A More Detailed Proof of Prop. 9.3.5

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In the slides we explained our goal was to show

$$\frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \xrightarrow{\text{P}} \nu L(f)$$

for any $f \in \tilde{\mathsf{C}}$. First, let's write

$$\left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \right\} + \left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \right\}$$

The first term converges to 0, and the second converges to $\mu(f)$.

1 The Trick

Recall in the slides we discussed that

$$\begin{aligned} L(\xi^{N,i}, f) &= \int L(\xi^{N,i}, dy) f(y) \\ &= \int R(\xi^{N,i}, dy) \frac{dL(\xi^{N,i}, \cdot)}{dR(\xi^{N,i}, \cdot)}(y) f(y) \\ &= E\left[\tilde{\omega}^{N,i} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N \right] \end{aligned}$$

The reason that this is useful is that it is a function of ν samples only. Therefore, as long as this function is sufficiently well-behaved, we only need to worry about ν- consistency.

Another thing: the sum over j (new samples) turns into a sum over i (old samples) because $\alpha_N M_N = \tilde{M}_N$:

$$\begin{aligned}
\tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] &= \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, f) \\
&= \alpha_N^{-1} M_N^{-1} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, f) \\
&= \alpha_N^{-1} M_N^{-1} \sum_{i=1}^{M_N} \alpha_N L(\xi^{N,i}, f) \\
&= M_N^{-1} \sum_{i=1}^{M_N} L(\xi^{N,i}, f),
\end{aligned}$$

So is $L(x, f) \in \mathsf{C}$? The answer is yes, as long as we pick $f \in \tilde{\mathsf{C}}$, and that follows just by the definition of it:

$$\tilde{\mathsf{C}} = \left\{ f \in L^1(\mathsf{X}, \mu) : x \mapsto L(x, |f|) \in \mathsf{C} \right\}.$$

2 Showing Convergence to Zero using Theorem 9.5.7

We showed the second part of

$$\left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) - \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] \right\} + \left\{ \frac{1}{\tilde{M}^N} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j}) \mid \mathcal{F}^N\right] \right\}$$

converges in probability to $\nu L(f)$ for any $f \in \tilde{\mathsf{C}}$. Now we must show the fist part converges to 0. To show this, we set $V_{N,j} = \frac{1}{\tilde{M}_N} \tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})$, and check the three conditions of 9.5.7.

2.1 First condition of 9.5.7

The first condition, translated into our notation, is: the triangular array $\{V_{N,j}\}_{1 \leq j \leq M_N}$ is conditionally independent given $\mathcal{F}^N$, and for any N, $j = 1, \ldots, \tilde{M}_N$, we have $E[|V_{N,j}| \mid \mathcal{F}^N] < \infty$.

The first condition of 9.5.7 is true because of the description of the algorithm used for 9.3.5, and because $f \in \tilde{\mathsf{C}}$ by assumption.

2.2 Second Condition of 9.5.7

The second condition is that the sequence (in N)

$$\left\{ \sum_{j=1}^{\tilde{M}_N} E[|V_{N,j}| \mid \mathcal{F}^N] \right\}_N$$

is bounded in probability. This is true because it converges in probability. To see this more easily, use the same trick we've been using a few times:

$$\begin{aligned}
\sum_{j=1}^{\tilde{M}_N} E[|V_{N,j}| \mid \mathcal{F}^N] &= \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E[|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| \mid \mathcal{F}^N] \\
&= \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} E[\tilde{\omega}^{N,j} |f(\tilde{\xi}^{N,j})| \mid \mathcal{F}^N] \\
&\qquad \text{(Assumption 9.3.3 positive RN deriv.)} \\
&= \frac{1}{\tilde{M}_N} \sum_{j=1}^{\tilde{M}_N} L(\xi^{N,i}, |f|) \xrightarrow{\text{P}} \nu L(|f|).
\end{aligned}$$

2.3 Third Condition of 9.5.7

The third condition is that for any $\epsilon > 0$ we have

$$\sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}| \mathbf{1}\left(|V_{N,j}| \geq \epsilon\right) \mid \mathcal{F}^N\right] \xrightarrow{\text{P}} 0.$$

Before, we show the main work, note that, after we pick $f \in \tilde{\mathsf{C}}$, then for any C,

$$L(x, |f|\mathbf{1}_{\{h(x,x') \geq C\}}) = \int L(x, dx') \, |f(x')| \, \mathbf{1}_{\{h(x,x') \geq C\}} \leq L(x, |f|).$$

This is useful because if we assume $f \in \tilde{\mathsf{C}}$, this will imply that $L(x, |f|) \in \mathsf{C}$, and by propriety, this will also imply that $L(x, |f|\mathbf{1}_{\{h(x,x') \geq C\}}) \in \mathsf{C}$. We need this in the last line of the following work:

$$
\begin{aligned}
\lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}| 1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] &= \lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})| 1_{\{|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\qquad\text{(defn of V)} \\
&= \lim_N \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} |f(\tilde{\xi}^{N,j})| 1_{\{|\tilde{M}_N^{-1}\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\qquad\text{(algebra)} \\
&\leq \lim_N \tilde{M}_N^{-1} \sum_{j=1}^{\tilde{M}_N} E\left[\tilde{\omega}^{N,j} |f(\tilde{\xi}^{N,j})| 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \mid \mathcal{F}^N\right] \\
&\qquad\text{(if } \tilde{M}_N\epsilon \geq C\text{)} \\
&= \lim_N M_N^{-1} \sum_{i=1}^{M_N} \int L(\xi^{N,i}, d\tilde{\xi}^{N,j}) \left| f(\tilde{\xi}^{N,j}) \right| 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad\text{(defn of expec. and the trick with replacing } i \text{ and } j\text{)} \\
&= \iint \nu(dx) L(x, dx') |f|(x') 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad\text{(discussion above)}
\end{aligned}
$$

The last term is bounded above by $\nu L(|f|) < \infty$, so we can use dominated convergence on the above work when we take the limit with $C \uparrow \infty$:

$$
\begin{aligned}
\lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}| 1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] &= \lim_C \lim_N \sum_{j=1}^{\tilde{M}_N} E\left[|V_{N,j}| 1_{\{|V_{N,j}|\geq\epsilon\}} \mid \mathcal{F}^N\right] \\
&\leq \lim_C \iint \nu(dx) L(x, dx') \left| f(x') \right| 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad\text{(above)} \\
&= \iint \nu(dx) L(x, dx') \left| f(x') \right| \lim_C 1_{\{|\tilde{\omega}^{N,j} f(\tilde{\xi}^{N,j})|\geq C\}} \\
&\qquad\text{(DCT)} \\
&= 0.
\end{aligned}
$$