} x^{2n}\frac{n\sqrt{n}}{n^2+1}$ $x>0$

✓9 $\sum\limits_{n\geq 1} \frac{2^n n!}{n^n}$ ✓10 $\sum\limits_{n\geq 1} \frac{4^n n!}{n^n}$

11 $\sum\limits_{n\geq 1} \frac{(n!)^2}{(2n)!}$

(c) Criteriul radicalului

✓12. $\sum\limits_{n\geq 1} x^n \cdot \left(1+\frac{1}{n}\right)^{n^2+n}$ $x>0$ 13. $\sum\limits_{n\geq 1} \left(\frac{3+(-1)^n\cdot 2}{4}\right)^n$

14. $\sum\limits_{n\geq 1} \left(\frac{an+1}{bn+1}\right)^n$ $a,b>0$ 15. $\sum\limits_{n\geq 1} n^{(-1)^{n+1}} a^n$ $a>0$.

d) Raabe-Duhamel

16). $\sum\limits_{n\geq 1} \dfrac{\sqrt{n!}}{(3+\sqrt{1})(3+\sqrt{2})\ldots(3+\sqrt{n})}$

17). $\sum\limits_{n\geq 1} \dfrac{1!+2!+\ldots+n!}{(n+2)!}$

18). $\sum\limits_{n\geq 1} \dfrac{1}{n^{a}}$ $\quad a\in\mathbb{R}$

19) $\sum\limits_{n\geq 1} \dfrac{a(a+1)\ldots(a+n)}{(n+6)!}$ $\quad a>0$

20) $\sum\limits_{n\geq 1} \left(\dfrac{1\cdot 4\cdot 7\cdot\ldots(3n-2)}{3\cdot 6\cdot 9\ldots(3n)}\right)^2$

e) Serii cu termeni oarecare:

21) $\sum\limits_{n\geq 1} x^n \sin\dfrac{1}{n^\alpha}$ $\quad \alpha>0$

22) $\sum\limits_{n\geq 1} x^n \operatorname{arctg}\dfrac{1}{n^\alpha}$ $\quad \alpha\in\mathbb{R}$

23) $\sum\limits_{n\geq 1} x^n \ln^\alpha\left(1+\dfrac{1}{n^2}\right)$

24) $\sum\limits_{n\geq 1} x^n\cdot\dfrac{(n+5)!}{a(a+1)\ldots(a+n)}$

$x\in\mathbb{R}$ $\quad a>0$

25) $\sum\limits_{n\geq 1} x^n\cdot\dfrac{\sqrt{n!}}{a(a+\sqrt{1})\ldots(a+\sqrt{n})}$ $\quad x\in\mathbb{R}$ $\quad a>0$

26) $\sum\limits_{n\geq 1} x^n\left(\dfrac{a}{2}-\sqrt{a}\right)\left(2-\sqrt[3]{a}\right)\ldots\left(2-\sqrt[n]{a}\right)$ $\quad a>0$, $x\in\mathbb{R}$.

27) $\sum\limits_{n\geq 1} \dfrac{x^n\cdot n!}{n^{n+\alpha}}$ $\quad x\in\mathbb{R}$ $\quad \alpha\in\{-2,5\}$.

Rezolvări

3) $$\lim_{n\to\infty} \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{\ln n} = 1$$

$$a_n = \frac{1+\frac{1}{2}+\dots+\frac{1}{n}}{n} \qquad b_n = \frac{\ln n}{n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \Rightarrow$$

$$\sum_{n\geq 1} a_n \sim \sum_{n\geq 1} b_n \geq \sum_{n\geq 1} \frac{1}{n} \text{ - divergentă}$$

9) $$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2^{n+1}(n+1)!}{(1+n)^{n+1}} \cdot \frac{n^n}{2^n n!} = 2\left(\frac{n}{n+1}\right)^n \to$$

$$\to \frac{2}{e} < 1 \Rightarrow \text{s. convergentă.}$$

14) $$\lim_{n\to\infty} \sqrt[n]{\left(\frac{an+1}{bn+1}\right)^n} = \lim_{n\to\infty} \frac{an+1}{bn+1} = \frac{a}{b}.$$

Dacă $a > b$ s divergentă. Dacă $a < b$ s convergentă.

Dacă $a = b \Rightarrow \left(\frac{an+1}{bn+1}\right)^n = 1 \not\to 0 \Rightarrow$ s. divergentă.

19) $\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{n+7}{a+n+1}-1\right) =$

④

$= \lim_{n\to\infty} n \frac{6-a}{a+n+1} = 6-a$

Dacă $6-a>1 \Leftrightarrow 5>a$ s divergentă

$6-a<1 \Leftrightarrow 5<a$ s convergentă

$a=5$ $\sum_{n\geq 1} \frac{5\cdot 6 \cdots (a+5)}{(n+6)!} = \sum_{n\geq 1} \frac{1}{4!(n+6)}$ divergentă.

22) $s=\sum_{n\geq 1} x^n \sin \frac{1}{n^\alpha}$ $\alpha>0$

studiem absolut convergenta (PAS1)

$\sum_{n\geq 1} |x|^n \cdot \sin \frac{1}{n^\alpha}$ $a_n := |x|^n \sin \frac{1}{n^\alpha}$

$x=0$; $s=\sum_{n\geq 1} 0 = 0 \Rightarrow$ seria este absolut convergentă.

$\frac{a_{n+1}}{a_n} = \frac{|x|^{n+1} \sin \frac{1}{(n+1)^2}}{\cancel{|x|^n} \sin \frac{1}{n^\alpha}} =$

$= |x| \underbrace{\frac{\sin \frac{1}{(n+1)^\alpha}}{\frac{1}{(n+1)^\alpha}}}_{\downarrow\, 1} \cdot \underbrace{\frac{\frac{1}{n^\alpha}}{\sin \frac{1}{n^\alpha}}}_{\downarrow\, 1} \cdot \underbrace{\frac{n^\alpha}{(n+1)^\alpha}}_{\downarrow\, 1} \to |x|$

Dacă $|t|<1$ seria este abs. conv

$|t|>1$ seria este divergentă $(a_n \to \infty)$

5

$|t|=1$ $(\Rightarrow x=1)$

$$\sum_{n\geq 1} \sin\frac{1}{n^\alpha} \sim \sum \frac{1}{n^\alpha}$$

$\overset{(1)}{}$

Conv $\alpha>1$

div $\alpha \leq 1$

$$\lim_{n\to\infty} \frac{\frac{1}{n^\alpha}}{\sin\frac{1}{n^\alpha}} = 1$$

$\underline{x=-1}$ $$\sum_{n\geq 1} (-1)^n \sin\frac{1}{n^\alpha}$$

$\frac{1}{n^\alpha} \downarrow 0 \Rightarrow \sin\frac{1}{n^\alpha} \downarrow 0 \Rightarrow$

$\alpha>1$ absolut convergentă

$0<\alpha\leq 1$ semiconvergentă.