Exploring finite representations of $sl(2, \mathbb{C})$

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1 Lie algebras and its representations

Definition 1.1. A Lie algebra is a vector space over a field k with a bilinear product $[x, y]$ such that the product satisfies the next two conditions:

- Alternativity:
  $[x, x] = 0$ for any $x \in \mathfrak{g}$
- Jacobi identity:
  $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ for any $x, y, z \in \mathfrak{g}$

*Alternativity condition is equivalent to $[x, y] = -[y, x]$ for any $x, y \in \mathfrak{g}$
For any algebra $\mathfrak{g}$ we have a linear map $ad : \mathfrak{g} \to End_k\mathfrak{g}$ given by

$$(ad\ x)(y) = [x, y]$$

Definition 1.2. Let V be a vector space over field $\mathbb{K}$. A representation of Lie algebra $\mathfrak{g}$ on V is a homomorphism of Lie algebras $\pi : \mathfrak{g} \to (End_{\mathbb{K}}V)^k$ which we can simply write as

$$\pi : \mathfrak{g} \to End_{\mathbb{K}}V$$

The way the bracket is defined in $End_{\mathbb{K}}V$ makes π k-linear and satisfies:

$$\pi([X, Y]) = \pi(X)\pi(Y) - \pi(Y)\pi(X)$$

2 $\mathfrak{sl}(2, \mathbb{C})$

$\mathfrak{sl}(n, \mathbb{F})$ is the special linear Lie algebra of order n over a field $\mathbb{F}$. It is comprised of $n \times n$ matrices with trace zero and with the Lie bracket defined like so:

$$[X, Y] = XY - YX$$

We are going to be talking about the scenario when $n = 2$ and $\mathbb{F} = \mathbb{C}$. We can describe elements of $\mathfrak{sl}(2, \mathbb{C})$ explicitly like so:

$$\mathfrak{sl}(2, \mathbb{C}) = \{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} | a, b, c \in \mathbb{C} \}$$

It is a 3 dimensional complex Lie algebra which has the basis:

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

This is the best basis to work with because of the convenient relations between them:

$$[h, e] = 2e$$
$$[h, f] = -2f$$
$$[e, f] = h$$

2.1 Irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$ on finite-dimensional vector spaces

Definition 2.1. An invariant subspace for a complex-linear representation of $sl(2, \mathbb{C})$ on a finite-dimensional vector space V is a complex vector subspace U such that $\phi(X)U \subset U$ for all $X \in sl(2, \mathbb{C})$.

Definition 2.2. We say that a representation on a nonzero space V is irreducible if the only invariant subspaces are 0 and V itself.

Now that we have established necessary definitions, we can begin to show the irreducible representations explicitly.

Theorem 2.1. *For each integer $m \geq 1$ there exists up to equivalence a unique irreducible complex-linear representation π of $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space V of dimension m. In V there is a basis $v_0, v_1, \cdots, v_m$ such that:*

- $\pi(h)v_j = (m - 1 - 2j)v_j$
- $\pi(f)v_j = v_{j-1}$ *with* $\pi(f)v_{-1} = 0$
- $\pi(e)v_j = j(m - 1 + i - 1)$ *with* $\pi(e)v_m = 0$

proof:

CONSTRUCTION AND UNIQUENESS:
Let π be a complex-linear irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ on a complex vector space V with dimension m. Since V is a complex vector space and the complex numbers is a closed field, then we can find a nonzero eigenvector v of $\pi(h)$ with eigenvalue λ, which by definition means that $\pi(h)v = \lambda v$. We can show that the vectors $\pi(e)v, \pi(e)^2v, \cdots$ are also eigenvectors of $\pi(h)$, by first using the fact that π is a representation and therefore $\pi([h, e]) = \pi(h)\pi(e) - \pi(e)\pi(h)$. Then we have:

$$\pi(h)\pi(e)v = \pi(e)\pi(h)v + \pi([h, e])v$$

Then we can use the fact that $\pi(h)v = \lambda v$ and the relations of the basis elements gives us $[h, e] = 2e$, therefore we have:

$$\pi(h)\pi(e)v = \pi(e)\lambda v + \pi(2e)v$$

$$\pi(h)\pi(e)v = (\lambda + 2)\pi(e)v$$

With a similar process, we can show the rest of the vectors $\pi(e)^2, \pi(e)^3, \cdots$ will be eigenvectors with eigenvalues $\lambda + 4, \lambda + 6, \cdots$, respectively. It is obvious that the eigenvalues we get are distinct and therefore the eigenvectors are independent.

By using the same method as above, using $\pi(f)$ instead of $\pi(e)$ will also give us new eigenvectors $\pi(f)v, \pi(f)^2 v, \cdots$ if we use the same eigenvector v of $\pi(h)$ with eigenvalue λ, except the eigenvalues will decrease by 2 in this case. So we will have $\pi(h)\pi(f)v = (\lambda - 2)\pi(f)v$.

We can see that we are essentially moving around in different eigenspaces. This is why the basis element e is called a raising operator, while f is called a lowering operator. In general, these are ladder operators. This is important to address since we are essentially going to be looking at V as the direct sum of the eigenspaces like so:

$$V = \bigoplus_{\alpha} V_\alpha$$

Since V is finite-dimensional, we can find a vector from V, let's call it v_0 for convenience, such that:

1. $v_0 \neq 0$
2. $\pi(h)v_0 = \lambda v_0$
3. $\pi(f)v_0 = 0$

Let us define[1] $v_j = \pi(e)^{j+1}v_0$. Since we know that using $\pi(e)$ increases the eigenvalue by 2, we can write it as

$$\pi(h)v_j = (\lambda + 2j)v_j$$

There is a minimum integer N such that $\pi(e)^{N+1}v_0 = 0$. By using the finite-dimensionality argument, $\pi(h)$ as a map of V of dimension m has at most m distinct eigenvalues. Then the vectors $v_0, v_1, \cdots, v_N$ (basically the list $v_0, ev_0, e^2v_0, \cdots, f^N v_0$) are independent and

- $\pi(h)v_j = (\lambda + 2j)v_j$
- $\pi(e)v_j = v_{j+1}$
- $\pi(f)v_0 = 0$

Now we can show that these list of vectors span V itself. For this, it is enough to show that the list of vectors is stable (closed under) $\pi(f)$. In fact, we can show that:

$$\pi(e)v_j = j(\lambda - j + 1)v_{j-1} \text{ with } v_{-1} = 0$$

We can prove this using induction, with the base case $j = m-1$ being $\pi(e)v_{m-1} = 0$. We assume that it is true for j and we are now going to prove it for v_{j+1}.

$$\pi(f)v_{j+1} = \pi(f)\pi(e)v_j = \pi([f,e])v_j + \pi(e)\pi(f)v_j =$$

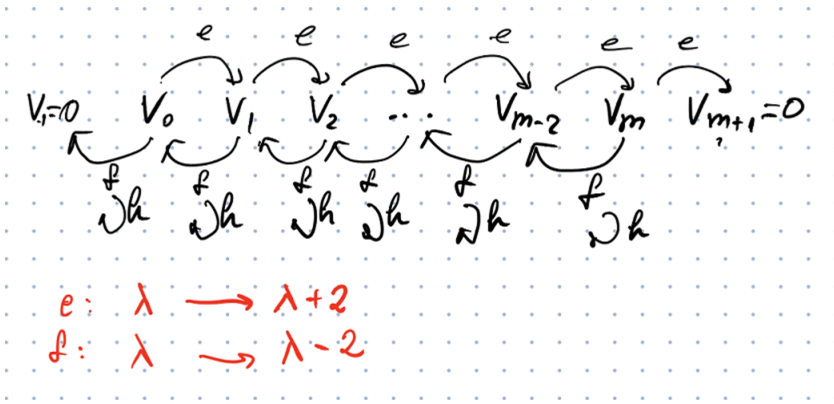
$$\pi(-h)v_j + \pi(e)\pi(f)v_j = -\pi(h)v_j + \pi(e)(j(-\lambda - j + 1)v_{j-1}) =$$

$$-(\lambda + 2j)v_j - (j(-\lambda - j + 1)\pi(e)v_{j-1} = -(\lambda + 2j)v_j - (j(-\lambda - j + 1)v_j$$

$$(-(\lambda + 2j) - (j(-\lambda - j + 1))v_j = (-\ ambda - j - j + j(-\lambda - j) + j)v_j =$$

$$(-\lambda - j + j(\lambda - j))v_j = (-\lambda - j)(1 + j)v_j$$

[1] Knapp's book actually uses $\pi(f)$ to cycle through the vectors, but this way it makes more sense for the picture.



To finish showing uniqueness, we show that $\lambda = m - 1$. We have $Tr\pi(h) = Tr(\pi(e)\pi(f) - \pi(f)\pi(e)) = 0$ therefore we will have:

$$\sum_{j=0}^{m}(\lambda - 2j) = 0$$

and we find that $\lambda = m$. This is really interesting, because this means the eigenvalues are integers!

EXISTENCE:
We define $\pi(h)$,$\pi(e)$, and $\pi(f)$ by the given the conditions given above and extend it linearly, by doing easy computation it is easy to see that:

$$\pi([h,e]) = \pi(h)\pi(e) - \pi(e)\pi(h)$$
$$\pi([h,f]) = \pi(h)\pi(f) - \pi(f)\pi(h)$$
$$\pi([e,f]) = \pi(e)\pi(f) - \pi(f)\pi(e)$$

This proves that π is a representation. To show that it is irreducible, let U be a nonzero invariant subspace. Since U is invariant under $\pi(h)$,U is spanned by a subset of the basis vectors $v_0, v_1, \cdots, v_{m-1}$. Let's take a v_k from the basis which is in U and let's apply $\pi(f)$ several times, we will see that V_0 is also in U. Repeated application of $\pi(e)$ then shows that $U = V$. Therefore by definition π is irreducible. ■

2.2 Representations of $\mathfrak{sl}(2,\mathbb{C})$

Definition 2.3. Let ϕ be a complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$ on a finite-dimensional complex vector space V. V is completely reducible if we can find invariant subspaces $V_1, V_2, \cdots V_r$ of V such that

$$V = V_1 \oplus V_2 \oplus \cdots V_r$$

and such that the restriction of the representation to each V_i is irreducible.

Theorem 2.2. *If V is a finite-dimensional complex vector space that has a complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$, then V is completely reducible.*

To prove this theorem we will require 4 lemmas.

Lemma 2.3. *If π is a representation of $\mathfrak{sl}(2,\mathbb{C})$, then $Z = \frac{1}{2}\pi(h)^2 + \pi(h) + 2\pi(f)\pi(e)$ commutes with each $\pi(X)$ for $X \in \mathfrak{sl}(2,\mathbb{C})$.*

Lemma 2.4 (Schur's Lemma)**.** *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$. If $\pi : \mathfrak{g} \to ENDV$ and $\pi' : \mathfrak{g} \to EndV'$ are irreducible finite-dimensional representations and if $L : V \to V'$ is a linear map such that $L\pi(X) = \pi'(X)L$ for all $X \in \mathfrak{g}$, then $L = 0$ or L is invertible. If $Z : V \to V$ is a linear map such that $Z\pi(X) = \pi(X)Z$ for all $X \in \mathfrak{g}$, then Z is scalar.*

Lemma 2.5. *If pi is an irreducible representation of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $n+1$, then the operator Z of Lemma 2.3 acts as the scalar $\frac{1}{2}n^2+n$, which is not 0 unless π is trivial.*

Lemma 2.6. *Let $\pi : \mathfrak{sl}(2,\mathbb{C}) \to EndV$ be a finite-dimensional representation, and let $U \subset V$ be an invariant subspace of codimension 1. Then there is a 1-dimensional invariant subspace W such that $V = U \oplus W$.*

So now that we have stated the lemmas, we can prove Theorem 2.2.
Let π be a representation of $\mathfrak{sl}(2,\mathbb{C})$ on M, and let $N \neq 0$ be an invariant subspace. Put

$$V = \{Y \in EndM | Y : M \to N \text{and} Y|_N \text{is a scalar}\}$$

Using linear algebra we can see that $V \neq 0$. Define a linear function $\sigma : \mathfrak{sl}(2,\mathbb{C}) \to End(EndM)$ by

$$\sigma(X)\gamma = \pi(X)\gamma - \gamma\pi(X)$$

for $\gamma \in EndM$ and $X \in \mathfrak{sl}(2,\mathbb{C})$.
We can check directly that $\sigma[X,Y] = \sigma(X)\sigma(Y) - \sigma(Y)\sigma(X)$, therefore σ is a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $EndM$.
So we can show that V as a subspace of $EndM$ is an invariant subspace under σ. In fact, let $\gamma(M) \subset N$ and $\gamma|_N = \lambda 1$. In the right side of the expression

$$\sigma(X)\gamma = \pi(X)\gamma - \gamma\pi(X)$$

the first term carries M to N since γ carries M to N and $\pi(X)$ carries N to N, and the second term carries M into N since $\pi(x)$ takes M to M and γ takes M to N. Thus $\sigma(X)\gamma$ takes M into N. On N, the action $\sigma(X)\gamma$ is given by

$$\sigma(X)\gamma(n) = \pi(X)\gamma(n) - \gamma\pi(X)(n) = \lambda\pi(X)(n) - \lambda\pi(X)(n) = 0$$

Thus V is an invariant subspace.
So, the argument above shows that the subspace U of V given by

$$U = \{\gamma \in V | \gamma = 0 \text{on} N\}$$

is an invariant subspace. So, it's easy to see that $dimV/U = 1$. By Lemma 2.6, $V = U \oplus W$ for a 1-dimensional invariant subspace $W = \mathbb{C}\gamma$, where γ is a nonzero scalar $\lambda 1$ on N. The invariance of W means that $\sigma(X)\gamma = 0$ since 1-dimensional representations are 0. Therefore γ commutes with $\pi(X)$ for all $X \in \mathfrak{sl}(2,\mathbb{C})$. But then the kernel of γ is a nonzero invariant subspace of M. Since γ is a nonsingular on N(being a nonzero scalar there), we must have $M = N \oplus ker\gamma$. ■

Corollary 2.6.1. *LIf π is a complex-linear representation of $\mathfrak{sl}(2,\mathbb{C})$ on a finite-dimensional complex vector space V, then $\pi(h)$ is diagonable, all of its eigenvalues are integers, and the multiplicity of an eigenvalue k equals the multiplicity of $-k$.*

Corollary 2.6.2. *If ϕ is a complex-linear representation(not necessarily an irredusible one) of $\mathfrak{sl}(2,\mathbb{C})$ on a finite-dimensional complex vector space V, and if each vector $v \in V$ is in an finite-dimensional invariant subspace of V, then V is the direct sum of finite-dimensional invariant subspaces on which $\mathfrak{sl}(2,\mathbb{C})$ acts irreducibly.*