Exploring finite representations of $sl(2, \mathbb{C})$

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1 Lie algebras and its representations

Definition 1.1. A Lie algebra (not necessarily associative) is a vector space over a field k with a bilinear product $[x, y]$ such that the product satisfies the next two conditions:

- Alternativity:
  $[x, x] = 0$ for any $x \in \mathfrak{g}$
- Jacobi identity:
  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for any $x, y, z \in \mathfrak{g}$

*Alternativity condition is equivalent to $[x, y] = -[y, x]$ for any $x, y \in \mathfrak{g}$
For any algebra $\mathfrak{g}$ we have a linear map $ad : \mathfrak{g} \to End_k \mathfrak{g}$ given by

$$(ad\ x)(y) = [x, y]$$

Definition 1.2. Let V be a vector space over field $\mathbb{K}$. A representation of Lie algebra $\mathfrak{g}$ on V is a homomorphism of Lie algebras $\pi : \mathfrak{g} \to (End_{\mathbb{K}} V)^k$ which we can simply write as

$$\pi : \mathfrak{g} \to End_{\mathbb{K}} V$$

The way the bracket is defined in $End_{\mathbb{K}} V$ makes π k-linear and satisfies:

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

2 $\mathfrak{sl}(2, \mathbb{C})$

$\mathfrak{sl}(n, \mathbb{F})$ is the special linear Lie algebra of order n over a field $\mathbb{F}$. It is comprised of $n \times n$ matrices with trace zero and with the Lie bracket defined like so:

$$[X, Y] = XY - YX$$

We are going to be talking about the scenario when $n = 2$ and $\mathbb{F} = \mathbb{C}$. We can describe elements of $\mathfrak{sl}(2, \mathbb{C})$ explicitly like so:

$$\mathfrak{sl}(2, \mathbb{C}) = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a, b, c \in \mathbb{C} \}$$

It is a 3 dimensional complex Lie algebra which has the basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

This is the best basis to work with because of the convenient relations between them:

$$[h, e] = 2e$$
$$[h, f] = -2f$$
$$[e, f] = h$$

2.1 Irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ on finite-dimensional vector spaces

Definition 2.1. An invariant subspace for a complex-linear representation of $sl(2, \mathbb{C})$ on a finite-dimensional vector space V is a complex vector subspace U such that $\phi(X)U \subset U$ for all $X \in sl(2, \mathbb{C})$.

Definition 2.2. We say that a representation on a nonzero space V is irreducible if the only invariant subspaces are 0 and V itself.

Now that we have established necessary definitions, we can begin to show the irreducible representations explicitly.

Theorem 2.1. *For each integer $m \geq 1$ there exists up to equivalence a unique irreducible complex-linear representation π of $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space V of dimension m. In V there is a basis $v_0, v_1, \cdots, v_m$ such that:*

2.2 Representations of $\mathfrak{sl}(2, \mathbb{C})$

Definition 2.3. Let ϕ be a complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$ on a finite-dimensional complex vector space V. V is completely reducible if we can find invariant subspaces $V_1, V_2, \cdots V_r$ of V such that

$$V = V_1 \oplus V_2 \oplus \cdots V_r$$

and such that the restriction of the representation to each V_i is irreducible.

Theorem 2.2. *If V is a finite-dimensional complex vector space that has a complex-linear representation of $\mathfrak{sl}(2, \mathbb{C})$, then V is completely reducible.*

3 su(2)

$su(2) \otimes \mathbb{C} \cong sl(2, \mathbb{C})$

3.1 su(2)?

3.2 Representations of su(2)

v with $\dim V = m$ $\quad \{v_0, \dots, v_{m-1}\}$ $\quad$ e, h, f

$$\pi(h) v_j = (\lambda + 2j) v_j$$
$$\pi(e) v_j = v_{j+1}$$
$$\pi(f) v_0 = 0$$
$$\pi(f) v_j = j(-\lambda - j + 1) v_{j-1}$$

π is irrep of $sl(2, \mathbb{C})$

$\pi(h) : V \to V$

v so $\pi(h) v = \lambda v$

$\pi(e) v, \pi(e)^2 v, \dots$

$$\pi([h,e]) = \pi(h)\pi(e) - \pi(e)\pi(h) \qquad 2e$$

$$\pi(h)\pi(e) v = \pi(e)\underbrace{\pi(h) v}_{\lambda v} + \pi([h,e]) v =$$
$$= (\lambda + 2)\pi(e) v$$

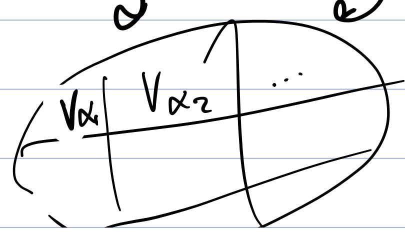
e raising $\quad$ f lower

$\lambda \to \lambda + 2 \quad \lambda \to \lambda - 2$

$\pi(e) v_j = v_{j+1}$

$$\begin{tikzcd}
v_{-1}=0 & v_0 \arrow[r, bend left, "e"] \arrow[l, bend left, "f"] \arrow[loop below, "h"] & v_1 \arrow[r, bend left, "e"] \arrow[l, bend left, "f"] \arrow[loop below, "h"] & v \arrow[r, bend left] \arrow[l, bend left, "f"] \arrow[loop below, "h"] & \cdots \arrow[r, bend left] \arrow[l, bend left, "f"] \arrow[loop below, "h"] & v_{m-1} \arrow[r, bend left] \arrow[l, bend left, "f"] \arrow[loop below, "h"] & v_m = 0
\end{tikzcd}$$

$$V = \bigoplus_\alpha V_\alpha$$



$\tilde{V}$ is completely reducible