A More Detailed Proof of the Second Half of Theorem 9.3.12

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1 What we have

Our assumptions are

1. Assumption 9.3.1: $0 < L(x, \mathsf{X}) < \infty$.
2. Assumption 9.3.2: $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ are consistent for (ν, C). $L(x, \mathsf{X}) \in \mathsf{C}$.
3. Assumption 9.3.3: $\forall x \in \mathsf{X}$, $L(x, \cdot) \ll R(x, \cdot)$, and there exists a strictly positive RN derivative: $\frac{dL(x,\cdot)}{dR(x,\cdot)}$.
4. Assumption 9.3.6 (new) $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ is asymptotically normal for $(\nu, \mathsf{A}, \sigma, \{M_N^{1/2}\})$, where A is a proper set, and σ is a nonnegative function on A.
5. $\{\alpha_N\}$ has a possibly-infinite limit α.

We also have the things we have shown already in the first half of the proof:

1. $$\tilde{\mathsf{A}} = \left\{ f \in L^2(\mathsf{X}, \mu) : x \mapsto L(x, |f|) \in \mathsf{A}, x \mapsto \int R(x, dx') \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \in \mathsf{C} \right\}$$

   is proper
2. $\{(\tilde{\xi}^{N,j}, \tilde{\omega}^{N,i})\}_{1 \le j \le \tilde{M}_N}$ is asymptotically normal for $(\mu, \tilde{\mathsf{A}}, \tilde{\sigma}, \{M_N^{1/2}\})$ with

   $$\tilde{\sigma}(f) = \frac{\sigma^2\left(L[f - \mu(f)]\right) + \alpha^{-1}\eta^2\left(f - \mu(f)\right)}{[\nu L(\mathsf{X})]^2}$$

2 Starting Off

We need to use the above to show that $\{(\check{\xi}^{N,i}, 1)\}_{1\le i\le \tilde{M}_N}$ is asymptotically normal for $(\mu, \tilde{\mathsf{A}}, \check{\sigma}, \{M_N^{1/2}\})$ with

$$\check{\sigma}(f) = \mathrm{Var}_\mu(f) + \tilde{\sigma}(f).$$

In other words, if we pick $f \in \tilde{\mathsf{A}}$, we want to show that

$$\sqrt{M_N}\left[M_N^{-1}\sum_{i=1}^{M_N}\{f(\check{\xi}^{N,i}) - \mu(f)\}\right]$$

converges in distribution to a mean-zero normal random variable.

The main trick is splitting it up into two pieces by adding and subtracting conditional expectations, which condition on all the information that is had just before resampling is conducted:

$$\begin{aligned}
&\sqrt{M_N}\left[M_N^{-1}\sum_{i=1}^{M_N}\Big\{f(\tilde{\xi}^{N,i}) - \mu(f)\Big\}\right]\\
&= \sqrt{\tilde{M}_N}\left[\tilde{M}_N^{-1}\sum_{i=1}^{\tilde{M}_N}\Big\{f(\tilde{\xi}^{N,i}) - E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N]\Big\} + \tilde{M}_N^{-1}\sum_{i=1}^{\tilde{M}_N}\Big\{E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f)\Big\}\right]\\
&= \sqrt{\tilde{M}_N}\,[B_N + A_N]
\end{aligned}$$

The book writes these A_N and B_N just as we do, but to understand the expression of A_N, we have to write out the average conditional expectation the long way:

$$\begin{aligned}
A_N &= \tilde{M}_N^{-1}\sum_{i=1}^{\tilde{M}_N}\Big\{E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f)\Big\}\\
&= E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f) && \text{(identicalness)}\\
&= \left[\sum_{i=1}^{M_N}\frac{\omega^{N,i}\frac{d\mu}{d\nu}(\xi^{N,i})}{\sum_j \omega^{N,j}\frac{d\mu}{d\nu}(\xi^{N,j})} f(\xi^{N,i})\right] - \mu(f)\\
&= \sum_{i=1}^{M_N}\frac{\omega^{N,i}\frac{d\mu}{d\nu}(\xi^{N,i})}{\sum_j \omega^{N,j}\frac{d\mu}{d\nu}(\xi^{N,j})}\left[f(\xi^{N,i}) - \mu(f)\right]
\end{aligned}$$

3 Handling A_N

Recall our assumptions

Assumption 9.1.1: $\mu \ll \nu$, and $d\mu/d\nu > 0$ almost surely.

Assumption 9.2.6: $\{(\xi^{N,i}, \omega^{N,i})\}_{1\le i\le M_N}$ is consistent for (ν, C) (a proper set), and $d\mu/d\nu \in \mathsf{C}$.

Assumption 9.2.10: The weighted sample $\{(\xi^{N,i}, \omega^{N,i})\}_{1\le i\le M_N}$ is asymptotically normal for $(\nu, \mathsf{A}, \sigma, \{a_N\})$, and $\frac{d\mu}{d\nu} \in \mathsf{A}$.

Last, $a_N^2/\tilde{M}_N \to \alpha$ and

$$\tilde{\mathsf{A}} \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathsf{X}, \mu) : |f| \frac{d\mu}{d\nu} \in \mathsf{A}, f^2 \frac{d\mu}{d\nu} \in \mathsf{C} \right\}.$$

Theorem 9.2.11 tells us that

$$a_N A_N \stackrel{\text{D}}{\to} \text{Normal}\left[0, \sigma^2\left(\frac{d\mu}{d\nu}\left[f - \mu(f)\right]\right)\right]$$

(note $|f|\frac{d\mu}{d\nu} \in \mathsf{A}$ because $\bar{\mathsf{A}} \subset \tilde{\mathsf{A}}$).

4 Bringing in B_N

In this section, we want to show the random vector

$$\begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix}$$

converges in distribution to a multivariate normal distribution. We will show this by showing that the joint characteristic function converges to the multivariate normal's characteristic function.

Pick $u, v \in \mathbb{R}$, and look at the bivariate characteristic function. The trick is to iterate expectations:

$$\begin{aligned} E\left[\exp\left(i\left\{u\sqrt{\tilde{M}_N}B_N + va_NA_N\right\}\right)\right] &= E\left[E\left[\exp\left(i\left\{u\sqrt{\tilde{M}_N}B_N + va_NA_N\right\}\right)\middle|\mathcal{F}^N\right]\right] \\ &= E\left[E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right)\exp\left(iva_NA_N\right)\middle|\mathcal{F}^N\right]\right] \\ &= E\left[\exp\left(iva_NA_N\right)E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right)\middle|\mathcal{F}^N\right]\right]. \end{aligned}$$

Then we take the limit as $N \to \infty$

$$
\begin{aligned}
&\lim_{N\to\infty} E\left[\exp\left(i\left\{u\sqrt{\tilde{M}_N}B_N + va_N A_N\right\}\right)\right] \\
&= \lim_{N\to\infty} E\left[\exp\left(iva_N A_N\right) E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right) \middle| \mathcal{F}^N\right]\right] && \text{(previous)} \\
&= E\left[\lim_{N\to\infty} \exp\left(iva_N A_N\right) E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right) \middle| \mathcal{F}^N\right]\right] && \text{(DCT)} \\
&= E\left[\lim_{N\to\infty} \exp\left(iva_N A_N\right) \lim_{N\to\infty} E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right) \middle| \mathcal{F}^N\right]\right] \\
& && \text{(limit properties)} \\
&= E\left[\lim_{N\to\infty} \exp\left(iva_N A_N\right) \exp\left[-\frac{u^2 \mathrm{Var}_\mu(f)}{2}\right]\right] && \text{(9.2.12)} \\
&= \exp\left[-\frac{u^2 \mathrm{Var}_\mu(f)}{2}\right] E\left[\lim_{N\to\infty} \exp\left(iva_N A_N\right)\right] && \text{(linearity of } E) \\
&= \exp\left[-\frac{u^2 \mathrm{Var}_\mu(f)}{2}\right] \lim_{N\to\infty} E\left[\exp\left(iva_N A_N\right)\right] && \text{(DCT)} \\
&= \exp\left[-\frac{u^2 \mathrm{Var}_\mu(f)}{2}\right] \exp\left[-\frac{u^2\sigma^2\left(\frac{d\mu}{d\nu}\left[f-\mu(f)\right]\right)}{2}\right]. && \text{(Theorem 9.2.11)}
\end{aligned}
$$

Two of those steps required separate theorems. We showed Theorem 9.2.11 in the previous section of this document:

$$
\exp\left[iva_N A_N\right] \xrightarrow{\mathrm{P}} \exp\left[-\frac{u^2\sigma^2\left(\frac{d\mu}{d\nu}\left[f-\mu(f)\right]\right)}{2}\right].
$$

And by Proposition 9.2.12

$$
E\left[\exp\left(iu\sqrt{\tilde{M}_N}B_N\right) \middle| \mathcal{F}^N\right] \xrightarrow{\mathrm{P}} \exp\left[-\frac{u^2 \mathrm{Var}_\mu(f)}{2}\right]
$$

(note $f^2 \frac{d\mu}{d\nu} \in \mathsf{C}$ because of the definition of $\tilde{\mathsf{A}}$.)

5 Final Step

Before, we showed

$$
\begin{bmatrix} \sqrt{\tilde{M}_N}B_N \\ a_N A_N \end{bmatrix} \xrightarrow{\mathrm{D}} \text{Normal}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \mathrm{Var}_\mu(f) & 0 \\ 0 & \sigma^2\left(\frac{d\mu}{d\nu}\left[f-\mu(f)\right]\right) \end{bmatrix}\right)
$$

If $\alpha < 1$, then $\frac{a_N^2}{\tilde{M}_N} \to \alpha < 1$, and we can take the following inner product and apply the multivariate delta method:

$$\begin{aligned}
a_N \left[B_N + A_N\right] &= \begin{bmatrix} a_N/\sqrt{\tilde{M}_N} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix} \\
&\xrightarrow{\text{D}} \text{Normal}\left(0, \lim_N \frac{a_N^2}{\tilde{M}_N} \text{Var}_\mu(f) + \sigma^2 \left(\frac{d\mu}{d\nu}\left[f - \mu(f)\right]\right)\right) \\
&= \text{Normal}\left(0, \alpha \text{Var}_\mu(f) + \sigma^2 \left(\frac{d\mu}{d\nu}\left[f - \mu(f)\right]\right)\right)
\end{aligned}$$

If $\alpha \geq 1$ (recall that it's possibly infinite-valued), then $\frac{a_N^2}{\tilde{M}_N} \to \alpha$, so the reciprocal of that goes to $1/\alpha$. We take the following inner product, and apply the multvariate delta method:

$$\begin{aligned}
\sqrt{\tilde{M}_N} \left[B_N + A_N\right] &= \begin{bmatrix} 1 & \sqrt{\tilde{M}_N}/a_N \end{bmatrix} \begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix} \\
&\xrightarrow{\text{D}} \text{Normal}\left(0, \text{Var}_\mu(f) + \lim_N \frac{\tilde{M}_N}{a_N^2} \sigma^2 \left(\frac{d\mu}{d\nu}\left[f - \mu(f)\right]\right)\right) \\
&= \text{Normal}\left(0, \text{Var}_\mu(f) + \frac{1}{\alpha} \sigma^2 \left(\frac{d\mu}{d\nu}\left[f - \mu(f)\right]\right)\right).
\end{aligned}$$