Assignment 7 Justin Kim ①

1. "All birds can fly!"

Penguin is a bird. Penguin cant fly. So above statement is false.

2. $(\forall x, y \in \mathbb{R})[(x-y)^2 > 0]$

$0 \in \mathbb{R}$, so let $x = 0 \wedge y = 0$.

Then $(0-0)^2 = 0^2 = 0 \not> 0$

So above statement is false.

3] Prove: $(\forall n_1 \in \mathbb{Q})(\forall n_2 \in \mathbb{Q})[n_1 < n_2 \Rightarrow \exists (n \in \mathbb{Q})(n_1 < n < n_2)]$

Proof: Suppose $n_1 \in \mathbb{Q}$, $n_2 \in \mathbb{Q}$, $n_1 < n_2$.

Then $(\exists a_1 \in \mathbb{Z})(\exists a_2 \in \mathbb{Z})(\exists b_1 \in \mathbb{Z}^+ \wedge b_1 \neq 0)(\exists b_2 \in \mathbb{Z}^+ \wedge b_2 \neq 0)\left[\frac{a_1}{b_1} = n_1 \wedge \frac{a_2}{b_2} = n_2\right]$

Then $\frac{a_1}{b_1} < \frac{a_2}{b_2} \Leftrightarrow \frac{a_1 b_2}{b_1 b_2} < \frac{a_2 b_1}{b_1 b_2}$

Then $a_1 b_2 < a_2 b_1$

Let $x = \frac{a_1 b_2 + a_2 b_1}{2}$

Then

$$a_1 b_2 < \frac{a_1 b_2 + a_2 b_1}{2} < a_2 b_1$$

$$\frac{a_1 b_2}{b_1 b_2} < \frac{a_1 b_2 + a_2 b_1}{2 b_1 b_2} < \frac{a_2 b_1}{b_1 b_2}$$

Letting $x = \frac{a_1 b_2 + a_2 b_1}{2 b_1 b_2}$, and doing math, we have:

$n_1 < x < n_2$. So

$(\forall n_1 \in \mathbb{Z})(\forall n_2 \in \mathbb{Z})[(n_1 < n_2) \Rightarrow \exists (x \in \mathbb{Q})(n_1 < x < n_2)]$

12

4]

ϕ	ψ	$\phi \Rightarrow \psi$	$\psi \Rightarrow \phi$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Looking at the Truth table, both $\phi \Rightarrow \psi$ and $\psi \Rightarrow \phi$ ~~implying~~ true implies both ϕ ψ are true, and both conditional being false implies both ϕ ψ are false. $\therefore$ in the case where both conditionals are true, ϕ and ψ are equivalent.

5]

ϕ	ψ	$\neg\phi$	$\neg\psi$	$(\neg\phi) \Rightarrow (\neg\psi)$	$\psi \Rightarrow \phi$
T	T	F	F	T	T
T	F	F	T	T	T
F	T	T	F	F	F
F	F	T	T	T	T

Looking at the truth table, $(\neg\phi) \Rightarrow (\neg\psi)$ is equivalent to $\psi \Rightarrow \phi$. So proving $\phi \Rightarrow \psi \wedge (\neg\phi) \Rightarrow (\neg\psi)$ is equivalent to proving $\phi \Rightarrow \psi \wedge \psi \Rightarrow \phi$, and by question #4 this establishes $\phi \Leftrightarrow \psi$.

5] Suppose by contradiction that all investors receive less than \$400,000 when \$2M is split between 5 investors. So suppose $a, b, c, d, e < 400{,}000$.

Then $a+b+c+d+e < 5 \cdot 400{,}000$

$a+b+c+d+e < 2{,}000{,}000$. So not all money was distributed, a contradiction.

$\therefore$ If 5 investors split \$2M, at least one receives at least \$400,000.

Assignment 7 Justin Kim (2)

7] Prove that $\sqrt{3}$ is irrational.

Proof: Suppose by contradiction that $\sqrt{3}$ is rational.

Then $(\exists a \in \mathbb{N})(\exists b \in \mathbb{N})\left[\frac{a}{b} = \sqrt{3}\right]$

where a,b have no common factors.

Then,

$$\frac{a^2}{b^2} = 3$$

$$a^2 = 3b^2$$

$\therefore$ a^2 has 3 has a factor. Then a has a common factor. $\therefore$ $a = 3c$ for some $c \in \mathbb{N}$. Then

$$(3c)^2 = 3b^2$$

$$9c^2 = 3b^2$$

$$3c^2 = b^2$$

Then b^2 has 3 as a factor. Then b has 3 as a factor.

So both a,b has 3 as a common factor, a contradiction.

$\therefore$ The ~~antecedent~~ statement is false,

So $\boxed{\sqrt{3} \text{ is irrational}}$.

8] (a) If the Yuan rise, the Dollar falls.

(b) If $-y < -x$, then $x < y$ (for $\mathbb{R}$)

(c) If two $\triangle$ have same area then they are $\cong$

(d) If $ax^2+bx+c=0 \wedge (a \neq 0)$ has a solution, then $b^2 \geq 4ac$.

(e) Let ABCD be a quad.

If opposite $\measuredangle$s are pairwise equal, then opposite sides are pairwise equal

(f) if all 4 $\measuredangle$ are equal, then all 4 sides are equal

(g) If $n^2+5 \mid 3$, then n is not divisible by 3.

9] (b) $\boxed{\text{True}}$ & $\boxed{\text{Equivalent}}$

$x < y$

$-1 \cdot x > -1 \cdot y$

$-x > -y \Leftrightarrow -y < -x$

(c) ~~Converse is fa~~

Original is true by defn, but converse is false. An isoceles $\triangle$ & equilateral $\triangle$ can have same area but not $\cong$

(d) $\boxed{\text{Equivalent}}$

$$ax^2+bx+c=0$$

$$\Longleftrightarrow$$

$$x^2+\frac{b}{a}x = \frac{-c}{a}$$

$$\Longleftrightarrow$$

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2} = \frac{-c}{a}+\frac{b^2}{4a^2}$$

$$\Longleftrightarrow$$

$$\left(x+\frac{b}{a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$\Longleftrightarrow$$

$$x+\frac{b}{a} = \frac{\pm\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Since only time x is not real is detm by $\sqrt{x}$, $b^2-4ac \geq 0$

$\boxed{b^2 \geq 4ac}$

(e) false converse. $\square \neq$ ▱

(f) false convers

(g) If n is not divisible by 3 then $n = 3a+1$ or $n = 3b+2$ for some $a,b \in \mathbb{Z}$.

Then $n^2+5 = (3a+1)^2+5$ or $(3b+2)^2+5$

$9a^2+6a+6$ or $9b^2+12b+9$

both $9a^2$ is divisible by 3. $6a$ & $12b$ is divisible by 3. 6 and 9 is so both are divisible by 3.

Assignment 7

Justin Kim

9a cont)

Now suppose n^2+5 div 3. Let $n=1$. Then $n^2+5=6$ is divisible by 3. But 1 is not divisible by 3. So converse is false.

10) "An integer n is divisible by 12 iff n^3 is divisible by 12" is only true if an integer n is divisible by 12 implies $12 \mid n^3$ and vice versa.

$$12 \mid n$$
$$\cancel{\Leftrightarrow} \Rightarrow$$
$$n = 12a \text{ for some } a \in \mathbb{N}$$
$$\cancel{\Leftrightarrow} \Rightarrow$$
$$n^3 = 12^3 a^3$$
$$\cancel{\Leftrightarrow} \Rightarrow$$
$$n^3 = 12(12^2 a^3)$$
$$\cancel{\Leftrightarrow} \Rightarrow$$
$$12 \mid n^3$$

So cond is true.

~~how to prove converse, I will attempt to prove by induction.~~

~~A=~~ Let $n=6$. Then $n^3 = 216$

$12 \mid 216$. but

$12 \nmid 6$ so converse is false.

11) (a) $r+3$

Suppose $r+3$ is rational. Then

$$(\exists a \in \mathbb{Z}) \wedge (\exists b \in \mathbb{N}) \left[r+3 = \frac{a}{b} \right]$$

Then $r = \dfrac{a-3b}{b}$.

Since $b \in \mathbb{N}$ and $a-3b \in \mathbb{Z}$, then r is a rational, a contradiction. $\therefore$ If ~~r+3~~ is irrational, $r+3$ is irrational.

(b) Suppose $5r$ is rational.

Then $5r = \frac{a}{b}$ $r = \frac{a}{b5}$ so r is rational, $\rightarrow\leftarrow$.

(c) ~~Suppose r+s is rational~~.

Let $r=\sqrt{2}$ and $s = 10-\sqrt{2}$.

Then ~~r+s =~~ both are irrational.

$s+r = 10$.

(d) $r=s=\sqrt{2}$ then $rs=2$.

5) Suppose $\sqrt{r}$ is rational. Then $\sqrt{r} = \frac{a}{b}$, $r = \frac{a^2}{b^2}$ So r is rational.

$\rightarrow\leftarrow$

6) Suppose r^s is rational.

Then $r^s = \frac{m}{n}$

$s \ln r = \ln\left(\frac{m}{n}\right)$

12) (a) $m=2a$ $n=2b$

$m+n = 2(a+b)$ so even

(b) $m=2a$ $n=2b$

$mn = 4ab$ so $4 \mid mn$

(c) $m=2a+1$ $n=2b-1$

$m+n = 2a+2b-\cancel{0}$

$= 2(a+b-1)$

So $m+n$ is even.

Assignment 7 Justin Kim (4)

12 (d) $m=2a \quad n=2b-1$

$m+b = 2(a+b)-1$ so odd.

(e) $m=2a \quad n=2b-1$

$(2a)(2b-1) = 2(2ab-a)$ so even.

Optional

(a) $\boxed{\text{True}}$ Let $x=0$ $y=0$, then $0+0=0$

(b) $\boxed{\text{True}}$ Additive Inverse

(c) $\boxed{\text{False}}$

$a=6 \quad b=2 \quad c=3$

6 divides $2\cdot3=6$ but not 2 or 3

(d) $\boxed{\text{True}}$ Suppose by contradiction that $x+y$ is rational.

Since x is rational, $\exists a\in\mathbb{Z}\ \exists b\in\mathbb{N}$ $x=\frac{a}{b}$ and $\exists c\in\mathbb{Z}\ \exists d\in\mathbb{N}$

$x+y=\frac{c}{d}$

$y=\frac{c}{d}-\frac{a}{b}=$

$y=\frac{bc-ad}{bd}$

So y is rational, a contradiction. So if x is rational and y is irrational, $x+y$ is irrational.

(e) $\boxed{\text{True}}$

By contrapositive, Suppose x,y both rational.

Then $x=\frac{a}{b}$ $y=\frac{c}{d}$

$x+y=\frac{ad+bc}{bd}\in\mathbb{Q}$ so $x+y$ is rational.

(f) $\boxed{\text{False}}$

$x=\sqrt{2} \quad y=10-\sqrt{2}$

neither are rational.