A More Detailed Proof of Prop. 9.5.9

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Take care to realize that the book has two i's. One represents the complex number $i = \sqrt{-1}$, and the other represents the index of the sum. To prevent confusion, the book changed the font for the complex number. However, I choose a different path, and I use j as the index of the sum.

The goal is to show that

$$E\left[\exp\left(iu\sum_j U_{N,j}\right)\middle|\mathcal{F}^N\right] - \exp(-\sigma^2u^2/2) \xrightarrow{p} 0.$$

However, the continuous mapping theorem, together with assumption (ii), implies that

$$\exp\left(-(u^2/2)\sum_j \sigma_{N,j}^2\right) - \exp(-\sigma^2u^2/2) \xrightarrow{p} 0,$$

so this explains why it suffices to prove 9.74.

We provded Lemma 9.5.8 for complex numbers with modulus less than 1. Real numbers are complex numbers, too, though.

$$
\begin{aligned}
&\left| E\left[\exp\left(iu\sum_j U_{N,j}\right) - \exp\left(-(u^2/2)\sum_j \sigma^2_{N,j}\right) \middle| \mathcal{F}^N\right]\right| \\
&= \left| E\left[\prod_j \exp\left(iuU_{N,j}\right) - \prod_j \exp\left(-(u^2/2)\sigma^2_{N,j}\right) \middle| \mathcal{F}^N\right]\right| && \text{(algebra)} \\
&= \left| \prod_j E\left[\exp\left(iuU_{N,j}\right) \middle| \mathcal{F}^N\right] - \prod_j E\left[\exp\left(-(u^2/2)\sigma^2_{N,j}\right) \middle| \mathcal{F}^N\right]\right| \\
& && \text{(independence, linearity)} \\
&\le \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - \exp\left(-(u^2/2)\sigma^2_{N,j}\right) \middle| \mathcal{F}^N\right]\right| && \text{(Lemma 9.5.8)} \\
&= \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma^2_{N,j}/2) + (1 - u^2\sigma^2_{N,j}/2) - \exp\left(-(u^2/2)\sigma^2_{N,j}\right) \middle| \mathcal{F}^N\right]\right| \\
& && \text{(algebra)} \\
&\le \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma^2_{N,j}/2) \mid \mathcal{F}^N\right]\right| \\
&\qquad + \sum_j \left| E\left[\exp\left(-(u^2/2)\sigma^2_{N,j}\right) - (1 - u^2\sigma^2_{N,j}/2) \mid \mathcal{F}^N\right]\right| \\
& && \text{(linearity, tri-ineq)} \\
&= A_N + B_N. && \text{(defn)}
\end{aligned}
$$

So, the overall goal is to show these two terms converge to 0. Focusing on A_N first, we will use the inequality we proved in the last section, namely

$$
-\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \le e^{iuU_{N,j}} - (1 + iuU_{N,j} - \frac{1}{2}u^2U^2_{N,j}) \le \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}.
$$

Take conditional expectations on all sides, then take the absolute value (in that order).

This gives you the book's inequality (the third one, unnumbered, on page 340):

$$
\begin{aligned}
\left| E\left[e^{iuU_{N,j}} - (1 + iuU_{N,j} - \frac{1}{2}u^2U^2_{N,j}) \mid \mathcal{F}^N\right]\right| &= \left| E\left[e^{iuU_{N,j}} \mid \mathcal{F}^N\right] - (1 - \frac{1}{2}u^2\sigma^2_{N,j})\right| \\
&\le E\left[\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \mid \mathcal{F}^N\right] \\
&\le E\left[|uU_{N,j}|^2 \mid \mathcal{F}^N\right] \\
&< \infty. && \text{(assumption)}
\end{aligned}
$$

We are using the fact that the mean of each $U_{N,j}$ is zero. The absolute value sign has to be on the outside.

Another trick we need is the following inequality. Let $\epsilon > 0$. Then

$$\begin{aligned}
&\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \\
&= \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}1(|U_{N,j}| > \epsilon) + \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}1(|U_{N,j}| \le \epsilon) \\
&\le |uU_{N,j}|^2 1(|U_{N,j}| > \epsilon) + \epsilon|u|^3|U_{N,j}|^2\mathbb{1}(|U_{N,j}| \le \epsilon) \\
&\le |uU_{N,j}|^2 1(|U_{N,j}| > \epsilon) + \epsilon|u|^3|U_{N,j}|^2.
\end{aligned}$$

This let's us take care of the first part:

$$\begin{aligned}
A_N &= \sum_j \left|E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma_{N,j}^2/2) \mid \mathcal{F}^N\right]\right| && \text{(defn.)} \\
&\le \sum_{j=1}^{M_N} E[\min\{|uU_{N,j}|^2, |uU_{N,i}|^3\} \mid \mathcal{F}^N] && \text{(Taylor's ineq)} \\
&\le u^2 \sum_{j=1}^{M_N} E[U_{N,j}^2 \mathbb{1}(|U_{N,j}| > \epsilon) \mid \mathcal{F}^N] + \epsilon|u|^3 \sum_{j=1}^{M_N} \sigma_{N,j}^2 && \text{(above inequality)} \\
&\to 0 + \epsilon|u|^3\sigma^2. && \text{(assumption ii. and iii.)}
\end{aligned}$$

Now we look at B_n. The real-valued version of Taylor's Inequality states

$$\left|e^{-x} - \sum_{i=0}^{n} \frac{x^i}{i!}\right| \le \frac{M}{(n+1)!}|x-a|^{n+1}$$

as long as $|f^{(n+1)}(x)| \le M$. Set $n = 1$, $a = 0$, and looking at $x \ge 0$, which means $|f^{(n+1)}(x)| = |f^{(2)}(x)| = |e^{-x}| \le 1$. This gives us,

$$|e^{-x} - 1 + x| \le x^2/2.$$

What does this mean for B_N? Well,

$$\begin{aligned}
B_N &= \sum_{i=1}^{M_N} |\exp(-u^2\sigma_{N,i}^2/2) - (1 - u^2\sigma_{N,i}^2/2)| && \text{(defn.)} \\
&\le \sum_{i=1}^{M_N} \left(u^2\sigma_{N,i}^2/2\right)^2/2 && \text{(real-valued Taylor's)} \\
&= \frac{u^4}{8}\sum_{i=1}^{M_N} \sigma_{N,i}^4 \\
&\le \frac{u^4}{8}\left(\max_i \sigma_{N,i}^2\right)\sum_{i=1}^{M_N} \sigma_{N,i}^2 && \text{(see below)} \\
&\to \frac{u^4}{8} \times 0 \times \sigma^2.
\end{aligned}$$

The last line follows because of assumption 2 and the Uniform Smallness Condition (Remark 9.5.10.)