CSIT113
Problem Solving

UNIT 3
REASONING USING LOGIC

UNIVERSITY OF WOLLONGONG AUSTRALIA

SIM GLOBAL EDUCATION

Overview

- Basic Notations and Operators
- Reasoning using Logic
- Apply basic to solve logic puzzle

Some introductory logic

- Logic dates back to ancient Greece.
- There are several different ways of looking at logic but all have a common set of concepts.
  - Propositions:
    - Statements which are either True or False
  - Axioms:
    - Propositions which are True by definition
  - Theorems:
    - Propositions which can be proved to be True
  - Universal Set:
    - The set of elements or objectors that propositions refer to

Logical Operators

- Logical operators combine one or two propositions to produce a new proposition. Such a proposition is also called a **compound proposition**.
- after introducing the truth table, we shall study the basic logical operators, ~, $\wedge$, $\vee$, $\Rightarrow$ and $\equiv$,

Truth tables

- A useful tool in understanding logic is the truth table.
- This sets out all possible results of propositions in tabular form
- The definitions of logical operators (~, ∧, ∨, ⇒, ≡) are often presented in truth tables.

Not (~)

- The Not operator operates on a single proposition.
- For a proposition P, Not P is written as ~P. It's truth value is the reverse of the truth value of P.

P	~P
T	F
F	T

And (∧)

- The And operator combines two propositions, P and Q.

P	Q	P ∧ Q
T	T	T
T	F	F
F	T	F
F	F	F

- And is true only if both P and Q are true

Or (∨)

- The Or operator combines two propositions, P and Q.

P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

- Or is true as long as at least one of P or Q is true

Equivalence (≡)

- The Equivalence operator combines two propositions, P and Q.

P	Q	P ≡ Q
T	T	T
T	F	F
F	T	F
F	F	T

- Equivalence is true if P and Q have the same truth value

Properties of logical Or

- Commutative: $p \lor q \equiv q \lor p$
- Associative: $p \lor (q \lor r) \equiv (p \lor q) \lor r$
- Idempotent: $p \lor p \equiv p$
- Has neutral element **false**: $p \lor \text{false} \equiv p$
- Distributes over boolean equality:
  $p \lor (q \equiv r) \equiv ((p \lor q) \equiv (p \lor r))$

All the equivalence relations can be proved by constructing a truth table, e.g. The following truth table shows the first relation:

p	q	p∨q	q ∨ p
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

Since for each case, the values of $p \lor q$ and $q \lor p$ are identical. Hence,

$p \lor q \equiv q \lor p$

Properties of logical And

- Commutative: $p \land q \equiv q \land p$
- Associative: $p \land (q \land r) \equiv (p \land q) \land r$
- Idempotent: $p \land p \equiv p$
- Has neutral element **true**: $p \land \text{true} \equiv p$

Implies (⇒)

- The Implies operator combines two propositions, P and Q.

P	Q	P ⇒ Q
T	T	T
T	F	F
F	T	T
F	F	T

- Implies is true unless P is true and Q is false

Order for Evaluating Logical Connectives

Order	Logical Connective
1	$\sim$
2	$\wedge$, $\vee$
3	$\Rightarrow$, $\equiv$

Compound statements in brackets must be evaluated first: from inner to outer. Then, evaluate $\sim$. Next, evaluate $\wedge$ and $\vee$. Last, last, evaluate $\Rightarrow$ and $\equiv$.

Some useful formulæ

- Inverse Laws:
  - $P \vee \sim P \equiv \mathbf{T}$
  - $P \wedge \sim P \equiv \mathbf{F}$
- Implies (if-then)
  - $P \Rightarrow Q \equiv \sim P \vee Q$
- De Morgan's laws
  - $\sim(p \vee q) \equiv \sim p \wedge \sim q$
  - $\sim(p \wedge q) \equiv \sim p \vee \sim q$
- We can show all of these with truth tables.

Testing propositions using Truth table

- We can use truth tables to test propositions to determine whether they are theorems.
- E.g. $P \Rightarrow (Q \Rightarrow P)$

P	Q	$Q \Rightarrow P$	$P \Rightarrow (Q \Rightarrow P)$
T	T	T	T
T	F	T	T
F	T	F	T
F	F	T	T

- Because the last column is all true, $P \Rightarrow (Q \Rightarrow P)$ is a theorem

Q

Simplifying Logic

- We can express all possible logical operators in terms of just two operators:
  - Not and And
  - Not and Or
- $P \vee Q \equiv \sim(\sim P \wedge \sim Q)$

P	Q	$P \vee Q$	$\sim P$	$\sim Q$	$\sim P \wedge \sim Q$	$\sim(\sim P \wedge \sim Q)$
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

- As the columns for $P \vee Q$ and $\sim(\sim P \wedge \sim Q)$ have identical values, hence, $P \vee Q \equiv \sim(\sim P \wedge \sim Q)$

Even simpler

- The two operators Nor and Nand make life even easier…

P	Q	P nor Q
T	T	F
T	F	F
F	T	F
F	F	T

P	Q	P nand Q
T	T	F
T	F	T
F	T	T
F	F	T

- Either of these can produce all operators on its own…

Nor

- $\sim P \equiv P \text{ nor } P$
- $P \wedge Q \equiv \sim P \text{ nor } \sim Q$
  $\equiv (P \text{ nor } P) \text{ nor } (Q \text{ nor } Q)$
- $P \vee Q \equiv \sim(P \text{ nor } Q)$
  $\equiv (P \text{ nor } Q) \text{ nor } (P \text{ nor } Q)$

Knights and Knaves

- Every inhabitant of a mythical island is either a knight or a knave.
- Knights always tell the truth.
- Knaves always lie.
- This forms the basis of several problems in logic puzzle.

Brute Force

- It is tempting to try to solve these problems by looking at all possible cases (using truth table).
- The problems here are:
  - the number of cases rapidly becomes too large
  - the answer is often still not clear
- Perhaps there is a better technique.
- One approach is *Calculational Logic*

Calculational Logic

- The basis of calculational logic is to calculate with Boolean expressions.
- These expression, called propositions, are either true or false.
- This method is less tedious than using truth table.
- In using calculational logic, we may need:
  - ➢ the basic Boolean equivalence and formalism introduced in the next few slides, and
  - ➢ the formulae introduced earlier (for example, the formula for associative and implies, and De Morgan's laws, etc.).

Boolean Equivalence

- The Boolean equivalence relation satisfies a number of properties:
  - Reflexive: **p** ≡ **p**
  - Symmetric: (**p** ≡ **q**) ≡ (**q** ≡ **p**).
  - Transitive: if **p** ≡ **q** and **q** ≡ **r** then **p** ≡ **r**.
  - Associative: (**p** ≡ (**q** ≡ **r**)) ≡ ((**p** ≡ **q**) ≡ **r**)
  - Substitution of equals for equals: if **p** ≡ **q** and *f* is a Boolean function then *f*(**p**) ≡ *f*(**q**).

Knights and Knaves

- If A is a native of the island the statement "A is a knight" is either true or false.
- So, the statement is a proposition.
- Let **A** represent the proposition "A is a knight".
- Suppose A makes some statement **S**.
- The truth or falsity of this statement is the same as the truth or falsity of **A**.

  A ≡ **S**

Knights and Knaves

- So if A says "the restaurant is to the left" then **A** ≡ **L**.
- In other words either A is a knight and the restaurant is to the left or A is not a knight and the restaurant is not to the left.
- If A says "I am a knight" we conclude that **A** ≡ **A** which tells us nothing!

Knights and Knaves

- If we ask A a Yes/No question, Q, the response will be the truth value of **A** ≡ **Q**.
- That is, if the response is "yes", either A is a knight and the answer to Q really is yes or A is a knave and the answer is really no.
- Otherwise the response will be "no".

Knights and Knaves

- Let's say we have two natives, A and B.
- A says "B is a knight"
  - What can we deduce?
- If **A** represents the proposition A is a knight and **B** represents the proposition B is a knight:

  A ≡ B.
- That is, A and B are of the same type.
- Note that we don't know *which* type.

Knights and Knaves Logic Puzzle: Problem 1

- It is rumoured there is gold on the island.
- A native tells you "The statement 'there is gold on the island' and the statement 'I am a knight' are either both true or both false".
  - Can you tell if the native is a knight?
  - Can you tell if there is gold on the island?

Knights and Knaves Logic Puzzle: Problem 1

- Since A says "The statement 'there is gold on the island' and the statement 'I am a knight' are either both true or both false", A is asserting **A** ≡ **G** where A is the proposition A is a knight and **G** the proposition there is gold on the Island.
- Note that A is NOT asserting $A \wedge G$.
- Any assertion by a native has the same truth value as **A** so:

  A ≡ (**A** ≡ **G**)

  (**A** ≡ **A**) ≡ **G**

  true ≡ **G**
- From this we can conclude that there is gold on the island, even though we have no idea if the native is a knight or a knave.

Knights and Knaves Logic Puzzle: Problem 1

- If we use brute force to solve problem 1, we will construct the following truth table:

A	G	A ≡ G	A ≡ (A ≡ G)
T	T	T	T
T	F	F	F
F	T	F	T
F	F	T	F

Since **A** ≡ (**A** ≡ **G**), it can only be the first case or third case. Hence, G is true and A can be true or false

Knights and Knaves Logic Puzzle: Problem 2

- You come across two natives.
- You ask each if the other is a knight.
  - Do you get the same answer from both of them?

Knights and Knaves Logic Puzzle: Problem 2

- A will answer "yes" if he is a knight and so is B or if he is a knave and so is B.
- In other words:
  - **A's answer** ≡ (**A** ≡ **B**)
  - **B's answer** ≡ (B ≡ A)
- Using the symmetry property:
  - (**A** ≡ **B**) ≡ (**B** ≡ **A**)
- So B's answer will be the same as A's.

Knights and Knaves Logic Puzzle: Problem 3

- There are three natives, A, B and C.
- A says "B and C are of the same type".
  - What can we conclude about the number of knights present?

Knights and Knaves Logic Puzzle: Problem 3

- A says $\mathbf{B} \equiv \mathbf{C}$ so:
  - $\mathbf{A} \equiv (\mathbf{B} \equiv \mathbf{C})$
- So
  - A is a knight and so are B and C

  or
  - A is a knight and B and C are knaves

  or
  - A is a knave and one of B and C is a knight
- There is an odd number of knights.

Another Way to Solve Problem 3: Using Truth Table

A	B	C	$\mathbf{B} \equiv \mathbf{C}$	$\mathbf{A} \equiv (\mathbf{B} \equiv \mathbf{C})$
T	T	T	T	T √
T	T	F	F	F
T	F	T	F	F
T	F	F	T	T √
F	T	T	T	F
F	T	F	F	T √
F	F	T	F	T √
F	F	F	T	F

- From the truth table, we conclude that the four possible cases are the 4 rows that have the value "T" for $\mathbf{A} \equiv (\mathbf{B} \equiv \mathbf{C})$ (indicated by "√").
- And, each of above-mentioned row has odd number of knights. Hence, we can conclude that there are odd number of knights.

Knights and Knaves Logic Puzzle: Problem 4

- There are three natives, A, B and C.
- A says "B and C are of the same type".
  - What question can we ask C to find out if A is telling the truth?

Knights and Knaves Logic Puzzle: Problem 4

- Let Q be the unknown question we must ask C, with truth value $\mathbf{Q}$.
- Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ denote the propositions A, B, C is a knight.
- The response we want is $\mathbf{A}$ so:
  - $(\mathbf{C} \equiv \mathbf{Q}) \equiv \mathbf{A}$
- Which we regroup to give:
  - $\mathbf{Q} \equiv (\mathbf{C} \equiv \mathbf{A})$
- But $\mathbf{A} \equiv (\mathbf{B} \equiv \mathbf{C})$ so substituting for A we get:
  - $\mathbf{Q} \equiv (\mathbf{C} \equiv (\mathbf{B} \equiv \mathbf{C}))$
- Which simplifies (after rearrangement) to:
  - $\mathbf{Q} \equiv \mathbf{B}$
- In other words, the question is "Is B a knight?".

Knights and Knaves Logic Puzzle: Problem 5

- There are two natives, A and B.
  - What question should you ask A to determine if B is a knight?

Knights and Knaves Logic Puzzle: Problem 5

- We want a question, Q, whose answer, when asked of A, is the type of B.
  - $(\mathbf{A} \equiv \mathbf{Q}) \equiv \mathbf{B}$
- Reorganising:
  - $\mathbf{Q} \equiv (\mathbf{A} \equiv \mathbf{B})$
- In other words "Is B of the same type as you?".

Knights and Knaves Logic Puzzle: Problem 6

- There are two natives, A and B.
  - What question should you ask A to determine whether A and B are of the same type?

Knights and Knaves Logic Puzzle: Problem 6

- We want a question, Q, which, when asked of A, determines if A and B are of the same type:
  - $(\mathbf{A} \equiv \mathbf{Q}) \equiv (\mathbf{A} \equiv \mathbf{B})$
- Regrouping and simplifying:
  - $\mathbf{Q} \equiv (\mathbf{A} \equiv (\mathbf{A} \equiv \mathbf{B}))$
  - $\mathbf{Q} \equiv ((\mathbf{A} \equiv \mathbf{A}) \equiv \mathbf{B})$
  - $\mathbf{Q} \equiv (\mathbf{true} \equiv \mathbf{B})$
  - $\mathbf{Q} \equiv \mathbf{B}$
- In other words, the question is "Is B a knight?".

Knights and Knaves Logic Puzzle: Problem 7

- You come to a fork in the road.
- There is a restaurant down one of the two branches.
- There is a native at the fork.
  - What question do you ask to find out if the restaurant lies down the left fork?

Knights and Knaves Logic Puzzle: Problem 7

- Following the same rules as before:
  - $(\mathbf{A} \equiv \mathbf{Q}) \equiv \mathbf{L}$
- Which we can rearrange as:
  - $\mathbf{Q} \equiv (\mathbf{A} \equiv \mathbf{L})$
- So our question is "Is the truth value of the statement 'you are a knight' the same as the truth value of 'the restaurant lies down the left fork'?"
- Our question can be rephrased as "Is it the case that the statement that the left fork leads to the restaurant is equivalent to your being a knight?"

*Verification of Correctness of the Deduction for Problem 7

- We can check the deduction is correct using brute force ☺
  - From the truth table, we can see that A's answer to Q is the same as L.

A	L	$\mathbf{A} \equiv \mathbf{L}$	A's answer to Q
T	T	T	T
T	F	F	F
F	T	F	T
F	F	T	F