THE NOETHERIAN PROPERTY OF INVARIANT RINGS

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1 Abstract

A Noetherian ring is a ring that has the ascending chain condition, which means that any ascending chain of ideals of the ring must stabilize after a finite number of steps. The concept of Noetherian rings came to be after the German mathematician, Emmy Noether, discovered in 1921 that primary decomposition of ideals is

a consequence of the ascending chain condition. It is known that for a graded ring over a field, the Noetherian property is equivalent to the ring being finitely generated over the field. Let k be a field. Noether proved in 1926 that the ring of invariants for the action of a finite group via k-algebra automorphisms on a finitely generated k-algebra is Noetherian. In a related direction, a Noetherian ring containing the field of rational numbers, will have a Noetherian invariant ring under a finite group action. The proof depends on the ring containing a characteristic zero field, because it uses an especially useful formula, the Reynolds operator, that involves the inverse of the order of the given finite group to define a map between the ring and its invariant ring that turns out to be a projection. This however is not necessarily true in other cases. We construct a class of Noetherian rings of characteristic p, for each prime integer p, with finite group actions such that the ring of invariants is not Noetherian. This class of rings is extended from a characteristic two counterexample due to Nagarajan from 1968, [NaK].

2 Literature Review

The proposed question first arose in invariant theory, a field that has a goal of exploring invariants of different mathematical objects, essentially trying to find all elements that are immovable after specific transformations, or group actions, as is the case here. The beginnings of our project can be found in Hilbert's Fourteenth Problem, [Hi, p. 462-464]. The question essentially asks whether certain algebras are finitely generated. Hilbert originally formulated the problem as follows: Consider a ring $R := k[x_1,...,x_n]$ which is a finitely generated algebra over a field k and is acted on linearly by a linear algebraic group over a field. This gives rise to the ring of invariants which is the target of investigation.

There were special cases and classes of rings for which the statement applied, such as the cases $n = 1$ and $n = 2$ proved by Oscar Zariski in 1954, [Za]. Specifically, we first assume that k is a field and the intermediate field K, a subfield of rational functions in n variables over k. We now consider the k-algebra R defined as the intersection of K and $k[x_1,\cdots,x_n]$. Hilbert conjectured that such algebras are finitely generated over k. However this turns out to be false in general. In the nineteenth century, mathematicians such as Cayley, Sylvester, Clebsch, Gordan, and Hilbert were invested in studying the invariants of binary forms in two variables with the natural action of the special linear group. Hilbert proved the finite generation of invariant rings in the case of the field of complex numbers for some classical semi-simple Lie groups, such as the general linear group over the complex numbers and specific linear actions on polynomial rings, the specific linear actions being actions coming from finite-dimensional representations of Lie groups. Hilbert's proof depends on the Hilbert basis theorem which is applied to the ideal inside the polynomial ring generated by the invariants.

The first counterexample was proposed by Masayoshi Nagata in which he constructed a ring of invariants for the action of a linear algebraic group, [NaM]. The counterexample is the polynomial ring in 32 variables under a field containing 48 elements that are algebraically independent over the prime field where the group acting on the ring is a 13-dimensional vector space over the given field. Then the ring invariant under the group action is not a finitely generated k-algebra and thus is not Noetherian. There is active research being done on the topic such as by Shigeru Kuroda who expanded on the existence of new counterexamples, including in dimensions 3 and 4, as well as his more recent contribution to Hilbert's Fourteenth Problem.[Ku]

3 Background

3.1 Noetherian Rings

If we start with an arbitrary partially ordered set Γ, then by taking an ascending chain $\gamma_1 < \gamma_2 < \cdots$ of elements of Γ, it can be shown that the ascending chain stabilizing after finite amounts of steps is equivalent to any nonempty subset of Γ having a maximal element. This property is called the ascending chain condition (a.c.c.), or the maximal condition. If we reverse the order, we get the descending chain condition (d.c.c.), or minimal condition.

Now if we choose our partially ordered set to be ideals of a ring or submodules of a module, it gets more interesting. In the case of a ring, the chains are made up of ideals of the ring, while for modules they are made up of submodules. The ascending chain condition in the case of rings and modules is called the Noetherian property and the descending chain condition is called the Artinian property. We will now show one of the most important interpretations of chain conditions in regards to finite generation of elements.

Proposition 3.1. *A ring A is Noetherian if and only if every ideal of A is finitely generated.*

Proof. Let A be a Noetherian ring and let I be an ideal of A. Let us define S as the set of finitely generated ideals contained in I. We note that S is not empty because the ideal I contains the zero ideal (0) and (0) is finitely generated. Since S is a subset of ideals of A and A is Noetherian, then S is guaranteed to have a maximal element, say I_l. The ideal I_l is finitely generated, say by the elements $x_1, x_2, \cdots, x_n$.

Now let x be an element of I. If we add this element to the generators of I_l, we are now considering the ideal $(x_1, \cdots, x_n, x)$. It is easy to see the inclusion $(x_1, \cdots, x_n, x) \subseteq I$ and $(x_1, \cdots, x_n, x)$ is finitely generated. Therefore $(x_1, \cdots, x_n, x) \in S$, however this contradicts I_l being maximal since $I_l \subseteq (x_1, \cdots, x_n)$ thus $I_n = (x_1, \cdots, x_n) = (x_1, \cdots, x_n, x)$ for any $x \in I$. Since $x \in I$, then $I \subseteq I_n$ and thus, $x \in I_l$. Additionally, $I_l \subseteq I$ because $I_l \in S$ by the hypothesis. We can conclude that $I_l = I$.

For the converse, assume every ideal of A is finitely generated. Let

$$I_1 \subseteq I_2 \subseteq \cdots$$

be an ascending chain of ideals of A. Take the union $I = \cup_{i=1}^{\infty} I_i$. Then I, as an ideal of A, is also finitely generated, say by $x_1, x_2, \cdots, x_n$. Since I is a union of the ideals from the given chain, the can find I_{k_i} for each generator x_i such that $x_i \in I_{k_i}$. Now let us take the largest ideal from the collection $\{I_{k_i}\}$ and call it I_m where $m = \max\{k_1, \cdots, k_n\}$. Then because $x \in \cup_{i=1}^{\infty} I_i$ then $x \in I_m$, we have $I_{k_i} \subseteq I_m$. It is easy to show that $I_m = \cup_{i=1}^{\infty} I_i$. First, $\cup_{i=1}^{\infty} I_i \subset I_m$ because $x \in \cup_{i=1}^{\infty} I_i$ then $x \in I_m$.

$I_m \subseteq \cup_{i=1}^{\infty} I_i$ is true from the definition of union.

Then this means that:

- $i < m$ we have $I_i \subseteq I_m$
- $i \leq m$ we have $I_i = I_m$

The chain stops at I_m, therefore A is Noetherian. $\square$

Example 3.1. *The simplest example of a Noetherian ring is a field, because it has only two ideals: the zero ideal and the field itself.*

Theorem 3.1 (Hilbert's Basis Theorem)**.** *If a ring A is Noetherian, then the polynomial ring $A[x]$ is also Noetherian.*

Corollary 3.1.1. *A polynomial ring in finitely many variables $k[x_1, x_2, \cdots, x_n]$ is Noetherian, where k is a field.*

Example 3.2. *A polynomial ring in infinitely many variables $k[x_1, x_2, \cdots]$ is not Noetherian because we can take the chain*

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$$

Lemma 3.2. *If R is a Noetherian ring, then $R[[x_1, x_2, \cdots, x_n]]$ is also Noetherian.*

Proof. Let us assume that R is a Noetherian ring.

Since it is easy to see that $R[[x_1, x_2, \cdots, x_n]] \cong R[[x_1, \cdots, x_{n-1}]][[x_n]]$, then it is sufficient to show that $R[[x]]$ is Noetherian.

Let $I \subseteq R[\![x]\!]$ be an ideal and denote $I_k := \{r \in R$ such that $rx^d + \cdots \in I\}$. Then $I_0 \subset I_1 \subset \cdots$ ascending chain of ideals will stabilize because R is Noetherian by hypothesis.

We can find d_0 such that $I_{d_0} = I_{d_0+1} = \cdots$ for every $d \leq d_0$. Then let us denote $f_{d,j}$ to be a power series in $R[\![x]\!]$ contained in $I \cap (x^d)$ for $j = 1, \cdots, n_d$ such that $f_{d,j} = a_{d,j}x^d + \cdots$. Then it can be implied that $I_d = (a_{d,j})$. Let I' be an ideal generated by $f_{d,j}$ for all $d = 0, \cdots, d_0$ and $j = 1, \cdots, n_d$. Then we can say that $I' \subset I$.

Let $f \in I$. We can find coefficients $c_{d,j} \in R$ such that

$$f - \sum_{i.j} c_{d,j} f_{d,j} \in (x^{d_0+1} \cap I.$$

Now we find coefficients $C_{i,1} \in R$ for $i = 1, \cdots n_{d_0}$ such that

$$f - \sum_i c_{d_1,i} f_{d_1,i} - \sum_i c_{i,1} x f_{d_0,i} \in (x^{d_0+2}) \cap I.$$

Next, we find coefficients $C_{i,2} \in R$ for $i = 1, \cdots n_{d_0}$ such that

$$f - \sum_i c_{d_2,i} f_{d_2,i} - \sum_i c_{i,2} x^2 f_{d_0,i} \in (x^{d_0+3}) \cap I.$$

We continue this process and can conclude that

$$f = \sum_i c_{d,i} f_{d,i} - \sum_i (\sum_j c_{i,j} x^j) f_{d_0,i} \in I'$$

We showed that $I = I'$ which demonstrates that every ideal of $R[\![x]\!]$ is finitely generated, thus $R[\![x]\!]$ is Noetherian. $\square$

Example 3.3. *A power series ring in finitely many variables $k[\![x_1, x_2, \cdots, x_n]\!]$ is Noetherian where k is a field.*

3.2 Group Actions

Definition 3.1. *A linear representation of a group G is a group homomorphism*

$$\phi : G \to GL(n, \mathbb{F})$$

where $\mathbb{F}$ is a field and $GL(n, \mathbb{F})$ is the general linear group consisting of all invertible $n \times n$ matrices with entries from $\mathbb{F}$.

Definition 3.2. *Let S be a set and G be a group. A (left) group action of G on S is a map*

$$G \times S \to S$$

$$(g, s) \mapsto gs$$

such that $1_G s = s$ and $(gh)s = g(hs)$ for any $g, h \in G$ and any $s \in S$.

Definition 3.3. *If we have a group G acting on a ring R by ring automorphisms which means that there is a group homomorphism*

$$G \to Aut R$$

then the invariant ring is

$$R^G := \{r \in R \mid gr = r \text{ for every } g \in G\}.$$

We can define an action of a group G on a vector space $V = \mathbb{F}^n$ where $\mathbb{F}$ is any field by using a linear representation

$$\rho : G \to GL(n, \mathbb{F}).$$

Definition 3.4. *A dual space V^* of a vector space consists of all linear transformations $\phi : V \to \mathbb{F}$. If $e_1, e_2, \cdots, e_n \in V$ is the standard basis of V, then we define the dual basis like so:*

$$x_i(e_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

The action of G on the dual space V^* of the given vector space V can be induced via the above linear representation by using the definition:

$$gx_i(e_j) = x_i(g^{-1}e_j).$$

3.3 Symmetric Polynomials

If we consider the ring $\mathbb{F}[V] = \mathbb{F}[x_1, x_2, \cdots, x_n]$ of polynomials in n variables, we can describe the ith elementary symmetric polynomial in the given variables $x_1, x_2, \cdots, x_n$ to be defined as polynomials that are fixed under the permutation operation on their subscripts. In other words, interchanging the variables gives us back the same polynomial. These polynomials, also sometimes called the *Viète polynomials*, in a more explicit manner can be described like so:

$$e_1(x_1,x_2,\cdots,x_n)=\sum_{1\leq j\leq n} x_j$$

$$e_2(x_1,x_2,\cdots,x_n)=\sum_{1\leq j<k\leq n} x_jx_k$$

$$e_3(x_1,x_2,\cdots,x_n)=\sum_{1\leq j<k<l\leq n} x_jx_kx_l$$

$$\cdots$$

$$e_n(x_1,x_2,\cdots,x_n)=\sum_{1\leq j_1<\cdots<j_k\leq n} x_{j_1}\cdots x_{j_k}$$

3.4 Integral Independence

Definition 3.5. *Let B be a ring and A a subring of B. An element b of B is said to be integral over A if b is a root of a monic polynomial with coefficients in A, that is if there exists a polynomial $f(t)\in A[t]$, say $f(t)=t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0$, such that $f(b)=0$.*

Proposition 3.2. *The following are equivalent:*

1. *$x\in B$ is integral over A.*

2. *$A[x]$ is a finitely generated A-module.*

3. *$A[x]$ is contained in a subring C of B such that C is a finitely generated*

A-module.

4. *There exists a faithful $A[x]$-module that is finitely generated as an A-module.*

4 Noetherian Invariant Rings

Proposition 4.1. *Let G be a finite group and let R be a Noetherian ring which contains $\mathbb{Q}$. Then if G acts on R, the invariant ring R^G is also Noetherian.*

Proof. Let $\rho : R \to R^G$ be the map $\rho(r) := \frac{1}{|G|}\sum_{g\in G} g(r)$ called the *Reynolds operator*.

For $a \in R^G$ we have

$$\rho(a) = \frac{1}{|G|}\sum_{g\in G} g(a) = \frac{1}{|G|}\sum_{g\in G} a = \frac{1}{|G|}|G|a = a.$$

therefore $\rho|_{R^G} = id_{R^G}$.

Let $h \in G$. Then

$$h(\rho(r)) = h(\frac{1}{|G|}\sum_{g\in G} g(r)) = \frac{1}{|G|}\sum_{g\in G} hg(r) = \frac{1}{|G|}\sum_{g\in G} g(r) = \rho(r).$$

Since $\rho(r) \in R^G$ for all $r \in R$, the map ρ is surjective. Since ρ is a surjection and $\rho|_{R^G} = id_{R^G}$, then it is a projection.

We will show that ρ is an R^G-linear map. Let $a \in R^G$ and $r \in R$. Then

$$\rho(ar) = \frac{1}{|G|}\sum_{g\in G} g(ar) = \frac{1}{|G|}\sum_{g\in G} ag(r) = \frac{a}{|G|}\sum_{g\in G} g(r) = a\rho(r).$$

Thus ρ is a surjective R^G-linear projection.

Now we will show that R^G is Noetherian, which by definition means that all ascending chains of ideals of the invariant ring R^G must stabilize. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an ascending chain of ideals of R^G. Then let us look at the ascending chain $I_1R \subseteq I_2R \subseteq I_3R \subseteq \cdots$ in R. Since R is Noetherian, then $I_kR = I_{k+1}R$ for all $k \geq N$ for some $N > 0$. We will show that $I_kR \cap R^G = I_k$ for all k. It is clear that

$$I_k \subseteq I_kR \cap R^G,$$

therefore the only thing left to show is that $I_kR \cap R^G \subseteq I_k$. Let $x \in I_kR \cap R^G$. Then $x = \sum_{i=1}^m a_ir_i$ where $a_i \in I_k$, $r_i \in R$ and $\rho(x) = x$.

$$\rho(\sum_{i=1}^m a_ir_i) = \sum_{i=1}^m a_ir_i = \sum_{i=1}^m a_i\rho(r_i).$$

Since $a_i \in I_k$ and $\rho(r_i) \in R^G$, then $\sum_{i=1}^m a_i\rho(r_i) \in I_k$ by the property of I_k being an ideal. Then $x \in I_k$ and therefore indeed $I_kR \cap R^G \subseteq I_k$. Now that we know that $I_kR \cap R^G = I_k$, then the chain

$$I_1R \subseteq I_2R \subseteq I_3R \subseteq \cdots \subseteq I_NR \subseteq I_NR \subseteq \cdots$$

can be written as a chain in R^G like so:

$$I_1R \cap R^G \subseteq I_2R \cap R^G \subseteq I_3R \cap R^G \subseteq \cdots \subseteq I_NR \cap R^G \subseteq I_NR \cap R^G \subseteq \cdots$$

Thus the above chain is equivalent to the ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_N \subseteq I_N \subseteq \cdots$$

showing that the chain in R^G stabilizes, therefore R^G is Noetherian. $\square$

Proposition 4.2. *Let $A \subseteq B \subseteq C$ be rings. Suppose that A is Noetherian, that C is finitely generated as an A-algebra and that C is either finitely generated as a B-module or integral over B. Then B is finitely generated as an A-algebra, [AM].*

Lemma 4.1. *Let G be a finite group of automorphisms of a ring A, then A is integral over the invariant ring A^G.*

Proof. Let $G = \{g_1, g_2, \cdots, g_n\}$. Let us fix an element $a \in A$. Then we take the polynomial $f(t)$ given by

$$\prod_{g \in G}(t - ga) = t^n - e_1(g_1a, g_2a, \cdots, g_na)t^{n-1} + \cdots + (-1)^n e_n(g_1a, g_2, \cdots g_na)$$

which we consider over the polynomial ring $A[t]$. Notice that the constant term is $\pm g_1a \cdot g_2a \cdots g_na$ and in general, the coefficients of this polynomial are elementary symmetric polynomials using $ga \in G$. This means that using any $g \in G$ on $f(t)$ will fix the coefficients because it will just reorder the ga's, resulting in the same polynomial. Therefore, we can conclude that f is a polynomial in $A^G[t]$. We can see that a is a root because one of the factors of f is $(t-a)$ since one of the g's is the identity of group G. This shows that A is integral over the invariant ring.

Theorem 4.2 (Emmy Noether, 1926)**.** *Let A be a finitely generated algebra over a field k. If a finite group G acts k-linearly on A, then the invariant ring A^G is Noetherian, [No2].*

Proof. From the above lemma we can say that A is integral over A^G. Additionally, we know that A is a finitely generated algebra. We can see that

$$k \subset A^G \subset A$$

which, using Proposition 4.1, shows that, A^G is a finitely generated k-algebra. □

5 Counterexamples

5.1 Characteristic Two Counterexample

Let $k := \mathbb{F}_2(a_1, b_1, a_2, b_2, \cdots)$ be the field of fractions of the polynomial ring $\mathbb{F}[a_1, b_1, a_2, b_2, \cdots]$. Now we take the polynomial ring $R := k[[x,y]]$ of two variables over the given field. Let $\sigma : k[x,y] \to k[x,y]$ be a ring homomorphism defined

$$\sigma : \begin{cases} x & \mapsto & x, \\ y & \mapsto & y, \\ a_n & \mapsto & a_n + yP_{n+1}, \\ b_n & \mapsto & b_n + xP_{n+1}. \end{cases}$$

with $P_n := a_n x + b_n y$. Then we can consider the group of actions $< \sigma >$, [NaK].

5.2 Extension to Nagarajan's Example

Now, we will extend the previous example due to Nagarajan to prime characteristic p. We define the ring like so:

$$k := \mathbb{F}_p(a_1, b_1, a_2, b_2, \cdots)$$

$$R := k[[x,y]]$$

Define an $\mathbb{F}$-algebra endomorphism σ of R as follows:

$$\sigma : \begin{cases} x & \mapsto & x, \\ y & \mapsto & y, \\ a_n & \mapsto & a_n + yP_{n+1}, \\ b_n & \mapsto & b_n - xP_{n+1}. \end{cases}$$

where $P_n = a_n y - b_n x$.

Then R^G is not Noetherian under the group action of $\langle \sigma \rangle$.

Lemma 5.1. *The group $\langle \sigma \rangle$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}$ when $p > 0$, otherwise it is isomorphic to $\mathbb{Z}$ when $p = 0$.*

Proof. First, we note that $\sigma(P_n) = P_n$.

$$\sigma(P_n) = \sigma(a_n x + b_n y)$$
$$= \sigma(a_n)\sigma(x) + \sigma(b_n)\sigma(y) = (a_n + yP_{n+1})x + (b_n - xP_{n+1})y$$
$$= a_n x + b_n y = P_n$$

Next, notice that $\sigma^2(a_n) = a_n$ and $\sigma^2(b_n) = b_n$. Indeed,

$$\sigma^2(a_n) = \sigma(a_n + yP_{n+1}) = \sigma(a_n) + \sigma(y)\sigma(P_n + 1)$$
$$= a_n + yP_{n+1} + yP_{n+1} = a_n + 2yP_{n+1}$$

We continue this process recursively, to get

$$\sigma^k(a_n) \;=\; a_n + kyP_{n+1}.$$

Similarly for b_n, we get

$$\sigma^k(b_n) \;=\; b_n - kxP_{n+1}. \quad \square$$

Thus the group G generated by σ has order p precisely if k has positive characteristic p, and is infinite cyclic otherwise.

It is important to note that σ is not an automorphism of $k[x,y]$, because $\sigma(\frac{1}{a_n}) = \frac{1}{a_n+yP_{n+1}}$ is not in $k[x,y]$. This element is in $k[\![x,y]\!]$ though since

$$\sigma(\frac{1}{a_n}) = \frac{1}{a_n}(\frac{1}{1-\frac{-yP_{n+1}}{a_n}}) = \frac{1}{a_n}\sum_i(\frac{-yP_{n+1}}{a_n})^i$$

this gives us an intuition that we need to choose the formal power series in two variables as the ring instead of just a polynomial ring.

Lemma 5.2. *Let $\lambda \in k$. Then $\sigma(\lambda) = \lambda \mod m^2$ where $m = (x,y)R$*

Proof. Let $\lambda \in k$ such that $\lambda \neq 0$. Note that $m^2 = (x^2, y^2, xy)$. Then

$$\begin{aligned}
\sigma(\lambda) &= \sigma(\sum_i c_i a_i + \sum_i d_i b_i) = \sum_i c_i \sigma(a_i) + \sum_i d_i \sigma(b_i) \\
&= \sum_i c_i (a_i + yP_{i+1}) + \sum_i d_i (b_i + xP_{i+1}) \\
&= \sum_i c_i a_i + \sum_i c_i y P_{i+1} + \sum_i d_i b_i + \sum_i d_i x P_{i+1} \\
&= \sum_i (c_i a_i + d_i b_i) + \sum_i (c_i a_{i+1} + d_i b_{i+1} xy + \sum (d_i a_i a_i x^2 + c_i b_i y^2) \\
&= \sum_i (c_i a_i + d_i b_i) \mod (x^2, y^2, xy) \quad \square
\end{aligned}$$

Corollary 5.2.1. *Let $r \in R$. We define $\bar{r}$ as the constant term of r. Then if $r \in R^G$ where $G =< \sigma >$ we have $\sigma(\bar{r}) = \bar{r} \mod (x^2, y^2)$.*

Proof. If $r \in R^G$, then $\sigma(r) = r$. Write r as $r = \bar{r} + \alpha x + \beta y + \gamma xy \mod (x^2, y^2)$ where $\bar{r}, \alpha, \beta, \gamma \in k$. Then

$$\begin{aligned}
\sigma(r) &= \sigma(\bar{r} + \alpha x + \beta y + \gamma xy) \mod (x^2, y^2) \\
&= \sigma(\bar{r}) + \sigma(\alpha)x + \sigma(\beta)y + \sigma(\gamma)xy \mod (x^2, y^2) \\
&= r = \bar{r} + \alpha x + \beta y + \gamma xy \mod (x^2, y^2)
\end{aligned}$$

since $\alpha \in k$, then $\sigma(\alpha) = \alpha \mod (x^2, y^2, xy)$. The same goes for β and γ, therefore we can conclude that $\sigma(\bar{r}) = \bar{r} \mod (x^2, y^2)$. $\square$

Proposition 5.1. *Let $R := k[\![x,y]\!]$ and $G :=< \sigma >$. Then the ring of invariants R^G is not Noetherian.*

Proof. If we take $P_k = a_n x + b_n y$, we can show that the chain of ideals in R^G

$$(P_1) \subsetneq (P_1, P_2) \subsetneq (P_1, P_2, P_3) \subsetneq \cdots \subsetneq (P_1, P_2, \cdots, P_n) \subsetneq \cdots$$

does not terminate by showing first that $P_{n+1} \notin (P_1, P_2, \cdots, P_n)R^G$.

Suppose, to the contrary, that there exists an integer n such that

$$P_{n+1} \ = \ \sum_{k=1}^{n} r_k P_k$$

where $r_k \in R^G$ for each k with $1 \leq k \leq n$. The above may be written as

$$a_{n+1}x + b_{n+1}y \ = \ \sum_{k=1}^{n} r_k (a_k x + b_k y),$$

so comparing the coefficients of x yields

$$a_{n+1} \ = \ \sum_{k=1}^{n} \bar{r}_k a_k. \tag{1}$$

Applying σ to the above gives

$$a_{n+1} + yP_{n+2} \ = \ \sum_{k=1}^{n} \sigma(\bar{r}_k)(a_k + yP_{k+1}),$$

i.e.,

$$a_{n+1} + a_{n+2}xy + b_{n+2}y^2 \;=\; \sum_{k=1}^{n} \sigma(\bar{r}_k)(a_k + a_{k+1}xy + b_{k+1}y^2).$$

Since $\sigma(\bar{r}_k) \equiv \bar{r}_k \mod (x^2, y^2)R$ for each k, one obtains

$$a_{n+1} + a_{n+2}xy \;\equiv\; \sum_{k=1}^{n} \bar{r}_k(a_k + a_{k+1}xy) \mod (x^2,\ y^2)R,$$

and, in light of (1),

$$a_{n+2}xy \;\equiv\; \sum_{k=1}^{n} \bar{r}_k a_{k+1}xy \mod (x^2,\ y^2)R.$$

But then

$$a_{n+2} \;=\; \sum_{k=1}^{n} \bar{r}_k a_{k+1}. \tag{2}$$

Repeating the argument that (1) implies (2) gives

$$a_{n+m+1} \;=\; \sum_{k=1}^{n} \bar{r}_k a_{k+m} \qquad \text{for each } m \geq 1.$$

Since $\bar{r}_1, \ldots, \bar{r}_n$ are finitely many elements of the field K, this contradicts the assumption that $a_1, a_2, \ldots$ are infinitely many elements algebraically independent over $\mathbb{F}$. $\square$

This indeed shows that R^G is non-Noetherian. A surprising result is that this counterexample applies to characteristic zero as well, except, in this case we would get an infinite cyclic group acting on the ring instead of a finite one.

6 References

[AM] M.F. Atiyah and I.G. MacDonald, *Introduction To Commutative Algebra*, Avalon Publishing, 1994.

[Ba] K. Baba, *An example of the non-Noetherian invariant subring of a Noetherian domain*, Res. Bull. Fac. Ed. Oita Univ. **12** (1990), 267–277.

[CL] S. L. Chuang and P. H. Lee, *Noetherian rings with involution*, Chinese J. Math. **5** (1977), 15–19.

[Fo] J. Fogarty, *Kähler differentials and Hilbert's fourteenth problem for finite groups*, Amer. J. Math. **102** (1980), 1159–1175.

[Hi] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313–373.

[Ko] J. Kollár, *Quotients by finite equivalence relations*, with an appendix by C. Raicu, in: *Current developments in algebraic geometry*, Math. Sci. Res. Inst. Publ. **59**, 227–256, Cambridge Univ. Press, Cambridge, 2012.

[Ku] S. Kuroda, *Hilbert's fourteenth problem and field modifications*, in: *Journal of Algebra*,pp. 93–105, 2020

[Mu] D. Mumford, *Hilbert's fourteenth problem — the finite generation of subrings such as rings of invariants*, in: *Mathematical developments arising from Hilbert problems*, Proc. Sympos. Pure Math. **28**, (Northern Illinois Univ. 1974), pp. 431–444, 1976.

[NaK] K. R. Nagarajan, *Groups acting on Noetherian rings*, Nieuw Arch. Wisk. (3) **16** (1968), 25–29.

[NaM] M. Nagata, *Some questions on rational actions of groups*, in: Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res. 1968) pp. 323–334 Oxford Univ. Press, London, 1969.

[No1] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1915), 89–92.

[No2] E. Noether, *Der Endlichkeitsatz der Invarianten endlicher linearer Gruppen der Charakteristik p*, Nachr. Ges. Wiss Göttingen (1926), 28–35.

[Za] O. Zariski, *Interpretations algebrico-geometriques du quatorzieme probleme de Hilbert*, Bulletin des Sciences Mathematiques **78**, (1954), pp. 155–168.

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