A More Detailed Proof of Prop. 9.3.7

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1 Statement

Recall $\tilde{\mathsf{C}} = \left\{ f \in L^1(\mathsf{X}, \mu) : x \mapsto L(x, |f|) \in \mathsf{C} \right\}$ and assume

1. Assumption 9.3.1: $0 < L(x, \mathsf{X}) < \infty$.
2. Assumption 9.3.2: $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ are consistent for (ν, C). $L(x, \mathsf{X}) \in \mathsf{C}$.
3. Assumption 9.3.3: $\forall x \in \mathsf{X}$, $L(x, \cdot) \ll R(x, \cdot)$, and there exists a strictly positive RN derivative: $\frac{dL(x,\cdot)}{dR(x,\cdot)}$.
4. Assumption 9.3.6: the weighted sample $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ is asymptotically normal for $(\nu, \mathsf{A}, \sigma, M_N^{1/2})$, where A is proper, and σ is some nonnegative function on A.

Then $\tilde{\mathsf{A}}$ is proper, and $\{(\tilde{\xi}^{N,j}, \tilde{\omega}^{N,j})\}_{1 \le j \le M_N}$ is asymptotically normal for $(\mu, \tilde{\mathsf{A}}, \tilde{\sigma}, M_N^{1/2})$.

2 Overall Strategy

We already showed that $\tilde{\mathsf{A}}$ is proper in class. Now we want to show that asymptotic normality part. So we're going to use Slutsky's to show that

$$\sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \Big/ \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}$$

converges to a normal distribution (non mean zero). The denominator is consistent, the numerator is asymptotically normal.

Unlike the book, I am not assuming that, WLOG, the mean is 0.

The denominator is the easiest part. $L(x, |1|) = L(x, \mathsf{X}) \in \mathsf{C}$ by assumption 9.3.2, so this means $1 \in \tilde{\mathsf{C}}$. Theorem 9.3.5 from last class gives us that the denominator converges in probability to $\nu L(\mathsf{X})$.

Regarding the numerator, write it as

$$\begin{aligned}
&\sqrt{M_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N]\right\} \\
&\qquad + \sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \\
&= \sqrt{M}_N B_N + \sqrt{M}_N A_N.
\end{aligned}$$

3 The second piece of the numerator

Looking at the second piece, recall that

$$\begin{aligned}
\sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] &= \sqrt{M_N}M_N^{-1}\sum_{i=1}^{M_N}L(\xi^{N,i}, f) \\
&\xrightarrow{D} N(\nu L(f), \sigma^2(Lf)) \\
&= N(\nu L(f), \nu\left[(L(x,f) - \nu L(f))^2\right])
\end{aligned}$$

The convergence takes place because f is assumed to be from $\tilde{\mathsf{A}}$, and this implies that, by definition, $L(x, f) \in \mathsf{A}$.

Note that this variance is taken with respect to the previous time's measure, and it is the variance of a conditional expectation. This is half of the law of total variance.

4 The first piece of the numerator

The second part of the numerator is

$$\sqrt{\tilde{M}_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N]\right\} = \sqrt{\tilde{M}_N}B_N$$

We're going to apply Proposition 9.5.12 to it. We make the substitution $V_{N,j} = \tilde{M}_N^{-1/2}\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})$, and verify the assumptions one by one.

4.1 First Condition

The triangular array is conditionally independent given $\mathcal{F}^N$, and for any row/column, $E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] < \infty$

This is true because we are only looking at functions f in

$$\tilde{\mathsf{A}} \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathsf{X}, \mu) : L(x,f) \in \mathsf{A}, x \mapsto \int R(x, dx') \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \in \mathsf{C} \right\}.$$

4.2 Second Condition

There exists a constant $\sigma^2 > 0$ such that

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \mathrm{Var}\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \xrightarrow{\text{P}} \sigma^2$$

This is easy to show if you split up the variance into the mean of the square minus the square of the mean:

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N \right] - \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{ E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \}^2$$

First, $f \in \tilde{\mathsf{A}}$, so

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N \right] \xrightarrow{\text{P}} \iint \nu(dx) R(x, dx') \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2$$

it might be helpful to write that last piece as an expectation:

$$\iint \nu(dx) R(x, dx') \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 = [\nu \otimes R] \left\{ \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \right\}$$

The second part also converges too. This is because

$$x \mapsto \left[\int R(x, dx') \frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \in \mathsf{C}$$

because

$$\left[\int R(x, dx') \frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \leq \int R(x, dx') \left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2$$

of Jensen's inequality. The right hand side is in C, and because it is proper, the left hand side is too. This means that

$$\begin{aligned} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{ E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \}^2 &\xrightarrow{\text{P}} \nu \left[\left[\int R(x, dx') \frac{dL(x,\cdot)}{dR(x,\cdot)}(x') f(x') \right]^2 \right] \\ &= \nu \left([L(x,f)]^2 \right) \end{aligned}$$

4.3 Third Condition

The third condition of 9.5.12 is the Lindberg condition: for all $\epsilon > 0$,

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} [\tilde{\omega}^{N,j}]^2 [f(\tilde{\xi}^{N,j})]^2 \mathbf{1} \left\{ |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \epsilon \sqrt{\tilde{M}_N} \right\}$$

This is true, and we're going to use the dominated convergence argument again here. After picking an ϵ, pick an arbitrary $C > 0$ and notice that

$$\begin{aligned}
&\lim_{\tilde{M}_N \to \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \sqrt{\tilde{M}_N \epsilon} \right) \mid \mathcal{F}^N] \\
&\leq \lim_{\tilde{M}_N \to \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \\
&\hfill \text{(if } C \leq \sqrt{\tilde{M}_N \epsilon}) \\
&= \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\}
\end{aligned}$$

The last line is true because $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \in \mathsf{C}$. Why? Well by assumption, $f \in \tilde{\mathsf{A}}$, and by definition of that set of functions, $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] \in \mathsf{C}$. This one is larger than the other, so proprietary guarantees it works.

Following up on that we have

$$\begin{aligned}
0 &\leq \lim_{M_N \to \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 1 \left(|V_{N,j}| \geq \epsilon \right) \mid \mathcal{F}^N] \\
&= \lim_{C \to \infty} \lim_{M_N \to \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 1 \left(|V_{N,j}| \geq \epsilon \right) \mid \mathcal{F}^N] \\
&\hfill \text{(no } C \text{ so limit doesn't matter)} \\
&\leq \lim_{C \to \infty} \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \qquad \text{(above work)} \\
&= \nu \left\{ \lim_{C \to \infty} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) 1 \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \qquad \text{(DCT)} \\
&= 0.
\end{aligned}$$

Therefore condition iii of 9.5.12 is satisfied, and we have the desired result:

$$
\sqrt{\tilde{M}_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})-\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]\right\}
$$

$$
\xrightarrow{\text{D}}\text{Normal}\left(0,\iint\nu(dx)R(x,dx')\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2-\nu\left([L(x,f)]^2\right)\right)
$$

$$
=\text{Normal}\left(0,\eta^2(f)\right)
$$

4.4 Putting it All Together

So far we have shown that the following two are asymptotically normal:

1. $M_N^{1/2}A_n=\sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\left[E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]-\nu L(f)\right]$
2. $\tilde{M}_N^{1/2}B_n=\sqrt{\tilde{M}_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})-\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]\right\}$.

To show their asymptotic *joint* distribution, we look at the limit of their joint characteristic functions, and we use the law of total expectation:

$$
\begin{aligned}
\lim_{N\to\infty}E\left[\exp\left[i\left(sM_N^{1/2}A_n+t\tilde{M}_N^{1/2}B_n\right)\right]\right]&=\lim_{N\to\infty}E\left[\exp\left[isM_N^{1/2}A_n+it\tilde{M}_N^{1/2}B_n\right]\right]\\
&=\lim_{N\to\infty}E\left[\exp\left[isM_N^{1/2}A_n\right]\exp\left[it\tilde{M}_N^{1/2}B_n\right]\right]\\
&=\lim_{N\to\infty}E\left[\exp\left[isM_N^{1/2}A_n\right]E\left(\exp\left[it\tilde{M}_N^{1/2}B_n\right]\mid\mathcal{F}^N\right)\right]\\
&=E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\lim_{N\to\infty}E\left(\exp\left[it\tilde{M}_N^{1/2}B_n\right]\mid\mathcal{F}^N\right)\right]\\
&\qquad\text{(DCT)}\\
&=E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\exp\left(-.5t^2\eta^2(f)\right)\right]\\
&=\exp\left(-.5t^2\eta^2(f)\right)E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\right]\\
&=\exp\left(-.5t^2\eta^2(f)\right)\lim_{N\to\infty}E\left[\exp\left[isM_N^{1/2}A_n\right]\right]\\
&\qquad\text{(DCT)}\\
&=\exp\left(-.5t^2\eta^2(f)\right)\exp\left(is\nu L(f)-.5s^2\sigma^2(Lf)\right)
\end{aligned}
$$

The second to last move should seem familiar:

$$
M_N^{1/2}(A_N+B_N)=M_N^{1/2}A_N+\frac{1}{\sqrt{\alpha_N}}\tilde{M}_N^{1/2}B_N\xrightarrow{\text{D}}\text{Normal}\left[\nu L(f),\sigma^2(Lf)+\frac{1}{\alpha}\eta^2(f)\right].
$$

Using Slutsky's theorem is our final move:

$$
\begin{aligned}
\frac{M_N^{1/2}(A_N + B_N)}{\frac{1}{\tilde{M}_N}\sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}} &\xrightarrow{\mathrm{D}} \mathrm{Normal}\left[\frac{\nu L(f)}{\nu L(X)}, \frac{\sigma^2(Lf) + \frac{1}{\alpha}\eta^2(f)}{[\nu L(X)]^2}\right] \\
&= \mathrm{Normal}\left[\mu(f), \tilde{\sigma}^2(f)\right].
\end{aligned}
$$