# 矩阵理论与方法

12月

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  **第4章 矩阵分解**

# 第4章 矩阵分解

1、LU分解

2、QR分解

3、满秩分解

4、SVD分解

**定义 4.11** 设 $\boldsymbol{A} \in \mathbf{C}_r^{m\times n}(r > 0)$，$\boldsymbol{A}^{\mathrm{H}}\boldsymbol{A}$ 的特征值为

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > \lambda_{r+1} = \cdots = \lambda_n = 0$$

则称 $\sigma_i = \sqrt{\lambda_i}\ (i = 1, 2, \cdots, n)$ 为 $\boldsymbol{A}$ 的**奇异值**；当 $\boldsymbol{A}$ 为零矩阵时，它的奇异值都是 0.

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$$\boldsymbol{U}^{\mathrm{H}}\boldsymbol{A}\boldsymbol{V} = \begin{bmatrix} \boldsymbol{\Sigma} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} = D \tag{4.4.4}$$

其中 $\boldsymbol{\Sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \cdots, \sigma_r)$，而 $\sigma_i (i=1, 2, \cdots, r)$ 为矩阵 $\boldsymbol{A}$ 的全部非零奇异值.

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$V=(v_1,\dots,v_N)$是酉矩阵：$V^H V=I$

$BV=V\Lambda \Rightarrow A^H AV=V\Lambda \Rightarrow A^H AV_1=V_1\Sigma^2$

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$\Rightarrow U_1^H A = \Sigma V_1^H \Rightarrow A = U_1\Sigma V_1^H$

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3、将$U_1$扩充成酉矩阵$U=[U_1 \vdots U_2]$

$A = UDV^H$

例10：称 $A=\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}$，求 $A=UDV^T$

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解： $A^TA=\begin{bmatrix}1&0&1\\0&1&1\\1&1&2\end{bmatrix}=B, |\lambda I-B|=\lambda(\lambda-1)(\lambda-3)$

$$\lambda_1=3: \quad 3I-B=\begin{bmatrix}2&0&-1\\0&2&-1\\-1&-1&1\end{bmatrix}, \xi_1=\begin{bmatrix}1\\1\\2\end{bmatrix}$$

$$\lambda_2=1: \quad 1I-B=\begin{bmatrix}0&0&-1\\0&0&-1\\-1&-1&-1\end{bmatrix}, \xi_2=\begin{bmatrix}1\\-1\\0\end{bmatrix}$$

$$\lambda_3 = 0: \quad 0I - B = \begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ -1 & -1 & -2 \end{bmatrix}, \xi_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$r_A = 2: \quad \Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}$$

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$$V = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{2}{\sqrt{6}} & 0 & -\frac{1}{\sqrt{3}} \end{bmatrix}, \quad V_1 = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{2}{\sqrt{6}} & 0 \end{bmatrix}$$

$$U_1 = AV_1\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}$$

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$$U_1 = AV_1\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & 0 \end{bmatrix}，取 U_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}，则 U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$U^{\mathrm{T}}AV = \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = D, \quad A = UDV^{\mathrm{T}}$$

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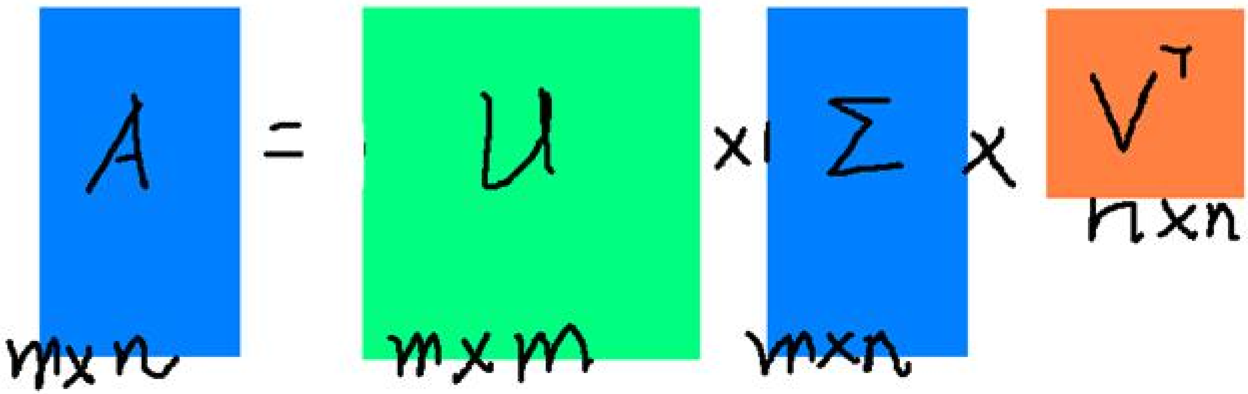
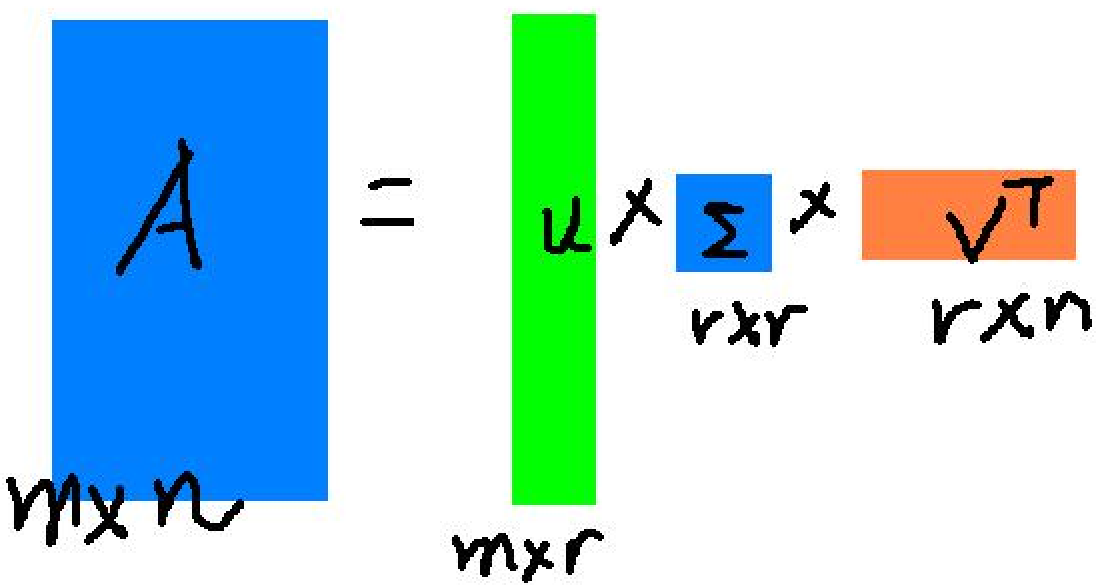
$$A = U\Sigma V^T$$




$$A_{m\times n} \approx U_{m\times r}\Sigma_{r\times r}V^T{}_{r\times n}$$



求 $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}$ 的奇异值分解

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$$U=\frac{1}{\sqrt{5}}\begin{bmatrix}1 & -2\\ 2 & 1\end{bmatrix}, V=I_3, A=U\begin{bmatrix}\sqrt{5} & 0 & 0\\ 0 & 0 & 0\end{bmatrix}V^{H}$$

# 作业（第五版）

1、定义：4.11

2、定理：4.15、4.16

3、例题：4.14、4.15

4、习题4.4：2、4

# 作业（第三版）

1、定义：4.11

2、定理：4.15、4.16

3、例题：4.14、4.15

4、习题4.4：2、4

下课，谢谢大家！