24-677: Homework 7

Exercise 1

a) $x(k+1)=\begin{bmatrix} 1 & 0 \\ -0.5 & 0.5 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u(k)$

$$\Delta(\lambda)=\begin{vmatrix} 1-\lambda & 0 \\ -0.5 & 0.5-\lambda \end{vmatrix} = (1-\lambda)(0.5-\lambda)=0 \Rightarrow \begin{matrix}\lambda_1=0.5 \\ \lambda_2=1\end{matrix}$$

Since $|\lambda_2|=1$, the system is NOT asymptotically stable. But the eigenvalues are distinct, so they are not defective. Therefore, since $|\lambda_1|\leq 1$, $|\lambda_2|\leq 1$ and $\lambda_{1,2}$ non-defective, the system is stable i.s.L.

b) $\dot{x}=\begin{bmatrix} -7 & -2 & 6 \\ 2 & -3 & -2 \\ -2 & -2 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix} u$

$$\Delta(\lambda)=\begin{vmatrix} -7-\lambda & -2 & 6 \\ 2 & -3-\lambda & -2 \\ -2 & -2 & 1-\lambda \end{vmatrix} = (-7-\lambda)\left[(-3-\lambda)(1-\lambda)-4\right]+2\left[2(1-\lambda)-4\right] + 6\left[-4+2(-3-\lambda)\right]=$$

$$=(-7-\lambda)(-3+3\lambda-\lambda+\lambda^2-4)+2(2-2\lambda-4)+6(-4-6-2\lambda)=$$

$$=(-7-\lambda)(\lambda^2+2\lambda-7)+2(-2\lambda-2)+6(-10-2\lambda)=$$

$$=-7\lambda^2-14\lambda+49-\lambda^3-2\lambda^2+7\lambda-4\lambda-4-60-12\lambda=$$

$$=-\lambda^3-9\lambda^2-23\lambda-15 \Rightarrow \lambda^3+9\lambda^2+23\lambda+15=0$$

$\lambda=-1$ is a root of the polynomial

Dividing the polynomial with $(\lambda+1)$:

1	9	23	15	-1
$\cancel{1}$	-1	-8	-15	
1	8	15	0	

So $\Delta(\lambda) = (\lambda+1)(\lambda^2+8\lambda+15)$

$\lambda_1 = -1$ $\quad \lambda_{2,3} = \frac{-8 \pm 2}{2}$ -3, -5

For $\begin{cases} \lambda_1 = -1 \\ \lambda_2 = -3 \\ \lambda_3 = -5 \end{cases}$, Real$(\lambda_i) < 0$ so the system is:

Asymptotically stable $\Rightarrow$ stable i.s.L.

Exercise 2

1) $A=\begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$ We know that $\|A\|_2 = \sigma_1$ (= max. singular value)

$$A^T A = \begin{bmatrix} 2 & 1 & -1 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & 2 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 6 \\ 2 & 6 & 2 \\ 6 & 2 & 6 \end{bmatrix}$$

$$\Delta(\lambda) = \begin{vmatrix} 6-\lambda & 2 & 6 \\ 2 & 6-\lambda & 2 \\ 6 & 2 & 6-\lambda \end{vmatrix} = (6-\lambda)\left[(6-\lambda)^2 - 4\right] - 2\left[2(6-\lambda) - 12\right] + 6\left[4 - 6(6-\lambda)\right]$$

$$= (6-\lambda)(\lambda^2 - 12\lambda + 32) + 40\lambda - 192 \Rightarrow \lambda^3 - 18\lambda^2 + 64\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 18\lambda + 64) = 0$$

$\Delta = 68$, so $\lambda_1 = 0$ ~~and~~ $\lambda_{2,3} = \dfrac{18 \pm \sqrt{68}}{2} = 9 \pm \sqrt{17} = \begin{cases} 13.123 \\ 4.877 \end{cases}$

Max singular value $= \sqrt{\lambda_2} = \sqrt{13.123} \simeq 3.62$

2) $A = \begin{bmatrix} 10 & 2 \\ 0 & -3 \end{bmatrix}$ $\|A\|_N = \sigma_1 + \dots + \sigma_n$

$$A^T A = \begin{bmatrix} 10 & 0 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 10 & 2 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 100 & 20 \\ 20 & 13 \end{bmatrix}$$

$$\Delta(\lambda) = \begin{vmatrix} 100-\lambda & 20 \\ 20 & 13-\lambda \end{vmatrix} = (100-\lambda)(13-\lambda) - 400 = 1300 - 100\lambda - 13\lambda + \lambda^2 - 400$$

$$\Rightarrow \lambda^2 - 113\lambda + 900 = 0$$

$\Delta = 9169$ and $\lambda_{1,2} = \frac{113 \pm 95.755}{2} \approx \begin{cases} 104.3775 \\ \frac{17.245}{2} = 8.6225 \end{cases}$

So $\sigma_1 = \sqrt{\lambda_1} = \sqrt{104.3775} = 10.2165$

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{8.6225} = 2.9364$

So $\|A\|_N = \sigma_1 + \sigma_2 = 13.1529 \simeq 13.15$

$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2} = \sqrt{\lambda_1 + \lambda_2} = \sqrt{104.3775 + 8.6225} \simeq 10.63$

3) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $\|A\|_1 = \max\limits_{j} \sum\limits_{i=1}^{m} |a_{ij}|$ (max column sum)

so $\|A\|_1 = \max(4,6) = 6$

$A^T A = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$

$\Delta(\lambda) = \begin{vmatrix} 10-\lambda & 14 \\ 14 & 20-\lambda \end{vmatrix} = (10-\lambda)(20-\lambda) - 196 = 200 - 10\lambda - 20\lambda + \lambda^2 - 196$

$\Rightarrow \lambda^2 - 30\lambda + 4 = 0$

$\Delta = 884$, so $\lambda_{1,2} = \frac{30 \pm 29.732}{2} \approx \begin{cases} 29.866 \\ 0.134 \end{cases}$

So $\sigma_1 = \sqrt{29.866} \simeq 5.46 = \|A\|_2$

$\sigma_2 = \sqrt{0.134} \simeq 0.37$ $\|A\|_\infty = $ max row sum $= \max(3,7) = 7$

Exercise 3

$$K = \frac{1}{N}\|y - XB\|_2^2 + \lambda \|B\|_1$$

$$y - XB = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 5 & -1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$(y - XB)^T (y - XB) = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$$

$$\Delta(\lambda) = \begin{vmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{vmatrix} = (20-\lambda)(5-\lambda) - 100 = 100 - 20\lambda - 5\lambda + \lambda^2 - 100 =$$

$$\Rightarrow \lambda^2 - 25\lambda = 0 \Rightarrow \lambda(\lambda - 25) = 0$$

$$\left.\begin{matrix} \lambda_1 = 0 \\ \lambda_2 = 25 \end{matrix} \Rightarrow \begin{matrix} \sigma_1 = 0 \\ \sigma_2 = 5 \end{matrix}\right\} \Rightarrow \|y - XB\|_2 = 5$$

$$\|B\|_1 = \text{max column sum}(B) = \max(5, 6) = 6$$

$$\text{So } K = \frac{1}{2} \cdot 5^2 + 20 \cdot 6 = \frac{25}{2} + 120 = 132.5$$

Exercise 4

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

So we have 3 1×1 and 1 3×3 Jordan blocks.

$\lambda_1 = 2,\ m_1 = q_1 = 1$

$\lambda_2 = 1,\ m_2 = 3,\ q_2 = 1$

$\lambda_3 = 0,\ m_3 = 1,\ q_3 = 1$

$\lambda_4 = -1,\ m_4 = 1,\ q_4 = 1$

$$A = \underbrace{\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{N}$$

$$e^{tD} = \begin{bmatrix} e^{2t} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{t} & 0 & 0 & 0 & 0 \\ 0 & 0 & e^{t} & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{t} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-t} \end{bmatrix}$$

and N is nilpotent. Precisely $N^3 = 0$.

So we can write e^{tN} as:

$$e^{tN} = \sum_{k=0}^{\infty} \frac{t^k N^k}{k!} = I + tN + \frac{t^2 N^2}{2} =$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & t & t^2/2 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So $e^{At} = e^{tD} \cdot e^{tN} \underset{\boxed{t=2}}{=} \begin{bmatrix} e^{4} & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{2} & 2e^{2} & 2e^{2} & 0 & 0 \\ 0 & 0 & e^{2} & 2e^{2} & 0 & 0 \\ 0 & 0 & 0 & e^{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{-2} \end{bmatrix}$

Exercise 5

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ e^{-3t} & 0 \end{bmatrix} x$$

$$\dot{x}_1 = -x_1 \Rightarrow \frac{dx_1}{dt} = -x_1 \Rightarrow \int \frac{1}{x_1} dx_1 = \int -dt \Rightarrow \Big[\ln x_1\Big]_{x_1(0)}^{x_1(t)} = \Big[-t\Big]_0^t \Rightarrow$$

$$\dot{x}_2 = e^{-3t} x_1$$

$$\ln x_1(t) - \ln x_1(0) = -t \Rightarrow \ln \frac{x_1(t)}{x_1(0)} = -t$$

$$\Rightarrow x_1(t) = e^{-t} x_1(0)$$

So $\frac{dx_2}{dt} = e^{-3t} e^{-t} x_1(0) \Rightarrow \int dx_2 = \int e^{-4t} x_1(0)\, dt \Rightarrow \Big[x_2\Big]_{x_2(0)}^{x_2(t)} = \Big[-\frac{e^{-4t}}{4} x_1(0)\Big]_0^t$

$$\Rightarrow x_2(t) - x_2(0) = -\frac{x_1(0)}{4} e^{-4t} \Rightarrow x_2(t) = -\frac{x_1(0)}{4} e^{-4t} + x_2(0)$$

Assuming $x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x(t) = \begin{bmatrix} e^{-t} \\ -\frac{e^{-4t}}{4} \end{bmatrix}$

and $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Fundamental Matrix: $X = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^{-4t}}{4} & 1 \end{bmatrix}$

State Transition Matrix:

$$\Phi(t, t_0) = X(t) X^{-1}(t_0) = \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^{-4t}}{4} & 1 \end{bmatrix} \frac{1}{e^{-t_0}} \begin{bmatrix} 1 & 0 \\ \frac{e^{-4t_0}}{4} & e^{-t_0} \end{bmatrix} =$$

$$= \begin{bmatrix} e^{-t} & 0 \\ -\frac{e^{-4t}}{4} & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{e^{-t_0}} & 0 \\ \frac{e^{-3t_0}}{4} & 1 \end{bmatrix} = \begin{bmatrix} e^{t_0-t} & 0 \\ \frac{-e^{t_0-4t}+e^{-3t_0}}{4} & 1 \end{bmatrix}$$

$$\|\Phi\|_\infty = \max\left\{ e^{t_0-t}+0, \frac{-e^{t_0-4t}+e^{-3t_0}}{4}+1 \right\} = \frac{-e^{t_0-4t}+e^{-3t_0}}{4}+1, \text{ for } t \geq t_0 \geq 0$$

For $t \to \infty$, $\|\Phi(t,t_0)\|_\infty \to \lim_{t\to\infty}\left(\frac{-e^{t_0}}{e^{4t}} + \frac{1}{4e^{3t_0}} + 1\right) \to \frac{1}{4e^{3t_0}} + 1$

So the equation is NOT asymptotically stable since $\lim_{t\to\infty} \|\Phi(t,t_0)\|_\infty \neq 0$ but it is still bounded, so it is stable i.s.L.

Exercise 6

Using SVD, A can be written as $A = USV^T$,
where S is diagonal and contains the singular values σ_i of A.

We know that $\|A\|_2 = \sigma_1$, where σ_1 is the maximum singular value of A.

A invertible $\Rightarrow A^{-1} = (USV^T)^{-1} = (V^T)^{-1} S^{-1} U^{-1} = V S^{-1} U^T$

A^{-1} has its singular values in S^{-1}, and its norm will be $\max(\mathrm{diag}(S^{-1}))$.

S is diagonal, so its inverse can be found if we invert every one of its element.

So: $\|A^{-1}\|_2 = \sigma_{max}(A^{-1}) = \max \frac{1}{\sigma_i} = \frac{1}{\min \sigma_i} = \frac{1}{\sigma_n}$

Exercise 7

$$\dot{x} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} x + \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} u \quad \text{and} \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x$$

Controllability Matrix:

$$P = \left[B \,\vdots\, AB \,\vdots\, A^2B \right] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 & 1 \end{bmatrix} \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 0 & -2 & 0 & 2 \end{bmatrix} \begin{array}{l} \\ \\ R_3 - 2R_2 \end{array} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(P) = 2 < n = 3$

uncontrollable system

Let's define $M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ where the first $n_c = 2$ columns are the first 2 columns of P and the third is an arbitrary one such that $\text{rank}(M) = n = 3$

$$M^{-1}AM = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 1 \\ 0 & -1 & 0 \\ \hline 0 & 0 & -2 \end{array}\right]$$

$$M^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] \qquad CM = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} 3 & 3 & 1 \end{array}\right]$$

Controllable Decomposition:

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}}_{A_c} x + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B_c} u \quad \text{and} \quad y = \underbrace{\begin{bmatrix} 3 & 3 \end{bmatrix}}_{C_c} u$$

Observability Matrix:

$$Q_C = \begin{bmatrix} C_c \\ C_cA_c \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}$$ $\text{rank}(Q_C) = 1 < n_c = 2$ so the reduced equation is NOT observable

The controllable decomposition has no uncontrollable modes, therefore the reduced equation IS stabilizable.
We can confirm our decomposition is controllable by taking the new Controllability Matrix.

$$P_C = [B_c \mid A_cB_c] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$ $\text{rank}(P_C) = 2 = n$ ✓

We observe that for $A_c \rightarrow \Delta(\lambda) = -\lambda(-1-\lambda) \Rightarrow \lambda = 0,\ \lambda = -1$

But for $\lambda = 0$, the e-value is not defective since it's distinct.
So since the system is stable i.s.L. we can say it's stabilizable.
To determine if the system is detectable, we need to decompose it into observable and unobservable states.

Let $M_c^{-1} = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}$, then $A_{CO} = M_c^{-1}A_cM_c = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -1 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 0 \\ \hline \frac{1}{3} & -1 \end{array}\right]$

$M_c^{-1}B_c = \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{cc} 3 & 3 \\ \hline 1 & 0 \end{array}\right]$ and $C_cM_c = \begin{bmatrix} 3 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \frac{1}{3} & -1 \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \end{array}\right]$

The eigenvalue that corresponds to the unobservable mode of the system is $\lambda = -1 < 0$ so it is stable i.s.L.

Hence, the reduced system is detectable.