A More Detailed Proof of Proposition 9.5.7

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The idea is we set $U_{N,i} = V_{N,i} - E[V_{N,i} \mid \mathcal{F}^N]$ and check conditons 1-3 of Proposition 9.5.5.

Condition 1: The conditional independence thing is obvious. Regarding the conditional expectations, first

$$\begin{aligned}
E[|U_{N,i}| \mid \mathcal{F}^N] &= E[|V_{N,i} - E[V_{N,i} \mid \mathcal{F}^N]| \mid \mathcal{F}^N] \\
&\leq E[|V_{N,i}| \mid \mathcal{F}^N] + E[|E[V_{N,i} \mid \mathcal{F}^N])| \mid \mathcal{F}^N] \\
&\hspace{10em} \text{(tri- ineq and linearity)} \\
&= E[|V_{N,i}| \mid \mathcal{F}^N] + |E[V_{N,i} \mid \mathcal{F}^N]| \\
&\leq E[|V_{N,i}| \mid \mathcal{F}^N] + E[|V_{N,i}| \mid \mathcal{F}^N] \qquad \text{(Jensen's)} \\
&< \infty. \qquad \text{(assumption i)}
\end{aligned}$$

Secondly,

$$E[U_{N,i} \mid \mathcal{F}^N] = E[V_{N,i} \mid \mathcal{F}^N] - E[V_{N,i} \mid \mathcal{F}^N] = 0. \tag{1}$$

Condition 2:
We showed in conditon 1 that $E[|U_{N,i}| \mid \mathcal{F}^N] \leq 2E[|V_{N,i}| \mid \mathcal{F}^N]$, so

$$\sum_i E[|U_{N,i}| \mid \mathcal{F}^N] \leq 2 \sum_i E[|V_{N,i}| \mid \mathcal{F}^N]. \tag{2}$$

The right hand side is bounded in probability, so the left hand side is as well.

Condition 3:
First, a little about the two tools used in this part, the verification of condition 3 in 9.5.5. The first trick is similar in spirit to that one used in 9.5.4. Recall that the triangle inequality gives us

$$|U_{N,i}| \leq |V_{N,i}| + E[|V_{N,i}| \mid \mathcal{F}^N]$$

which implies (check contrapositive) that

$$\{|U_{N,i}| \geq \epsilon\} \subset \{|V_{N,i}| \geq \epsilon/2\} \cup \{E[|V_{N,i}| \mid \mathcal{F}^N] \geq \epsilon/2\},$$

which implies further that

$$\mathbb{1}_{\{|U_{N,i}|\geq\epsilon\}} \leq \mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}\cup\{E[|V_{N,i}||\mathcal{F}^N]\geq\epsilon/2\}} \leq \mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}}+\mathbb{1}_{\{E[|V_{N,i}||\mathcal{F}^N]\geq\epsilon/2\}}.$$

The second trick is a way to bound the maximum.

$$\begin{aligned}
\max_i |V_{N,i}| &\leq \max_i |V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} + \max_i |V_{N,i}|\mathbb{1}_{\{|V_{N,i}|<\epsilon/2\}} \\
&\leq \max_i |V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} + \epsilon/2 \\
&\leq \sum_i |V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} + \epsilon/2.
\end{aligned}$$

This will be used to say things like

$$\{\max_i |V_{N,i}| \geq \epsilon\} \subseteq \left\{\sum_i |V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} \geq \epsilon/2\right\}.$$

Going back to the proof, we want to show that $\sum_i |U_{N,i}|\mathbb{1}_{\{|U_{N,i}|\geq\epsilon\}}$ converges to 0 in probability:

$$\begin{aligned}
&\sum_i E[|U_{N,i}|\mathbb{1}(|U_{N,i}| \geq \epsilon) \mid \mathcal{F}^N] \\
&= \sum_i E[|V_{N,i} - E[V_{N,i} \mid \mathcal{F}^N]|\mathbb{1}(|U_{N,i}| \geq \epsilon) \mid \mathcal{F}^N] && \text{(defn.)} \\
&\leq \sum_i E[|V_{N,i} - E[V_{N,i} \mid \mathcal{F}^N]|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} + \mathbb{1}_{\{E[|V_{N,i}||\mathcal{F}^N]\geq\epsilon/2\}} \mid \mathcal{F}^N] \\
& && \text{(first trick)} \\
&\leq \sum_i E[\{|V_{N,i}| + |E[V_{N,i} \mid \mathcal{F}^N]|\}\left\{\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} + \mathbb{1}_{\{E[|V_{N,i}||\mathcal{F}^N]\geq\epsilon/2\}}\right\} \mid \mathcal{F}^N] \\
& && \text{(tri-neq)} \\
&\leq \sum_i E[|V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} \mid \mathcal{F}^N] + 2\sum_i E[|V_{N,i}| \mid \mathcal{F}^N]\mathbb{1}_{\{E[|V_{N,i}||\mathcal{F}^N]\geq\epsilon/2\}} \\
&\qquad + \sum_i E[|V_{N,i}| \mid \mathcal{F}^N]P(|V_{N,i}| \geq \epsilon \mid \mathcal{F}^N) \\
&= \sum_i E[|V_{N,i}|\mathbb{1}_{\{|V_{N,i}|\geq\epsilon/2\}} \mid \mathcal{F}^N] + 2B_N + A_N. && \text{(defns in 9.69 and 9.70)}
\end{aligned}$$

The first term goes to 0 by eqn. 9.6.8 in assumption (iii). This explains why we are to focus on proving 9.69 and 9.70.

To prove $A_N \xrightarrow{p} 0$:

$$
\begin{aligned}
A_n &= \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] P(|V_{N,i}| \geq \epsilon/2 \mid \mathcal{F}^N) \\
&\leq P(\max_j |V_{N,j}| \geq \epsilon/2 \mid \mathcal{F}^N) \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] && \text{(max)} \\
&\leq P\left(\left\{\epsilon/4 + \sum_{j=1}^{M_N} |V_{N,j}| \mathbb{1}(|V_{N,j}| \geq \epsilon/4)\right\} \geq \epsilon/2 \middle| \mathcal{F}^N\right) \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] \\
& && \text{(second trick)} \\
&= P\left(\sum_{j=1}^{M_N} |V_{N,j}| \mathbb{1}(|V_{N,j}| \geq \epsilon/4) \geq \epsilon/4 \middle| \mathcal{F}^N\right) \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] \\
&\leq (4/\epsilon) \sum_j E\left[|V_{N,j}| \mathbb{1}(|V_{N,j}| \geq \epsilon/4) \mid \mathcal{F}^N\right] \sum_i E[|V_{N,i}| \mid \mathcal{F}^N]. && \text{(Markov's)}
\end{aligned}
$$

$\sum_i E\left[|V_{N,i}| \mathbb{1}(|V_{N,i}| \geq \epsilon/4) \mid \mathcal{F}^N\right]$ goes to 0 by hypothesis. The other term is bounded in probability by hypothesis. Thus, the whole thing goes to 0 by Lemma 9.5.3 (3).

Proving $B_N \xrightarrow{p} 0$:

This uses the second inequality trick again:

$$
\begin{aligned}
B_N &= \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] \mathbb{1}(E[|V_{N,i}| \mid \mathcal{F}^N] \geq \epsilon/2) \\
&\leq \mathbb{1}(\max_i E[|V_{N,i}| \mid \mathcal{F}^N] \geq \epsilon/2) \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] && \text{(max)} \\
&\leq \mathbb{1}\left\{\sum_i E[|V_{N,i}| \mathbb{1}(|V_{N,i}| \geq \epsilon/4) \mid \mathcal{F}^N] \geq \epsilon/4\right\} \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] \\
& && \text{(second trick)} \\
&\leq (4/\epsilon) \sum_i E[|V_{N,i}| \mathbb{1}(|V_{N,i}| \geq \epsilon/4) \mid \mathcal{F}^N] \sum_i E[|V_{N,i}| \mid \mathcal{F}^N] && \text{(logic)} \\
&\xrightarrow{p} 0
\end{aligned}
$$

where the last line follows because $\sum_i E[|V_{N,i}| \mathbb{1}(|V_{N,i}| \geq \epsilon/4) \mid \mathcal{F}^N]$ goes to 0 by hypothesis, and the second is bounded in probability by hypothesis, which allows us to use Lemma 9.5.3(3) again.

In the second to last line, I say "logic" because this sum is nonnegative, and either it is bigger than $\epsilon/4$, or it isn't–verify the inequality holds in these two cases.