Linear Control Systems

Homework 2

Exercise 1

1) The given set of vectors is linearly independent if the matrix containing them as columns has non-zero determinant

$$\det\begin{bmatrix} -1 & 1 & 2 \\ -9 & 3 & -2 \\ 0 & 0 & 1 \end{bmatrix} = -1(3\cdot1-0)-1(-9-0)+2\cdot0=-3+9=6\neq0$$

Therefore, they are linearly independent

2) $x_1=\begin{bmatrix} 2-i \\ -i \end{bmatrix}$, $x_2=\begin{bmatrix} 1+2i \\ -i \end{bmatrix}$, $x_3=\begin{bmatrix} -i \\ 3+4i \end{bmatrix}$

The given vectors are linearly independent if for

$a,b,c\in\mathbb{R}$, $a\underline{x_1}+b\underline{x_2}+c\underline{x_3}=0 \Leftrightarrow a=b=c=0$

$$\left.\begin{aligned} a(2-i)+b(i+2i)+c(-i)=0 \\ a(-i)+b(-i)+c(3+4i)=0 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} 2a-ai+bi+2bi-ci=0 \\ -ai-bi+3c+4ci=0 \end{aligned}\right\}$$

$$\Rightarrow \left.\begin{aligned} (2a)+(-a+3b-c)i=0 \\ (3c)+(-a-b+4c)i=0 \end{aligned}\right\} \Rightarrow \begin{aligned} &2a=0\Rightarrow a=0 \\ &3c=0\Rightarrow c=0 \\ &-\underset{0}{a}+3b-\underset{0}{c}=0\Rightarrow b=0 \end{aligned}$$

Hence, the vectors are linearly independent.

3) $x_1 = 2s^2 + 2s - 1$, $x_2 = -2s^2 + 2s + 1$, $x_3 = s^2 - s - 5$

x_1, x_2, x_3 are linearly independent if for $a, b, c \in \mathbb{R}$:

$$a x_1 + b x_2 + c x_3 = 0 \Leftrightarrow a = b = c = 0$$

$a(2s^2+2s-1) + b(-2s^2+2s+1) + c(s^2-s-5) = 0 \Rightarrow$

$2as^2 + 2as - a - 2bs^2 + 2bs + b + cs^2 - cs - 5c = 0 \Rightarrow$

$(2a-2b+c)s^2 + (2a+2b-c)s + (-a+b-5c) = 0$

$$\Rightarrow \begin{cases} 2a-2b+c=0 \Rightarrow c = 2b-2a \\ 2a+2b-c=0 \Rightarrow 2a+\cancel{2b}-\cancel{2b}+2a=0 \Rightarrow a=0 \text{ and } c=2b \\ -\underset{0}{a}+b-5c=0 \Rightarrow b-10b=0 \Rightarrow 9b=0 \Rightarrow b=0 \end{cases}$$

Because $c = 2b \overset{b=0}{=\!=} 0$

So, the given vectors are linearly independent.

4) The given vectors are linearly independent if:

$$\det\begin{bmatrix} | & | & | & | \\ x_1 & x_2 & x_3 & x_4 \\ | & | & | & | \end{bmatrix} \neq 0$$

$$\det\begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 1 & 5 \\ 1 & 4 & 2 & 7 \end{bmatrix} = 1\underbrace{\begin{vmatrix} 2 & 2 & 5 \\ 3 & 1 & 5 \\ 4 & 2 & 7 \end{vmatrix}}_{d_1} - 1\underbrace{\begin{vmatrix} 1 & 2 & 5 \\ 1 & 1 & 5 \\ 1 & 2 & 7 \end{vmatrix}}_{d_2} + 1\underbrace{\begin{vmatrix} 1 & 2 & 5 \\ 1 & 3 & 5 \\ 1 & 4 & 7 \end{vmatrix}}_{d_3} - 3\underbrace{\begin{vmatrix} 1 & 2 & 2 \\ 1 & 3 & 1 \\ 1 & 4 & 2 \end{vmatrix}}_{d_4}$$

$$d_1 = 2(7-10) - 2(21-20) + 5(6-4) = -6-2+10 = 2$$

$$d_2 = (7-10) - 2(7-5) + 5(2-1) = -3-4+5 = -2$$

$$d_3 = (21-20) - 2(7-5) + 5(4-3) = 1-4+5 = 2$$

$$d_4 = (6-4) - 2(2-1) + 2(4-3) = 2-2+2 = 2$$

So $\det = 1\cdot 2 - 1(-2) + 1\cdot 2 - 3\cdot 2 = 2+2+2-6 = 0$

Therefore, the vectors are linearly dependent.

Exercise 2

$$x_1 = \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}, \quad x_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Knowing that $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{\frac{1}{p}}$,

- $\|x_1\|_1 = |2| + |-3| + |5| = 10$
- $\|x_1\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{|2|^2 + |-3|^2 + |5|^2} = \sqrt{38} \simeq 6.16$
- $\|x_1\|_\infty = \max |x_i| = 5$

- $\|x_2\|_1 = |1| + |1| + |-1| = 3$
- $\|x_2\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{\frac{1}{2}} = \sqrt{|1|^2 + |1|^2 + |-1|^2} = \sqrt{3} \simeq 1.73$
- $\|x_2\|_\infty = \max |x_i| = 1$

Exercise 3

1) We are given two bases, $B = \{b_1, b_2\}$, $C = \{c_1, c_2\}$, of a vector space V.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Knowing that $\begin{cases} b_1 = \overbrace{6}^{a_{11}} c_1 \overbrace{-2}^{a_{12}} c_2 \\ b_2 = \underbrace{9}_{a_{21}} c_1 \underbrace{-4}_{a_{22}} c_2 \end{cases}$ we can form the change-of-basis matrix from B to C.

$$\Rightarrow \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 6 & -2 \\ 9 & -4 \end{bmatrix}}_{A_{B \to C}} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

2) $x = -3b_1 + 2b_2$. Let's transform $\underline{x}$ to C-system coord

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 9 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}_{\text{coords in B}} = \underbrace{\begin{bmatrix} 0 \\ -2 \end{bmatrix}}_{\text{coords in C}}$$

Exercise 4

To find the basis for W, we will transform the matrix that consists of the given vectors as columns into reduced row echelon form. Then, the vectors corresponding to the pivot rows are going to be the basis vectors that we are looking for.

$$\begin{bmatrix} -1 & 2 & 3 & 5 \\ 1 & 1 & 2 & 5 \\ -5 & 7 & 10 & 15 \\ 7 & -8 & -11 & -15 \end{bmatrix} \xrightarrow{R_1\cdot(-1)} \begin{bmatrix} 1 & -2 & -3 & -5 \\ 1 & 1 & 2 & 5 \\ -5 & 7 & 10 & 15 \\ 7 & -8 & -11 & -15 \end{bmatrix} \begin{array}{l} \\ R_2-R_1 \\ R_3+5R_1 \\ R_4-7R_1 \end{array} \Rightarrow$$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 3 & 5 & 10 \\ 0 & -3 & -5 & -10 \\ 0 & 6 & 10 & 20 \end{bmatrix} \xrightarrow{R_2\cdot\left(\frac{1}{3}\right)} \begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 1 & 5/3 & 10/3 \\ 0 & -3 & -5 & -10 \\ 0 & 6 & 10 & 20 \end{bmatrix} \begin{array}{l} \\ \\ R_3+3R_2 \\ R_4-6R_2 \end{array} \Rightarrow$$

$$\begin{bmatrix} 1 & -2 & -3 & -5 \\ 0 & 1 & 5/3 & 10/3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1+2R_2} \begin{bmatrix} 1 & 0 & 0.32 & 1.66 \\ 0 & 1 & 1.66 & 3.33 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, the initial vectors corresponding to the pivot columns of the rref matrix are:

$$\underbrace{\begin{bmatrix} -1 \\ 1 \\ -5 \\ 7 \end{bmatrix}}_{y_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 7 \\ -8 \end{bmatrix}}_{y_2}$$ and they constitute the basis for W.

To determine if z, u and v are part of subspace W, we will solve the system $ay_1 + by_2 = x$, ($x = z$ or u or v)

3) $$\left(\begin{array}{cc|c} -1 & 2 & 3 \\ 1 & 1 & -1 \\ -5 & 7 & 13 \\ 7 & -8 & -17 \end{array}\right) \xrightarrow{R_1\cdot(-1)} \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 1 & 1 & -1 \\ -5 & 7 & 13 \\ 7 & -8 & -17 \end{array}\right) \begin{array}{l} \\ R_2 - R_1 \\ R_3 + 5R_1 \\ R_4 - 7R_1 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 3 & 2 \\ 0 & -3 & -2 \\ 0 & 6 & 4 \end{array}\right) \cdot \frac{1}{3}$$

$$\Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2/3 \\ 0 & -3 & -2 \\ 0 & 6 & 4 \end{array}\right) \begin{array}{l} \\ \\ R_3 + 3R_2 \\ R_4 - 6R_2 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -3 \\ 0 & 1 & 2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} a - 2b = -3 \Rightarrow a = 2b - 3 \\ b = \frac{2}{3} = 0.66 \end{cases}$$

$$a = \frac{4}{3} - \frac{9}{3} = -\frac{5}{3} = -1.66$$

Confirming,

$$-\frac{5}{3}\begin{bmatrix} -1 \\ 1 \\ -5 \\ 7 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ 1 \\ 7 \\ -8 \end{bmatrix} = \begin{pmatrix} 5/3 \\ -5/3 \\ 25/3 \\ -35/3 \end{pmatrix} + \begin{pmatrix} 4/3 \\ 2/3 \\ 14/3 \\ -16/3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 13 \\ -17 \end{pmatrix} \equiv z$$

So $z \in W$.

u) $\left(\begin{array}{cc|c} -1 & 2 & 4 \\ 1 & 1 & 9 \\ -5 & 7 & 12 \\ 7 & -8 & -8 \end{array}\right) \xrightarrow{R_1\cdot(-1)} \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 1 & 1 & 9 \\ -5 & 7 & 12 \\ 7 & -8 & -8 \end{array}\right) \begin{array}{l} \\ R_2-R_1 \\ R_3+5R_1 \\ R_4-7R_1 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 3 & 13 \\ 0 & -3 & -8 \\ 0 & 6 & 20 \end{array}\right) R_2\cdot\frac{1}{3}$

$\Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 13/3 \\ 0 & -3 & -8 \\ 0 & 6 & 20 \end{array}\right) \begin{array}{l} \\ \\ R_3+3R_2 \\ R_4-6R_2 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & -4 \\ 0 & 1 & 13/3 \\ 0 & 0 & 5 \\ 0 & 0 & -6 \end{array}\right) \Rightarrow$ ~~$a-2b=-4$~~, ~~$b=13/3$~~, ~~$a+b=5$~~, ~~$a+b=$~~

Inconsistent system, no solution $\Rightarrow u \notin W$. (*)

v) $\left(\begin{array}{cc|c} -1 & 2 & -1 \\ 1 & 1 & -1 \\ -5 & 7 & -3 \\ 7 & -8 & 3 \end{array}\right) \xrightarrow{R_1\cdot(-1)} \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 1 & 1 & -1 \\ -5 & 7 & -3 \\ 7 & -8 & 3 \end{array}\right) \begin{array}{l} \\ R_2-R_1 \\ R_3+5R_1 \\ R_4-7R_1 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 3 & -2 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{array}\right) R_2\cdot\frac{1}{3}$

$\Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -2/3 \\ 0 & -3 & 2 \\ 0 & 6 & -4 \end{array}\right) \begin{array}{l} \\ \\ R_3+3R_2 \\ R_4-6R_2 \end{array} \Rightarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} a-2b=1 \\ b=-\frac{2}{3}=-0.66 \\ a=-\frac{4}{3}+1=-\frac{1}{3}=-0.3 \end{cases}$

Confirmation:

$$-\frac{1}{3}\begin{bmatrix} -1 \\ 1 \\ -5 \\ 7 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 2 \\ 1 \\ 7 \\ -8 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 5/3 \\ -7/3 \end{bmatrix} + \begin{bmatrix} -4/3 \\ -2/3 \\ -14/3 \\ 16/3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -3 \\ 3 \end{bmatrix} = V, \text{ so } V \in W.$$

(*) Substituting $a=\frac{14}{3}$, $b=\frac{13}{3}$ does not give us u, so $u \notin W$.

Exercise 5

$\underline{x}_1 = [1, 12, 18]^T$

$\underline{x}_2 = [25, 37, 11]^T$

$\langle x_1, x_2 \rangle = x_1^T x_2 = 1 \cdot 25 + 12 \cdot 37 + 18 \cdot 11 = 667$