

A More Detailed Proof of Prop. 9.3.7

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1 Statement

Recall $\tilde{\mathbf{C}} = \{f \in L^1(\mathbf{X}, \mu) : x \mapsto L(x, |f|) \in \mathbf{C}\}$ and assume

1. Assumption 9.3.1: $0 < L(x, \mathbf{X}) < \infty$.
2. Assumption 9.3.2: $\{(\xi^{N,i}, 1)\}_{1 \leq i \leq M_N}$ are consistent for (ν, \mathbf{C}) . $L(x, \mathbf{X}) \in \mathbf{C}$.
3. Assumption 9.3.3: $\forall x \in \mathbf{X}$, $L(x, \cdot) \ll R(x, \cdot)$, and there exists a strictly positive RN derivative: $\frac{dL(x, \cdot)}{dR(x, \cdot)}$.
4. Assumption 9.3.6: the weighted sample $\{(\xi^{N,i}, 1)\}_{1 \leq i \leq M_N}$ is asymptotically normal for $(\nu, \mathbf{A}, \sigma, M_N^{1/2})$, where \mathbf{A} is proper, and σ is some nonnegative function on \mathbf{A} .

Then $\tilde{\mathbf{A}}$ is proper, and $\{(\tilde{\xi}^{N,j}, \tilde{\omega}^{N,j})\}_{1 \leq j \leq M_N}$ is asymptotically normal for $(\mu, \tilde{\mathbf{A}}, \tilde{\sigma}, M_N^{1/2})$.

2 Overall Strategy

We already showed that $\tilde{\mathbf{A}}$ is proper in class. Now we want to show that asymptotic normality part. So we're going to use Slutsky's to show that

$$\sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} \left[f(\tilde{\xi}^{N,i}) - \nu L(f) \right] \bigg/ \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}$$

converges to a normal distribution. The denominator is consistent, the numerator is asymptotically normal.

Unlike the book, I am not assuming that, WLOG, the mean is 0.

The denominator is the easiest part. $L(x, |1|) = L(x, \mathbf{X}) \in \mathbf{C}$ by assumption 9.3.2, so this means $1 \in \tilde{\mathbf{C}}$. Theorem 9.3.5 from last class gives us that the denominator converges in probability to $\nu L(\mathbf{X})$.

Regarding the numerator, write it as

$$\begin{aligned} & \sqrt{M_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} \\ & \quad + \sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \left\{ E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \nu L(f) \right\} \\ & = \sqrt{M_N} B_N + \sqrt{M_N} A_N. \end{aligned}$$

3 The second piece of the numerator

Looking at the second piece, recall that

$$\begin{aligned} \sqrt{M_N} \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \left[E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \nu L(f) \right] &= \sqrt{M_N} M_N^{-1} \sum_{i=1}^{M_N} [L(\xi^{N,i}, f) - \nu L(f)] \\ &\xrightarrow{D} N(\cdot, \sigma^2(Lf)). \end{aligned}$$

The convergence takes place because f is assumed to be from $\tilde{\mathbf{A}}$, and this implies that, by definition, $L(x, f) \in \mathbf{A}$.

Note that this variance is taken with respect to the previous time's measure, and it is the variance of a conditional expectation. This is half of the law of total variance.

4 The first piece of the numerator

The second part of the numerator is

$$\sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} = \sqrt{\tilde{M}_N} B_N$$

We're going to apply Proposition 9.5.12 to it. We make the substitution $V_{N,j} = \tilde{M}_N^{-1/2} \tilde{\omega}^{N,j} f(\tilde{\omega}^{N,j})$, and verify the assumptions one by one.

4.1 First Condition

The triangular array is conditionally independent given \mathcal{F}^N , and for any row/column, $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\omega}^{N,j})\}^2 \mid \mathcal{F}^N] < \infty$

This is true because we are only looking at functions f in

$$\tilde{\mathbf{A}} \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathbf{X}, \mu) : L(x, f) \in \mathbf{A}, x \mapsto \int R(x, dx') \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \in \mathbf{C} \right\}.$$

4.2 Second Condition

There exists a constant $\sigma^2 > 0$ such that

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \text{Var} [\tilde{\omega}^{N,j} f(\tilde{\omega}^{N,j}) \mid \mathcal{F}^N] \xrightarrow{\mathbb{P}} \sigma^2$$

This is easy to show if you split up the variance into the mean of the square minus the square of the mean:

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E [\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\omega}^{N,j})\}^2 \mid \mathcal{F}^N] - \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{E [\tilde{\omega}^{N,j} f(\tilde{\omega}^{N,j}) \mid \mathcal{F}^N]\}^2$$

First, $f \in \tilde{\mathbf{A}}$, so

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E [\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\omega}^{N,j})\}^2 \mid \mathcal{F}^N] \xrightarrow{\mathbb{P}} \iint \nu(dx) R(x, dx') \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2$$

it might be helpful to write that last piece as an expectation:

$$\iint \nu(dx) R(x, dx') \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 = [\nu \otimes R] \left\{ \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \right\}$$

The second part also converges too. This is because

$$x \mapsto \left[\int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \in \mathbb{C}$$

because

$$\left[\int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \leq \int R(x, dx') \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2$$

of Jensen's inequality. The right hand side is in \mathbb{C} , and because it is proper, the left hand side is too. This means that

$$\begin{aligned} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} \{E [\tilde{\omega}^{N,j} f(\tilde{\omega}^{N,j}) \mid \mathcal{F}^N]\}^2 &\xrightarrow{\mathbb{P}} \nu \left[\left[\int R(x, dx') \frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 \right] \\ &= \nu \left([L(x, f)]^2 \right) \end{aligned}$$

4.3 Third Condition

The third condition of 9.5.12 is the Lindberg condition: for all $\epsilon > 0$,

$$\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} [\tilde{\omega}^{N,j}]^2 [f(\tilde{\xi}^{N,j})]^2 \mathbf{1} \left\{ |\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \epsilon \sqrt{\tilde{M}_N} \right\}$$

This is true, and we're going to use the dominated convergence argument again here. After picking an ϵ , pick an arbitrary $C > 0$ and notice that

$$\begin{aligned} & \lim_{\tilde{M}_N \rightarrow \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq \sqrt{\tilde{M}_N} \epsilon \right) \mid \mathcal{F}^N] \\ & \leq \lim_{\tilde{M}_N \rightarrow \infty} \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \\ & \quad \text{(if } C \leq \sqrt{\tilde{M}_N} \epsilon) \\ & = \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \end{aligned}$$

The last line is true because $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \in \mathbb{C}$. Why? Well by assumption, $f \in \tilde{\mathbf{A}}$, and by definition of that set of functions, $E[\{\tilde{\omega}^{N,j}\}^2 \{f(\tilde{\xi}^{N,j})\}^2 \mid \mathcal{F}^N] \in \mathbb{C}$. This one is larger than the other, so propriety guarantees it works.

Following up on that we have

$$\begin{aligned} 0 & \leq \lim_{\tilde{M}_N \rightarrow \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 \mathbf{1}(|V_{N,j}| \geq \epsilon) \mid \mathcal{F}^N] \\ & = \lim_{C \rightarrow \infty} \lim_{\tilde{M}_N \rightarrow \infty} \sum_{j=1}^{\tilde{M}_N} E[V_{N,j}^2 \mathbf{1}(|V_{N,j}| \geq \epsilon) \mid \mathcal{F}^N] \\ & \quad \text{(no } C \text{ so limit doesn't matter)} \\ & \leq \lim_{C \rightarrow \infty} \nu \left\{ E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \quad \text{(above work)} \\ & = \nu \left\{ \lim_{C \rightarrow \infty} E[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mathbf{1} \left(|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \geq C \right) \mid \mathcal{F}^N] \right\} \quad \text{(DCT)} \\ & = 0. \end{aligned}$$

Therefore condition iii of 9.5.12 is satisfied, and we have the desired result:

$$\sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} \\ \xrightarrow{D} \text{Normal} \left(0, \iint \nu(dx) R(x, dx') \left[\frac{dL(x, \cdot)}{dR(x, \cdot)}(x') f(x') \right]^2 - \nu([L(x, f)]^2) \right)$$

4.4 Putting it All Together

So far we have shown that the following two are asymptotically normal:

1. $M_N^{1/2} A_n = \sqrt{M_N} \frac{1}{M_N} \sum_{i=1}^{\tilde{M}_N} \left[E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \nu L(f) \right]$
2. $\tilde{M}_N^{1/2} B_n = \sqrt{\tilde{M}_N} \left\{ \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} \tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) - \frac{1}{\tilde{M}_N} \sum_{i=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\}.$

To show their asymptotic *joint* distribution, we have to use the law of total expectation:

$$\begin{aligned} \lim_{N \rightarrow \infty} E \left[\exp \left[i \left(s M_N^{1/2} A_n + t \tilde{M}_N^{1/2} B_n \right) \right] \right] &= \lim_{N \rightarrow \infty} E \left[\exp \left[i s M_N^{1/2} A_n + i t \tilde{M}_N^{1/2} B_n \right] \right] \\ &= \lim_{N \rightarrow \infty} E \left[\exp \left[i s M_N^{1/2} A_n \right] \exp \left[i t \tilde{M}_N^{1/2} B_n \right] \right] \\ &= \lim_{N \rightarrow \infty} E \left[\exp \left[i s M_N^{1/2} A_n \right] E \left(\exp \left[i t \tilde{M}_N^{1/2} B_n \right] \mid \mathcal{F}^N \right) \right] \\ &= E \left[\lim_{N \rightarrow \infty} \exp \left[i s M_N^{1/2} A_n \right] \lim_{N \rightarrow \infty} E \left(\exp \left[i t \tilde{M}_N^{1/2} B_n \right] \mid \mathcal{F}^N \right) \right] \\ &\quad \text{(DCT)} \\ &= E \left[\lim_{N \rightarrow \infty} \exp \left[i s M_N^{1/2} A_n \right] \exp \left(-.5 t^2 \eta^2(f) \right) \right] \\ &= \exp \left(-.5 t^2 \eta^2(f) \right) E \left[\lim_{N \rightarrow \infty} \exp \left[i s M_N^{1/2} A_n \right] \right] \\ &= \exp \left(-.5 t^2 \eta^2(f) \right) \lim_{N \rightarrow \infty} E \left[\exp \left[i s M_N^{1/2} A_n \right] \right] \\ &\quad \text{(DCT)} \\ &= \exp \left(-.5 t^2 \eta^2(f) \right) \exp \left(-.5 s^2 \sigma^2(Lf) \right) \end{aligned}$$

The second to last move should seem familiar:

$$M_N^{1/2} (A_N + B_N) = M_N^{1/2} A_N + \frac{1}{\sqrt{\alpha_N}} \tilde{M}_N^{1/2} B_N \xrightarrow{D} \text{Normal} \left[0, \sigma^2(Lf) + \frac{1}{\alpha} \eta^2(f) \right].$$

The final move is using Slutsky's:

$$\frac{M_N^{1/2} (A_N + B_N)}{\frac{1}{M_N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j}} \xrightarrow{D} \text{Normal} \left[0, \frac{\sigma^2(Lf) + \frac{1}{\alpha} \eta^2(f)}{[\nu L(f)]^2} \right].$$