

# A More Detailed Proof of Prop. 9.2.14

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## 1 Starting Off

This doesn't go into the details of properness of  $\tilde{\mathbf{A}}$ .

First, pick  $f \in \tilde{\mathbf{A}}$ . We want to show that

$$\sqrt{\tilde{M}_N} \left[ \tilde{M}_N^{-1} \sum_{i=1}^{\tilde{M}_N} \left\{ f(\tilde{\xi}^{N,i}) - \mu(f) \right\} \right]$$

converges in distribution to a normal random variable. Well, the main trick is splitting it up into two pieces by adding and subtracting conditional expectations, which condition on all the information that is had just before resampling is conducted:

$$\begin{aligned} & \sqrt{\tilde{M}_N} \left[ \tilde{M}_N^{-1} \sum_{i=1}^{\tilde{M}_N} \left\{ f(\tilde{\xi}^{N,i}) - \mu(f) \right\} \right] \\ &= \sqrt{\tilde{M}_N} \left[ \tilde{M}_N^{-1} \sum_{i=1}^{\tilde{M}_N} \left\{ f(\tilde{\xi}^{N,i}) - E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] \right\} + \tilde{M}_N^{-1} \sum_{i=1}^{\tilde{M}_N} \left\{ E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f) \right\} \right] \\ &= \sqrt{\tilde{M}_N} [B_N + A_N] \end{aligned}$$

The book writes these  $A_N$  and  $B_N$  just as we do, but to understand the expression of  $A_N$ , we have to write out the average conditional expectation the long way:

$$\begin{aligned}
A_N &= \tilde{M}_N^{-1} \sum_{i=1}^{\tilde{M}_N} \left\{ E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f) \right\} \\
&= E[f(\tilde{\xi}^{N,i}) \mid \mathcal{F}^N] - \mu(f) \quad (\text{identicalness}) \\
&= \left[ \sum_{i=1}^{M_N} \frac{\omega^{N,i} \frac{d\mu}{d\nu}(\xi^{N,i})}{\sum_j \omega^{N,j} \frac{d\mu}{d\nu}(\xi^{N,j})} f(\xi^{N,i}) \right] - \mu(f) \\
&= \sum_{i=1}^{M_N} \frac{\omega^{N,i} \frac{d\mu}{d\nu}(\xi^{N,i})}{\sum_j \omega^{N,j} \frac{d\mu}{d\nu}(\xi^{N,j})} [f(\xi^{N,i}) - \mu(f)]
\end{aligned}$$

## 2 Handling $A_N$

Recall our assumptions

Assumption 9.1.1:  $\mu \ll \nu$ , and  $d\mu/d\nu > 0$  almost surely.

Assumption 9.2.6:  $\{(\xi^{N,i}, \omega^{N,i})\}_{1 \leq i \leq M_N}$  is consistent for  $(\nu, \mathbb{C})$  (a proper set), and  $d\mu/d\nu \in \mathbb{C}$ .

Assumption 9.2.10: The weighted sample  $\{(\xi^{N,i}, \omega^{N,i})\}_{1 \leq i \leq M_N}$  is asymptotically normal for  $(\nu, \mathbb{A}, \sigma, \{a_N\})$ , and  $\frac{d\mu}{d\nu} \in \mathbb{A}$ .

Last,  $a_N^2/\tilde{M}_N \rightarrow \alpha$  and

$$\tilde{\mathbb{A}} \stackrel{\text{def}}{=} \left\{ f \in L^2(\mathbb{X}, \mu) : |f| \frac{d\mu}{d\nu} \in \mathbb{A}, f^2 \frac{d\mu}{d\nu} \in \mathbb{C} \right\}.$$

Theorem 9.2.11 tells us that

$$a_N A_N \xrightarrow{D} \text{Normal} \left[ 0, \sigma^2 \left( \frac{d\mu}{d\nu} [f - \mu(f)] \right) \right]$$

(note  $|f| \frac{d\mu}{d\nu} \in \mathbb{A}$  because  $\bar{\mathbb{A}} \subset \tilde{\mathbb{A}}$ ).

## 3 Bringing in $B_N$

In this section, we want to show the random vector

$$\begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix}$$

converges in distribution to a multivariate normal distribution. We will show this by showing that the joint characteristic function converges to the multivariate normal's characteristic function.

Pick  $u, v \in \mathbb{R}$ , and look at the bivariate characteristic function. The trick is to iterate expectations:

$$\begin{aligned}
E \left[ \exp \left( i \left\{ u \sqrt{\tilde{M}_N} B_N + v a_N A_N \right\} \right) \right] &= E \left[ E \left[ \exp \left( i \left\{ u \sqrt{\tilde{M}_N} B_N + v a_N A_N \right\} \right) \middle| \mathcal{F}^N \right] \right] \\
&= E \left[ E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \exp (i v a_N A_N) \middle| \mathcal{F}^N \right] \right] \\
&= E \left[ \exp (i v a_N A_N) E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \middle| \mathcal{F}^N \right] \right].
\end{aligned}$$

Then we take the limit as  $N \rightarrow \infty$

$$\begin{aligned}
&\lim_{N \rightarrow \infty} E \left[ \exp \left( i \left\{ u \sqrt{\tilde{M}_N} B_N + v a_N A_N \right\} \right) \right] \\
&= \lim_{N \rightarrow \infty} E \left[ \exp (i v a_N A_N) E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \middle| \mathcal{F}^N \right] \right] \quad (\text{previous}) \\
&= E \left[ \lim_{N \rightarrow \infty} \exp (i v a_N A_N) E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \middle| \mathcal{F}^N \right] \right] \quad (\text{DCT}) \\
&= E \left[ \lim_{N \rightarrow \infty} \exp (i v a_N A_N) \lim_{N \rightarrow \infty} E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \middle| \mathcal{F}^N \right] \right] \\
&\quad \quad \quad (\text{limit properties}) \\
&= E \left[ \lim_{N \rightarrow \infty} \exp (i v a_N A_N) \exp \left[ -\frac{u^2 \text{Var}_\mu(f)}{2} \right] \right] \quad (9.2.12) \\
&= \exp \left[ -\frac{u^2 \text{Var}_\mu(f)}{2} \right] E \left[ \lim_{N \rightarrow \infty} \exp (i v a_N A_N) \right] \quad (\text{linearity of } E) \\
&= \exp \left[ -\frac{u^2 \text{Var}_\mu(f)}{2} \right] \lim_{N \rightarrow \infty} E [\exp (i v a_N A_N)] \quad (\text{DCT}) \\
&= \exp \left[ -\frac{u^2 \text{Var}_\mu(f)}{2} \right] \exp \left[ -\frac{u^2 \sigma^2 \left( \frac{d\mu}{dv} [f - \mu(f)] \right)}{2} \right]. \quad (\text{Theorem 9.2.11})
\end{aligned}$$

Two of those steps required separate theorems. We showed Theorem 9.2.11 in the previous section of this document:

$$\exp [i v a_N A_N] \xrightarrow{p} \exp \left[ -\frac{u^2 \sigma^2 \left( \frac{d\mu}{dv} [f - \mu(f)] \right)}{2} \right].$$

And by Proposition 9.2.12

$$E \left[ \exp \left( i u \sqrt{\tilde{M}_N} B_N \right) \middle| \mathcal{F}^N \right] \xrightarrow{p} \exp \left[ -\frac{u^2 \text{Var}_\mu(f)}{2} \right]$$

(note  $f^2 \frac{d\mu}{dv} \in \mathbb{C}$  because of the definition of  $\tilde{A}$ .)

## 4 Final Step

Before, we showed

$$\begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix} \xrightarrow{D} \text{Normal} \left( \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \text{Var}_\mu(f) & 0 \\ 0 & \sigma^2 \left( \frac{d\mu}{d\nu} [f - \mu(f)] \right) \end{bmatrix} \right)$$

If  $\alpha < 1$ , then  $\frac{a_N^2}{\tilde{M}_N} \rightarrow \alpha < 1$ , and we can take the following inner product and apply the multivariate delta method:

$$\begin{aligned} a_N [B_N + A_N] &= \begin{bmatrix} a_N / \sqrt{\tilde{M}_N} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix} \\ &\xrightarrow{D} \text{Normal} \left( 0, \lim_N \frac{a_N^2}{\tilde{M}_N} \text{Var}_\mu(f) + \text{Var}_\mu(f) \right) \\ &= \text{Normal} (0, \alpha \text{Var}_\mu(f) + \text{Var}_\mu(f)) \end{aligned}$$

If  $\alpha \geq 1$  (recall that it's possibly infinite-valued), then  $\frac{a_N^2}{\tilde{M}_N} \rightarrow \alpha$ , so the reciprocal of that goes to  $1/\alpha$ . We take the following inner product, and apply the multivariate delta method:

$$\begin{aligned} \sqrt{\tilde{M}_N} [B_N + A_N] &= \begin{bmatrix} 1 & \sqrt{\tilde{M}_N} / a_N \end{bmatrix} \begin{bmatrix} \sqrt{\tilde{M}_N} B_N \\ a_N A_N \end{bmatrix} \\ &\xrightarrow{D} \text{Normal} \left( 0, \text{Var}_\mu(f) + \lim_N \frac{\tilde{M}_N}{a_N^2} \text{Var}_\mu(f) \right) \\ &= \text{Normal} \left( 0, \text{Var}_\mu(f) + \frac{1}{\alpha} \text{Var}_\mu(f) \right) \end{aligned}$$

where the last line follows because