A More Detailed Proof of Prop. 9.5.9

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Take care to realize that the book has two i's. One represents the complex number $i = \sqrt{-1}$, and the other represents the index of the sum. To prevent confusion, the book changed the font for the complex number. However, I choose a different path, and I use j as the index of the sum.

The goal is to show that

$$E\left[\exp\left(iu\sum_j U_{N,j}\right)\Bigg|\mathcal{F}^N\right] - \exp(-\sigma^2 u^2/2) \xrightarrow{p} 0.$$

It is equivalent to show

$$\underbrace{E\left[\exp\left(iu\sum_j U_{N,j}\right)\Bigg|\mathcal{F}^N\right] - \exp\left(-(u^2/2)\sum_j \sigma_{N,j}^2\right)}_{\text{first chunk}} + \underbrace{\exp\left(-(u^2/2)\sum_j \sigma_{N,j}^2\right) - \exp(-\sigma^2 u^2/2)}_{\text{second chunk}}$$

converges in probability to 0.

However, the second chunk is easy...by the continuous mapping theorem, together with assumption (ii), we have

$$\exp\left(-(u^2/2)\sum_j \sigma_{N,j}^2\right) - \exp(-\sigma^2 u^2/2) \xrightarrow{p} 0.$$

So this explains why it suffices to prove 9.74 (the left chunk).

We provided Lemma 9.5.8 for complex numbers with modulus less than 1. Keep in mind that real numbers are complex numbers, too, so we can apply this theorem there as well.

$$
\begin{aligned}
&\left| E\left[\exp\left(iu\sum_j U_{N,j}\right) - \exp\left(-(u^2/2)\sum_j \sigma_{N,j}^2\right) \middle| \mathcal{F}^N\right]\right| \\
&= \left| E\left[\prod_j \exp\left(iuU_{N,j}\right) - \prod_j \exp\left(-(u^2/2)\sigma_{N,j}^2\right) \middle| \mathcal{F}^N\right]\right| && \text{(algebra)} \\
&= \left| \prod_j E\left[\exp\left(iuU_{N,j}\right) \middle| \mathcal{F}^N\right] - \prod_j E\left[\exp\left(-(u^2/2)\sigma_{N,j}^2\right) \middle| \mathcal{F}^N\right]\right| \\
& && \text{(independence, linearity)} \\
&\le \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - \exp\left(-(u^2/2)\sigma_{N,j}^2\right) \middle| \mathcal{F}^N\right]\right| && \text{(Lemma 9.5.8, Linearity)} \\
&= \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma_{N,j}^2/2) + (1 - u^2\sigma_{N,j}^2/2) - \exp\left(-(u^2/2)\sigma_{N,j}^2\right) \middle| \mathcal{F}^N\right]\right| \\
& && \text{(algebra)} \\
&\le \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma_{N,j}^2/2) \mid \mathcal{F}^N\right]\right| \\
&\qquad + \sum_j \left| E\left[\exp\left(-(u^2/2)\sigma_{N,j}^2\right) - (1 - u^2\sigma_{N,j}^2/2) \mid \mathcal{F}^N\right]\right| \\
& && \text{(linearity, tri-ineq)} \\
&\le \sum_j \left| E\left[\exp\left(iuU_{N,j}\right) - (1 - u^2\sigma_{N,j}^2/2) \mid \mathcal{F}^N\right]\right| \\
&\qquad + \sum_j \left| \exp\left(-(u^2/2)\sigma_{N,j}^2\right) - (1 - u^2\sigma_{N,j}^2/2) \right| \\
&= A_N + B_N. && \text{(defn)}
\end{aligned}
$$

So, the overall goal is to show these two terms converge to 0. Focusing on A_N first, we will use the inequality we proved in the last section, namely

$$
-\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \le e^{iuU_{N,j}} - (1 + iuU_{N,j} - \frac{1}{2}u^2U_{N,j}^2) \le \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}.
$$

Take conditional expectations on all sides, then take the absolute value (in that order).

This gives you the book's inequality (the third one, unnumbered, on page 340):

$$\left|E\left[e^{iuU_{N,j}} - (1 + iuU_{N,j} - \frac{1}{2}u^2U_{N,j}^2) \mid \mathcal{F}^N\right]\right| = \left|E\left[e^{iuU_{N,j}} \mid \mathcal{F}^N\right] - (1 - \frac{1}{2}u^2\sigma_{N,j}^2)\right|$$
$$\leq E\left[\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \mid \mathcal{F}^N\right]$$
$$\leq E\left[|uU_{N,j}|^2 \mid \mathcal{F}^N\right]$$
$$< \infty. \qquad \text{(assumption)}$$

We are using the fact that the mean of each $U_{N,j}$ is zero. The absolute value sign has to be on the outside.

Another trick we need is the following inequality. Let $\epsilon > 0$. Then

$$\begin{aligned}
&\min\{|uU_{N,j}|^2, |uU_{N,j}|^3\} \\
&= \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}1(|U_{N,j}| > \epsilon) + \min\{|uU_{N,j}|^2, |uU_{N,j}|^3\}1(|U_{N,j}| \leq \epsilon) \\
&\leq |uU_{N,j}|^2 1(|U_{N,j}| > \epsilon) + \epsilon|u|^3|U_{N,j}|^2 \mathbb{1}(|U_{N,j}| \leq \epsilon) \\
&\leq |uU_{N,j}|^2 1(|U_{N,j}| > \epsilon) + \epsilon|u|^3|U_{N,j}|^2.
\end{aligned}$$

This let's us take care of the first part:

$$\begin{aligned}
A_N &= \sum_j \left|E\left[\exp(iuU_{N,j}) - (1 - u^2\sigma_{N,j}^2/2) \mid \mathcal{F}^N\right]\right| && \text{(defn.)} \\
&\leq \sum_{j=1}^{M_N} E[\min\{|uU_{N,j}|^2, |uU_{N,i}|^3\} \mid \mathcal{F}^N] && \text{(Taylor's ineq)} \\
&\leq u^2 \sum_{j=1}^{M_N} E[U_{N,j}^2 \mathbb{1}(|U_{N,j}| > \epsilon) \mid \mathcal{F}^N] + \epsilon|u|^3 \sum_{j=1}^{M_N} \sigma_{N,j}^2 && \text{(above inequality)} \\
&\to 0 + \epsilon|u|^3\sigma^2. && \text{(assumption ii. and iii.)}
\end{aligned}$$

Now we look at B_n. The real-valued version of Taylor's Inequality states

$$\left|e^{-x} - \sum_{i=0}^{n} \frac{(-x)^i}{i!}\right| \leq \frac{M}{(n+1)!}|x|^{n+1}$$

as long as $|f^{(n+1)}(x)| \leq M$. Set $n = 1$, and looking at $x \geq 0$, which means $|f^{(n+1)}(x)| = |f^{(2)}(x)| = |e^{-x}| \leq 1$. This gives us,

$$|e^{-x} - 1 + x| \leq x^2/2.$$

What does this mean for B_N? Well,

$$\begin{aligned}
B_N &= \sum_{i=1}^{M_N} |\exp(-u^2\sigma_{N,i}^2/2) - (1 - u^2\sigma_{N,i}^2/2)| && \text{(defn.)} \\
&\leq \sum_{i=1}^{M_N} \left(u^2\sigma_{N,i}^2/2\right)^2/2 && \text{(real-valued Taylor's)} \\
&= \frac{u^4}{8}\sum_{i=1}^{M_N} \sigma_{N,i}^4 \\
&\leq \frac{u^4}{8}\left(\max_i \sigma_{N,i}^2\right)\sum_{i=1}^{M_N} \sigma_{N,i}^2 && \text{(see below)} \\
&\to \frac{u^4}{8} \times 0 \times \sigma^2.
\end{aligned}$$

The last line follows because of assumption 2 and the Uniform Smallness Condition (Remark 9.5.10.)