Quantum Information With Discrete Variables

1. Qubits

1. Spin qubits
2. Photonic qubits
3. Two-level atoms as a qubits

2. EPR

3. Quantum gates

Basic Literature

1. Michael A. Nielsen, Isaac L. Chuang, Quantum Computation and Quantum Information Cambridge University Press, 2000
2. Christopher Gerry, Peter Knight, Introductory Quantum Optics, Cambridge University Press, 2005
3. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, 1995.
4. D. F. Walls, Gerard J. Milburn, Quantum Optics Springer Science & Business Media, 2008.
5. S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics,* Oxford, 1997
6. Anthony Sudbery. Quantum Mechanics and the Particles of Nature: An Outline for Mathematicians Cambridge University Press 1986

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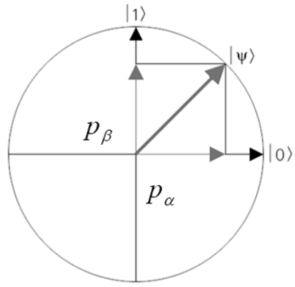
Discrete variables

The Qubits

Def. $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

$|0\rangle$ **is logical «0», and** $|1\rangle$ **is logical «1» in D Hilbert space**



$p_\alpha = |\alpha|^2$ **- Probability to obtain** $|0\rangle$

$p_\beta = |\beta|^2$ **- probability to obtain** $|1\rangle$

Normalization is $|\alpha|^2 + |\beta|^2 = 1$

Classical computing

○ ○ … ○ **Bits**

$x_1 \ x_2 \ \dots \ x_n \ \ x_j \in \mathbb{B}$

$\mathbb{B}^n = \{0,1\}^n$

Quantum computing

○ ○ … ○ **Q-bits**

$|x\rangle$≡ичное: $|\, x_1,\ x_2,\ \dots,\ x_n \rangle$, $\quad x_j \in \mathbb{B}$

$x = \sum_{i=1}^{L} x_i 2^i$ $\quad L$ – Number of Q-bits

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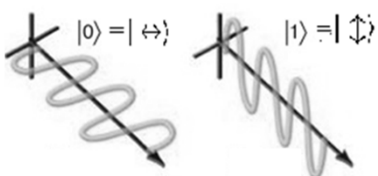
Two-Level Systems As Qubits

1. **Photon polarization and path entangled qubits (flight q-bit)**

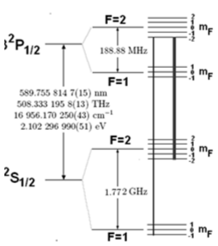
$$|\psi\rangle = \alpha|\leftrightarrow\rangle + \beta|\updownarrow\rangle$$

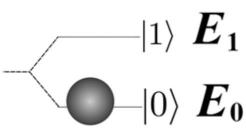


H-polarization ***V- polarization***

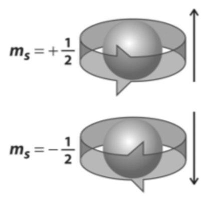
2. Two-level atom

Energy levels





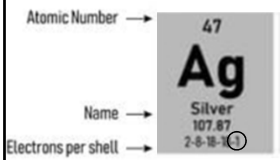
3. Electron spin

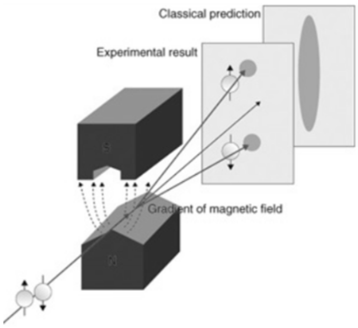


(spin up)

(spin down)

Spin: Stern-Gerlach Experiment





Ag atoms

- A beam of neutral silver atoms is split into two components by a non-uniform magnetic field
- The atoms experienced a force due to their magnetic moments
- The beam had **two distinct** components in contrast to the classical prediction

Total momentum

If the electron had spin, it might explain this

J=L+S

Electron Spin Description

Wolfgang Pauli

Pauli Matrixes

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Lets $\vec{S} = \frac{\hbar}{2}\vec{\sigma}$ $\quad \widehat{S}_x = \frac{\hbar}{2}\sigma_x,\ \widehat{S}_y = \frac{\hbar}{2}\sigma_y,\ \widehat{S}_z = \frac{\hbar}{2}\sigma_z,\ \widehat{S}^2 = \frac{3\hbar^2}{4}\sigma_0.$

Commutation relations

$$\boxed{[S_i, S_j] = i\hbar\epsilon_{ijk}S_k \qquad [S_i, S_0] = \mathbf{0}}$$

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), \text{ or } (3,1,2), \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), \text{ or } (2,1,3), \\ 0 & \text{if } i=j, \text{ or } j=k, \text{ or } k=i \end{cases}$$

Is Levi-Civita symbol

Properties of the Pauli Spin Matrices:

$$\sigma_j\sigma_k = i\epsilon_{jkl}\sigma_l \qquad i \neq j$$

$$Tr(\sigma_i) = 0$$

$$\sigma_i^2 = \mathbf{1}$$

Uncertainty relations for spin

$$\Delta S_x\, \Delta S_y \geq \tfrac{1}{2}\left|\langle S_z \rangle\right| \qquad \Delta S_x\, \Delta S_z \geq \tfrac{1}{2}\left|\langle S_y \rangle\right| \qquad \Delta S_y\, \Delta S_z \geq \tfrac{1}{2}\left|\langle S_x \rangle\right|$$

where $\Delta S_j = \sqrt{\langle (\Delta S_j)^2 \rangle}$

This means that we cannot simultaneously and exactly measure all component of spin-vector

Electron Spin Eigenstates

Pauli Matrixes

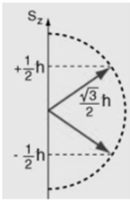
$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

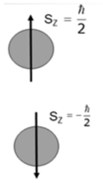
Spin wave function $u = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

Norm $u^+ u = |c_1|^2 + |c_2|^2 = 1.$

Eigenstates eigenvalues equation

$$\hat{S}_z u = S_z u \quad \Rightarrow \quad u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$





Qubit Properties

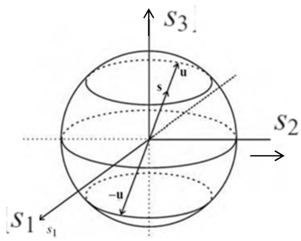
Definition $|\psi\rangle = a|0\rangle + b|1\rangle \qquad |a|^2 + |b|^2 = 1$

Where coding is $|0\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Density operator $\rho = |\psi\rangle\langle\psi|$

Density matrix $\hat{\rho} = |\psi\rangle\langle\psi| = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$

Bloch sphere



$$\rho = \frac{1}{2}\begin{pmatrix} 1 + s_3 & s_1 + i\, s_2 \\ s_1 - i\, s_2 & 1 - s_3 \end{pmatrix} = \frac{1}{2}(\hat{I}_2 + \mathbf{s}\cdot\sigma)$$

$\mathbf{s} = \{s_1, s_2, s_3\}$ **Is Bloch vector** $\sum_i |s_i|^2 = 1.$

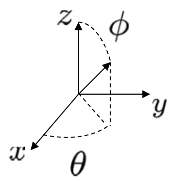
$$\boxed{s = \mathrm{Sp}(\widehat{\rho\sigma})}$$

Bloch Sphere Representation

Felix Bloch

Basis: $|0\rangle, |1\rangle$

Qubit state: $|\psi\rangle = \cos\left(\frac{\phi}{2}\right)|0\rangle + e^{i\theta}\sin\left(\frac{\phi}{2}\right)|1\rangle$

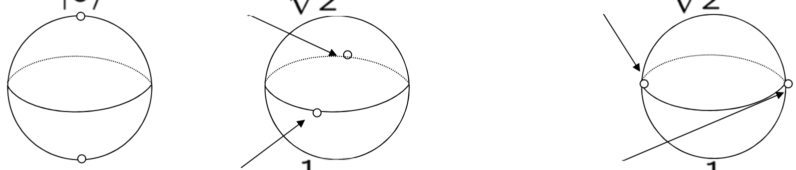


$0 \le \theta < 2\pi$

$0 \le \phi \le \pi$

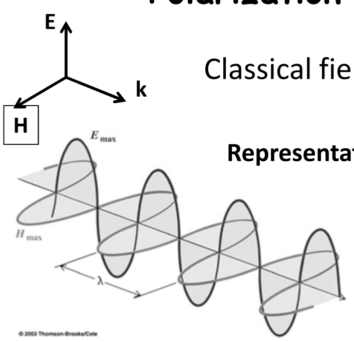
$|0\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ $\quad |-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$



$|1\rangle$ $\quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ $\quad |+i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

Polarization of Classical Electromagnetic Wave



Classical field is a vector $\quad \vec{E}^{(+)}(t) = \vec{E}_0(t)e^{ikz-i\omega t}$ + C.C.

Representation of vector field amplitude $\quad \vec{E}_0(t) = \vec{E}_x(t) + \vec{E}_y(t)$

where $\quad \vec{E}_x = \varepsilon_x \vec{e}_x \quad \vec{E}_y = \varepsilon_y \vec{e}_y$

$\vec{e}_x \vec{e}_y = \delta_{xy}$

Jones vector

$$\vec{e}(t) = \begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}\frac{E_x(t)}{\sqrt{S_0}} \\ \frac{E_y(t)}{\sqrt{S_0}}\end{pmatrix} \qquad S_0 \equiv |E_x(t)|^2 + |E_y(t)|^2.$$

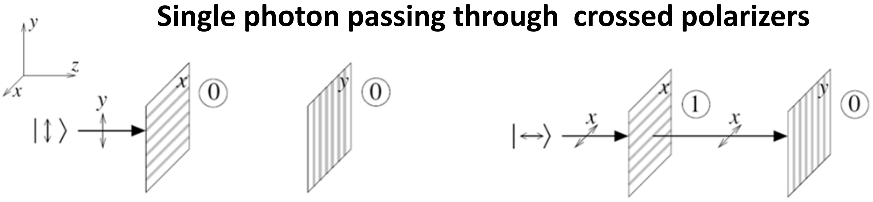
For elliptically polarized light we have

$$\hat{\mathbf{e}} = e_x \hat{\mathbf{x}} + e_y \hat{\mathbf{y}} = e_+ \hat{\mathbf{e}}^{(+)} + e_- \hat{\mathbf{e}}^{(-)}$$

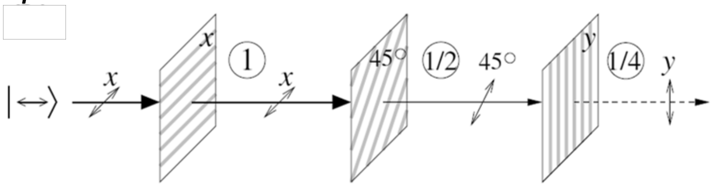
Jones R.C. New calculus for the treatment of optical systems [Text] / R.C. Jones / J. Opt. Soc. Am. 31. – 1941. – P. 488–493.

Single photon polarization: vital peculiarities

Single photon passing through crossed polarizers



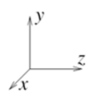
Neither a vertically nor a horizontally polarized photon can go through two crossed polarizers



Inserting between the crossed polarizers another polarizer at 45° allows the horizontally polarized photon to pass with probability 1/4 .

Single Photon Qubit

Encoding $|\updownarrow\rangle \equiv |1\rangle_y |0\rangle_x \rightarrow |1\rangle \qquad |\leftrightarrow\rangle \equiv |0\rangle_y |1\rangle_x \rightarrow |0\rangle$



$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

$$\sigma_x|\psi\rangle = \alpha|1\rangle + \beta|0\rangle \qquad \sigma_y|\psi\rangle = i\alpha|1\rangle - i\beta|0\rangle$$

$$\sigma_z|\psi\rangle = \alpha|0\rangle - \beta|1\rangle \qquad \langle\sigma_z\rangle = |\alpha|^2 - |\beta|^2$$

Pauli Matrixes

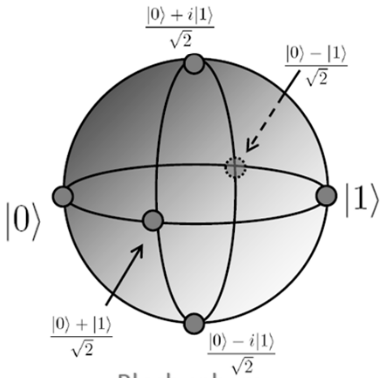
$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_z \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_x \qquad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_y,$$

Poincare sphere vs Bloch sphere

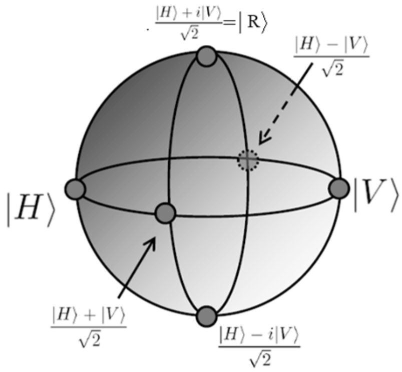
$\sigma_x|\psi_x\rangle = s_x|\psi_x\rangle$ $\quad$ $\sigma_y|\psi_y\rangle = s_y|\psi_y\rangle$ $\quad$ $\sigma_z|\psi_z\rangle = s_z|\psi_z\rangle$

$s_x = 1 \quad |\nearrow\rangle$ $\quad$ $s_y = 1 \quad |\circlearrowleft\rangle \equiv |\mathrm{R}\rangle$ $\quad$ $s_z = 1 \quad |\leftrightarrow\rangle$

$s_x = -1 \quad |\searrow\rangle$ $\quad$ $s_y = -1 \quad |\circlearrowright\rangle = |\mathrm{L}\rangle$ $\quad$ $s_z = -1 \quad |\updownarrow\rangle$



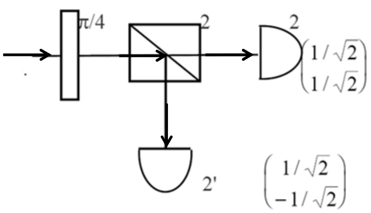
Bloch sphere



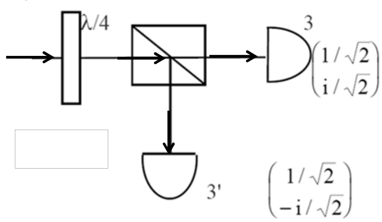
Poincare sphere

Measurement of Stokes parameters

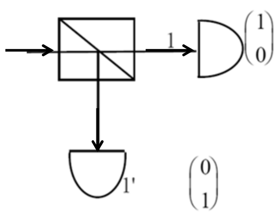
S_x ***- measurement***



S_y ***- measurement***



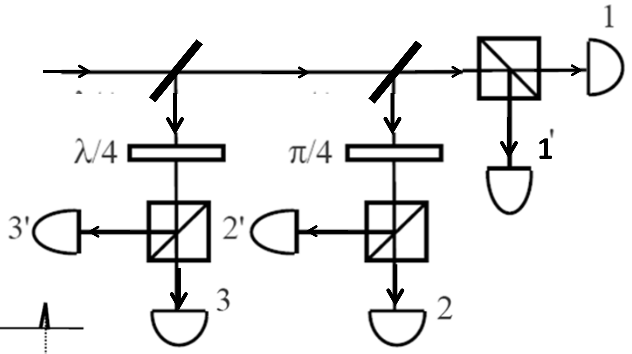
S_z ***- measurement***



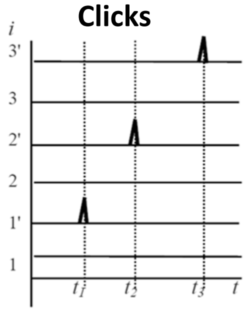
For ideal detectors we obtain only one click per one input photon

Single Photon Polarization Measurement

Scheme



Clicks

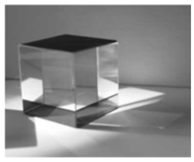


Single photon evokes only one detector click!

For example, at the moment t_1 ***we know that photon is NOT polarized along*** $|\leftrightarrow\rangle$

For example, at the moment t_2 ***we know that photon is NOT polarized along*** $|\swarrow\!\!\nearrow\rangle$

Beam Splitter

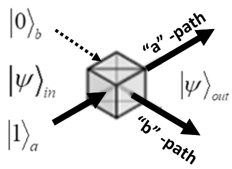


BS matrix B= $\begin{pmatrix} \cos(\Theta/2)\, e^{i(\Psi+\Phi)/2} & \sin(\Theta/2)\, e^{i(\Psi-\Phi)/2} \\ -\sin(\Theta/2)\, e^{i(\Phi-\Psi)/2} & \cos(\Theta/2)\, e^{-i(\Psi+\Phi)/2} \end{pmatrix}$

Φ, Θ, Ψ ***are real parameters which define rotational symmetries***

Conditions $|B_{11}|^2 + |B_{12}|^2 = 1, \quad |B_{21}|^2 + |B_{22}|^2 = 1, \quad B_{11}^* B_{21} + B_{12}^* B_{22} = 0\,.$

Single photon path entangled qubit



$|\psi\rangle_{in} = |1\rangle_a |0\rangle_b$

$$\boxed{|\psi\rangle_{out} = \alpha|0\rangle_a|1\rangle_b + \beta|1\rangle_a|0\rangle_b}$$

$$\Downarrow$$

$$\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$$

Encoding $|0\rangle = |0\rangle_a|1\rangle_b, \; |1\rangle = |1\rangle_a|0\rangle_b$

Density Matrix formalism; pure state

Definition $|\psi\rangle = a|0\rangle + b|1\rangle \quad |a|^2 + |b|^2 = 1$

Density operator $\rho = |\psi\rangle\langle\psi|$

Matrix representation $\widehat{\rho} = \begin{pmatrix} |a|^2 & ab^* \\ ba^* & |b|^2 \end{pmatrix} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$; $\rho_{nn}\rho_{mm} = |\rho_{nm}|^2$.

Master equation **Schrodinger Equation**

$$\boxed{\dot{\rho} = -\frac{i}{\hbar}[\mathscr{H}, \rho]} \Leftrightarrow \boxed{|\dot{\psi}\rangle = -\frac{i}{\hbar}\mathscr{H}|\psi\rangle}$$

$$\dot{\rho}_{ij} = -\frac{i}{\hbar}\sum_k (\mathscr{H}_{ik}\rho_{kj} - \rho_{ik}\mathscr{H}_{kj})$$

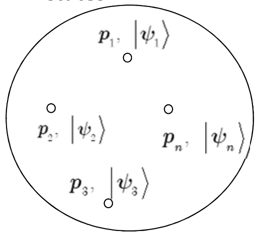
Mixed States

It often happens that exact quantum state of the system is unknown.

States that cannot be described by state vectors are said to be in *mixed* states. Mixed states are described by the density operator

$$\boxed{\hat{\rho} = \sum_i |\psi_i\rangle\, p_i\, \langle\psi_i| = \sum_i p_i\, |\psi_i\rangle\langle\psi_i|,}$$

Ensemble of quantum states

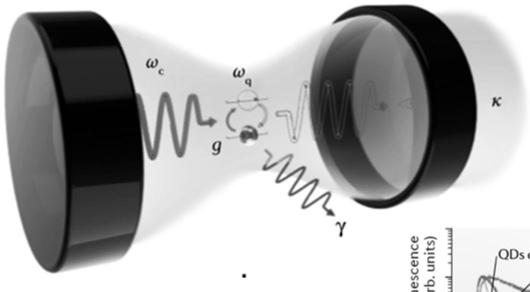


where the sum is over an ensemble (in the sense of statistical mechanics) where p_i is the probability of the system being in the *i*th state of the ensemble $|\psi_i\rangle$

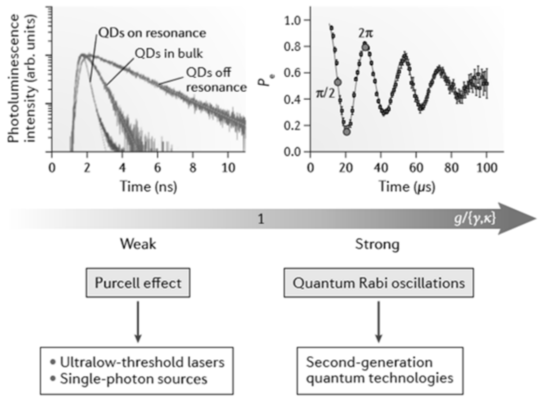
The probabilities satisfy the relations $0 \le p_i \le 1, \quad \sum_i p_i = 1, \quad \sum_i p_i^2 \le 1.$

$$\boxed{\rho_{nn}\,\rho_{mm} > |\rho_{nm}|^2}$$

If $p_i = \delta_{ij}$ $\quad \hat{\rho} = |\psi_j\rangle\langle\psi_j|$. for pure state

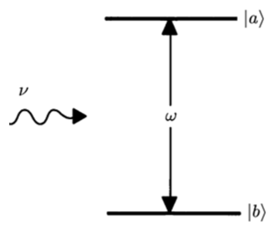


Cavity QED



Anton Frisk Kockum, Adam Miranowicz, Simone De Liberato, Salvatore Savasta & Franco Nori, Nature Reviews Physics ***volume 1****, pages19–40 (2019)*

Two –level atom



Pure state is $|\psi(t)\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle$

Hamiltonian $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_1$

$$\mathscr{H}_0 = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b|,$$

$$\mathscr{H}_1 = -exE(t) = -(\wp_{ab}|a\rangle\langle b| + \wp_{ba}|b\rangle\langle a|)E(t),$$

$\wp_{ab} = \wp_{ba}^* = e\langle a|x|b\rangle$ **Is dipole matrix element**

$E(t) = \mathscr{E}\cos\nu t$ **Is classical field**

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Solution of Schrodinger Equation

$$|\psi(t)\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle \quad \Rightarrow \quad |\dot{\psi}\rangle = -\frac{i}{\hbar}\mathscr{H}|\psi\rangle$$

$$c_a(t) = \left\{ c_a(0)\left[\cos\left(\frac{\Omega t}{2}\right) - \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] + i\frac{\Omega_R}{\Omega}\, e^{-i\phi} c_b(0)\sin\left(\frac{\Omega t}{2}\right)\right\} e^{i\Delta t/2},$$

$$c_b(t) = \left\{ c_b(0)\left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right] + i\frac{\Omega_R}{\Omega}\, e^{i\phi} c_a(0)\sin\left(\frac{\Omega t}{2}\right)\right\} e^{-i\Delta t/2}.$$

$$c_a = C_a\, e^{i\omega_a t}, \qquad c_b = C_b\, e^{i\omega_b t}.$$

$\Delta = \omega - \nu$; **Is detuning**

$$\Omega = \sqrt{\Omega_R^2 + (\omega-\nu)^2}$$

$\Omega_R = \dfrac{\wp_{ba}\mathscr{E}}{\hbar}$; **Is Rabi frequency**

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Rabi oscillations

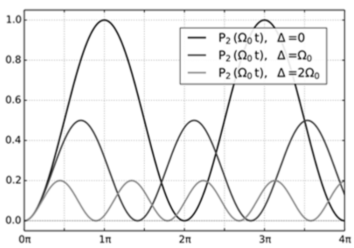
Isaac Rabi

Population imbalance (for $c_a(0) = 1,\ c_b(0) = 0$**)**

$$W(t) = |c_a(t)|^2 - |c_b(t)|^2 = \left(\frac{\Delta^2 - \Omega_R^2}{\Omega^2}\right)\sin^2\left(\frac{\Omega t}{2}\right) + \cos^2\left(\frac{\Omega t}{2}\right)$$

Polarization

$$P(t) = e\langle\psi(t)|r|\psi(t)\rangle = C_a^* C_b \wp_{ab} + \text{к.с.} = c_a^* c_b \wp_{ab}\, e^{i\omega t} + \text{ H.C. } =$$



$$2\mathrm{Re}\left\{\frac{i\Omega_R}{\Omega}\wp_{ab}\left[\cos\left(\frac{\Omega t}{2}\right) + \frac{i\Delta}{\Omega}\sin\left(\frac{\Omega t}{2}\right)\right]\sin\left(\frac{\Omega t}{2}\right) e^{i\phi}\, e^{i\nu t}\right\}$$

$$\Omega = \sqrt{\Omega_R^2 + (\omega-\nu)^2}$$

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Master Equation

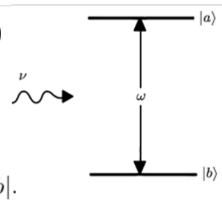
$$\boxed{\dot{\rho} = -\frac{i}{\hbar}[\mathscr{H}, \rho] - \frac{1}{2}\{\Gamma, \rho\}}$$

Where $\{\Gamma, \rho\} = \Gamma\varrho + \varrho\Gamma.$ **describes decoherence**

$$\dot{\rho}_{ij} = -\frac{i}{\hbar}\sum_k (\mathscr{H}_{ik}\rho_{kj} - \rho_{ik}\mathscr{H}_{kj}) - \frac{1}{2}\sum_k (\Gamma_{ik}\rho_{kj} + \rho_{ik}\Gamma_{kj})$$

•

Two –Level Atom (D.M. approach)



Pure state is $|\psi(t)\rangle = C_a(t)|a\rangle + C_b(t)|b\rangle$

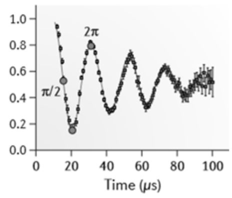
Density operator

$$\rho = |\psi\rangle\langle\psi| = |C_a|^2|a\rangle\langle a| + C_a C_b^*|a\rangle\langle b| + C_b C_a^*|b\rangle\langle a| + |C_b|^2|b\rangle\langle b|.$$

$$\rho = \begin{pmatrix} \rho_{aa} & \rho_{ab} \\ \rho_{ba} & \rho_{bb} \end{pmatrix} \quad \rho_{aa} = \langle a|\rho|a\rangle = |C_a(t)|^2, \quad \rho_{ab} = \langle a|\rho|b\rangle = C_a(t)C_b(t)^*, \quad \rho_{ba} = \rho_{ab}^*, \quad \rho_{bb} = \langle b|\rho|b\rangle = |C_b(t)|^2.$$

Atomic polarization $P(z,t) = C_a C_b^* \wp_{ba} + \text{к.с.} = \rho_{ab}(z,t)\wp_{ba} +$ H.C.

Equations for density matrix elements



$$\dot{\rho}_{aa} = -\gamma_a\rho_{aa} + \frac{i}{\hbar}[\wp_{ab}E\rho_{ba} - \text{к.с.}],$$

$$\dot{\rho}_{bb} = -\gamma_b\rho_{bb} - \frac{i}{\hbar}[\wp_{ab}E\rho_{ba} - \text{к.с.}],$$

$$\dot{\rho}_{ab} = -(i\omega + \gamma)\rho_{ab} - \frac{i}{\hbar}\wp_{ab}E(z,t)(\rho_{aa} - \rho_{bb}), \quad \gamma_{ab} = (\gamma_a + \gamma_b)/2 \quad \gamma = \gamma_{ab} + \gamma_{\text{ph}}$$

The $\gamma_{\rm ph}$ >> γ_{ab} , Ω_R limit

Wojciech Żurek

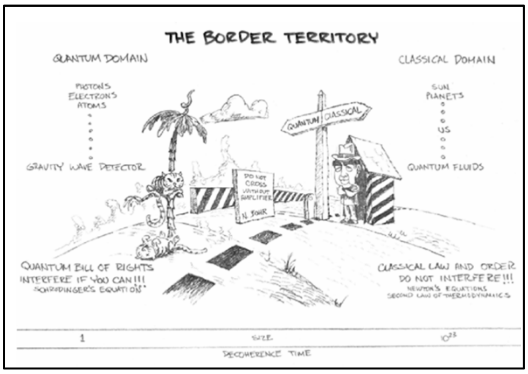
$$\rho_{ab} \approx 0 \quad \rho_{aa}, \rho_{bb} \neq 0$$

$$\rho = |\psi\rangle\langle\psi|$$
$$= |C_a|^2|a\rangle\langle a| + C_a C_b^*|a\rangle\langle b| + C_b C_a^*|b\rangle\langle a| + |C_b|^2|b\rangle\langle b|.$$

$$\Downarrow$$

$$\rho = |C_a|^2|a\rangle\langle a| + |C_b|^2|b\rangle\langle b|.$$

Classical state !



W. Zurek, Decoherence and the Transition From Quantum to Classical, Phys. Today 44 (10), 36 (1991)

The von Neumann Entropy

Benjamin Schumacher

Definition

$$S(\rho) = -\operatorname{Sp}\widehat{\rho}\log_2\widehat{\rho} \geqslant 0, \qquad \operatorname{Sp}\widehat{\rho} = 1$$

For pure state $S(\rho) = 0$.

Entropy is minimal that means complete information about qubit

The maximal value of $S(\rho)$ is determined by equal probability distribution of D density matrix eigenvalues

$$S(\rho) \leqslant \log_2 D$$

Fidelity of quantum state transfer is $F = \operatorname{Sp}\widehat{\rho}_{in}\,\widehat{\rho}_{out} \leqslant 1.$

For pure (noise less) state $\widehat{\rho}_{out} = \widehat{\rho}_{in}$,

$$F = 1.$$

The Shannon Entropy

Definition $H = -\sum_n \rho_{nn} \log_2 \rho_{nn} \geqslant 0, \qquad \sum_n \rho_{nn} = \mathrm{Sp}\,\widehat{\rho}(n) = 1,$

$$S(\rho) \leqslant H.$$

Quantum information admits dense coding!

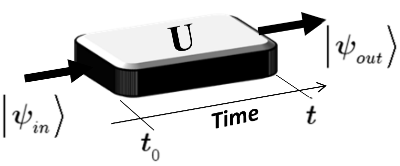
Shannon entropy for qubit state $|\psi\rangle = a|\uparrow\rangle + b|\downarrow\rangle$

$$H(|a|^2) = -|a|^2 \log_2 |a|^2 - (1-|a|^2)\log_2(1-|a|^2)$$

The maximum is $H = \log_2 2$ **and obtained for** $|a|^2 = |b|^2 = 1/2$.

Quantum Gates and Circuits

Lets consider some quantum device



Time Dependent Schrodinger Equation

$$\boxed{i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H|\psi(t)\rangle}$$

H *is Hamiltonian of processes which are happening in the device*

Formal solution
(without losses of generally)

$$\boxed{|\psi(t)\rangle = U(t, t_0)\,|\psi(t_0)\rangle}$$

$$\boxed{U(t, t_0) = \exp\left[-\frac{iH(t-t_0)}{\hbar}\right]}$$

is **unitary evolution operator** characterizing device performance

Main Properties of $U(t, t_0)$

$U^{+}(t, t_0) = \exp\left[\frac{iH(t-t_0)}{\hbar}\right] = U^{-1}(t, t_0), \qquad U(t_0, t_0) = I \qquad U^{+}(t, t_0)\; U(t, t_0) = 1$

Single Qubit Quantum Gates

NOT gate $\equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} : \widehat{\sigma_x}$

Single qubit rotations

$$U_X(\theta) = \exp(-i\theta X) = \begin{pmatrix} \cos\theta & -i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

$$U_Y(\phi) = \exp(-i\phi Y) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$U_Z(\varphi) = \exp(-i\varphi Z) = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

U

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hadamard Gate

Definition of Hadamard operator

$$H \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{\sqrt{2}}\left(\widehat{\sigma}_z + \widehat{\sigma}_x\right)$$

$$\widehat{H}|0\rangle = \widehat{H}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{1/2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sqrt{1/2}\left(|0\rangle + |1\rangle\right)$$

$$\widehat{H}|1\rangle = \widehat{H}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{1/2}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sqrt{1/2}\left(|0\rangle - |1\rangle\right)$$

$$\boxed{\widehat{H}|x\rangle = \sqrt{1/2}\sum_{y=0,1}(-1)^{x\cdot y}|y\rangle}$$

Jacques Hadamard

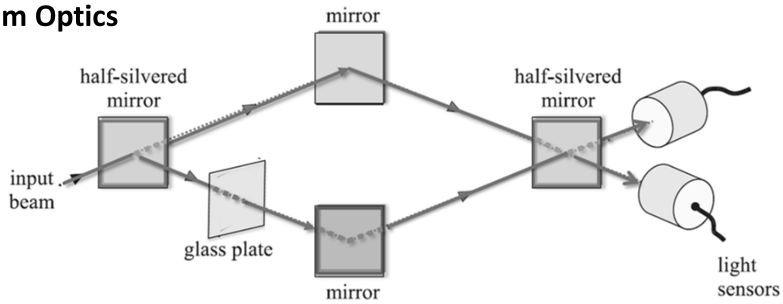
Hadamard operation for *L* qubits

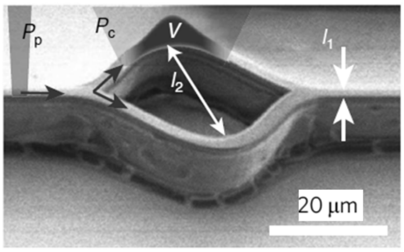
$$\boxed{\widehat{W}|x\rangle = \sqrt{1/N}\sum_{y=0}^{N-1}(-1)^{x\cdot y}|y\rangle,} \qquad N = 2^L.$$

$x\cdot y \equiv \sum_{n=0}^{N-1}(x_n \cap y_n)$ **Is bitwise operation**

Mach-Zehnder interferometer (MZI)

In Quantum Optics



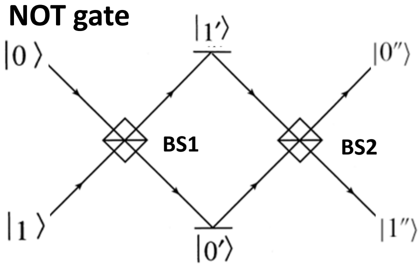


In Semiconductor physics – exciton polariton MZI

C. Sturm, et al, Nat. Comms. 5:3278, 2014

Gate Realization with MZI

NOT gate

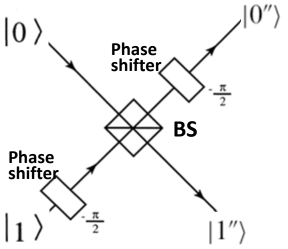


After **BS1**
$$|0'\rangle = 2^{-1/2}|0\rangle + 2^{-1/2}i|1\rangle,$$
$$|1'\rangle = 2^{-1/2}i|0\rangle + 2^{-1/2}|1\rangle,$$

After **BS2**
$$|0''\rangle = 2^{-1/2}|0'\rangle + 2^{-1/2}i|1'\rangle,$$
$$|1''\rangle = 2^{-1/2}i|0'\rangle + 2^{-1/2}|1'\rangle.$$

In matrix form
$$\boxed{\begin{pmatrix}|0''\rangle\\|1''\rangle\end{pmatrix} = i\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}\begin{pmatrix}|0\rangle\\|1\rangle\end{pmatrix}}$$

Hadamard gate



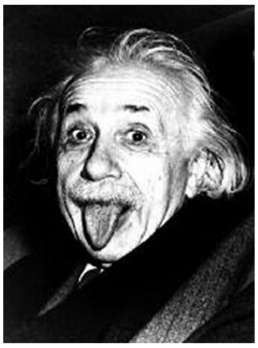
Output states
$$|0'\rangle = 2^{-1/2}|0\rangle + e^{-i\pi/2}2^{-1/2}i|1\rangle = 2^{-1/2}(|0\rangle + |1\rangle);$$
$$|1'\rangle = 2^{-1/2}ie^{-i\pi/2}|0\rangle + e^{-i\pi/2}2^{-1/2}e^{-i\pi/2}|1\rangle = 2^{-1/2}(|0\rangle - |1\rangle).$$

In matrix form
$$\boxed{\begin{pmatrix}|0'\rangle\\|1'\rangle\end{pmatrix} = 2^{-1/2}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}|0\rangle\\|1\rangle\end{pmatrix}.}$$

The Entanglement….

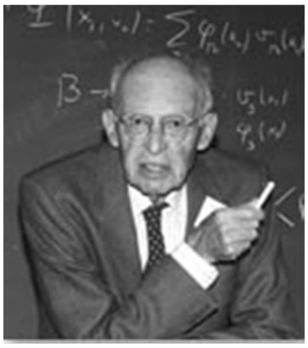
Einstein, Podolsky, Rosen (EPR) Paradox

Albert Einstein



Boris Podolsky

Nathan Rosen



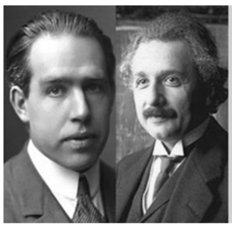
"If, without in any way disturbing a system, we can predict with certainty ... the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity."

The Einstein-Podolsky-Rosen Paper

- In 1935, Einstein, working with physicists Boris Podolsky and Nathan Rosen, published the paper, "*Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*"
- Einstein believed that, while quantum mechanics could be used to make highly accurate **statistical** predictions about experiments, it's an **incomplete theory** of physical reality.
- In this paper, they devised a clever thought experiment that "beat" the Uncertainty Principle. So they concluded that there must be more going on than quantum mechanics knew about, concluding:

> *The quantum-mechanical description of reality given by the wave function is not complete, that is, there must be **Hidden Variables** that we don't know about and hence don't measure that cause the uncertainty.*

Entanglement

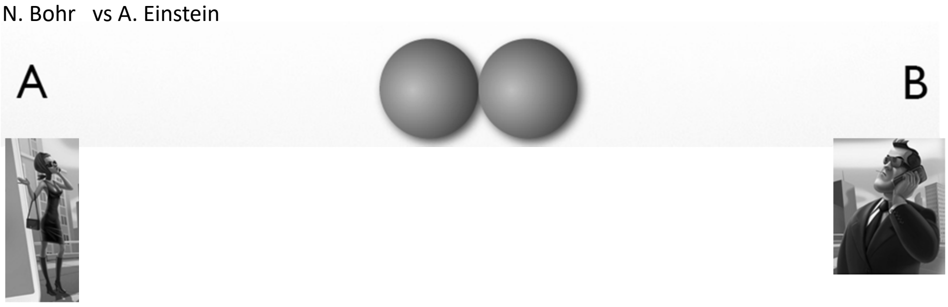


N. Bohr vs A. Einstein

Suppose that we have two-atom molecule that decays in two atoms

The spin of molecule is $\vec{S} = 0$

Two-atom molecule



Quantum Entanglement as Resource of QI

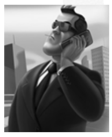
After decay each of the atom possess some spin

Question: What is the state of atoms after decay?

Total spin of two atoms is $\vec{S}_A + \vec{S}_B = 0$

A   B

$$|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}\Big(\underbrace{|\bigcirc\rangle_A|\bigcirc\rangle_B}_{|0\rangle} + \underbrace{|\bigcirc\rangle_A|\bigcirc\rangle_B}_{|1\rangle}\Big)$$



One qubit (ebit) shared between two channel

Measurement Results

A   B

$$|\psi\rangle_{AB} \xrightarrow{\text{collapse}} |\bigcirc\rangle_A |\bigcirc\rangle_B$$

or,

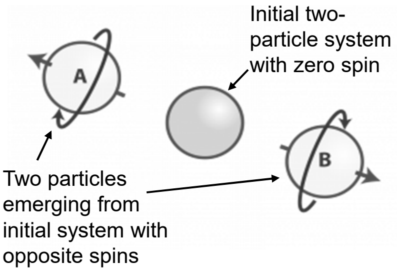
A   B

$$|\psi\rangle_{AB} \xrightarrow{\text{collapse}} |\bigcirc\rangle_A |\bigcirc\rangle_B$$

It appears that particles simply do not have properties until we measure them.

<u>However!</u> ***When I measure particle A, I cannot modify instantaneously the result of measuring particle B. There is no action at distance (faster than light)***

EPR paradox



Since initial molecule has spin $\vec{S} = 0$,

We have $\underbrace{0}_{\text{Before decay}} = \underbrace{\vec{S}_A + \vec{S}_B}_{\text{After decay}}$

i.e. $\vec{S}_A = -\vec{S}_B$

Measure the vertical spin $|\updownarrow\rangle_A$ component of particle A and the horizontal spin component $|\leftrightarrow\rangle_B$ of particle B

Because the particle A measurement determines **both** particles' **vertical** spin components, and the particle B measurement determines **both** particles' **horizontal** spin components, haven't we determined two components of each particle's spin? And beaten the Quantum Mechanics?

$\Psi = \frac{1}{\sqrt{2}}|\uparrow\rangle_A|\downarrow\rangle_B + \frac{1}{\sqrt{2}}|\downarrow\rangle_A|\uparrow\rangle_B$ or $\Psi = \frac{1}{\sqrt{2}}|\rightarrow\rangle_A|\leftarrow\rangle_B + \frac{1}{\sqrt{2}}|\leftarrow\rangle_A|\rightarrow\rangle_B$

Local realism is out

John Bell showed in a 1964 paper entitled "On the Einstein Podolsky Rosen paradox," that local realism leads to a series of requirements—known as **Bell's inequalities**.

John Bell (1928-1990)

Alain Aspect, 1947

Alain Aspect has performed numerous beautiful experiments, proving conclusively that our universe violates Bell's Inequalities big time. And quantum mechanics explains the effects quite nicely.

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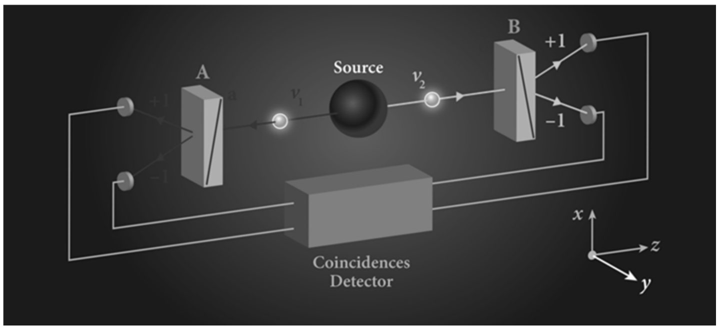
EPR Scheme With Photons



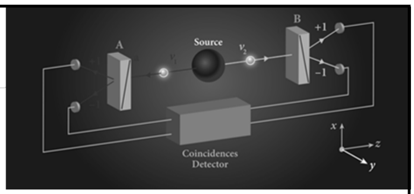
Alain Aspect; Philippe Grangier; Gérard Roger (1981), "Experimental Tests of Realistic Local Theories via Bell's Theorem", Phys. Rev. Lett., 47 (7): 460–3,

Bell parameter: $S = \langle AB\rangle - \langle AB'\rangle + \langle A'B\rangle + \langle A'B'\rangle$

where $A, A', B, B' , = \pm 1$ dichotomy variables,

•

EPR Analysis; Classical Picture



Predictions of classical theory (if classical variables are statistically independent)

$$|\boldsymbol{S}| = | < \boldsymbol{A} > (< \boldsymbol{B} > - < \boldsymbol{B'} >) \ + < \boldsymbol{A'} > (< \boldsymbol{B} > + < \boldsymbol{B'} >)|$$

Hidden variable theory predicts that (CHSC inequality)

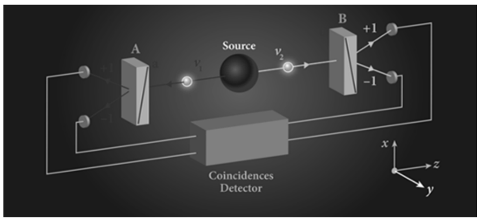
J.F. Clauser; M.A. Horne; A. Shimony; R.A. Holt (1969)

$$|\boldsymbol{S}| \leq \boldsymbol{2}$$

Proof

$$|S| = |\langle AB\rangle - \langle AB'\rangle + \langle A'B\rangle + \langle A'B'\rangle| =$$
$$= |\sum_{A,A',B,B'} p(A,A',B,B')(A(B-B') \ + A'(B+B'))| \ \leq$$
$$= \sum_{A,A',B,B'} p(A,A',B,B')|A(B-B') \ + A'(B+B')| \ =$$
$$=2 \sum_{A,A',B,B'} p(A,A',B,B') = 2$$

EPR Analysis; Quantum Picture



Quantum theory predicts (Tsirelson theorem) B. Cirel'son, 1980

$$|\boldsymbol{S}| \leq \mathbf{2\sqrt{2}}$$

Two particles are in entangled states

$$|\boldsymbol{\Psi}> = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}} \,|\updownarrow>_{\boldsymbol{B}}|\leftrightarrow>_{\boldsymbol{A}} + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\,|\leftrightarrow>_{\boldsymbol{B}}|\updownarrow>_{\boldsymbol{A}} \neq |\psi>_{\boldsymbol{B}}|\psi>_{\boldsymbol{A}}$$

Practice

Problem: Prove Tsirelson Theorem for two-particle quantum state

$$|\boldsymbol{\Psi}> = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\,|\updownarrow>_{\boldsymbol{B}}|\leftrightarrow>_{\boldsymbol{A}} + \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\,|\leftrightarrow>_{\boldsymbol{B}}|\updownarrow>_{\boldsymbol{A}} \equiv > = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}|\mathbf{0}>_{\boldsymbol{B}}|\mathbf{1}>_{\boldsymbol{A}} - \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\,|\mathbf{1}>_{\boldsymbol{B}}|\mathbf{0}>_{\boldsymbol{A}}$$

Bob measures $\hat{B} = \sigma_x^{(B)}$ ***and*** $\hat{B}' = \sigma_z^{(B)}$

Alice measures $\hat{A} = \frac{1}{\sqrt{2}}\left(\sigma_z^{(A)} - \sigma_x^{(A)}\right), \quad \hat{A}' = -\frac{1}{\sqrt{2}}\left(\sigma_z^{(A)} + \sigma_x^{(A)}\right)$

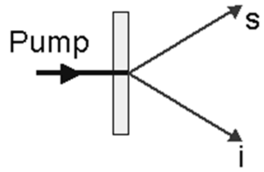
Now we should calculate $S = \langle AB \rangle - \langle AB' \rangle + \langle A'B \rangle + \langle A'B' \rangle$

After some algebra we obtain $|\boldsymbol{S}| = \mathbf{2\sqrt{2}}$

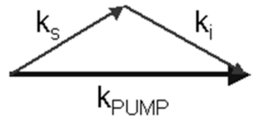
Parametric Downconversion: Type I

Entangled Photons in Quantum Optics

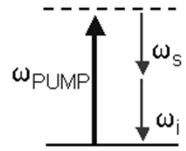
Downconversion



Momentum is conserved..



..as well as energy



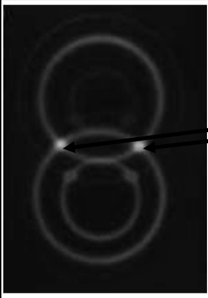
$\varphi_{PUMP} = \varphi_s + \varphi_i$

- A pump photon is spontaneously converted into two lower frequency photons in a material with a nonzero $\chi^{(2)}$

Y. R. Shen, The Principles of Nonlinear Optics , 2002

Parametric Downconversion: Type II

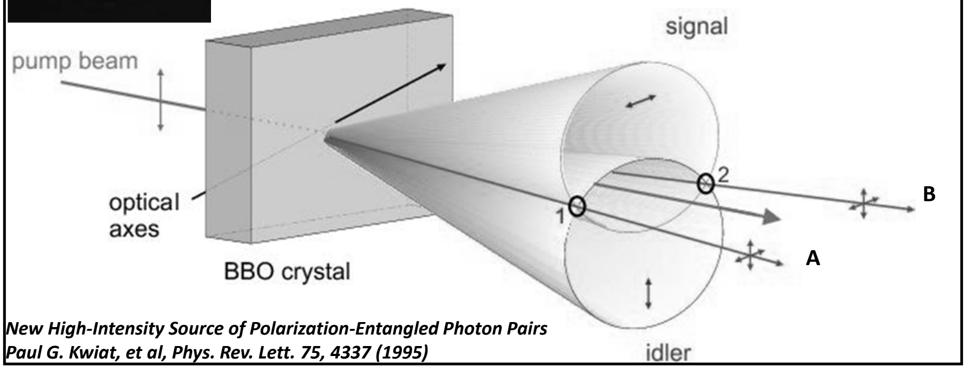
Photon Polarization Entanglement



Degenerate (Entangled) Case: $\omega_s=\omega_i$

$$|H\rangle_A |V\rangle_B + |V\rangle_A |H\rangle_B$$

Anton Zeilinger



New High-Intensity Source of Polarization-Entangled Photon Pairs
Paul G. Kwiat, et al, Phys. Rev. Lett. 75, 4337 (1995)

Two-qubit states

<u>Example</u>: state of two quantum subsystems

A

B

$$|\psi_A\rangle = \alpha_A|0\rangle + \beta_A|1\rangle \qquad |\psi_B\rangle = \alpha_B|0\rangle + \beta_B|1\rangle$$

AB

Tensor product

$$|\psi_{AB}\rangle = |\psi_A\rangle\otimes|\psi_B\rangle = (\alpha_A|0\rangle+\beta_A|1\rangle)\otimes(\alpha_B|0\rangle+\beta_B|1\rangle)$$
$$= \alpha_A\alpha_B|00\rangle + \alpha_A\beta_B|01\rangle + \beta_A\alpha_A|10\rangle + \beta_A\beta_B|11\rangle$$

Separable state

Non-separable, or, Entangled States

If joint state of two systems cannot be expressed as

$$|\psi_{AB}\rangle \neq |\psi_A\rangle \otimes |\psi_B\rangle$$

Such states are called *entangled state (of two qubits)*

Example: Bell states

$$|\psi_{AB}\rangle_{\pm} = \frac{1}{\sqrt{2}}\left[|0\rangle_A|1\rangle_B \pm |1\rangle_A|0\rangle_B\right] \qquad |\phi_{AB}\rangle_{\pm} = \frac{1}{\sqrt{2}}\left[|0\rangle_A|0\rangle_B \pm |1\rangle_A|1\rangle_B\right]$$

In general - bipartite state is $|\Psi\rangle = \sum_{i}^{d_A}\sum_{j}^{d_B}\alpha_{ij}|i\rangle_A|j\rangle_B$

How we can quantify how much entanglement there is between two quantum systems?

Or,

How many 'Bell pairs of entanglement' does a given state correspond to?

Schmidt Decomposition

Erhard Schmidt

Lets suppose that we have bipartite state

$$|\Psi\rangle = \sum_{i}^{d_A}\sum_{j}^{d_B} \alpha_{ij} |i\rangle_A |j\rangle_B \implies |\Psi_\lambda\rangle = \sum_{k}^{\min\{d_B, d_A\}} \lambda_k |v_k\rangle |w_k\rangle$$

where $d_{A,B} = \dim(H_{A,B})$ $\quad \sum_{k}^{\min\{d_B, d_A\}} \lambda_k^2 = 1,$ $\quad \lambda_k^2$ are Schmidt numbers

We can always find a change of basis on Alice's side, and another change of basis on Bob's side, that puts the state into the simpler form.

Schmidt decomposition is SVD *(that presumes factorization of a matrix)*

$$A = \begin{bmatrix} \alpha_{11} & & \alpha_{1n} \\ & \ddots & \\ \alpha_{n1} & & \alpha_{nn} \end{bmatrix}:$$

We can multiply by two unitary matrices, one on each side, to get a diagonal matrix: $UAV = \Lambda$

U and **V** matrixes can be find out applying linear algebra procedures.

Measuring in the $\{|v_k\rangle|w_k\rangle\}$ basis would then yield the probability distribution. $\begin{bmatrix} |\lambda_1|^2 \\ \vdots \\ |\lambda_n|^2 \end{bmatrix}$

Shannon Entropy vs von Neumann Entropy

$$H(D) = \sum_i |\lambda_i|^2 \log \frac{1}{|\lambda_i|^2}$$

Von Neumann entropy

$$S(\rho) = \sum_{i=1}^{n} \lambda_i \log_2 \frac{1}{\lambda_i}$$

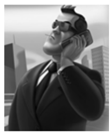
- One can say that von Neumann entropy *is* the Shannon entropy of the vector of eigenvalues of the density matrix of ρ.
- *If we diagonalize the density matrix, the diagonal represents probability distribution over possible outcomes, and taking the Shannon entropy of that distribution gives us the von Neumann entropy of our quantum state.*

Entropy Entanglement

defines degree of quantum entanglement measure for two- and more body quantum state

Suppose Alice and Bob share a **bipartite pure** state

$$|\Psi\rangle = \sum_{i}^{d_A}\sum_{j}^{d_B} \alpha_{ij} |i\rangle_A |j\rangle_B$$



To quantify the **entanglement entropy**, we'll trace out Bob's part, and look at the von Neumann entropy of Alice's side, $E(\rho)$, in effect asking: if Alice made an optimal measurement, how much could she learn about Bob's state?

Definition Von Neumann Entropy

$$E(\rho) = -\,\mathrm{Sp}\,\widehat{\rho}_A \log_2 \widehat{\rho}_A = -\,\mathrm{Sp}\,\widehat{\rho}_B \log_2 \widehat{\rho}_B.$$

Reduced density matrixes $\widehat{\rho}_A = \mathrm{Sp}_B\,|\psi_{AB}\rangle\langle\psi_{AB}|,\qquad \widehat{\rho}_B = \mathrm{Sp}_A\,|\psi_{AB}\rangle\langle\psi_{AB}|,$

Entropy Entanglement

<u>Example.</u> Spin qubit

Definition Von Neumann Entropy

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Reduced density matrixes $\widehat{\rho}_A = \mathrm{Sp}_B\,|\psi_{AB}\rangle\langle\psi_{AB}|,\qquad \widehat{\rho}_B = \mathrm{Sp}_A\,|\psi_{AB}\rangle\langle\psi_{AB}|,$

(i) $|\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$

$\Rightarrow \rho_A = \mathrm{Tr_B}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right]\cdot\left[\langle\uparrow|_A + \langle\downarrow|_A\right]$

Not Entangled

$E = 0$

(ii) $|\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B\right] / \sqrt{2}$

$\Rightarrow \rho_A = \mathrm{Tr_B}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A\langle\uparrow|_A + |\downarrow\rangle_A\langle\downarrow|_A\right]$

Entangled

$E = \log 2 = 1$

Maximally Entangled Qubits

Schmidt decomposition $|\psi_{AB}\rangle = \sum_{i=1,2} c_i |\alpha_i\rangle \otimes |\beta_i\rangle, \qquad \sum_{i=1,2} c_i^2 = 1,$

Considering $\widehat{\rho}_A = \mathrm{Sp}_B\, |\psi_{AB}\rangle\langle\psi_{AB}|, \qquad \widehat{\rho}_A = \sum_{i=1,2} c_i^2 |\alpha_i\rangle\langle\alpha_i|.$

c_i^2 *are eigenvalues of* $\widehat{\rho}_A$

Entropy entanglement (for pure state)

$$E(\rho) = -\,\mathrm{Sp}\,\widehat{\rho}_A \log_2 \widehat{\rho}_A = -\,\mathrm{Sp}\,\widehat{\rho}_B \log_2 \widehat{\rho}_B = -\sum_{i=1,2} c_i^2 \log_2 c_i^2$$

If $c_1^2 = 0$ ***or*** $c_2^2 = 0$ ***the state*** $|\psi_{AB}\rangle$ ***is separable, and***

$$|\psi_{AB}\rangle\langle\psi_{AB}| = |\alpha_1\rangle\langle\alpha_1| \otimes |\beta_1\rangle\langle\beta_1|$$

Concurrence

William Wootters

Lets consider two qubit (pure) state

$$|\psi_{AB}\rangle = a|0_A 0_B\rangle + b|0_A 1_B\rangle + c|1_A 0_B\rangle + d|1_A 1_B\rangle = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$\langle\Psi_{AB}|\Psi_{AB}\rangle = |a|^2 + |b|^2 + |c|^2 + |d|^2 = 1$$

Definition $\boxed{C = 2(ad - bc) = 2c_1c_2 = 2c_1\sqrt{1 - c_1^2}.}$

and $c_1^2 = \dfrac{1 + \sqrt{1 - C^2}}{2}, \qquad 0 \leqslant C \leqslant 1$

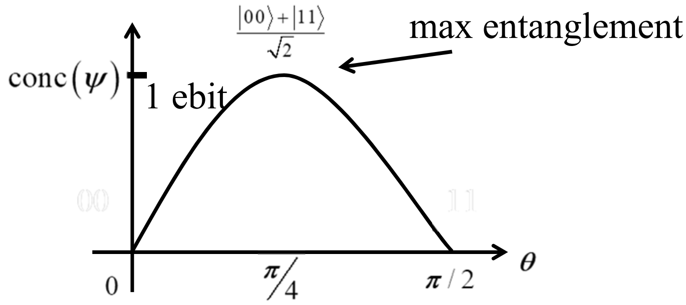
For $c_2 = c_1 = \sqrt{1/2}.$ ***we obtain maximally entangled state with*** $C = 1$

$E_{\max}(\rho) = \log_2 2 = 1$ ***ebit***

$$|\psi_{AB}\rangle = \frac{1}{\sqrt{2(1 + |\gamma|^2)(1 + |\delta|^2)}}\Big\{(1 + \gamma\delta)(|0_A 0_B\rangle + |1_A 1_B\rangle) + \\ +(\gamma - \delta)(|1_A 0_B\rangle - |0_A 1_B\rangle)\Big\}.$$

Entanglement vs Concurrence

$$|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle \qquad \Rightarrow \mathrm{conc}(\psi) = 2|\sin(\theta)\cos(\theta)| = |\sin(2\theta)|$$



✓entangled spins are non-separable
✓entanglement is base-independent

Entanglement with Mixed States

Entanglement of formation
is the number of ebits that Alice and Bob need to create one copy of the state ρ_{AB} , in the limit where they're creating many copies of it, and assuming they're allowed unlimited local operations and classical communication (called "LOCC") for free

$$E_F(\rho) = \min_{\rho} \sum_n p_n E(\rho_n),$$

$$\hat{\rho} = \sum_n p_n \hat{\rho}_n$$

$$E_F = H\Big(\frac{1+\sqrt{1-C^2}}{2}\Big)$$

Where $H(x) = -x\log_2 x - (1-x)\log_2(1-x)$ ***is Shannon entropy for two qbits***

Entanglement purification $E_D(\rho_{AB})$
is the number of ebits that Alice and Bob can extract per copy of ρ_{AB} , again in the limit where they're given many copies of it, and assuming local operations and classical communication are free

$$E_F \geq E_D$$

If you're given as input a density matrix for a bipartite state, then deciding whether represents a separable or entangled state is an NP-hard problem!

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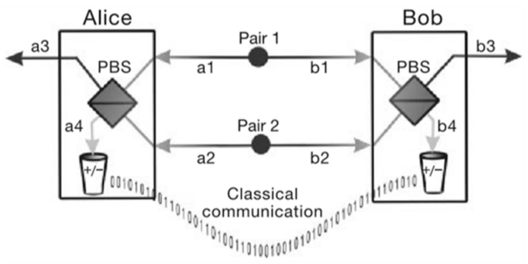
Purification Protocol

Jian-Wei Pan, S. Gasparoni, R. Ursin, G. Weihs & A. Zeilinger, *Nature,* vol. 423, p. 417–422 (2003)

Entanglement purification aimed at extracting a smaller number of highly entangled pairs out of a large number of less-entangled pairs using only local operations and classical communication.



Alice and Bob shares the state

$$\rho_{ab} = F|\Phi^+\rangle_{ab}\langle\Phi^+| + \underbrace{(1-F)|\Psi^-\rangle_{ab}\langle\Psi^-|}_{\textbf{unwanted admixture}}$$

$F = \langle\Phi^+|\rho_{ab}|\Phi^+\rangle$ is fidelity in respect of state

After purification

$$\boxed{\begin{aligned}\rho'_{ab} &= F'|\Phi^+\rangle_{ab}\langle\Phi^+| + (1-F')|\Psi^-\rangle_{ab}\langle\Psi^-| \\ F' &= F^2/[F^2+(1-F)^2] > F \text{ (for } F > 1/2)\end{aligned}}$$

Scheme of the principle of entanglement purification using linear optics.

We start with two less entangled pairs shared by Alice and Bob who superpose their photons on a polarizing beam splitter(PBS). Alice and Bob keep only those cases where there is exactly one photon in each output mode ('four-mode cases'). They perform a polarization measure-ment in the + ,- basis in modes a4 and b4, where $|+\rangle = (1/\sqrt{2})(|H\rangle + |V\rangle)$ and $|-\rangle = (1/\sqrt{2})(|H\rangle - |V\rangle)$. : Depending on the results, Alice performs a specific operation on the photon in mode a3. After this procedure, the remaining pair in modes a3 and b3 will have a higher degree of entanglement than the two original pairs.

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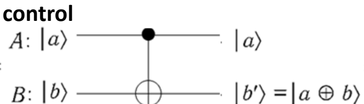
Two Qubit CNOT Gates



$$\text{CNOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

<u>**Computational rule**</u> If **control** qubit is in $|0\rangle$, **target** qubit **does not** change the state

If **control** qubit is in $|1\rangle$, target qubit **change** the state

Truth table

Before		After	
Control	Target	Control	Target
$\|0\rangle$	$\|0\rangle$	$\|0\rangle$	$\|0\rangle$
$\|0\rangle$	$\|1\rangle$	$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$	$\|1\rangle$	$\|1\rangle$
$\|1\rangle$	$\|1\rangle$	$\|1\rangle$	$\|0\rangle$

Reversible gate

Example

Input state $|\psi\rangle_{in} = \left(\alpha|0\rangle_a + \beta|1\rangle_a\right)|0\rangle_b$

Output state

$$\boxed{|\psi\rangle_{out} = CNOT|\psi\rangle_{in} = \alpha|0\rangle_a|0\rangle_b + \beta|1\rangle_a|1\rangle_b}$$

Output state is entangled state!

No-cloning Theorem

Wojciech Żurek

Theorem

Unknown quantum state cannot be cloned

Wootters W.K., Zurek W.H. A Single Quantum Cannot Be Cloned // Nature, 1982, v. 299, № 1982, pp. 802-803.

Proof *Suppose that we have cloning machine.*

For two arbitrary states $|a_m\rangle$ *and* $|b_m\rangle$ *we should have simultaneously*

$$|a_m 0_x\rangle \Rightarrow \sqrt{1/2}|a_m a_x\rangle$$
$$|b_m 0_x\rangle \Rightarrow \sqrt{1/2}|b_m b_x\rangle.$$

$|\psi_{in}\rangle$ $|\psi_{out}\rangle$

On the other hand for superposition state

$$|c_m\rangle = \frac{1}{\sqrt{2}}\left(|a_m\rangle + |b_m\rangle\right).$$

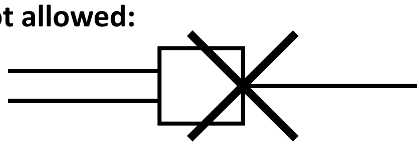
$$|c_m 0_x\rangle = \sqrt{1/2}\Big(|a_m 0_x\rangle + |b_m 0_x\rangle\Big) \Rightarrow 1/2\Big(|a_m a_x\rangle + |b_m b_x\rangle\Big) \neq$$
$$\neq |c_m c_x\rangle = 1/2\Big(|a_m a_x\rangle + |b_m b_x\rangle + \boxed{|a_m b_x\rangle + |b_m a_x\rangle}\Big).$$

Non-zero "extra" terms for arbitrary (non-orthogonal) states!

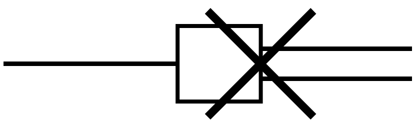
Features of Quantum Circuits

1. **No loops are allowed; quantum circuits are acyclic**

2. Fan-in is not allowed:

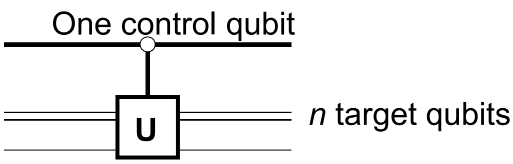


3. Fan-out is not allowed:



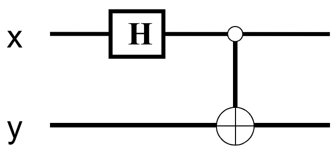
Generalised Control Gate

- Any quantum gate **U** can be converted into a controlled gate:



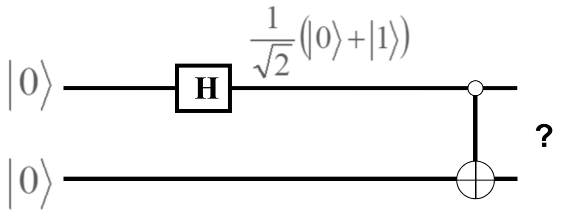
If the control qubit is "high," **U** is applied to the targets. **CNOT** is the Controlled-**X** gate!

The Bell State Circuit



x	y	Output (Bell States)
$\|0\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle+\|11\rangle)$
$\|0\rangle$	$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle+\|10\rangle)$
$\|1\rangle$	$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|00\rangle-\|11\rangle)$
$\|1\rangle$	$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|01\rangle-\|10\rangle)$

The Bell State Circuit By Example

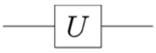


$$\mathbf{CNOT}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\cdot|0\rangle\right)=\mathbf{CNOT}\left(\frac{|00\rangle+|10\rangle}{\sqrt{2}}\right)$$

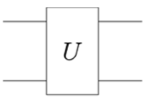
$$=\frac{1}{\sqrt{2}}\left(\mathbf{CNOT}|00\rangle+\mathbf{CNOT}|10\rangle\right)$$

$$=\frac{1}{\sqrt{2}}\left(|00\rangle+|11\rangle\right)$$

Quantum Circuits



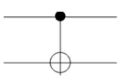
single qubit rotations



two qubit rotations

control

target

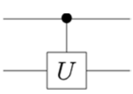


controlled-NOT $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

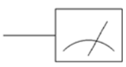
$|00\rangle \rightarrow |00\rangle$
$|01\rangle \rightarrow |01\rangle$
$|10\rangle \rightarrow |11\rangle$
$|11\rangle \rightarrow |10\rangle$

control

target



controlled-U $= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{00} & U_{01} \\ 0 & 0 & U_{10} & U_{11} \end{bmatrix}$

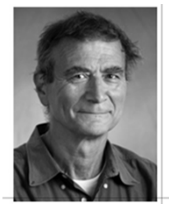


measurement in the $|0\rangle, |1\rangle$ basis

Universality

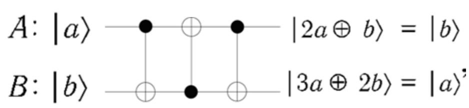
- Any quantum computation can be performed by a circuit consisting of Hadamard, phase, rotation by $\pi/8$ and controlled NOT gates.

Quantum Circuits



Tomasso Toffoli

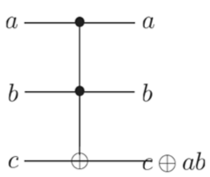
Scheme



Matrix operator

$$\mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

CCNOT (Toffoli) three qubit gate



Scheme

$$\mathrm{CCNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Matrix operator

Inputs			Outputs		
a	b	c	a'	b'	c'
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

Truth table

Practice

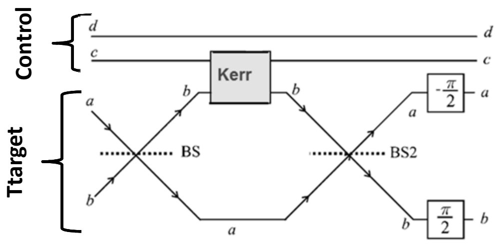
Worked Exercise: Show that all permutation matrices are unitary. Use this to show that any classical reversible gate has a corresponding unitary quantum gate.

Challenge exercise: Show that the Toffoli gate can be built up from controlled-not and single-qubit gates.

Cf. the classical case: it is not possible to build up a Toffoli gate from reversible one- and two-bit gates.

Optical Realization of CNOT gate $|\psi\rangle_{out} = U_{CNOT}|\psi\rangle_{in}$

Scheme



Encoding

$$|0\rangle|0\rangle = |0\rangle_a|1\rangle_b|0\rangle_c|1\rangle_d$$
$$|0\rangle|1\rangle = |0\rangle_a|1\rangle_b|1\rangle_c|0\rangle_d$$
$$|1\rangle|0\rangle = |1\rangle_a|0\rangle_b|0\rangle_c|1\rangle_d$$
$$|1\rangle|1\rangle = |1\rangle_a|0\rangle_b|1\rangle_c|0\rangle_d$$

Where $\boxed{\hat{U}_{\text{C-NOT}} = \exp\left(-i\frac{\pi}{2}\hat{a}^\dagger\hat{a}\right)\exp\left(i\frac{\pi}{2}\hat{b}^\dagger\hat{b}\right)\hat{U}_{\text{F}}(\pi)}$ and $U_F = U_{BS2}U_{Kerr}U_{BS1}$

We take $\hat{U}_{\text{BS1}} = \exp\left[i\frac{\pi}{4}(\hat{a}^\dagger\hat{b} + \hat{a}\,\hat{b}^\dagger)\right]$ $\quad U_{BS2} = U_{BS1}^\dagger$ $\quad \hat{U}_{\text{Kerr}}(\eta) = \exp(i\,\eta\,\hat{b}^\dagger\hat{b}\,\hat{d}^\dagger\hat{d})$,

With

$$\hat{U}_{\text{F}}(\eta) = \exp\left[i\frac{\eta}{2}\hat{c}^\dagger\hat{c}(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b})\right]\exp\left[\frac{\eta}{2}\hat{c}^\dagger\hat{c}(\hat{a}^\dagger\hat{b} - \hat{b}^\dagger\hat{a})\right].$$

$$\hat{U}_{\text{F}}(\eta)|0\rangle|0\rangle = |0\rangle|0\rangle,$$
$$\hat{U}_{\text{F}}(\eta)|1\rangle|0\rangle = |1\rangle|0\rangle,$$
$$\hat{U}_{\text{F}}(\eta)|0\rangle|1\rangle = i|1\rangle|1\rangle,$$
$$\hat{U}_{\text{F}}(\eta)|1\rangle|1\rangle = -i|0\rangle|1\rangle,$$

Photon state switching requires $\eta = \pi$ nonlinear phase shift per photon that is practically impossible up to now

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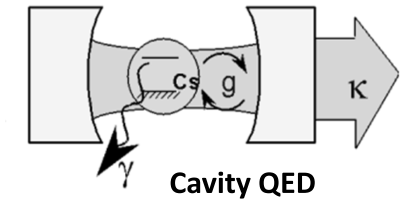
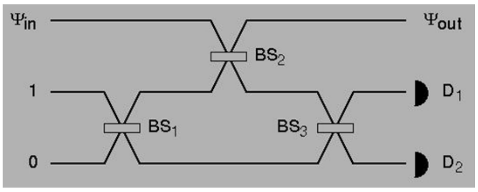
Two ways to Optical Quantum Computing

1. Enhance Nonlinear Interaction with a Cavity or EIT, photon blocade — Kimble, Lukin, *et al.*



Cavity QED

II. Exploit Nonlinearity of Measurement — Knill, Laflamme, Milburn, Nemoto, *et al.*



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What is KLM ?

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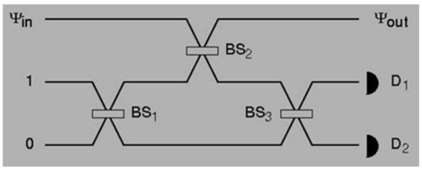
Measurement-based Quantum computing (MBQC)

KLM approach for Linear optics quantum computing

E. Knill, R. Lafamme & G. J. Milburn, Nature, 2001

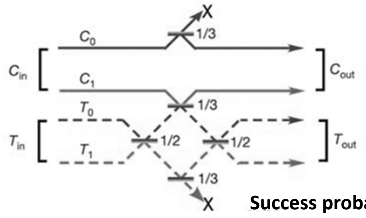
BS + Phase Shifters + Detectors can be Used to Construct 2 X CSIGN = CNOT Gate and a Quantum Computer

Non-deterministic sign (NS) gate



$$\alpha|0\rangle + \beta|1\rangle + \gamma|2\rangle \implies \alpha|0\rangle + \beta|1\rangle - \gamma|2\rangle$$

CNOT Gate



Non-unitary operation of photon detection posses to avoid strong photon-photon interaction

Success probability: p=1/9

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The End