A More Detailed Proof of Prop. 9.3.7

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1 Statement

Recall $\tilde{\mathsf{C}} = \left\{f \in L^1(\mathsf{X}, \mu) : x \mapsto L(x, |f|) \in \mathsf{C}\right\}$ and assume

1. Assumption 9.3.1: $0 < L(x, \mathsf{X}) < \infty$.
2. Assumption 9.3.2: $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ are consistent for (ν, C). $L(x, \mathsf{X}) \in \mathsf{C}$.
3. Assumption 9.3.3: $\forall x \in \mathsf{X}$, $L(x, \cdot) \ll R(x, \cdot)$, and there exists a strictly positive RN derivative: $\frac{dL(x,\cdot)}{dR(x,\cdot)}$.
4. Assumption 9.3.6: the weighted sample $\{(\xi^{N,i}, 1)\}_{1 \le i \le M_N}$ is asymptotically normal for $(\nu, \mathsf{A}, \sigma, M_N^{1/2})$, where A is proper, and σ is some nonnegative function on A.

Then $\tilde{\mathsf{A}}$ is proper, and $\{(\tilde{\xi}^{N,j}, \tilde{\omega}^{N,j})\}_{1 \le j \le M_N}$ is asymptotically normal for $(\mu, \tilde{\mathsf{A}}, \tilde{\sigma}, M_N^{1/2})$.

2 propriety

Let's show $\tilde{\mathsf{A}}$ is proper. Pick any $f, g \in \tilde{\mathsf{A}}$ and $\alpha, \beta \in \mathbb{R}$. Then

1. $L(x, f), L(x, g) \in \mathsf{A}$, and
2. $x \mapsto \int_{\mathsf{X}} R(x, dx')[h_f(x, x')]^2, x \mapsto \int_{\mathsf{X}} R(x, dx')[h_g(x, x')]^2 \in \mathsf{C}$.

Because A and C are proper,

1. $L(x, \alpha f + \beta g) = \alpha L(x, f) + \beta L(x, g) \in \mathsf{A}$, and
2. (same idea)

So $\alpha f + \beta g \in \tilde{\mathsf{A}}$.

Next take $g \in \tilde{\mathsf{A}}$, and f such that $|f| \le |g|$. Same idea, look at them in A and C, and then show this implies $f \in \tilde{\mathsf{A}}$.

3 Overall Strategy

Now we want to show that asymptotic normality part. So we're going to use Slutsky's to show that

$$\sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}\left[f(\tilde{\xi}^{N,i})-\nu L(f)\right]\Big/\frac{1}{\tilde{M}_N}\sum_{j=1}^{\tilde{M}_N}\tilde{\omega}^{N,j}$$

converges to a normal distribution. The denominator is consistent, the numerator is asymptotically normal.

Unlike the book, I am not assuming that, WLOG, the mean is 0.

The denominator is the easiest part. $L(x,|1|)=L(x,\mathsf{X})\in\mathsf{C}$ by assumption 9.3.2, so this means $1\in\tilde{\mathsf{C}}$. Theorem 9.3.5 from last class gives us that the denominator converges in probability to $\nu L(\mathsf{X})$.

Regarding the numerator, write it as

$$\sqrt{M_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})-\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]\right\}$$
$$+\sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\left\{E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]-\nu L(f)\right\}.$$

4 The second piece of the numerator

Looking at the second piece, recall that

$$\sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\left[E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]-\nu L(f)\right]=\sqrt{M_N}M_N^{-1}\sum_{i=1}^{M_N}\left[L(\xi^{N,i},f)-\nu L(f)\right]$$
$$\xrightarrow{D}N(,\sigma^2(Lf)).$$

The convergence takes place because f is assumed to be from $\tilde{\mathsf{A}}$, and this implies that, by definition, $L(x,f)\in\mathsf{A}$.

Note that this variance is taken with respect to the previous time's measure, and it is the variance of a conditional expectation. This is half of the law of total variance.

5 The first piece of the numerator

The second part of the numerator is

$$\sqrt{M_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})-\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]\right\}$$

We're first going to take a look at

$$\sqrt{\tilde{M}_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})-\frac{1`}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}E[\tilde{\omega}^{N,i}f(\tilde{\xi}^{N,i})\mid\mathcal{F}^N]\right\}=\sqrt{\tilde{M}_N}B_N$$

and apply Proposition 9.5.12 to it. We make the substitution $V_{N,j} = \tilde{M}_N^{-1/2}\tilde{\omega}^{N,j}f(\tilde{\omega}^{N,j})$, and verify the assumptions one by one.

5.1 First Condition

The triangular array is conditionally independent given $\mathcal{F}^N$, and for any row/column, $E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\omega}^{N,j})\}^2\mid\mathcal{F}^N]<\infty$

This is true because we are only looking at functions f in

$$\tilde{\mathsf{A}}\stackrel{\text{def}}{=}\left\{f\in L^2(\mathsf{X},\mu):L(x,f)\in\mathsf{A},x\mapsto\int R(x,dx')\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2\in\mathsf{C}\right\}.$$

5.2 Second Condition

There exists a constant $\sigma^2>0$ such that

$$\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}\mathrm{Var}\left[\tilde{\omega}^{N,j}f(\tilde{\omega}^{N,j})\mid\mathcal{F}^N\right]\xrightarrow{\text{p}}\sigma^2$$

This is easy to show if you split up the variance into the mean of the square minus the square of the mean:

$$\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}E\left[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\omega}^{N,j})\}^2\mid\mathcal{F}^N\right]-\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}\{E\left[\tilde{\omega}^{N,j}f(\tilde{\omega}^{N,j})\mid\mathcal{F}^N\right]\}^2$$

First, $f\in\tilde{\mathsf{A}}$, so

$$\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}E\left[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\omega}^{N,j})\}^2\mid\mathcal{F}^N\right]\xrightarrow{\text{p}}\iint\nu(dx)R(x,dx')\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2$$

it might be helpful to write that last piece as an expectation:

$$\iint\nu(dx)R(x,dx')\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2=\nu R\left\{\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2\right\}$$

The second part also converges too. This is because

$$x\mapsto\left[\int R(x,dx')\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2\in\mathsf{C}$$

because

$$\left[\int R(x,dx')\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2 \le \int R(x,dx')\left[\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2$$

of Jensen's inequality. The right hand side is in C, and because it is proper, the left hand side is too. This means that

$$\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}\{E\left[\tilde{\omega}^{N,j}f(\tilde{\omega}^{N,j}) \mid \mathcal{F}^N\right]\}^2 \mapsto \nu\left[\left[\int R(x,dx')\frac{dL(x,\cdot)}{dR(x,\cdot)}(x')f(x')\right]^2\right]$$

5.3 Third Condition

The third condition of 9.5.12 is the Lindberg condition: for all $\epsilon > 0$,

$$\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}[\tilde{\omega}^{N,j}]^2[f(\tilde{\xi}^{N,j})]^2\mathbf{1}\left\{|\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})| \ge \epsilon\right\}$$

This is true, and we're going to use the dominated convergence argument again here. After picking an ϵ, pick an arbitrary $C > 0$ and n otice that

$$\begin{aligned}
&\lim_{\tilde{M}_N\to\infty}\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\xi}^{N,j})\}^2 1\left(|\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})| \ge \sqrt{\tilde{M}_N}\epsilon\right) \mid \mathcal{F}^N] \\
&\le \lim_{\tilde{M}_N\to\infty}\tilde{M}_N^{-1}\sum_{j=1}^{\tilde{M}_N}E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\xi}^{N,j})\}^2 1\left(|\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})| \ge C\right) \mid \mathcal{F}^N] \\
&\hfill (\text{if } C \le \sqrt{\tilde{M}_N}\epsilon) \\
&= \nu\left\{E[\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})1\left(|\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})| \ge C\right) \mid \mathcal{F}^N]\right\}
\end{aligned}$$

The last line is true because $E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\xi}^{N,j})\}^2 1\left(|\tilde{\omega}^{N,j}f(\tilde{\xi}^{N,j})| \ge C\right) \mid \mathcal{F}^N] \in \mathsf{C}$. Why? Well by assumption, $f \in \tilde{\mathsf{A}}$, and by definition of that set of functions, $E[\{\tilde{\omega}^{N,j}\}^2\{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] \in \mathsf{C}$. This one is larger than the other, so proprietary guarantees it works.

Following up on that we have

$$
\begin{aligned}
&\lim_{M_N\to\infty}\sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 1\left(|V_{N,j}|\geq \epsilon\right) \mid \mathcal{F}^N] \\
&= \lim_{C\to\infty}\lim_{M_N\to\infty}\sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 1\left(|V_{N,j}|\geq \epsilon\right) \mid \mathcal{F}^N] \quad \text{(no C so limit doesn't matter)} \\
&\leq \lim_{C\to\infty} \nu\left\{E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) 1\left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C\right) \mid \mathcal{F}^N]\right\} \quad \text{(above work)} \\
&= \nu\left\{\lim_{C\to\infty} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) 1\left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C\right) \mid \mathcal{F}^N]\right\} \quad \text{(DCT)} \\
&= 0.
\end{aligned}
$$

Therefore condition iii of 9.5.12 is satisfied, and we have the desired result.

5.4 Putting it All Together

So far we have shown that the following two are asymptotically normal:

1. $M_N^{1/2} A_n = \sqrt{M_N}\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\left[E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \nu L(f)\right]$
2. $\tilde{M}_N^{1/2} B_n = \sqrt{\tilde{M}_N}\left\{\frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N}\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N}\sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N]\right\}$.

To show their asymptotic *joint* distribution, we have to use the law of total expectation:

$$
\begin{aligned}
\lim_{N\to\infty} E\left[\exp\left[i\left(sM_N^{1/2}A_n + t\tilde{M}_N^{1/2}B_n\right)\right]\right] &= \lim_{N\to\infty} E\left[\exp\left[isM_N^{1/2}A_n + it\tilde{M}_N^{1/2}B_n\right]\right] \\
&= \lim_{N\to\infty} E\left[\exp\left[isM_N^{1/2}A_n\right]\exp\left[it\tilde{M}_N^{1/2}B_n\right]\right] \\
&= \lim_{N\to\infty} E\left[\exp\left[isM_N^{1/2}A_n\right] E\left(\exp\left[it\tilde{M}_N^{1/2}B_n\right] \mid \mathcal{F}^N\right)\right] \\
&= E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\lim_{N\to\infty} E\left(\exp\left[it\tilde{M}_N^{1/2}B_n\right] \mid \mathcal{F}^N\right)\right] \\
&\qquad \text{(DCT)} \\
&= E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\exp\left(-.5t^2\eta^2(f)\right)\right] \\
&= \exp\left(-.5t^2\eta^2(f)\right) E\left[\lim_{N\to\infty}\exp\left[isM_N^{1/2}A_n\right]\right] \\
&= \exp\left(-.5t^2\eta^2(f)\right)\lim_{N\to\infty} E\left[\exp\left[isM_N^{1/2}A_n\right]\right] \\
&\qquad \text{(DCT)} \\
&= \exp\left(-.5t^2\eta^2(f)\right)\exp\left(-.5s^2\sigma^2(Lf)\right)
\end{aligned}
$$

The second to last move should seem familiar:

$$
M_N^{1/2}(A_N + B_N) = M_N^{1/2}A_N + \frac{1}{\sqrt{\alpha_N}}\tilde{M}_N^{1/2}B_N \xrightarrow{\text{D}} \text{Normal}\left[0, \sigma^2(Lf) + \eta^2(f)\right].
$$

The final move is using Slutsky's:

$$\frac{M_N^{1/2}(A_N + B_N)}{\frac{1}{\tilde{M}_N}\sum_{j=1}^{\tilde{M}_N}\tilde{\omega}^{N,j}} \xrightarrow{\mathrm{D}} \mathrm{Normal}\left[0, \frac{\sigma^2(Lf) + \eta^2(f)}{[\nu L(f)]^2}\right].$$