24-677 Homework 8

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Exercise 1

Let

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Since $V = x_1^2 + x_2^2$,

$$\dot{V} = 2x_1\dot{x_1} + 2x_2\dot{x_2} = 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = 2x^TAx$$

In order to show that $\dot{V} < 0$ we need to show that $2x^TAx < 0 \Rightarrow x^TAx < 0$
This means we have to show that A is negative definite. All we need to do is guarantee that A's eigenvalues are negative.

$$A - \lambda I = \begin{bmatrix} \alpha - \lambda & 0 \\ 1 & -1 - \lambda \end{bmatrix}$$

So $\lambda_1 = \alpha$ and $\lambda_2 = -1$.
Since $\lambda_2 < 0$, A is negative definite for $\alpha < 0$ and therefore the system is asymptotically stable for all negative values of the parameter α.

Exercise 2

(a) We can find the linearized system by setting $\dot{x_1} = 0$ and $\dot{x_2} = 0$

$$\dot{x_1} = 0 \Rightarrow x_2 - x_1 x_2^2 = 0 \Rightarrow x_2(1 - x_1 x_2) = 0 \Rightarrow x_2 = 0 \text{ and } x_1 x_2 = 0$$
$$\dot{x_2} = 0 \Rightarrow -x_1^3 = 0 \Rightarrow x_1 = 0 \text{ and } x_2 = 0$$

So,

$$A = \frac{\partial f}{\partial x} = \begin{bmatrix} -x_2^2 & 1 - 2x_1 x_2 \\ -3x_1^2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ for } x = \bar{x}$$

Linearized system:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

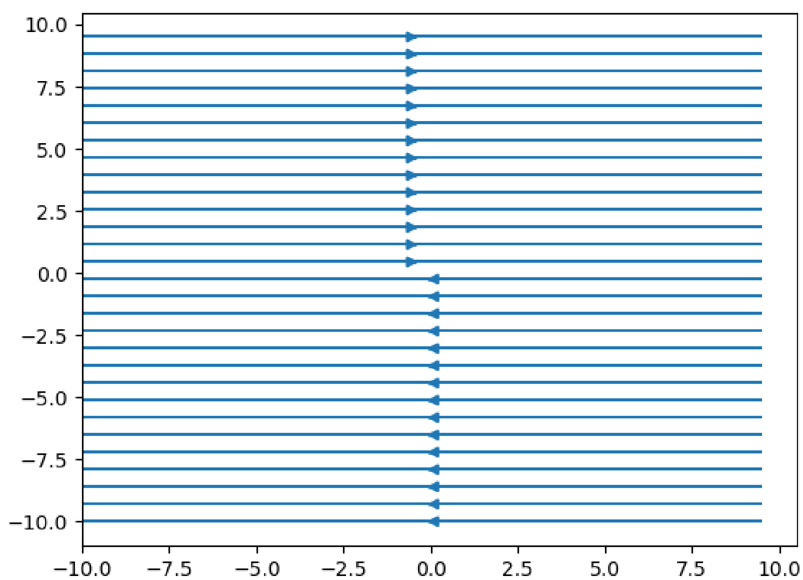
$\Delta\lambda = (-\lambda)^2$ so $\lambda = 0$ with algebraic multiplicity $m = 2$, hence the linearized system is not stable.

(b) With $\dot{x_1} = x_2 - x_1 x_2^2$ and $\dot{x_2} = -x_1^3$, we can compute the time derivative of the given Lyapunov function:

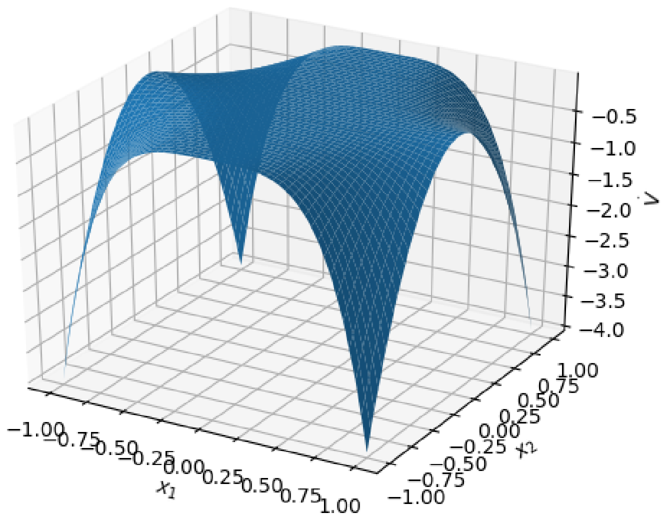
$$\dot{V} = 4x_1^3 \dot{x_1} + 4x_2 \dot{x_2} = 4x_1^3(x_2 - x_1 x_2^2) + 4x_2(-x_1^3) = -4x_1^4 x_2^2 < 0 \text{ for } x \neq 0$$

So the system is globally asymptotically stable.

(c) The following image shows the phase portrait for the linearized system of 2.a.



(d) The following 3D plot shows the variation of $\dot{V}$ with respect to x_1 and x_2.



Here is the code used for plotting the figures above.

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d import Axes3D

#code for plotting the phase portrait
w = 10
y, x = np.mgrid[-w:w:100j, -w:w:100j]
x_dot = y
y_dot = np.zeros(x.shape)
plt.streamplot(x, y, x_dot, y_dot)
plt.show()

#code for plotting the derivative of the energy function
#with respect to x_1 and x_2
x1 = np.linspace(-1, 1, 1000)
x2 = np.linspace(-1, 1, 1000)
x1, x2 = np.meshgrid(x1, x2)
v = -4 * np.multiply(x1**4, x2**2)

fig = plt.figure()
ax = fig.gca(projection='3d')
ax.plot_surface(x1, x2, v)
ax.set_xlabel('$x_1$')
ax.set_ylabel('$x_2$')
ax.set_zlabel('$\dot{V}$')
plt.show()
```

Exercise 3

(a)

$$G(s) = C(zI - A)^{-1}B + D = \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} z-1 & 0 \\ 0.5 & s-0.5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \end{bmatrix} \frac{1}{(z-1)(z-0.5)} \begin{bmatrix} z-0.5 & 0 \\ -0.5 & z-1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} & 0 \\ -\frac{0.5}{(z-1)(z-0.5)} & \frac{1}{z-0.5} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 & 5 \end{bmatrix} \begin{bmatrix} \frac{1}{z-1} \\ \frac{-z+0.5}{(z-1)(z-0.5)} \end{bmatrix} = \frac{5}{z-1} + \frac{-5z+2.5}{(z-1)(z-0.5)}$$

$$= \frac{5z-2.5}{(z-1)(z-0.5)} + \frac{-5z+2.5}{(z-1)(z-0.5)} = \frac{0}{(z-1)(z-0.5)}$$

The transfer function is zero, which means that the output is bounded. However, the poles are $z_1 = 1$ and $z_2 = 0.5$, therefore the system is not stable, since pole z_1 lies exactly on the unit cycle.

(b)

$$\Delta(\lambda) = \begin{bmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{bmatrix} = \lambda^3 + 9\lambda^2 + 23\lambda + 15$$

We observe that $\lambda_1 = -1$ is a root of the polynomial and dividing with $\lambda + 1$ we get the other two solutions $\lambda_2 = -3$ and $\lambda_3 = -5$. We need to show that $\int_{-\infty}^{t} Ce^{A(t-\tau} d\tau$ is finite in order to show that the system is BIBO stable. Let's calculate $e^A(t-\tau)$ using a similarity transformation. To define M, we have to find the eigenspace for each unique eigenvalue of A.

$$A + I = \begin{bmatrix} -6 & -2 & 6 \\ 2 & -2 & -2 \\ -2 & -2 & 2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{3} & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} -4 & -2 & 6 \\ 2 & 0 & -2 \\ -2 & -2 & 4 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$A + 5I = \begin{bmatrix} -2 & -2 & 6 \\ 2 & 2 & -2 \\ -2 & -2 & 6 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 1 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow x_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \text{ and we calculate } M^-1 = \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$e^{A(t-\tau)} = Me^{J(t-\tau)}M^-1 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-t+\tau} & 0 & 0 \\ 0 & e^{-3t+3\tau} & 0 \\ 0 & 0 & e^{-5t+5\tau} \end{bmatrix} \begin{bmatrix} -1 & -1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-3t+3\tau} + e^{-t+\tau} + e^{-5t+5\tau} & e^{-3t+3\tau} - e^{-t+\tau} & 2e^{-t+\tau} - e^{-3t+3\tau} - e^{-5t+5\tau} \\ e^{-3t+3\tau} - e^{-5t+5\tau} & e^{-3t+3\tau} & e^{-5t+5\tau} - e^{-3t+3\tau} \\ e^{-3t+3\tau} - e^{-t+\tau} & e^{-3t+3\tau} - e^{-t+\tau} & 2e^{-t+\tau} - e^{-3t+3\tau} \end{bmatrix}$$

$$\int_{-\infty}^{t} Ce^{A(t-\tau} d\tau = \int_{-\infty}^{t} \begin{bmatrix} 0 & 0 \\ e^{-3t+3\tau} & 0 \end{bmatrix} d\tau = \frac{1}{3}$$

Since the limit is finite, the system is BIBO stable.

Exercise 4

$$\frac{Y(s)}{U(s)} = \frac{s+3}{s^2+3s+2} = \frac{Y(s)}{x_1}\frac{x_1}{U(s)} = \frac{s+3}{1}\frac{1}{s^2+3s+2}$$

Setting $x_2 = \dot{x_1}$, we have:

$$Y(s) = (s+3)x_1 = sx_1 + 3x_1 = \dot{x_1} + 3x_1 = x_2 + 3x_1$$

$$U(s) = (s^2+3s+2)x_1 = s^2x_1 + 3sx_1 + 2x_1 = s\dot{x_1} + 3\dot{x_1} + 2x_1 = \dot{x_2} + 3x_2 + 2x_1$$

Solving for $\dot{x_2} = -2x_1 - 3x_2 + u$, we can write the controllable canonical form state space representation:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u$$

Exercise 5

$$D = G_1\hat{(}\infty) = \begin{bmatrix} \frac{1}{\infty} & 1 \\ \frac{1}{\infty} & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

So,

$$\hat{G_s p} = G(s) - D = \begin{bmatrix} \frac{1}{s} & \frac{2}{s+1} \\ \frac{1}{s+3} & -\frac{1}{s+1} \end{bmatrix}$$

We can find the least common denominator:

$$d(s) = s(s+1)(s+3) = (s^2+s)(s+3) = s^3 + 4s^2 + 3s$$

$$G_s p = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} (s+1)(s+3) & 2s(s+3) \\ s(s+1) & -s(s+3) \end{bmatrix} = \frac{1}{s^3+4s^2+3s} \begin{bmatrix} s^2+4s+3 & 2s^2+6s \\ s^2+s & -s^2-3s \end{bmatrix}$$

Then we can define $N_1(s)$, $N_2(s)$ and $N_3(s)$:

$$N_1(s) = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$N_2(s) = \begin{bmatrix} 4 & 6 \\ 1 & -3 \end{bmatrix}$$

$$N_3(s) = \begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$$

Then, having $p = 2$, $\alpha_1 = 4$, $\alpha_2 = 3$ and $\alpha_3 = 0$ we can define all matrices:

$$A = \begin{bmatrix} -4 & 0 & -3 & 0 & 0 & 0 \\ 0 & -4 & 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 4 & 6 & 3 & 0 \\ 1 & -1 & 1 & -3 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Exercise 6

1st system:

$$sI - A = C = \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}$$

with $\Delta(s) = (s-2)(s-1)$

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 2 & 2 \end{bmatrix} \frac{1}{(s-1)(s-2)} \begin{bmatrix} s-1 & 0 \\ 1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & \frac{1}{(s-1)(s-2)} \\ 0 & \frac{1}{s-1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & 0 \end{bmatrix} = \frac{2}{s-2}$$

Controllability Matrix:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$Rank(P) = 1 < n = 2$ so the 1st system is NOT controllable. We can also check the observability with the observability matrix:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix}$$

$Rank(Q) = 1 < n = 2$ so the system is also NOT observable. Since the system is neither controllable nor observable, this realization is not a minimal realization based on the theorem.

2nd system:

$$sI - A = C = \begin{bmatrix} s-2 & 0 \\ 1 & s+1 \end{bmatrix}$$

with $\Delta(s) = (s+1)(s-2)$

$$G(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 2 & 0 \end{bmatrix} \frac{1}{(s+1)(s-2)} \begin{bmatrix} s+1 & 0 \\ -1 & s-2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{s-2} & 0 \\ -\frac{1}{(s+1)(s-2)} & \frac{1}{s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2}{s-2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{2}{s-2}$$

We notice that the two systems are equivalent, since they have the same transfer function. Controllability Matrix:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -3 \end{bmatrix}$$

$Rank(P) = 2 = n$ so the 2nd system is controllable. We also have to check the observability with the observability matrix:

$$Q = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

$Rank(Q) = 1 < n = 2$ so the system is also NOT observable. Since the system is controllable but NOT observable, this realization is not a minimal realization based on the theorem.