A More Detailed Proof of Proposition 9.5.1

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Overall, we have

$$\sum_i U_{N,i} = \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] + \left(\sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right) + \left(\sum_i U_{N,i} - \sum_i \bar{U}_{N,i} \right).$$

We want to show each of these pieces on the right hand side converges to 0. If they all do, then we can use triangle inequality to show the left hand side converges to 0.

Part 1

Starting with the first summand on the right hand side, we have

$$0 = E[U_{N,i} \mid \mathcal{F}^N] = E[U_{N,i} \mathbb{1}(U_{N,i} \geq \epsilon) \mid \mathcal{F}^N] + E[\bar{U}_{N,i} \mid \mathcal{F}^N]$$

summing both sides gives us

$$0 = \sum_{i=1}^{M_N} E[U_{N,i} \mathbb{1}(U_{N,i} \geq \epsilon) \mid \mathcal{F}^N] + \sum_{i=1}^{M_N} E[\bar{U}_{N,i} \mid \mathcal{F}^N].$$

Subtracting the rightmost portion from both sides and then applying an absolute value to both sides gives us

$$\left| \sum_{i=1}^{M_N} E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right| = \left| \sum_{i=1}^{M_N} E[U_{N,i} 1(U_{N,i} \geq \epsilon) \mid \mathcal{F}^N] \right| \leq \sum_{i=1}^{M_N} E[|U_{N,i}| \mathbb{1}(U_{N,i} \geq \epsilon) \mid \mathcal{F}^N].$$

The last inequality arises after we use the triangle inequality, and Jensen's inequality.

Then, taking the limit on both sides and using assumption 9.6.3,

$$\sum_{i=1}^{M_N} E[\bar{U}_{N,i} \mid \mathcal{F}^N] \xrightarrow{\text{p}} 0 \tag{1}$$

as $N \to \infty$.

Part 2

Expanding a little on the Chebyshev's thing at the top of page 334:

$$
\begin{aligned}
0 &\le A_N(\delta) \\
&= P\left(\left|\sum_{i=1}^{M_N} \bar{U}_{N,i} - \sum_{i=1}^{M_N} E\left[\bar{U}_{N,i} \mid \mathcal{F}^N\right]\right| \ge \delta \middle| \mathcal{F}^N\right) && \text{(defn.)} \\
&\le \delta^{-2} \mathrm{Var}\left(\sum_{i}^{M_N} \bar{U}_{N,i} \mid \mathcal{F}^N\right) && \text{(Chebyshev's)} \\
&= \delta^{-2} \sum_i \sum_j \mathrm{Cov}\left(\bar{U}_{N,i}, \bar{U}_{N,j} \mid \mathcal{F}^N\right) \\
&= \delta^{-2} \sum_i \mathrm{Var}\left(\bar{U}_{N,i} \mid \mathcal{F}^N\right) && \text{(independence)} \\
&= \delta^{-2} \sum_i \left\{E\left(\bar{U}_{N,i}^2 \mid \mathcal{F}^N\right) - \left[E\left(\bar{U}_{N,i} \mid \mathcal{F}^N\right)\right]^2\right\} \\
&\le \delta^{-2} \sum_i E\left(\bar{U}_{N,i}^2 \mid \mathcal{F}^N\right) \\
&\xrightarrow{p} 0. && (9.62)
\end{aligned}
$$

So this is where 9.6.2 applies, giving us $A_N(\delta) \to 0$ in probability.

Because $A_N(\delta) \le 1$ (it's a probability), we have by dominated convergence

$$\lim_N E[A_N(\delta)] = E[\lim_N A_N(\delta)] = 0.$$

This is equivalent to (9.65) on page 334, and so

$$\sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \xrightarrow{\mathrm{p}} 0.$$

If you're thinking that we don't need dominated convergence, notice that this last statement is about unconditional expectations, not conditional expectations, which is what $A_N(\delta)$ is.

Part 3

$$
\begin{aligned}
P\left(\left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| \geq \delta \middle| \mathcal{F}^N\right) &= P\left(\left|\sum_i U_{N,i} \mathbb{1}(U_{N,i} \geq \epsilon)\right| \geq \delta \middle| \mathcal{F}^N\right) \\
&\leq P\left(\sum_i \left|U_{N,i}\right| \mathbb{1}(U_{N,i} \geq \epsilon) \geq \delta \middle| \mathcal{F}^N\right) \\
&\qquad \text{(tri-ineq)} \\
&\leq \delta^{-1} \sum_i E\left[\left|U_{N,i}\right| \mathbb{1}(U_{N,i} \geq \epsilon) \middle| \mathcal{F}^N\right] \\
&\qquad \text{(Markov's and linearity)} \\
&\to 0. \qquad (9.63)
\end{aligned}
$$

So the *conditional* probailities go to 0. To show that the unconditional ones do, as well, we need to iterate the expectations, and use the same argument of dominated convergence as before:

$$
\begin{aligned}
\lim_N P\left(\left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| \geq \delta\right) &= \lim_N E\left[\mathbb{1}\left(\left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| \geq \delta\right)\right] \\
&= \lim_N E\left[E\left\{\mathbb{1}\left(\left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| \geq \delta\right) \mid \mathcal{F}^N\right\}\right] \\
&\qquad \text{(law total expec.)} \\
&= E\left[\lim_N E\mathbb{1}\left(\left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| \geq \delta \mid \mathcal{F}^N\right)\right] \\
&\qquad \text{(DCT)} \\
&= 0. \qquad \text{(previous)}
\end{aligned}
$$

Tying it all together

Recall that

$$
\sum_i U_{N,i} = \left(\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right) + \left(\sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N]\right) + \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N].
$$

Using the triangle inequality we have that

$$
\left|\sum_i U_{N,i}\right| \leq \left|\sum_i U_{N,i} - \sum_i \bar{U}_{N,i}\right| + \left|\sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N]\right| + \left|\sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N]\right|.
$$

so

$$
\begin{aligned}
\left\{ \left| \sum_i U_{N,i} \right| \geq \delta \right\} \subseteq & \left\{ \left| \sum_i U_{N,i} - \sum_i \bar{U}_{N,i} \right| \geq \delta/3 \right\} \\
& \cup \left\{ \left| \sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right| \geq \delta/3 \right\} \\
& \cup \left\{ \left| \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right| \geq \delta/3 \right\}.
\end{aligned}
$$

so

$$
\begin{aligned}
P\left(\left| \sum_i U_{N,i} \right| \geq \delta \right) \leq & \, P\left(\left| \sum_i U_{N,i} - \sum_i \bar{U}_{N,i} \right| \geq \delta/3 \right) \\
& + P\left(\left| \sum_i \bar{U}_{N,i} - \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right| \geq \delta/3 \right) \\
& + P\left(\left| \sum_i E[\bar{U}_{N,i} \mid \mathcal{F}^N] \right| \geq \delta/3 \right).
\end{aligned}
$$

by sub-additivity of P, and because probability of supersets is greater than subsets.

Use parts (1), (2) and (3), the three terms on the right hand side all go to 0 in probability.