

# A More Detailed Proof of Prop. 9.3.7

April 15, 2020

## 1 Statement

Recall  $\tilde{C} = \{f \in L^1(X, \mu) : x \mapsto L(x, |f|) \in C\}$  and assume

1. Assumption 9.3.1:  $0 < L(x, X) < \infty$ .
2. Assumption 9.3.2:  $\{(\xi^{N,i}, 1)\}_{1 \leq i \leq M_N}$  are consistent for  $(\nu, C)$ .  $L(x, X) \in C$ .
3. Assumption 9.3.3:  $\forall x \in X$ ,  $L(x, \cdot) \ll R(x, \cdot)$ , and there exists a strictly positive RN derivative:  $\frac{dL(x, \cdot)}{dR(x, \cdot)}$ .
4. Assumption 9.3.6: the weighted sample  $\{(\xi^{N,i}, 1)\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\nu, A, \sigma, M_N^{1/2})$ , where  $A$  is proper, and  $\sigma$  is some nonnegative function on  $A$ .

Then  $\tilde{A}$  is proper, and  $\{(\tilde{\xi}^{N,j}, \tilde{\omega}^{N,j})\}_{1 \leq j \leq M_N}$  is asymptotically normal for  $(\mu, \tilde{A}, \tilde{\sigma}, M_N^{1/2})$ .

## 2 Overall Strategy

We already showed that  $\tilde{A}$  is proper in class. Now we want to show that asymptotic normality part. So we're going to use Slutsky's to show that

$$\sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \bigg/ \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}$$

converges to a normal distribution (non mean zero). The denominator is consistent, the numerator is asymptotically normal.

Unlike the book, I am not assuming that, WLOG, the mean is 0.

The denominator is the easiest part.  $L(x, |1|) = L(x, X) \in C$  by assumption 9.3.2, so this means  $1 \in \tilde{C}$ . Theorem 9.3.5 from last class gives us that the denominator converges in probability to  $\nu L(X)$ .

Regarding the numerator, write it as

$$\begin{aligned}
& \sqrt{M_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} \\
& \quad + \sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \\
& = \sqrt{M_N} B_N + \sqrt{M_N} A_N.
\end{aligned}$$

### 3 The second piece of the numerator

Looking at the second piece, recall that

$$\begin{aligned}
\sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] &= \sqrt{M_N} M_N^{-1} \sum_{i=1}^{M_N} L(\xi^{N,i}, f) \\
&\xrightarrow{D} N(\nu L(f), \sigma^2(Lf)) \\
&= N(\nu L(f), \nu \left[ (L(x, f) - \nu L(f))^2 \right])
\end{aligned}$$

The convergence takes place because  $f$  is assumed to be from  $\tilde{\mathbf{A}}$ , and this implies that, by definition,  $L(x, f) \in \mathbf{A}$ .

Note that this variance is taken with respect to the previous time's measure, and it is the variance of a conditional expectation. This is half of the law of total variance.

### 4 The first piece of the numerator

The second part of the numerator is

$$\sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} = \sqrt{\tilde{M}_N} B_N$$

We're going to apply Proposition 9.5.12 to it. We make the substitution  $V_{N,j} = \tilde{M}_N^{-1/2} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})$ , and verify the assumptions one by one.

#### 4.1 First Condition

The triangular array is conditionally independent given  $\mathcal{F}^N$ , and for any row/column,  $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] < \infty$

This is true because we are only looking at functions  $f$  in

$$\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathbf{X}, \mu) : L(x, f) \in \mathbf{A}, x \mapsto \int R(x, dx') \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \in \mathbf{C} \right\}.$$

## 4.2 Second Condition

There exists a constant  $\sigma^2 > 0$  such that

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \text{Var} \left[ \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \xrightarrow{\mathbb{P}} \sigma^2$$

This is easy to show if you split up the variance into the mean of the square minus the square of the mean:

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E \left[ \{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N \right] - \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{E [\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N]\}^2$$

First,  $f \in \tilde{\mathbf{A}}$ , so

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E \left[ \{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N \right] \xrightarrow{\mathbb{P}} \iint \nu(dx) R(x, dx') \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2$$

it might be helpful to write that last piece as an expectation:

$$\iint \nu(dx) R(x, dx') \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 = [\nu \otimes R] \left\{ \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \right\}$$

The second part also converges too. This is because

$$x \mapsto \left[ \int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \in \mathbf{C}$$

because

$$\left[ \int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \leq \int R(x, dx') \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2$$

of Jensen's inequality. The right hand side is in  $\mathbf{C}$ , and because it is proper, the left hand side is too. This means that

$$\begin{aligned} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{E [\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N]\}^2 &\xrightarrow{\mathbb{P}} \nu \left[ \left[ \int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \right] \\ &= \nu \left( [L(x, f)]^2 \right) \end{aligned}$$

### 4.3 Third Condition

The third condition of 9.5.12 is the Lindberg condition: for all  $\epsilon > 0$ ,

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} [\tilde{\omega}^{N,j}]^2 [f(\tilde{\xi}^{N,j})]^2 \mathbf{1} \left\{ |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \epsilon \sqrt{\tilde{M}_N} \right\}$$

This is true, and we're going to use the dominated convergence argument again here. After picking an  $\epsilon$ , pick an arbitrary  $C > 0$  and notice that

$$\begin{aligned} & \lim_{\tilde{M}_N \rightarrow \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \sqrt{\tilde{M}_N} \epsilon \right) \mid \mathcal{F}^N] \\ & \leq \lim_{\tilde{M}_N \rightarrow \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \\ & \quad \text{(if } C \leq \sqrt{\tilde{M}_N} \epsilon) \\ & = \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \end{aligned}$$

The last line is true because  $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \in \mathbb{C}$ . Why? Well by assumption,  $f \in \tilde{\mathbf{A}}$ , and by definition of that set of functions,  $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] \in \mathbb{C}$ . This one is larger than the other, so propriety guarantees it works.

Following up on that we have

$$\begin{aligned} 0 & \leq \lim_{\tilde{M}_N \rightarrow \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 \mathbf{1}(|V_{N,j}| \geq \epsilon) \mid \mathcal{F}^N] \\ & = \lim_{C \rightarrow \infty} \lim_{\tilde{M}_N \rightarrow \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 \mathbf{1}(|V_{N,j}| \geq \epsilon) \mid \mathcal{F}^N] \\ & \quad \text{(no } C \text{ so limit doesn't matter)} \\ & \leq \lim_{C \rightarrow \infty} \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \quad \text{(above work)} \\ & = \nu \left\{ \lim_{C \rightarrow \infty} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left( |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \quad \text{(DCT)} \\ & = 0. \end{aligned}$$

Therefore condition iii of 9.5.12 is satisfied, and we have the desired result:

$$\begin{aligned}
& \sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} \\
& \xrightarrow{D} \text{Normal} \left( 0, \iint \nu(dx) R(x, dx') \left[ \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 - \nu \left( [L(x, f)]^2 \right) \right) \\
& = \text{Normal} (0, \eta^2(f))
\end{aligned}$$

#### 4.4 Putting it All Together

So far we have shown that the following two are asymptotically normal:

1.  $M_N^{1/2} A_n = \sqrt{M_N} \frac{1}{M_N} \sum_{i=1}^{\tilde{M}_N} \left[ E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right]$
2.  $\tilde{M}_N^{1/2} B_n = \sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\}.$

To show their asymptotic \*joint\* distribution, we look at the limit of their joint characteristic functions, and we use the law of total expectation:

$$\begin{aligned}
\lim_{N \rightarrow \infty} E \left[ \exp \left[ i \left( s M_N^{1/2} A_n + t \tilde{M}_N^{1/2} B_n \right) \right] \right] &= \lim_{N \rightarrow \infty} E \left[ \exp \left[ i s M_N^{1/2} A_n + i t \tilde{M}_N^{1/2} B_n \right] \right] \\
&= \lim_{N \rightarrow \infty} E \left[ \exp \left[ i s M_N^{1/2} A_n \right] \exp \left[ i t \tilde{M}_N^{1/2} B_n \right] \right] \\
&= \lim_{N \rightarrow \infty} E \left[ \exp \left[ i s M_N^{1/2} A_n \right] E \left( \exp \left[ i t \tilde{M}_N^{1/2} B_n \right] \mid \mathcal{F}^N \right) \right] \\
&= E \left[ \lim_{N \rightarrow \infty} \exp \left[ i s M_N^{1/2} A_n \right] \lim_{N \rightarrow \infty} E \left( \exp \left[ i t \tilde{M}_N^{1/2} B_n \right] \mid \mathcal{F}^N \right) \right] \\
&\quad \text{(DCT)} \\
&= E \left[ \lim_{N \rightarrow \infty} \exp \left[ i s M_N^{1/2} A_n \right] \exp \left( -.5 t^2 \eta^2(f) \right) \right] \\
&= \exp \left( -.5 t^2 \eta^2(f) \right) E \left[ \lim_{N \rightarrow \infty} \exp \left[ i s M_N^{1/2} A_n \right] \right] \\
&= \exp \left( -.5 t^2 \eta^2(f) \right) \lim_{N \rightarrow \infty} E \left[ \exp \left[ i s M_N^{1/2} A_n \right] \right] \\
&\quad \text{(DCT)} \\
&= \exp \left( -.5 t^2 \eta^2(f) \right) \exp \left( i s \nu L(f) - .5 s^2 \sigma^2(Lf) \right)
\end{aligned}$$

The second to last move should seem familiar:

$$M_N^{1/2} (A_N + B_N) = M_N^{1/2} A_N + \frac{1}{\sqrt{\alpha_N}} \tilde{M}_N^{1/2} B_N \xrightarrow{D} \text{Normal} \left[ \nu L(f), \sigma^2(Lf) + \frac{1}{\alpha} \eta^2(f) \right].$$

Using Slutsky's theorem is our final move:

$$\begin{aligned}
\frac{M_N^{1/2}(A_N + B_N)}{\frac{1}{M_N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}} &\xrightarrow{D} \text{Normal} \left[ \frac{\nu L(f)}{\nu L(X)}, \frac{\sigma^2(Lf) + \frac{1}{\alpha} \eta^2(f)}{[\nu L(X)]^2} \right] \\
&= \text{Normal} [\mu(f), \tilde{\sigma}^2(f)] .
\end{aligned}$$