24-677 Project 1

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1 Model Linearization

System 1

:

$$s_1 = \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix}, u = \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{1}$$

Differentiating, we get:

$$\frac{dy}{dt} = \dot{y} \tag{2}$$

$$\frac{d\dot{y}}{dt} = -\dot{\psi}\dot{x} + \frac{2C_a}{m}(cos\delta\left(\delta - \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}\right) - \frac{\dot{y} - l_r\dot{\psi}}{\dot{x}} \tag{3}$$

$$\frac{d\psi}{dt} = \dot{\psi} \tag{4}$$

$$\frac{d\dot{\psi}}{dt} = \frac{2l_fC_a}{I_z}\left(\delta - \frac{\dot{y} + l_f\dot{\psi}}{\dot{x}}\right) - \frac{2l_rC_a}{I_z}\left(-\frac{\dot{y} - l_r\dot{\psi}}{\dot{x}}\right) \tag{5}$$

Equilibrium point:

$$\dot{y} = 0 \tag{6}$$

$$\ddot{y} = 0 \Rightarrow \frac{2C_a}{m}\delta cos\delta = 0 \Rightarrow \delta = 0 \tag{7}$$

$$\dot{\psi} = 0 \tag{8}$$

$$\ddot{\psi} = 0 \Rightarrow \frac{2l_fC_a}{I_z}\delta = 0 \Rightarrow \delta = 0 \tag{9}$$

Jacobian matrix:

$$A_1 = \begin{bmatrix} \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial \dot{y}} & \frac{\partial f_1}{\partial \psi} & \frac{\partial f_1}{\partial \dot{\psi}} \\ \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial \dot{y}} & \frac{\partial f_2}{\partial \psi} & \frac{\partial f_2}{\partial \dot{\psi}} \\ \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial \dot{y}} & \frac{\partial f_3}{\partial \psi} & \frac{\partial f_3}{\partial \dot{\psi}} \\ \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial \dot{y}} & \frac{\partial f_4}{\partial \psi} & \frac{\partial f_4}{\partial \dot{\psi}} \end{bmatrix} \tag{10}$$

Partial derivatives:

$$\frac{\partial \dot{y}}{\partial y} = 0, \frac{\partial \dot{y}}{\partial \dot{y}} = 1, \frac{\partial \dot{y}}{\partial \psi} = 0, \frac{\partial \dot{y}}{\partial \dot{\psi}} = 0, \frac{\partial \dot{y}}{\partial \delta} = 0, \frac{\partial \dot{y}}{\partial F} = 0 \tag{11}$$

$$\frac{\partial \ddot{y}}{\partial y} = 0, \frac{\partial \ddot{y}}{\partial \psi} = 1, \frac{\partial \ddot{y}}{\partial F} = 0 \tag{12}$$

$$\frac{\partial \ddot{y}}{\partial \dot{y}} = \frac{-2C_a(cos\delta + 1)}{m\dot{x}} \tag{13}$$

$$\frac{\partial \ddot{y}}{\partial \dot{\psi}} = -\dot{x} + \frac{2C_a(l_r - l_f cos\delta)}{m\dot{x}} \tag{14}$$

$$\frac{\partial \ddot{y}}{\partial \delta} = \frac{2C_a(cos\delta - \delta sin\delta)}{m} + \frac{2C_a(\dot{y} + l_f\dot{\psi})}{\dot{x}} sin\delta \tag{15}$$

$$\frac{\partial \dot{\psi}}{\partial y} = 0, \frac{\partial \dot{\psi}}{\partial \dot{y}} = 0, \frac{\partial \dot{\psi}}{\partial \psi} = 0, \frac{\partial \dot{\psi}}{\partial \dot{\psi}} = 1 \tag{16}$$

$$\frac{\partial \ddot{\psi}}{\partial y} = 0, \frac{\partial \ddot{\psi}}{\partial \psi} = 0, \frac{\partial \ddot{\psi}}{\partial F} = 0 \tag{17}$$

$$\frac{\partial \ddot{\psi}}{\partial \dot{y}} = \frac{2C_a(l_r - l_f)}{I_z\dot{x}} \tag{18}$$

$$\frac{\partial \ddot{\psi}}{\partial \dot{\psi}} = -\frac{2C_a(l_f^2 + l_r^2)}{I_z\dot{x}} \tag{19}$$

$$\frac{\partial \ddot{\psi}}{\partial \delta} = \frac{2l_fC_a}{I_z} \tag{20}$$

Substituting $\delta = 0$, which corresponds to the equilibrium point, we get:

$$\frac{\partial \ddot{y}}{\partial \dot{y}} = \frac{-4C_a}{m\dot{x}} \tag{21}$$

$$\frac{\partial \ddot{y}}{\partial \dot{\psi}} = -\dot{x} + \frac{2C_a(l_r - l_f)}{m\dot{x}} \tag{22}$$

and

$$\frac{\partial \ddot{y}}{\partial \delta} = \frac{2C_a}{m} \tag{23}$$

Hence, the linearized system 1 can be written as:

$$\dot{s_1} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-4C_a}{m\dot{x}} & 0 & -\dot{x} + \frac{2C_a(l_r - l_f)}{m\dot{x}} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{2C_a(lr - l_f)}{I_z\dot{x}} & 0 & \frac{-2C_a(l_r^2 + l_f^2)}{I_z\dot{x}} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \frac{2C_a}{m} & 0 \\ 0 & 0 \\ \frac{2l_fC_a}{I_z} & 0 \end{bmatrix} \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{24}$$

Using: $C_a = 15000N, m = 2000kg, l_r = 1.7m, l_f = 1.1m, I_z = 3344kgm^2$ we can write the above system as:

$$\dot{s_1} = \begin{bmatrix} \dot{y} \\ \ddot{y} \\ \dot{\psi} \\ \ddot{\psi} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \frac{-30}{\dot{x}} & 0 & -\dot{x} + \frac{9}{\dot{x}} \\ 0 & 0 & 0 & 1 \\ 0 & \frac{5.383}{\dot{x}} & 0 & \frac{-36.782}{\dot{x}} \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \\ \psi \\ \dot{\psi} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 15 & 0 \\ 0 & 0 \\ 9.868 & 0 \end{bmatrix} \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{25}$$

1.1 System 2

$$s_2 = \begin{bmatrix} x \\ \dot{x} \end{bmatrix}, u = \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{26}$$

Equilibrium:

$$\dot{x} = 0, \ddot{x} = 0 \Rightarrow F = (fg - \dot{\psi}\dot{y})m \tag{27}$$

Jacobian Matrix:

$$A_2 = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \dot{x}} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \dot{x}} \end{bmatrix} \tag{28}$$

Partial Derivatives:

$$\frac{\partial \dot{x}}{\partial x} = 0, \frac{\partial \dot{x}}{\partial \dot{x}} = 1, \frac{\partial \dot{x}}{\partial \delta} = 0, \frac{\partial \dot{x}}{\partial F} = 0 \tag{29}$$

$$\frac{\partial \ddot{x}}{\partial x} = 0, \frac{\partial \ddot{x}}{\partial \dot{x}} = 0, \frac{\partial \ddot{x}}{\partial \delta} = 0, \frac{\partial \ddot{x}}{\partial F} = \frac{1}{m} \tag{30}$$

Therefore, we can write the linearized system as:

$$\dot{s_2} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{bmatrix} \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{31}$$

Substituting $m = 2000N$, we get:

$$\dot{s_2} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.0005 \end{bmatrix} \begin{bmatrix} \delta \\ F \end{bmatrix} \tag{32}$$