16-741 Mechanics of Manipulation 10-17-2018

Problem Statement 3

1. For null rotation, rotation matrix with respect to ~~for~~ each of x, y and z axes is represented by an Identity matrix.

$$R_{x,y,z} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

also if we consider the angle of rotation $\theta = 0°$ to determine the quaternion, q;

$$q = \cos\frac{\theta}{2} + \hat{n}\sin\frac{\theta}{2}$$ where θ is the rotation angle and $\hat{n}$ is the axis of rotation.

$$\approx q_0 + \vec{q}$$

$$= \cos 0 + \hat{n}\sin 0$$

$$= 1.$$

$\therefore$ The null quaternion is simply the scalar component with $q = q_0 = 1$.

For rotation $\theta = \pi/2$;

$$q = \cos\frac{\pi}{4} + \hat{n}\sin\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\hat{n}$$

$$\therefore q = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j + \frac{1}{\sqrt{2}}k.$$

for this, the unit quaternion can be determined as $|q| = 1$; $\sqrt{(1/2) + 3a^2} = 1$; $a = \pm\sqrt{\frac{0.5}{3}} = \pm 0.4082$

$\therefore$ unit quaternion for $\theta = \pi/2$ is

$$q = \frac{1}{\sqrt{2}} \pm 0.4082i \pm 0.4082j \pm 0.4082k.$$

Now,

Quaternion for $\theta = \pi$ is

$$q = \cos\frac{\pi}{2} + \hat{n}\sin\frac{\pi}{2}$$

$$= 0 + \hat{n}\,\cancel{\sin\theta} \times 1.$$

$$= \hat{n} = i+j+k.$$

for us to get a unit quaternion,

$$|q| = 1;$$

$$\sqrt{3a^2} = 1$$

$$a^2 = \frac{1}{3} \quad ; \quad a = \pm\sqrt{\frac{1}{3}} = \pm 0.5773$$

$$\therefore \quad q = \pm 0.5773i \pm 0.5773j \pm 0.5773k.$$

2. As stated in the textbook, a simple method for converting a rotation matrix to an axis-angle representation is to first convert the matrix to a quaternion, and then convert the quaternion to axis-angle form.

Matrix → Unit Quaternion

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} -2/3 & -2/3 & 1/3 \\ 2/3 & -1/3 & 2/3 \\ -1/3 & 2/3 & 2/3 \end{pmatrix}$$

First, we will use equations (3.89-3.92) to find the largest q_i^2.

$(3.89) \Rightarrow q_0^2 = \frac{1}{4}(1+r_{11}+r_{22}+r_{33}) = \frac{1}{6}$

$(3.90) \Rightarrow q_1^2 = \frac{1}{4}(1+r_{11}-r_{22}-r_{33}) = 0$

$(3.91) \Rightarrow q_2^2 = \frac{1}{4}(1-r_{11}+r_{22}-r_{33}) = \frac{1}{6}$

$(3.92) \Rightarrow q_3^2 = \frac{1}{4}(1-r_{11}-r_{22}+r_{33}) = \frac{2}{3}$ ~ largest q_i^2

Since either sign will do for the square root, $q_3 = \sqrt{\frac{2}{3}} = 0.8165$

Now, we can use equations (3.93-3.98) to find the other three components of the quaternion.

$(3.95) \Rightarrow q_0 q_3 = \frac{1}{4}(r_{21} - r_{12}) \Rightarrow q_0 = 0.4082$

$(3.93) \Rightarrow q_0 q_1 = \frac{1}{4}(r_{32} - r_{23}) \Rightarrow q_1 = 0$ (we also knew that from before, since $q_1^2 = 0$)

$(3.94) \Rightarrow q_0 q_2 = \frac{1}{4}(r_{13} - r_{31}) \Rightarrow q_2 = 0.4082$

Unit Quaternion: $q = \underset{q_0\text{: scalar part}}{0.4082} + \underbrace{0 \cdot i + 0.4082 j + 0.8165 k}_{q\text{: vector part}}$

Quaternion → Axis-Angle

To obtain the axis and angle from the quaternion, we need to calculate θ and $\hat{n}$, using the following formulas:

$$\theta = 2 \tan^{-1}(|q|, q_0)$$

$$\hat{n} = q / |q|$$

$|q| = \sqrt{0.4082^2 + 0.8165^2} = 0.9128$,

$\cancel{\frac{q_0}{|q|} = \frac{0.4082}{0.9128} = 0.4472}$ $\quad \frac{|q|}{q_0} = \frac{0.9128}{0.4082} = 2.236$

so $\theta = 2\arctan(2.236) \Rightarrow \theta = 131.8^\circ$ or $\theta = 2.3$ rad

$\hat{n} = \frac{q}{|q|} = \frac{0i + 0.4082j + 0.8165k}{0.9128} \Rightarrow \hat{n} = 0i + 0.4472j + 0.8945k$

Axis-Angle: $\mathrm{rot}(\hat{n}, \theta)$, where $\begin{cases} \theta = 131.8^\circ \\ \hat{n} = 0.4472j + 0.8945k \end{cases}$

Rotation Matrix to Euler angles

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{pmatrix}$$

$$a = \tan^{-1}(r_{23}, r_{13}) = \tan^{-1}\left(\frac{2}{3}, \frac{1}{3}\right) = \arctan(2) = 63.43^\circ \text{ or } 1.107 \text{ rad}$$

$$\gamma = \tan^{-1}(r_{32}, -r_{31}) = \tan^{-1}\left(\frac{2}{3}, \frac{1}{3}\right) = \arctan(2) = 63.43^\circ \text{ or } 1.107 \text{ rad}$$

and $\beta = \cos^{-1}(r_{33}) = \cos^{-1}\left(\frac{2}{3}\right) = 48.24^\circ$ or 0.84 rad

3.

Please find the code 3.py attached in the angelosm_schakra1_spokhare_ps3.zip folder for this problem

When using the method described in 3 (by discarding every quadruple with magnitude greater than 1), we get the mean angle as 124.6˚ (2.175rad).
When using the procedure mentioned in this website http://planning.cs.uiuc.edu/node198.html, we get the mean angle as 126.5˚(2.208rad).

The histogram showing the distribution of uniformly distributed unit quaternions.