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1. (a) Lec. 3 Ex. 1,2

   i. Define $e \in \text{Halt}_{X \times Y}$ if e halts and $(fst(e) \in \text{Halt}_X) \wedge (snd(e) \in \text{Halt}_Y)$

   ii. Define $e \in \text{Halt}_{X+Y}$ if e halts and $((\forall e''' \in \text{Halt}_X.[e'''/x]e' \in \text{Halt}_Z) \wedge (\forall e''' \in \text{Halt}_Y.[e'''/y]e'' \in \text{Halt}_Z)) \to$ `case(e, Lx -> e', Ry -> e'')` $\in \text{Halt}_Z$

   (b) Lec 5. Ex

   i. Regularity Lemma

Regularity Theorem

If $\Theta \vdash \Gamma$ ctx and $\Theta ; \Gamma \vdash e : A$ then $\Theta \vdash A$ type

Induction on $\Theta ; \Gamma \vdash e : A$

case 1 $\dfrac{x : A \in \Gamma}{\Theta ; \Gamma \vdash x : A}$ and $\Theta \vdash \Gamma$ ctx

Since $x : A \in \Gamma$ then $\Gamma = \Gamma', x : A$ so the well-formedness of term variable context gives

$$\frac{\Theta \vdash \Gamma' \text{ ctx} \qquad \Theta \vdash A \text{ type}}{\Theta \vdash \Gamma', x : A \text{ ctx}}$$

Case 2

$$\frac{\Theta \vdash A \text{ type} \qquad \Theta ; \Gamma, x : A \vdash e : B}{\Theta ; \Gamma \vdash \lambda x : A.\, e : A \to B}$$, RTP $\Theta \vdash A \to B$ type

By inductive hypothesis : $\Theta \vdash B$ type

By assumption $\Theta \vdash A$ type

So $\Theta \vdash A \to B$ type follows.

Case ③.

$$\frac{\Theta; \Gamma \vdash e : A \to B \qquad \Theta; \Gamma \vdash e' : A}{\Theta; \Gamma \vdash e e' : B}$$ RTP $\Theta \vdash B$ type

$\Theta \vdash A \to B$ type (Ind. hyp.)

~~$\Theta \vdash A$ type (Ind. hyp.)~~

Then $\Theta \vdash B$ type from $$\frac{\Theta \vdash A \text{ type} \qquad \Theta \vdash B \text{ type}}{\Theta \vdash A \to B \text{ type}}$$

Case ④. $$\frac{\Theta, \alpha; \Gamma \vdash e : B}{\Theta; \Gamma \vdash \Lambda \alpha. e : \forall \alpha . B}$$ RTP $\Theta \vdash \forall \alpha . B$ type

RTP. $\Theta, \alpha \vdash B$ type (inversion on well-formedness of types)
which is true by inductive hypothesis.

Case ⑤. $$\frac{\Theta; \Gamma \vdash e : \forall \alpha . B \qquad \Theta \vdash A \text{ type}}{\Theta; \Gamma \vdash e A : [A/\alpha] B}$$ RTP $\Theta \vdash [A/\alpha]B$ type

RTP $\Theta \vdash A$ type and $\Theta, \alpha \vdash B$ type (type substitution)

$\Theta \vdash A$ type true by assumption.

$\Theta \vdash \forall \alpha . B$ type (by inductive hypothesis)

So $\Theta, \alpha \vdash B$ type (by inversion).

ii. Church encoding for unit type: The identity function's type $\forall \alpha.(\alpha \to \alpha)$

iii. Church encoding for empty type: $\forall \alpha . \alpha$

iv. Church encoding for binary trees: $\forall \alpha . \alpha \to (X \to \alpha \to \alpha \to \alpha) \to \alpha$

2. PYP 2019.9.14

(a)

$$\begin{aligned} \text{Boolean: } & \forall\alpha.\, \alpha \to \alpha \to \alpha \\ \text{True: } & \Lambda\alpha.\, \lambda x : \alpha.\, \lambda y : \alpha.\, x \\ \text{False: } & \Lambda\alpha.\, \lambda x : \alpha.\, \lambda y : \alpha.\, y \end{aligned}$$