Noetherian Property of Invariant Rings

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Motivation

If we have a group G acting on a ring R by ring automorphisms which means that there is a group homomorphism

$$G \to AutR$$

then the invariant ring is

$$R^G := \{r \in R \ | \ gr = r \text{ for every } g \in G\}$$

Question: if a group G acts on a Noetherian ring R, then is the invariant ring R^G also going to be Noetherian?

This is a version of Hilbert's Fourteenth Problem which leads to many interesting results. The answer is yes in cases. As an example, Emmy Noether proved in 1926 that if A is a finitely generated algebra over a field k and a finite group G acts k-linearly on A, then the invariant ring A^G is Noetherian. However in general, the statement is not true. We construct a class of Noetherian rings of characteristic p, for each prime integer p, with finite group actions such that the ring of invariants is not Noetherian. This class of rings is extended from a characteristic two counterexample due to Nagarajan from 1968

Noetherian Invariant Rings

Let G be a finite group and let R be a Noetherian ring which contains $\mathbb{Q}$. Then if G acts on R, the invariant ring R^G is also Noetherian. Reynold's Operator is defined as the map $\rho : R \to R^G$ such that

$$\rho(r) := \frac{1}{|G|}\sum_{g \in G} g(r)$$

Now we will show that R^G is Noetherian. Let

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$$

be an ascending chain of ideals of R^G. Then let us look at the ascending chain $I_1R \subseteq I_2R \subseteq I_3R \subseteq \cdots$ in R. Since R is Noetherian, then $I_kR = I_{k+1}R$ for all $k \geq N$ for some $N > 0$. We will show that $I_kR \cap R^G = I_k$ for all k. It is clear that

$$I_k \subseteq I_kR \cap R^G,$$

therefore the only thing left to show is that $I_kR \cap R^G \subseteq I_k$. Let $x \in I_kR \cap R^G$. Then $x = \sum_{i=1}^m a_ir_i$ where $a_i \in I_k, r_i \in R$ and $\rho(x) = x$.

$$\rho(\sum_{i=1}^m a_ir_i) = \sum_{i=1}^m a_ir_i = \sum_{i=1}^m a_i\rho(r_i).$$

Since $a_i \in I_k$ and $\rho(r_i) \in R^G$, then $\sum_{i=1}^m a_i\rho(r_i) \in I_k$ by the property of I_k being an ideal. Then $x \in I_k$ and therefore indeed $I_kR \cap R^G \subseteq I_k$. Now that we know that $I_kR \cap R^G = I_k$, then the chain

$$I_1R \subseteq I_2R \subseteq I_3R \subseteq \cdots \subseteq I_NR \subseteq I_NR \subseteq \cdots$$

can be written as a chain in R^G like so:

$$I_1R \cap R^G \subseteq I_2R \cap R^G \subseteq I_3R \cap R^G \subseteq \cdots \subseteq I_NR \cap R^G \subseteq I_NR \cap R^G \subseteq \cdots$$

Thus the above chain is equivalent to the ascending chain

$$I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_N \subseteq I_N \subseteq \cdots$$

showing that the chain in R^G stabilizes, therefore R^G is Noetherian.

Positive Characteristic Counterexample

Let p be a prime integer.

$$k := \mathbb{F}_p(a_1, b_1, a_2, b_2, \cdots)$$
$$R := k[\![x, y]\!]$$

We define an automorphism $\sigma : R \to R$ where

- $\sigma(x) = x$
- $\sigma(y) = y$
- $\sigma(a_n) = a_n + yP_{n+1}$
- $\sigma(b_n) = b_n - xP_{n+1}$

$$P_n := a_nx - b_ny$$

Then R^G is Noetherian under the group action of $\langle\sigma\rangle$. σ generates a finite group from these properties:

- $\sigma(P_n) = P_n$
- $\sigma^p(a_n) = a_n$
- $\sigma^p(b_n) = b_n$

$$\langle\sigma\rangle \cong \mathbb{Z}/p\mathbb{Z}$$

Let $R := k[\![x, y]\!]$ and $G := <\sigma>$. Then the ring of invariants R^G is not Noetherian.

If we take $P_k = a_kx + b_ky$, we can show that the chain of ideals in R^G

$$(P_1) \subsetneq (P_1, P_2) \subsetneq (P_1, P_2, P_3) \subsetneq \cdots \subsetneq (P_1, P_2, \cdots, P_n) \subsetneq \cdots$$

does not terminate by showing first that $P_{n+1} \notin (P_1, P_2, \cdots, P_n)R^G$.

Suppose, to the contrary, that there exists an integer n such that

$$\begin{aligned} P_{n+1} &= \sum_{k=1}^n r_kP_k \\ a_{n+1}x + b_{n+1}y &= \sum_{k=1}^n r_k(a_kx + b_ky), \\ a_{n+1} &= \sum_{k=1}^n \bar{r}_ka_k \end{aligned}$$

Applying σ to the above gives

$$a_{n+1} + a_{n+2}xy + b_{n+2}y^2 = \sum_{k=1}^n \sigma(\bar{r}_k)(a_k + a_{k+1}xy + b_{k+1}y^2).$$

Since $\sigma(\bar{r_k}) \equiv \bar{r}_k \mod (x^2, y^2)R$ for each k, one obtains

$$\begin{aligned} a_{n+1} + a_{n+2}xy &\equiv \sum_{k=1}^n \bar{r}_k(a_k + a_{k+1}xy) \mod (x^2, \ y^2)R, \\ a_{n+2}xy &\equiv \sum_{k=1}^n \bar{r}_ka_{k+1}xy \mod (x^2, \ y^2)R. \end{aligned}$$

But then

$$a_{n+2} = \sum_{k=1}^n \bar{r}_ka_{k+1}. \tag{1}$$

implies that

$$a_{n+m+1} = \sum_{k=1}^n \bar{r}_ka_{k+m} \qquad \text{for each } m \geq 1.$$

Since $\bar{r}_1, \ldots, \bar{r}_n$ are finitely many elements of the field K, this contradicts the assumption that $a_1, a_2, \ldots$ are infinitely many elements algebraically independent over $\mathbb{F}$.

Remarks

It is important to note that σ is not an automorphism of $k[x, y]$, because $\sigma(\frac{1}{a_n}) = \frac{1}{a_n + yP_{n+1}}$ is not in $k[x, y]$. this gives us an intuition that we need to choose the formal power series in two variables as the ring instead of just a polynomial ring.

The group G generated by σ has order p precisely if k has positive characteristic p, and is infinite cyclic otherwise.One of the most interesting aspects of exploring this class of counterexamples is the fact that characteristic 0 also applies, however in that case the group is infinite.

The reason why the proof using Reynold's operator works is specifically because it depends on the ring containing a characteristic zero field, because that involves the inverse of the order of the given finite group. However, we cannot use the same method when the characteristic of the given ring divides the order of the group, as seen here. Additionally, one can also construct a similar example that results in a Noetherian invariant ring for a characteristic $p > 0$ ring, as long as the group acting on it has finite order that cannot be divided by the characteristic of the ring.

Acknowledgements

Working on this project with Professor Anurag Singh has been an incredible honor.

References

References

[AM] M.F. Atiyah and I.G. MacDonald, *Introduction To Commutative Algebra*, Avalon Publishing, 1994.

[Ba] K. Baba, *An example of the non-Noetherian invariant subring of a Noetherian domain*, Res. Bull. Fac. Ed. Oita Univ. **12** (1990), 267–277.

[CL] S. L. Chuang and P. H. Lee, *Noetherian rings with involution*, Chinese J. Math. **5** (1977), 15–19.

[Hi] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), 313–373.

[Mu] D. Mumford, *Hilbert's fourteenth problem — the finite generation of subrings such as rings of invariants*, in: *Mathematical developments arising from Hilbert problems*, Proc. Sympos. Pure Math. **28**, (Northern Illinois Univ. 1974), pp. 431–444, 1976.

[NaK] K. R. Nagarajan, *Groups acting on Noetherian rings*, Nieuw Arch. Wisk. (3) **16** (1968), 25–29.

[NaM] M. Nagata, *Some questions on rational actions of groups*, in: Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res. 1968) pp. 323–334 Oxford Univ. Press, London, 1969.

[No1] E. Noether, *Der Endlichkeitssatz der Invarianten endlicher Gruppen*, Math. Ann. **77** (1915), 89–92.

[No2] E. Noether, *Der Endlichkeitsatz der Invarianten endlicher linearer Gruppen der Charakteristik p*, Nachr. Ges. Wiss Göttingen (1926), 28–35.