Quantum Information Axiomatics

Course Short Content

Introduction

1. **Quantum approach to information,**
2. **Superpositions, Uncertainties, probabilities ,etc**
3. **Quantum gates,**
4. **Quantum algorithms,**
5. **Quantum information with continuous variables,**
6. **Quantum computers and current quantum technologies**

Basic Literature

1. Michael A. Nielsen, Isaac L. Chuang, Quantum Computation and Quantum Information Cambridge University Press, 2000
2. Gregg Jaeger, Quantum Information, An Overview, Springer Science+Business Media, LLC, 2007
3. Christopher Gerry, Peter Knight, Introductory Quantum Optics, Cambridge University Press, 2005
4. L. Mandel and E. Wolf, Optical Coherence and Quantum Optics, Cambridge University Press, 1995.
5. D. F. Walls, Gerard J. Milburn, Quantum Optics Springer Science & Business Media, 2008.
6. S. M. Barnett and P. M. Radmore, *Methods in Theoretical Quantum Optics,* Oxford, 1997
7. Anthony Sudbery. Quantum Mechanics and the Particles of Nature: An Outline for Mathematicians Cambridge University Press 1986

QC; Some historical retrospective

1961 — 1982 — 1994 — 2011 → ?

1961 Landauer Principle

Rolf Landauer

Energy losses for one bit

$$W = k_B T \ln 2$$

k_B **Is Boltzamann constant**

T - **Is temperature**

При T = 300 K

W=0,017 eV

1982 Quantum Simulators

Richard Feynman proposed to use quantum particle system to solve computational tasks

1994 First quantum algorithm

P. Shor proved that the factorization problem of large numbers by a quantum computer can be solved in significantly less time than a classical computer.

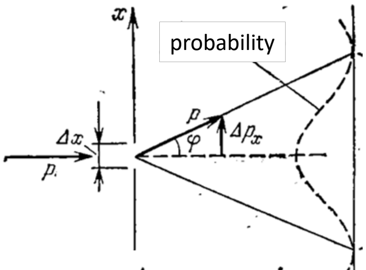
2011 Hybrid (Quantum-classical) simulator

D-Wave The Quantum Computing Company™

128-qubits "Quantum" simulation of D-Wave to solve the discrete optimization problem

Wave-Particle Dualism

Particle diffraction (wave-particle duality)



$$\Delta p_x = p \sin \varphi.$$

$$\boxed{p = \frac{2\pi\hbar}{\lambda_{dB}}}$$

Lui de Broglie

The most probable speed is $v_p = \sqrt{\frac{2kT}{m}}$

Particle wave properties

$$\sin \varphi = \lambda / \Delta x \qquad \Delta p_x = p\lambda/\Delta x.$$

$$\boxed{\lambda_{dB} = \frac{2\pi\hbar}{\sqrt{2mkT}}}$$

$$\boxed{\Delta x \cdot \Delta p_x = p\lambda = 2\pi\hbar}$$

Hilbert Space

David Hilbert (1862 – 1943)

Definition:

H is a complete infinite-dimensional linear vector space with a definite complex scalar product and finite norm.

Paul Adrien Maurice Dirac (1902 – 1984)

Dirac notation

Bracket = "bra" x "ket"

< ....> = <... |... > = "<... |" x "|... >"

Notation od quantum state (vector in H) is a ket vector $\left|\psi\right\rangle$

Any element of the dual space we will call a bra vector $\left\langle\psi\right|$

2.B.2

Hilbert space properties

1. The norm is $\langle\psi|\psi\rangle = 1$

2. Scalar product for any two vectors $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

3. Superposition principle

If $|\phi\rangle, |\psi\rangle$ are vectors from H , the linear combination is also vector from H

$$|\Psi\rangle = C_1|\phi\rangle + C_2|\psi\rangle$$

C_1, C_2 *are coefficients which are not equal to 0.*

4. Any vector state can be expanded as $|\psi\rangle = \sum_{j=1}^{D} C_j|j\rangle$

Collection of linearly independent vectors $\{|j\rangle\}$ form a basis of H

$$\langle j|k\rangle = \delta_{jk} \begin{cases} 0, \text{ if } j \neq k, \\ 1, \text{ if } j = k. \end{cases}$$

Matrix representations of H-space elements

It is often convenient to think of $|\psi\rangle$ as represented by a column vector

$$|\psi\rangle \equiv \begin{pmatrix} C_1 \\ C_2 \\ . \\ . \\ . \\ C_n \\ \end{pmatrix} = C_1\begin{pmatrix} 1 \\ 0 \\ . \\ . \\ . \\ 0 \\ . \end{pmatrix} + C_2\begin{pmatrix} 0 \\ 1 \\ . \\ . \\ . \\ 0 \\ . \end{pmatrix} + C_n\begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \\ . \end{pmatrix} + \ldots$$

$$\langle\psi| = \begin{pmatrix} C_1^* & C_2^* & \ldots & C_n^* & \ldots \end{pmatrix}$$

Orthonormal States

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \\ \vdots \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \\ \vdots \end{pmatrix} \ldots$$

$$\langle j|k\rangle = \delta_{jk}$$

Continuous variables in H space

We can associate kets with (wave) functions as in Quantum Mechanics

$$|\psi\rangle \Leftrightarrow \psi(\vec{r})$$ **Continuously vary in space!** $$\psi(\vec{r}) = |\psi(\vec{r})|\, e^{i\theta}$$

$dP = |\psi(\vec{r})|^2\, d\vec{r}$ **Is probability density to find particle in elementary volume** $d\vec{r}$

$$P = \int\limits_{Volume} dP = \int\limits_{Volume} |\psi(\vec{r})|^2\, d\vec{r} = 1$$

Properties of complex wave function

Norm $(\psi,\psi) = \int \psi^*(\vec{r})\psi(\vec{r})d\vec{r} = \int |\psi(\vec{r})|^2 d\vec{r}$

Inner product $\langle\phi|\psi\rangle = \int \phi^*(\vec{r})\psi(\vec{r})d\vec{r}$

Continuous variable representation

Decomposition $$\psi(\vec{r}) = \sum_i c_i u_i(\vec{r})$$

Where $u_i(\vec{r})$ **are orthonormal functions** $\sum_i u_i^*(\vec{r}\,')u_i(\vec{r}) = \delta(\vec{r}-\vec{r}\,')$

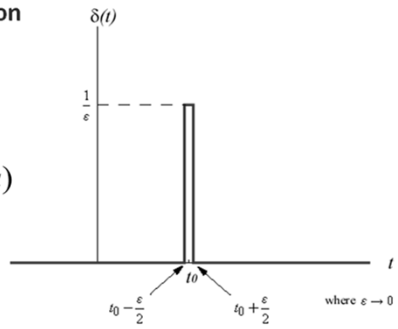
$\delta(\vec{r}-\vec{r}\,')$ **is Dirac Delta function**

Definition of Delta function

$$\delta(t-t_0) = \begin{cases} 0, & t \neq t_0 \\ \infty, & t = t_0 \end{cases}$$

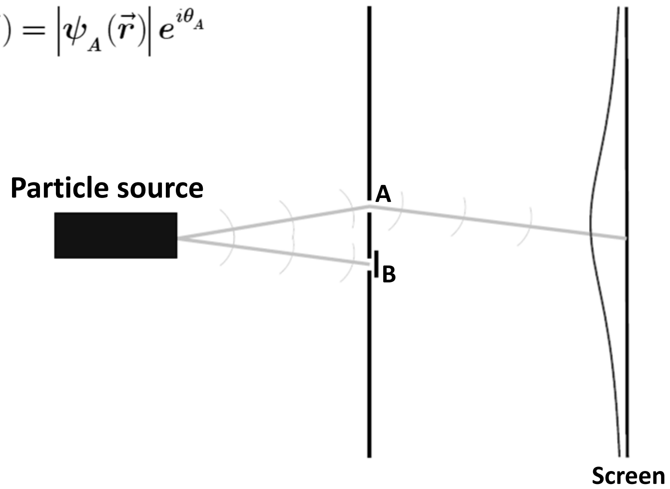
$$\int f(x)\delta(x-a)dx = f(a)$$

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{iku}dk = \delta(u)$$



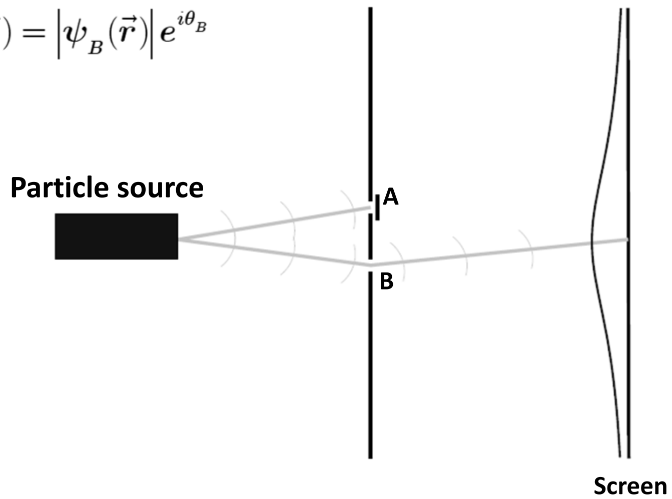
Two –slit experiment

$$\psi_A(\vec{r}) = \left|\psi_A(\vec{r})\right| e^{i\theta_A}$$



Two-slit experiment

$$\psi_B(\vec{r}) = \left|\psi_B(\vec{r})\right| e^{i\theta_B}$$

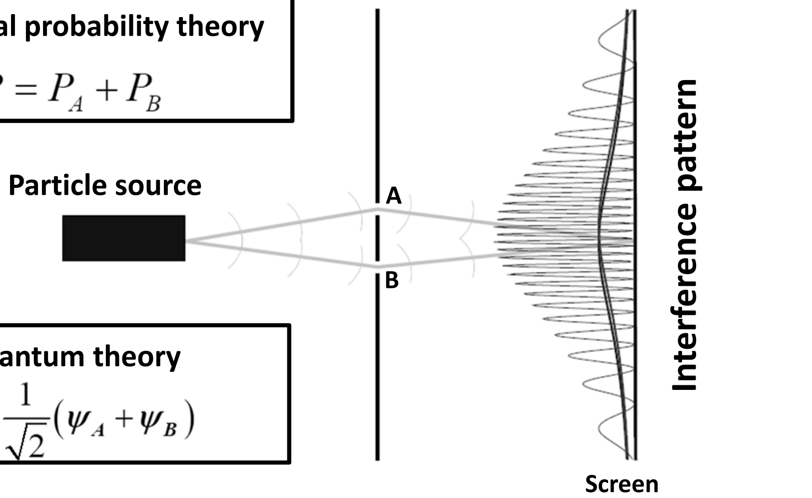


Quantum interference

$$P = \frac{1}{2}\left(P_A + P_B + 2\sqrt{P_A P_B}\cos(\theta_A - \theta_B)\right)$$

Classical probability theory

$$P = P_A + P_B$$



Quantum theory

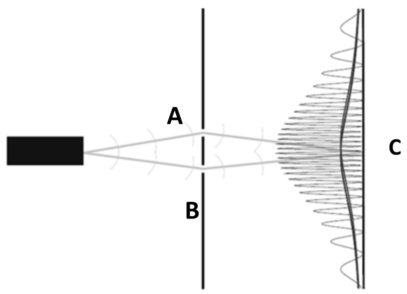
$$\psi = \frac{1}{\sqrt{2}}\left(\psi_A + \psi_B\right)$$

Quantum logic

G. BIRKHOFF, .J. VON NEUMANN, ANNALS OP MATHEMATICS, 1936

Predicates

« **A –** particle goes through slit A»,
« **B**- particle goes through slit B»,
« **C** – there is point like trace at the screen»



If two slits are open $(A\wedge C)\vee(B\wedge C)$ **is false!**

If only one slit is open we have $(A\vee B)\wedge C$

Distributivity $\boxed{(A\vee B)\wedge C \neq (A\wedge C)\vee(B\wedge C)}$ **is violated !**

A. Grib, W. Rodrigues ., Nonlocality in Quantum Physics, Springer, 1999

Linear operators

■ *Operators* are linear maps of the Hilbert space $\mathcal{H}$ onto itself. If A is an operator, then for any $|\psi\rangle$ in $\mathcal{H}$, $A|\psi\rangle$ is another element in $\mathcal{H}$, and linearity means that

- **Linear operator *A* is defined as:** $A|\psi\rangle = |\psi'\rangle$

$$A\left[c_1\left|\psi_1\right\rangle + c_2\left|\psi_2\right\rangle\right] = c_1 A\left|\psi_1\right\rangle + c_2 A\left|\psi_2\right\rangle$$

- **Matrix element of operator *A*:** $\langle\varphi|\left(A|\psi\rangle\right)$
- **Hermitian operator:** $A = A^\dagger$

$$\langle\varphi|A|\psi\rangle = \langle\psi|A|\varphi\rangle^*$$

- **Hermitian operators play a fundamental role in quantum mechanics (we'll see later)**

Charles Hermite (1822 – 1901)

How we can represent Hermitian operators?

Lets consider matrix A

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

Hermitian conjugate is $A^\dagger = [A^*]^{tr} = \begin{pmatrix} a_{11}^* & a_{21}^* & a_{31}^* \\ a_{12}^* & a_{22}^* & a_{32}^* \\ a_{13}^* & a_{23}^* & a_{33}^* \end{pmatrix}$

A Is Hermitian if

$$[A] = [A^*]^{tr}$$

It means that a_{ik} is real

Lets consider an operator $A = i\dfrac{d}{dx}$

Pls, prove that it is Hermitian

In particular, we should prove that

$$\int \psi^*(x)\,\widehat{A}\varphi(x)\,dx = \int [\widehat{A}\psi(x)]^*\,\varphi(x)\,dx.$$

Solution

$$\int_{-\infty}^{\infty} \psi^*\widehat{A}\varphi dx = i\int_{-\infty}^{\infty} \psi^*\frac{d\varphi}{dx}dx =$$

$$= i\psi^*\varphi \Big|_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} \varphi\frac{d\psi^*}{dx}dx = i\psi^*\varphi \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \varphi(\widehat{A}\psi)^* dx$$

$$\boxed{\int \psi^*(x)\,\widehat{A}\varphi(x)\,dx = \int [\widehat{A}\psi(x)]^*\,\varphi(x)\,dx.}$$

Eigenvalue Equation

The $|n\rangle$ is called an eigenvector of a linear operator if:

$$\boxed{A|n\rangle = \lambda_n|n\rangle} \quad \text{and} \quad \langle n|A^\dagger = \langle n|\lambda_n{}^*$$

•This is called an eigenvalue equation (EVEq)

$|0\rangle, |1\rangle, \ldots, |n\rangle, \ldots$ are eigenstates , $\{\lambda_n\}$ is spectrum of operator A

- ***Prove that if two arbitrary vectors obey EVEq then*** $\langle n|k\rangle = \delta_{nk}$
- ***Prove that all*** $\{\lambda_n\}$ ***are real if*** A ***is Hermitian***

Eigenvalue Equation

Proof. **Lets** A **is Hermitian**

$$\int \varphi^* \hat{A} \varphi dx = \int (\hat{A}\varphi)^* \varphi dx,$$

$$a \int |\varphi|^2 dx = a^* \int |\varphi|^2 dx,$$

$$a = a^*.$$

<u>Remark</u> *If* $A \neq A^\dagger$ *, the* $\{\lambda_n\}$ *could be real!*

In this case quantum system possess PT (parity-time) symmetry!

C.M. Bender, S. Boettcher, P.N. Meisinger, J. Math. Phys. 40, 2201 (1999) .

Schrodinger Equation

Schrodinger thinking about his equation.

Schrodinger: If electrons are waves, their position and motion in space must obey a wave equation.

Solutions of wave equations yield wavefunctions, Ψ, which contain the information required to describe ALL of the properties of the wave.

Lets consider EVEq for Hamilton operator

$$H\psi(r) = E\psi(r)$$

It Is stationary Schrodinger Equation ⇨

$$\boxed{\frac{\hbar^2}{2m}\Delta\psi(r) + (E - U(r))\psi(r) = 0}$$

If we take Hamiltonian operator as $H = -\frac{\hbar^2}{2m}\Delta + U(r)$

Quantum Theory postulates

Second postulate

Every observable attribute of a physical system is described by an Hermitian operator A that acts on the kets $|\psi\rangle$ that describe the system.

The only possible result of the measurement of an observable A is one of the eigenvalues of the corresponding operator A

$$A|n\rangle = \lambda_n |n\rangle$$

$\{|n\rangle\}$ **are eigenstates ,**

set of $\{\lambda_n\}$ represents measurement outputs of A

Practice

Eigenvalue equations

• Lets for a certain eigenvalue λ :

$$A\left|n\right\rangle = \lambda_n\left|n\right\rangle$$

Prove that expansion state $\left|\psi\right\rangle = \sum_i c_i\left|\psi_i\right\rangle$

is also eigenvector of the operator A corresponding to the eigenvalue λ for any c_i:

Practice

Eigenvalue equations

• Lets for a certain eigenvalue λ :

$$A\left|n\right\rangle = \lambda\left|n\right\rangle$$

Prove that the state $\left|\psi\right\rangle = \sum_n c_n\left|n\right\rangle$

is also eigenvector of the operator A corresponding to the eigenvalue λ for any c_i:

$$A\left|\psi\right\rangle = A\sum_n c_n\left|n\right\rangle = \sum_n c_n A\left|n\right\rangle = \sum_i c_n\lambda\left|n\right\rangle = \lambda\sum_n c_n\left|n\right\rangle$$

$$= \lambda\left|\psi\right\rangle$$

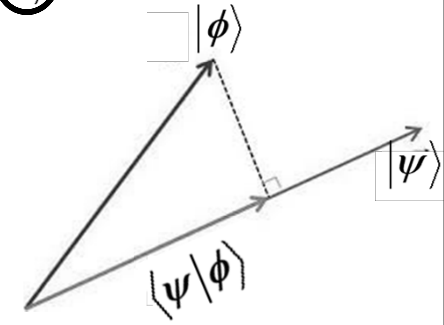
2.B.3

Density operator vs Projector

- Let consider arbitrary state $|\psi\rangle$
- Projector operator $P_\psi = |\psi\rangle\langle\psi| = \begin{pmatrix} |C_1|^2 & C_1C_2^* & \dots \\ C_2C_1^* & |C_2|^2 & \dots \\ \dots & \dots & \dots \end{pmatrix}$

$$P_\psi|\phi\rangle = |\psi\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$$

- It projects one ket onto another



In Quantum theory $\rho = |\psi\rangle\langle\psi|$
Is calling density operator

$$P_\psi^{\,2} = P_\psi$$

Quantum Theory postulates

Suppose that we would like to measure observable A
of arbitrary quantum system that possess quantum state $|\psi\rangle$
1. We should find solution of Eq. $A|n\rangle = \lambda_n|n\rangle$
2. We should consider expansion $|\psi\rangle = \sum C_n|n\rangle$

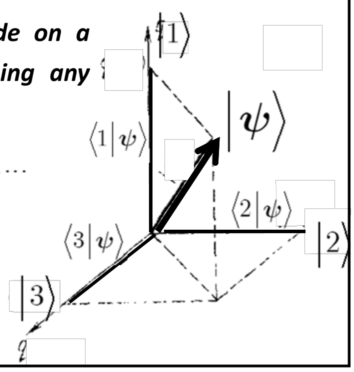
Max Born

Born rule

When a measurement of an observable A is made on a arbitrary state $|\psi\rangle$ ***, the probability of obtaining any eigenvalue*** λ_j ***from the spectrum of A is given by***

$$P_j \equiv |C_j|^2 = |\langle j|\psi\rangle|^2 \qquad j = 0,1,2,\dots n,\dots$$

$C_j = \langle j|\psi\rangle$ Is projection of $|\psi\rangle$ onto $|j\rangle$



Magic Coin

$|\Psi\rangle = \alpha|$  $\rangle + \beta|$  $\rangle$

$p_\alpha = |\alpha|^2$ - **Is probability to obtain heads under the measurement,**

$p_\beta = |\beta|^2$ - **is probability to obtain tails under the measurement**

No tails , No heads before the measurement !

Is classical coin really random?

If we know initial conditions we can solve Newton's equation for determining position of the coin at final state

Conclusion:
Classical uncertainty can be removed in principle, Quantum – NOT!!

Утверждение. Среднее значение оператора

в состоянии

Quantum averages

Average value of some observable is determined by $\boxed{\langle \hat{A}\rangle = \langle\psi|\hat{A}|\psi\rangle}$

$$\langle\psi|\hat{A}|\psi\rangle = \langle\psi|\hat{A}\sum|i\rangle\langle i|\psi\rangle = \langle\psi|\hat{A}\sum\langle i|\psi\rangle|i\rangle = \langle\psi|\sum\langle i|\psi\rangle\hat{A}|i\rangle =$$

$$= \langle\psi|\sum\langle i|\psi\rangle\alpha_i|i\rangle = \sum\alpha_i\langle i|\psi\rangle\langle\psi|i\rangle = \sum|\langle i|\psi\rangle|^2\alpha_i = \sum W_i\alpha_i = \langle\hat{A}\rangle$$

Here we use that

$I = \sum_j |j\rangle\langle j|.$ I is the identity operator, $I|\psi\rangle = |\psi\rangle$ for any $|\psi\rangle$

Утверждение. Среднее значение оператора

в состоянии

Quantum Fluctuations

Fluctuation of A observable is $\Delta\hat{A} = \hat{A} - \langle\hat{A}\rangle$

Dispersion $\left(A - \langle A\rangle\right)^2$. ⇨

$$\boxed{\left\langle\left(\Delta\hat{A}\right)^2\right\rangle = \langle\psi|\hat{A}^2|\psi\rangle - \langle\psi|\hat{A}|\psi\rangle^2}$$

$$\left\langle\left(\Delta\hat{A}\right)^2\right\rangle = \langle\psi|\left(\hat{A} - \langle\hat{A}\rangle\right)^2|\psi\rangle = \langle\psi|\hat{A}^2|\psi\rangle - 2\langle\hat{A}\rangle\langle\psi|\hat{A}|\psi\rangle + \langle\hat{A}\rangle^2\langle\psi|\psi\rangle = \langle\psi|\hat{A}^2|\psi\rangle - \langle\hat{A}\rangle^2$$

$\left\langle\left(\Delta\hat{A}\right)^2\right\rangle = 0$ If $|\psi\rangle$ is eigenstate

Compatible and incompatible observables

Lets consider problem of measurement of two observables determined by operators A and **B**

- **Product of operators:** $(AB)|\psi\rangle = A\left[B|\psi\rangle\right]$

We can also consider $(\boldsymbol{BA})|\psi\rangle = \boldsymbol{B}\left[\boldsymbol{A}|\psi\rangle\right]$

Or, $\frac{1}{2}(\boldsymbol{BA} + \boldsymbol{AB})|\psi\rangle$

Lets define commutation relation for two operators

$$[A, B] \equiv AB - BA$$

A and **B** are compatible if $\boldsymbol{AB} = \boldsymbol{BA}$, and $[\boldsymbol{A}, \boldsymbol{B}] = 0$

A and **B** are **in**compatible if $\boldsymbol{AB} \neq \boldsymbol{BA}$, and $[\boldsymbol{A}, \boldsymbol{B}] \neq 0$

(1.34a)

Practice

Some operator algebra

Prove that $\left[\hat{B},\hat{A}\right]=-\left[\hat{A},\hat{B}\right]$ $\left[\hat{A},\hat{B}\hat{C}\right]=\left[\hat{A},\hat{B}\right]\hat{C}+\hat{B}\left[\hat{A},\hat{C}\right]$

If $A^{\dagger}=A$ **and** $B^{\dagger}=B$ **, and** $\left[A,B\right]=0$ $\Rightarrow$ $\boxed{\left(AB\right)^{\dagger}=AB}$

$$\left(AB\right)^{\dagger}=B^{\dagger}A^{\dagger}=BA=AB$$

If two operators commute, there is an orthonormal basis with eigenvectors common to both operators

IF $\left[A,B\right]=0$ **and** $A\left|\psi_1\right\rangle=a_1\left|\psi_1\right\rangle$ $A\left|\psi_2\right\rangle=a_2\left|\psi_2\right\rangle$ $a_1\neq a_2$

Then $\left\langle\psi_1\right|AB\left|\psi_2\right\rangle=a_1\left\langle\psi_1\right|B\left|\psi_2\right\rangle$ $\left\langle\psi_1\right|BA\left|\psi_2\right\rangle=a_2\left\langle\psi_1\right|B\left|\psi_2\right\rangle$

$$\left\langle\psi_1\right|AB\left|\psi_2\right\rangle-\left\langle\psi_1\right|BA\left|\psi_2\right\rangle=\left(a_1-a_2\right)\left\langle\psi_1\right|B\left|\psi_2\right\rangle$$

$$\left\langle\psi_1\right|B\left|\psi_2\right\rangle=0$$

Heisenberg's Uncertainty Principle

$$\text{Variable A:}\quad \mathrm{A}\cdot\boldsymbol{\psi}_i=\boldsymbol{a}_i\cdot\boldsymbol{\psi}_i$$
$$\text{Variable B:}\quad \mathrm{B}\cdot\Phi_i=\boldsymbol{b}_i\cdot\Phi_i$$

$$[\boldsymbol{A},\boldsymbol{B}]\neq 0$$

Werner Heisenberg

A fundamental incompatibility exists in the measurement of physical variables that are represented by non-commuting operators:

"A measurement of one causes an uncertainty in the other."

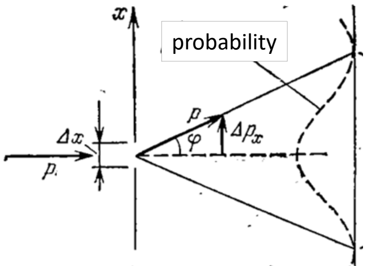
The Uncertainty Relation

$$\boxed{\boldsymbol{\delta A}\cdot\boldsymbol{\delta B}\geq\frac{1}{2}\left|<[\boldsymbol{A},\boldsymbol{B}]>\right|}$$

$$\boldsymbol{\delta A}\equiv\sqrt{\left\langle(\Delta\boldsymbol{A})^2\right\rangle},\qquad \boldsymbol{\delta B}\equiv\sqrt{\left\langle(\Delta\boldsymbol{B})^2\right\rangle}$$

Example: Momentum and coordinate

Wave-particle duality $p = 2\pi\hbar / \lambda_{dB}$



$$\Delta p_x = p \sin \varphi.$$

Particle wave properties

$$\sin \varphi = \lambda / \Delta x \qquad \Delta p_x = p\lambda / \Delta x.$$

$$\boxed{\Delta x \cdot \Delta p_x = p\lambda = 2\pi\hbar}$$

Coordinate operator

$$\hat{x} = x$$

Momentum operator

$$\hat{p} = -i\hbar \frac{d}{dx}$$

Commutator relation

$$\left[x, p_x\right] = i\hbar$$

$$\boxed{\delta x \delta p \geq \frac{\hbar}{2}}$$

Eigenvalue Problem for Momentum

Eigenvalue equation for momentum operator

$$\boxed{\hat{p}_x \varphi_p(x) = p_x \varphi_p(x).} \Rightarrow -i\hbar \frac{\partial \varphi_p}{\partial x} = p_x \varphi_p$$

Solution $\boxed{\varphi_p = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_x x}.}$ where $\boxed{\frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} e^{i\frac{x}{\hbar}(p'-p)} dx = \delta(p - p')}$

We can expand any quantum state $\psi(x)$ ***as***

$$\psi(x) = \int_{-\infty}^{+\infty} \bar{\psi}(p) \varphi_p(x) dp = \boxed{\frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \bar{\psi}(p) e^{\frac{ipx}{\hbar}} dp}$$

This is a Fourier transform !

Lets consider inverse Fourier transform

$$\boxed{\bar{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{+\infty} \psi(x) e^{\frac{-ipx}{\hbar}} dx}$$

is wavefunction in momentum space!

Eigenvalue problem for coordinate

Eigenvalue equation for momentum operator $\boxed{\hat{x}\,\varphi_{x_0}(x) = x_0\,\varphi_{x_0}(x)}$

Since $x\delta(x - x_0) = x_0\delta(x - x_0).$ *Pls, prove that*

Solution $\boxed{\varphi_{x_0}(x) = \delta(x - x_0),}$

$$\boxed{\int \delta(x - x_0)\delta(x - x_0')dx = \delta(x_0 - x_0')}$$

- orthonormal basis

We can expand any quantum state $\psi(x)$ ***as***

$$\psi(x) = \int \delta(x - x_0)\psi(x_0)dx_0$$

The End