A More Detailed Proof of Lemma 9.5.4

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Assume

$$|U_N| \leq \eta V_N + W_N(\eta).$$

Then, for any $\delta > 0$ and $\eta > 0$

$$\{|U_N| < \delta\} \supset \{V_N < \delta/2\eta\} \cap \{W_N(\eta) < \delta/2\}.$$

Look at the contrapositive:

$$\{|U_N| \geq \delta\} \subset \{V_N \geq \delta/2\eta\} \cup \{W_N(\eta) \geq \delta/2\}.$$

That means

$$\begin{aligned} P(|U_N| \geq \delta) &\leq P(\{V_N \geq \delta/2\eta\} \cup \{W_N(\eta) \geq \delta/2\}) \\ &\leq P(V_N \geq \delta/2\eta) + P(W_N(\eta) \geq \delta/2) \qquad \text{(sub-additivity)} \end{aligned}$$

which is the first equation of the authors' proof.

Taking the $\limsup_{N\to\infty}$ on both sides we have

$$\begin{aligned} \limsup_N P(|U_N| \geq \delta) &\leq \limsup_N \{P(V_N \geq \delta/2\eta) + P(W_N(\eta) \geq \delta/2)\} \\ &\leq \limsup_N P(V_N \geq \delta/2\eta) + \limsup_N P(W_N(\eta) \geq \delta/2) \\ &\qquad \text{(properties of sup)} \\ &= \limsup_N P(V_N \geq \delta/2\eta) + \lim_N P(W_N(\eta) \geq \delta/2) \\ &\qquad \text{(assumption)} \\ &= \limsup_N P(V_N \geq \delta/2\eta) + 0 \qquad \text{(assumption)} \\ &\leq \sup_N P(V_N \geq \delta/2\eta). \end{aligned}$$

As $\eta \to 0$, $\delta/2\eta \to \infty$, and $\sup_N P(V_N \geq \delta/2\eta) \to 0$ by $\{V_N\}$'s boundedness in probability.