INTEGRALA MULTIPLĂ

TEOREMA FUBINI.

Fie $f : [a, b] \times [c, d] \to \mathbb{R}$ continuă. Atunci

$$\int_{[a b] \times [c, d]} f(x, y)\,dx\,dy = \int_a^b \left(\int_c^d f(x, y)\,dy \right) dx$$

<u>Exemplu</u>

Fie $f : [0, 1] \times [1 2] \to \mathbb{R}$ $f(x, y) = x^2 y + y^2$

$$I = \int_{[0, 1] \times [1, 2]} f(x, y)\,dx\,dy = \int_0^1 \left(\int_1^2 x^2 y + y^2 dy \right) dx$$

$$= \int_0^1 \left(\frac{x^2 y^2}{2} + \frac{y^3}{3} \right) \Bigg|_{y=1}^{y=2} dx$$

$$= \int_0^1 \left(2x^2 + \frac{8}{3} - \frac{x^2}{2} - \frac{1}{3} \right) dX = \frac{\cancel{3}}{2} \cdot \frac{1}{\cancel{3}} + \frac{7}{3} = \frac{17}{6}$$

$$I = \int_1^2 \left(\int_0^1 (x^2 y + y^2) dx \right) dy = \int_1^2 \frac{y}{3} + y^2 dy =$$

$$= \frac{y^2}{6} \Bigg|_1^2 + \frac{y^3}{3} \Bigg|_1^2 = \frac{4}{6} - \frac{1}{6} + \frac{8}{3} - \frac{1}{3} = \frac{17}{6}.$$

Calculati:

1) $\int_{[0,1]\times[0,2]} e^x y^3 \, dx\, dy$

2) $\int_{[0,1]\times[0,3]} \frac{1}{(1+x+y)^2} \, dx\, dy$

3) $\int_{[0,\frac{\pi}{2}]\times[0,\frac{\pi}{2}]} \cos(x+y) \, dx\, dy$

4) $\int_{[0,\frac{\pi}{2}]\times[1,2]} y^3 \sin^2 x \, dx\, dy$

5) $\int_{[0,1]\times[0,2]\times[0,3]} x^2 y + y^3 z \, dx\, dy\, dz$

6) $\int_{[0,2]\times[0,\frac{\pi}{2}]\times[1,2]} e^x \sin y \cdot z^3 \, dx\, dy\, dz.$

7) Fie $f:[a,b]\to\mathbb{R}$ continuă, $g:[c,d]\to\mathbb{R}$ continuă, şi $h:[a,b]\times[c,d]\to\mathbb{R}$. Arătaţi că (unde $h(x,y)=f(x)\cdot g(y)$)

$$\int_{[a,b]\times[c,d]} h(x,y)\, dx\, dy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy.$$

8) Fie $h : [a,b] \times [c,d] \to \mathbb{R}$ a.î.

$\exists \dfrac{\partial^2 h}{\partial x \partial y}$ și este continuă. Atunci:

$$\iint\limits_{[a,b]\times[c,d]} \frac{\partial^2 h}{\partial x \partial y}(x,y)\, dx\, dy = h(a,c) - h(a,d) + h(c,b) + h(b,d).$$

T. Fubini

Fie $g, h : [a,b] \to \mathbb{R}$ continue cu $g \le h$,

$\Gamma_{g,h} = \{(x,y) \mid x \in [a,b] \quad y \in [g(x), h(x)]\}$ și

$f : \Gamma_{g,h} \to \mathbb{R}$ continuă atunci

$$\int_{\Gamma_{g,h}} f(x,y)\, dx\, dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\, dy \right) dx.$$

$$\text{aria}(\Gamma_{g,h}) = \int_{\Gamma_{g,h}} 1\, dx\, dy = \int_a^b h(x)\, dx - \int_a^b g(x)\, dx.$$

Exemplu

Fie $A = \{(x,y) \mid x^2 \le y \le x\}$ şi $f : A \to \mathbb{R}$ continuă dată de $f(x,y) = xy$. Se cere

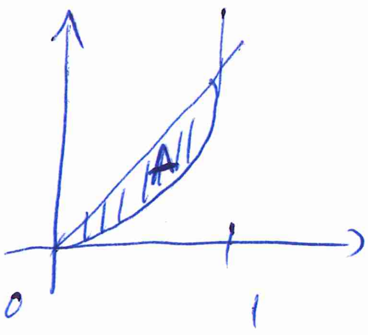
a) să se deseneze mulţimea A

b) Să se calculeze $\int_A f(x,y)\,dx\,dy$.

Fie $g, h : [0,1] \to \mathbb{R}$ $\quad g(x) = x^2$ şi $h(x) = x$.

$A = \Gamma_{g,h}$.

$x^2 \le y \le x \Rightarrow x \ge 0,\ y \ge 0 \quad x^2 - x \le 0 \Rightarrow x \in [0,1]$.



$$I = \int_A f(x,y)\,dx\,dy =$$
$$= \int_0^1 \left(\int_{x^2}^{x} xy\,dy \right) dx =$$

$$= \int_0^1 \frac{xy^2}{2} \Bigg|_{y=x^2}^{y=x} dx = \int_0^1 \frac{x^3}{2} - \frac{x^5}{2}\,dx = {}^{3)}\frac{1}{8} - {}^{2)}\frac{1}{12} = \frac{1}{24}.$$

$$I = \int_0^1 \left(\int_{y}^{\sqrt{y}} xy\,dx \right) dy = \int_0^1 \frac{x^2 y}{2} \Bigg|_{x=y}^{x=\sqrt{y}} dy$$

$$= \int_0^1 \frac{y^2}{2} - \frac{y^3}{2}\,dy = {}^{4)}\frac{1}{6} - {}^{3)}\frac{1}{8} = \frac{1}{24}.$$

<u>Propoziție</u> Dacă A și B au arie și

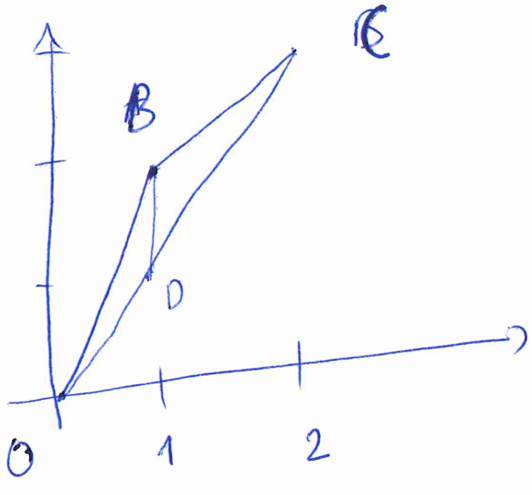
$\text{arie}(A \cap B) = \emptyset$, $f: A \cup B \to \mathbb{R}$ integrabilă $\Rightarrow$

$$\int_{A \cup B} f = \int_A f + \int_B f.$$

<u>Exercițiu</u> Să se calculeze integrala

$\int_A xy \, dx dy$ unde A este triunghiul cu

vârfurile $O(0,0)$ $B(1,2)$ $C=(2,3)$



Fie $D\left(1, \frac{3}{2}\right)$

OC $y = \frac{3}{2}x$

$x = 1$

$B = \Delta OBD$ $C = \Delta BDC$ $A = B \cup C$

$B \cap C = [B, D]$

Ec_{BC} $\dfrac{x - x_B}{x_C - x_B} = \dfrac{y - y_B}{y_C - y_B}$ $\dfrac{x-1}{2-1} = \dfrac{y-2}{3-2}$

$x = y - 1$ $y = x + 1$

$$I \int_A xy\,dx\,dy = \int_B xy\,dx\,dy + \int_C xy\,dx\,dy$$

$$I_1 = \int_B xy\,dx\,dy = \int_0^1 \left(\int_{\frac{3}{2}x}^{2x} xy\,dy \right) dx =$$

$$\Downarrow$$

$\Delta\, OBD$

$$= \int_0^1 \frac{xy^2}{2} \Bigg|_{y=\frac{3}{2}x}^{y=2x} dx = \int_0^1 \left(\frac{4x^3}{2} - \frac{9}{8}x^3 \right) dx$$

$$= \left(2 - \frac{9}{8}\right) \int_0^1 x^3\,dx = \frac{7}{8} \cdot \frac{1}{4} = \frac{7}{32}$$

$$I_2 \int_C xy\,dx\,dy = \int_1^2 \left(\int_{\frac{3}{2}x}^{x+1} xy\,dy \right) dx$$

$$= \int_1^2 \frac{xy^2}{2} \Bigg|_{y=\frac{3}{2}x}^{y=(x+1)} dx$$

$$= \frac{1}{2} \int_1^2 x^3 + 2x^2 + x - \frac{9}{4}x^3\,dx$$

$$= \frac{1}{2} \int_1^2 \left(-\frac{5}{4}x^3 + 2x^2 + x \right) dx = \dots\dots$$

$$I = I_1 + I_2.$$

1) Să se calculeze $\iint_A x^2 y \, dx\, dy$ unde

A este mulțimea delimitată de curbele

$xy=1 \quad x+y=\frac{5}{2}$

2) $\iint_A (|x|+|y|)\, dx\, dy$ unde $A=\{x\geq 0, |x|+|y|\leq 1\}$

3) $\iint_A \frac{1}{\sqrt{x}}\, dx\, dy$ unde A este mulțimea

$A=\{y^2\leq 8x, \; y\leq 2x, \; y+4x\leq 24\}$.

4) $\iint_A xy\, dx\, dy$ unde $A=\{x^2\leq y\leq 3x-2\}$.

5) $\iint_A x^2y^2\, dx\, dy$ unde $A=\{0\leq y\leq 1-|x|\}$.

6) $\iint_D x\, dx\, dy$ D este domeniul mărginit

de parabola $y^2=2x$, cercul $x^2+y^2=2x$, și

dreapta $x=2$.

7) a) Fie $A = \{(x_1, x_2) \mid x_1 \geq 0,\ x_2 \geq 0\ \ \frac{x_1}{a_1} + \frac{x_2}{a_2} \leq 1\}$ unde $a_1, a_2 > 0$

Se cere 1) $\text{aria}(A)$

2) $\iint_A x_1 x_2 \, dx_1 \, dx_2$

b) Fie $B = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \geq 0\ \ \frac{x_1}{a_1} + \frac{x_2}{a_2} + \frac{x_3}{a_3} \leq 1\}$

unde $a_1, a_2, a_3 > 0$. Se cere:

1) $\text{vol}(B) = ?$

2) $\iiint_B x_1 x_2 x_3 \, dx_1 \, dx_2 \, dx_3$

c) generalizare.