TAYLOR RESOLUTIONS (EXPOSITORY)

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Motivation

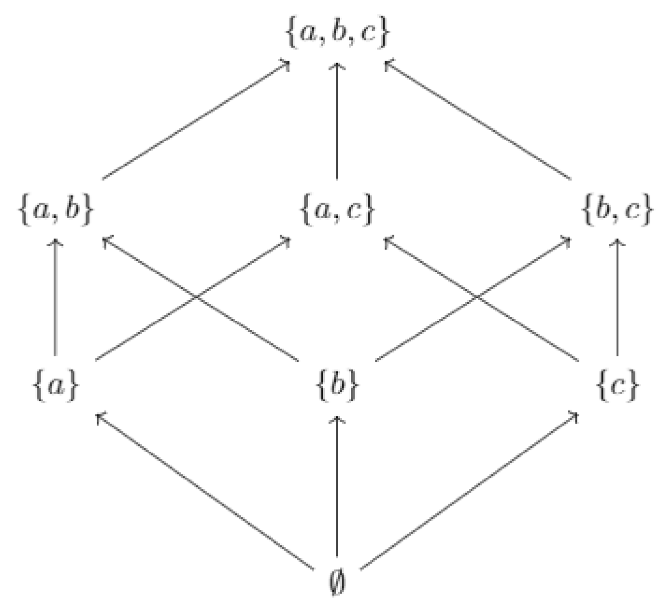
The study of (minimal) free resolutions is an important area in commutative algebra because free resolutions provide a method for describing the structure of modules. By introducing simplicial complexes in the context of free resolutions of monomial ideals, we get a combinatorial perspective on the topic. We can construct important simplicial resolutions that resolve monomial ideals. One such resolution is the Taylor resolution. We will also talk about some corollaries related to Betti numbers and regularity of monomial ideals.

The Taylor resolution is usually highly non-minimal. It is useful because of its simple structure, which is similar to that of the Koszul complex. Taylor's resolution was first constructed by Taylor in her Ph.D Thesis.

The following theorem illustrates an application of Taylor resolutions.

Simplicial Complexes

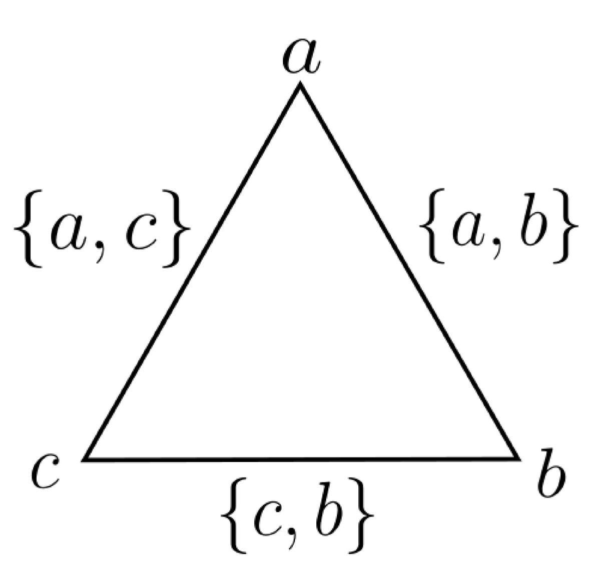
From a combinatorial standpoint, we will define simplicial complexes, an important topic in algebraic topology. We define terminology through an example. Suppose we are given a set of monomials M, say $\{a, b, c\} \subset k[a, b, c]$ for a field k. Each element is called a **vertex** (see figure further below). We will say that a **simplex** on M is the power set of M (see first image below), denoted Δ_M.



A simplicial complex of M is a subset of Δ_M that is closed under taking subsets. In our example below, we are looking at the set

$$\{\{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \emptyset\}.$$

Given a simplicial complex of M, each element is called a **face**, and any subset from it obtained by removing a single element is called a **facet**. The **order** of a face F is its order as a subset. In the figure below, $\{a, b\}$ is a face, and a is one of its facets.



Constructing Taylor Resolution through an example

Let $I = (a, b^2, c^3)$. Then we will construct its Taylor resolution with these steps. First let us list all of the faces based on their dimension using the given generating monomials of ideal I: $\{F : |F| = 0\} = a, b^2, c^3$, $\{F : |F| = 1\} = b^2c^3, ac^3, ab^2$,$\{F : |F| = 2\} = ab^2c^3$

Then the Taylor resolution of I will look like this:

$$0 \to S \xrightarrow{\phi_2} S^3 \xrightarrow{\phi_1} S^3 \xrightarrow{\phi_0} S \to S/I \to 0$$

Now we can define the maps in the given resolution:

$$\phi_0 : S[a] \oplus S[b^2] \oplus S[c^3] \to S[\emptyset]$$

We will define the map for each component essentially and keep track of the monomial components using the $[G] notation$. Here we have 3 faces $\{a\}, \{b^2\}, \{c^2\}$:

$\phi_0([a]) = 1 \cdot \frac{lcm(a)}{lcm(\emptyset)}[\emptyset] = a[\emptyset]$, $\phi_0([b^2]) = 1 \cdot \frac{lcm(b^2)}{lcm(\emptyset)}[\emptyset] = b^2[\emptyset]$, $\phi_0([[c^3]]) = 1 \cdot \frac{lcm(c^3)}{lcm(\emptyset}[\emptyset] = c^3[\emptyset]$

Therefore if we had an element $(f_1, f_2, f_3) \in S[a] \oplus S[b^2] \oplus S[c^3]$, then

$$\phi_0(f_1, f_2, f_3) = f_1 \cdot a + f_2 \cdot b^2 + f_3 \cdot c^3$$

We can represent ϕ_0 as a matrix multiplication by $\begin{pmatrix} a & b^2 & c^3 \end{pmatrix}$ Now we calculate

$$\phi_1 : S[b^2, c^3] \oplus S[a, c^3] \oplus S[a.b^2] \to S[a] \oplus S[b^2] \oplus S[c^3]$$

Notice that it is extremely important to keep track of the order of the monomials, which means that we can internally number the subsets of the given faces when choosing which elements to remove in order. As an example if we take the face b^2, c^3, then we can number b^2 to be 1 and c^3 to be 2, so this tells us what sign to give to the coefficient in the sum. This is why we choose to remove b^2 first and the sign is positive and when we remove c^3, then we get a negative sign so we multiply by -1 as you see below.

$$\phi_1([b^2, c^3]) = 1 \cdot \frac{lcm(b^2, c^3)}{lcm(c^3)}[c^3] + (-1)\frac{lcm(b^2, c^3}{lcm(b^2)}[b^2] =, b^2[c^3] + (-c^3)[b^2]$$

$$\phi_1([a, c^3]) = 1 \cdot \frac{lcm(a, c^3)}{lcm(c^3)}[c^3] + (-1)\frac{lcm(a, c^3)}{lcm(a)}[a] = a[c^3] + (-c^3)[a]$$

$$\phi_1([a.b^2]) = 1 \cdot \frac{lcm(a, b^2)}{lcm(b^2)}[b^2] + (-1)\frac{a, b^2}{a}[a] = a[b^2] + (-1)b^2[a]$$

From each face calculation, we can view it as columns of a matrix which represent the map ϕ_1. This matrix is:

$$\begin{pmatrix} 0 & -c^3 & -b^2 \\ -c^3 & 0 & a \\ b^2 & a & 0 \end{pmatrix}$$

The final map left to calculate is

$$\phi_2 : S[a, b^2, c^3] \to S[b^2, c^3] \oplus S[a, c^3] \oplus S[a.b^2]$$

Similarly to before

$$\phi([a, b^2, c^3]) = 1 \cdot \frac{a, b^2, c^3}{lcm(b^2, c^3)}[b^2, c^3] + \frac{a, b^2, c^3}{lcm(a, c^3)}[a, c^3] + 1 \cdot \frac{lcm(a, b^2, c^3)}{lcm(a, b^2)}[a, b^2]$$

$$\phi([a, b^2, c^3]) = a[b^2, c^3] + (-b^2)[a, c^3] + c^3[a, b^2]$$

ϕ_2 can be represented as multiplication by the matrix $\begin{pmatrix} a \\ -b^2 \\ c^3 \end{pmatrix}$

Checking that Resolution is minimal

One can check if a given resolution is minimal by the following theorem:

Theorem. *Let I be a homogeneous ideal of a polynomial ring $S = k[x_1, \ldots, x_n]$, and let $\mathbb{F}$ be a resolution of S/I,*

$$\ldots \to F_{i+1} \xrightarrow{\phi_i} F_i \xrightarrow{\phi_{i-1}} F_{i-1} \to \ldots \to F_1 \xrightarrow{\phi_0} F_0 \to \frac{S}{I} \to 0$$

Let $m = (x_1, \ldots, x_n)$ be the maximal ideal. Then $\mathbb{F}$ is minimal if and only if $\phi_i(F_{i+1}) \subset mF_i$ for all i.

We can use this to see that the given resolution above is indeed minimal.

Example of non-minimal Taylor resolution

Most Taylor resolutions are in fact non-minimal. The next example is a good demonstration of that fact. Consider polynomial ring $k[a, b, c]$, and let $I = (a^2, ab, b^3)$. Then the Taylor resolution of I is

$$0 \to T_3 \xrightarrow{\begin{pmatrix} a \\ -1 \\ b^2 \end{pmatrix}} T_2 \xrightarrow{\begin{pmatrix} 0 & -b^3 & -b \\ -b^2 & 0 & a \\ a^2 & a & 0 \end{pmatrix}} T_1 \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} T_0 \to S/I \to 0$$

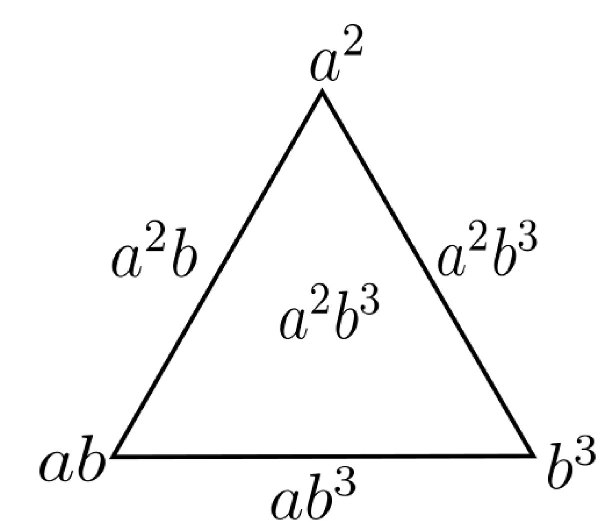
Where

$T_0 := S[\emptyset]$,

$T_1 := S[a^2] \oplus S[ab] \oplus S[b^3]$,

$T_2 := S[ab, b^3] \oplus S[a^2, b^3] \oplus S[a^2, ab]$,

$T_3 := S[a^2, ab, b^3]$



Acknowledgements

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References

[1] Jeff Mermin. *Three simplicial resolutions.* 2011. arXiv: 1102.5062 [math.AC].

[2] Irena Peeva. *Graded syzygies.* Springer, 2011.