A More Detailed Proof of Proposition 9.5.5

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This is a revision of 9.5.1. It produces the same result with weakened assumptions.

We want to show that the assumptions of 9.5.5 imply all of the assumptions, collectively, of 9.5.1. Many of the assumptions are the same, so those don't need to be dealt with by us.

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First, let's show that the assumptions of 9.5.5 imply: for some $\epsilon > 0$, we have

$$\sum_{i=1}^{M_N} E[U_{N,i}^2 \mathbf{1}(|U_{N,i}| < \epsilon) \mid \mathcal{F}^N] \xrightarrow{\text{p}} 0.$$

Fix an $\epsilon > 0$ and notice that for $0 < \eta_1 < \epsilon$ we have

$$\begin{aligned}
&\sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(|U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] \\
&= \sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(|U_{N,i}| < \eta_1) \mid \mathcal{F}^N\right] + \sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(\eta_1 \leq |U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] \\
&\leq \eta_1 \sum_{i=1}^{M_N} E\left[|U_{N,i}| 1(|U_{N,i}| < \eta_1) \mid \mathcal{F}^N\right] + \sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(\eta_1 \leq |U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] && (1) \\
&\leq \eta_1 \sum_{i=1}^{M_N} E\left[|U_{N,i}| \mid \mathcal{F}^N\right] + \sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(\eta_1 \leq |U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] && \text{(more space)} \\
&\leq \eta_1 \sum_{i=1}^{M_N} E\left[|U_{N,i}| \mid \mathcal{F}^N\right] + \epsilon \sum_{i=1}^{M_N} E\left[|U_{N,i}| 1(\eta_1 \leq |U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] && (2) \\
&\leq \eta_1 \underbrace{\sum_{i=1}^{M_N} E\left[|U_{N,i}| \mid \mathcal{F}^N\right]}_{V_N} + \underbrace{\epsilon \sum_{i=1}^{M_N} E\left[|U_{N,i}| 1(\eta_1 \leq |U_{N,i}|) \mid \mathcal{F}^N\right]}_{W_N(\eta_1)} && \text{(more space)}
\end{aligned}$$

First, $\epsilon \sum_{i=1}^{M_N} E\left[|U_{N,i}| 1(\eta_1 \leq |U_{N,i}|) \mid \mathcal{F}^N\right]$ goes to 0 because $\sum_{i=1}^{M_N} E\left[|U_{N,i}| 1(\eta_1 \leq |U_{N,i}|) \mid \mathcal{F}^N\right]$ does by assumption. Then we just apply Lemma 9.5.4 to get

$$\sum_{i=1}^{M_N} E\left[U_{N,i}^2 1(|U_{N,i}| < \epsilon) \mid \mathcal{F}^N\right] \xrightarrow{p} 0.$$

NB: we won't use the "bar" notation here, because there are two thresholds, ϵ and η_1, and we don't want to get confused.

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Now we need to show that for the same $\epsilon > 0$ as above, that

$$\sum_{i=1}^{M_N} E[|U_{N,i}| \mathbf{1}(|U_{N,i}| \geq \epsilon) \mid \mathcal{F}^N] \xrightarrow{p} 0$$

This one is easy. We can just pick $\eta_2 = \epsilon$, and use assumption 3 of 9.5.5 again.