A More Detailed Proof of Lemma 9.5.4

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Assume

$$|U_N| \le \eta V_N + W_N(\eta).$$

Then, for any $\epsilon > 0$ and $\eta > 0$

$$\{|U_N| \le \epsilon\} \supset \{V_N \le \epsilon/2\eta\} \cap \{W_N(\eta) \le \epsilon/2\}.$$

Look at the contrapositive:

$$\{|U_N| > \epsilon\} \subset \{V_N > \epsilon/2\eta\} \cup \{W_N(\eta) > \epsilon/2\}.$$

That means

$$\begin{aligned} P(|U_N| > \epsilon) &\le P(\{V_N > \epsilon/2\eta\} \cup \{W_N(\eta) > \epsilon/2\}) \\ &\le P(V_N > \epsilon/2\eta) + P(W_N(\eta) > \epsilon/2) && \text{(sub-additivity)} \end{aligned}$$

which is the first equation of the authors' proof.

Taking the $\limsup_{N\to\infty}$ on both sides we have

$$\begin{aligned} \limsup_N P(|U_N| > \epsilon) &\le \limsup_N \{P(V_N > \epsilon/2\eta) + P(W_N(\eta) > \epsilon/2)\} \\ &\le \limsup_N P(V_N > \epsilon/2\eta) + \limsup_N P(W_N(\eta) > \epsilon/2) \\ & && \text{(properties of sup)} \\ &= \limsup_N P(V_N > \epsilon/2\eta) + \lim_N P(W_N(\eta) > \epsilon/2) \\ & && \text{(assumption)} \\ &= \limsup_N P(V_N > \epsilon/2\eta) + 0 && \text{(assumption)} \\ &= 0. \end{aligned}$$