# A More Detailed Proof of Prop. 9.3.5

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In the slides we explained our goal was to show

$$\frac{1}{\tilde{M}^N}\sum_{j=1}^{\tilde{M}_N}\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})\xrightarrow{\text{P}}\nu L(f)$$

for any $f \in \tilde{\mathsf{C}}$. First, let's write

$$\left\{\frac{1}{\tilde{M}^N}\sum_{j=1}^{\tilde{M}_N}\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})-\frac{1}{\tilde{M}^N}\sum_{j=1}^{\tilde{M}_N}E\left[\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})\mid\mathcal{F}^N\right]\right\}+\left\{\frac{1}{\tilde{M}^N}\sum_{j=1}^{\tilde{M}_N}E\left[\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})\mid\mathcal{F}^N\right]\right\}$$

The first term converges to 0, and the second converges to $\mu(f)$.

## 1 The Trick

Recall in the slides we discussed that

$$\begin{aligned}
L(\xi^{N,i},f) &= \int L(\xi^{N,i},dy)f(y)\\
&= \int R(\xi^{N,i},dy)\frac{dL(\xi^{N,i},\cdot)}{dR(\xi^{N,i},\cdot)}(y)f(y)\\
&= E\left[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,j})\mid\mathcal{F}^N\right]
\end{aligned}$$

The reason that this is useful is that it is a function of $\nu$ samples only. Therefore, as long as this function is sufficiently well-behaved, we only need to worry about $\nu$- consistency.

Another thing: the sum over $j$ (new samples) turns into a sum over $i$ (old samples) because $\alpha_N M_N = \tilde{M}_N$:

$$\begin{aligned}
\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] &= \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, f) \\
&= \alpha_N^{-1} M_N^{-1} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, f) \\
&= \alpha_N^{-1} M_N^{-1} \sum_{i=1}^{M_N} \alpha_N L(\xi^{N,i}, f) \\
&= M_N^{-1} \sum_{i=1}^{M_N} L(\xi^{N,i}, f),
\end{aligned}$$

So is $L(x, f) \in \mathsf{C}$? The answer is yes, as long as we pick $f \in \tilde{\mathsf{C}}$, and that follows just by the definition of it:

$$\tilde{\mathsf{C}} = \left\{ f \in L^1(\mathsf{X}, \mu) : x \mapsto L(x, |f|) \in \mathsf{C} \right\}.$$

# 2 Showing Convergence to Zero with 9.5.7

We showed the second part of

$$\left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] \right\} + \left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] \right\}$$

converges in probability to $\nu L(f)$ for any $f \in \tilde{\mathsf{C}}$. To show this, we set $V_{N,j} = \frac{1}{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})$, and check the three conditions of 9.5.7.

## 2.1 First condition of 9.5.7

The first condition, translated into our notation, is: the triangular array is conditionally independent given $\mathcal{F}^N$, and for any $N$, $i = 1, \ldots, \tilde{M}_N$, we have $E[|V_{N,j}| \mid \mathcal{F}^N] < \infty$.

Recall our assumptions:

1. Assumption 9.3.1: $0 < L(x, \mathsf{X}) < \infty$.
2. Assumption 9.3.2: $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ are consistent for $(\nu, \mathsf{C})$. $L(x, \mathsf{X}) \in \mathsf{C}$.
3. Assumption 9.3.3: $\forall x \in \mathsf{X}$, $L(x, \cdot) \ll R(x, \cdot)$, and there exists a strictly positive RN derivative: $\frac{dL(x,\cdot)}{dR(x,\cdot)}$.

So, the first condition of 9.5.7 is true because of the description of the algorithm, and because $f \in \tilde{\mathsf{C}}$.

## 2.2 Second Condition of 9.5.7

The second condition is that the sequence (in $N$)

$$\left\{ \sum_{j=1}^{\tilde{M}_N} E[|V_{N,j}| \mid \mathcal{F}^N] \right\}_N$$

is bounded in probability. This is true because it converges in probability. To see this more easily, use the same trick we've been using a few times:

$$\sum_{j=1}^{\tilde{M}_N} E[|V_{N,j}| \mid \mathcal{F}^N] = \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, |f|) \xrightarrow{\text{P}} \nu L(|f|).$$

## 2.3 Third Condition of 9.5.7

The third condition is that for any $\epsilon > 0$ we have

$$\sum_{j=1}^{\tilde{M}_N} E\left[ |V_{N,j}| \mathbf{1}\left(|V_{N,j}| \geq \epsilon\right) \mid \mathcal{F}^N \right] \xrightarrow{\text{P}} 0.$$

Translate a little, and you'll see it's true. First, pick $f \in \tilde{\mathsf{C}}$. Notice that, for any $C$,

$$L(x, |f| \mathbf{1}_{\{h(x,x') \geq C\}}) = \int L(x, dx') |f|(x') \mathbf{1}_{\{h(x,x') \geq C\}} \leq L(x, |f|).$$

We assume $f \in \tilde{\mathsf{C}}$, so $L(x, |f|) \in \mathsf{C}$ so the left hand side of the above is in $\mathsf{C}$ as well, by propriety and the above inequality. This is important because we don't need any further assumptions to get convergence of average of expectations that involve indicator random variables.

So

$$\begin{aligned}
\lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}|1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] &= \lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|1_{\{|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\qquad \text{(defn of V)} \\
&= \lim_N \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|1_{\{|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\qquad \text{(algebra)} \\
&\leq \lim_N \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \mid \mathcal{F}^N\right] \\
&\qquad \text{(if } \tilde{M}_N\epsilon \geq C\text{)} \\
&= \lim_N M_N^{-1} \sum_{i=1}^{M_N} \int L(\xi^{N,i}, d\tilde{\xi}^{N,j})|f|(\tilde{\xi}^{N,j})1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad \text{(defn of expec. and the trick with replacing } i \text{ and } j\text{)} \\
&= \iint \nu(dx)L(x,dx')|f|(x')1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad \text{(discussion above)}
\end{aligned}$$

The last term is bounded above by $\nu L(|f|) < \infty$, so we can use dominated convergence on the above work when we take the limit with $C \uparrow \infty$:

$$\begin{aligned}
\lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}|1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] &= \lim_C \lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}|1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\leq \lim_C \iint \nu(dx)L(x,dx')|f|(x')1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad \text{(above)} \\
&= \iint \nu(dx)L(x,dx')|f|(x') \lim_C 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad \text{(DCT)} \\
&= 0.
\end{aligned}$$