A More Detailed Proof of Prop. 9.5.12

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This applies 9.5.9 to random variables that no longer have a (conditional) mean of 0. Set $U_{N,j} = V_{N,j} - E[V_{N,j} \mid \mathcal{F}^N]$ and use 9.5.9. The first two conditions are easily checked. To check the third, we must show that

$$\sum_{j=1}^{M_N} E\left[U_{N,j}^2 \mathbb{1}\left(|U_{N,j}| \geq \epsilon\right) \mid \mathcal{F}^N\right] \xrightarrow{P} 0.$$

When they say "by Jensen's," there's a little more to it.

$$\begin{aligned}
U_{N,j}^2 &= \left(V_{N,j} - E[V_{N,j} \mid \mathcal{F}^N]\right)^2 \\
&= V_{N,j}^2 - 2V_{N,j}E[V_{N,j} \mid \mathcal{F}^N] + \left(E[V_{N,j} \mid \mathcal{F}^N]\right)^2 \\
&= 2V_{N,j}^2 + 2\left(E[V_{N,j} \mid \mathcal{F}^N]\right)^2 - \left(V_{N,j} + E[V_{N,j} \mid \mathcal{F}^N]\right)^2 \\
&\leq 2V_{N,j}^2 + 2\left(E[V_{N,j} \mid \mathcal{F}^N]\right)^2 \\
&\leq 2V_{N,j}^2 + 2E[V_{N,j}^2 \mid \mathcal{F}^N]. && \text{(Jensen's)}
\end{aligned}$$

This implies that

$$\begin{aligned}
\{|U_{N,j}| \geq \epsilon\} = \{U_{N,j}^2 \geq \epsilon^2\} & \\
&\subset \{V_{N,j}^2 + E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/2\} \\
&\subset \{V_{N,j}^2 \geq \epsilon^2/4\} \cup \{E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\}
\end{aligned}$$

which in turn implies that

$$\mathbb{1}\left(|U_{N,j}| \geq \epsilon\right) \leq \mathbb{1}\left(V_{N,j}^2 \geq \epsilon^2/4\right) + \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right)$$

and taking these two inequalities together we have

$$U_{N,j}^2 \mathbb{1}\left(|U_{N,j}| \geq \epsilon\right) \leq 2\left(V_{N,j}^2 + E[V_{N,j}^2 \mid \mathcal{F}^N]\right)\left(\mathbb{1}\left(\{V_{N,j}^2 \geq \epsilon^2/4\right) + \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right)\right).$$

Take the conditional expectation on both sides, then sum.

$$
\begin{aligned}
&\sum_j E[U_{N,j}^2 \mathbb{1}\left(|U_{N,j}| \geq \epsilon\right)] \\
&\leq 2\sum_j E[V_{N,j}^2 \mathbb{1}\left(V_{N,j}^2 \geq \epsilon^2/4\right) \mid \mathcal{F}^N] + 2\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right) \\
&\qquad + 2\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] P(V_{N,j}^2 \geq \epsilon^2/4 \mid \mathcal{F}^N) + 2\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right) \\
&= 2\sum_j E[V_{N,j}^2 \mathbb{1}\left(V_{N,j}^2 \geq \epsilon^2/4\right) \mid \mathcal{F}^N] + 4\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right) \\
&\qquad + 2\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] P(V_{N,j}^2 \geq \epsilon^2/4 \mid \mathcal{F}^N) \\
&= 2\sum_j E[V_{N,j}^2 \mathbb{1}\left(V_{N,j}^2 \geq \epsilon^2/4\right) \mid \mathcal{F}^N] + 4B_N + 2A_N && \text{(defns)} \\
&\to 0. && \text{(9.5.7 and assumption)}
\end{aligned}
$$

The first term goes to 0 assumption (iii) of this proposition. The other two terms go to 0 using logic that is similar to the tools we used at the end of proof 9.5.7. We will reproduce those details here, but recall that in 9.5.7, there was an inequality trick we discussed. For details on that inequality, see `prof_9.5.7_proof.pdf`.

To prove $A_N \xrightarrow{p} 0$:

$$
\begin{aligned}
A_n &= \sum_i E[V_{N,i}^2 \mid \mathcal{F}^N] P(V_{N,i}^2 \geq \epsilon^2/4 \mid \mathcal{F}^N) \\
&\leq P(\max_j V_{N,j}^2 \geq \epsilon^2/4 \mid \mathcal{F}^N) \sum_i E[V_{N,i}^2 \mid \mathcal{F}^N] && \text{(max)} \\
&\leq P\left(\left\{\epsilon^2/8 + \sum_{j=1}^{M_N} V_{N,j}^2 \mathbb{1}(V_{N,j}^2 \geq \epsilon^2/8)\right\} \geq \epsilon^2/4 \middle| \mathcal{F}^N\right) \sum_i E[V_{N,i}^2 \mid \mathcal{F}^N] \\
& && \text{(9.5.7's second trick)} \\
&= P\left(\sum_{j=1}^{M_N} V_{N,j}^2 \mathbb{1}(V_{N,j}^2 \geq \epsilon^2/8) \geq \epsilon^2/8 \middle| \mathcal{F}^N\right) \sum_i E[V_{N,i}^2 \mid \mathcal{F}^N] \\
&\leq (8/\epsilon^2) \sum_j E\left[V_{N,j}^2 \mathbb{1}(V_{N,j}^2 \geq \epsilon^2/8) \mid \mathcal{F}^N\right] \sum_i E[V_{N,i}^2 \mid \mathcal{F}^N]. && \text{(Markov's)}
\end{aligned}
$$

$\sum_i E\left[V_{N,i}^2 \mathbb{1}(V_{N,i}^2 \geq \epsilon^2/8) \mid \mathcal{F}^N\right]$ goes to 0 by hypothesis. If we can show that the other term is bounded in probability, then the whole thing goes to 0 by Lemma 9.5.3 (3). TODO

To show B_N goes to zero:

$$\begin{aligned}
B_N &= \sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \mathbb{1}\left(E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right) \\
&\leq \mathbb{1}\left(\max_j E[V_{N,j}^2 \mid \mathcal{F}^N] \geq \epsilon^2/4\right) \sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \\
&\leq \mathbb{1}\left(\sum_j E[V_{N,j}^2 1(|V_{N,j}| > \epsilon^2/8) \mid \mathcal{F}^N] \geq \epsilon^2/8\right) \sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \\
&\leq (8/\epsilon^2) \sum_j E[V_{N,j}^2 \mathbb{1}\left(|V_{N,j}| > \epsilon^2/8\right) \mid \mathcal{F}^N] \sum_j E[V_{N,j}^2 \mid \mathcal{F}^N] \\
&\to 0. \qquad\qquad \text{(assumption iii and ii)}
\end{aligned}$$

The last line follows because $\sum_j E[V_{N,j}^2 \mathbb{1}\left(|V_{N,j}| > \epsilon^2/8\right) \mid \mathcal{F}^N]$ converges to 0 by hypothesis, and because we showed in the previous step that $\sum_j E[V_{N,j}^2 \mid \mathcal{F}^N]$ is bounded in probability.

This ensures condition (iii) of 9.5.9, and therefore the conclusion holds.