

FOUR COLOR THEOREM AND CONVEX RELAXATION FOR IMAGE SEGMENTATION WITH ANY NUMBER OF REGIONS

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ABSTRACT. Image segmentation is an essential problem in imaging science. One of the most successful segmentation models is the piecewise constant Mumford-Shah minimization model. This minimization problem is however difficult to carry out, mainly due to the non-convexity of the energy. Recent advances based on convex relaxation methods are capable of estimating almost perfectly the geometry of the regions to be segmented when the mean intensity and the number of segmented regions are known a priori. The next important challenge is to provide a tight approximation of the optimal geometry, mean intensity and the number of regions simultaneously while keeping the computational time and memory usage reasonable. In this work, we propose a new algorithm that combines convex relaxation methods with the four color theorem to deal with the unsupervised segmentation problem. More precisely, the proposed algorithm can segment any a priori unknown number of regions with only four intensity functions and four indicator (“labeling”) functions. The number of regions in our segmentation model is decided by one parameter that controls the regularization strength of the geometry, i.e., the total length of the boundary of all the regions. The segmented image function can take as many constant values as needed.

1. Introduction. Image segmentation is one of the most fundamental and challenging problems in image processing and computer vision. An ideal image segmentation algorithm should be able to partition any image into several desired objects with well-defined boundaries. This problem can be formulated as an energy minimization problem, which usually consists of a data term and a regularization term as commonly proposed in the context of inverse problems. Geman and Geman [17] introduced a generic segmentation model in the discrete setting, while Mumford and Shah [30] proposed the continuous counterpart. These two models are among the most influential models in imaging. Many existing segmentation models in the

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literature are influenced by these methods. However, finding a solution to these minimization problems can be numerically challenging.

The level set method [31] has been very popular for approximating solutions of the Mumford-Shah problem, such as the Chan-Vese model [14, 39]. But the level set method can be easily trapped in local minima, depending on the choice of the initialization. Besides, the level set method is rather slow, compared to other related methods such as graph cut method [8]. Graph cut method is a combinatorial method that can solve exactly two-phase segmentation problems and it is significantly faster than the level set method. However, graph cut method is limited by the metrication error and it can only use anisotropic operators. This produces less accurate results than continuous methods such as the level set method and also weaker approximations of solution for multi-phase problems. Finally, the memory requirement and the difficulty to parallelize can become issues for 3D applications, unlike continuous approaches. Recently, a new alternative to the level set method and the graph cut method has been introduced to overcome the previous limitations. This new approach is based on continuous convex relaxation methods [13, 9, 45, 32, 25, 24, 7, 10] or continuous max-flow approaches [13, 44, 6, 43, 42]. The main advantages is to produce exact or tight approximations of solutions for two-phase and multi-phase problems, provide arbitrary accurate results, and benefit from fast convex optimization algorithms that can be parallelized on graphics processing unit (GPU). The original paper of Strang [37] introduced the theoretical basis of this new method. More recently, it was used in imaging problems [13, 9] to solve exactly and quickly the two-phase active contour segmentation problem when the mean intensity is known. It was then extended to the multi-phase segmentation problem (also known as the Potts problem [33] in the discrete setting) in [45, 32, 25, 24, 7, 10, 44, 6, 43, 42]. For multiphase segmentation problems, it is NP-hard if we need to regularize the problem by the length of the interfaces. Thus, it may be difficult to find exact global minimizer even if we use convex relaxation methods. However, if one is allowed to “relax” the regularization term, then exact global minimization methods are available. For examples, if we use the Chan-Vese regularization which is not the length of the interface, exact and global minimizer has been given in [2, 3]. If we use one label function as in [26] and the total variation of this label function as the regularizer, then global minimization method is also available in [5].

Although the recent multi-phase segmentation methods based on convex relaxation methods are almost optimal, fast and accurate, they also have three main limitations. Firstly, they assume that the mean intensity of each segmented region is known (usually k-means algorithm is used to estimate the mean intensity). Secondly, they are supposed to know a priori the number of regions to be segmented. Finally, they require one function per region to be segmented, which can therefore be time and memory consuming when dealing with a large number of regions. Recent papers started to tackle these limitations in the context of convex relaxation methods. In the case of two-phase, [36] introduced a branch and bound method to compute a global solution to the two-phase Chan-Vese model when both the geometry and the mean intensity are unknown. In [11], the authors introduced a convex relaxation method to compute tight approximations of the multi-phase Chan-Vese model when both the geometry and the mean intensity are unknown. However, the number of regions must be known a priori. In [4], the authors also introduced a convex relaxation method to compute tight approximations of the multi-phase

Chan-Vese model without knowing a priori the mean intensity and the number of regions. However, the computational time and memory usage for these methods are significant. For example, in the recent work [4], the number of indicator functions K is equal to the number of discretized intensity levels s.a. $K = 256$. Note also that other works s.a. [20, 21, 22, 35] have been proposed to solve the unsupervised segmentation method but not in the context of geometric convex relaxation methods. These other methods may be more influenced by local minima e.g. due to noise.

In this work, we propose a new unsupervised segmentation method based on convex relaxation method that can compute simultaneously the geometry, the mean intensity and the number of regions. The number of regions is decided by one parameter that controls the regularization strength of the geometry, i.e. the total length of the boundary of all the regions. The proposed energy minimization problem is convex w.r.t. the geometry and the mean intensity separately, but not globally convex (i.e. considering geometry and mean intensity together). The separate convex optimization problems can be solved efficiently based on augmented Lagrangian methods. The number of regions to be segmented is not given a priori but learned during the segmentation task, i.e. the number of regions is automatically determined. Moreover, we only need four indicator or so-called “labeling” functions to represent these regions of which could be any number. Experiments are conducted on synthetic images to illustrate our method and real-world images.

2. The new model and its convex relaxation.

2.1. The Mumford-Shah model. The model we are going to use is closed related to the piecewise constant Mumford-Shah model. The Mumford-Shah (MS) model is an optimization problem:

$$(1) \quad \min_{C,s} \mu|C| + \frac{\alpha}{2} \int_{\Omega} (s - s_0)^2 + \frac{\gamma}{2} \int_{\Omega \setminus C} |\nabla s|^2,$$

where μ, α and $\gamma > 0$, $|C|$ is the length of C , and $s_0 : \Omega \subset \mathbb{R}^N \rightarrow \mathbb{R}$ is the original N -dimensional “image” to be segmented. The solution of the MS problem is a piecewise smooth approximation $s : \Omega \rightarrow \mathbb{R}$ of the original image s_0 with a curve C representing image edges with minimal length. After twenty years of analysis, the well-posedness of the MS model is still an open problem. However, there is a conjecture that there exists (at least) one minimizer (s, C) of (1) where 1) $s \in C^1(\Omega \setminus C)$ and 2) C is made up of a finite union of C^1 -regular arcs, with two special cases: crack tip and triple junction (where three branches meet with 120° angles). The original MS model (1) is actually not used in most real-world segmentation problems as real applications look for objects with closed boundary s.a. anatomical structures or moving objects. However, the original MS model is not designed to detect closed boundary only, but also open boundary. Therefore in the context of segmentation, a variant of the MS model is preferred. This variant/simplified version is called the piecewise constant Mumford-Shah model and can be obtained with $\gamma \rightarrow \infty$ in (1). As its name suggests, this model looks for a piecewise constant approximation $u(x) = \sum_{i=1}^r c_i 1_{\Gamma_i}(x)$ of the original image s_0 , Figure 1(a), as follows:

$$(2) \quad \begin{aligned} & \min_r \min_{\{\Gamma_i\}_{i=1}^r} \min_{\{c_i\}_{i=1}^r} \sum_{i=1}^r (\mu|\partial\Gamma_i| + \frac{\alpha}{2} \int_{\Gamma_i} (c_i - s_0)^2) \quad \text{s.t.} \\ & \Omega = \cup_{i=1}^r \Gamma_i \text{ and } \Gamma_i \cap \Gamma_j = \emptyset \quad \forall i \neq j \end{aligned}$$

where $\partial\Gamma_i$ is the boundary of the region Γ_i , $|\partial\Gamma_i|$ is the length of $\partial\Gamma_i$, c_i are some constants representing the mean intensity in region Γ_i and the constraints $\Omega = \cup_{i=1}^r \Gamma_i$ and $\Gamma_i \cap \Gamma_j = \emptyset \ \forall i \neq j$ guarantee no overlapping and vacuum between the regions (i.e. all Γ_i are pairwise disjoint). This segmentation problem is shown to be well-posed. Also, when the number of regions r and the mean intensities c_i are fixed, then the optimization problem (2) reduces to the Potts problem [33], which is also known as the minimal partition problem in the continuous setting. The Potts problem was proven to be NP-hard. As discussed previously, approximate solutions for the Potts problem exist both in the discrete setting [8] and the continuous setting [45, 32, 25, 24, 7, 10, 44, 6, 43, 42].

2.2. Our new model. In this work, we propose a new model which looks like the PCMS model (2), but it is essentially different in the way on how to define the smooth image functions. This new model offers some inherent advantages in dealing with the topology of the segmented regions, unlike many other approaches where the “sensitivity analysis” with the topological changes is very difficult. For the new model, we propose to solve:

$$(3) \quad \min_{\{\Omega_i\}_{i=1}^n} \min_{\{s_i\}_{i=1}^n} \sum_{i=1}^n \mu |\partial\Omega_i| + \frac{\alpha}{2} \int_{\Omega_i} (s_i - s_0)^2 \text{ s.t.} \\ \Omega = \cup_{i=1}^n \Omega_i, \ \Omega_i \cap \Omega_j = \emptyset \ \forall i \neq j \text{ and } |\nabla s_i(x)|^2 = 0 \ \forall x \in \{\Omega_i\}_{i=1, \dots, n},$$

where we now consider phases Ω_i instead of regions Γ_i . Each phase Ω_i can contain many simple disconnected sub-regions Γ_k . The proposed variant (3) of the piecewise constant Mumford-Shah model (2) (used together with the four color theorem) is the essential contribution of our work. Observe that, unlike the formulation (2), the functions s_i is piecewise constant due to the constraint

$$|\nabla s_i(x)|^2 = 0 \ \forall x \in \{\Omega_i\}_{i=1, \dots, n}.$$

However, this constraint only guarantees that s_i is a constant inside each connected sub-region of Ω_i . There may exist many connected sub-regions in Ω_i and thus s_i can take many different constant values in each phase $\Omega_i, i = 1, 2, \dots, n$. Figure 1(b) illustrates this idea where each $\{\Omega_i\}_{i=1}^n$ represents one phase and each phase contains a number of disconnected sub-regions Γ_k . In other words, the proposed model requires less than the original number r of functions Γ_i to represent the segmented objects. What is the minimum number of phases that must be used? The number of phases for a two-dimensional image can actually be reduced to $n = 4$ using the four color theorem. Indeed, the four color theorem states that any planar graph (s.a. the image grid) can be colored with at most four colors such that no adjacent nodes have the same color [1]. We can now represent any number r of regions Γ_i s.a. $r = 10, 100, 1000$ with only $n = 4$ phases Ω_i , each phase Ω_i is a union of multiple simple connected regions, each simple connected regions could have a different mean intensity value due to constraint $|\nabla s_i(x)|^2 = 0$. This is illustrated on Figure 1(c). Note that segmenting $r \gg 2$ regions (e.g. 1000) with the original piecewise constant Mumford-Shah model (2), i.e. with [45, 32, 25, 24, 7, 10], would be time and memory intractable. The idea to use the four color theorem has already been used in imaging and segmentation [39, 19] in the context of the level set method. This work proposes to use the new model (3) which is especially suitable for four-color painting of the segmented regions. Moreover, more efficient algorithms

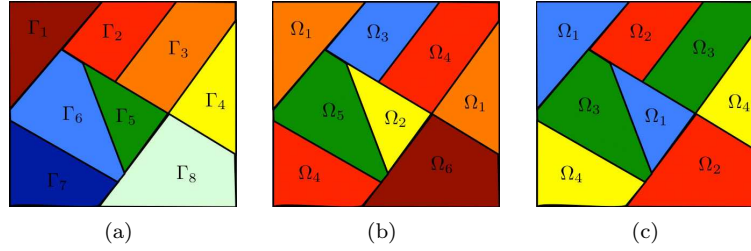


FIGURE 1. (a) Standard representation of r regions Γ_i with the piecewise constant Mumford-Shah model, Eq. (2). (b) Representation of r regions with n phases Ω_i (that include regions Γ_i), Eq. (3). (c) Four color theorem: no more than $n = 4$ phases Ω_i are needed to represent any number of regions, Eq. (4).

based on the recent convex relaxation methods for ℓ^1 problems will give some fast numerical schemes.

Based on the explanation given above, the main minimization problem of this paper is:

$$(4) \quad \min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \mu |\partial \Omega_i| + \frac{\alpha}{2} \int_{\Omega_i} (s_i - s_0)^2 \quad \text{s.t.} \\ \Omega = \cup_{i=1}^4 \Omega_i, \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \quad \text{and} \quad |\nabla s_i(x)|^2 = 0 \quad \forall x \in \{\Omega_i\}_{i=1,\dots,4}.$$

The constraint $|\nabla s_i(x)|^2 = 0 \quad \forall x \in \Omega_i$ is essential to guarantee that functions s_i are constant in each simple connected sub-region of phase Ω_i and the constant value is equal to the mean intensity inside this simple connected sub-region. Another point we want to emphasize is that each functions $\{s_i\}_{i=1}^4$ is defined in the whole domain Ω , and not only restricted to the phase Ω_i . Therefore another term is necessary to ensure the smoothness of the function s_i outside Ω_i . This can be easily done by adding a regularization term to the energy (last term):

$$(5) \quad \min_{\{\Omega_i\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \mu |\partial \Omega_i| + \frac{\alpha}{2} \int_{\Omega_i} (s_i - s_0)^2 + \frac{\sigma}{2} \int_{\Omega} |\nabla s_i|^2 \quad \text{s.t.} \\ \Omega = \cup_{i=1}^4 \Omega_i, \Omega_i \cap \Omega_j = \emptyset \quad \forall i \neq j \quad \text{and} \quad |\nabla s_i(x)|^2 = 0 \quad \forall x \in \{\Omega_i\}_{i=1,\dots,4}.$$

The last term not only regularizes the values of s_i outside Ω_i , but also inside.

2.3. Convex relaxation of the new model. The next step is to design an optimization algorithm for solving (5). This is difficult as the global energy is not convex and the sub-optimization problem w.r.t. Ω_i is also not convex. We proceed as follows. Firstly, the sub-problem w.r.t. Ω_i is going to be relaxed by using indicator functions u_i of the regions Ω_i :

$$(6) \quad u_i(x) = \begin{cases} 1, & \forall x \in \Omega_i, \\ 0, & \text{otherwise.} \end{cases}$$

The problem (5) is thus equivalent to

$$(7) \quad \min_{\{u_i \in \{0,1\}\}_{i=1}^4} \min_{\{s_i\}_{i=1}^4} \sum_{i=1}^4 \int_{\Omega} \mu |\nabla u_i| + \frac{\alpha}{2} \int_{\Omega} u_i (s_i - s_0)^2 + \frac{\sigma}{2} \int_{\Omega} |\nabla s_i|^2 \quad \text{s.t.} \\ \sum_{i=1}^4 u_i(x) = 1 \quad \forall x \in \Omega \text{ and } u_i(x) |\nabla s_i(x)|^2 = 0 \quad \forall x \in \Omega, i = 1, \dots, 4.$$

The sub-problem w.r.t. u_i is convex w.r.t. the energy but not convex w.r.t. constraints (which are non-convex sets of binary functions). The standard approach is to relax function $u_i \in \{0,1\}$ to the closest convex set, i.e. $u_i \in [0,1]$. Besides, the constraint $u_i(x) |\nabla s_i(x)|^2 = 0, \forall x \in \Omega$ will be enforced with a Lagrangian multiplier [41]. The corresponding iterative updating of the solutions is:

$$(8) \quad (u_i^{k+1}, s_i^{k+1}) = \arg \min_{\{u_i \in [0,1]\}_{i=1}^4, \{s_i\}_{i=1}^4} \sum_{i=1}^4 \int_{\Omega} w_b |\nabla u_i| + \frac{\alpha}{2} \int_{\Omega} u_i (s_i - s_0)^2 \\ + \frac{\sigma}{2} \int_{\Omega} |\nabla s_i|^2 + \int_{\Omega} u_i (\lambda_i^k |\nabla s_i|^2 + \frac{r}{2} |\nabla s_i|^2) \quad \text{s.t.} \quad \sum_{i=1}^4 u_i(x) = 1 \quad \forall x \in \Omega \\ \lambda_i^{k+1} = \lambda_i^k + r u_i |\nabla s_i|^2$$

where $w_b(x)$ can be an edge detector s.a. $w_b(x) = \frac{\mu}{1+|\nabla s_0(x)|^2}$ in [12]. By doing so, we are regularizing the problem by the weighted length from the “edge detector” instead of the conventional length of the interface. The minimization problem (8) is still not convex w.r.t. (u_i, s_i) together, but it is now convex w.r.t. u_i and s_i separately. Therefore, there is a guarantee to solve each sub-minimization problem exactly and even efficiently with augmented Lagrangian methods [41]. This naturally leads to an algorithm that solves the segmentation problem (2) by solving each minimization problem w.r.t. u_i and s_i separately and alternately until convergence. Although this kind of minimization procedure does not guarantee convergence to a global minimum (even the question of convergence to a local minimum is open), it usually produces good approximate solutions in practice.

3. Efficient algorithms. In this section, we derive efficient algorithms to solve the two sub-minimization problems of (8). The algorithms are based on operator splitting and augmented Lagrangian methods, along the same line of [18, 41].

3.1. Sub-minimization w.r.t. u_i (geometry). The minimization problem w.r.t. u_i is

$$(9) \quad \min_{u_i \in [0,1]} \int_{\Omega} w_b |\nabla u_i| + \int_{\Omega} u_i f_i \quad \text{s.t.} \quad \sum_{i=1}^4 u_i(x) = 1 \quad \forall x \in \Omega$$

where $f_i = \frac{\alpha}{2} (s_i - s_0)^2 + |\nabla s_i|^2 (\lambda_i^k + \frac{r}{2})$. There exist several efficient approaches to deal with this problem [45, 32, 25, 24, 7, 10, 44, 6, 43, 42]. We propose in this work a simple and fast algorithm based on split-Bregman [18] and the simplex projection [29], such as in [25, 24]. Firstly, the original minimization problem is split into three sub-problems, which are easier to solve by introducing new variables $v_i \in \mathbb{R}, d_i \in \mathbb{R}^N$:

$$(10) \quad \min_{u_i \in [0,1], v_i, d_i} \int_{\Omega} w_b |d_i| + \int_{\Omega} v_i f_i \quad \text{s.t.} \quad \sum_{i=1}^4 u_i = 1, \quad d_i = \nabla u_i, \quad v_i = u_i,$$

Then, the previous multiple-constraint minimization problem is converted back to a simplex constraint minimization problem using the augmented Lagrangian method:

$$\begin{aligned}
 (u_i^{m+1}, v_i^{m+1}, d_i^{m+1}) &= \arg \min_{u_i \in [0,1], v_i, d_i} \int_{\Omega} w_b |d_i| + \int_{\Omega} v_i \cdot f_i \\
 &+ \lambda_{d_i}^m \cdot (d_i - \nabla u_i) + \frac{r_d}{2} |d_i - \nabla u_i|^2 + \lambda_{v_i}^m \cdot (v_i - u_i) + \frac{r_v}{2} (v_i - u_i)^2 \\
 \text{s.t. } &\sum_{i=1}^4 u_i(x) = 1, \quad \forall x \in \Omega \\
 (11) \quad &\lambda_{d_i}^{m+1} = \lambda_{d_i}^m + r_d(d_i^{m+1} - \nabla u_i^{m+1}), \quad \lambda_{v_i}^{m+1} = \lambda_{v_i}^m + r_v(v_i^{m+1} - u_i^{m+1})
 \end{aligned}$$

Approximate solution of the minimization problem w.r.t. u_i , i.e. $\min_{u_i \in [0,1]} \int_{\Omega} \frac{r_v}{2} (u_i - (v_i + \frac{\lambda_{v_i}}{r_v}))^2 + \frac{r_d}{2} |\nabla u_i - (d_i + \frac{\lambda_{d_i}}{r_d})|^2$ s.t. $\sum_{i=1}^4 u_i = 1$, can be efficiently computed as follows. Consider the Euler-Lagrange equations without the simplex constraint:

$$(12) \quad (-r_d \Delta + r_v) u_i = -r_d \operatorname{div}(d_i + \frac{\lambda_{d_i}}{r_d}) + r_v(v_i + \frac{\lambda_{v_i}}{r_v})$$

The solution can be quickly computed by FFT or DCT depending on the boundary condition (i.e. periodic or Neumann-type). We refer the reader to [40, 41, 16] for more details. Once the solution of (12) has been computed, call it \hat{u}_i , then the simplex constraint is enforced according to Michelot's method [29], which is fast and easy to implement. We get:

$$(13) \quad \{u_i\}_{i=1}^4 = \Pi_s(\{\hat{u}_i\}_{i=1}^4)$$

where Π_s is the Michelot's projector operator on the linear simplex constraint.

The minimization problem w.r.t. d_i , i.e. $\min_{d_i} \int_{\Omega} w_b |d_i| + \frac{r_d}{2} |d_i - (\nabla u_i - \frac{\lambda_{d_i}}{r_d})|^2$ is fast to solve as the solution is given by soft-threshold with the shrinkage operator [15]:

$$(14) \quad d_i = \max(|z_i| - \frac{w_b}{r_d}, 0) \frac{z_i}{|z_i|}, \quad z_i = \nabla u_i - \frac{\lambda_{d_i}}{r_d}$$

The minimization problem w.r.t. v_i , i.e. $\min \int_{\Omega} \frac{r_v}{2} (v_i - (u_i - \frac{\lambda_{v_i}}{r_v}))^2 + v_i \cdot f_i$ is simply given by:

$$(15) \quad v_i = u_i - \frac{\lambda_{v_i}}{r_v} - \frac{f_i}{r_v}$$

Finally, let us remind that the solution of the geometric problem (9) is not guaranteed to produce a global minimizer of the original non-convex geometric problem. A global minimizer is obtained if and only if the computed solution u_i is binary. Although experiments suggest good approximate solutions for this multi-phase problem, the computed solutions are not exactly binary. Hence, as in [45, 32, 25, 24, 7, 10] a threshold/binarization step is required. The simplest thresholding technique is

$$(16) \quad \hat{u}_i^*(x) = \begin{cases} 1 & \text{if } i = \arg \max_{j \in \{1, \dots, 4\}} u_j(x) \\ 0 & \text{otherwise} \end{cases} \quad \forall x \in \Omega$$

where $\{\hat{u}_i^*\}_{i=1}^4$ are binary functions satisfying $\sum_{i=1}^4 \hat{u}_i^*(x) = 1, \quad \forall x \in \Omega$.

To summarize the optimization algorithm to solve the geometric problem in (8), the pseudo-algorithm is given in Algorithm 1.

Algorithm 1 Fast algorithm for the geometric problem (9).

Given that s_i^k and u_i^{k+1} are given by the inner loop:

while inner loop not converged **do**

u_i^{m+1} computed with (12) and (13)

d_i^{m+1} computed with (14)

v_i^{m+1} computed with (15)

$\lambda_{d_i}^{m+1} = \lambda_{d_i}^m + r_d(d_i^{m+1} - \nabla u_i^{m+1})$

$\lambda_{v_i}^{m+1} = \lambda_{v_i}^m + r_v(v_i^{m+1} - u_i^{m+1})$

end while

Threshold/binarization with (16)

3.2. Sub-minimization w.r.t. s_i (mean intensity). The minimization problem w.r.t. s_i is

$$(17) \quad \min_{s_i} \int_{\Omega} \frac{h_i}{2} (s_i - s_0)^2 + \frac{g_i}{2} |\nabla s_i|^2$$

where $h_i = \alpha u_i$ and $g_i = \sigma + (2\lambda_i^k + r)u_i$. There exist different methods to solve this optimization problem. Multi-grid and other fast linear solvers can be easily used for this subproblem. In our tests, we investigated an AOS [28] and LOD methods to solve (17) s.a. in [34]. We also studied a splitting and augmented Lagrangian method, which outperformed the AOS [28] and LOD approaches [34] in terms of speed. We present this method here. The minimization problem (17) is split into three sub-minimization problems, which are easier to solve by introducing new variables $t_i \in \mathbb{R}, p_i \in \mathbb{R}^N$:

$$(18) \quad \min_{s_i} \int_{\Omega} \frac{h_i}{2} (t_i - s_0)^2 + \frac{g_i}{2} |p_i|^2, \quad p_i = \nabla s_i, \quad t_i = s_i$$

Then, the previous multiple-constraint minimization problem is converted back to an unconstrained minimization problem using the augmented Lagrangian method:

$$(19) \quad \begin{aligned} (s_i^{m+1}, t_i^{m+1}, p_i^{m+1}) &= \arg \min_{s_i, t_i, p_i} \int_{\Omega} \frac{h_i}{2} (t_i - s_0)^2 + \frac{g_i}{2} |p_i|^2 \\ &\quad + \lambda_{p_i}^m \cdot (p_i - \nabla s_i) + \frac{r_p}{2} |p_i - \nabla s_i|^2 + \lambda_{t_i}^m \cdot (t_i - s_i) + \frac{r_t}{2} (t_i - s_i)^2 \\ \lambda_{p_i}^{m+1} &= \lambda_{p_i}^m + r_p (p_i^{m+1} - \nabla s_i^{m+1}), \quad \lambda_{t_i}^{m+1} = \lambda_{t_i}^m + r_t (t_i^{m+1} - s_i^{m+1}) \end{aligned}$$

As before, the minimization problem w.r.t. s_i can be efficiently computed by FFT or DCT as the Euler-Lagrange equations of the problem $\min_{u_i} \int_{\Omega} \frac{r_t}{2} (s_i - (t_i + \frac{\lambda_{t_i}}{r_t}))^2 + \frac{r_p}{2} |\nabla s_i - (p_i + \frac{\lambda_{p_i}}{r_p})|^2$ is:

$$(20) \quad (-r_p \Delta + r_t) s_i = -r_p \operatorname{div} (p_i + \frac{\lambda_{p_i}}{r_p}) + r_t (t_i + \frac{\lambda_{t_i}}{r_t}).$$

The minimization problem w.r.t. p_i , i.e. $\min_{p_i} \int_{\Omega} \frac{g_i}{2} |p_i|^2 + \frac{r_p}{2} |p_i - (\nabla s_i - \frac{\lambda_{p_i}}{r_p})|^2$ has an analytical solution

$$(21) \quad p_i = \frac{r_p}{r_p + g_i} (\nabla s_i - \frac{\lambda_{p_i}}{r_p})$$

The minimization problem w.r.t. t_i , i.e. $\min_{t_i} \int_{\Omega} \frac{h_i}{2} (t_i - s_0)^2 + \frac{r_t}{2} (t_i - (s_i - \frac{\lambda_{t_i}}{r_t}))^2$ has also a simple analytical solution

$$(22) \quad t_i = \frac{h_i s_0 + r_t s_i - \lambda_{t_i}}{h_i + r_t}$$

To summarize the optimization algorithm to solve the image feature/region intensity problem in (17), the pseudo-algorithm is given in Algorithm 2.

Algorithm 2 Fast algorithm for the image feature problem (17).

Given that u_i^{k+1} and s_i^{k+1} are given by the inner loop:

while inner loop not converged **do**

s_i^{m+1} computed with (20)

p_i^{m+1} computed with (21)

t_i^{m+1} computed with (22)

$\lambda_{p_i}^{m+1} = \lambda_{p_i}^n + r_p (p_i^{m+1} - \nabla s_i^{m+1})$

$\lambda_{t_i}^{m+1} = \lambda_{t_i}^n + r_t (t_i^{m+1} - s_i^{m+1})$

end while

3.3. Alternate algorithm. In summary, the proposed algorithm for solving the unsupervised segmentation problem (5) is described in the pseudo-algorithm 3.

Algorithm 3 Algorithm for the unsupervised image segmentation model (5) using the four color theorem (with a priori unknown number of regions).

Initialize u_i (given by random initialization or k-means)

while not converged **do**

s_i^{k+1} computed with Algorithm 2

u_i^{k+1} computed with Algorithm 1

$\lambda_i^{k+1} = \lambda_i^k + r u_i^{k+1} |\nabla s_i^{k+1}|^2$

end while

Preliminary experiments showed that Algorithm 3 can produce under segmentation, as it can be trapped in some (trivial) local minima such as Figure 2(b). To avoid these local minima, we propose to use a two-level recursive method as follows. First, the image is segmented into four phases $\{\Omega_i\}_{i=1}^4$ using Algorithm 3, Figure 2(a). Each phase may contain multiple regions, Figure 2(b). However, it is possible that two distinct regions are merged together s.a. the central disk and the upper right part of the image. In order to separate the two distinct regions, Algorithm 3 is applied again to each of the four phases $\{\Omega_i\}_{i=1}^4$ (simply by using an indicator function of each phase in Algorithm 3 s.t. $\int_{\Omega} \rightarrow \int_{\Omega} 1_{\Omega_i}$). This produces sixteen regions $\{\Omega_{i,j}\}_{i,j=1}^4$. These sixteen regions are then recolored into four phases, Figure 2(c), using any fast coloring algorithm s.a. Recursive Large First (RLF) algorithm[23], which only requires $O(n^2)$ time and $O(n^2)$ space, where n here is the number of regions. The implementation of RLF algorithm is available at codeproject¹. This two-level recursive method is iterated until the energy (4) does not decrease anymore. Algorithm 4 presents the final proposed unsupervised segmentation method

¹<http://www.codeproject.com/Articles/88674/Graph-coloring-using-Recursive-Large-First-RLF-alg>

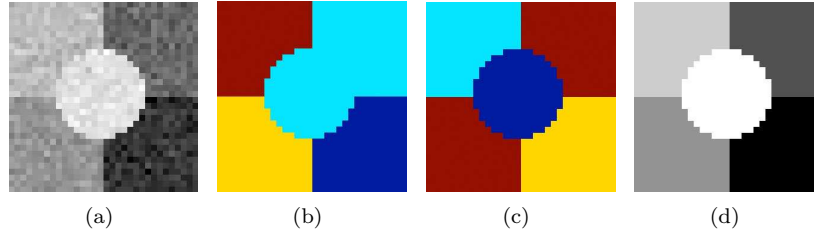


FIGURE 2. (a) Original image. (b) segmentation with four phases $\{\Omega_i\}_{i=1}^4$ (two distinct regions, central disk and upper right part are merged). (c) segmentation result after segmenting each phase into four sub-phase (this produces sixteen sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$) and recoloring into four phases (correct result). (d) piecewise constant approximation of (a).

that estimates the geometry, the mean intensity and the number of regions simultaneously.

Algorithm 4 Two-level recursive algorithm for the unsupervised image segmentation model (2) using the four color theorem (with a priori unknown number of regions).

Initialize u_i (given by random initialization or k-means), select the scale parameter α which controls the number of regions
while not converged **do**
 Compute four phases $\{\Omega_i\}_{i=1}^4$ with Algorithm (3)
 Partition each phase $\{\Omega_i\}_{i=1}^4$ into 4 sub-phases $\{\Omega_{i,j}\}_{i,j=1}^4$ with Algorithm (3)
 Recolor the 16 sub-phases into 4 phases
end while

4. Experiments. In this section, we conduct several experiments on synthetic and natural images to demonstrate the proposed algorithm. In all experiments, the inputs of the segmentation model are the image to be segmented and the value of the parameter α in (5), which determines the final number of segmented regions as it controls the total length of the boundary of all regions. Other parameters are fixed for all experiments: if the original image $s_0 \in [0, 255]$ then the parameter values are $r_v = 1e1, r_b = 1e2, r_p = 1e1/255, r_t = 1e2/255, \sigma = 1e4/255^2, \mu = 1e2$.

Let us start with a simple image composed of three regions, Figure 3(a). The objective of this figure is to compare our algorithm with the standard recursive bi-partitioning approach used in several segmentation problems. The recursive bi-partitioning method consists in segmenting the image into two regions, then segment each of the two regions into two sub-regions, and so on. The main limitation of this approach is illustrated on Figure 3(b-c). If the recursive bi-partitioning fails at some certain levels of recursion (here at the first level, Figure 3(b)) then it can never fix its mistake and carry it on to the next levels (Figure 3(c)). Our segmentation algorithm is more flexible and can correct some segmentation mistakes previously made during the segmentation task.

The objective of Figure 4 is to illustrate the influence of the parameter α on the segmentation result. We do not investigate the explicit relationship between the

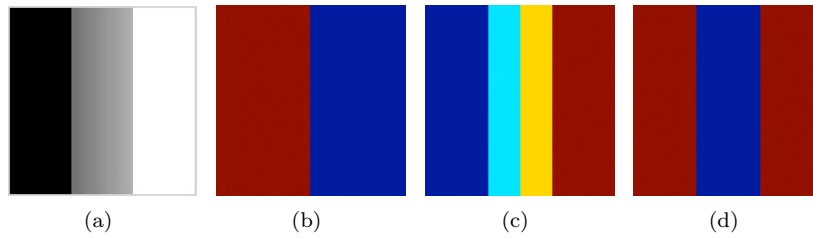


FIGURE 3. Comparison between the standard recursive bi-partitioning method and our method: (a) Original image. (b) segmentation after 1st bi-partitioning. (c) segmentation after 2nd/final bi-partitioning (over-segmentation). (d) Our algorithm (correct segmentation).

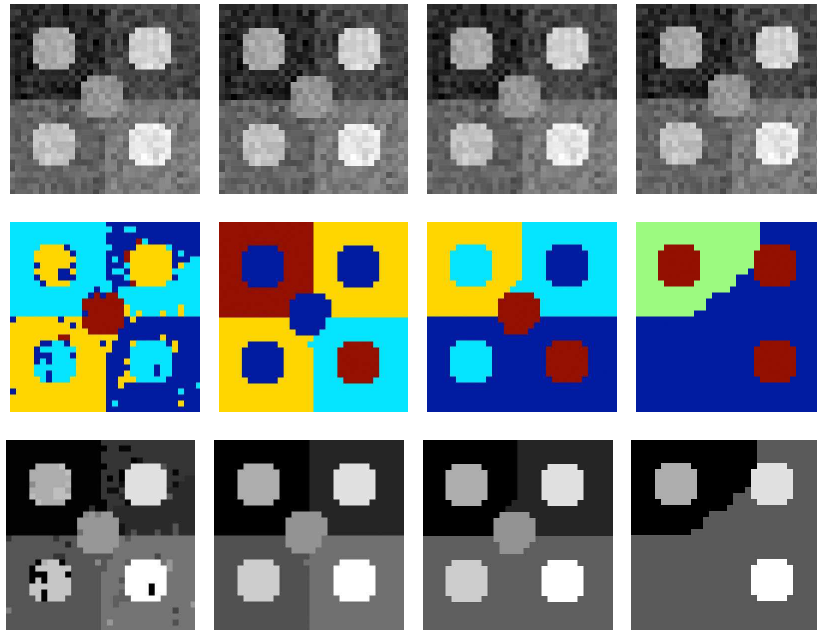


FIGURE 4. Influence of the regularization parameter α . First row is the original image. Second row is the four-color segmentation result. Third row is the piecewise constant approximation of the image. First column $\alpha = 1.5e5/255^2$, second column $\alpha = 3e4/255^2$, third column $\alpha = 1e4/255^2$, fourth column $\alpha = 1e3/255^2$.

choice of α and the number of regions in this work. Instead, we empirically selected several values of α to show the influence of α on the total number of final regions (each column presents the segmentation result with a different value of α). Observe that for large α the number of segmented regions is large, and as the value of α gets smaller and smaller then the number of regions decreases.

Figure 5 presents the segmentation of well-known real-world images with our algorithm.

Finally, a related segmentation method has been proposed recently in [27], where the authors also use the four color theorem to deal with any number of regions as in [19, 38]. We want to emphasize that the model we use here has some essential advantages over the traditional Mumford-Shah model. This new model offers some essential advantages in dealing with topological changes of the segmented regions (or phases). The approach of [27] has several limitations. Firstly, the segmentation task is carried out with graph cut method, which is known to produce weaker approximations of the solution than continuous relaxation methods, as they suffer from metrication errors and anisotropic operators. Secondly, the method [27] requires to know a priori the number of sub-regions for partitioning each of the four preliminary phases. In addition, some of the region boundary are regularized while other have no regularization at all. Our model is not influenced by these limitations as it is based on continuous optimization methods and the number of sub-regions is automatically estimated. Even more, we regularize all the region boundaries.

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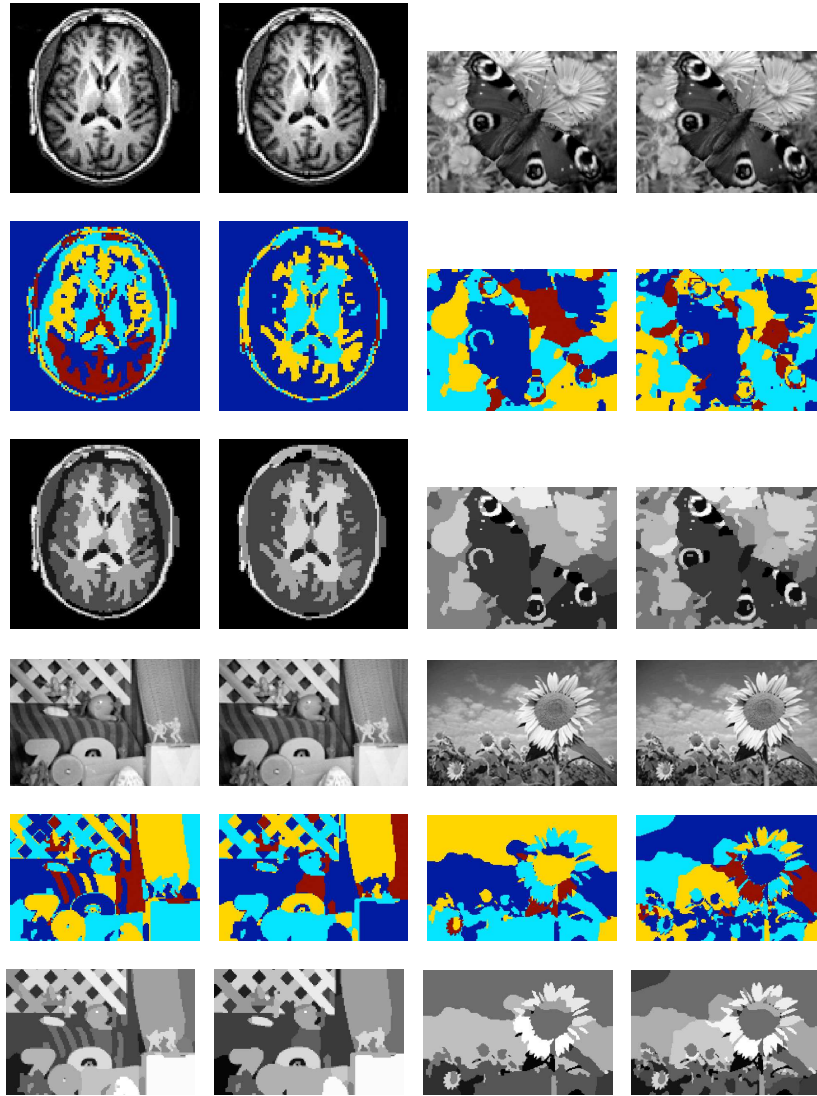


FIGURE 5. First and fourth rows present the original image. Second and fifth rows show the four-color segmentation result. Third and last rows display the piecewise constant approximation of the image s_0 . Each column present a different value of α , which controls the number of final segmented regions.

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