Exercise 1

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B} u \quad \text{and} \quad y = \underbrace{\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}}_{C} x$$

Controllability Matrix

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$\text{Rank}(P) = 3 = n$ so the system is controllable.

Observability Matrix

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix} \begin{matrix} \\ R_2+R_1 \\ R_3-R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(Q) = 1 < n = 3$ so the system is <u>not</u> observable.

Exercise 2

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}}_{B} u \quad \text{and} \quad y = \underbrace{\begin{bmatrix} 1 & 0 & 1 \end{bmatrix}}_{C} x$$

$$P = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & -1 & -1 \end{bmatrix} R_3 \cdot \tfrac{1}{2}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 2 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1/2 & -1/2 & -1/2 \end{bmatrix}$$

$\text{Rank}(P) = 3 = n$ so Controllable ✓

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & -1 \\ -1 & -1 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - R_1 \\ R_3 + R_1 \end{matrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & -2 \\ 0 & -1 & 5 \end{bmatrix} R_2 \cdot \left(\tfrac{1}{3}\right) \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2/3 \\ 0 & -1 & 5 \end{bmatrix} R_3 + R_2$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2/3 \\ 0 & 0 & 13/3 \end{bmatrix}$$

$\text{Rank}(Q) = 3 = n$ so Observable ✓

Exercise 3

$$\dot{x} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 2 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 2 & 2 & 1 & 3 & -1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

$$\lambda = 2 \to B^2 = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \begin{matrix} R_2-2R_1 \\ R_3-3R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -2 \end{bmatrix} R_3+R_2$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{Rank}=3 \to \text{full rank}$$

$$\lambda = 1 \to B^1 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \xrightarrow{R_2-R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \text{Rank}=2 \\ \text{full rank} \end{matrix}$$

So the system is controllable.

$$\lambda = 2 \to C^2 = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 0 & 1 & 1 \end{bmatrix} R_2-2R_1 \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} R_3-R_2$$

$$\to \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{Rank}=2 \text{ so } \underline{\text{not}} \text{ full column rank}$$

We don't need to check C^1's rank. Since $\text{rank}(C^2) = 2 < n = 3$, the system is not observable.

Exercise 4

$$\dot{x} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} \end{bmatrix} x$$

$$B^1 = \begin{bmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \\ b_{51} & b_{52} \end{bmatrix}$$

This matrix has to be of full row rank for the state equation to be controllable.

$$C^1 = \begin{bmatrix} c_{11} & c_{13} & c_{15} \\ c_{21} & c_{23} & c_{25} \\ c_{31} & c_{33} & c_{35} \end{bmatrix}$$

This matrix has to be of full column rank for the state equation to be observable.

Controllability: B^1 has $n=3$ rows and $m=2$ columns, which means $\text{rank}(B^1) \leq \min(m,n) = m = 2$, so the maximum rank is 2. This means that it cannot have full rank, so it's <u>uncontrollable.</u>

Observability: C^1 has $n=3$ rows and $m=3$ columns, so we can find three linearly independent columns so the system is <u>observable.</u>

Exercise 5

$\dot{x} = Ax + Bu$

with $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 1 \\ -1 & 1-\lambda & 1 & 0 \\ 0 & 0 & 1-\lambda & 1 \\ 2 & 0 & -2 & -\lambda \end{bmatrix}$$

$$\det(A-\lambda I) = -1 \cdot \begin{vmatrix} -1 & 1 & 0 \\ 0 & 1-\lambda & 1 \\ 2 & -2 & -\lambda \end{vmatrix} + (1-\lambda) \cdot \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 1 \\ 2 & -2 & -\lambda \end{vmatrix} =$$

$$= -1 \cdot \left[(-1) \begin{vmatrix} 1-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 2 & -\lambda \end{vmatrix} \right] + (1-\lambda) \left[-\lambda \begin{vmatrix} 1-\lambda & 1 \\ -2 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1-\lambda \\ 2 & -2 \end{vmatrix} \right] =$$

$$= -1 \left[-1 \cdot \left((1-\lambda)(-\lambda) + 2 \right) - 1 \cdot (-2) \right] + (1-\lambda) \left[-\lambda \left((1-\lambda)(-\lambda) + 2 \right) + 1 \cdot \left[-2(1-\lambda) \right] \right] =$$

$$= -1 \cdot (+\lambda - \lambda^2 - 2 + 2) + (1-\lambda)(+\lambda^2 - \lambda^3 - 2\lambda - 2 + 2\lambda) =$$

$$= -\lambda + \lambda^2 + (1-\lambda)(+\lambda^2 - \lambda^3 - 2) = -\lambda + \lambda^2 + \lambda^2 - \lambda^3 - 2 - \lambda^3 + \lambda^4 + 2\lambda =$$

$$= +\lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 2 \Rightarrow \lambda^4 - 2\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

Solving this 4th order polynomial yields the following solutions:

$\lambda_1 = -0.811$

$\lambda_2 = 0.905 + 1.284i$

$\lambda_3 = 0.905 - 1.284i$

$\lambda_4 = 1$

PBH test

<u>$\lambda_1 = -0.811$</u>

$$A-\lambda_1 I \vdots B = \left[\begin{array}{cccc|c} 0.811 & 1 & 0 & 1 & 0 \\ -1 & 1.811 & 1 & 0 & -1 \\ 0 & 0 & 1.811 & 1 & 0 \\ 2 & 0 & -2 & 0.811 & 2 \end{array}\right]$$

$\text{Rank}(A-\lambda_1 I \vdots B) = 4 = n$, so $\lambda_1 = -0.811$ corresponds to a controllable mode of the system

<u>$\lambda_2 = 0.905 + 1.284i$</u>

$$A-\lambda_2 I \vdots B = \left[\begin{array}{cccc|c} -0.905-1.284i & 1 & 0 & 1 & 0 \\ -1 & 0.095-1.284i & 1 & 0 & -1 \\ 0 & 0 & 0.095-1.284i & 1 & 0 \\ 2 & 0 & -2 & -0.905-1.284i & 2 \end{array}\right]$$

$\text{Rank}(A-\lambda_2 I \vdots B) = 4 = n$, so $\lambda_2 \rightarrow$ controllable mode.

<u>$\lambda_3 = 0.905 - 1.284i$</u>

$$A-\lambda_3 I \vdots B = \left[\begin{array}{cccc|c} -0.905+1.284i & 1 & 0 & 1 & 0 \\ -1 & 0.095+1.284i & 1 & 0 & -1 \\ 0 & 0 & 0.095+1.284i & 1 & 0 \\ 2 & 0 & -2 & -0.905+1.284i & 2 \end{array}\right]$$

$\text{Rank}(A-\lambda_3 I \vdots B) = 4 = n$, so $\lambda_3 \rightarrow$ controllable mode

$\underline{\lambda_4 = 1}$

$$A-\lambda_4 I : B = \begin{bmatrix} -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & -1 & 2 \end{bmatrix} \cdot(-1) \rightarrow \begin{array}{c} \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & -1 & 2 \end{bmatrix} \end{array} \begin{array}{l} \\ R_2+R_1 \\ \\ R_4-2R_1 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 & 2 \end{bmatrix} \cdot(-1) \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 & 2 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4-2R_2 \end{array}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{array}{l} \\ \\ \\ R_4+R_3 \end{array} \rightarrow \begin{bmatrix} 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathrm{Rank}(A-\lambda_4 I : B) = 3 < n = 4$, so the eigenvalue $1 = \lambda_4$ corresponds to the uncontrollable mode of the system.

Exercise 6

$$\dot{x} = \underbrace{\begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u \quad \text{and} \quad y = \underbrace{\begin{bmatrix} 1 & 1 \end{bmatrix}}_{C} x$$

Controllability Matrix:

$$P = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 1 & 3 \end{bmatrix} \xrightarrow[R_2 - R_1]{} \begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \quad \text{Rank}(P) = 1 < n = 2$$

Therefore we can define M as:

$$M = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$ where $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the first $/ n_c = r(P) = 1$ column of P

and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is arbitrarily chosen in order to maintain $r(M) = n = 2$

Transformed system matrices:

$$M^{-1}AM = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 3 & -1 \end{bmatrix} = \left[\begin{array}{c|c} 3 & 4 \\ \hline 0 & -5 \end{array}\right]$$

$$M^{-1}B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left[\begin{array}{c} 1 \\ \hline 0 \end{array}\right] \quad \text{and} \quad CM = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \left[\begin{array}{c|c} 2 & 1 \end{array}\right]$$

So, Controllable Decomposition:

$$\dot{x} = \begin{bmatrix} 3 \end{bmatrix} x + \begin{bmatrix} 1 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 2 \end{bmatrix}$$

The reduced system has the following observability matrix:

$$Q_C = \begin{bmatrix} C_c \\ C_c A_c \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \end{bmatrix} \xrightarrow{\cdot\left(\frac{1}{2}\right)} \begin{bmatrix} 1 \\ 6 \end{bmatrix} \xrightarrow{R_2 - 6R_1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{Rank}(Q_c) = 1 = n_c$$

Hence, the reduced form is controllable and observable.

Exercise 7

$$\dot{x} = \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \end{bmatrix} x$$

Controllability Matrix

$$P = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2\lambda_1 & 3\lambda_1^2 & 4\lambda_1^3 \\ 1 & \lambda_1 & \lambda_1^2 & \lambda_1^3 & \lambda_1^4 \\ 1 & \lambda_2 & \lambda_2^2 & \lambda_2^3 & \lambda_2^4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

P has 2 zero rows, so P has rank = 3, hence it is uncontrollable.

Let's define M using P's first 3 columns and 2 arbitrary last columns so that rank(M) = n = 5

$$M = \begin{bmatrix} 0 & 1 & 2\lambda_1 & 0 & 0 \\ 1 & \lambda_1 & \lambda_1^2 & 0 & 0 \\ 1 & \lambda_2 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Transformed System

M^{-1}

$$M^{-1}AM = \begin{bmatrix} \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2} & \frac{\lambda_2^2 - 2\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2} & \frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2} & 0 & 0 \\ -\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} & \frac{2\lambda_1}{(\lambda_1-\lambda_2)^2} & \frac{-2\lambda_1}{(\lambda_1-\lambda_2)^2} & 0 & 0 \\ \frac{1}{\lambda_1-\lambda_2} & -\frac{1}{(\lambda_1-\lambda_2)^2} & \frac{1}{(\lambda_1-\lambda_2)^2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} x$$

$$\times \begin{bmatrix} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & \lambda_2 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 2\lambda_1 & 0 & 0 \\ 1 & \lambda_1 & \lambda_1^2 & 0 & 0 \\ 1 & \lambda_2 & \lambda_2^2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= M^{-1} \times \begin{bmatrix} 1 & 2\lambda_1 & 3\lambda_1^2 & 0 & 0 \\ \lambda_1 & \lambda_1^2 & \lambda_1^3 & 0 & 0 \\ \lambda_2 & \lambda_2^2 & \lambda_2^3 & 1 & 0 \\ 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}+\frac{\lambda_1\lambda_2^2-2\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^2}+\frac{\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)^2} & \frac{2\lambda_1^2\lambda_2}{\lambda_1-\lambda_2}+\frac{\lambda_1^2\lambda_2^2-2\lambda_1^3\lambda_2}{(\lambda_1-\lambda_2)^2}+\frac{\lambda_1^2\lambda_2^2}{(\lambda_1-\lambda_2)^2} & \frac{3\lambda_1^3\lambda_2}{(\lambda_1-\lambda_2)}+\frac{\lambda_1^3\lambda_2^2-2\lambda_1^4\lambda_2+\lambda_1\lambda_2^2}{(\lambda_1-\lambda_2)^2} & \frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2} & 0 \\ -\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}+\frac{2\lambda_1^2}{(\lambda_1-\lambda_2)^2}-\frac{2\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2} & \frac{-2\lambda_1^2+2\lambda_1\lambda_2}{\lambda_1-\lambda_2}+\frac{2\lambda_1^3-2\lambda_1\lambda_2^2}{(\lambda_1-\lambda_2)^2} & \frac{-3\lambda_1^3-3\lambda_1^2\lambda_2}{(\lambda_1-\lambda_2)}+\frac{2\lambda_1^4-2\lambda_1\lambda_2^3}{(\lambda_1-\lambda_2)^2} & \frac{-2\lambda_1}{(\lambda_1-\lambda_2)^2} & 0 \\ \frac{1}{\lambda_1-\lambda_2}-\frac{\lambda_1}{(\lambda_1-\lambda_2)^2}+\frac{\lambda_2}{(\lambda_1-\lambda_2)^2} & \frac{2\lambda_1}{\lambda_1-\lambda_2}+\frac{-\lambda_1^2+\lambda_2^2}{(\lambda_1-\lambda_2)^2} & \frac{3\lambda_1^2}{\lambda_1-\lambda_2}+\frac{-\lambda_1^3+\lambda_2^3}{(\lambda_1-\lambda_2)^2} & \frac{1}{(\lambda_1-\lambda_2)^2} & 0 \\ 0 & 0 & & \lambda_2 & 1 \\ 0 & 0 & & 0 & \lambda_2 \end{bmatrix}$$

$$M^{-1}B = \begin{bmatrix} \frac{\lambda_2^2-2\lambda_1\lambda_2}{(\lambda_1-\lambda_2)^2}+\frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2} \\ \frac{2\lambda_1}{(\lambda_1-\lambda_2)^2}+\frac{-2\lambda_1}{(\lambda_1-\lambda_2)^2} \\ \frac{-1}{(\lambda_1-\lambda_2)^2}+\frac{1}{(\lambda_1-\lambda_2)^2} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and $CM = \begin{bmatrix} 2 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & 0 & 1 \end{bmatrix}$

Simplifying $M^{-1}AM$, we can write the transformed system as:

$$M^{-1}AM = \left[\begin{array}{ccc|cc} 0 & 0 & \lambda_1^2\lambda_2 & \frac{\lambda_1^2}{(\lambda_1-\lambda_2)^2} & 0 \\ 1 & 0 & -\lambda_1(\lambda_1+2\lambda_2) & \frac{-2\lambda_1}{(\lambda_1-\lambda_2)^2} & 0 \\ 0 & 1 & 2\lambda_1+\lambda_2 & 1/(\lambda_1-\lambda_2)^2 & 0 \\ \hline 0 & 0 & 0 & \lambda_2 & 1 \\ 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

A_c

$$M^{-1}B = \left[\begin{array}{c} 1 \\ 0 \\ 0 \\ \hline 0 \\ 0 \end{array}\right] \quad \text{and} \quad CM = \left[\begin{array}{ccc|cc} 2 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & 0 & 1 \end{array}\right]$$

Controllable Decomposition:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 0 & \lambda_1^2\lambda_2 \\ 1 & 0 & -\lambda_1(\lambda_1+2\lambda_2) \\ 0 & 1 & 2\lambda_1+\lambda_2 \end{bmatrix}}_{A_c} x + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{B_c} u$$

and $y = \underbrace{\begin{bmatrix} 2 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \end{bmatrix}}_{C_c}$

The reduced system has observability matrix:

$$Q_c = \begin{bmatrix} C_c \\ C A_c \\ C A_c^2 \end{bmatrix} = \begin{bmatrix} 2 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \\ \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & \lambda_1^3+\lambda_2^3 \\ \lambda_1^2+\lambda_2^2 & \lambda_1^3+\lambda_2^3 & \lambda_1^4+\lambda_2^4 \end{bmatrix} \cdot\left(\tfrac{1}{2}\right)$$

$$\rightarrow \begin{bmatrix} 1 & (\lambda_1+\lambda_2)/2 & (\lambda_1^2+\lambda_2^2)/2 \\ \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & \lambda_1^3+\lambda_2^3 \\ \lambda_1^2+\lambda_2^2 & \lambda_1^3+\lambda_2^3 & \lambda_1^4+\lambda_2^4 \end{bmatrix} \begin{matrix} \\ R_2-(\lambda_1+\lambda_2)R_1 \\ R_3-(\lambda_1^2+\lambda_2^2)R_1 \end{matrix} \rightarrow$$

$$\begin{bmatrix} 1 & \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1^2+\lambda_2^2}{2} \\ 0 & \frac{(\lambda_1-\lambda_2)^2}{2} & \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2} \\ 0 & \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2} & \frac{(\lambda_1^2-\lambda_2^2)^2}{2} \end{bmatrix} \cdot \frac{2}{(\lambda_1-\lambda_2)^2} \rightarrow \begin{bmatrix} 1 & \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1^2+\lambda_2^2}{2} \\ 0 & 1 & \lambda_1+\lambda_2 \\ 0 & \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2} & \frac{(\lambda_1^2-\lambda_2^2)^2}{2} \end{bmatrix}$$

$$R_3 - \frac{(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2} \rightarrow \begin{bmatrix} 1 & \frac{\lambda_1+\lambda_2}{2} & \frac{\lambda_1^2+\lambda_2^2}{2} \\ 0 & 1 & \lambda_1+\lambda_2 \\ 0 & 0 & 0 \end{bmatrix}$$

since $\frac{(\lambda_1+\lambda_2)(\lambda_1+\lambda_2)(\lambda_1-\lambda_2)^2}{2} = \frac{(\lambda_1+\lambda_2)^2(\lambda_1-\lambda_2)^2}{2} = \frac{(\lambda_1^2-\lambda_2^2)^2}{2}$

$\mathrm{Rank}(Q_c) = 2 < n = 3$ so the reduced system is not observable. Therefore, we will proceed with "breaking" the reduced system to an observable and unobservable part, and retain the observable one.

Let's define $M_C^{-1} = \begin{bmatrix} 2 & \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 \\ \lambda_1+\lambda_2 & \lambda_1^2+\lambda_2^2 & \lambda_1^3+\lambda_2^3 \\ 0 & 0 & 1 \end{bmatrix}$ where the first

2 rows are linearly independent rows of $\mathcal{C}_c$ and the 3rd one is arbitrary but able to maintain $\text{rank}(M^{-1}c) = 3$

Then $M_C^{-1} A_C M_C = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 & 0 \\ \hline -\frac{\lambda_1+\lambda_2}{(\lambda_1-\lambda_2)^2} & \frac{2}{(\lambda_1-\lambda_2)^2} & \lambda_1 \end{array}\right]$

$M_C^{-1} B = \left[\begin{array}{c} 2 \\ \lambda_1+\lambda_2 \\ \hline 0 \end{array}\right]$ and $C_c M_c = \left[\begin{array}{cc|c} 1 & 0 & 0 \end{array}\right]$

The final form that is both controllable and observable is:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\lambda_1\lambda_2 & \lambda_1+\lambda_2 \end{bmatrix}}_{A_{co}} x + \underbrace{\begin{bmatrix} 2 \\ \lambda_1+\lambda_2 \end{bmatrix}}_{B_{co}} u$$

and $$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C_{co}} x$$