Fejér's Theorem for Fourier Series

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Introduction

Lipót Fejér

On the 9th of February 1880, Lipót Féjer was born as Leopold Weiss in Pécs, Hungary. Around 1900, he abandoned his Germanic name in favor of a Hungarian equivalent so as to express his solidarity with Hungarian culture. Remarkably, at the age of 19 years, he proved what is now called *Fejér's Theorem* which may be used to approximate continuous, periodic functions of one variable defined over the real numbers $\mathbb{R}$.

Definitions

Let f be a locally integrable, 2π-periodic function over $\mathbb{R}$.

- **Uniform Convergence**: We say that a sequence of functions $\{f_n(x)\}_{n\in\mathbb{N}}$ converges uniformly to f on $[-\pi,\pi]$ if

$$\lim_{n\to\infty}\left(\sup_{x\in[-\pi,\pi]}|f_n(x)-f(x)|\right)=0.$$

- **Fourier Coefficient** $\hat{f}_k$: Corresponding to a given integer k, this is a number given by $\hat{f}_k=\frac{1}{2\pi}\int_0^{2\pi}e^{-ikt}f(t)\text{d}t$.
- **Fourier Series N-th Partial Sum**: Given a natural number $N\geq 0$ and x in the interval $[-\pi,\pi]$, this is the finite sum $S_N(x)=\sum_{k=-N}^{N}\hat{f}_k e^{ikx}$.
- **n-th Cesáro average**: For any non-negative integer n, this is the average of the first $n+1$ partial sums $\{S_N(x)\}_{N=0}^{n}$. We denote it by $\sigma_n(x)=\frac{1}{n+1}\sum_{N=0}^{n}S_N(x)$.

Fejér's Theorem

If $f:[-\pi,\pi]\to\mathbb{C}$ is continuous at $x\in[-\pi,\pi]$ and f is Riemann integrable, then:

$$\sigma_n(x) \text{ converges to } f(x) \text{ as } n\to\infty.$$

Furthermore, if f is continuous, then:

$$\sigma_n \text{ converges to } f \text{ uniformly on } [-\pi,\pi].$$

The first part of the theorem gives conditions which ensure that the sequence of Cesáro averages give good approximations of the value of the function f wherever it is continuous. The final result of the theorem follows quite naturally if f is continuous everywhere on its domain.

Consequences

While Fejér's theorem has uses in advanced topics, such as Functional Analysis (see [4]), here three relatively elementary consequences are shared:

- If $f:[-\pi,\pi]\to\mathbb{C}$ is continuous, then there exists a sequence of trigonometric polynomials which converge uniformly to f on $[-\pi,\pi]$. An example of a sequence of trigonometric polynomials would be $\{S_N(x)\}_{N=0}^{\infty}$.
- If $f,g:[-\pi,\pi]\to\mathbb{C}$ are continuous and

$$\int_0^{2\pi}e^{-ikt}f(t)dt=\int_0^{2\pi}e^{-ikt}g(t)dt$$

for all $k\in\mathbb{Z}$, then $f\equiv g$. In other words, any two continuous functions which have identical Fourier coefficients, describe the same function.
- Fejér's theorem can be used to derive *Weierstrass' approximation theorem* for continuous functions on $[-\pi,\pi]$, which states that there exists a sequence of polynomials which converge uniformly to f.

Example

Consider the function:

$$h(x)=\frac{\pi}{2}-|x| \qquad x\in[-\pi,\pi].$$

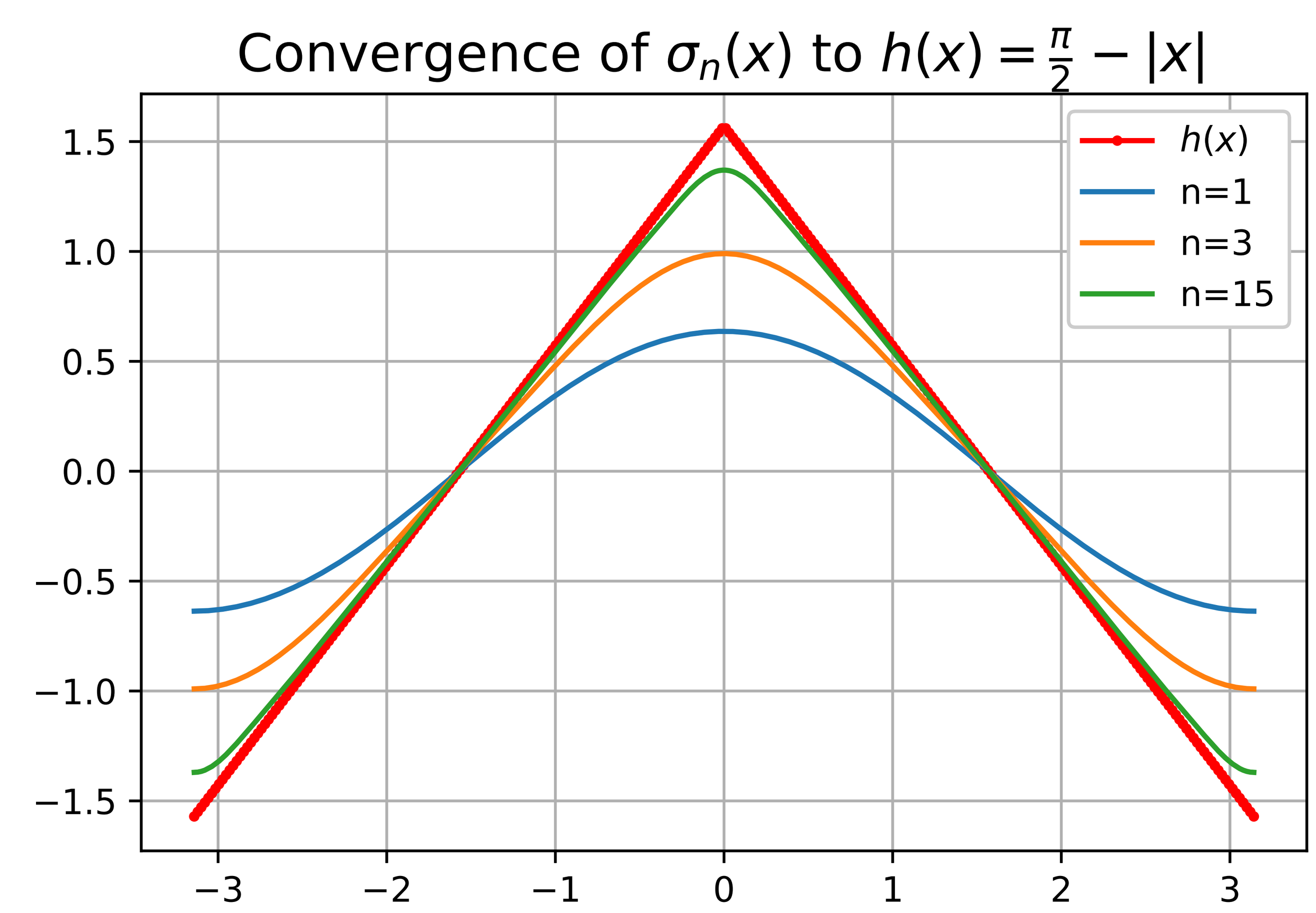
We may express the N-th partial sum of the Fourier series for h and the associated n-th Cesáro average as follows:

$$S_N(x)=\sum_{k=0}^{N}\frac{4\cos[(2k+1)x]}{\pi(2k+1)^2},$$

$$\sigma_n(x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{\pi}{2}-|t|\right)\frac{1}{n+1}\left(\frac{\sin[(n+1)\frac{(x-t)}{2}]}{\sin(\frac{x-t}{2})}\right)^2 dt.$$

Since h is continuous on $[-\pi,\pi]$, we can apply Fejér's theorem to deduce that

$$\sigma_n \text{ converges to } h \text{ uniformly as } n\to\infty.$$



This is an illustration of uniform convergence as it shows how σ_n better approximates h, over all of $[-\pi,\pi]$, as n is increased.

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References

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