A More Detailed Proof of Taylor's Inequality (page 340 in IHMM)

December 26, 2019

IHMM's "Taylor's Inequality" is a special case of the following, which is from page 343 of Billingsley:

$$\left| e^{ix} - \sum_{k=0}^{n} \frac{(ix)^k}{k!} \right| \le \min\left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}.$$

The main tool we use to prove this inequality is the following result that comes from integration by parts. To reiterate what Billingsley says,

$$\begin{aligned}
\int_0^x (x-s)^n e^{is} ds &= -\frac{(x-s)^{n+1}}{n+1} e^{is}\Big|_{s=0}^{s=x} + \int_0^x \frac{i(x-s)^{n+1}}{n+1} e^{is} ds \\
&= \left[0 + \frac{x^{n+1}}{n+1}\right] + \int_0^x \frac{i(x-s)^{n+1}}{n+1} e^{is} ds \\
&= \frac{x^{n+1}}{n+1} + \frac{i}{n+1} \int_0^x (x-s)^{n+1} e^{is} ds.
\end{aligned}$$

To prove this statement is true for any $n \ge 0$, use induction. Start at $n = 0$

and then go up. To illustrate that induction, consider the following:

$$\begin{aligned}
i^{-1}(e^{ix}-1) &= \int_0^x e^{is} ds \\
&= x + i\int_0^x (x-s)e^{is} ds && \text{(IBP)} \\
&= x + i\left(\frac{x^2}{2} + \frac{i}{2}\int_0^x (x-s)^2 e^{is}\right) ds && \text{(IBP)} \\
&= x + i\frac{x^2}{2} - \frac{1}{2}\int_0^x (x-s)^2 e^{is} ds \\
&= x + i\frac{x^2}{2} - \frac{1}{2}\left(\frac{x^3}{3} + \frac{i}{3}\int_0^x (x-s)^3 e^{is} ds\right) && \text{(IBP)} \\
&= x + i\frac{x^2}{2} - \frac{x^3}{3!} - \frac{i}{3!}\int_0^x (x-s)^3 e^{is} ds \\
&= i^0 x + i^1\frac{x^2}{2} + i^2\frac{x^3}{3!} + \frac{i^3}{3!}\int_0^x (x-s)^3 e^{is} ds \\
&\vdots \\
&= \sum_{k=1}^{n} \frac{i^{k-1}x^k}{k!} + \frac{i^n}{n!}\int_0^x (x-s)^n e^{is} ds.
\end{aligned}$$

Equation (26.2) in Billingsley comes from the above after solving for e^{ix}:

$$\begin{aligned}
e^{ix} &= 1 + \sum_{k=1}^{n} \frac{i^k x^k}{k!} + \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds \\
&= \sum_{k=0}^{n} \frac{i^k x^k}{k!} + \frac{i^{n+1}}{n!}\int_0^x (x-s)^n e^{is} ds. && (1)
\end{aligned}$$

We also need a variant of this equation:

$$
\begin{aligned}
e^{ix} &= \sum_{k=0}^{n-1} \frac{i^k x^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1}(e^{is}-1+1)ds \\
&\hspace{20em} \text{(replace } n \text{ with } n-1) \\
&= \sum_{k=0}^{n-1} \frac{i^k x^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1}(e^{is}-1)ds + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1} ds \\
&= \sum_{k=0}^{n-1} \frac{i^k x^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1}(e^{is}-1)ds + \frac{i^n}{(n-1)!} \frac{x^n}{n} \\
&\hspace{20em} \text{(substitution)} \\
&= \sum_{k=0}^{n} \frac{i^k x^k}{k!} + \frac{i^n}{(n-1)!} \int_0^x (x-s)^{n-1}(e^{is}-1)ds. \qquad (2)
\end{aligned}
$$

We use (1) when $x \geq 0$, and we use (2) when $x < 0$. When $x \geq 0$:

$$
\begin{aligned}
\left| e^{ix} - \sum_{k=0}^{n} \frac{i^k x^k}{k!} \right| &= \left| \frac{i^{n+1}}{n!} \int_0^x (x-s)^n e^{is} ds \right| \\
&\leq \frac{1}{n!} \int_0^x |(x-s)^n e^{is}| ds && \text{(Jensen's)} \\
&\leq \frac{1}{n!} \int_0^x (x-s)^n ds \\
&= \frac{x^{n+1}}{(n+1)!}. && \text{(substitution)}
\end{aligned}
$$

When $x < 0$,

$$
\begin{aligned}
\left| e^{ix} - \sum_{k=0}^{n} \frac{i^k x^k}{k!} \right| &\leq \frac{1}{(n-1)!} \int_0^x |(x-s)^{n-1}||e^{is}-1| ds && \text{(Jensen's)} \\
&= \frac{1}{(n-1)!} \int_0^x (s-x)^{n-1} |e^{is}-1| ds && (x < 0) \\
&\leq \frac{2}{(n-1)!} \int_0^x (s-x)^{n-1} ds && \text{(tri-ineq)} \\
&= -\frac{2}{(n-1)!} \int_x^0 (s-x)^{n-1} ds \\
&= \frac{2|x|^n}{n!} && \text{(substitution)}
\end{aligned}
$$

Since they are both true

$$
\left| e^{ix} - \sum_{k=0}^{n} \frac{i^k x^k}{k!} \right| \leq \min \left\{ \frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!} \right\}
$$

and in the case of when $n = 2$

$$\left|e^{ix} - (1 + ix - x^2/2)\right| \le \min\left\{\frac{|x|^3}{(3)!}, |x|^2\right\} \le \min\left\{|x|^3, |x|^2\right\}.$$

This inequality will be used on each random variable in a row of the array. We will replace x with particular $uU_{N,j}$.