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A diagram-free approach to the stochastic estimates in regularity structures

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Abstract

In this paper, we explore the version of Hairer's regularity structures based on a greedier index set than trees, as introduced in (Otto et al. in A priori bounds for quasi-linear SPDEs in the full sub-critical regime, 2021, arXiv:2103.11039) and algebraically characterized in (Linares et al. in Comm. Am. Math. Soc. 3:1–64, 2023). More precisely, we construct and stochastically estimate the renormalized model postulated in (Otto et al. in A priori bounds for quasi-linear SPDEs in the full sub-critical regime, 2021, arXiv:2103.11039), avoiding the use of Feynman diagrams but still in a fully automated, i. e. inductive way. This is carried out for a class of quasi-linear parabolic PDEs driven by noise in the full singular but renormalizable range. We assume a spectral gap inequality on the (not necessarily Gaussian) noise ensemble. The resulting control on the variance of the model naturally complements its vanishing expectation arising from the BPHZ-choice of renormalization. We capture the gain in regularity on the level of the Malliavin derivative of the model by describing it as a modelled distribution. Symmetry is an important guiding principle and built-in on the level of the renormalization Ansatz. Our approach is analytic and top-down rather than combinatorial and bottom-up.

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1 Introduction

We continue the program started in [57] of replacing trees by multi-indices as a more parsimonious but equally natural index set, within the framework of Hairer's regularity structures. Like in [53], we implement this for quasi-linear parabolic[1] equations of the form

$$(\partial_2 - \partial_1^2)u = a(u)\partial_1^2 u + \xi, \tag{1.1}$$

driven by a stationary[2] noise ξ in the regime where the product $a(u)\partial_1^2 u$ is singular but renormalizable. This is the case when[3] a general solution u to (1.1) with $a \equiv 0$ is Hölder continuous with an exponent $\alpha \in (0, 1)$, which means that ξ is in the (negative) Hölder space $C^{\alpha-2}$. However, we believe that our program applies to a much larger class[4] of nonlinear PDEs.

The investigation of (1.1) started in [55] sparked some activity, at first in the mildly singular range [5, 25] of $\alpha \in (\frac{2}{3}, 1)$, and then up to the white-noise level ($\alpha > \frac{1}{2}$) [30, 56], and finally including the white-noise level: [30] actually deals with the regime $\alpha > 0$, however the obtained counterterm is only shown to be local up to $\alpha > \frac{1}{2}$, which was extended in [29] to $\alpha > \frac{1}{3}$; [17] put the calculations performed in [29] on an algebraic footing as a step towards an extension to $\alpha > 0$; [4] deals with the analytic part of the solution theory up to $\alpha > \frac{2}{5}$, and so does [6] down to $\alpha > 0$, however again with the caveat that the obtained counterterm can only be shown to be local up to $\alpha > \frac{2}{5}$. In [57], local Schauder estimates were established for $\alpha \in (0, 1)$, based on the notion of modelled distributions, postulating the existence and estimation of a suitable renormalized model. In [53], the Hopf-theoretic nature of the structure group based on multi-indices was uncovered, which is rather Lie-geometric

[1] treating (1.1) as an (anisotropic) elliptic equation, we denote by x_1 the space-like variable and by x_2 the time-like variable; this allows us to use t for the semi-group convolution (4.1) parameter.

[2] i. e. invariant in law under space-time shifts.

[3] as a consequence of the single space dimension, there is an additional constraint.

[4] in particular, the restriction to a single spatial variable is just for convenience, and has the advantage of making white noise renormalizable with $\alpha = \frac{1}{2}-$. However, as discussed after Assumption 2.1 in Sect. 2.1, there is a didactic advantage in allowing for general space dimension, replacing ∂_1^2 by the Laplacian, as done in [57].

than combinatorial in the sense that it provides a representation of natural actions on the solution manifold. In this paper, we construct the BPHZ-renormalized model and provide its stochastic estimates, as the input to [57]. Our multi-index approach is analytic, meaning that it is based on taking derivatives w. r. t. the nonlinearity a and the noise ξ, whereas the tree-based approach is combinatorial, using Feynman diagrams. Our approach is fully automated since it proceeds by induction over the index set, with all induction steps having the same structure.[5] Loosely speaking, [57] can be seen as an analogue[6] of[7] [38] in the sense that it deals with the analytic solution theory, this paper corresponds to [20] by establishing the stochastic estimates on the model, while [53] works out the Hopf-algebraic structure in the spirit of [14, 15]. Let us stress that this paper is essentially self-contained w. r. t. [57] and [53], and can be read and appreciated independently: It serves as an input for [57], while [53] provides a deeper algebraic understanding not required in this paper.[8]

In terms of renormalization, the multi-index approach is top-down rather than bottom-up, in the sense that for the renormalized equation

$$(\partial_2 - \partial_1^2)u + h = a(u)\partial_1^2 u + \xi \tag{1.2}$$

we postulate on the counter term h:

- h is local, i. e. it depends on the solution u only through its value[9] $u(x)$ at the current space-time point x and
- h is homogeneous, i. e. not explicitly dependent on x, and thus[10] deterministic, i. e. not explicitly dependent on ξ. Both conditions imply that $h = h(u(x))$ for some deterministic nonlinearity h.
- The most subtle postulate relates a to h: If we replace a by $a(\cdot + u)$ for some u-shift $u \in \mathbb{R}$, h is replaced by $h(\cdot + u)$. This means that the renormalization is independent of the choice of the origin in u-space. It implies that

  $$h(u) = c[a(\cdot + u)] \quad \text{for } u \in \mathbb{R} \tag{1.3}$$

  for some deterministic functional[11] c on a-space, which typically diverges as the regularization of ξ fades away.

As opposed to the tree-based treatment of quasi-linear equations by regularity structures [29, 30], we thus do not have to show a posteriori that h is local. Some symmetries are built-in, like the independence from the choice of the origin in u-space and a-space. Other symmetries, like the invariance in law under $\xi \mapsto -\xi$, are easily seen to transmit to the model.[12] As a consequence, our more greedy approach based on

[5] up to the distinction between singular and regular multi-indices.

[6] however partial and restricted to (1.1).

[7] including aspects of [15] in the sense that the notion of the structure group is less generic than in [38].

[8] besides a few finiteness properties and identities explicitly spelled out.

[9] this particularly simple form relies on the assumption that ξ is invariant in law under spatial reflection.

[10] in view of the stationarity of the noise.

[11] here and in the sequel, $[\cdot]$ denotes the functional dependence on a field.

[12] however, our approach is oblivious to symmetries arising from a Gaussian nature of the noise.

multi-indices rather than trees reduces the number of divergent constants contained in c. The comparison for the quasi-linear equation (1.1) is complicated by the non-local nature of the tree-based treatment in [29, 30], and the fact that the divergent constants are better treated as functions of a place-holder a_0 for the elliptic coefficient $1+a$, see Sect. 2.7. The comparison is easier for the semi-linear multiplicative stochastic heat equation $(\partial_2-\partial_1^2)u=a(u)\xi$, as treated by tree-based regularity structures in [42] for $\alpha=\frac{1}{2}-$. Here it is clear that for, e. g., $\alpha=\frac{1}{4}+$, the number of divergent constants decreases from 85 to 30, see also [53, Sect. 7] and [11, Sect. 5].

The two main conceptual merits of our approach are:

- Spectral gap (SG) inequality. Our main assumption on the law of ξ, next to invariance under translation[13] and reflection[14] is a spectral gap inequality. The SG inequality is specified by a Hilbert norm on ξ-space, which provides the analogue of the Cameron-Martin space from the Gaussian case, and here is $L^2(\mathbb{R}^2)$-based. This Hilbert norm is chosen in agreement with $\xi\in C^{\alpha-2}$. While this includes non-Gaussian ensembles, the main benefit is that the SG inequality naturally complements the BPHZ-choice of renormalization: On the one hand, a SG inequality, which we apply to the negative-homogeneity part Π^- of the model, estimates the variance of a random variable. On the other hand, the BPHZ-choice of renormalization is just made to annihilate the expectation $\mathbb{E}\Pi^-$. We refer to Sects. 4.3 and 5.1 for the details.
- Malliavin derivative as modelled distribution. The use of a SG inequality requires the control of the (first-order) Malliavin derivative of Π^-, which is the Fréchet derivative of $\Pi^-=\Pi^-[\xi]$ w. r. t. the noise ξ. It is convenient to think of it in terms of the directional derivative $\delta\Pi^-$ for some arbitrary infinitesimal noise perturbation[15] $\delta\xi$. Since Π^- is multi-linear in ξ, passing to $\delta\Pi^-$ amounts to replacing one of the instances of ξ by $\delta\xi$. This leads only to a subtle gain in regularity, which is conveniently expressed after integration,[16] i. e. on the level of $\delta\Pi$. It is captured by describing $\delta\Pi$ as a modelled distribution[17] $\mathrm{d}\Gamma^*$ w. r. t. Π itself. Hence surprisingly, the notion of a modelled distribution with its intrinsic continuity property, which was introduced in [38, Definition 3.1] for the deterministic Schauder theory given the model, here plays a role in the stochastic estimation of the model itself. Crucially, as opposed to Π^- itself, the representation for $\delta\Pi^-$, or rather of its rough-path increment $\delta\Pi^- - \mathrm{d}\Gamma^*\Pi^-$, in terms of Π and $\delta\Pi-\mathrm{d}\Gamma^*\Pi$ does not involve the divergent c. This ultimately allows for reconstruction of $\delta\Pi^-$. We refer to Sect. 4.5 for details.

Two more technical merits of our approach are:

[13] i. e. stationarity.

[14] in the spatial variable(s).

[15] an element of what would be the Cameron-Martin space in the Gaussian case.

[16] in Hairer's jargon.

[17] we use the notation $\mathrm{d}\Gamma^*$ to indicate that it has some structural similarities with the Malliavin derivative $\delta\Gamma^*$ of the change-of-base-point transformation Γ^*; the star is a reminder of the fact that Γ^* is the transpose of Hairer's, see the discussion at the beginning of Sect. 2.2.6.

- Scaling as a guiding principle. In order not to break it, we work on the whole space-time, which because of potential infra-red divergences is interesting in its own right. As a collateral damage, in order to avoid critical cases, we have to generalize from white noise to a more general noise ξ with a fractional (negative) Sobolev norm playing the role of the Cameron-Martin space. Like in [21], this has the positive side effect of allowing to explore the limits of the approach. In order not to break scaling, we work with annealed instead of quenched estimates. By this jargon[18] we mean that the inner norm is an L^p-norm in probability, while the outer norm is a Hölder norm in space-time; Sect. 4.3 is the place where this transition is made on the level of $\delta\xi$. Estimates in annealed norms have the advantage of coming without a (marginal) loss in the exponent.[19]
- Hölder vs. L^2-topologies. The SG inequality and Malliavin calculus rely on L^2-based space-time norms (the analogue of the Cameron-Martin space is an L^2-norm of a negative fractional derivative of $\delta\xi$) whereas the Schauder calculus of modelled distributions like our $d\Gamma^*$ is based on Hölder norms. We introduce a weight that is singular (but integrable) in a secondary base point z into the L^2-based norms to emulate a Hölder norm localized in z. Averaging over z recovers the original Cameron-Martin norm. We refer to Sect. 4.5 for details.

We now comment on related work. Hairer's regularity structures triggered a rapid development in the field of singular SPDEs. They provide a framework for local well-posedness for a large class of semi-linear SPDEs, as worked out in [14, 15, 20, 38]. As mentioned, this approach has been extended to quasi-linear SPDEs[20] in [29, 30]. Gubinelli, Imkeller and Perkowski's paracontrolled calculus [35] provides an alternative approach, based on Littlewood-Paley decomposition, to (stochastic) estimates and renormalization. While it does not provide a general framework by itself, see however [2], paracontrolled calculus has been efficiently applied to a variety of singular SPDEs [19, 34]. Furthermore, it naturally extends to quasi-linear equations, see [5, 25], and to dispersive equations, see [36]. Kupiainen appealed to Wilsonian renormalization [49, 50] to treat some semi-linear SPDEs. Duch [23] used Wilsonian renormalization in the continuum form of the Polchinski flow equation, see below. Our approach has also similarities with the one of Epstein and Glaser [59, Sect. 3.1] in the sense that it is inductive, and that it uses a coarser index set than trees, meaning that it only monitors specific linear combinations of trees. With its modern version [44], it shares the guiding principle of scaling and symmetries (covariance).

Although the stochastic estimates on the centered model obtained in [20] are identical to the ones in the present paper, the assumption of cumulant bounds on the noise, and the methodology to obtain stochastic estimates via multi-scale analysis differ radically from our approach. As opposed to our method, the analysis in [20] builds up on the well-established physics approach to Feynman diagrams into which it incorporates positive renormalization. Problems like overlapping sub-divergences, or counter terms necessary to cure divergences at one scale that could potentially interfere at another scale, do not come up in our recursive approach.

[18] from statistical physics and ultimately from metallurgy.

[19] as is well-known from Brownian motion and its law of iterated logarithm.

[20] at the expense of loosing the local nature of the problem inside the proof.

The approach of [23, 49] is not in the framework of regularity structures, with its conceptual separation between the task of constructing and stochastically estimating the centered model on the one hand, and a deterministic fixed point argument to solve a given initial/boundary value problem on the other. It directly constructs solutions for given (periodic) boundary conditions, while the centered model constructed and estimated in [14, 20] and in this paper describes the entire solution manifold. Nevertheless [23] has similarities with the present work in the sense that it recursively constructs and estimates multi-linear functionals of the noise in such a way that overlapping sub-divergences do not play a role. However, the assumption of cumulant bounds on the noise is more closely related to [20] than to the present work. The recursive structure in [23] arises from a book-keeping parameter in front of the non-linearity, which leads to a formal power series expansion in this parameter, into which polynomials are incorporated like in the present work. This leads to an even more parsimonious index set than in the present work.

In the use of Malliavin calculus, the spectral gap inequality, and annealed estimates, this paper is inspired by recent developments in quantitative stochastic homogenization [24, 31, 47]. Malliavin calculus has been used, within the framework of regularity structures, in [18, 28, 60], however in its original purpose, namely for the existence of probability densities. In the context of stochastic estimates for singular SPDEs, Malliavin calculus has been used in [26] to estimate non-polynomial functionals of the noise through the Wiener chaos decomposition, and in combination with the spectral gap inequality in [45] to estimate the first non-linear term in case of a non-local operator.

Since posting this work, the spectral gap inequality has been used in several works to establish stochastic estimates. In [48, Appendix C] the authors obtained stochastic estimates for the simple case of Φ^4_2. First algebraic steps towards extending this work to the tree-based setting were made in [12], whereas [43] obtained stochastic estimates in the tree-based setting for a large class of semi-linear equations. Also [3] revisited the arguments of the present work and applied it to the tree-based setting, and [1] obtained estimates in the tree-based setting for the generalized KPZ equation in one spatial dimension by appealing to a higher order version of the spectral gap inequality, however still relying on diagrammatic tools to estimate higher order Malliavin derivatives. The present work has been extended in [37] and [8] (within the multi-index setting) to a fourth-order quasi-linear equation with multiplicative noise, and a semi-linear equation with polynomial non-linearity and additive noise, respectively. Furthermore, [61] used Malliavin calculus and a spectral gap assumption to give a characterization of models. The spectral gap inequality has also been applied in a more classical setting of rough paths [27].

On the algebraic side, the approach based on multi-indices has been generalized to a large class of semi-linear SPDEs in [11], which also attempts to systematize the algebraic structure of the top-down approach to renormalization developed here (see also [51] for an algebraic construction based on multi-indices consistent with [11] and connected to the algebraic renormalization of rough paths [13]). Moreover, algebraic structures based on multi-indices have triggered the study of post-Lie and Novikov algebras in the context of regularity structures [9, 10, 46] and, more recently, they have been introduced in numerical analysis [16].

2 Assumptions and statement of result

2.1 Spectral gap inequality

In this subsection, we motivate and state our assumptions on the law of random Schwartz distributions ξ on space-time $\mathbb{R}^2$. The crucial assumption is that of a spectral gap (SG) inequality. The structure underlying a SG inequality is a Hilbert norm on the space of space-time fields, which plays the role of the Cameron-Martin norm from the Gaussian case. In the same way white noise has $L^2(\mathbb{R}^2)$ as Cameron-Martin space, our norm will be an $L^2(\mathbb{R}^2)$-based norm. Because our law is shift-invariant, it is natural to choose a translation-invariant norm. Since we are aiming at ξ's that are almost surely in the negative Hölder space $C^{\alpha-2-}$, by Kolmogorov's criterion, it should be an L^2-based Sobolev norm of the fractional order $\frac{D}{2}+\alpha-2$, where D is the effective dimension (see [38, Lemma 10.2]). In view of the parabolic nature, both Hölder and Sobolev norms need to be anisotropic: If the spatial variable x_1 sets the unit, the time variable x_2 is worth two units. In particular, we have for the effective dimension $D = 1 + 2 = 3$ so that the order of fractional derivative should be $\alpha - \frac{1}{2}$. For the Hölder norm anisotropy means that it is based on the parabolic Carnot-Carathéodory distance

$$|y - x| := |y_1 - x_1| + \sqrt{|y_2 - x_2|}. \tag{2.1}$$

It is convenient to express the anisotropic version of the Sobolev norm in terms of the space-time elliptic operator $\partial_1^4 - \partial_2^2$, which is of order four:

$$\Big(\int_{\mathbb{R}^2} dx \big((\partial_1^4 - \partial_2^2)^{\frac{1}{4}(\alpha-\frac{1}{2})}\delta\xi\big)^2\Big)^{\frac{1}{2}}, \tag{2.2}$$

where here and in the sequel, we think of $\delta\xi$ as an infinitesimal perturbation of ξ.

Having motivated the Hilbert norm (2.2), we return to the notion of a SG inequality. A SG inequality amounts to a Poincaré inequality (with mean value zero) on the space of space-time fields endowed with a probability measure and a Hilbert norm.[21] It is formulated in terms of a generic random variable F, which is an integrable function(al) on the space of ξ's, i. e. $F = F[\xi]$. The notion of a gradient of F and its (squared) norm relies on the Hilbertian structure (2.2). We momentarily consider $F = F[\xi]$ that are Fréchet differentiable w. r. t. (2.2); meaning that the differential $dF[\xi]$ in a configuration ξ, which is a linear form on the space of infinitesimal perturbation $\delta\xi$'s, is bounded w. r. t. (2.2). Representing the differential $dF[\xi]$ in terms of

$$\delta F := dF[\xi].\delta\xi = \int_{\mathbb{R}^2} dx\ \delta\xi\ \frac{\partial F}{\partial \xi}[\xi], \tag{2.3}$$

[21] In analogy to the standard Poincaré inequality providing a lower bound on the spectral gap of the (Neumann) Laplacian, the SG inequality provides a lower bound on the spectral gap of the generator of the stochastic process defined through the Dirichlet form arising from the Hilbertian structure, for which the given ensemble is in detailed balance by construction.

this means that the $L^2(\mathbb{R}^2)$-dual norm of the Malliavin derivative $\frac{\partial F}{\partial \xi} = \frac{\partial F}{\partial \xi}[\xi](x)$ is finite:[22]

$$\|\frac{\partial F}{\partial \xi}[\xi]\|_* := \Big(\int_{\mathbb{R}^2} dx\big((\partial_1^4 - \partial_2^2)^{\frac{1}{4}(\frac{1}{2}-\alpha)}\frac{\partial F}{\partial \xi}[\xi]\big)^2\Big)^{\frac{1}{2}} < \infty. \tag{2.4}$$

A functional-analytic subtlety arises from the fact that Fréchet differentiability of F is not enough to give an a priori meaning to (2.4) for almost every realization[23] ξ. Therefore one restricts to cylindrical functionals

$$F[\xi] = \bar{F}((\xi, \zeta_1), \ldots, (\xi, \zeta_N)) \tag{2.5}$$

for some smooth function $\bar{F}$ on $\mathbb{R}^N$ and Schwartz functions $\zeta_1, \ldots, \zeta_N$, where $(\cdot, \cdot)$ here stands for the pairing between a Schwartz distribution and a Schwartz function. These cylindrical functionals are obviously Fréchet differentiable (even on the space of Schwartz distributions) with

$$\frac{\partial F}{\partial \xi}[\xi] = \sum_{n=1}^{N} \partial_n \bar{F}((\xi, \zeta_1), \ldots, (\xi, \zeta_N))\zeta_n. \tag{2.6}$$

Assumption 2.1 The law $\mathbb{E}$ of the Schwartz distribution ξ is invariant under space-time shift and spatial reflection.[24] It is centered[25] and for an $\alpha \in (\frac{1}{4}, \frac{1}{2}) - \mathbb{Q}$ it satisfies the spectral gap inequality

$$\mathbb{E}(F - \mathbb{E}F)^2 \leq \mathbb{E}\|\frac{\partial F}{\partial \xi}\|_*^2, \tag{2.7}$$

for all integrable cylindrical functionals F. In addition, we assume that the operator (2.6) is closable[26] with respect to the topologies[27] of $\mathbb{E}^{\frac{1}{2}}|\cdot|^2$ and $\mathbb{E}^{\frac{1}{2}}\|\cdot\|_*^2$; here and in the sequel we use for $p \geq 1$ the shorthand notation $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$ to denote the stochastic $\mathbb{L}^p$-norm.

We learn from a (parabolic) rescaling of space-time that there is no loss in generality in assuming that the constant in (2.7) is unity. We note that any Gaussian ensemble with a Cameron-Martin norm that dominates (2.2) satisfies (2.7), see [7, Theorem 5.5.1, Eq. (5.5.2)]. In particular, this applies to any stationary Gaussian ensemble with a covariance function of which the Fourier transform satisfies

[22] To ease the notation we suppress the dependence of $\|\cdot\|_*$ on α.

[23] consider $F[\xi] = \frac{1}{2}\int dx \xi^2$; on the one hand, F is Fréchet differentiable for $\alpha = \frac{1}{2}$ with $\frac{\partial F}{\partial \xi}[\xi] = \xi$; on the other hand, white noise almost surely has infinite $L^2(\mathbb{R}^2)$-norm.

[24] meaning $x_1 \mapsto -x_1$.

[25] meaning $\mathbb{E}\xi(x) = 0$, an assumption just made for convenience and w. l. o. g.

[26] in the Gaussian case, this is automatic [54, Proposition 1.2.1].

[27] meaning that the closure of the graph of $F \mapsto \frac{\partial F}{\partial \xi}$ on the space of cylindrical functionals w. r. t. the product topology remains a graph.

$\mathcal{F}c(k) \leq (k_1^4 + k_2^2)^{\frac{1}{2}(\frac{1}{2}-\alpha)}$. Let us comment on the constraints on α: Recall that white noise corresponds to $\alpha = \frac{1}{2}$; however, because of the Schauder theory involved in integration, we need to avoid rational α. For convenience, we restrict ourselves to the more singular side by assuming $\alpha < \frac{1}{2}$. This does include white noise, provided it is tamed by an infra-red cut-off, which can be achieved by cutting off the large-scale Fourier modes to satisfy the above-mentioned estimate (while preserving stationarity). In the case of rational α, like $\alpha = \frac{1}{3}$, and without an infra-red cut-off, logarithms in the estimates are unavoidable; these are not captured in this work. Reconstruction imposes $\alpha > \frac{1}{4}$, a constraint already present on the level of the first counter term, see [55, Proposition 4.2], and specific to the single space dimension; a similar restriction arises already in rough path theory, where fractional Brownian motion can be (canonically) lifted to a rough path only for Hurst parameter $H > \frac{1}{4}$ [22]. We will discuss in Sect. 4.5 that this is the only reason for restricting the α-range away from $\alpha = 0$ and thus the limit of renormalizability.

We note that assumption (2.7) implies that ξ has an annealed (parabolic) Hölder regularity of exponent $\alpha - 2$, as expressed by (2.64) for $\beta = 0$, and thus almost every realization ξ has a quenched local Hölder regularity for any exponent $< \alpha - 2$, but not better. In order to give a classical sense to the model, ξ will enter its definition only after mollification,[28] see (2.18). The key insight is that the estimates of Theorem 2.2 do not depend on the scale of this ultra-violet cut-off. The companion work [61] builds upon the tools developed here to also pass to the limit of vanishing mollification, and to give an independent characterization of the latter. This uniqueness, which relies on the assumption $\alpha \notin \mathbb{Q}$, amounts to universality in the spirit of [45, Proposition 1.9]: The limit is independent of the specific regularization. The assumption of reflection invariance in law is crucial for the (simple) form of the renormalized equation; if omitted, we expect an additional counter term of the form $\tilde{h}(u)\partial_1 u$.

2.2 Definition of the model $(\boldsymbol{\Pi}_x, \boldsymbol{\Gamma}_{yx})$

In this subsection, we motivate and define the objects of our main result, Theorem 2.2. We first develop an algebraic perspective on the counter-term, then introduce coordinates on the solution manifold, then define the centered model Π_x proper, as a linear map on the abstract model space T, introduce the grading of T by the homogeneity $|\cdot|$, and finally the re-centering transformations $\Gamma_{yx} \in \mathrm{End}(\mathsf{T})$. We follow the concepts, language, and notation of regularity structures. A more in-depth treatment is provided in [53], more motivations are provided in [52, 58], but this text is self-contained.

2.2.1 The counter term c as an element of $\mathbb{R}[[\mathsf{z}_k]]$

On the space of analytic functions a in the variable u, coordinates are given by

$$\mathsf{z}_k[a] = \frac{1}{k!}\frac{d^k a}{du^k}(0) \quad \text{for } k \in \mathbb{N}_0. \tag{2.8}$$

[28] the resulting pathwise smoothness is not used.

If a is a polynomial, as denoted by $a \in \mathbb{R}[u]$, we obtain by Taylor

$$a(u) = \sum_{k \geq 0} u^k \mathsf{z}_k[a]. \tag{2.9}$$

For a multi-index β, which associates a frequency $\beta(k) \in \mathbb{N}_0$ to every $k \geq 0$ such that all but finitely many $\beta(k)$'s vanish, the monomial $\mathsf{z}^\beta = \prod_{k \geq 0} \mathsf{z}_k^{\beta(k)}$ defines a (nonlinear) functional on the space of analytic a's via (2.8). In fact, (2.8) naturally extends to the space $\mathbb{R}[[u]]$ of formal power series a in u. Hence we may identify the algebra $\mathbb{R}[\mathsf{z}_k]$ of polynomials $\sum_\beta \pi_\beta \mathsf{z}^\beta$ in the variables $\{\mathsf{z}_k\}_{k \geq 0}$ with a sub-algebra of the algebra of function(al)s on $\mathbb{R}[[u]]$.

In view of (1.3), we are interested in the action of u-shift $a \mapsto a(\cdot + v)$ for $v \in \mathbb{R}$, which is also well-defined as an endomorphism of $\mathbb{R}[[u]]$. By pull back, this endomorphism of $\mathbb{R}[[u]]$ lifts to an endomorphism of the algebra of functionals π on $\mathbb{R}[[u]]$. We consider its infinitesimal generator $D^{(\mathbf{0})}$ defined on the sub-algebra $\mathbb{R}[\mathsf{z}_k]$ through $(D^{(\mathbf{0})}\pi)[a] := \frac{d}{dv}_{|v=0} \pi[a(\cdot + v)]$ and claim that

$$D^{(\mathbf{0})} = \sum_{k \geq 0} (k+1) \mathsf{z}_{k+1} \partial_{\mathsf{z}_k}, \tag{2.10}$$

noting that the sum is effectively finite when applied to $\pi \in \mathbb{R}[\mathsf{z}_k]$. The elementary argument is given in Sect. 8.

Returning to (1.3), we note that the pull back of $a \mapsto a(\cdot + v)$ can be expressed in terms of its infinitesimal generator $D^{(\mathbf{0})}$ via the exponential formula

$$c[a(\cdot + v)] = \Big(\sum_{l \geq 0} \frac{1}{l!} v^l (D^{(\mathbf{0})})^l c \Big)[a]. \tag{2.11}$$

In view of (2.10), $\sum_{l \geq 0} \frac{1}{l!} v^l (D^{(\mathbf{0})})^l$ maps z_k onto the *infinite*[29] linear combination $\sum_{l \geq 0} \binom{l+k}{l} \, v^l \, \mathsf{z}_{l+k}$. This motivates to pass from the algebra of polynomials $\mathbb{R}[\mathsf{z}_k]$ to the algebra of formal power series $\mathbb{R}[[\mathsf{z}_k]]$. The matrix representation of (2.10) w. r. t. the monomial basis $\{\mathsf{z}^\beta\}_\beta$ is given by[30]

$$(D^{(\mathbf{0})})_\beta^\gamma = (D^{(\mathbf{0})} \mathsf{z}^\gamma)_\beta = \sum_{k \geq 0} \left\{ \begin{array}{cc} (k+1)\gamma(k) & \text{provided } \gamma + e_{k+1} = \beta + e_k \\ 0 & \text{else} \end{array} \right\}, \tag{2.12}$$

and has the property that for every multi-index β, it vanishes for all but finitely many multi-indices γ. Hence (2.10) extends to an endomorphism[31] on $\mathbb{R}[[\mathsf{z}_k]]$. Moreover, for the scaled length $[\beta] = \sum_{k \geq 0} k\beta(k)$ we learn from (2.12) by induction in l that

$$((D^{(\mathbf{0})})^l)_\beta^\gamma = 0 \quad \text{unless} \quad [\beta] = [\gamma] + l, \tag{2.13}$$

[29] which is still effectively finite when evaluated at an $a \in \mathbb{R}[u]$.

[30] e_k denotes the multi-index with $e_k(l) = \delta_k^l$.

[31] in fact, a derivation.

so that the sum

$$\sum_{l\geq 0}\frac{1}{l!}v^l(D^{(\mathbf{0})})^l c \tag{2.14}$$

is effectively finite, meaning that it is finite on the level of a component β. A collateral damage of this extension from $\mathbb{R}[\mathsf{z}_k]$ to $\mathbb{R}[[\mathsf{z}_k]]$ is that $c \in \mathbb{R}[[\mathsf{z}_k]]$ can no longer be identified with a functional on $\mathbb{R}[u]$, so that (2.11) becomes formal.

2.2.2 Coordinates z_k and $\mathsf{z}_\mathbf{n}$ for the solution manifold

The next (heuristic) task is to endow the solution manifold for (1.2) with coordinates. In case of $a \equiv 0$, which by (1.3) entails $h \equiv const$, the manifold of solutions u of (1.2) obviously is an affine space over the linear space of space-time functions p with $(\partial_2 - \partial_1^2)p = 0$; those functions p are analytic.[32] It is convenient to free oneself from the constraint $(\partial_2 - \partial_1^2)p = 0$ by extending the manifold to all space-time functions u that satisfy (1.2) up to a space-time analytic function.[33] In view of the Cauchy-Kovalevskaya theorem, one expects that for analytic a, the space of analytic space-time functions p still provides a parameterization of the (extended) nonlinear solution manifold – at least for sufficiently small a and locally near a base-point $x \in \mathbb{R}^2$.

According to (1.3), if u solves (1.2), then for any constant v, $u - v$ solves (1.2) with a replaced by $a(\cdot + v)$. Hence the manifold of all space-time functions u modulo constants that satisfy (1.2) up to a space-time analytic function – for some analytic nonlinearity a – is parameterized by the tuple (a, p) with p modulo constants. We think of p as providing a germ at the base-point $x = 0$ so that

$$\mathsf{z}_\mathbf{n}[p] = \frac{1}{\mathbf{n}!}\frac{\partial^\mathbf{n} p}{\partial y^\mathbf{n}}(0) \quad \text{for } \mathbf{n} \in \mathbb{N}_0^2 - \{(0,0)\}, \tag{2.15}$$

which are coordinates for $\{p \in \mathbb{R}[[x_1, x_2]] \,|\, p(0) = 0\}$, are natural for the parameterization near $x = 0$. Hence the union of (2.8) and (2.15) is expected to provide coordinates for the above solution manifold.

2.2.3 Definition of Π_x

The coordinates allow us to identify the general solution u with an element $\Pi_x \in C^2[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$, where $C^2[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$ denotes the space of formal power series in z_k, $\mathsf{z}_\mathbf{n}$ with coefficients given by space-time functions that are twice continuously differentiable in y_1 and continuously differentiable in y_2. On a formal level, the relationship between u and Π_x is the following: On the one hand, u can be recovered from Π_x via the series[34]

$$u(\cdot) - u(x) = \sum_\beta \Pi_{x\beta}(\cdot)\mathsf{z}^\beta[a, p], \tag{2.16}$$

[32] in fact, entire.

[33] it turns out that this extension is not explored, see (2.35).

[34] which has no reason to converge; and with p depending on x.

where the sum runs over all multi-indices β that associate a frequency to both $k \geq 0$ and $\mathbf{n} \neq \mathbf{0}$, and the monomial $\mathsf{z}^\beta = \prod_{k\geq 0, \mathbf{n}\neq\mathbf{0}} \mathsf{z}_k^{\beta(k)} \mathsf{z}_\mathbf{n}^{\beta(\mathbf{n})}$ is evaluated at (a, p) according to (2.8) and (2.15). On the other hand, $\Pi_{x\beta}$ can be recovered from u by taking the partial derivative w. r. t. the variables z_k, $\mathsf{z}_\mathbf{n}$ corresponding to the multi-index β, and then evaluating at $\mathsf{z}_k = \mathsf{z}_\mathbf{n} = 0$. In particular, we have

$$\Pi_{x\beta}(x) = 0. \tag{2.17}$$

The expansion (2.16) can be seen as a PDE version of a Butcher series[35] as extended to rough paths in [33, Sect. 5].

We note that by (2.9), $a(u)\partial_1^2 u$ formally corresponds to $\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x$, an effectively finite sum. Likewise, in view of (1.3) and (2.11), the counter term h formally corresponds to $\sum_{l\geq 0} \frac{1}{l!} \Pi_x^l (D^{(\mathbf{0})})^l c$. For any $c \in \mathbb{R}[[\mathsf{z}_k]]$, this sum is effectively finite according to (2.13). As announced in Sect. 2.1, we replace ξ by a mollified version $\xi_\tau \in C^0$. Hence to $\Pi_x \in C^2[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$ we associate the r. h. s.

$$\Pi_{x\beta}^- := \Big(\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x - \sum_{l\geq 0} \frac{1}{l!} \Pi_x^l (D^{(\mathbf{0})})^l c + \xi_\tau \mathsf{1}\Big)_\beta \in C^0, \tag{2.18}$$

where $\mathsf{1}$ is the neutral element of the algebra $\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$.

2.2.4 Definition of $\bar{\mathsf{T}}$ and T, purely polynomial and populated multi-indices β

A special role is played by the multi-indices of the form

$$\beta = e_\mathbf{n} \quad \text{for some } \mathbf{n} \neq \mathbf{0}, \quad \text{which we call "purely polynomial".} \tag{2.19}$$

In view of (2.15), the corresponding linear subspace

$$\bar{\mathsf{T}}^* = \{\, \pi \in \mathbb{R}[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]] \mid \pi_\beta = 0 \text{ unless } \beta \text{ is purely polynomial} \,\} \tag{2.20}$$

is the algebraic dual of $\bar{\mathsf{T}} \cong \mathbb{R}[y_1, y_2]/\mathbb{R}$, the space of space-time polynomials (i. e. the polynomial sector in [38, Remark 2.23]) modulo constants. By definitions (2.8) & (2.15), for $a \equiv 0$, (2.16) collapses to $u(\cdot) - u(x) = \Pi_{x0}(\cdot) + \sum_{\mathbf{n}\neq\mathbf{0}} \Pi_{xe_\mathbf{n}}(\cdot) \frac{1}{\mathbf{n}!} \frac{\partial^\mathbf{n} p}{\partial y^\mathbf{n}}(0)$. Specifying the affine parameterization[36] to be $u = u(x) + p(\cdot - x) + \Pi_{x0}$ for $a \equiv 0$, this implies that $p(\cdot - x) = \sum_{\mathbf{n}\neq\mathbf{0}} \Pi_{xe_\mathbf{n}}(\cdot) \frac{1}{\mathbf{n}!} \frac{\partial^\mathbf{n} p}{\partial y^\mathbf{n}}(0)$. Reproducing [38, Assumption 5.3], this leads us to postulate

$$\Pi_{x\beta} = (\cdot - x)^\mathbf{n} \quad \text{for } \beta \text{ of the form (2.19).} \tag{2.21}$$

A special role is played by the additive function

$$[\beta] := \sum_{k\geq 0} k\beta(k) - \sum_{\mathbf{n}\neq\mathbf{0}} \beta(\mathbf{n}) \tag{2.22}$$

[35] with a more parsimonious index set, which arises from combining terms that depend on the nonlinearity a in the same way.

[36] which is consistent with $p(0) = 0$ and (2.17).

and by the subset of multi-indices

$$\beta = e_{\mathbf{n}} \text{ for some } \mathbf{n} \neq \mathbf{0} \quad \text{or} \quad [\beta] \geq 0, \tag{2.23}$$

which we subsume by saying "β is populated".

Indeed, we claim that from (2.18) and (2.21) we obtain

$$\Pi_{x\gamma} \equiv 0 \text{ unless } \gamma \text{ is populated}$$
$$\Longrightarrow \quad \Pi^{-}_{x\beta} \in \mathbb{R}[y_1, y_2] \text{ unless } [\beta] \geq 0. \tag{2.24}$$

For the reader's convenience, the elementary argument for (2.24) is provided in Sect. 8. Since we impose the PDE (1.2) only up to analytic space-time functions, we learn from (2.24) that it is self-consistent to postulate the l. h. s. of (2.24). Hence the space-time function Π_x will have values in

$$\mathsf{T}^* = \{\, \pi \in \mathbb{R}[[\mathsf{z}_k, \mathsf{z}_{\mathbf{n}}]] \mid \pi_\beta = 0 \text{ unless } \beta \text{ is populated} \,\}, \tag{2.25}$$

the (algebraic) dual of the direct sum indexed by all populated β's, which we assimilate to Hairer's abstract model space $\mathsf{T} \supset \bar{\mathsf{T}}$.

2.2.5 Homogeneity $|\beta|$ of a multi-index β

The homogeneity $|\beta|$ of a populated multi-index β is motivated by a scaling invariance in law of the manifold of solutions to (1.2): We start with a parabolic rescaling of space and time according to $x_1 = \lambda \hat{x}_1$ and $x_2 = \lambda^2 \hat{x}_2$. Our assumption (2.7) on the noise ensemble is consistent with[37] $\xi(x) =_{\mathrm{law}} \lambda^{\alpha-2}\hat{\xi}(\hat{x})$. This translates into the desired $u(x) =_{\mathrm{law}} \lambda^{\alpha}\hat{u}(\hat{x})$, provided we transform the nonlinearities according to $a(u) = \hat{a}(\lambda^{-\alpha}u)$ and $h(u) = \lambda^{\alpha-2}\hat{h}(\lambda^{-\alpha}u)$. On the level of the coordinates (2.8) the former translates into $\mathsf{z}_k = \lambda^{-\alpha k}\hat{\mathsf{z}}_k$. When it comes to the parameter p it is consistent with the above[38] that it scales like u, i. e. $p(x) = \lambda^{\alpha}\hat{p}(\hat{x})$, so that the coordinates (2.15) transform according to $\mathsf{z}_{\mathbf{n}} = \lambda^{\alpha-|\mathbf{n}|}\hat{\mathsf{z}}_{\mathbf{n}}$, where

$$|\mathbf{n}| = n_1 + 2n_2. \tag{2.26}$$

Hence we read off (2.16) that $\Pi_{x\beta}(y) =_{\mathrm{law}} \lambda^{|\beta|}\hat{\Pi}_{\hat{x}\beta}(\hat{y})$, where

$$|\beta| := \alpha(1 + [\beta]) + |\beta|_p \quad \text{with} \quad |\beta|_p := \sum_{\mathbf{n}\neq\mathbf{0}} |\mathbf{n}|\beta(\mathbf{n}). \tag{2.27}$$

In agreement with regularity structures [38, Definition 2.1], because of $\alpha > 0$, the set A of homogeneities is locally finite and bounded from below, in fact by α in our range of $\alpha < 1$. Moreover, (2.26) and (2.27) are consistent in the sense of

$$|e_{\mathbf{n}}| = |\mathbf{n}| \quad \text{for } \mathbf{n} \neq \mathbf{0}. \tag{2.28}$$

[37] which for $\alpha = \frac{1}{2}$ turns into the well-known invariance of white noise.

[38] in particular $u = u(x) + p(\cdot - x) + \Pi_{x0}$ for $a \equiv 0$.

Thanks to our assumption of $\alpha \notin \mathbb{Q}$, there is a reverse of (2.28):

$$|\beta| \in \mathbb{N} \quad \Longrightarrow \quad \beta \text{ is purely polynomial.} \tag{2.29}$$

Both reproduce [38, Assumption 5.3]; the implication (2.29), in its negated form, plays a crucial role in the Liouville argument Proposition 5.3.

2.2.6 Postulates on Γ_{yx}

While Hairer thinks of Π_x as a linear map from the abstract model space T into the space of functions[39] of space-time, we equivalently interpret Π_x as a space-time function with values in T^*. Next to Π_x, Hairer's notion of a model features $\Gamma_{yx} \in \mathrm{End}(\mathsf{T})$ encoding the transformation from base-point y to base-point x in the sense of $\Pi_x = \Pi_y \Gamma_{yx}$. We express this in terms of the (algebraic) transpose[40] $\Gamma^*_{xy} \in \mathrm{End}(\mathsf{T}^*)$ as[41]

$$\Pi_x = \Gamma^*_{xy} \Pi_y \quad \text{modulo a space-time constant,} \tag{2.30}$$

where the modulo is a consequence of (2.17).

In particular, (2.30) suggests transitivity $\Gamma_{yx} = \Gamma_{yz}\Gamma_{zx}$, which following [38, Definition 2.17] we postulate:

$$\Gamma^*_{xy} = \Gamma^*_{xz}\Gamma^*_{zy} \quad \text{and} \quad \Gamma^*_{xx} = \mathrm{id}. \tag{2.31}$$

Likewise, following [39, Assumption 3.20] we postulate that Γ_{yx} preserves the polynomial sector $\bar{\mathsf{T}} \cong \mathbb{R}[[x_1, x_2]]/\mathbb{R}$ and acts on it according to $(\cdot - y)^{\mathbf{n}}\Gamma_{yx} = (\cdot - x)^{\mathbf{n}}$. We claim that this postulate translates into

$$(\Gamma^*_{xy})^\gamma_\beta = \left\{ \begin{array}{ll} \binom{\mathbf{n}}{\mathbf{m}}(y-x)^{\mathbf{n}-\mathbf{m}} & \text{if } \gamma = e_{\mathbf{m}} \text{ for some } \mathbf{m} \neq \mathbf{0} \\ 0 & \text{else} \end{array} \right\}, \tag{2.32}$$

provided that β is of the form (2.19), with the understanding that $\binom{\mathbf{n}}{\mathbf{m}}$ vanishes if the componentwise $\mathbf{m} \leq \mathbf{n}$ is violated. We refer to Sect. 8 for the argument. Note that (2.32) is consistent with (2.21) and (2.30).

Finally, the strict triangularity of $\Gamma_{yx} - \mathrm{id}$ with respect to the grading of T induced by the homogeneity (2.27), as postulated in [38, Definition 2.1], amounts in our setting to

$$(\Gamma^*_{xy} - \mathrm{id})^\gamma_\beta = 0 \quad \text{unless } |\gamma| < |\beta|. \tag{2.33}$$

Note that in view of (2.28), (2.32) is consistent with (2.33).

[39] in his case rather distributions, so linear forms themselves.

[40] We note that since T is infinite-dimensional, not every element of $\mathrm{End}(\mathsf{T}^*)$ is a transpose of an element in $\mathrm{End}(\mathsf{T})$. We will construct Γ^*_{xy} via its matrix representation with the property that for every β, $(\Gamma^*_{xy})^\gamma_\beta := (\Gamma^*_{xy}\mathsf{z}^\gamma)_\beta = 0$ for all but finitely many γ's, which means that we have access to Γ_{yx} even if it is not used in our approach.

[41] Note the notational abuse of swapping xy when transposing, which is done here to obtain more intuitive re-centering formulas.

2.3 Statement of main result: construction and estimates on (Π_x, Γ_{yx})

Our main result Theorem 2.2 provides a construction of a model (Π_x, Γ_{yx}) that satisfies all the postulates of the previous Sect. 2.2.

Theorem 2.2 *Under Assumption 2.1 the following holds*:
For every populated β, there exists[42]

$$\begin{aligned} &\text{a deterministic } c_\beta \in \mathbb{R} \text{ satisfying} \quad c_\beta = 0 \\ &\quad \text{unless} \quad |\beta| < 2 \quad \text{and} \quad \beta(\mathbf{n}) = 0 \text{ for all } \mathbf{n} \neq \mathbf{0}, \end{aligned} \tag{2.34}$$

and for every $x \in \mathbb{R}^2$ there exists a random $\Pi_{x\beta} \in C^2(\mathbb{R}^2)$ such that almost surely

$$(\partial_2 - \partial_1^2)\Pi_{x\beta} = \Pi^-_{x\beta} \quad \text{unless } \beta \text{ is purely polynomial}, \tag{2.35}$$

with $\Pi^-_{x\beta}$ defined in (2.18), and which is given by (2.21) for β purely polynomial.

Moreover, for every $x, y \in \mathbb{R}^2$ there exists a random $\Gamma_{yx} \in \mathrm{End}(\mathsf{T})$ such that almost surely we have (2.30), (2.31), (2.32), and (2.33).

Finally, we have for all $p < \infty$

$$\mathbb{E}^{\frac{1}{p}} |\Pi_{x\beta}(y)|^p \lesssim |y - x|^{|\beta|}, \tag{2.36}$$

$$\mathbb{E}^{\frac{1}{p}} |(\Gamma^*_{xy})^\gamma_\beta|^p \lesssim |y - x|^{|\beta| - |\gamma|} \quad \text{for all populated } \gamma, \tag{2.37}$$

where here and in the sequel, $\lesssim$ means $\le C$ with a constant C only depending[43] *on α, β and p.*

The crucial point is that the estimates (2.36) and (2.37) are independent of the mollification ξ_τ of ξ in (2.18). For pure convenience, we fix the mollification to be the semi-group convolution $(\cdot)_\tau$, so that the ultra-violet cut-off scale is given by $\sqrt[4]{\tau}$, see Sect. 4.1. Our proof extends to more general classes of ultra-violet cut-off: For instance, instead of the mollification, one could strengthen the norm (2.2) by adding, on the level of the squared norm, the term $\tau \int_{\mathbb{R}^2} dx ((\partial_1^4 - \partial_2^2)^{\frac{1}{4}(\alpha + \frac{3}{2})} \delta\xi)^2$. It would even be sufficient to impose the annealed Hölder regularity (6.1) on the level of ξ itself by means of an additional approximation argument. Since $\Pi_{x\beta} \in C^2$ almost surely, it follows from (2.36) together with $|\beta| \ge \alpha > 0$ that (2.17) holds.

2.4 BPHZ-choice of c and divergent bounds

In this subsection, we argue that the infra-red part of (2.36) enforces a canonical choice of c, for given regularization parameter τ. In fact, our inductive construction algorithm for (Π_x, c) is unique, as a benefit of working on the whole space-time and

[42] the last condition amounts to $c \in \mathbb{R}[[\mathsf{z}_k]]$.

[43] note that there is no dependence on the law, since we normalized the constant in (2.7); the dependence on α could be subsumed into the one on β.

avoiding a non-canonical infra-red truncation of the heat kernel $(\partial_2 - \partial_1^2)^{-1}$. See [61, Theorem 1.3] for a much stronger uniqueness statement.

For the rest of this subsection we fix a populated and not purely polynomial β. We start by observing that the PDE (2.35) combined with the estimate (2.36) uniquely determines $\Pi_{x\beta}$ given $\Pi^-_{x\beta}$. Indeed, the difference w of two solutions satisfies $(\partial_2 - \partial_1^2)w = 0$. Appealing to estimate (2.36) for $|y-x| \uparrow \infty$, a Liouville argument shows that w is a (random) polynomial of (parabolic) degree $\leq |\beta|$. By (2.29), this strengthens to being a polynomial of degree $< |\beta|$. Appealing to (2.36) for $|y-x| \downarrow 0$, we learn that this polynomial must vanish. We refer to the proof of Proposition 5.3 for details on an annealed version of this Liouville argument.

Our inductive algorithm is based on an ordering $\prec$ on multi-indices, see Sect. 3.5; we now argue that c_β is unique given $(\Pi_{x\gamma}, c_\gamma)$ for $\gamma \prec \beta$. Indeed, we obtain from the main estimate (2.36), with help of the kernel estimate (4.3), that the ensemble and space-time average $\lim_{t\uparrow\infty} \mathbb{E}(\partial_2 - \partial_1^2)\Pi_{x\beta t}(x)$ vanishes for $|\beta| < 2$, where 2 is the order of the differential operator $\partial_2 - \partial_1^2$; here $(\cdot)_t$ denotes the semi-group convolution, see Sect. 4.1. By the PDE (2.35) this implies

$$\lim_{t\uparrow\infty} \mathbb{E}\Pi^-_{x\beta t}(x) = 0 \quad \text{for } |\beta| < 2. \tag{2.38}$$

Writing $\Pi^-_{x\beta} = -c_\beta + \tilde{\Pi}^-_{x\beta}$, where $\tilde{\Pi}^-_{x\beta}$ does not contain the $(l=0)$-term in (2.18), we learn from (2.38) and the fact that c_β is deterministic (and a space-time constant)

$$c_\beta = \lim_{t\uparrow\infty} \mathbb{E}\tilde{\Pi}^-_{x\beta t}(x) \quad \text{for } |\beta| < 2. \tag{2.39}$$

Note that by the first part of the population condition (2.34), we don't have to consider $|\beta| \geq 2$. It thus remains to note that $\tilde{\Pi}^-_{x\beta}$ depends on $(\Pi_{x\gamma}, c_\gamma)$ only through $\gamma \prec \beta$, which we shall establish in Sect. 8. This shows that our algorithm uniquely determines c, in the spirit of a BPHZ-choice of renormalization, see [14, Theorem 6.18 and Eq. (6.25)] for the form BPHZ renormalization takes within regularity structures.

Let us sketch why (2.39) is consistent with the second part of the population condition (2.34) on c, referring to Sect. 5.1 for details. The shift invariance in law of ξ, see Assumption 2.1, which by the above uniqueness transmits to $\Pi_{x\beta}$, ensures that the r. h. s. of (2.39) is independent of x. The reflection parity in law of ξ, which transmits to $\Pi_{x\beta}$, ensures that the r. h. s. of (2.39) vanishes for odd $\sum_{\mathbf{n}\neq\mathbf{0}} n_1\beta(\mathbf{n})$, which in view of $|\beta|_p \leq |\beta| < 2$, cf. (2.27), implies the second population constraint in (2.34).

Theorem 2.2 only states those estimates that are independent of the ultra-violet cut-off provided by the convolution of the driver ξ in (2.18). In particular, c is expected to diverge as the cut-off scale[44] $\sqrt[4]{\tau}$ goes to zero. The following proposition provides the scaling-wise natural estimate on c_β, an annealed and weighted $C^{2,\alpha}$-estimate on $\Pi_{x\beta}$, and a similar C^α-estimate on $\Pi^-_{x\beta}$. By Kolmogorov's criterion, see for instance [55, proof of Lemma 4.1], the latter ensures that $\Pi_{x\beta} \in C^2$ almost surely, in line with Theorem 2.2. We stress that these estimates are not used in a quantitative

[44] as always, measured in units of x_1.

way in the proof of Theorem 2.2 – the estimates of Theorem 2.2 are orthogonal to the divergence of c. However, qualitative boundedness of c plays a role when addressing analyticity in z_0, see Sect. 2.7, and qualitative continuity is used in reconstruction, see Proposition 4.2 and Proposition 4.12. Moreover, qualitative boundedness of $\partial_1^2\Pi_x$ is convenient when rigorously establishing Malliavin differentiability of Π_x^-, see Sect. 7.2.

Proposition 2.3 (Divergent bounds I) *Under Assumption 2.1 the following holds for every populated β:*

$$|c_\beta| \lesssim (\sqrt[4]{\tau})^{|\beta|-2}, \tag{2.40}$$

$$\mathbb{E}^{\frac{1}{p}}|\partial_1^2\Pi_{x\beta}(y)|^p + \mathbb{E}^{\frac{1}{p}}|\partial_2\Pi_{x\beta}(y)|^p \lesssim (\sqrt[4]{\tau})^{\alpha-2}(\sqrt[4]{\tau}+|y-x|)^{|\beta|-\alpha}. \tag{2.41}$$

Furthermore, we have

$$\mathbb{E}^{\frac{1}{p}}|\partial_1^2\Pi_{x\beta}(y)-\partial_1^2\Pi_{x\beta}(z)|^p + \mathbb{E}^{\frac{1}{p}}|\partial_2\Pi_{x\beta}(y)-\partial_2\Pi_{x\beta}(z)|^p \tag{2.42}$$

$$+\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}^-(y)-\Pi_{x\beta}^-(z)|^p \tag{2.43}$$

$$\lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau}+|y-z|+|z-x|)^{|\beta|-\alpha}|y-z|^\alpha.$$

The proof of Proposition 2.3 is given in Sect. 6.

2.5 Exponential formula for Γ^*_{xy} and structure group G, definition of $\pi^{(\mathbf{n})}_{xy}$

In Sect. 5.3, we will construct the change-of-basepoint transformation $\Gamma^*_{xy} \in \mathrm{End}(\mathsf{T}^*)$, see (2.30), by inductively tilting the T^*-valued model Π_x with help of a space-time polynomial in order to achieve the appropriate order of vanishing of Π_y in y. The coefficients of these space-time polynomials are collected in $\{\pi^{(\mathbf{n})}_{xy}\}_{\mathbf{n}} \subset \mathsf{T}^*$, so that the $\mathbf{n}$'s range[45] over $\mathbb{N}_0^2$.

Following Hairer we adopt a more abstract point of view on the purely algebraic map $\{\pi^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^* \in \mathrm{End}(\mathsf{T}^*)$. It is given by the exponential-type formula

$$\Gamma^* = \sum_{k\geq 0}\tfrac{1}{k!}\sum_{\mathbf{n}_1,\ldots,\mathbf{n}_k}\pi^{(\mathbf{n}_1)}\cdots\pi^{(\mathbf{n}_k)}D^{(\mathbf{n}_1)}\cdots D^{(\mathbf{n}_k)}, \tag{2.44}$$

where we have set for convenience

$$D^{(\mathbf{n})} := \partial_{\mathsf{z}_{\mathbf{n}}} \quad \text{for } \mathbf{n}\neq\mathbf{0}, \tag{2.45}$$

which like $D^{(\mathbf{0})}$ defines a derivation on the algebra $\mathbb{R}[[\mathsf{z}_k,\mathsf{z}_{\mathbf{n}}]]$. We note that (2.44) is an extension of (2.11) from $\mathbb{R}[[\mathsf{z}_k]]$ to $\mathbb{R}[[\mathsf{z}_k,\mathsf{z}_{\mathbf{n}}]]$ and from $v\in\mathbb{R}$ to $\pi^{(\mathbf{0})}\in\mathsf{T}^*$. A collateral damage of the latter is that (2.44) can no longer be interpreted as a standard matrix exponential, since multiplication by $\pi^{(\mathbf{0})}$ and the derivation $D^{(\mathbf{0})}$ do not commute.[46]

[45] we will always state explicitly if we only mean $\mathbf{n}\neq\mathbf{0}$, unless it is used in $\mathbb{R}[[\mathsf{z}_k,\mathsf{z}_{\mathbf{n}}]]$.

[46] however, the derivations commute among themselves, like of course the multiplication operators do.

In line with the order of vanishing of $\Pi_{y\beta}$ to be achieved, and thus the degree of the tilting space-time polynomial, one imposes the population condition

$$\pi_{\beta}^{(\mathbf{n})} = 0 \quad \text{unless } |\mathbf{n}| < |\beta|. \tag{2.46}$$

The constraint (2.46) also ensures that the sum (2.44) over k and $\mathbf{n}_1, \ldots, \mathbf{n}_k$ is effectively[47] finite, which is not obvious, see [53, Sect. 5.1 and Eq. (5.16)] or [52, Lemma 3.12].

As shown in [53, Sect. 5.1], provided the purely polynomial part of the $\pi^{(\mathbf{n})}$'s is of the form[48]

$$\pi_{e_{\mathbf{m}}}^{(\mathbf{n})} = \left\{ \begin{array}{ll} \binom{\mathbf{m}}{\mathbf{n}} h^{\mathbf{m}-\mathbf{n}} & \text{provided } \mathbf{m} > \mathbf{n} \\ 0 & \text{else} \end{array} \right\} \tag{2.47}$$

for some space-time shift vector $h \in \mathbb{R}^2$, the corresponding set of transposed endomorphisms $\Gamma \in \mathrm{End}(\mathsf{T})$ can be assimilated to the structure group $\mathsf{G} \subset \mathrm{Aut}(\mathsf{T})$ in the sense of Hairer. In particular, such Γ's meet the postulates of regularity structures: They are strictly triangular in the sense of (2.33), see [53, Eq. (5.10)], and they respect the polynomial sector $\bar{\mathsf{T}}$ in the sense of [39, Assumption 3.20], i. e. $\Gamma(\cdot)^{\mathbf{n}} = (\cdot + h)^{\mathbf{n}}$ (modulo constants) with the h from (2.47), see [53, Eq. (5.11)]; in particular we have

$$\Gamma \bar{\mathsf{T}} \subset \bar{\mathsf{T}}. \tag{2.48}$$

Our dual perspective has the advantage of revealing that Γ^* is multiplicative:[49]

$$\Gamma^* \pi \pi' = (\Gamma^* \pi)(\Gamma^* \pi') \quad \text{provided } \pi \pi' \in \mathsf{T}^*, \quad \text{and} \quad \Gamma^* \mathsf{1} = \mathsf{1}, \tag{2.49}$$

see [53, Proposition 5.1 (ii)]. Hence[50] Γ^* is determined by its value on the coordinates z_k, $\mathsf{z}_{\mathbf{n}}$; it is straightforward to check from (2.10) and (2.45) that those are given by[51]

$$\Gamma^* \mathsf{z}_k = \sum_{l \geq 0} \binom{k+l}{k} (\pi^{(\mathbf{0})})^l \mathsf{z}_{k+l} \quad \text{for } k \geq 0, \tag{2.50}$$

$$\Gamma^* \mathsf{z}_{\mathbf{n}} = \mathsf{z}_{\mathbf{n}} + \pi^{(\mathbf{n})} \quad \text{for } \mathbf{n} \neq \mathbf{0}. \tag{2.51}$$

Moreover, the group structure becomes more apparent on the dual level:[52]

$$\{\pi^{(\mathbf{n})} + \Gamma^* \bar{\pi}^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^* \bar{\Gamma}^* \quad \text{provided} \quad \{\pi^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^* \text{ and } \{\bar{\pi}^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \bar{\Gamma}^*, \tag{2.52}$$

---

[47] i. e. on the level of the matrix entries.

[48] where $\mathbf{m} > \mathbf{n}$ means ($\mathbf{m} \geq \mathbf{n}$ and $\mathbf{m} \neq \mathbf{n}$), so that (2.47) is consistent with (2.46) in view of (2.28).

[49] due to the presence of the polynomial sector $\bar{\mathsf{T}}$, cf. (2.20), T^* is not closed under multiplication, hence Γ^* cannot be called an algebra endomorphism.

[50] also appealing to the finiteness properties that ensure the existence of the transpose.

[51] a version of (2.50) already appeared in the discussion of Sect. 2.2.1.

[52] we recall that by $\{\pi^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^*$ we understand that $\{\pi^{(\mathbf{n})}\}_{\mathbf{n}}$ gives rise to Γ^* via (2.44).

see [53, Proposition 5.1 (iii)]. For the purpose of establishing transitivity (2.31) we retain that because of the obvious $\{0\}_{\mathbf{n}} \mapsto \mathrm{id}$, we obtain from (2.52) and (qualitative) invertibility in a first stage that $\{-\Gamma^{-*}\pi^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^{-*}$, and in a second stage that

$$\begin{aligned} &\{\pi^{(\mathbf{n})} - \Gamma^*\bar{\Gamma}^{-*}\bar{\pi}^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^*\bar{\Gamma}^{-*} \quad \text{provided} \\ &\{\pi^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \Gamma^* \text{ and } \{\bar{\pi}^{(\mathbf{n})}\}_{\mathbf{n}} \mapsto \bar{\Gamma}^*. \end{aligned} \tag{2.53}$$

The construction of $\{\pi^{(\mathbf{n})}_{xy}\}_{\mathbf{n}}$ that gives rise to Γ^*_{xy} which satisfy (2.30) and (2.31) is carried out in Sect. 5.3. The purely polynomial part of $\pi^{(\mathbf{n})}_{xy}$ is forced upon us: By (2.47) only h needs to be chosen, and by (2.48) and (2.51) we see that the choice $h = y - x$ is necessary and sufficient to obtain (2.32):

$$\pi^{(\mathbf{n})}_{xye_{\mathbf{m}}} = \left\{ \begin{array}{cl} \binom{\mathbf{m}}{\mathbf{n}} (y-x)^{\mathbf{m}-\mathbf{n}} & \text{provided } \mathbf{m} > \mathbf{n} \\ 0 & \text{else} \end{array} \right\}. \tag{2.54}$$

Our estimate (2.36) of Γ^*_{xy} will be derived from an estimate of $\{\pi^{(\mathbf{n})}_{xy}\}_{\mathbf{n}}$, see the upcoming proposition, which crucially uses the effective finiteness of the sum in (2.44).

Proposition 2.4 *Under Assumption 2.1 the following holds for every populated β:*

$$\mathbb{E}^{\frac{1}{p}} |\pi^{(\mathbf{n})}_{xy\beta}|^p \lesssim |y-x|^{|\beta|-|\mathbf{n}|}. \tag{2.55}$$

Note that (2.54) is consistent with (2.55) in view of (2.28).

2.6 Augmenting the model space with $\tilde{\mathsf{T}}$ and the model with Π^-_x

There are essentially only semantic differences between our results and Hairer's postulates, as stated in [38, Definition 2.17]. The algebraic aspects of this are worked out in [53, Sect. 5.3], of which we now give a synopsis: Hairer's abstract model space actually corresponds to what in our notation is

$$\mathbb{R} \oplus \mathsf{T} \oplus \tilde{\mathsf{T}} = \mathbb{R} \oplus \bar{\mathsf{T}} \oplus \tilde{\mathsf{T}} \oplus \tilde{\mathsf{T}}, \tag{2.56}$$

where $\tilde{\mathsf{T}}$ is the direct sum over all populated, not purely polynomial multi-indices, cf. (2.23), a linear complement to $\bar{\mathsf{T}}$ in T. Hairer's abstract integration operator $\mathcal{I}$, cf. [38, Definition 5.7], is in our notation given by the identification of the second $\tilde{\mathsf{T}}$-component with the first $\tilde{\mathsf{T}}$-component in (2.56), see also [53, (5.36)]. Endowing the β-component of the second $\tilde{\mathsf{T}}$-contribution with the homogeneity $|\beta| - 2$ meets [38, Definition 5.7] for our second-order integration operator $(\partial_2 - \partial_1^2)^{-1}$. The $\mathbb{R}$-component, which is endowed with homogeneity 0, is the placeholder for the constant functions factored out in our approach to $\bar{\mathsf{T}} \cong \mathbb{R}[y_1, y_2]/\mathbb{R}$. Loosely speaking, our set-up is minimalistic.

To be consistent with (2.56), our model Π_x needs to be extended by the constant function of value 1, and by Π_x^-, which in agreement with (2.18) is given by[53]

$$\Pi_x^- = P\sum_{k\geq 0}\mathsf{z}_k\Pi_x^k\partial_1^2\Pi_x - \sum_{l\geq 0}\frac{1}{l!}\Pi_x^l(D^{(\mathbf{0})})^l c + \xi_\tau\mathsf{1} \;\in\; \tilde{\mathsf{T}}^*, \tag{2.57}$$

where P denotes the projection[54]

$$P \quad \text{of } \mathbb{R}[[\mathsf{z}_k,\mathsf{z}_{\mathbf{n}}]] \text{ on } \tilde{\mathsf{T}}^*, \tag{2.58}$$

on the algebraic dual[55] $\tilde{\mathsf{T}}^*$ of $\tilde{\mathsf{T}}$. Let us point out that the role of P in (2.57) is rather to project $\mathsf{z}_k\Pi_x^k\partial_1^2\Pi_x$ from $\mathbb{R}[[\mathsf{z}_k,\mathsf{z}_{\mathbf{n}}]]$ into T^*; the further restriction to $\tilde{\mathsf{T}}^*$ is automatic due to the presence of z_k. The second and third contributions to (2.57) belong to $\tilde{\mathsf{T}}^*$, which is obvious for the third contribution and follows from (8.3) for the second contribution. For later use, we note that by (2.48) and the definitions (2.10) & (2.45)

$$\Gamma^*\tilde{\mathsf{T}}^* \subset \tilde{\mathsf{T}}^* \quad \text{and} \quad D^{(\mathbf{n})}\mathsf{T}^* \subset \tilde{\mathsf{T}}^*. \tag{2.59}$$

Like (2.30) for Π_x we obtain for Π_x^- that

$$(\Pi_x^- - \Gamma_{xy}^*\Pi_y^-)_\beta \quad \text{is a random polynomial of degree } \leq |\beta|-2, \tag{2.60}$$

see (2.62) below for a more specific form. To be consistent with (2.56), also Γ needs to be extended to an endomorphism of (2.56). The extension of Γ in [53, (5.34)] precisely incorporates the polynomial correction terms in (2.30) & (2.60), in now full agreement with [38, Definition 2.17].

Proposition 2.5 *The transformations* (2.30) *and* (2.60) *specify to*

$$\Pi_x = \Gamma_{xy}^*\Pi_y + \pi_{xy}^{(\mathbf{0})}, \tag{2.61}$$

$$\begin{aligned}\Pi_x^- &= \Gamma_{xy}^*\Pi_y^- \\ &\quad + P\sum_{k\geq 0}\mathsf{z}_k\big(\Gamma_{xy}^*(\mathrm{id}-P)\Pi_y + \pi_{xy}^{(\mathbf{0})}\big)^k\partial_1^2\big(\Gamma_{xy}^*(\mathrm{id}-P)\Pi_y + \pi_{xy}^{(\mathbf{0})}\big).\end{aligned} \tag{2.62}$$

where $\pi_{xy}^{(\mathbf{n})}$ *is related to* Γ_{xy}^* *via* (2.44).

The re-centering properties (2.61) and (2.62) are established in Propositions 5.3 and 5.2, respectively.

An alternative strengthening of (2.60), which we will not make any use of, is given in [53, (5.29)] in the form

$$\Pi_x^- = \Gamma_{xy}^*\Pi_y^- + \sum_{\mathbf{n}\neq\mathbf{0}}(P\pi_{xy}^{(\mathbf{n})})(\partial_2-\partial_1^2)(\cdot - y)^{\mathbf{n}}.$$

---

[53] with the abuse of notation $\Pi_x^- \in \tilde{\mathsf{T}}^*$, meaning that Π_x^- is a $\tilde{\mathsf{T}}^*$-valued function.

[54] note that (2.58) differs from [57] and [53] where P denotes the projection onto $\bar{\mathsf{T}}^*$.

[55] which as a linear space is the direct product over the populated not purely polynomial multi-indices.

To show this, first, in view of (2.21), (2.35) may be extended to all multi-indices:

$$(\partial_2 - \partial_1^2)\Pi_y = \Pi_y^- + \sum_{\mathbf{n}\neq\mathbf{0}} \mathsf{z}_{\mathbf{n}}(\partial_2 - \partial_1^2)(\cdot - y)^{\mathbf{n}}.$$

We apply Γ_{xy}^* to this identity, and use (2.30) on the l. h. s. and (2.51) on the r. h. s. We then apply P to the resulting identity, and use (2.35) on the l. h. s. and the first item of (2.59) on the r. h. s.

We learn from evaluating (2.61) and using (2.17) that

$$\Pi_x(y) = \pi_{xy}^{(\mathbf{0})}, \tag{2.63}$$

which will play a major role in the proof. With help of (2.35), it is easy to extend (2.36) from Π_x to[56] Π_x^-:

Proposition 2.6 *Under Assumption 2.1 the following holds for every populated β:*

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta t}^-(y)|^p \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t} + |y-x|)^{|\beta|-\alpha}. \tag{2.64}$$

The estimate (2.64) actually holds for any convolution kernel in the sense of [38, Definition 2.17], as can be seen from the upcoming argument. The setting of [38, Definition 2.17] corresponds to $|y-x| \le \sqrt[4]{t}$ here, in which case the r. h. s. of (2.64) is $\sim (\sqrt[4]{t})^{|\beta|-2}$, in line with the postulated homogeneity $|\beta| - 2$ of the β-component of $\tilde{\mathsf{T}}$ at the beginning of this subsection. For (2.64) (and thus β not purely polynomial thanks to P in (2.57)), we appeal to (2.35), to which we apply the convolution operator and then $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$:

$$\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta t}^-(y)|^p \le \int dz |(\partial_2 - \partial_1^2)\psi_t(y-z)| \mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(z)|^p.$$

Hence (2.64) follows from (2.36) via the moment bounds (4.3) on ψ_t.

2.7 Relation to model in [57]

In this subsection we connect our notion of model constructed in Theorem 2.2 to the one postulated in [57]. The following is specific to the quasi-linear problem and unrelated to the proof of our main result, and thus may be skipped.

The quasi-linear equation (1.2) differs from a semi-linear equation, like the multiplicative heat equation $(\partial_2 - \partial_1^2)u + h(u) = a(u)\xi$, by the absence of the invariance in law $u = \lambda\hat{u}$, $a = \lambda\hat{a}$, $h = \lambda\hat{h}$. This scale invariance would be encoded by the tighter population condition $\sum_k (k-1)\beta(k) + \sum_{\mathbf{n}} \beta(\mathbf{n}) = -1$, compare with (2.23), see [53, Eq. (7.2)] for details on the latter. However, this lack of the tighter population condition for the quasi-linear equation (1.2) is compensated by the special role of z_0: A priori only formal power series in z_0 are convergent power series. Indeed, in view of

[56] In fact, the actual proof proceeds the other way around, namely passing from Π_x^- to Π_x, cf. Proposition 4.3 below.

(2.8), changing the origin of a-space from 0 to some $a_0 - 1$ amounts to the actual replacement

$$\partial_2 - \partial_1^2 \quad \rightsquigarrow \quad \partial_2 - a_0\partial_1^2 \tag{2.65}$$

next to the formal replacement $\mathsf{z}_0 \rightsquigarrow \mathsf{z}_0 + (a_0 - 1)$. Hence we expect analyticity as long as the real part of $a_0 \in \mathbb{C}$ is positive. Replacing the formal variable z_0 by the actual variable $a_0 - 1$ will allow us to eventually restrict to multi-indices β with

$$\beta(k = 0) = 0. \tag{2.66}$$

Let us be more precise: Any formal power series π in the variables $\{\mathsf{z}_k\}_{k\geq 1}$ and $\{\mathsf{z}_\mathbf{n}\}_{\mathbf{n}\neq\mathbf{0}}$ with coefficients that are analytic functions in $a_0 \in \mathbb{C}$ with positive real part can be identified with an element of $\mathbb{C}[[\{\mathsf{z}_k\}_{k\geq 0}, \{\mathsf{z}_\mathbf{n}\}_{\mathbf{n}\neq\mathbf{0}}]]$ via

$$\pi_{\beta+le_0} = \frac{1}{l!}\frac{d^l\pi_\beta}{da_0^l}(a_0 = 1) \quad \text{for } \beta \text{ satisfying (2.66) and } l \geq 0. \tag{2.67}$$

Hence this new algebra is canonically (and strictly) embedded into our $\mathbb{C}[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$. Hence we will pass from T^* to its intersection with this new algebra. As a linear space, this smaller T^* amounts to the direct product of the space of analytic functions in a_0, indexed by populated multi-indices satisfying (2.66). Under the identification (2.67), the derivation $D^{(\mathbf{0})}$ defined in (2.10) restricts to a derivation on the new algebra (formally) given by

$$D^{(\mathbf{0})} = \mathsf{z}_1\partial_{a_0} + \sum_{k\geq 1}(k+1)\mathsf{z}_{k+1}\partial_{\mathsf{z}_k}. \tag{2.68}$$

Analogous to (2.12), as an endomorphism on the tighter version of T^*, it is (rigorously) given through its matrix entries

$$(D^{(\mathbf{0})})_\beta^\gamma = \left\{\begin{array}{cc} \partial_{a_0} & \text{provided } \gamma + e_1 = \beta \\ 0 & \text{else} \end{array}\right\} + \sum_{k\geq 1}\left\{\begin{array}{cc} (k+1)\gamma(k) & \text{provided } \gamma + e_{k+1} = \beta + e_k \\ 0 & \text{else} \end{array}\right\}, \tag{2.69}$$

where both β and γ satisfy (2.66). Given elements $\pi^{(\mathbf{n})}$ of the tighter version of T^* satisfying the population condition (2.46), (2.44) then defines a triangular automorphism on the tighter version of T^*. This gives rise to a re-interpretation of $\mathsf{G}^* \in \mathrm{End}(\mathsf{T}^*)$ as the dual of the structure group.

Proposition 2.7 *When replacing* (2.35) *by*

$$(\partial_2 - a_0\partial_1^2)\Pi_{x\beta} = \Pi^-_{x\beta} \quad \textit{for } \beta \textit{ not purely polynomial} \tag{2.70}$$

with $a_0 \in \mathbb{C}$ of positive real part, all statements of Theorem 2.2 hold locally uniformly in a_0. Also the estimates (2.40) & (2.41) on c & $\partial_1^2\Pi_x$, as well as the estimate (2.55) on $\pi_{xy}^{(\mathbf{n})}$ hold locally uniformly.

For any base point x, y, any $\mathbf{n}$ and any populated multi-index β,

$$c_\beta \textit{ is analytic}, \tag{2.71}$$

$$\Pi_{x\beta} \textit{ is analytic w. r. t. the norm given by (2.36)}, \tag{2.72}$$

$$\pi^{(\mathbf{n})}_{xy\beta} \textit{ is analytic w. r. t. the norm given by (2.55)}, \tag{2.73}$$

all w. r. t. the variable a_0.

For these three objects $\pi = c, \pi^{(\mathbf{n})}_{xy}, \Pi_x$, (2.67) holds for all a_0:

$$\pi_{\beta+le_0} = \frac{1}{l!}\frac{d^l\pi_\beta}{da_0^l} \quad \textit{for } \beta \textit{ satisfying (2.66) and } l \geq 0. \tag{2.74}$$

Noting that (2.18) takes the form

$$\Pi^-_{x\beta} = \Big(\sum_{k\geq 1} \mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x - \sum_{k\geq 0} \frac{1}{k!} \Pi_x^k (D^{(\mathbf{0})})^k c + \xi_\tau \mathsf{1}\Big)_\beta \quad \text{for } \beta \text{ satisfying (2.66)}, \tag{2.75}$$

we learn from (2.74) that when equipped with the interpretation (2.69) of $D^{(\mathbf{0})}$ and the corresponding interpretation of Γ^*_{xy} via (2.44), the restriction of the a_0-dependent model to multi-indices of the form (2.66) is closed. Together with (2.70), this is the input for [57].

On the above class (2.66) of β's, the homogeneity $|\cdot|$ defined in (2.27) is actually coercive. In the case of $\alpha = \frac{1}{2}-$ relevant for white noise, we are left with 10 multi-indices satisfying $|\beta| < 2$. Ordered by increasing homogeneity they are given in Table 1.

Table 1 Multi-indices β with $|\beta| < 2$ for $\alpha \in (\frac{2}{5}, \frac{1}{2})$

Homogeneity	Multi-indices
α	0
2α	e_1
3α	$e_2, 2e_1$
$\alpha+1$	$e_1 + e_{(1,0)}$
4α	$e_3, e_1+e_2, 3e_1$
$2\alpha+1$	$e_2+e_{(1,0)}, 2e_1+e_{(1,0)}$

For the 10 multi-indices in Table 1, the combination of (2.70) and (2.75) takes the form

$$(\partial_2 - a_0\partial_1^2)\Pi_{x0} = \xi_\tau - c_0,$$

$$(\partial_2 - a_0\partial_1^2)\Pi_{xe_1} = \Pi_{x0}\partial_1^2\Pi_{x0} - c_{e_1},$$

$$
\begin{aligned}
(\partial_2 - a_0\partial_1^2)\Pi_{xe_2} &= \Pi_{x0}^2\partial_1^2\Pi_{x0} - (c_{e_2} + 2\Pi_{x0}c_{e_1}),\\
(\partial_2 - a_0\partial_1^2)\Pi_{x2e_1} &= \Pi_{x0}\partial_1^2\Pi_{xe_1} + \Pi_{xe_1}\partial_1^2\Pi_{x0} - (c_{2e_1} + \Pi_{x0}\partial_{a_0}c_{e_1}),\\
(\partial_2 - a_0\partial_1^2)\Pi_{xe_3} &= \Pi_{x0}^3\partial_1^2\Pi_{x0} - (c_{e_3} + 3\Pi_{x0}c_{e_2} + 3\Pi_{x0}^2c_{e_1}),\\
(\partial_2 - a_0\partial_1^2)\Pi_{xe_1+e_2} &= \Pi_{x0}\partial_1^2\Pi_{xe_2} + \Pi_{xe_2}\partial_1^2\Pi_{x0}\\
&\quad + 2\Pi_{x0}\Pi_{xe_1}\partial_1^2\Pi_{x0} + \Pi_{x0}^2\partial_1^2\Pi_{xe_1}\\
&\quad - (c_{e_1+e_2} + 2\Pi_{xe_1}c_{e_1} + \Pi_{x0}\partial_{a_0}c_{e_2}\\
&\quad + 4\Pi_{x0}c_{2e_1} + 3\Pi_{x0}^2\partial_{a_0}c_{e_1}),\\
(\partial_2 - a_0\partial_1^2)\Pi_{x3e_1} &= \Pi_{x0}\partial_1^2\Pi_{x2e_1} + \Pi_{x2e_1}\partial_1^2\Pi_{x0} + \Pi_{xe_1}\partial_1^2\Pi_{xe_1}\\
&\quad - (c_{3e_1} + \Pi_{xe_1}\partial_{a_0}c_{e_1} + \Pi_{x0}\partial_{a_0}c_{2e_1} + \Pi_{x0}^2\tfrac{1}{2}\partial_{a_0}^2c_{e_1}),\\
(\partial_2 - a_0\partial_1^2)\Pi_{xe_1+e_{(1,0)}} &= \Pi_{xe_{(1,0)}}\partial_1^2\Pi_{x0},\\
(\partial_2 - a_0\partial_1^2)\Pi_{xe_2+e_{(1,0)}} &= 2\Pi_{xe_{(1,0)}}\Pi_{x0}\partial_1^2\Pi_{x0} - 2\Pi_{xe_{(1,0)}}c_{e_1},\\
(\partial_2 - a_0\partial_1^2)\Pi_{x2e_1+e_{(1,0)}} &= \Pi_{x0}\partial_1^2\Pi_{xe_1+e_{(1,0)}} + \Pi_{xe_1+e_{(1,0)}}\partial_1^2\Pi_{x0}\\
&\quad + \Pi_{xe_{(1,0)}}\partial_1^2\Pi_{xe_1} - \Pi_{xe_{(1,0)}}\partial_{a_0}c_{e_1}.
\end{aligned}
$$

Together with the BPHZ-choice of renormalization contained in the large-$\sqrt[4]{t}$ behavior imposed on $\Pi_{x\beta}$ through (2.36), this inductively determines the functions $c_\beta(a_0)$. Equipped with these, (1.3) takes the form

$$
h(u) = \sum_{\beta:|\beta|<2} c_\beta(a(u)) \prod_{k\geq 1}\Big(\frac{1}{k!}\frac{d^k a}{du^k}(u)\Big)^{\beta(k)},
$$

which reproduces [57, Eq. (15)] in the present paper's notation.

It is in this form we may connect to [57, Assumptions 1 and 2]. Loosely speaking, the assumptions in [57] are contained in the output of Theorem 2.2, as upgraded through Proposition 2.7. More precisely, [57, Eq. (5)] is covered by (2.35) in the form of (2.70), and [57, Eq. (6)] is covered by (2.21). The estimates [57, Eq. (7) and (8)] are covered by (2.64) and (2.36), with the difference that in [57] (like in [38]), they are formulated in terms of a general (though fixed) convolution kernel, and that they are pathwise, with a constant absorbed into a single scaling factor N_0 and, as mentioned above, locally uniform in a_0, cf. [57, Eq. (30)]. The transformation [57, Eq. (9)] is reproduced by (2.61). Estimate [57, Eq. (10)] is covered by (2.37); however, in view of (2.68), the entries of Γ_{xy} are differential operators in a_0. Finally [57, Eq. (11)] follows from evaluating (2.75) at x while appealing to (2.17). The crucial population condition (2.34) on c is contained in the text just above [57, Eq. (11)], and re-formulated as $D^{(\mathbf{n})}c = 0$ for $\mathbf{n} \neq \mathbf{0}$. The polynomial correction in (2.60) does not appear in [57], since there, the model is (implicitly) truncated beyond $|\beta| < 2$. Because of this truncation, only $\mathbf{n} = \mathbf{0}, (1,0)$ matter in [57]; however, since [57] considers d space dimensions, there are d versions of $\mathbf{n} = (1,0)$. There are some differences in notation: When it comes to Γ, [57] omits the $*$ but exchanges the order in xy, while c is called q.

3 Structure of proof

In this section, all multi-indices β, β', γ are implicitly assumed to be populated, cf. (2.23). The induction runs over populated multi-indices β which are not purely polynomial. The purely polynomial multi-indices are treated before the inductive proof: If $\beta = e_{\mathbf{n}}$ for $\mathbf{n} \neq \mathbf{0}$, we must set $c_\beta = 0$ and $\Pi^-_{x\beta} = 0$ by (2.34) and (2.57), respectively, and define $\Pi_{x\beta}$ by (2.21) and $\pi^{(\mathbf{n})}_{xy\beta}$ by (2.54). We note that these definitions are consistent with covariance under shift (5.1) and reflection (5.2). Finally, (2.36) and (2.55) are satisfied by (2.28).

3.1 Intertwining of estimates and constructions in induction proof

Working on the whole space-time $\mathbb{R}^2$ instead of a torus, as we do, has many advantages. The most obvious is that we do not introduce an artificial scale, namely the size of the torus, that would break scaling. Another advantage is that the inversion of $(\partial_2 - \partial_1^2)$ does not require a Fredholm alternative and thus a Lagrange parameter.[57] However, an inconvenience is that we can't separate the construction from the estimates: Because of possible infra-red divergences, we need the large-$\sqrt[4]{t}$ part of the estimate (2.64) on Π^-_x to uniquely solve (2.35) for Π_x within the growth and anchoring expressed by (2.36). In fact, it is not clear whether one can construct a pre-model in the sense of [39, Sect. 4.2] on the whole space.

While construction and estimates are logically intertwined, as explained in Sect. 3.4 (see also Appendix B), we choose to separate them in presentation: Sect. 4 contains the uniform in the ultra-violet cut-off estimates and their proofs, while Sect. 5 contains the construction. Section 6 contains the proof of the divergent (in the ultra-violet cut-off) estimates of Proposition 2.3 and the proof of Proposition 2.7. In fact, also the proof of Malliavin differentiability is intertwined logically; it is given in Sect. 7, while definitions and properties of the Malliavin–Sobolev spaces can be found in Appendix A. Moreover, the order we present the estimates within the Sects. 4 and 6 and the Malliavin differentiability in Sect. 7 is not strictly by logical order, but according to the objects estimated. We explain this inherent structure in Sects. 3.2 and 3.3. Section 8 establishes the various triangular structures important for the inductive construction.

3.2 The five loops of an induction step: original quantities, expectation, Malliavin derivative, modelled distribution, and back

The structure of an induction step requires the distinction of two cases:[58]

[57] which is polynomial when working on the tensor space of periodic functions and polynomials, as in [52, Sect. 3.1].

[58] even if we were just interested in singular β's, the structure of $d\Gamma^*$ requires the estimate of regular β's.

- the regular case of $|\beta| \geq 2$, and
- the singular case of $|\beta| < 2$.

It is thus convenient to introduce the following projection

$$Q \quad \text{is the projection onto the direct product indexed by } \beta\text{'s with } |\beta| < 2. \tag{3.1}$$

The proof of the estimates in Sect. 4 is structured into five subsections, each having the structure of a loop, and which we order by increasing complexity,

- Original quantities: In Sect. 4.2, we estimate Γ^*_{xy}, Π^-_x, and Π_x, assuming control of $Q\Pi^-_x$.
- Expectation: In Sect. 4.3, we show that the BPHZ-choice of renormalization gives control of $\mathbb{E}Q\Pi^-_x$. By the SG inequality, this gives control of Π^-_x, assuming control of the (directional) Malliavin derivative $Q\delta\Pi^-_x$.
- Malliavin derivatives: In Sect. 4.4, we estimate the Malliavin derivatives $Q\delta\Gamma^*_{xy}P$ and $Q\delta\Pi_x$, assuming control of $Q\delta\Pi^-_x$.
- Modelled distribution: In Sect. 4.5, we introduce $\mathrm{d}\Gamma^*_{xz} \in \mathrm{Hom}(\mathsf{T}^*, \tilde{\mathsf{T}}^*)$, of which we show that it is continuous in[59] z in the sense of controlling $Q(\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy})PQ$, that it describes $Q\delta\Pi^-_x$ in terms of $Q\Pi^-_z$ in the sense of controlling $Q(\delta\Pi^-_x - \mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)$, and that it describes $Q\delta\Pi_x$ in terms of $Q\Pi_z$ in the sense of controlling the rough-path increments $Q(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz}Q\Pi_z)$. This subsection is the core of the proof.
- Back to the estimate of $Q\delta\Pi^-_x$ itself. In Sect. 4.6, we provide control of $Q\mathrm{d}\Gamma^*_{xz}P$ and then of $Q\delta\Pi^-_x$, assuming control of $Q(\delta\Pi^-_x - \mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)$.

3.3 The four types of tasks in a loop: algebraic argument, reconstruction, integration, three-point argument

Sections 4.2, 4.4, 4.5, and, to some extent, Sect. 4.6 and Sect. 7 involve the same type of tasks, arranged in a similar type of loop (see Table 2). An important role is played by the $\pi^{(\mathbf{n})}_{xy}$'s that determine the Γ^*_{xy} via the exponential formula (2.44), and their counterparts $\mathrm{d}\pi^{(\mathbf{n})}_{xz}$ for $\mathrm{d}\Gamma^*_{xz}$, see (4.40). The four tasks are:

- Algebraic argument. All four subsections start[60] with an "algebraic argument" (called like this because it is based on an exponential-type formula) to estimate $\Gamma^*_{xy}P$, $Q\delta\Gamma^*_{xy}P$, $Q(\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy})PQ$, and $Q\mathrm{d}\Gamma^*_{xz}P$, in terms of $\pi^{(\mathbf{n})}_{xy}$, $Q\delta\pi^{(\mathbf{n})}_{xy}$, $Q(\mathrm{d}\pi^{(\mathbf{n})}_{xy} - \mathrm{d}\pi^{(\mathbf{n})}_{xz} - \mathrm{d}\Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})$, and $Q\mathrm{d}\pi^{(\mathbf{n})}_{xz}$, respectively.
- Reconstruction. Sections 4.2 and 4.5 feature a reconstruction argument in order to control $(\mathrm{id} - Q)\Pi^-_x$ and $Q(\delta\Pi^-_x - \mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)$. By a reconstruction argument, we understand that for a family $\{F_x\}_{x\in\mathbb{R}^2}$ of Schwartz distributions on $\mathbb{R}^2$ that satisfy a continuity condition in the base point x, we estimate $F_{xt}(x)$ in terms of the diagonal[61] $F_x(x)$.

[59] what we call the secondary base point.

[60] and for Sect. 4.2 ends.

[61] in the sense of $\lim_{t\downarrow 0} F_{xt}(x)$ with the understanding that this limit exists.

- Integration. Sections 4.2, 4.4, 4.5 involve an integration argument to pass from Π_x^-, $Q\delta\Pi_x^-$, and $Q(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)$, to Π_x, $Q\delta\Pi_x$, and $Q(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^* Q\Pi_z)$, respectively. By an integration argument, we mean that we pass an annealed Hölder norm anchored in a base point x through an integral representation. It amounts to a Schauder estimate.
- Three-point argument. All four subsections appeal to a "three-point argument" (called like this because it involves varying an additional third point in order to control polynomial coefficients) to pass from the estimate of Π_x, $Q\delta\Pi_x$, $Q(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^* Q\Pi_z)$, and $Q\mathrm{d}\Gamma_{xz}^* P$, to the estimate of $\pi_{xy}^{(\mathbf{n})}$, $Q\delta\pi_{xy}^{(\mathbf{n})}$, $Q(\mathrm{d}\pi_{xy}^{(\mathbf{n})} - \mathrm{d}\pi_{xz}^{(\mathbf{n})} - \mathrm{d}\Gamma_{xz}^*\pi_{zy}^{(\mathbf{n})})$, and $Q\mathrm{d}\pi_{xz}^{(\mathbf{n})}$, respectively.

3.4 The logical order of loops and tasks in one induction step

In the course of Sects. 4 and 5, we will add a fairly large number of auxiliary statements. Some have to be logically included into the induction statement, because we refer to them as part of the induction hypothesis. For the convenience of the reader, we list all statements of the induction hypothesis here:

- The transitivity of Γ_{xy}^* (2.31) and $\pi_{xy}^{(\mathbf{n})}$ (5.10), and the recentering of Π_x^- (2.60) and Π_x (2.61),
- the estimates of the main objects Π_x (2.36), Γ_{xy}^* (2.37), $\pi_{xy}^{(\mathbf{n})}$ (2.55), and Π_x^- (2.64),
- the boundedness of Malliavin derivatives $\delta\Pi_x^-$ (4.22), $\delta\pi_{xy}^{(\mathbf{n})}$ (4.25), $\delta\Gamma_{xy}^*$ (4.27), and $\delta\Pi_x$ (4.34), and the modeledness of the latter (4.89),
- the boundedness (4.107), (4.108), and continuity (4.44), (4.47) of the modelled distribution $\mathrm{d}\Gamma_{xz}^*$,
- the divergent bounds of c (2.40), $\partial^{\mathbf{m}}\Pi_x$ (2.41), (2.42), (6.5), Π_x^- (2.43), and $\partial^{\mathbf{m}}\delta\Pi_x$ (4.53), (4.54), (6.11),
- symmetries, i. e. shift (5.1) and reflection (5.2) covariances of Π_x,
- and the Malliavin differentiability of Π_x, $\partial_1^2\Pi_x$, $\partial_2\Pi_x$, Γ_{xy}^*, and $\pi_{xy}^{(\mathbf{n})}$.

The remaining additional statements are just used inside one induction step and are therefore not listed above, for example: the estimates (4.18) and (4.20) on the expectation, the rough-path estimates on $\delta\Pi^-$ (4.52), the shift and reflection covariances (5.3) and (5.4) on the level of Π^- and Malliavin differentiability of the singular components of Π^-, see Item 2 in Sect. 7.

The logical order of one induction step depends on whether β is regular or singular. For regular β, we just run through the first Sect. 4.2 (but still most of the construction):

1. By the induction hypothesis, $\Gamma_{xy}^* P$ is constructed and estimated via an algebraic argument, see Proposition 4.1.
2. Because of $(\mathrm{id} - Q)c = 0$, we construct Π_x^- and show
   a) its continuity, see (2.43) in Proposition 2.3,
   b) its shift and reflection covariance, see the first part of Proposition 5.1,
   c) and the recentering property (2.62), see Proposition 5.2.
3. We then estimate Π_x^- by regular reconstruction, see Proposition 4.2.
4. We then may

   a) construct and estimate Π_x in an integration step, see Proposition 4.3,
   b) and show shift and reflection covariance, see Sect. 5.2.
5. Next we construct $\pi_{xy}^{(\mathbf{n})}$, and thus the full Γ_{xy}^*,
   a) obtain the recentering property (2.61), see Proposition 5.3,
   b) establish (5.10), and therefore (2.31), see Proposition 5.4,
   c) and estimate $\pi_{xy}^{(\mathbf{n})}$ via a three-point argument, see Proposition 4.4, and consequently Γ_{xy}^*, see Proposition 4.5.
6. We conclude showing boundedness and continuity of $\partial_1^2\Pi_x$ and $\partial_2\Pi_x$, see (2.41) and (2.42) in Proposition 2.3.

The logical order is much more complex for singular β:

1. Algebraic arguments. By induction hypothesis:
   a) $\Gamma_{xy}^* P$ is constructed and estimated, see Proposition 4.1.
   b) We establish the Malliavin differentiability of $\Gamma_{xy}^* P$, see Sect. 7.4.
   c) This allows to estimate $Q\delta\Gamma_{xy}^* P$, see Proposition 4.8,
   d) $Q\mathrm{d}\Gamma_{xy}^* P$, see Proposition 4.16,
   e) and $Q(\mathrm{d}\Gamma_{xy}^* - \mathrm{d}\Gamma_{xz}^*\Gamma_{zy}^*)PQ$, see Proposition 4.11.
2. Next we establish some properties which are independent of the specific value of c, and thus can be shown before the BPHZ choice of renormalization constants. In particular, we show that for any c satisfying (2.34),
   a) $Q\Pi_x^-$ satisfies the recentering property (2.62), see Proposition 5.2,
   b) $Q\Pi_x^-$ is shift and reflection covariant, see Sect. 5.1,
   c) and $\frac{d}{dt}\mathbb{E}Q\Pi_{xt}^-$ is estimated in Proposition 4.6.
3. We now choose the BPHZ renormalization constant c and estimate $\mathbb{E}\Pi_x^-$.
   a) We show that $\lim_{t\uparrow\infty}\mathbb{E}\Pi_{xt}^-(x)$ exists and therefore we can choose c so that (2.38) holds, see Proposition 5.1,
   b) and show the divergent bound (2.40), see Step 2 of the proof of Proposition 2.3.
   c) We then estimate $\mathbb{E}Q\Pi_x^-$, see Proposition 4.7.
4. Next we study the Malliavin derivative of Π_x^-.
   a) By the induction hypothesis, we show Malliavin differentiability of $Q\Pi_x^-$, see Sect. 7.2.
   b) We show continuity of $Q\delta\Pi_x^-$, see (4.55) in Proposition 4.13.
   c) Next we estimate $Q(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)$, see Proposition 4.12,
   d) and finally estimate $Q\delta\Pi_x^-$ itself, see Proposition 4.18.
5. Equipped with the estimates of $\mathbb{E}Q\Pi_x^-$ and $Q\delta\Pi_x^-$, we use the SG inequality to gain control of $Q\Pi_x^-$.
6. At this stage we may show the results of our main theorem.
   a) We construct and estimate Π_x, see Proposition 4.3,
   b) and show its shift and reflection covariance, see Sect. 5.2.
   c) Next we construct $\pi_{xy}^{(\mathbf{n})}$ and thus the full Γ_{xy}^*, obtaining the recentering property (2.61), see Proposition 5.3,
   d) establish the recentering property (5.10), and therefore (2.31), see Proposition 5.4,
   e) and finally estimate $\pi_{xy}^{(\mathbf{n})}$, see Proposition 4.4, and consequently Γ_{xy}^*, see Proposition 4.5.

7. Next we turn to the Malliavin derivatives of our main objects.
   a) We establish Malliavin differentiability of Π_x, see Sect. 7.3,
   b) and estimate $Q\delta\Pi_x$, see Proposition 4.9.
   c) We show that $\pi_{xy}^{(\mathbf{n})}$ is Malliavin differentiable, see Sect. 7.4, and so is Γ_{xy}^*,
   d) and estimate $\delta\pi_{xy}^{(\mathbf{n})}$, see Proposition 4.10, which in turn implies the estimate of the full $\delta\Gamma_{xy}^*$.
8. Next we show the bounds divergent in the ultra-violet cut-off.
   a) We first show the continuity of Π_x^-, see (2.43) in Proposition 2.3.
   b) Next we establish boundedness and continuity of $\partial_1^2\Pi_x$ and $\partial_2\Pi_x$, see (2.41) and (2.42) in Proposition 2.3.
   c) Finally we obtain boundedness and continuity of $\partial_1^2\delta\Pi_x$ and $\partial_2\delta\Pi_x$, see (4.53) and (4.54) in Proposition 4.13.
9. Finally we establish the modeledness estimates.
   a) We construct $\mathrm{d}\pi_{xy}^{(1,0)}$ and $\mathrm{d}\Gamma_{xy}^*$, see Sect. 5.4.
   b) We estimate $Q(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^* Q\Pi_z)$, see Proposition 4.14,
   c) $Q(\mathrm{d}\pi_{xy}^{(\mathbf{n})} - \mathrm{d}\pi_{xz}^{(\mathbf{n})} - \mathrm{d}\Gamma_{xz}^*\pi_{zy}^{(\mathbf{n})})$, see Proposition 4.15, which in turn implies the continuity estimate of the full $\mathrm{d}\Gamma_{xz}^*$,
   d) and $\mathrm{d}\pi_{xz}^{(1,0)}$, see Proposition 4.17, which in turn implies the boundedness estimate of the full $\mathrm{d}\Gamma_{xz}^*$.

The logical order of the core estimates, from Proposition 4.1 to Proposition 4.18, corresponding to the five loops of an induction step is indicated in Table 2 by the small number in the lower right corner of each field. A detailed overview of the logical order in the regular and singular case including precise input and output of each statement is given in Tables 4 and 5, respectively; see Appendix B.

3.5 The ordering relation $\prec$ for the induction

At first sight, one would hope for an induction in the homogeneity $|\beta|$; the set A of homogeneities, being locally finite and bounded from below, lends itself to an induction argument. However, this is not possible because as opposed to Γ_{xy}^*, or the structurally closer $\delta\Gamma_{xy}^*$, $\mathrm{d}\Gamma_{xz}^*$ is *not* triangular w. r. t. $|\cdot|$. As we shall see in Sect. 4.5, this lies in the nature of $\mathrm{d}\Gamma_{xz}^*$: $\delta\Pi_{x\beta}$ is modelled (almost) to order $\frac{3}{2}+\alpha$, independently of β. Here is a simple example for the failure of triangularity: On the one hand we have[62] $(\mathrm{d}\Gamma_{xz}^*)_{e_1}^{e_1+e_{(1,0)}} = \partial_1\delta\Pi_{x0}(z)$, which does not vanish for generic z, while on the other hand, $|e_1| = 2\alpha < \alpha+1 = |e_1+e_{(1,0)}|$. Even block triangularity with respect to the threshold homogeneity 2 fails: $(\mathrm{d}\Gamma_{xz}^*)_{2e_1+e_{(1,0)}}^{2e_1+2e_{(1,0)}} = 2\partial_1\delta\Pi_{x0}(z) \neq 0$, while $|2e_1+e_{(1,0)}| = 2\alpha+1 < 2+\alpha = |2e_1+2e_{(1,0)}|$. This however does not create any problems in the induction: Since $\mathrm{d}\Gamma^*$ is never applied to the objects $\delta\Pi$, $\delta\Pi^-$, $\delta\pi^{(\mathbf{n})}$, $\mathrm{d}\pi^{(\mathbf{n})}$ and $\delta\Gamma^*$, we will never appeal to statements involving them for multi-indices $|\beta| \geq 2$.

[62] as a consequence of the definition (4.40) of $\mathrm{d}\Gamma_{xz}$, and using $\mathrm{d}\pi_{xz0}^{(1,0)} = \partial_1\delta\Pi_{x0}(z)$, as a consequence of the definition (5.29) of $\mathrm{d}\pi_{xz0}^{(1,0)}$ and the triangular structure (8.11) of $\mathrm{d}\Gamma_{xz}^*$ w. r. t. $\prec$.

Table 2 Columns correspond to the five loops of an induction step, cf. Sect. 3.2, rows correspond the four types of tasks in a loop, cf. Sect. 3.3. The numbers in the lower right corner indicate the logical order within an induction step for singular multi-indices, cf. Sect. 3.4. The minor tasks of Malliavin differentiability and divergent bounds in the ultra-violet cut-off are not included

Original quantities	Expectation	Malliavin derivatives	Path increments	Back to $\delta\Pi_x^-$
$\Gamma^*_{xy}, \Pi_x^-, \Pi_x, \pi^{(\mathbf{n})}_{xy}$	$\mathbb{E}\Pi_x^-$	$\delta\Gamma^*_{xy}, \delta\Pi_x^-, \delta\Pi_x, \delta\pi^{(\mathbf{n})}_{xy}$	$d\Gamma^*_{xy} - d\Gamma^*_{xz}\Gamma^*_{zy}, \delta\Pi_x^- - d\Gamma^*_{xz}\Pi_z^-,$ $\delta\Pi_x - \delta\Pi_x(z) - d\Gamma^*_{xz}\Pi_z$	$d\Gamma^*_{xy}, d\pi^{(\mathbf{n})}_{xy}$, and averaging
Sect. 4.2	Sect. 4.3	Sect. 4.4	Sect. 4.5	Sect. 4.6
Algebraic arg. I (i) Proposition 4.1 1 Algebraic arg. I (ii) Proposition 4.5 11		Algebraic arg. II Proposition 4.8 2	Algebraic arg. III Proposition 4.11 3	Algebraic arg. IV Proposition 4.16 4
Reconstruction I Proposition 4.2			Reconstruction III Proposition 4.12 7	
Integration I Proposition 4.3 9		Integration II Proposition 4.9 12	Integration III Proposition 4.14 14	
Three-point arg. I Proposition 4.4 10		Three-point arg. II Proposition 4.10 13	Three-point arg. III Proposition 4.15 15	Three-point arg. IV Proposition 4.17 16
	Proposition 4.6 5 Proposition 4.7 6			Averaging Proposition 4.18 8

Fortunately, it turns out that $d\Gamma^*_{xz}$ does have a triangular structure w. r. t. an ordering that involves $[\beta]$, cf. (2.23), as a "first digit" and $|\beta|_p$, cf. (2.27), as a second digit, in the spirit of [53, Sect. 3.5]. In the reconstruction argument, based on the structure of the term $\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x$ (or rather its Malliavin derivative), we need a finer ordering, which involves the component $\beta(0)$ (to which the two other digits are oblivious) as a third digit. We shall argue in Sect. 8 that at least the triangular effect of this ordering can be captured by the following ordinal

$$|\beta|_\prec := [\beta] + \frac{1}{2}|\beta|_p + \frac{1}{4}\beta(0), \tag{3.2}$$

where the weights in this combination are fixed for convenience but could be replaced by any three strictly ordered positive numbers.

For notational convenience, we define

$$\beta' \prec \beta \quad \Leftrightarrow \quad |\beta'|_\prec < |\beta|_\prec, \tag{3.3}$$

$$\beta' \preccurlyeq \beta \quad \Leftrightarrow \quad (\beta' \prec \beta \ \text{ or } \ \beta' = \beta). \tag{3.4}$$

A further benefit of (3.2) is that $|\cdot|_\prec$ is coercive, meaning that the set $\{\,\beta \,|\, |\beta|_\prec \leq M\,\}$ is finite for every finite M (which would not be true when $|\cdot|_\prec$ is replaced by $|\cdot|$ because both $[\cdot]$ and $|\cdot|_p$ are oblivious to the $\beta(0)$ component). This is crucial in the induction since at every step, the stochastic integrability deteriorates due to the unavoidable use of Hölder's inequality in probability when estimating products of random variables.

While in general, we only[63] have

$$[\beta] \geq -1 \quad \text{and} \quad |\beta|_\prec \geq -\frac{1}{2}, \tag{3.5}$$

it follows from (2.23) that $|\beta|_\prec \geq 0$ for all β not purely polynomial with equality iff $\beta = 0$. In view of (3.3) and the additivity of $|\cdot|_\prec$ this implies compatibility of $\prec$ and summation:

$$\beta_1 \preccurlyeq \beta_1 + \beta_2 \quad \text{provided } \beta_2 \text{ is not purely polynomial}, \tag{3.6}$$

which we shall repeatedly use. In fact, the ordering is de facto irrelevant on the purely polynomial β's, which have been treated. Among the non-purely polynomial β's, the base case is given by $\beta = 0$.

While $|\cdot|_\prec$ is additive (but negative on some purely polynomial indices), the homogeneity $|\cdot|$ is not (but it is strictly positive, in fact, $\geq \alpha$). We will often use that

$$|\cdot| - \alpha \overset{(2.27)}{=} \alpha[\cdot] + |\cdot|_p \quad \text{is additive and non-negative} \tag{3.7}$$

on all populated multi-indices.

In order to make the value of the multi-index explicit when referring to a statement like (2.36), we write $(2.36)_\beta$ with the understanding that we refer to the corresponding statement for the multi-index β. When a statement involves two multi-indices

[63] consider $\beta = e_{(1,0)}$.

like (2.37), we write $(2.37)_\beta^\gamma$ when we want to specify also the second multi-index, and $(2.37)_\beta^{\gamma\neq\text{p.p.}}$ when we only mean it for γ's which are not purely polynomial. All statements of the induction hypothesis will be implicitly assumed to hold for every integrability exponent $p<\infty$, for every space-time points $x, y, z \in \mathbb{R}^d$, and every convolution parameter $t \in (0,\infty)$, if applicable. For example, when we state $(2.36)_{\prec\beta}$, we mean the estimate for every $p<\infty$ and $x, y \in \mathbb{R}^2$ and for all multi-indices $\beta' \prec \beta$.

3.6 The base case $\beta = 0$

In fact, the argument for the base case w. r. t. the ordering $\prec$, which reduces to $\beta = 0$, is contained in the argument for the induction step, as we shall explain now, referring to the logical order of the induction in the singular case outlined in Sect. 3.4.

First note that, due to the triangularity properties (8.9), (8.10), (8.11) and (8.13), all estimates of Item 1 (namely $(2.37)_0^{\gamma\neq\text{p.p.}}$, $(4.27)_0^{\gamma\neq\text{p.p.}}$, $(4.108)_0^{\gamma\neq\text{p.p.}}$ and $(4.47)_0^{\gamma\neq\text{p.p.}}$) are void for $\beta = 0$.

In Item 2, the recentering property (2.62) is trivial for $\xi_\tau + c_0$ for any $c_0 \in \mathbb{R}$, and so is the shift $(5.3)_0$ and reflection $(5.4)_0$ covariance. By stationarity, which is contained in Assumption 2.1, $\mathbb{E}\xi_t(y)$ is constant in t and y, and by centeredness (also in Assumption 2.1) it is equal to 0; in particular, estimate $(4.20)_0$ is trivially satisfied. We choose $c_0 = 0$, so that $(2.38)_0$ holds (Item 3). Note that this choice is consistent with the population condition (2.34) and makes the estimate $(2.40)_0$ hold trivially.

Next we turn to Item 4, i. e. the Malliavin derivative $\delta\xi_\tau$ (see Sect. 7.1 for Malliavin differentiability). Continuity $(4.55)_0$ is contained in Step 1 of the proof of Proposition 4.13, modeledness $(4.52)_0$ is established in the proof of Proposition 4.12 in form of (4.38), and boundedness $(4.22)_0$ is a consequence of modeledness and contained in the proof of Proposition 4.18. The combination of $\mathbb{E}\xi_t(y) = 0$ with $(4.22)_0$ via the SG inequality yields $(2.64)_0$ (Item 5), which takes the form

$$\mathbb{E}^{\frac{1}{p}}|\xi_t(y)|^p \lesssim (\sqrt[4]{t})^{\alpha-2}.$$

Moreover, the divergent bound $(2.43)_0$, i. e. the continuity of ξ_τ (Item 8. (a)), is shown in Step 1 of the proof of Proposition 2.3, cf. Sect. 6.1 and in particular (6.1).

Equipped with the estimates of ξ and $\delta\xi$, and the triangularity properties, the rest of the base case follows from the same procedure as in any induction step, namely Items 6 to 9.

4 Estimates

In this section, we establish the stochastic estimates (2.36), (2.37), (2.55) and (2.64) for a fixed non-purely polynomial multi-index β.

4.1 Semi-group convolution

Following [55, Sect. 2], we use the space-time elliptic operator $\partial_1^4 - \partial_2^2$ to introduce the family $\{(\cdot)_t\}_{t\in(0,\infty)}$ of convolution operators that respect the parabolic scaling and satisfy the semi-group property

$$f_{t+s} = (f_t)_s \tag{4.1}$$

convenient for the dyadic nature of reconstruction arguments. It is given by the convolution with the Schwartz kernel ψ_t defined through

$$\partial_t \psi_t + (\partial_1^4 - \partial_2^2)\psi_t = 0 \quad \text{and} \quad \psi_{t=0} = \text{Dirac at origin.} \tag{4.2}$$

Note the scaling $\psi_t(x) = (\sqrt[4]{t})^{-3}\,\psi_{t=1}(\frac{x_1}{\sqrt[4]{t}}, \frac{x_2}{\sqrt[2]{t}})$, so that the x_1-scale is $\sqrt[4]{t}$, which explains the appearance of $\sqrt[4]{\tau}$ in Proposition 2.3. One reads off (4.2) that the Fourier transform of $\psi_{t=1}$ is given by the Schwartz function $\exp(-k_1^4 - k_2^2)$, so that $\psi_{t=1}$ is a Schwartz function itself. In view of the scaling of ψ_t in terms of t, this implies the moment bound

$$\int dz |\partial^{\mathbf{n}} \psi_t(z-y)| (\sqrt[4]{t} + |y-z| + |z-x|)^\theta \lesssim (\sqrt[4]{t})^{-|\mathbf{n}|} (\sqrt[4]{t} + |y-x|)^\theta, \tag{4.3}$$

where we recall that $|z-y|$ is the anisotropic distance function (2.1). We will need that (4.3) holds for all $\theta > -3$.

Finally, because of the factorization

$$\partial_1^4 - \partial_2^2 = -(\partial_2 - \partial_1^2)(\partial_2 + \partial_1^2), \tag{4.4}$$

the convolution $(\cdot)_t$ will also be helpful for integration by providing the kernel representation in form of

$$(\partial_2 - \partial_1^2)^{-1} f = -\int_0^\infty dt (\partial_2 + \partial_1^2) f_t. \tag{4.5}$$

4.2 Estimate of the original quantities Γ^*_{xy}, Π^-_x, Π_x, and $\pi^{(\mathbf{n})}_{xy}$

The first task is estimate (2.37) on Γ^*_{xy}, based on the exponential formula (2.44), estimate (2.55) of $\pi^{(\mathbf{n})}_{xy}$ and the population constraint (2.46). We split this first task into a first half, where we treat $\Gamma^*_{xy} P$, see Proposition 4.1, and a second half, where we tackle the full Γ^*_{xy}, see Proposition 4.5. The reason for this splitting is that according to (8.1), provided γ is not purely polynomial, the matrix entry $(\Gamma^*_{xy})^\gamma_\beta$ depends on $\pi^{(\mathbf{n})}_{xy}$ only through $\pi^{(\mathbf{n})}_{xy\beta'}$ with $\beta' \prec \beta$. Following the elementary proof of [52, Lemma 4.3] we obtain from Hölder's inequality in probability:

Proposition 4.1 (Algebraic argument I, first half) *Assume that* $(2.55)_{\prec\beta}$ *holds. Then* $(2.37)^\gamma_\beta$ *holds for all* γ *not purely polynomial.*

The second task is the estimate (2.64) of Π^-_x, based on the output of Proposition 4.1. Note that by (2.17), definition (2.57), when evaluated at the base point x, collapses to

$$\Pi^-_x(x) = P\mathsf{z}_0 \partial_1^2 \Pi_x(x) - c + \xi_\tau(x)\mathsf{1}. \tag{4.6}$$

From (2.36) we obtain with help of the semi-group property (4.1) and the moment bounds (4.3)

$$\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\Pi_{x\beta\, t}(x)|^p \lesssim (\sqrt[4]{t})^{|\beta|-|\mathbf{n}|}. \tag{4.7}$$

Using this for $\mathbf{n}=(2,0)$ and $\beta-e_0\prec\beta$ we learn from the (annealed) continuity (2.42) of $\partial_1^2\Pi_x$ that $\partial_1^2\Pi_{x\,\beta-e_0}(x)=0$ (almost surely) for $|\beta-e_0|=|\beta|>2$. Since $c_\beta=0$ for $|\beta|\geq 2$, cf. (2.34), and $|0|=\alpha\leq 2$, we get from (4.6) that $\Pi^-_{x\beta}(x)=0$. By the continuity $(2.43)_\beta$ of $\Pi^-_{x\beta}$ we may convert this back into the more robust

$$\lim_{t\downarrow 0}\mathbb{E}^{\frac{1}{p}}|\Pi^-_{x\beta\, t}(x)|^p=0 \quad \text{provided } |\beta|>2. \tag{4.8}$$

Proposition 4.2 (Reconstruction I) *Assume* $|\beta|>2$, *that* $(2.36)_{\prec\beta}$, $(2.42)_{\prec\beta}$, $(2.43)_\beta$, $(2.55)_{\prec\beta}$, $(2.62)_\beta$ *and* $(2.64)_{\prec\beta}$ *hold, and that* $(2.37)_\beta^\gamma$ *holds for all* γ *not purely polynomial. Then* $(2.64)_\beta$ *holds.*

Proof By general reconstruction, see e. g. [56, Proposition 1] or [52, Lemma 4.8], the estimate on $\Pi^-_{x\beta}$ follows from (4.8) (which as explained above is a consequence of $(2.36)_{\prec\beta}$, $(2.42)_{\prec\beta}$ and $(2.43)_\beta$) once we establish its continuity in the base point x to the order $|\beta|-2>0$:

$$\mathbb{E}^{\frac{1}{p}}|(\Pi^-_y-\Pi^-_x)_{\beta t}(x)|^p \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}. \tag{4.9}$$

Estimate (4.9) in turn relies on (2.62), which we rearrange to

$$\begin{aligned}\Pi^-_x-\Pi^-_y&=(\Gamma^*_{xy}-\mathrm{id})\Pi^-_y\\&\quad+P\sum_{k\geq 0}\mathsf{z}_k\big(\Gamma^*_{xy}(\mathrm{id}-P)\Pi_y+\pi^{(\mathbf{0})}_{xy}\big)^k\partial_1^2\big(\Gamma^*_{xy}(\mathrm{id}-P)\Pi_y+\pi^{(\mathbf{0})}_{xy}\big),\\&\text{where}\quad \Gamma^*_{xy}(\mathrm{id}-P)\Pi_y+\pi^{(\mathbf{0})}_{xy}\overset{(2.21),(2.51)}{=}\sum_{\mathbf{n}}(\mathbf{1}_{\mathbf{n}\neq\mathbf{0}}\mathsf{z}_{\mathbf{n}}+\pi^{(\mathbf{n})}_{xy})(\cdot-y)^{\mathbf{n}}.\end{aligned} \tag{4.10}$$

Note that by the strict triangularity (8.1) of $\Gamma^*_{xy}-\mathrm{id}$ w. r. t. $\prec$, the first r. h. s. term of (4.10) involves $\Pi^-_{y\beta'}$ only for $\beta'\prec\beta$. We observe that by Hölder's inequality in probability, the $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$-norm of each constituent $(\Gamma^*_{xy}-\mathrm{id})_\beta^{\beta'}\Pi^-_{y\beta' t}(x)$ to the matrix-vector product is estimated by $|y-x|^{|\beta|-|\beta'|}(\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta'|-\alpha}$, which because of $|\beta|-|\beta'|\geq 0$ (by the triangularity (8.1) of Γ^*_{xy} w. r. t. $|\cdot|$) is dominated by the r. h. s. of (4.9). By the structure (8.6) of the expression $\sum_{k\geq 0}\mathsf{z}_k\pi^k\pi'$, the second r. h. s. term of (4.10) involves $\mathbf{1}_{\mathbf{n}\neq\mathbf{0}}\mathsf{z}_{\mathbf{n}}+\pi^{(\mathbf{n})}_{xy\beta'}$ only for $\beta'\prec\beta$. Note that both the population condition (2.46) and the estimate (2.55) extend from $\pi^{(\mathbf{n})}_{xy}$ to $\mathbf{1}_{\mathbf{n}\neq\mathbf{0}}\mathsf{z}_{\mathbf{n}}+\pi^{(\mathbf{n})}_{xy}$, provided one relaxes $|\beta|>|\mathbf{n}|$ to $|\beta|\geq|\mathbf{n}|$. Hence as a function of the active variable, the second r. h. s. of $(4.10)_\beta$ is a linear combination of monomials

$(\cdot - y)^{\mathbf{n}}$ with $|\mathbf{n}| \le |\beta| - 2$ with a coefficient estimated by $|y-x|^{|\beta|-|\mathbf{n}|-2}$, where we used (8.7).

We now apply the convolution $(\cdot)_t$ to this polynomial in (4.10) and evaluate in x. Using (4.3) in form of $|((\cdot - y)^{\mathbf{n}})_t(x)| \lesssim (\sqrt[4]{t} + |y-x|)^{|\mathbf{n}|}$, we see that each summand[64] of the second r. h. s. term of (4.10) is estimated by $|y-x|^{|\beta|-|\mathbf{n}|-2}(\sqrt[4]{t} + |y-x|)^{|\mathbf{n}|}$, which obviously is dominated by the r. h. s. of (4.9). □

The third task is the estimate (2.36) of Π_x, based on the output of Proposition 4.2. It relies on the construction of $\Pi_{x\beta}$ in terms of $\Pi^-_{x\beta}$ with help of the semi-group kernel, see (4.5):

$$\Pi_{x\beta} = -\int_0^\infty dt (1 - \mathrm{T}_x^{|\beta|})(\partial_2 + \partial_1^2)\Pi^-_{x\beta\, t}. \tag{4.11}$$

Here and in the sequel, for $\theta > 0$ and $x \in \mathbb{R}^2$, T_x^θ denotes the operation of taking the Taylor polynomial of degree $< \theta$ in the base point x, i. e. $\mathrm{T}_x^\theta f(y) = \sum_{|\mathbf{n}|<\theta} \frac{1}{\mathbf{n}!}\partial^{\mathbf{n}} f(x)(y-x)^{\mathbf{n}}$.

As specific to integration in regularity structures, the estimate $(2.36)_\beta$ can be seen as a Schauder estimate anchored in the base point x, here on an annealed level. As is typical for Schauder theory, one has to avoid integer values, which is ensured by (2.29). Incidentally, (4.11) reproduces Hairer's form of integration [38, Eq. (8.19)]. However, since we are working on the whole space instead of the torus so that there is no a priori decay in t of the integrand, the polynomial $\mathrm{T}_x^{|\beta|}\int_0^\infty dt(\partial_2 + \partial_1^2)\Pi^-_{x\beta\, t}$ may not be well-defined by itself.

Proposition 4.3 (Integration I) *Suppose that $(2.64)_\beta$ holds. Then (4.11) defines[65] a[66] solution of (2.35) and $(2.36)_\beta$ holds.*

Proof We start with some preliminary estimates: Appealing to the semi-group property (4.1) in form of $\partial^{\mathbf{n}}\Pi^-_{x\beta\, t}(y) = \int dz \partial^{\mathbf{n}}\psi_{\frac{t}{2}}(y-z)\Pi^-_{x\beta\, \frac{t}{2}}(z)$, we gather from $(2.64)_\beta$ and (4.3)

$$\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\Pi^-_{x\beta\, t}(y)|^p \lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}(\sqrt[4]{t} + |y-x|)^{|\beta|-\alpha}. \tag{4.12}$$

Using (4.12) for $y = x$ we infer for the Taylor polynomial of (parabolic) order $< \theta$ that

$$\mathbb{E}^{\frac{1}{p}}|\mathrm{T}_x^\theta \partial^{\mathbf{n}}\Pi^-_{x\beta\, t}(y)|^p \lesssim \sum_{|\mathbf{m}|<\theta} (\sqrt[4]{t})^{|\beta|-2-|\mathbf{n}+\mathbf{m}|}|y-x|^{|\mathbf{m}|}. \tag{4.13}$$

[64] summed over $k \ge 0$ and $\mathbf{n}_1, \ldots, \mathbf{n}_k$.

[65] By which we mean that the r. h. s. of (4.11) makes sense in the Banach space with norm $\sup_x |y-x|^{-|\beta|}\mathbb{E}^{\frac{1}{p}}|\cdot|^p$.

[66] By a Liouville argument similar to the one of Proposition 5.3 this solution can be seen to be unique, cf. [58, Lemma 2.1], which however we will not make any use of.

Note that by definition, $(1-\mathrm{T}_x^\theta)\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}$ vanishes to order θ in $s=0$ along the "parabolic" curve $[0,1]\ni s\mapsto(sy_1+(1-s)x_1, s^2y_2+(1-s^2)x_2)$ connecting x to y. Hence the ensuing integral representation formula (4.11) only involves derivatives of order $\mathbf{m}$ with $|\mathbf{m}|\geq\theta$ of $\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}$, next to being constrained by $m_1+m_2\leq\theta+1$. Using (4.12) with y replaced by a point along this curve and $\mathbf{n}$ replaced by $\mathbf{n}+\mathbf{m}$ we obtain

$$\begin{aligned}&\mathbb{E}^{\frac{1}{p}}|(1-\mathrm{T}_x^\theta)\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}(y)|^p\\&\lesssim\sum_{\substack{|\mathbf{m}|\geq\theta\\ m_1+m_2\leq\theta+1}}(\sqrt[4]{t})^{\alpha-2-|\mathbf{n}+\mathbf{m}|}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}|y-x|^{|\mathbf{m}|}.\end{aligned}\tag{4.14}$$

With help of these three auxiliary estimates we now derive the central one, namely

$$\int_0^\infty dt\,\mathbb{E}^{\frac{1}{p}}|(1-\mathrm{T}_x^{|\beta|})(\partial_2+\partial_1^2)\Pi^-_{x\beta\,t}(y)|^p\lesssim|y-x|^{|\beta|}.\tag{4.15}$$

To this purpose, we split the integral into the near-field $t\leq|y-x|^4$ and the far-field $t\geq|y-x|^4$. For the latter, we use (4.14) with $\theta=|\beta|$ and $\mathbf{n}=(0,1),(2,0)$ (and thus $|\mathbf{n}|=2$), so that the growth rate of its r. h. s. in $\sqrt[4]{t}$ is given by $-4-|\mathbf{m}|+|\beta|$. Since by Assumption 2.1 α is irrational, $|\beta|$ is not an integer according to (2.29), so that the sum in (4.14) effectively restricts to $|\mathbf{m}|>|\beta|$. Hence the growth rate in t itself is <-1, so that the integral converges, and is estimated by the r. h. s. of (4.15).

We use the triangle inequality to split the near-field part of (4.15) into its contribution from $\mathrm{T}_x^{|\beta|}$ and from 1. On the former, we apply (4.13), again for $\theta=|\beta|$ and with $|\mathbf{n}|=2$. The order of vanishing of its r. h. s. in $\sqrt[4]{t}$ is given by $-4-|\mathbf{m}|+|\beta|$. Since the sum is restricted to $|\mathbf{m}|<|\beta|$, the order of vanishing in t is >-1, so that the integral converges at $t=0$, and is also estimated by the r. h. s. of (4.15). For the contribution from 1, we directly use (4.12) with $|\mathbf{n}|=2$; the order of vanishing is now given by $\alpha-4$, so that by $\alpha>0$ the same reasoning applies.

We finally argue that (4.11) satisfies (2.35). Estimate (4.15) implies that (4.11) is well-defined in the Banach space with norm $\sup_x|y-x|^{-|\beta|}\mathbb{E}^{\frac{1}{p}}|\cdot|^p$. It also implies that the version of (4.11) where $\int_0^\infty dt$ is replaced by $\int_s^T dt$ converges when $s\downarrow 0$ and $T\uparrow\infty$ in that topology. On the level of $\int_s^T dt$, we may exchange the differential operator $(\partial_2-\partial_1^2)$ with the integration, so that by (4.4),

$$\begin{aligned}&-(\partial_2-\partial_1^2)\int_s^T dt(1-\mathrm{T}_x^{|\beta|})(\partial_2+\partial_1^2)\Pi^-_{x\beta\,t}\\&=(1-\mathrm{T}_x^{|\beta|-2})\Pi^-_{x\beta\,s}-(1-\mathrm{T}_x^{|\beta|-2})\Pi^-_{x\beta\,T}.\end{aligned}$$

We learn from (4.13) and (4.14), both for $\theta=|\beta|-2\notin\mathbb{Z}$ and $\mathbf{n}=\mathbf{0}$, that $\mathrm{T}_x^{|\beta|-2}\Pi^-_{x\beta\,s}$ vanishes for $s\downarrow 0$, and that $(1-\mathrm{T}_x^{|\beta|-2})\Pi^-_{x\beta\,T}$ vanishes for $T\uparrow\infty$. $\square$

We now turn to the fourth task of estimating $\pi_{xy}^{(\mathbf{n})}$. This relies on the identity

$$\sum_{\mathbf{n}} \pi_{xy}^{(\mathbf{n})}(z-y)^{\mathbf{n}} = \Pi_x(z) - \Pi_y(z) - (\Gamma_{xy}^* - \mathrm{id}) P \Pi_y(z) \tag{4.16}$$

involving the three points x, y, and z. Identity (4.16) follows from (2.61), using (2.51) and (2.21) for $\mathbf{n} \neq \mathbf{0}$, and (2.63) for $\mathbf{n} = \mathbf{0}$. Choosing $\#\{\mathbf{n} \,|\, |\mathbf{n}| < |\beta|\}$ pairwise distinct space-time points in place of z we may interpret the β component of the l. h. s. of (4.16) as a Vandermonde matrix applied to the vector $\{\pi_{xy\beta}^{(\mathbf{n})}\}_{|\mathbf{n}|<|\beta|}$. Then the invertibility of the Vandermonde matrix yields the equivalence of annealed norms

$$\max_{\mathbf{n}:|\mathbf{n}|<|\beta|} |y-x|^{|\mathbf{n}|} \mathbb{E}^{\frac{1}{p}} |\pi_{xy\beta}^{(\mathbf{n})}|^p \sim \sup_{z:|z-x|\le|y-x|} \mathbb{E}^{\frac{1}{p}} \Big| \sum_{\mathbf{n}} \pi_{xy\beta}^{(\mathbf{n})}(z-y)^{\mathbf{n}} \Big|^p;$$

formula $(4.16)_\beta$ allows to estimate $\pi_{xy\beta}^{(\mathbf{n})}$ by the outputs of Propositions 4.1 and 4.3:

Proposition 4.4 (Three-point argument I) *Assume that* $(2.36)_{\preccurlyeq\beta}$ *and* $(2.61)_\beta$ *hold and that* $(2.37)_\beta^\gamma$ *holds for all* γ *not purely polynomial. Then* $(2.55)_\beta$ *holds.*

Equipped with the output of Proposition 4.4, we now may complete our first task. By the same argument as for Proposition 4.1, we have

Proposition 4.5 (Algebraic argument I, second half) $(2.55)_{\prec\beta}$ *implies* $(2.37)_\beta$.

4.3 Estimate of the expectation: BPHZ-choice of renormalization, SG inequality, and dualization of Malliavin derivative estimate

For this and the next three subsections, we restrict to singular and not purely polynomial β and start addressing the challenging part of the proof, namely the estimate $(2.64)_\beta$ of $\Pi_{x\beta}^-$ in this singular case. We will use what is called the $\mathbb{L}^p$-version,[67] for $p \ge 2$, of the SG inequality

$$\mathbb{E}^{\frac{1}{p}} |F|^p \lesssim |\mathbb{E}F| + \mathbb{E}^{\frac{1}{p}} \|\frac{\partial F}{\partial \xi}\|_*^p. \tag{4.17}$$

for all cylindrical functionals F as in (2.5). It extends by continuity to the classical Malliavin–Sobolev space $\mathbb{H}^p$, see Appendix A. This $\mathbb{L}^p$-version is a simple consequence of (2.7), using the chain rule for the Malliavin derivative and Hölder's estimate in probability, and is oblivious to the nature of the underlying Hilbert norm (2.4), see for instance [47, Step 2 in the proof of Lemma 3.1]; the result is classical in the Gaussian case [7, Theorem 5.5.11]. As we argue in Sect. 7.2, for fixed t and x, $F = \Pi_{x\beta t}^-(y) \in \mathbb{H}^p$, to which we will apply (4.17).

We now argue that the first r. h. s. term $\mathbb{E}\Pi_{x\beta t}^-(y)$ in (4.17) is estimated as a consequence of the BPHZ-choice of renormalization from Sect. 5.1, cf. (4.20). In order to pass from the limit $\lim_{t\uparrow\infty}$ in Proposition 5.1 to a finite value $\sqrt[4]{t}$ of the

[67] In the sequel $\mathbb{L}^p$ denotes the space of p-integrable random variables w. r. t. $\mathbb{E}$.

(spatial) convolution scale, we need the following proposition; note that the statement (and the proof) is oblivious to the specific value of c_β and thus it can be shown before the BPHZ choice $(2.38)_\beta$ has been made.

Proposition 4.6 *For $|\beta| < 2$, suppose that $(2.60)_\beta$, $(2.64)_{\prec\beta}$ and $(5.3)_\beta$ holds, and that $(2.37)^\gamma_\beta$ holds for all γ not purely polynomial. Then we have*

$$\int_T^\infty dt \Big|\frac{d}{dt}\mathbb{E}\Pi^-_{x\beta\,t}(y)\Big| \lesssim (\sqrt[4]{T})^{\alpha-2}(\sqrt[4]{T}+|y-x|)^{|\beta|-\alpha}. \tag{4.18}$$

Proof For $|\beta| < 2$, $(2.60)_\beta$ reduces to

$$\Pi^-_{x\beta} = (\Gamma^*_{xz}\Pi^-_z)_\beta.$$

Moreover, by $(5.3)_\beta$, $\mathbb{E}\Pi^-_{z\beta s}(z)$ is independent of z. These two facts combined yield

$$\frac{d}{dt}\mathbb{E}\Pi^-_{x\beta\,t}(y) = -\int_{\mathbb{R}^2} dz(\partial_1^4-\partial_2^2)\psi_{t-s}(y-z)\mathbb{E}\big((\Gamma^*_{xz}-\mathrm{id})\Pi^-_{z\,s}\big)_\beta(z), \tag{4.19}$$

for all $s \in (0,t)$. The merit is that as a consequence of the strict triangularity (8.1) of $\Gamma^*_{xy}-\mathrm{id}$ w. r. t. $\prec$, (4.19) only features $\{\Pi^-_{z\beta'}\}_{\beta'\prec\beta}$. Since by definition (2.57), $\Pi^-_z \in \tilde{\mathsf{T}}^*$, (4.19) only features $(\Gamma^*_{xz}-\mathrm{id})^{\beta'}_\beta$ for β' not purely polynomial.

We use (4.19) with $s = \frac{t}{2}$. By the Cauchy-Schwarz inequality in probability, the contribution from $(\Gamma^*_{xz}-\mathrm{id})^{\beta'}_\beta\Pi_{z\beta'\frac{t}{2}}(z)$ is estimated in expectation by $|z-x|^{|\beta|-|\beta'|}(\sqrt[4]{t})^{|\beta'|-2} \lesssim (|y-z|+|y-x|)^{|\beta|-|\beta'|}\ (\sqrt[4]{t})^{|\beta'|-2}$. After integration in z, by the moment bounds (4.3) on ψ_s, this contribution to (4.19) is controlled by $t^{-1}(\sqrt[4]{t}+|y-x|)^{|\beta|-|\beta'|}(\sqrt[4]{t})^{|\beta'|-2}$. Since $|\beta|<2$, integration in $t \geq T$ yields control by $(\sqrt[4]{T}+|y-x|)^{|\beta|-|\beta'|}(\sqrt[4]{T})^{|\beta'|-2}$. By $|\beta'| \geq \alpha$, (4.18) follows. □

Equipped with Proposition 4.6 and the choice $(2.38)_\beta$ of c_β, the qualitative Proposition 5.1 instantly upgrades to an estimate of $\mathbb{E}\Pi^-_{x\beta\,t}(x)$; using $(2.60)_\beta$ together with Hölder's inequality and $(2.37)^\gamma_\beta$ for γ not purely polynomial and $(2.64)_{\prec\beta}$ yields the desired estimate of $\mathbb{E}\Pi^-_{x\beta\,t}(y)$:

Proposition 4.7 *For $|\beta| < 2$, suppose that $(2.38)_\beta$, $(2.60)_\beta$, $(2.64)_{\prec\beta}$, and $(4.18)_\beta$ hold, and that $(2.37)^\gamma_\beta$ holds for all γ not purely polynomial. Then we have*

$$|\mathbb{E}\Pi^-_{x\beta\,t}(y)| \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}. \tag{4.20}$$

The remaining task of this and the next three subsections is thus to estimate the Malliavin derivative of $\Pi^-_{x\beta\,t}(y)$, in the norm given by (4.17), by the r. h. s. of (4.20):

$$\mathbb{E}^{\frac{1}{p}}\Big|\int_{\mathbb{R}^2}\big((\partial_1^4-\partial_2^2)^{\frac14(\frac12-\alpha)}\frac{\partial}{\partial\xi}\Pi^-_{x\beta\,t}(y)\big)^2\Big|^{\frac{p}{2}} \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}. \tag{4.21}$$

It is convenient to undo the Riesz representation (2.3) and to return to the derivative $\delta\Pi^-_{x\beta t}(y)$ of $\Pi^-_{x\beta t}(y)$ in direction of the space-time field $\delta\xi$. It is a straightforward consequence of $\mathbb{L}^p$-duality, with q denoting the conjugate exponent of p, that (4.21) is equivalent to

$$|\mathbb{E}\delta\Pi^-_{x\beta\, t}(y)| \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}\mathbb{E}^{\frac{1}{q}}\Big|\int_{\mathbb{R}^2}\big((\partial_1^4-\partial_2^2)^{\frac{1}{4}\left(\alpha-\frac{1}{2}\right)}\delta\xi\big)^2\Big|^{\frac{q}{2}},$$

provided that this is established for an arbitrary $\delta\xi$ that is allowed to be random in order to pull the supremum over $\delta\xi$ out of the $\mathbb{L}^q$-norm. For the base case and when introducing a weight, both in Sect. 4.5, it will be convenient to replace the $L^2(\mathbb{R}^2)$-based fractional Sobolev norm of $\delta\xi$ by its equivalent $L^2(\mathbb{R}^2)$-based Besov norm. This equivalence is obvious when the Besov side is formulated in terms of our semi-group, since it then follows by Plancherel from the elementary scaling identity $(k_1^4+k_2^2)^{\frac{1}{2}(\alpha-\frac{1}{2})} \sim \int_0^\infty \frac{ds}{s}s^{\frac{1}{2}(\frac{1}{2}-\alpha)}\exp(-2s(k_1^4+k_2^2))$ in terms of the wave vector $k=(k_1,k_2)$ (and relies on $\alpha<\frac{1}{2}$):

$$|\mathbb{E}\delta\Pi^-_{x\beta t}(y)| \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}\mathbb{E}^{\frac{1}{q}}\Big|\int_0^\infty \frac{ds}{s}(\sqrt[4]{s})^{2(\frac{1}{2}-\alpha)}\int_{\mathbb{R}^2}(\delta\xi_s)^2\Big|^{\frac{q}{2}}.$$

In fact, we will establish a stronger version of this estimate: It is strengthened on the l. h. s. by replacing the expectation by a $\mathbb{L}^{q'}$-norm, and it is strengthened on the r. h. s. by exchanging the spatial and probabilistic norm (which by Minkowski's inequality is a strengthening due to $q\le 2$)

$$\mathbb{E}^{\frac{1}{q'}}|\delta\Pi^-_{x\beta\, t}(y)|^{q'} \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}\bar{w}, \tag{4.22}$$

where we introduced the following abbreviation for a norm of $\delta\xi$

$$\bar{w} := \Big(\int_0^\infty \frac{ds}{s}(\sqrt[4]{s})^{2(\frac{1}{2}-\alpha)}\int_{\mathbb{R}^2}\mathbb{E}^{\frac{2}{q}}|\delta\xi_s|^q\Big)^{\frac{1}{2}}. \tag{4.23}$$

Estimate (4.22) is an annealed estimate, where an annealed norm of $\delta\Pi^-_{x\beta}$ is controlled by the annealed norm $\bar{w}$ of $\delta\xi$. We shall establish (4.22) for all $q'<q\le 2$. Hence $\lesssim$ now also acquires a dependence on $q'<q\le 2$ (next to α and β) when it comes up in the estimate of a Malliavin derivative. As for the integrability exponent $2\le p<\infty$, all estimates will be implicitly assumed to hold for all $q'<q\le 2$. The strengthening from $q'=1$ to $q'>1$ is important when using (4.22) in the induction: One often needs to estimate products where (at most) one of the factors comes from a (directional) Malliavin derivative of the model (as indicated by the appearance of the symbol δ or d) whereas the other factors are one of the model components (i. e. Π^-_x, Π_x, Γ^*_{xy}). Since the other factors ask for a stochastic $\mathbb{L}^p$-norm with $p<\infty$, by Hölder's inequality, we need a stochastic $\mathbb{L}^{q'}$-norm with $q'>1$ on the Malliavin factor, see for instance (4.30) in the proof of Proposition 4.8 below. This reflects a deterioration in the stochastic integrability, which is unavoidable since the homogeneity of $\Pi^-_{x\beta}$ in ξ is $[\beta]+1$.

4.4 Estimate of Malliavin derivatives: $\delta\Gamma^*_{xy}$, $\delta\Pi^-_x$, $\delta\Pi_x$, and $\delta\pi^{(\mathbf{n})}_{xy}$

For this and the next two subsections, we fix a random $\delta\xi$. This subsection is an interlude: As will become clear only in Sect. 4.6, next to $Q\delta\Pi^-_x$, we also need to estimate the directional Malliavin derivatives $Q\delta\Pi_x$ and $Q\delta\Gamma^*_{xy}P$. In fact, in this subsection, we shall proceed like in Sect. 4.2 and assume the estimate (4.22) on $Q\delta\Pi^-_x$ in order to (inductively) derive the estimates on the remaining objects $Q\delta\Gamma^*_{xy}P$, $Q\delta\Pi_x$, and $Q\delta\pi^{(\mathbf{n})}_{xy}$.

Taking the (directional) Malliavin derivative of the exponential formula (2.44) applied to $\Gamma^* = \Gamma^*_{xy}$, we obtain by Leibniz' rule, see Sect. 7.4,

$$\delta\Gamma^*_{xy} = \sum_{\mathbf{n}} \delta\pi^{(\mathbf{n})}_{xy}\Gamma^*_{xy}D^{(\mathbf{n})}, \tag{4.24}$$

which motivates to include the estimates

$$\mathbb{E}^{\frac{1}{q'}}|\delta\pi^{(\mathbf{n})}_{xy\beta}|^{q'} \lesssim |y-x|^{|\beta|-|\mathbf{n}|}\bar{w} \quad \text{for all } |\mathbf{n}| < |\beta|. \tag{4.25}$$

Note that in view of (2.47) we have

$$\delta\pi^{(\mathbf{n})}_{xy} \in \tilde{\mathsf{T}}^*. \tag{4.26}$$

In analogy to Proposition 4.1 we have

Proposition 4.8 (Algebraic argument II) *Assume that* $(4.25)_{\prec\beta}$ *holds, and that* $(2.37)^{\gamma}_{\preccurlyeq\beta}$ *holds for all* γ *not purely polynomial. Then we have for all* γ *not purely polynomial*

$$\mathbb{E}^{\frac{1}{q'}}|(\delta\Gamma^*_{xy})^{\gamma}_{\beta}|^{q'} \lesssim |y-x|^{|\beta|-|\gamma|}\bar{w}. \tag{4.27}$$

Proof We distinguish the contributions $\mathbf{n}=\mathbf{0}$ and $\mathbf{n}\neq\mathbf{0}$ to (4.24). By definition (2.10) of $D^{(\mathbf{0})}$, all contributions to $(4.24)^{\gamma}_{\beta}$ from $\mathbf{n}=\mathbf{0}$ are of the form[68]

$$\delta\pi^{(\mathbf{0})}_{xy\beta_1}(\Gamma^*_{xy})^{\gamma-e_k+e_{k+1}}_{\beta_2} \tag{4.28}$$

for some $k\geq 0$ and (populated) multi-indices β_1, β_2 with $\beta_1+\beta_2=\beta$ and $k\geq 0$. Note that by (4.26), β_1 is not purely polynomial; likewise, since $\gamma-e_k+e_{k+1}$ is obviously neither purely polynomial nor vanishing, this transmits to β_2 by (2.59) and (8.12). Hence we may apply (3.6) to the desired effect of

$$\beta_1 \prec \beta \text{ and } \beta_2 \preccurlyeq \beta. \tag{4.29}$$

By Hölder's inequality in probability, we estimate the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm of the product (4.28) by the product of the $\mathbb{E}^{\frac{1}{q}}|\cdot|^{q}$-norm of the first factor and the $\mathbb{E}^{\frac{1}{p}}|\cdot|^{p}$-norm of

[68] with the implicit understanding that this term vanishes if $\gamma(k)=0$.

the second factor; recall that

$$\frac{1}{q}=\frac{1}{q'}+\frac{1}{p} \quad \text{and thus requires } q'<q \text{ because of } p<\infty. \tag{4.30}$$

By $(4.25)_{\prec\beta}$ and $(2.37)^{\gamma}_{\preccurlyeq\beta}$ for γ not purely polynomial, we thus obtain an estimate by $|y-x|^{|\beta_1|}\bar{w}|y-x|^{|\beta_2|-|\gamma-e_k+e_{k+1}|}$. Since $|\gamma-e_k+e_{k+1}|=|\gamma|+\alpha$ by definition (2.27), we learn from (3.7) that as desired

$$|\beta_1|+|\beta_2|-|\gamma-e_k+e_{k+1}|=|\beta|-|\gamma|. \tag{4.31}$$

We now address the contributions to $(4.24)^{\gamma}_{\beta}$ from some $\mathbf{n}\neq\mathbf{0}$, which in view of definition (2.45) of $D^{(\mathbf{n})}$ are of the form

$$\delta\pi^{(\mathbf{n})}_{xy\beta_1}(\Gamma^*_{xy})^{\gamma-e_{\mathbf{n}}}_{\beta_2}. \tag{4.32}$$

Note that once more, $\gamma-e_{\mathbf{n}}$ is neither purely polynomial nor, by assumption, vanishing. Like above, this transmits to β_2, and yields (4.29). Now (4.32) is estimated by $|y-x|^{|\beta_1|-|\mathbf{n}|}\bar{w}\ |y-x|^{|\beta_2|-|\gamma-e_{\mathbf{n}}|}$. Since $|\gamma-e_{\mathbf{n}}|=|\gamma|+\alpha-|\mathbf{n}|$ by definition (2.27), we get once more from (3.7)

$$|\beta_1|-|\mathbf{n}|+|\beta_2|-|\gamma-e_{\mathbf{n}}|=|\beta|-|\gamma|. \tag{4.33}$$

□

We now pass from $\delta\Pi^-_{x\beta}$ to $\delta\Pi_{x\beta}$. To this purpose, we take the Malliavin derivative of (4.11), see Sect. 7.3. By an almost identical integration argument to Proposition 4.3, we obtain

Proposition 4.9 (Integration II) *Assume that $(4.22)_\beta$ holds. Then we have*

$$\mathbb{E}^{\frac{1}{q'}}|\delta\Pi_{x\beta}(y)|^{q'}\lesssim|y-x|^{|\beta|}\bar{w}. \tag{4.34}$$

We finally return from $\delta\Pi_{x\beta}$ to $\delta\pi^{(\mathbf{n})}_{xy}$, by taking the Malliavin derivative of the three-point identity (4.16) (which as shown above follows from (2.61)), which by Leibniz' rule, see Sect. 7.4, assumes the form

$$\sum_{\mathbf{n}}\delta\pi^{(\mathbf{n})}_{xy}(z-y)^{\mathbf{n}}=\delta\Pi_x(z)-\Gamma^*_{xy}P\delta\Pi_y(z)-\delta\Gamma^*_{xy}P\Pi_y(z). \tag{4.35}$$

We obtain quite analogously to Propositions 4.4 and 4.5, using Hölder's inequality in probability like in the proof of Proposition 4.8

Proposition 4.10 (Three-point argument II) *Assume that $(2.37)^{\gamma}_{\beta}$ and $(4.27)^{\gamma}_{\beta}$ hold for γ not purely polynomial, and that $(2.36)_{\prec\beta}$, $(2.61)_\beta$, and $(4.34)_{\preccurlyeq\beta}$ hold. Then $(4.25)_\beta$ and $(4.27)_\beta$ hold.*

4.5 Estimate of modelled distributions: $\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy}$, $\delta\Pi^-_x - \mathrm{d}\Gamma^*_{xz}\Pi^-_z$, $\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz}\Pi_z$, and $\mathrm{d}\pi^{(\mathrm{n})}_{xy} - \mathrm{d}\pi^{(\mathrm{n})}_{xz} - \mathrm{d}\Gamma^*_{xz}\pi^{(\mathrm{n})}_{zy}$

This subsection is at the heart of our proof. We return to the estimate $(4.22)_\beta$ on $\delta\Pi^-_x$. Because of a lack of regularity in the singular case, (4.22) cannot be inferred from the estimate (4.34) on $\delta\Pi_x$ via the Malliavin derivative of the formula (2.18). In addition, such a formula would involve the divergent constants $c_{\beta'}$, at least for $\beta' \prec \beta$. Instead, we have to capitalize on the gain in regularity that comes with the passage from Π_x to $\delta\Pi_x$, which arises from replacing one of the instances of ξ in this multi-linear expression by a $\delta\xi$. However, this gain is subtle for two reasons:

- In terms of derivative count, the passage from ξ to $\delta\xi$ amounts to a gain in regularity by $\frac{D}{2} = \frac{3}{2}$, namely from $\alpha - 2$ to $\alpha - \frac{1}{2}$. However, due to the presence of the other instances of ξ in the multi-linear $\delta\Pi_x$, this does not translate into a plain gain of $\frac{3}{2}$ derivatives when passing from Π_x to $\delta\Pi_x$. Still, the degree of modeledness (cf. (4.89) below) of $\delta\Pi_x$ has a boost from α-Hölder continuity to $(\alpha + \frac{3}{2})$-modeledness. This modeledness w. r. t. Π_z is described by a modelled distribution[69] $\mathrm{d}\Gamma^*_{xz}$. Indeed, we shall control the rough-path increment $\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz}\Pi_z(y)$ and the continuity expression $\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy}$ to order $\alpha + \frac{3}{2}-$, the former in the sense of Gubinelli's controlled rough paths [32, Definition 1], see (4.89), the latter in the sense of [38, Definition 3.1], see (4.47).[70]
- In terms of scaling, there is – actually by construction – no difference between the $(\alpha - 2)$-Hölder norm relevant for ξ and the L^2-based Sobolev norm (2.2) of (fractional) order $\alpha - 2 + \frac{D}{2} = \alpha - \frac{1}{2}$ relevant for $\delta\xi$, or its (scaling-wise identical) annealed Besov version (4.23). Hence we will resort to a trick that appears like a cheat: In order to control the rough-path increments $\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz}\Pi_z(y)$ in terms of the (parabolic) distance $|y - z|$ of the active variable y to the secondary base point z to the desired power of $\alpha + \frac{D}{2}$, we will replace the norm (4.23) of $\delta\xi$ by a norm that involves a weight that diverges in z. In order to recover the full (nominal) gain of order of derivatives of $\frac{D}{2}$, one would be tempted to replace $\delta\xi_s$ in (4.23) by its weighted version $|\cdot - z|^{-\frac{D}{2}}\delta\xi_s$, which after squaring would result in the weighted integral $\int_{\mathbb{R}^2} |\cdot - z|^{-D}$. Recalling the definition (2.1) of the Carnot-Carathéodory distance, we however see that this integral is borderline divergent. This would, in Sect. 4.6, make it impossible to return from the weighted to the unweighted norm $\bar{w}$ by averaging in the base point z. Hence we have to marginally tame the weight by replacing $\frac{D}{2} = \frac{3}{2}$ by some exponent

$$\kappa < \frac{3}{2} \quad \text{(and thus in particular } \kappa + \alpha < 2 \text{ by } \alpha \leq \frac{1}{2}) \tag{4.36}$$

[69] which arises from $\mathrm{d}\pi^{(1,0)}_{xz}$.

[70] We remark that this separation between a controlled rough path condition (4.89) and the continuity condition (4.47) is once more due to the fact that our abstract model space T needs to be complemented by a copy of $\mathbb{R}$ capturing constant functions in order to reproduce Hairer's abstract model space, see Sect. 2.6. If this is done, [38, Definition 3.1] corresponds to the combination of (4.47) and (4.89).

and define

$$w(z) := \Big(\int_0^\infty \frac{ds}{s}(\sqrt[4]{s})^{2(\frac{1}{2}-\alpha)}\int_{\mathbb{R}^2} dy\,|y-z|^{-2\kappa}\,\mathbb{E}^{\frac{2}{q}}|\delta\xi_s(y)|^q\Big)^{\frac{1}{2}}. \tag{4.37}$$

On the one hand, $w(z)$ is strong enough to control an only slightly negative Hölder norm (however quenched and localized in z) of $\delta\xi_\tau$

$$\mathbb{E}^{\frac{1}{q}}|(\delta\xi_\tau)_t(z)|^q \lesssim (\sqrt[4]{t})^{\alpha-2+\kappa}\,w(z), \tag{4.38}$$

as we shall show in the proof of Proposition 4.12 in the context of the base case. On the other hand, because of (4.36) we have that (even square) averages of $w(z)$ reduce to $\bar{w}$:

$$\fint_{z:|z-x|\le\lambda} dz\,w(z) \le \Big(\fint_{z:|z-x|\le\lambda} dz\,w^2(z)\Big)^{\frac{1}{2}} \lesssim \lambda^{-\kappa}\,\bar{w}. \tag{4.39}$$

It is conceivable that one could carry out the tasks of this subsection on the level of Besov spaces, possibly appealing to [41].[71] However working on the (positive) Hölder level has the advantage that it is well-behaved under taking products, which is amply used in reconstruction.

This stronger norm (4.37) will indeed result in an (annealed) controlled rough-path estimate of $\delta\Pi_x$ of order $\kappa+\alpha$, see Proposition 4.14. This provides sufficient regularity in reconstruction when passing from the rough-path increments of $\delta\Pi_x$ to those of $\delta\Pi_x^-$, see Proposition 4.12. When it comes to the above-mentioned presence of the divergent c in the formula relating $\delta\Pi_x^-$ to $\delta\Pi_x$, we are saved by the fact that the c drops out when relating the rough-path increment $\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^*\Pi_z^-$ of $\delta\Pi_x^-$ to the (second derivative of the) rough-path increment $\partial_1^2(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^*\Pi_z)$ of $\delta\Pi_x$, see the crucial formula (4.50). We refer the reader to [8] for a more geometric intuition of $\mathrm{d}\Gamma^*$.

We defer the (inductive) construction of $\mathrm{d}\Gamma_{xz}^*$ to Sect. 5.4 and mention here just what is necessary to explain the estimates: In terms of its form, $\mathrm{d}\Gamma_{xz}^*$ is quite similar to $\delta\Gamma_{xy}^*$, see (4.24), but truncated beyond $\mathbf{n}=\mathbf{0}$, $(1,0)$, and with the Malliavin derivative $\delta\pi_{xy}^{(1,0)}$ replaced by some[72] $\mathrm{d}\pi_{xz}^{(1,0)} \in Q\tilde{\mathsf{T}}^*$:

$$\mathrm{d}\Gamma_{xz}^* = \sum_{\mathbf{n}=\mathbf{0},(1,0)} \mathrm{d}\pi_{xz}^{(\mathbf{n})}\Gamma_{xz}^* D^{(\mathbf{n})} \quad \text{with} \quad \mathrm{d}\pi_{xz}^{(\mathbf{0})} := \delta\pi_{xz}^{(\mathbf{0})}. \tag{4.40}$$

In line with $\mathrm{d}\pi_{xz}^{(\mathbf{0})} = \delta\pi_{xz}^{(\mathbf{0})} \in \tilde{\mathsf{T}}^*$, see (4.26), we impose

$$\mathrm{d}\pi_{xz}^{(1,0)} \in Q\tilde{\mathsf{T}}^* \quad \text{so that by (2.59)} \quad \mathrm{d}\Gamma_{xz}^*\mathsf{T}^* \subset \tilde{\mathsf{T}}^*. \tag{4.41}$$

[71] Note added in revision: this has been implemented in [8, 43].

[72] like for $\mathrm{d}\Gamma_{xy}^*$, the pre-fix d does not refer to an operation like the directional Malliavin derivative δ, but is part of the symbol.

Table 3 Population pattern of $(\mathrm{d}\Gamma^*)_\beta$ for $\beta = 0, e_1, 2e_1, 3e_1$

β	γ's for which $(\mathrm{d}\Gamma^*)_\beta^\gamma \neq 0$
0	$e_{(1,0)}$
e_1	$e_0, e_{(1,0)}, e_1 + e_{(1,0)}$
$2e_1$	$e_0, 2e_0, e_0 + e_1, e_{(1,0)}, e_1 + e_{(1,0)}, e_0 + e_1 + e_{(1,0)}, 2e_1 + e_{(1,0)}$
$3e_1$	$e_0, 2e_0, 3e_0, e_0 + e_1, 2e_0 + e_1, e_{(1,0)}, e_0 + 2e_1, e_1 + e_{(1,0)}$, $e_0 + e_1 + e_{(1,0)}, 2e_0 + e_1 + e_{(1,0)}, 2e_1 + e_{(1,0)}, e_0 + 2e_1 + e_{(1,0)}, 3e_1 + e_{(1,0)}$

In Sect. 5.4, we argue that $\mathrm{d}\pi_{xz}^{(1,0)}$ is determined by imposing qualitative first-order vanishing on every singular component

$$\mathbb{E}^{\frac{1}{q'}}|Q\big(\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^*\Pi_z(y)\big)|^{q'} = o(|y-z|). \tag{4.42}$$

We note that the population pattern of $(\mathrm{d}\Gamma_{xz}^*)_\beta^\gamma$ quickly gains in complexity as the homogeneity of β increases, see Table 3.

Up to these differences, the type and order of tasks will be as in Sect. 4.2. The first task is the algebraic argument relying on the following analogue of (4.24):

$$(\mathrm{d}\Gamma_{xy}^* - \mathrm{d}\Gamma_{xz}^*\Gamma_{zy}^*)Q = \sum_{\mathbf{n}=\mathbf{0},(1,0)} \big(\mathrm{d}\pi_{xy}^{(\mathbf{n})} - \mathrm{d}\pi_{xz}^{(\mathbf{n})} - \mathrm{d}\Gamma_{xz}^*\pi_{zy}^{(\mathbf{n})}\big)\Gamma_{xy}^* D^{(\mathbf{n})} Q. \tag{4.43}$$

This formula, which will be established in the proof of the upcoming Proposition 4.11, suggests to introduce the following estimate on the rough-path increments of $\mathrm{d}\pi_{xz}^{(\mathbf{n})}$:

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}\big|\big(\mathrm{d}\pi_{xy}^{(\mathbf{n})} - \mathrm{d}\pi_{xz}^{(\mathbf{n})} - \mathrm{d}\Gamma_{xz}^*\pi_{zy}^{(\mathbf{n})}\big)_\beta\big|^{q'}\\ &\lesssim |y-z|^{\kappa+\alpha-|\mathbf{n}|}(|y-z| + |z-x|)^{|\beta|-\alpha}(w_x(y) + w_x(z)) \quad \text{for } \mathbf{n} = \mathbf{0},\ (1,0),\end{aligned} \tag{4.44}$$

which is the analogue of (2.55). We note that (4.51), which we need to impose below, implies in particular[73]

$$\kappa > 1 - \alpha, \tag{4.45}$$

so that the first exponent in (4.44) is strictly positive. As we shall discuss at the beginning of Sect. 4.6, in (4.44) and in this subsection, we do not just need the norm $w(y) + w(z)$ with a singular weight at the two active points, y and z, but also a contribution from the unweighted norm $\bar{w}$. We combine weighted and unweighted norms through

$$w_x(z) := w(z) + |z-x|^{-\kappa}\bar{w}. \tag{4.46}$$

[73] using $\alpha \leq 1$.

While this inclusion of $\bar{w}$ is dimensionally correct, it has the irritating effect of introducing an artificial singularity at $z = x$, which however does not create problems.

Proposition 4.11 (Algebraic argument III) *Assume that* (4.44)$_{\prec\beta}$ *and* (5.10)$_{\prec\beta}$ *hold, and that* (2.37)$^{\gamma}_{\preccurlyeq\beta}$ *holds for all* γ *not purely polynomial. Then we have for all* γ *not purely polynomial*

$$\mathbb{E}^{\frac{1}{q'}}\big|\big((\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy})Q\big)^{\gamma}_{\beta}\big|^{q'} \lesssim \Big(\mathbf{1}_{\gamma(1,0)=0}\,|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha}$$

$$+\mathbf{1}_{\gamma(1,0)=1}\,|y-z|^{\kappa+\alpha-1}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha+1}\Big)(w_x(y)+w_x(z)), \quad (4.47)$$

with the implicit understanding that all exponents are non-negative unless the l. h. s. vanishes.

Before embarking on the proof, let us comment on how to interpret (4.47). For this, it is convenient to write the l. h. s. in the component-wise fashion of $(\mathrm{d}\Gamma^*_{xy})^{\gamma}_{\beta} - \sum_{\beta'}(\mathrm{d}\Gamma^*_{xz})^{\beta'}_{\beta}(\Gamma^*_{zy})^{\gamma}_{\beta'}$, to fix β and x, and to think of γ, β' and z, y as instances of the active index and variable, respectively. First let y and z be such that $|y-z| \ll |z-x|$. Then the r. h. s. of (4.47)$_\beta$ reduces, up to a multiplicative constant depending on x and z, to $\mathbf{1}_{\gamma(1,0)=0}\,|y-z|^{\kappa+\alpha} + \mathbf{1}_{\gamma(1,0)=1}\,|y-z|^{\kappa+\alpha-1}$, and therefore can be read as a modelled continuity condition [38, (3.1)] of degree $\kappa+\alpha$, provided the degree of γ is given by[74] $|\gamma|_p$. There is a second more subtle interpretation which becomes apparent when setting $z = x$ (and ignoring that $w_x(z=x) = \infty$): Then the r. h. s. of (4.47)$_\beta$ reduces to $|y-x|^{\kappa+|\beta|-|\gamma|}$, which expresses a modelled continuity condition of order $\kappa+|\beta|$ provided γ is graded by $|\gamma|$.

Proof We start with the argument for formula (4.43): Applying one of the commuting derivations $D \in \{D^{(\mathbf{m})}\}_{\mathbf{m}}$ to (2.44) yields by Leibniz' rule the operator identity

$$D\Gamma^* = \Gamma^* D + \sum_{\mathbf{n}}(D\pi^{(\mathbf{n})})\Gamma^* D^{(\mathbf{n})} \quad \text{and thus}$$

$$D\Gamma^* Q = \Gamma^* D Q + \sum_{\mathbf{n}=\mathbf{0},(1,0)}(D\pi^{(\mathbf{n})})\Gamma^* D^{(\mathbf{n})} Q.$$

Using this for $\Gamma^* = \Gamma^*_{zy}$ and then applying Γ^*_{xz} to it, we obtain by multiplicativity (2.49), which we may apply since D and $\Gamma^*_{zy}D^{(\mathbf{n})}$ map T^* into the sub-algebra $\tilde{\mathsf{T}}^*$ by (2.59), and by transitivity (5.11) (which can be applied since once more D maps T^* into $\tilde{\mathsf{T}}^*$; furthermore we will need only (5.11)$_{\beta_2}$ for $\beta_2 \preccurlyeq \beta$ as we shall see below, which follows from (5.10)$_{\prec\beta}$)

$$\Gamma^*_{xz} D\Gamma^*_{zy} Q = \Gamma^*_{xy} D Q + \sum_{\mathbf{n}=\mathbf{0},(1,0)}(\Gamma^*_{xz} D\pi^{(\mathbf{n})}_{zy})\Gamma^*_{xy} D^{(\mathbf{n})} Q.$$

[74] Note that for $|\gamma| < 2$, $|\gamma|_p = 0$ if $\gamma(1,0) = 0$ and $|\gamma|_p = 1$ for $\gamma(1,0) = 1$.

We now specify to $D=D^{(\mathbf{m})}$, (left-)multiply by $d\pi^{(\mathbf{m})}_{xz}$, and sum over $\mathbf{m}=\mathbf{0},(1,0)$ to obtain by definition (4.40),

$$\begin{aligned} d\Gamma^*_{xz}\Gamma^*_{zy}Q &= \sum_{\mathbf{m}=\mathbf{0},(1,0)} d\pi^{(\mathbf{m})}_{xz}\Gamma^*_{xy}D^{(\mathbf{m})}Q + \sum_{\mathbf{n}=\mathbf{0},(1,0)} (d\Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})\Gamma^*_{xy}D^{(\mathbf{n})}Q \\ &= \sum_{\mathbf{n}=\mathbf{0},(1,0)} \big(d\pi^{(\mathbf{n})}_{xz} + (d\Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})\big)\Gamma^*_{xy}D^{(\mathbf{n})}Q. \end{aligned}$$

Subtracting this from (4.40) (with z replaced by y and multiplied by Q from the right) yields (4.43).

We now turn to the estimate (4.47) proper. Since up to the indicator functions, the first r. h. s. term is dominated by the second one, it is enough to establish (4.47) without the first indicator function $\mathbf{1}_{\gamma(1,0)=0}$. Like in the proof of Proposition 4.8, we distinguish the contributions $\mathbf{n}=\mathbf{0}$ and $\mathbf{n}=(1,0)$. In particular, the $(\mathbf{n}=\mathbf{0})$-contribution to $(4.43)^\gamma_\beta$ gives rise to terms of the form

$$\big(d\pi^{(\mathbf{0})}_{xy} - d\pi^{(\mathbf{0})}_{xz} - d\Gamma^*_{xz}\pi^{(\mathbf{0})}_{zy}\big)_{\beta_1}(\Gamma^*_{xy})^{\gamma-e_k+e_{k+1}}_{\beta_2} \tag{4.48}$$

for some $k\geq 0$ and $\beta_1+\beta_2=\beta$. Since by (4.26) and (4.41), the first factor vanishes when β_1 is purely polynomial, like in the proof of Proposition 4.8, we effectively have $\beta_1\prec\beta$, $\beta_2\preccurlyeq\beta$, and $\gamma-e_k+e_{k+1}$ not purely polynomial. By Hölder's inequality in probability space, the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm of (4.48) is estimated by

$$|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta_1|-\alpha}(w_x(y)+w_x(z))\,|y-x|^{|\beta_2|-|\gamma-e_k+e_{k+1}|}.$$

Using $|y-x|\leq|y-z|+|z-x|$ on the last factor, and appealing to (4.31), we see that this terms is contained in the first r. h. s. term of (4.47).

The terms coming from the $(\mathbf{n}=(1,0))$-contribution to $(4.43)^\gamma_\beta$ are of the form

$$\big(d\pi^{(1,0)}_{xy} - d\pi^{(1,0)}_{xz} - d\Gamma^*_{xz}\pi^{(1,0)}_{zy}\big)_{\beta_1}(\Gamma^*_{xy})^{\gamma-e_{(1,0)}}_{\beta_2} \tag{4.49}$$

for some $\beta_1+\beta_2=\beta$. They are only present for $\gamma(1,0)\geq 1$; the presence of Q on the l. h. s. of (4.47) amounts to the restriction to $|\gamma|<2$, which in view of (2.27) only leaves $\gamma(1,0)=1$, giving rise to the characteristic function $\mathbf{1}_{\gamma(1,0)=1}$ in the second r. h. s. contribution to (4.47). Again, as in the proof of Proposition 4.8, we effectively have $\beta_1\prec\beta$, $\beta_2\preccurlyeq\beta$, and $\gamma-e_{\mathbf{n}}$ not purely polynomial. By Hölder's inequality in probability space, the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm of (4.49) is estimated by

$$|y-z|^{\kappa+\alpha-1}(|y-z|+|z-x|)^{|\beta_1|-\alpha}(w_x(y)+w_x(z))\,|y-x|^{|\beta_2|-|\gamma-e_{(1,0)}|}.$$

As before, this time appealing to (4.33), we see that this term is contained in the second r. h. s. term of (4.47). □

The second task is to estimate the rough-path increments of $\delta\Pi^-_x$ based on the estimate of the rough-path increments of $\delta\Pi_x$ stated in Proposition 4.14. It relies on the c-free formula

$$\begin{aligned} &Q\big(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-\big)(z) \\ &\quad = Q\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k(z)\partial_1^2\big(\delta\Pi_x - \mathrm{d}\Gamma_{xz}^* Q\Pi_z\big)(z) + \delta\xi_\tau(z)\mathsf{1}, \end{aligned} \tag{4.50}$$

which is the analogue of (4.6), and the argument for which will be given in the proof of Proposition 4.12. In order to actually pass from $(4.89)_{\prec\beta}$ to $(4.52)_\beta$, we need to free ourselves from the evaluation at z in $(4.50)_\beta$, which will be done by a reconstruction argument. Reconstruction requires that the sum of α (the bare regularity of the first factor Π_x) and of $\kappa+\alpha$ (the degree of modeledness of the second factor $\delta\Pi_x$) is larger than 2 (due to the presence of the second spatial derivatives). This enforces the lower bound

$$\kappa > 2 - 2\alpha, \tag{4.51}$$

which together with the upper bound (4.36) is the sole reason for our assumption $\alpha > \frac{1}{4}$. Incidentally, an identity analogous to (4.50) would hold for the non-centered model Π, i. e. for the first base point x omitted, and would presumably allow for reconstruction. However, it would not allow us to derive the estimates of the right homogeneity, but rather the plain Hölder estimates quite similar to [40].

In terms of Q, there is a mismatch between the output of Proposition 4.11 and the ideal input for the upcoming Proposition 4.12. Handling the mismatch requires estimating $\mathrm{d}\Gamma_{xz}^*(\mathrm{id}-Q)$, which follows from the boundedness – as opposed to continuity – of $\mathrm{d}\Gamma_{xz}^*$ (see (4.108) in Sect. 4.6, where it plays a more important role). Analogously to the second task of Sect. 4.2 we obtain

Proposition 4.12 (Reconstruction III) *Assume that* $(2.36)_{\prec\beta}$, $(2.37)_{\prec\beta}$, $(2.42)_{\prec\beta}$, $(2.43)_{\prec\beta}$, $(2.60)_{\prec\beta}$, $(2.61)_{\prec\beta}$, $(2.64)_{\prec\beta}$, $(4.54)_{\prec\beta}$, $(4.55)_\beta$, *and* $(4.89)_{\prec\beta}$ *hold, assume that* $(4.47)^\gamma_{\preccurlyeq\beta}$ *and* $(4.108)^\gamma_{\preccurlyeq\beta}$ *hold, both for all* γ *not purely polynomial. Then we have*

$$\begin{aligned} &\mathbb{E}^{\frac{1}{q'}}|(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'} \\ &\lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\kappa}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)). \end{aligned} \tag{4.52}$$

We remark that these three exponents are natural: The first exponent $\alpha-2$ captures the bare (distributional) regularity of $\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^*\Pi_z^-$ at an arbitrary point, which is not better than the one of $\delta\Pi_x^-$ or Π_z^-, and does not depend on β. If $y=z$, the sum $\kappa+\alpha-2$ of the two first exponents emerges and describes the regularity of the expression $\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^*\Pi_z^-$ near the secondary base point z, which does not depend on β; it makes the gain of κ appear that arises from the weight in $w(z)$. Finally, the sum $\kappa+|\beta|-2$ is dictated by scaling; passing from $w(z)$ back to $\bar{w}$ removes a length to the power κ, leads to $|\beta|-2$, in line with (4.22) and ultimately (2.64).

As in the proof of Proposition 4.2, for the reconstruction of the rough-path increments of $\delta\Pi_x^-$ we require (weak) continuity in the active variable. Therefore, we need

– in a purely qualitative way – the analogue of Proposition 2.3 on the level of Malliavin derivatives, the proof of which follows along the same lines and is postponed to Sect. 6.

Proposition 4.13 (Divergent bounds II) *Under Assumption 2.1 the following holds for every populated* $|\beta| < 2$:

$$\mathbb{E}^{\frac{1}{q'}}|\partial_1^2\delta\Pi_{x\beta}(y)|^{q'} + \mathbb{E}^{\frac{1}{q'}}|\partial_2\delta\Pi_{x\beta}(y)|^{q'} \lesssim (\sqrt[4]{\tau})^{\alpha-2}(\sqrt[4]{\tau}+|y-x|)^{|\beta|-\alpha}\bar{w}. \quad (4.53)$$

Furthermore, we have

$$\mathbb{E}^{\frac{1}{q'}}|\partial_1^2\delta\Pi_{x\beta}(y) - \partial_1^2\delta\Pi_{x\beta}(z)|^{q'} + \mathbb{E}^{\frac{1}{q'}}|\partial_2\delta\Pi_{x\beta}(y) - \partial_2\delta\Pi_{x\beta}(z)|^{q'} \quad (4.54)$$

$$+ \mathbb{E}^{\frac{1}{q'}}|\delta\Pi^-_{x\beta}(y) - \delta\Pi^-_{x\beta}(z)|^{q'} \quad (4.55)$$

$$\lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau}+|y-x|+|z-x|)^{|\beta|-\alpha}|y-z|^{\alpha}\bar{w}.$$

By duality, (4.54) implies an annealed and weighted $C^{2,\alpha}$-estimate for the Malliavin derivate $\frac{\partial}{\partial\xi}\Pi_{x\beta}$. Kolmogorov's criterion[75] then ensures that $\frac{\partial}{\partial\xi}\Pi_{x\beta} \in C^2(H^*)$ almost surely, where H^* denotes the Hilbert space with norm $\|\cdot\|_*$ defined in (2.4). Likewise, we have $\frac{\partial}{\partial\xi}\Pi^-_{x\beta} \in C^0(H^*)$. This justifies evaluating $\delta\Pi^-_x$ and $\partial_1^2\delta\Pi_x$ in a point z, as done in (4.50).

Proof of Proposition 4.12 We start with the proof of (4.50). On the one hand, we apply Q to (4.6); on the other hand, we take the Malliavin derivative of (2.57), see Sect. 7.2:

$$Q\Pi^-_z(z) = Q\big(P\mathsf{z}_0\partial_1^2\Pi_z - c + \xi_\tau\mathsf{1}\big)(z), \quad (4.56)$$

$$\delta\Pi^-_x = P\Big(\sum_{k\ge0}\mathsf{z}_k\Pi^k_x\partial_1^2\delta\Pi_x + \sum_{k\ge0}(k+1)\mathsf{z}_{k+1}\Pi^k_x\delta\Pi_x\partial_1^2\Pi_x\Big)$$
$$-\sum_{k\ge0}\frac{1}{k!}\Pi^k_x\delta\Pi_x(D^{(\mathbf{0})})^{k+1}c + \delta\xi_\tau\mathsf{1}. \quad (4.57)$$

Since by definition (2.27), the $\beta(k=0)$-component has no effect on $|\beta|$, we have $QP\mathsf{z}_0 = P\mathsf{z}_0Q$, so that in view of

$$Q\partial_1^2\Pi_x \in \tilde{\mathsf{T}}^*, \quad (4.58)$$

which follows from (2.21) and (2.28), P is inactive in (4.56). Moreover, by the first item in (2.34), Q is inactive on c and of course on $\mathsf{1}$. By (4.26) and (8.8), P is also inactive in (4.57). Hence we learn (using also $\mathrm{d}\Gamma^*_{xz}\mathsf{1} = 0$) that (4.50) follows from

$$Q\mathrm{d}\Gamma^*_{xz}\mathsf{z}_0Q\partial_1^2\Pi_z(z)$$
$$= Q\Big(\sum_{k\ge0}\mathsf{z}_k\Pi^k_x\mathrm{d}\Gamma^*_{xz}Q\partial_1^2\Pi_z + \sum_{k\ge0}(k+1)\mathsf{z}_{k+1}\Pi^k_x\delta\Pi_x\partial_1^2\Pi_x\Big)(z), \quad (4.59)$$

[75] for random fields taking values in a Hilbert space.

$$\mathrm{d}\Gamma^*_{xz}c = \big(\delta\Pi_x \sum_{k\geq 0}\frac{1}{k!}\Pi_x^k(D^{(\mathbf{0})})^{k+1}c\big)(z). \tag{4.60}$$

We start by arguing that (4.59) follows from

$$\mathrm{d}\Gamma^*_{xz}\mathsf{z}_0\pi' = \sum_{k\geq 0}\mathsf{z}_k\Pi_x^k(z)\mathrm{d}\Gamma^*_{xz}\pi' + \delta\Pi_x(z)\sum_{k\geq 0}(k+1)\mathsf{z}_{k+1}\Pi_x^k(z)\Gamma^*_{xz}\pi' \\ \text{for } \pi' \in \tilde{\mathsf{T}}^*. \tag{4.61}$$

Indeed, we use (2.61) in form of $\partial_1^2\Pi_x = \Gamma^*_{xz}\partial_1^2\Pi_z$ and the triangularity properties (8.7) and (8.9) w. r. t. $|\cdot|$ to see $Q\mathsf{z}_{k+1}\Pi_x^k(z)\delta\Pi_x(z)\Gamma^*_{xz}\partial_1^2\Pi_z(z) = Q\mathsf{z}_{k+1}\Pi_x^k(z)\times \delta\Pi_x(z)\Gamma^*_{xz}Q\partial_1^2\Pi_z(z)$. Hence (4.59) indeed follows from (4.61) for $\pi' = Q\partial_1^2\Pi_z(z)$. By (2.63) and the second item in (4.40) in form of $\delta\Pi_x(z) = \mathrm{d}\pi^{(\mathbf{0})}_{xz}$, (4.60) and (4.61) take the form of

$$\mathrm{d}\Gamma^*_{xz}\mathsf{z}_0\pi' = \sum_{k\geq 0}\mathsf{z}_k(\pi^{(\mathbf{0})}_{xz})^k\mathrm{d}\Gamma^*_{xz}\pi' + \mathrm{d}\pi^{(\mathbf{0})}_{xz}\sum_{k\geq 0}(k+1)\mathsf{z}_{k+1}(\pi^{(\mathbf{0})}_{xz})^k\Gamma^*_{xz}\pi', \tag{4.62}$$

$$\mathrm{d}\Gamma^*_{xz}c = \mathrm{d}\pi^{(\mathbf{0})}_{xz}\sum_{k\geq 0}\frac{1}{k!}(\pi^{(\mathbf{0})}_{xz})^k(D^{(\mathbf{0})})^{k+1}c. \tag{4.63}$$

Because of the second item in the population condition (2.34), which we may rewrite as $D^{(\mathbf{n})}c = 0$ for $\mathbf{n} \neq \mathbf{0}$, it follows from (2.44) that

$$\Gamma^*(D^{(\mathbf{0})})^{k'}c = \sum_{k\geq 0}\frac{1}{k!}(\pi^{(\mathbf{0})})^k(D^{(\mathbf{0})})^{k+k'}c. \tag{4.64}$$

Together with (2.50) for $k = 0, 1$, we see that the two identities (4.62) & (4.63) can be written as

$$\mathrm{d}\Gamma^*_{xz}\mathsf{z}_0\pi' = (\Gamma^*_{xz}\mathsf{z}_0)\mathrm{d}\Gamma^*_{xz}\pi' + \mathrm{d}\pi^{(\mathbf{0})}_{xz}(\Gamma^*_{xz}\mathsf{z}_1)\Gamma^*_{xz}\pi' \quad \text{and} \quad \mathrm{d}\Gamma^*_{xz}c = \mathrm{d}\pi^{(\mathbf{0})}_{xz}\Gamma^*_{xz}D^{(\mathbf{0})}c.$$

By definition (4.40), the first identity follows from the fact that $D^{(\mathbf{n})}$ is a derivation, mapping T^* into $\tilde{\mathsf{T}}^*$, see (2.59), from how it acts on z_k, see (2.10), and the multiplicativity (2.49) of Γ^*_{xz} together with the fact that $\tilde{\mathsf{T}}^*$ is closed under multiplication. The second identity follows by definition (4.40) using once more that $D^{(1,0)}c = 0$.

Before starting with the estimates, we note that not only $w(y)$, cf. (4.39), but also $w_x(y)$ behaves well under (square) averaging in y, in particular by $(\sqrt[4]{t})^{|\mathbf{n}|}|\partial^{\mathbf{n}}\psi_t|$:

$$(\sqrt[4]{t})^{|\mathbf{n}|}\int dy'|\partial^{\mathbf{n}}\psi_t(y-y')|w_x^2(y') \lesssim w_x^2(z) \quad \text{provided } |y-z| \leq \sqrt[4]{t}. \tag{4.65}$$

Indeed, in view of the definitions (4.37) and (4.46), (4.65) follows from the elementary fact that also negative moments are preserved by averaging with $(\sqrt[4]{t})^{|\mathbf{n}|}|\partial^{\mathbf{n}}\psi_t|$:

$$
\begin{aligned}
&(\sqrt[4]{t})^{|\mathbf{n}|}\int dy'|\partial^{\mathbf{n}}\psi_t(y-y')||x'-y'|^{-2\kappa}\\
&\quad\lesssim(\sqrt[4]{t}+|x'-y|)^{-2\kappa}\\
&\quad\sim(2\sqrt[4]{t}+|x'-y|)^{-2\kappa}\overset{|y-z|\leq\sqrt[4]{t}}{\lesssim}(\sqrt[4]{t}+|x'-z|)^{-2\kappa}.
\end{aligned}
\tag{4.66}
$$

We now introduce the family $\{F_{xz}\}_{x,z}$ of random space-time Schwartz distributions

$$
\begin{aligned}
F_{xz}:=&\big(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^*Q\Pi_z^-\big)\\
&-\Big(\sum_{k\geq 0}\mathsf{z}_k\Pi_x^k(z)\partial_1^2\big(\delta\Pi_x-\mathrm{d}\Gamma_{xz}^*Q\Pi_z\big)+\delta\xi_\tau\mathsf{1}\Big),
\end{aligned}
\tag{4.67}
$$

and note that (4.50) amounts to

$$
F_{xz\beta}(z)=0. \tag{4.68}
$$

In fact, we need (4.68) in the more robust form

$$
\lim_{t\downarrow 0}\mathbb{E}^{\frac{1}{q'}}|F_{xz\beta\,t}(z)|^{q'}=0. \tag{4.69}
$$

In order to pass from (4.68) to (4.69) it is sufficient to argue that the β-component of the r. h. s. of (4.67) is continuous in the active variable w. r. t. $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$, which we do term by term. The continuity of $\delta\Pi_{x\beta}^-$ is stated in $(4.55)_\beta$; the continuity of $\delta\xi_\tau$ amounts to $(4.55)_{\beta=0}$. Appealing to (8.11), the continuity of $(\mathrm{d}\Gamma_{xz}^*Q\Pi_z^-)_\beta$ follows from $(4.108)_\beta^{\gamma\neq\mathrm{p.p.}}$ and $(2.43)_{\prec\beta}$. For the term $\big(\sum_{k\geq0}\mathsf{z}_k\Pi_x^k(z)\partial_1^2(\delta\Pi_x - \mathrm{d}\Gamma_{xz}^*Q\Pi_z)\big)_\beta$ we appeal to (8.5) & (8.6). Hence, for the term involving $\partial_1^2\delta\Pi_x$, continuity follows from boundedness $(2.36)_{\prec\beta}$ and the continuity $(4.54)_{\prec\beta}$. For the term involving $\partial_1^2(\mathrm{d}\Gamma_{xz}^*Q\Pi_z)$, continuity follows from boundedness $(2.36)_{\prec\beta}$, combined with (8.11) & $(4.108)_{\beta'}^{\gamma\neq\mathrm{p.p.}}$ and the continuity $(2.42)_{\prec\beta}$.

We shall establish the following continuity condition of $\{F_{xz}\}_{x,z}$ in the secondary base point z:

$$
\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|(F_{xy}-F_{xz})_{\beta t}(y)|^{q'}\\
&\lesssim(\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\theta-\alpha}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|+\kappa-\theta}\\
&\quad\times(w_x(z)+w_x(y)),
\end{aligned}
\tag{4.70}
$$

where

$$
\theta:=\min\{\kappa+2\alpha,\inf(\mathsf{A}\cap(2,\infty))\}\overset{(4.51),\ \mathsf{A}\text{ is locally finite}}{>}2. \tag{4.71}
$$

Before proceeding, we note that the l. h. s. of (4.70) vanishes for $\beta \in \mathbb{N}_0 e_0$. Indeed, we start by observing that $k \neq 0$ cannot contribute to such a component of (4.67). Because of the difference on the l. h. s. of (4.70), only the contributions involving $\mathrm{d}\Gamma^*_{x\cdot}$ contribute. Hence in view of (4.58), the claim follows from (8.14). In particular, by (8.15), the l. h. s. of (4.70) vanishes unless $|\beta| \geq 2\alpha$, which implies for the last exponent in (4.70) that $|\beta| + \kappa - \theta \geq 0$ by definition (4.71) of θ.

Since in view of (4.71), the sum of the first two exponents is positive, (4.70) provides a continuity condition of positive order. Using the reconstruction argument in [52, Lemma 4.8] in combination with the averaging property of the weight (4.65), (4.69) and (4.70) upgrade to

$$\mathbb{E}^{\frac{1}{q'}} |F_{xz\beta\, t}(z)|^{q'} \lesssim (\sqrt[4]{t})^{\theta-2} (\sqrt[4]{t} + |z-x|)^{|\beta|+\kappa-\theta} w_x(z). \tag{4.72}$$

Using once more (4.70), this yields

$$\begin{aligned}\mathbb{E}^{\frac{1}{q'}} |F_{xz\beta\, t}(y)|^{q'} &\lesssim (\sqrt[4]{t})^{\alpha-2} (\sqrt[4]{t} + |y-z|)^{\theta-\alpha} (\sqrt[4]{t} + |y-z| + |z-x|)^{|\beta|+\kappa-\theta} \\ &\quad \times (w_x(z) + w_x(y)),\end{aligned}$$

which we just use in the weakened form of (since $\kappa \leq \theta - \alpha$ by (4.36) and (4.71))

$$\begin{aligned}\mathbb{E}^{\frac{1}{q'}} |F_{xz\beta\, t}(y)|^{q'} &\lesssim (\sqrt[4]{t})^{\alpha-2} (\sqrt[4]{t} + |y-z|)^{\kappa} (\sqrt[4]{t} + |y-z| + |z-x|)^{|\beta|-\alpha} \\ &\quad \times (w_x(z) + w_x(y)).\end{aligned}$$

Once we establish the boundedness of the second contribution to (4.67) in form of

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}} \Big| \Big(\sum_{k\geq 0} \mathsf{z}_k \Pi_x^k(z) \partial_1^2 \big(\delta\Pi_x - \mathrm{d}\Gamma^*_{xz} Q \Pi_z\big) + \delta\xi_\tau \mathsf{1}\Big)_{\beta\, t}(y)\Big|^{q'} \\ &\lesssim (\sqrt[4]{t})^{\alpha-2} (\sqrt[4]{t} + |y-z|)^{\kappa} (\sqrt[4]{t} + |y-z| + |z-x|)^{|\beta|-\alpha} (w_x(z) + w_x(y)),\end{aligned} \tag{4.73}$$

we obtain the desired (4.52).

The remainder of the proof is devoted to the estimates (4.70) and (4.73). By the triangle inequality, we split (4.70) into

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}} |(\mathrm{d}\Gamma^*_{xy} Q \Pi^-_y - \mathrm{d}\Gamma^*_{xz} Q \Pi^-_z)_{\beta\, t}(y)|^{q'} \\ &\lesssim (\sqrt[4]{t})^{\alpha-2} (\sqrt[4]{t} + |y-z|)^{\theta-\alpha} (\sqrt[4]{t} + |y-z| + |z-x|)^{|\beta|+\kappa-\theta} (w_x(z) + w_x(y)),\end{aligned} \tag{4.74}$$

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}} \Big| \Big(\mathsf{z}_k \big(\Pi_x^k(y) \partial_1^2 (\delta\Pi_x - \mathrm{d}\Gamma^*_{xy} Q \Pi_y) - \Pi_x^k(z) \partial_1^2 (\delta\Pi_x - \mathrm{d}\Gamma^*_{xz} Q \Pi_z)\big)\Big)_{\beta\, t}(y)\Big|^{q'} \\ &\lesssim (\sqrt[4]{t})^{\alpha-2} (\sqrt[4]{t} + |y-z|)^{\theta-\alpha} (\sqrt[4]{t} + |y-z| + |z-x|)^{|\beta|+\kappa-\theta} (w_x(z) + w_x(y)).\end{aligned} \tag{4.75}$$

Note that as opposed to (4.70), (4.73) does see $\delta\xi$, which we thus split into

$$\mathbb{E}^{\frac{1}{q'}}\big|\big(\mathsf{z}_k\Pi_x^k(z)\partial_1^2(\delta\Pi_x - \mathrm{d}\Gamma_{xz}^*Q\Pi_z)\big)_{\beta\,t}(y)\big|^{q'}$$
$$\lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\kappa}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)) \quad (4.76)$$

and the base case, meaning (4.73) for $\beta=0$. Due to the presence of z_k, the l. h. s. of $(4.73)_{\beta=0}$ collapses to $(\delta\xi_\tau)_t(y)$; we shall establish the stronger version of (4.38). By the semi-group property (4.1) and $\alpha-2+\kappa<0$, see (4.36), it is enough to establish (4.38) for $\tau=0$. In order to do so, we appeal again to the semi-group property in form of $\delta\xi_t(y)=\int dz\psi_{t-s}(y-z)\delta\xi_s(z)$ for $s\in(0,t)$, so that by the triangle inequality w. r. t. $\mathbb{E}^{\frac{1}{q}}|\cdot|^q$ we have $\mathbb{E}^{\frac{1}{q}}|\delta\xi_t(y)|^q\le\int dz|\psi_{t-s}(y-z)|\mathbb{E}^{\frac{1}{q}}|\delta\xi_s(z)|^q$, and thus by Cauchy-Schwarz in z (and in view of the obvious sup-bound $|\psi_{t-s}|\lesssim(\sqrt[4]{t-s})^{-3}$)

$$\begin{aligned}\mathbb{E}^{\frac{2}{q}}|\delta\xi_t(y)|^q &\le \int dz\psi_{t-s}^2(y-z)|z-y|^{2\kappa}\int dz|z-y|^{-2\kappa}\mathbb{E}^{\frac{2}{q}}|\delta\xi_s(z)|^q\\ &\overset{(4.3)}{\lesssim} (\sqrt[4]{t-s})^{-3+2\kappa}\int dz|z-y|^{-2\kappa}\mathbb{E}^{\frac{2}{q}}|\delta\xi_s(z)|^q.\end{aligned}$$

Applying $\int_{\frac{t}{4}}^{\frac{t}{2}}\frac{ds}{s}$, so that $(\sqrt[4]{t-s})^{-3+2\kappa}\sim(\sqrt[4]{t})^{2(\alpha-2+\kappa)}\,(\sqrt[4]{s})^{2(\frac{1}{2}-\alpha)}$, we obtain the square of (4.38) with $\tau=0$ by definition (4.37).

Turning to (4.76) we note that the l. h. s. has the product structure $(\mathsf{z}_k\pi\pi')_\beta=\sum_{\beta_1+\beta_2=\beta}(\mathsf{z}_k\pi)_{\beta_1}\pi'_{\beta_2}$. In view of the presence of z_k, β_1 is neither purely polynomial nor 0; by (4.26) and (4.41) also β_2 is not purely polynomial. Hence by (3.6) we have $\beta_1\preccurlyeq\beta$ and $\beta_2\prec\beta$. Therefore $(4.76)_\beta$ follows from the two estimates $(4.85)_{\prec\beta}$ and $(4.87)_{\preccurlyeq\beta}$ stated below via Hölder's inequality in probability and (3.7).

We turn to (4.74), which relies on $(2.60)_{\prec\beta}$ in form of $Q\Pi_z^-=Q\Gamma_{zy}^*\Pi_y^-$. In preparation for the use of (4.47), we use the triangularity property (2.33) in form of

$$Q\Gamma^*=Q\Gamma^*Q \quad (4.77)$$

to split the increment:

$$\begin{aligned}&(\mathrm{d}\Gamma_{xy}^*Q\Pi_y^- - \mathrm{d}\Gamma_{xz}^*Q\Pi_z^-)_t(y)\\ &\overset{(4.77)}{=}(\mathrm{d}\Gamma_{xy}^*-\mathrm{d}\Gamma_{xz}^*\Gamma_{zy}^*)Q\Pi_{y\,t}^-(y)+\mathrm{d}\Gamma_{xz}^*(\mathrm{id}-Q)\Gamma_{zy}^*Q\Pi_{y\,t}^-(y).\end{aligned} \quad (4.78)$$

By the triangular structure (8.9) of Γ^*, and the strict triangular structure (8.11) of $\mathrm{d}\Gamma^*$ and (8.13) of its increments, we learn that in order to estimate $(4.78)_\beta$, we only need $(2.64)_{\prec\beta}$ and $(2.37)_{\prec\beta}$. Because of the presence of P in the definition (2.57) and by the consistency (2.48) of elements $\Gamma\in\mathsf{G}$ with the polynomial sector, which for later purpose we write in the compact form

$$\Gamma^* P = P\Gamma^* P, \tag{4.79}$$

we only need $(2.37)^{\gamma\neq\mathrm{p.p.}}_{\prec\beta}$, $(4.47)^{\gamma\neq\mathrm{p.p.}}_{\beta}$, and $(4.108)^{\gamma\neq\mathrm{p.p.}}_{\beta}$. We thus have term-by-term

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|(\mathrm{d}\Gamma^*_{xy}Q\Pi^-_y - \mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)_{\beta\,t}(y)|^{q'}\\
&\lesssim \sum_{\substack{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|-\alpha]\\ \gamma(1,0)=0}} |y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha}(w_x(z)+w_x(y))(\sqrt[4]{t})^{|\gamma|-2}\\
&+ \sum_{\substack{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|-\alpha+1]\\ \gamma(1,0)=1}} |y-z|^{\kappa+\alpha-1}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha+1}\\
&\quad\times (w_x(z)+w_x(y))(\sqrt[4]{t})^{|\gamma|-2}\\
&+ \sum_{|\gamma'|\in\mathsf{A}\cap[2,\kappa+|\beta|]} |z-x|^{\kappa+|\beta|-|\gamma'|}w_x(z) \sum_{|\gamma|\in\mathsf{A}\cap[\alpha,2]} |y-z|^{|\gamma'|-|\gamma|}(\sqrt[4]{t})^{|\gamma|-2}\\
&\lesssim \Big(\sum_{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|-\alpha]} (\sqrt[4]{t})^{|\gamma|-2}(\sqrt[4]{t}+|y-z|)^{\kappa+\alpha}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha}
\end{aligned} \tag{4.80}$$

$$\begin{aligned}
&+ \sum_{\substack{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|-\alpha+1]\\ \gamma(1,0)=1}} (\sqrt[4]{t})^{|\gamma|-2}\\
&\quad\times (\sqrt[4]{t}+|y-z|)^{\kappa+\alpha-1}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-(|\gamma|-1)-\alpha}
\end{aligned} \tag{4.81}$$

$$\begin{aligned}
&+ \sum_{|\gamma|\in\mathsf{A}\cap[\alpha,2]} (\sqrt[4]{t})^{|\gamma|-2}\\
&\quad\times \sum_{|\gamma'|\in\mathsf{A}\cap[2,\kappa+|\beta|]} (\sqrt[4]{t}+|y-z|)^{|\gamma'|-|\gamma|}(\sqrt[4]{t}+|y-z|+|z-x|)^{\kappa+|\beta|-|\gamma'|}\Big)
\end{aligned} \tag{4.82}$$

$$\times (w_x(z)+w_x(y)).$$

The term (4.80) is absorbed into the r. h. s. of (4.74) because the first exponent decreases from $|\gamma|-2$ to $\alpha-2$, because the sum of the first two exponent decreases from $|\gamma|-2+\kappa+\alpha$ to $\theta-2$ by definition (4.71) of θ, and because the sum $|\beta|-2+\kappa$ of the three exponents agrees. The term (4.81) is also absorbed because once more, the first exponent decreases and the sum of all three exponents agrees, and because for our not purely polynomial γ, the constraint $\gamma(1,0)=1$ implies $|\gamma|\geq 1+\alpha$ by definition (2.27) of $|\cdot|$, leading to $|\gamma|-2+\kappa+\alpha-1\geq\theta-2$ on the sum of the first two exponents. Finally, the term (4.82) is also absorbed, since for the not purely polynomial γ', the constraint $|\gamma'|\geq 2$ implies $|\gamma'|>2$ by

(2.29) and thus $|\gamma'| \geq \theta$ by definition (4.71) of θ, leading once more to $|\gamma'| - 2 \geq \theta - 2$.

We now turn to (4.75). As usual, we will estimate this difference of a product by the sum of two products, where each summand contains a difference as one of its factors. For the same reason as for (4.76) we have the structure $(\mathsf{z}_k\pi\pi')_\beta = \sum_{\beta_1+\beta_2=\beta}(\mathsf{z}_k\pi)_{\beta_1}\pi'_{\beta_2}$ with $\beta_1 \preccurlyeq \beta$ and $\beta_2 \prec \beta$. Hence we see that $(4.75)_\beta$ follows from the four estimates $(4.83)_{\prec\beta}$, $(4.84)_{\preccurlyeq\beta}$, $(4.85)_{\prec\beta}$, and $(4.87)_{\preccurlyeq\beta}$, stated below, by the triangle inequality and Hölder's inequality in probability and (3.7).

As the first ingredient to (4.75), we estimate the components $\prec \beta$ of $\mathrm{d}\Gamma^*_{xy}Q\partial_1^2\Pi_y - \mathrm{d}\Gamma^*_{xz}Q\partial_1^2\Pi_z$. In view of (4.58), the argument is similar to (4.74). More precisely, considering[76]

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|(\mathrm{d}\Gamma^*_{xy}Q\partial_1^2\Pi_y - \mathrm{d}\Gamma^*_{xz}Q\partial_1^2\Pi_z)_{\beta\, t}(y)|^{q'} \\
&\lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\theta-\alpha}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|+\kappa-\theta}(w_x(z)+w_y(z)),
\end{aligned} \tag{4.83}$$

$(4.83)_\beta$ follows from $(2.36)_{\prec\beta}$, $(2.37)^{\gamma\neq\mathrm{p.p.}}_{\prec\beta}$, $(4.47)^{\gamma\neq\mathrm{p.p.}}_{\beta}$, and $(4.108)^{\gamma\neq\mathrm{p.p.}}_{\beta}$. As for (4.70), we note that the l. h. s. of $(4.83)_\beta$ vanishes unless $|\beta| \geq 2\alpha$ because of (8.14), so that $|\beta| + \kappa - \theta > 0$.

As the second ingredient to (4.75), we need to estimate all $\preccurlyeq \beta$-components of the increment of the factor $\mathsf{z}_k\Pi_x^k$. Considering for all $k \geq 0$

$$\mathbb{E}^{\frac{1}{p}}\big|\big(\mathsf{z}_k\Pi_x^k(y) - \mathsf{z}_k\Pi_x^k(z)\big)_\beta\big|^p \lesssim |y-z|^\alpha(|y-z|+|z-x|)^{|\beta|-2\alpha}, \tag{4.84}$$

we claim that $(2.36)_{\prec\beta}$, $(2.37)^{\gamma}_{\prec\beta}$ (including the purely polynomial γ's) and $(2.61)_{\prec\beta}$ imply $(4.84)_\beta$. We still have the implicit understanding that the l. h. s. vanishes unless $|\beta| - 2\alpha \geq 0$. Indeed, in view of (8.15), $|\beta| - 2\alpha < 0$ implies $\beta \in \mathbb{N}_0 e_0$; due to the presence of z_k, the term $(\mathsf{z}_k\Pi_x^k)_\beta$ vanishes unless $k = 0$, in which case the l. h. s. obviously vanishes. In particular, in establishing (4.84), we may restrict to $k \geq 1$ and write with the help of (2.61) and (2.63)

$$\Pi_x^k(y) - \Pi_x^k(z) = \sum_{k'+k''=k-1}\Pi_x^{k'}(y)\Pi_x^{k''}(z)\Gamma^*_{xz}\Pi_z(y),$$

so that we obtain componentwise

$$\begin{aligned}
&\big(\mathsf{z}_k\Pi_x^k(y) - \mathsf{z}_k\Pi_x^k(z)\big)_\beta \\
&= \sum_{k'+k''=k-1}\ \sum_{e_k+\beta_1+\cdots+\beta_k=\beta}\Pi_{x\beta_1}(y)\cdots\Pi_{x\beta_{k'}}(y)\Pi_{x\beta_{k'+1}}(z)\cdots\Pi_{x\beta_{k-1}}(z) \\
&\quad\times\sum_\gamma(\Gamma^*_{xz})^\gamma_{\beta_k}\Pi_{z\gamma}(y).
\end{aligned}$$

[76] where β here is generic and in fact will be applied to a preceding multi-index.

By (8.6) we have $\beta_1,\ldots,\beta_k \prec \beta$, and then by (8.9) also $\gamma \prec \beta$, so that $(2.36)_{\prec\beta}$ and $(2.37)_{\prec\beta}$ are indeed sufficient to conclude by Hölder's inequality in probability

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{p}}\big|\big(\mathsf{z}_k\Pi_x^k(y)-\mathsf{z}_k\Pi_x^k(z)\big)_\beta\big|^p\\
&\lesssim \sum_{k'+k''=k-1}\ \sum_{e_k+\beta_1+\cdots+\beta_k=\beta} |y-x|^{|\beta_1|+\cdots+|\beta_{k'}|}|z-x|^{|\beta_{k'+1}|+\cdots+|\beta_{k-1}|}\\
&\quad\times \sum_{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta_k|]} |z-x|^{|\beta_k|-|\gamma|}|y-z|^{|\gamma|}.
\end{aligned}$$

Using $|y-x|\le|y-z|+|z-x|$ and (3.7), this collapses to (4.84).

As the third ingredient to (4.75), and the first ingredient to (4.76), we estimate the $\prec\beta$-components of $\partial_1^2(\delta\Pi_x-\mathrm{d}\Gamma_{xz}^*Q\Pi_z)_t(y)$. Introducing[77]

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\big|\partial_1^2(\delta\Pi_x-\mathrm{d}\Gamma_{xz}^*Q\Pi_z)_{\beta\,t}(y)\big|^{q'}\\
&\lesssim\ (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\kappa}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y))
\end{aligned}\tag{4.85}$$

$$\begin{aligned}
&\overset{(4.71)}{\le}\ (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\theta-2\alpha}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|+\kappa-\theta+\alpha}\\
&\quad\times(w_x(z)+w_x(y)),
\end{aligned}$$

we claim that $(4.89)_\beta$ implies $(4.85)_\beta$. Indeed by (4.83) in the weakened form of

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\big|(\mathrm{d}\Gamma_{xy}^*Q\partial_1^2\Pi_y-\mathrm{d}\Gamma_{xz}^*Q\partial_1^2\Pi_z)_{\beta\,t}(y)\big|^{q'}\\
&\lesssim(\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{\kappa}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)),
\end{aligned}$$

it is enough to establish (4.85) for $z=y$,

$$\mathbb{E}^{\frac{1}{q'}}\big|\partial_1^2(\delta\Pi_x-\mathrm{d}\Gamma_{xy}^*Q\Pi_y)_{\beta\,t}(y)\big|^{q'}\lesssim(\sqrt[4]{t})^{\alpha-2+\kappa}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}w_x(y).\tag{4.86}$$

Writing

$$\begin{aligned}
&\partial_1^2(\delta\Pi_x-\mathrm{d}\Gamma_{xy}^*Q\Pi_y)_{\beta\,t}(y)\\
&\quad=\int dy'\partial_1^2\psi_t(y-y')\big(\delta\Pi_x-\delta\Pi_x(y)-\mathrm{d}\Gamma_{xy}^*Q\Pi_y\big)_\beta(y'),
\end{aligned}$$

we obtain $(4.86)_\beta$ from $(4.89)_\beta$ via Hölder's inequality in y' and the moment bounds (4.3) and (4.65) (both with $\mathbf{n}=(2,0)$ and the latter with $z=y$).

The last ingredient to (4.75), and the second ingredient for (4.76), is the estimate of the $\preccurlyeq\beta$-components $\mathsf{z}_k\Pi_x^k(y)$. Assuming just $(2.36)_{\prec\beta}$ and with a subset of the

[77] once more β here denotes a generic multi-index.

arguments for (4.84), we obtain $(4.87)_\beta$, where

$$\mathbb{E}^{\frac{1}{p}}\big|\big(\mathsf{z}_k\Pi_x^k(z)\big)_\beta|^p \lesssim |z-x|^{|\beta|-\alpha}. \tag{4.87}$$

□

The third task is to pass from the estimate (4.52) of the rough-path increment of $\delta\Pi_x^-$ to the estimate of the rough-path increment of $\delta\Pi_x$. The crucial ingredient is the representation

$$\begin{aligned}(\delta\Pi_x &- \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}^* Q\Pi_z)_\beta \\ &= -\int_0^\infty dt(1-\mathrm{T}_z^2)(\partial_2+\partial_1^2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t},\end{aligned} \tag{4.88}$$

which is the analog of (4.11) in the integration task of Proposition 4.3. Compared to this previous integration task, there are now three length scales involved, namely $\sqrt[4]{t}$, $|y-z|$, and $|z-x|$. In the near-field range $\sqrt[4]{t}\le|y-z|$, we split $1-\mathrm{T}_z^2$; in the far-field range $\sqrt[4]{t}\ge\max\{|y-z|,|z-x|\}$, we split $\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-$. Only on the intermediate range $|y-z|\le\sqrt[4]{t}\le\max\{|y-z|,|z-x|\}$ we use the cancellations in both the Taylor remainder and the rough-path increment.

Proposition 4.14 (Integration III) *Assume that* $(2.36)_{\prec\beta}$, $(2.64)_{\prec\beta}$, $(4.22)_\beta$, $(4.34)_\beta$, *and* $(4.52)_\beta$ *hold, and that* $(4.108)_\beta^\gamma$ *holds for all* γ *not purely polynomial. Then* $(4.88)_\beta$ *holds and we have*

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma_{xz}^* Q\Pi_z)_\beta(y)|^{q'} \\ &\quad\lesssim |y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)).\end{aligned} \tag{4.89}$$

Proof We first show that the r. h. s. of (4.88) is estimated by the r. h. s. of (4.89). In preparation for the near-field range $\sqrt[4]{t}\le|y-z|$, we pre-process $(4.52)_\beta$. By the semi-group property (4.1) followed by Jensen's inequality we have for any random space-time function f that $\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}f_t(y)|^{q'}\le\int dy'|\partial^{\mathbf{n}}\psi_{\frac{t}{2}}(y-y')\mathbb{E}^{\frac{1}{q'}}|f_{\frac{t}{2}}(y')|^{q'}$. We use this for $f=(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_\beta$, insert $(4.52)_\beta$ (with t replaced by $\frac{t}{2}$), and appeal to Hölder's inequality on the r. h. s. of $(4.52)_\beta$ (with y' replacing y), in order to use both the positive moment bounds (4.3) and the negative moment bounds (4.65) (for $z=y$ and with t replaced by $\frac{t}{2}$). This leads to

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'} \\ &\lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}(\sqrt[4]{t}+|y-z|)^\kappa(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)).\end{aligned} \tag{4.90}$$

We restrict (4.90), once to $y=z$, and once to the near-field range:

$$\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(z)|^{q'} \lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|+\kappa}(\sqrt[4]{t}+|z-x|)^{|\beta|-\alpha}w_x(z),$$

$$\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'}$$
$$\lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}|y-z|^{\kappa}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y))$$
$$\text{provided } \sqrt[4]{t} \le |y-z|.$$

We use this in two ways:

$$\mathbb{E}^{\frac{1}{q'}}|\mathrm{T}_z^2(\partial_1^2+\partial_2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'}$$
$$\lesssim t^{-1}\sum_{\mathbf{n}=\mathbf{0},(1,0)}|y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|+\kappa}(\sqrt[4]{t}+|z-x|)^{|\beta|-\alpha}w_x(z), \tag{4.91}$$
$$\mathbb{E}^{\frac{1}{q'}}|(\partial_1^2+\partial_2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'}$$
$$\lesssim t^{-1}(\sqrt[4]{t})^{\alpha}|y-z|^{\kappa}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y))$$
$$\text{provided } \sqrt[4]{t} \le |y-z|.$$

Applying $\int_0^{|y-z|^4} dt$, using $\alpha-1+\kappa>0$ by (4.45) on the first integral, and $\alpha>0$ on the second, we obtain

$$\mathbb{E}^{\frac{1}{q'}}\Big|\int_0^{|y-z|^4} dt\,\mathrm{T}_z^2(\partial_1^2+\partial_2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)\Big|^{q'}$$
$$\lesssim |y-z|^{\alpha+\kappa}(|y-z|+|z-x|)^{|\beta|-\alpha}w_x(z), \tag{4.92}$$
$$\mathbb{E}^{\frac{1}{q'}}\Big|\int_0^{|y-z|^4} dt\,(\partial_1^2+\partial_2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)\Big|^{q'}$$
$$\lesssim |y-z|^{\alpha+\kappa}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(z)+w_x(y)), \tag{4.93}$$

which takes care of the near-field contribution.

We now turn to the far-field contribution $\sqrt[4]{t} \ge \max\{|y-z|,|z-x|\}$, which we split into the one coming from $\mathrm{d}\Gamma_{xz}^* Q\Pi_z^-$ and the one from $\delta\Pi_x^-$. For the first one, we note that by $\Pi_z^- \in \tilde{\mathsf{T}}^*$, see (2.57), and the strict triangularity (8.11) of $\mathrm{d}\Gamma_{xz}^*$, only $(4.108)_\beta^{\gamma\neq\text{p.p.}}$ and $(2.64)_{\prec\beta}$ are needed for

$$\mathbb{E}^{\frac{1}{q'}}|(\mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}(y)|^{q'}$$
$$\lesssim \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)\cap[\alpha,\kappa+|\beta|]}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)(\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-z|)^{|\gamma|-\alpha},$$

which, by a similar but simpler argument as for (4.90), we pre-process to

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}(\mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)_{\beta t}(y)|^{q'}\\&\lesssim\sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)\cap[\alpha,\kappa+|\beta|]}(\sqrt[4]{t})^{|\gamma|-2-|\mathbf{n}|}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)\\&\quad\text{provided } |y-z|\le\sqrt[4]{t}.\end{aligned}\tag{4.94}$$

Representing Taylor's remainder in a way suitable for our parabolic scaling, namely

$$\begin{aligned}(1-\mathrm{T}^2_z)f(y)&=\int_0^1 ds(1-s)\frac{d^2h}{ds^2}(s)\\ \text{with}\quad h(s)&=f(sy_1+(1-s)z_1,s^2y_2+(1-s^2)z_2),\end{aligned}\tag{4.95}$$

so that the l. h. s. involves the four partial derivatives $\partial^{\mathbf{n}}f$ with $|\mathbf{n}|\ge2$ and $n_1+n_2\le 2$. Applying this to $f=(\partial_1^2+\partial_2)(\mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)_{\beta t}$, we learn from (4.94) that

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|(1-\mathrm{T}^2_z)(\partial_1^2+\partial_2)(\mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)_{\beta t}(y)|^{q'}\\&\lesssim t^{-1}\sum_{\substack{|\mathbf{n}|\ge2\\n_1+n_2\le2}}\sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)\cap[\alpha,\kappa+|\beta|]}|y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{|\gamma|-|\mathbf{n}|}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)\\&\quad\text{provided } |y-z|\le\sqrt[4]{t}.\end{aligned}$$

Applying $\int_{\max\{|y-z|^4,|z-x|^4\}}^\infty dt$ we obtain because of $|\gamma|-|\mathbf{n}|<2-2=0$

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}\Big|\int_{\max\{|y-z|^4,|z-x|^4\}}^\infty dt(1-\mathrm{T}^2_z)(\partial_1^2+\partial_2)(\mathrm{d}\Gamma^*_{xz}Q\Pi^-_z)_{\beta t}(y)\Big|^{q'}\\&\lesssim\sum_{\substack{|\mathbf{n}|\ge2\\n_1+n_2\le2}}\sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)\cap[\alpha,\kappa+|\beta|]}|y-z|^{|\mathbf{n}|}(|y-z|+|z-x|)^{|\gamma|-|\mathbf{n}|}\\&\quad\times|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)\\&\lesssim|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}w_x(z)\quad\text{by}\quad|\mathbf{n}|\ge2\overset{(4.36)}{\ge}\kappa+\alpha.\end{aligned}\tag{4.96}$$

For the second part of the integrand, we pre-process $(4.22)_\beta$ to

$$\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}\delta\Pi^-_{x\beta t}(y)|^{q'}\lesssim(\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}\bar{w},\tag{4.97}$$

which in turn implies by Taylor and $|y-x|+|z-x|\lesssim|y-z|+|z-x|$

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|(1-\mathrm{T}^2_z)(\partial_1^2+\partial_2)\delta\Pi^-_{x\beta t}(y)|^{q'}\\&\quad\lesssim t^{-1}\sum_{\substack{|\mathbf{n}|\ge2\\n_1+n_2\le2}}|y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}(\sqrt[4]{t}+|y-z|+|z-x|)^{|\beta|-\alpha}\bar{w}.\end{aligned}$$

Applying $\int_{\max\{|y-z|^4,|z-x|^4\}}^{\infty} dt$ we obtain because of $|\beta|-|\mathbf{n}|<2-2=0$ and as in (4.96)

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_{\max\{|y-z|^4,|z-x|^4\}}^{\infty} dt(1-\mathrm{T}_z^2)(\partial_1^2+\partial_2)\delta\Pi_{x\beta t}^{-}(y)\Big|^{q'}\\
&\lesssim \sum_{\substack{|\mathbf{n}|\geq 2\\ n_1+n_2\leq 2}} |y-z|^{|\mathbf{n}|}(|y-z|+|z-x|)^{\alpha-|\mathbf{n}|}(|y-z|+|z-x|)^{|\beta|-\alpha}\bar{w}\\
&\overset{(4.46)}{\lesssim} |y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}w_x(z).
\end{aligned} \tag{4.98}$$

In view of (4.92), (4.93), (4.96), and (4.98), it remains to consider the case $|y-z|\leq|z-x|$ and to estimate the intermediate range

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_{|y-z|^4}^{|z-x|^4} dt(1-\mathrm{T}_z^2)(\partial_1^2+\partial_2)(\delta\Pi_x^{-}-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z^{-})_{\beta t}(y)\Big|^{q'}\\
&\lesssim |y-z|^{\kappa+\alpha}|z-x|^{|\beta|-\alpha}w_x(z).
\end{aligned} \tag{4.99}$$

To this purpose, we pre-process (4.90): We apply the semi-group property (4.1) in form of $\mathbb{E}^{\frac{1}{q'}}|f_t(y)|^{q'}\leq\int dy'|\psi_{\frac{t}{2}}(y-y')|\mathbb{E}^{\frac{1}{q'}}|f_{\frac{t}{2}}(y')|^{q'}$ to $f=\partial^{\mathbf{n}}(\delta\Pi_x^{-}-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z^{-})_{\beta}$, insert (4.90), use Hölder's inequality in y' in order to access (4.3) and (4.65) (both with $\mathbf{n}=\mathbf{0}$ and t replaced by $\frac{t}{2}$), thereby obtaining

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}(\delta\Pi_x^{-}-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z^{-})_{\beta t}(y)|^{q'}\\
&\lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|+\kappa}(\sqrt[4]{t}+|z-x|)^{|\beta|-\alpha}w_x(z)\quad\text{provided } |y-z|\leq\sqrt[4]{t}.
\end{aligned} \tag{4.100}$$

By Taylor, this implies

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|(1-\mathrm{T}_z^2)(\partial_1^2+\partial_2)(\delta\Pi_x^{-}-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z^{-})_{\beta t}(y)|^{q'}\\
&\lesssim t^{-1}\sum_{\substack{|\mathbf{n}|\geq 2\\ n_1+n_2\leq 2}} |y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|+\kappa}|z-x|^{|\beta|-\alpha}w_x(z)\\
&\qquad\text{provided } |y-z|\leq\sqrt[4]{t}\leq|z-x|.
\end{aligned}$$

Integration gives (4.99).

We now argue in favor of (4.88). We obtain from $(2.35)_{\preccurlyeq\beta}$ and its Malliavin derivative that

$$(\partial_2-\partial_1^2)Q(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z)=Q(\delta\Pi_x^{-}-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z^{-}). \tag{4.101}$$

By (4.42), $Q(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z)_\beta$ vanishes to first order in z, where we could smuggle in the second Q by (2.36). Thanks to the presence of Q, and the control of $(2.36)_{\prec\beta}$, $(\mathrm{d}\Gamma_{xz}^{*}Q\Pi_z)_\beta$ grows sub-quadratically at infinity, by the control $(4.34)_\beta$, the same applies to $\delta\Pi_{x\beta}$. By the above established estimate, the t-integral in (4.88) vanishes to first order in z. Hence the annealed Liouville argument from the proof of Proposition 5.3 yields (4.88), provided the t-integral grows sub-quadratically at infinity as well, and is a solution to (4.101). To prove it is a solution to (4.101), we proceed as in the proof of Proposition 4.3, and replace $\int_0^\infty dt$ by $\int_s^T dt$ to obtain

$$\begin{aligned}
-(\partial_2-\partial_1^2)\int_s^T dt(1-\mathrm{T}_z^2)(\partial_2+\partial_1^2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta t}\\
=(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta s}-(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-)_{\beta T}.
\end{aligned}$$

The first term coincides with the r. h. s. of (4.101) as $s\downarrow 0$, the second term vanishes as $T\uparrow\infty$ by $(4.22)_\beta$, $(4.108)_\beta^{\gamma\neq\text{p.p.}}$ and $(2.64)_{\prec\beta}$ due to the presence of Q.

The rest of the proof is dedicated to establish sub-quadratic growth of the t-integral in (4.88), where we distinguish the near-field range $\sqrt[4]{t}\le|y-z|$ from the far-field range $\sqrt[4]{t}\ge|y-z|$. In the far field, we break up $\delta\Pi_x^-$ and $\mathrm{d}\Gamma_{xz}^* Q\Pi_z^-$. The intermediate estimate in (4.96) shows growth in y of order $|\gamma|<2$ of the latter, and the intermediate estimate in (4.98) shows growth in y of order $|\beta|<2$ of the former. We turn to the near field, where we break up 1 and T_z^2. For the former, we further break up $\delta\Pi_x^-$ and $\mathrm{d}\Gamma_{xz}^* Q\Pi_z^-$. By (4.97) we obtain

$$\begin{aligned}
\mathbb{E}^{\frac{1}{q'}}\Big|\int_0^{|y-z|^4} dt\,(\partial_1^2+\partial_2)\delta\Pi_{x\beta t}^-(y)\Big|^{q'} &\lesssim \int_0^{|y-z|^4} dt\,(\sqrt[4]{t})^{\alpha-4}(\sqrt[4]{t}+|x-y|)^{|\beta|-\alpha}\bar{w}\\
&\lesssim |y-z|^\alpha(|y-z|+|x-y|)^{|\beta|-\alpha}\bar{w},
\end{aligned}$$

which grows in y with order $|\beta|<2$. Similarly, by $(4.108)_\beta^{\gamma\neq\text{p.p.}}$ and $(2.64)_{\prec\beta}$ (actually in its strengthened version $(4.12)_{\prec\beta}$) we obtain

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_0^{|y-z|^4} dt\,(\partial_1^2+\partial_2)(\mathrm{d}\Gamma_{xz}^*\Pi_z^-)_{\beta t}(y)\Big|^{q'}\\
&\lesssim \int_0^{|y-z|^4} dt \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)} |x-z|^{|\beta|-|\gamma|+\kappa} w_x(z)(\sqrt[4]{t})^{\alpha-4}(\sqrt[4]{t}+|y-z|)^{|\gamma|-\alpha}\\
&\lesssim \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)} |x-z|^{|\beta|-|\gamma|+\kappa} w_x(z)|y-z|^{|\gamma|},
\end{aligned}$$

which once more grows sub-quadratically in y. For the contributions from T_z^2 we first consider the regime $\sqrt[4]{t}\le|x-z|$ (recall that we are interested in large y and we may therefore assume w. l. o. g. that $|x-z|<|y-z|$). By (4.91) we obtain

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_0^{|x-z|^4} dt\,\mathrm{T}_z^2(\partial_1^2+\partial_2)(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^*\Pi_z^-)_{\beta t}(y)\Big|^{q'}\\
&\lesssim \int_0^{|x-z|^4} dt \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|-4+\kappa}(\sqrt[4]{t}+|x-z|)^{|\beta|-\alpha} w_x(z)
\end{aligned}$$

$$\lesssim \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}|x-z|^{|\beta|-|\mathbf{n}|+\kappa} w_x(z),$$

where in the last inequality we used $\kappa+\alpha-1>0$ (see (4.45)), and hence obtain again sub-quadratic growth in y. In the regime $|x-z|\le\sqrt[4]{t}\le|y-z|$, we split $\delta\Pi^-_x$ and $\mathrm{d}\Gamma^*_{xz}\Pi^-_z$. On the one hand, we have by (4.97)

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_{|x-z|^4}^{|y-z|^4} dt\,\mathrm{T}_z^2(\partial_1^2+\partial_2)\delta\Pi^-_{x\beta\,t}(y)\Big|^{q'}\\
&\lesssim \int_{|x-z|^4}^{|y-z|^4} dt \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|-4}((\sqrt[4]{t})^{|\beta|-\alpha}+|x-z|^{|\beta|-\alpha})\bar{w}\\
&\lesssim \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}(|y-z|^{|\beta|-|\mathbf{n}|}+|x-z|^{|\beta|-|\mathbf{n}|}+|y-z|^{\alpha-|\mathbf{n}|}|x-z|^{|\beta|-\alpha})\bar{w},
\end{aligned}$$

which grows again sub-quadratically in y. On the other hand, by $(4.108)^{\gamma\neq\text{p.p.}}_{\beta}$ and $(2.64)_{\prec\beta}$ (actually in its strengthened version $(4.12)_{\prec\beta}$), we have

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}\Big|\int_{|x-z|^4}^{|y-z|^4} dt\,\mathrm{T}_z^2(\partial_1^2+\partial_2)(\mathrm{d}\Gamma^*_{xz}\Pi^-_z)_{\beta\,t}(y)\Big|^{q'}\\
&\lesssim \int_{|x-z|^4}^{|y-z|^4} dt \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)} |x-z|^{|\beta|-|\gamma|+\kappa} w_x(z) \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}(\sqrt[4]{t})^{|\gamma|-4-|\mathbf{n}|}\\
&\lesssim \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,2)} |x-z|^{|\beta|-|\gamma|+\kappa} w_x(z)\\
&\qquad\times \sum_{\mathbf{n}=\mathbf{0},(1,0)} |y-z|^{|\mathbf{n}|}(|y-z|^{|\gamma|-|\mathbf{n}|}+|x-z|^{|\gamma|-|\mathbf{n}|}),
\end{aligned}$$

which is of sub-quadratic growth in y. □

The fourth task is to pass from the output $(4.89)_\beta$ of Proposition 4.14 to $(4.44)_\beta$. The proof is based on formula

$$\begin{aligned}
&\big(\mathrm{d}\pi^{(\mathbf{0})}_{xy}-\mathrm{d}\pi^{(\mathbf{0})}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(\mathbf{0})}_{zy}\big)+\big(\mathrm{d}\pi^{(1,0)}_{xy}-\mathrm{d}\pi^{(1,0)}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(1,0)}_{zy}\big)(\cdot-y)_1\\
&=\big(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}Q\Pi_z\big)-\big(\delta\Pi_x-\delta\Pi_x(y)-\mathrm{d}\Gamma^*_{xy}Q\Pi_y\big)\\
&\quad-\big(\mathrm{d}\Gamma^*_{xy}Q-\mathrm{d}\Gamma^*_{xz}Q\Gamma^*_{zy}\big)P\Pi_y,
\end{aligned}\tag{4.102}$$

which we shall establish in the proof of the upcoming Proposition 4.15. Its merit is that it connects an affine polynomial with the coefficients given by the rough-path increments of $\{\mathrm{d}\pi^{(\mathbf{n})}_{xz}\}_{\mathbf{n}=\mathbf{0},(1,0)}$ to the rough-path increments of $\delta\Pi_x$ (in the secondary base points y and z) and the continuity expression for $\mathrm{d}\Gamma^*_{xz}$. In analogy to the fourth task in Sect. 4.2 we have

Proposition 4.15 (Three-point argument III) *Assume that* (2.36)$_{\prec\beta}$, (2.55)$_{\prec\beta}$, (2.61)$_{\prec\beta}$, *and* (4.89)$_\beta$ *hold, and that* (2.37)$^\gamma_{\prec\beta}$, (4.47)$^\gamma_\beta$, *and* (4.108)$^\gamma_\beta$ *hold for all* γ *not purely polynomial. Then* (4.44)$_\beta$ *holds.*

Proof The formula (4.102) follows from substituting Π_z according to (2.61) by $\Gamma^*_{zy}\Pi_y+\pi^{(\mathbf{0})}_{zy}$, and then appealing to

$$(\delta\Pi_x(y),\delta\Pi_x(z))\overset{(2.63)}{=}(\mathrm{d}\pi^{(\mathbf{0})}_{xy},\mathrm{d}\pi^{(\mathbf{0})}_{xz}),\tag{4.103}$$

$$Q\Gamma^*_{zy}(\mathrm{id}-P)\Pi_y\overset{(2.51),(2.21)}{=}(\mathsf{z}_{(1,0)}+Q\pi^{(1,0)}_{zy})(\cdot-y)_1,$$

$$\big(\mathrm{d}\Gamma^*_{xy}Q-\mathrm{d}\Gamma^*_{xz}Q\Gamma^*_{zy}\big)(\mathrm{id}-P)\Pi_y\overset{(4.40)}{=}\big(\mathrm{d}\pi^{(1,0)}_{xy}-\mathrm{d}\pi^{(1,0)}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(1,0)}_{zy}\big)(\cdot-y)_1.$$

Evaluating at y makes formula (4.102) collapse to

$$\mathrm{d}\pi^{(\mathbf{0})}_{xy}-\mathrm{d}\pi^{(\mathbf{0})}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(\mathbf{0})}_{zy}=\delta\Pi_x(y)-\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}Q\Pi_z(y).\tag{4.104}$$

Since $\mathrm{d}\Gamma^*_{xz}$ vanishes on $\mathsf{z}_\mathbf{n}$ unless $\mathbf{n}=(1,0)$, we may rewrite (4.104) as

$$\begin{aligned}\mathrm{d}\pi^{(\mathbf{0})}_{xy}-\mathrm{d}\pi^{(\mathbf{0})}_{xz}-\mathrm{d}\Gamma^*_{xz}\pi^{(\mathbf{0})}_{zy}\overset{(4.103)}{=}&\big(\delta\Pi_x(y)-\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}Q\Pi_z(y)\big)\\&-\mathrm{d}\Gamma^*_{xz}P(\mathrm{id}-Q)\Pi_z(y).\end{aligned}$$

Taking the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$ of the β-component of this identity, appealing to the strictly triangular structure (8.11) of $\mathrm{d}\Gamma^*_{xz}$ w. r. t. $\prec$, we obtain by our assumptions (2.36)$_{\prec\beta}$, (4.89)$_\beta$, and (4.108)$^{\gamma\neq\mathrm{p.p.}}_\beta$,

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}\big|\mathrm{d}\pi^{(\mathbf{0})}_{xy}-\mathrm{d}\pi^{(\mathbf{0})}_{xz}-\mathrm{d}\Gamma^*_{xz}\pi^{(\mathbf{0})}_{zy}\big|^{q'}\\&\lesssim|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(y)+w_x(z))\\&\quad+\sum_{|\gamma|\in\mathsf{A}\cap[2,\infty)}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)|y-z|^{|\gamma|}.\end{aligned}$$

By the second item in (4.36), the second r. h. s. term can be absorbed into the first. This establishes (4.44)$_\beta$ for $\mathbf{n}=\mathbf{0}$.

We now address the $\mathbf{n}=(1,0)$ contribution to (4.102). Appealing to (4.77), (4.79) and (4.104), we may rewrite (4.102) as

$$\begin{aligned}&\big(\mathrm{d}\pi^{(1,0)}_{xy}-\mathrm{d}\pi^{(1,0)}_{xz}-\mathrm{d}\Gamma^*_{xz}\pi^{(1,0)}_{zy}\big)(\cdot-y)_1\\&=\big(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}Q\Pi_z\big)-\big(\delta\Pi_x-\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}Q\Pi_z\big)(y)\\&\quad-\big(\delta\Pi_x-\delta\Pi_x(y)-\mathrm{d}\Gamma^*_{xy}Q\Pi_y\big)\\&\quad-\big(\mathrm{d}\Gamma^*_{xy}-\mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy}\big)PQ\Pi_y-\mathrm{d}\Gamma^*_{xz}P(\mathrm{id}-Q)\Gamma^*_{zy}QP\Pi_y\\&\quad-\mathrm{d}\Gamma^*_{xz}P(\mathrm{id}-Q)\pi^{(1,0)}_{zy}(\cdot-y)_1.\end{aligned}$$

We then take the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$ of the β-component of this identity. The first three r. h. s. terms are controlled by $(4.89)_\beta$. For the fourth term we use the strict triangularity (8.13), so that $(2.36)_{\prec\beta}$ is sufficient, next to $(4.47)_\beta^{\gamma\neq\text{p.p.}}$. For the fifth term we use the strict triangularity of $\mathrm{d}\Gamma^*_{xz}$ and the triangularity of Γ^*_{zy}, so that $(2.37)_{\prec\beta}^{\gamma\neq\text{p.p.}}$ and once more $(2.36)_{\prec\beta}$ are sufficient, next to $(4.108)_\beta^{\gamma\neq\text{p.p.}}$. For the sixth term we just use the strict triangularity of $\mathrm{d}\Gamma^*_{xz}$, so that $(2.55)_{\prec\beta}$ is sufficient, once more next to $(4.108)_\beta^{\gamma\neq\text{p.p.}}$. We obtain term by term, using that $\gamma(1,0)\neq 0$ implies $|\gamma|\geq 1$ on the fifth r. h. s. term below,

$$
\begin{aligned}
&|(\cdot-y)_1|\mathbb{E}^{\frac{1}{q'}}\left|\mathrm{d}\pi^{(1,0)}_{xy}-\mathrm{d}\pi^{(1,0)}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(1,0)}_{zy}\right|^{q'}\\
&\lesssim|\cdot-z|^{\kappa+\alpha}(|\cdot-z|+|z-x|)^{|\beta|-\alpha}(w_x(\cdot)+w_x(z))\\
&+|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(y)+w_x(z))\\
&+|\cdot-y|^{\kappa+\alpha}(|\cdot-y|+|y-x|)^{|\beta|-\alpha}(w_x(\cdot)+w_x(y))\\
&+\sum_{|\gamma|\in\mathsf{A}\cap[0,|\beta|-\alpha]}|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha}(w_x(z)+w_x(y))|\cdot-y|^{|\gamma|}\\
&+\sum_{|\gamma|\in\mathsf{A}\cap[1,|\beta|-\alpha+1]}|y-z|^{\kappa+\alpha-1}(|y-z|+|z-x|)^{|\beta|-|\gamma|-\alpha+1}\\
&\times(w_x(z)+w_x(y))|\cdot-y|^{|\gamma|}\\
&+\sum_{|\gamma|\in\mathsf{A}\cap[2,\kappa+|\beta|]}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)\sum_{|\gamma'|\in\mathsf{A}\cap[0,|\gamma|]}|y-z|^{|\gamma|-|\gamma'|}|\cdot-y|^{|\gamma'|}\\
&+\sum_{|\gamma|\in\mathsf{A}\cap[2,\kappa+|\beta|]}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)|y-z|^{|\gamma|-1}|\cdot-y|.
\end{aligned}
$$

Restricting the active variable to the (parabolic) ball $|\cdot-y|\leq|y-z|$, and using the second item in (4.36) on the last two terms, this estimate collapses to

$$
\begin{aligned}
&|(\cdot-y)_1|\mathbb{E}^{\frac{1}{q'}}\left|\mathrm{d}\pi^{(1,0)}_{xy}-\mathrm{d}\pi^{(1,0)}_{xz}-\mathrm{d}\Gamma^*_{xz}Q\pi^{(1,0)}_{zy}\right|^{q'}\\
&\lesssim|y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(\cdot)+w_x(y)+w_x(z)).
\end{aligned}\tag{4.105}
$$

We now average the active variable over this ball in order to recover $(4.44)_\beta$ for $\mathbf{n}=(1,0)$. Indeed, for the l. h. s. of (4.105) we appeal to the obvious $\fint_{|\cdot-y|\leq|y-z|}|(\cdot-y)_1|\sim|y-z|$. For the r. h. s. of (4.105), by definition (4.46) of w_x, it suffices to establish

$$
\fint_{|\cdot-y|\leq\lambda}|\cdot-x|^{-\kappa}\lesssim|y-x|^{-\kappa}\quad\text{for }\lambda=|y-z|,\tag{4.106}
$$

which is an easy version of (4.66). $\square$

4.6 From $\delta\Pi_x^- - \mathrm{d}\Gamma^*_{xz}\Pi_z^-$ back to $\delta\Pi_x^-$ via boundedness of $\mathrm{d}\Gamma^*_{xz}$ and $\mathrm{d}\pi^{(\mathbf{n})}_{xz}$, and by averaging in the secondary base point z

The aim of this last subsection is to pass from the estimate (4.52) of the rough-path increment of $\delta\Pi_x^-$ to the estimate (4.22) of $\delta\Pi_x^-$ itself. Clearly, in view of the structure of the rough-path increment, this will require an estimate of $\mathrm{d}\Gamma^*_{xz}P$, see Proposition 4.16. The proof of Proposition 4.16 will be similar to the first task of Sects. 4.2 and 4.5, and rely on the estimate of $\mathrm{d}\pi^{(\mathbf{n})}_{xz}$ for $\mathbf{n}=\mathbf{0}, (1,0)$. By (4.103) the estimate of $\mathrm{d}\pi^{(\mathbf{0})}_{xz}$ is already part of the induction hypothesis. However, the estimate of $\mathrm{d}\pi^{(1,0)}_{xz}$ needs to be included into the induction:

$$\mathbb{E}^{\frac{1}{q'}}|\mathrm{d}\pi^{(1,0)}_{xz\beta}|^{q'} \lesssim |z-x|^{\kappa+|\beta|-1}w_x(z). \tag{4.107}$$

Note that the exponent is strictly positive, see (4.45). The first task of this subsection is based on the formula (4.40) for $\mathrm{d}\Gamma^*_{xz}$.

Proposition 4.16 (Algebraic argument IV) *Assume that* (4.34)$_{\prec\beta}$ *and* (4.107)$_{\prec\beta}$ *hold, and assume that* (2.37)$^{\gamma}_{\preccurlyeq\beta}$ *holds for all γ not purely polynomial. Then we have for γ not purely polynomial*[78]

$$\mathbb{E}^{\frac{1}{q'}}|(\mathrm{d}\Gamma^*_{xz})^{\gamma}_{\beta}|^{q'} \lesssim |z-x|^{\kappa+|\beta|-|\gamma|}w_x(z). \tag{4.108}$$

This includes the statement that $(\mathrm{d}\Gamma^*_{xz}P)_\beta$ depends only on $\mathrm{d}\pi^{(1,0)}_{xz\beta'}$ and $\delta\Pi_{x\beta'}$ with $\beta'\prec\beta$, and on $(\Gamma^*_{xz}P)_{\beta'}$ with $\beta'\preccurlyeq\beta$.

Proof The structure is very similar to the proof of Proposition 4.11: Again, we distinguish the contributions from $\mathbf{n}=\mathbf{0}$ and $\mathbf{n}=(1,0)$ to (4.40)$^{\gamma}_{\beta}$. In view of (4.103), all terms in the ($\mathbf{n}=\mathbf{0}$)-contribution are of the form

$$\delta\Pi_{x\beta_1}(z)(\Gamma^*_{xz})^{\gamma-e_k+e_{k+1}}_{\beta_2}, \tag{4.109}$$

for some $k\geq 0$ and multi-indices β_1,β_2 constrained by $\beta_1\prec\beta$, $\beta_2\preccurlyeq\beta$, $\beta_1+\beta_2=\beta$, and noting that $\gamma-e_k+e_{k+1}$ is not purely polynomial. Hence under our assumptions, the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm of (4.109) is estimated by

$$|z-x|^{|\beta_1|}\bar{w}\,|z-x|^{|\beta_2|-|\gamma-e_k+e_{k+1}|} \overset{(4.46)}{\leq} |z-x|^{\kappa+|\beta_1|}w_x(z)\,|z-x|^{|\beta_2|-|\gamma-e_k+e_{k+1}|}.$$

It follows from (4.31) that this is contained in the r. h. s. of (4.108).

All terms in the ($\mathbf{n}=(1,0)$)-contribution are of the form

$$\mathrm{d}\pi^{(1,0)}_{xz\beta_1}(\Gamma^*_{xz})^{\gamma-e_{(1,0)}}_{\beta_2}, \tag{4.110}$$

[78] with the understanding that the l. h. s. vanishes when the exponent is non-positive.

for some multi-indices β_1, β_2 constrained by $\beta_1 \prec \beta$, $\beta_2 \preccurlyeq \beta$, $\beta_1 + \beta_2 = \beta$, and noting that $\gamma - e_{(1,0)}$ is not purely polynomial. Hence under our assumptions, the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm of (4.110) is estimated by

$$|z-x|^{\kappa+|\beta_1|-1} w_x(z)\,|z-x|^{|\beta_2|-|\gamma-e_{(1,0)}|}.$$

It follows from (4.33) that also this is contained in the r. h. s. of (4.108). □

We will establish (4.107) based on the three-point formula

$$\begin{aligned}(y-z)_1 \mathrm{d}\pi^{(1,0)}_{xz} = &-\big(\delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz} Q\Pi_z(y)\big)\\ &+ \delta\Pi_x(y) - \delta\Pi_x(z) - \mathrm{d}\Gamma^*_{xz} PQ\Pi_z(y),\end{aligned} \tag{4.111}$$

which follows from (2.21) in form of $(\mathrm{id}-P)Q\Pi_z(y) = (y-z)_1 \mathsf{z}_{(1,0)}$ and (5.28). By an argument similar to the fourth task of Sects. 4.2 and 4.5 (see Propositions 4.4, 4.5 and 4.15) we obtain

Proposition 4.17 (Three-point argument IV) *Assume that* $(2.36)_{\prec\beta}$, $(4.34)_\beta$ *and* $(4.89)_\beta$ *hold, and that* $(4.108)^\gamma_\beta$ *holds for all* γ *not purely polynomial. Then* $(4.107)_\beta$ *and* $(4.108)_\beta$ *hold.*

Proof Taking the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$ of $(4.111)_\beta$, and appealing to the strict triangularity (8.11) of $\mathrm{d}\Gamma^*$, we obtain by our assumptions

$$\begin{aligned}&|(y-z)_1|\mathbb{E}^{\frac{1}{q'}}|\mathrm{d}\pi^{(1,0)}_{xz\beta}|^{q'}\\ &\quad\lesssim\ |y-z|^{\kappa+\alpha}(|y-z|+|z-x|)^{|\beta|-\alpha}(w_x(y)+w_x(z))\\ &\quad+\ |y-x|^{|\beta|}\bar{w} + |z-x|^{|\beta|}\bar{w} + \sum_{|\gamma|\in\mathsf{A}\cap[|\alpha|,\kappa+|\beta|)} |z-x|^{\kappa+|\beta|-|\gamma|} w_x(z)|y-z|^{|\gamma|}\\ &\quad\overset{(4.46)}{\lesssim}\ (|y-z|+|z-x|)^{\kappa+|\beta|}\big(w(y)+|y-x|^{-\kappa}\bar{w}+w_x(z)\big).\end{aligned} \tag{4.112}$$

We now average over all y with $|y-z| \le \frac{1}{2}|z-x|$, where the factor of $\frac{1}{2}$ ensures that $|y-x| \ge \frac{1}{2}|z-x|$, so that in this range (4.112) simplifies to

$$|(y-z)_1|\mathbb{E}^{\frac{1}{q'}}|\mathrm{d}\pi^{(1,0)}_{xz\beta}| \lesssim |z-x|^{\kappa+|\beta|}\big(w(y)+|z-x|^{-\kappa}\bar{w}+w_x(z)\big). \tag{4.113}$$

The averaging of (4.113) over $\{y : |y-z| \le \frac{1}{2}|z-x|\}$ ensures that on the one hand, we have for the l. h. s. that $\fint_{y:|y-z|\le\frac{1}{2}|z-x|} dy\,|(y-z)_1| \sim |z-x|$, and that on the other hand, we may use (4.39) for the r. h. s. to the effect of $\fint_{y:|y-z|\le\frac{1}{2}|z-x|} dy\, w(y) \lesssim |z-x|^{-\kappa}\bar{w}$. Hence, once more appealing to definition (4.46) of $w_x(z)$, we see that (4.113) turns into $(4.107)_\beta$. □

Passing from (4.52) to (4.22) also means replacing the weighted norms of $\delta\xi$ by the original norm $\bar{w}$. This will be done starting from the obvious identity

$$\delta\Pi_x^- = (\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}^* Q\Pi_z^-) + \mathrm{d}\Gamma_{xz}^* Q\Pi_z^- \tag{4.114}$$

and averaging in the secondary base point z. In particular, we have the following proposition

Proposition 4.18 *Assume that* $(2.60)_\beta$, $(2.64)_{\prec\beta}$, $(4.22)_{\prec\beta}$ *and* $(4.52)_\beta$ *hold, assume that* $(2.37)_\beta^\gamma$, $(4.27)_\beta^\gamma$ *and* $(4.108)_\beta^\gamma$ *for* γ *not purely polynomial hold. Then* $(4.22)_\beta$ *holds.*

Proof We apply $(\cdot)_t$ to $(4.114)_\beta$ and then the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm; on the first term, we use the upgrade $(4.100)_\beta$ (with $\mathbf{n}=\mathbf{0}$) of $(4.52)_\beta$; on the second term, we use $(4.108)_\beta^{\gamma\neq\text{p.p.}}$, and $(2.64)_{\prec\beta}$ (thanks to the strict triangularity (8.11) of $\mathrm{d}\Gamma_{xz}^*$ w. r. t. $\prec$), obtaining

$$\begin{aligned}
&\mathbb{E}^{\frac{1}{q'}}|\delta\Pi_{x\beta t}^-(y)|^{q'} \lesssim (\sqrt[4]{t})^{\alpha+\kappa-2}(\sqrt[4]{t}+|z-x|)^{|\beta|-\alpha}w_x(z)\\
&+\sum_{|\gamma|\in\mathsf{A}\cap[\kappa+|\beta|,2]}|z-x|^{\kappa+|\beta|-|\gamma|}w_x(z)(\sqrt[4]{t})^{|\gamma|-2}\quad \text{provided } |y-z|\le\sqrt[4]{t}.
\end{aligned}$$

By definition (4.46), this yields

$$\begin{aligned}
\mathbb{E}^{\frac{1}{q'}}|\delta\Pi_{x\beta t}^-(y)|^{q'} &\lesssim (\sqrt[4]{t})^{|\beta|+\kappa-2}(w(z)+(\sqrt[4]{t})^{-\kappa}\bar{w})\\
&\text{provided } |y-z|\le\sqrt[4]{t} \text{ and } \frac{1}{2}\sqrt[4]{t}\le|z-x|\le\sqrt[4]{t}.
\end{aligned}$$

Using this for $y=x$ and averaging over the z-annulus $\frac{1}{2}\sqrt[4]{t}\le|z-x|\le\sqrt[4]{t}$ yields by (4.39)

$$\mathbb{E}^{\frac{1}{q'}}|\delta\Pi_{x\beta t}^-(x)|^{q'} \lesssim (\sqrt[4]{t})^{|\beta|-2}\bar{w}. \tag{4.115}$$

In order to replace in (4.115) the evaluation at x by a generic point y, we take the Malliavin derivative of $(2.60)_\beta$, see Sect. 7.2, noting that there is no polynomial defect because of $|\beta|<2$:

$$\delta\Pi_{x\beta}^- = \left(\delta\Gamma_{xy}^*\Pi_y^- + \Gamma_{xy}^*\delta\Pi_y^-\right)_\beta. \tag{4.116}$$

Applying $(\cdot)_t$, evaluating at y, and taking the $\mathbb{E}^{\frac{1}{q'}}|\cdot|^{q'}$-norm we obtain using $(4.27)_\beta^{\gamma\neq\text{p.p.}}$, $(2.64)_{\prec\beta}$, $(2.37)_\beta^{\gamma\neq\text{p.p.}}$, $(4.22)_{\prec\beta}$, and $(4.115)_\beta$ (with x replaced by y)

that

$$\mathbb{E}^{\frac{1}{q'}}|\delta\Pi^{-}_{x\beta t}(y)|^{q'} \lesssim \sum_{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|]} |x-y|^{|\beta|-|\gamma|}\bar{w}(\sqrt[4]{t})^{|\gamma|-2} + \sum_{|\gamma|\in\mathsf{A}\cap[\alpha,|\beta|]} |x-y|^{|\beta|-|\gamma|}(\sqrt[4]{t})^{|\gamma|-2}\bar{w} \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}\bar{w}. \quad \square$$

5 Constructions

In this section, we carry out one step of the inductive construction of c, Π^{-}_{x}, Π_x, and Γ^{*}_{xy} such that, next to the population properties, the axioms (2.18), (2.30), (2.31), (2.35), and (2.60) are satisfied. In fact, instead of (2.30) and (2.60) we shall establish the stronger (2.61) and (2.62). In line with the logical order of an induction step, see Sect. 3.4, we proceed as follows

- In Sect. 5.1, we construct Π^{-}_{x} by the choice of c that amounts to the BPHZ-choice of renormalization.
- In Sect. 5.2, we construct Π_x by integration of Π^{-}_{x}.
- In Sect. 5.3, we construct Γ^{*}_{xy} via the re-centering when passing from Π_y to Π_x by a polynomial with coefficients $\pi^{(\mathbf{n})}_{xy}$.
- In Sect. 5.4, we construct $\mathrm{d}\Gamma^{*}_{xz}$ via $\mathrm{d}\pi^{(\mathbf{n})}_{xz}$ (with $\mathbf{n}=(1,0)$).

5.1 Construction of c and thus Π^{-}_{x} via BPHZ-choice of renormalization

According to (8.4) and (8.6), the r. h. s. of (2.18) depends on Π_x only through $\Pi_{x\beta'}$ with $\beta'\prec\beta$, and on c only through $c_{\beta'}$ with $\beta'\preccurlyeq\beta$. In addition, the levels $c_{\beta'}$ with $\beta'\prec\beta$ have stabilized; our task now it to choose c_β such that $(2.38)_\beta$ is satisfied.

As alluded to at the end of Sect. 2.2, we have to include shift covariance and reflection parity of Π_x into the induction in order for $(2.38)_\beta$ to hold with a c_β independent of x and meeting the population condition (2.34). By covariance under a space-time shift $z+\cdot$, we mean

$$\Pi_{z+x\,\beta}[\xi](z+y)=\Pi_{x\,\beta}[\xi(z+\cdot)](y). \tag{5.1}$$

By parity, we understand the following covariance under the spatial reflection $Rx=(-x_1,x_2)$:

$$\Pi_{Rx\,\beta}[\xi](Ry)=(-1)^{\sum_{\mathbf{n}}n_1\beta(\mathbf{n})}\Pi_{x\,\beta}[\xi(R\cdot)](y). \tag{5.2}$$

Note that in view of (2.21), $(5.1)_\beta$ and $(5.2)_\beta$ are tautologically satisfied for β purely polynomial. For any c_β satisfying the population constraint $(2.34)_\beta$, properties $(5.1)_{\prec\beta}$ and $(5.2)_{\prec\beta}$ automatically upgrade to $(5.3)_\beta$ and $(5.4)_\beta$, respectively, where

$$\Pi^{-}_{z+x\,\beta}[\xi](z+y)=\Pi^{-}_{x\,\beta}[\xi(z+\cdot)](y), \tag{5.3}$$

$$\Pi^{-}_{Rx\,\beta}[\xi](Ry)=(-1)^{\sum_{\mathbf{n}}n_1\beta(\mathbf{n})}\Pi^{-}_{x\,\beta}[\xi(R\cdot)](y). \tag{5.4}$$

Proposition 5.1 *Assume that* $(4.18)_\beta$, $(5.3)_\beta$, *and* $(5.4)_\beta$ *hold.*[79] *Then there exists a choice of* c_β *satisfying* $(2.34)_\beta$ *such that* $(2.38)_\beta$ *holds.*

Proof Let us denote[80]

$$\tilde\Pi^-_x := P\sum_{k\geq 1}\mathsf{z}_k\Pi^k_x\partial_1^2\Pi_x - \sum_{l\geq 1}\frac{1}{l!}\Pi^l_x(D^{(\mathbf{0})})^l c, \tag{5.5}$$

which is defined such that

$$\Pi^-_{x\beta} = \partial_1^2\Pi_{x\beta-e_0} + \tilde\Pi^-_{x\beta} - c_\beta + \xi_\tau\delta^0_\beta, \tag{5.6}$$

with the understanding that the first r. h. s. term is absent unless $\beta(0)\geq 1$. We use (5.6) in form of $\frac{d}{dt}\Pi^-_{x\beta t}(x) = \frac{d}{dt}(\partial_1^2\Pi_{x\beta-e_0} + \tilde\Pi^-_{x\beta} + \xi\delta^0_\beta)_t(x)$. By stationarity of ξ, we may rewrite

$$\lim_{t\uparrow\infty}\mathbb{E}(\partial_1^2\Pi_{x\beta-e_0} + \tilde\Pi^-_{x\beta})_t(x) = \mathbb{E}(\partial_1^2\Pi_{x\beta-e_0} + \tilde\Pi^-_{x\beta})_\tau(x) + \int_\tau^\infty dt\frac{d}{dt}\mathbb{E}\Pi^-_{x\beta t}(x). \tag{5.7}$$

Here the last term of the r. h. s. is well-defined thanks to $(4.18)_\beta$. Therefore, $\lim_{t\uparrow\infty}\mathbb{E}\Pi^-_{x\beta t}(x)$ exists for any $c_\beta\in\mathbb{R}$. By $(5.3)_\beta$ and the invariance in law of ξ under space-time shift in Assumption 2.1, it is independent of x; in addition, by $(5.4)_\beta$ and the invariance in law of ξ under spatial reflection in Assumption 2.1, $\lim_{t\uparrow\infty}\mathbb{E}\Pi^-_{x\beta t}(x) = 0$ if $\sum_{\mathbf{n}} n_1\beta(\mathbf{n})$ is odd (recall c_β satisfies (2.34) by assumption). If $\sum_{\mathbf{n}} n_1\beta(\mathbf{n})$ is even, then in view of $|\beta|<2$ we necessarily have $\beta(\mathbf{n})=0$ for all $\mathbf{n}\neq\mathbf{0}$. Consequently, we may choose c_β such that $(2.38)_\beta$ holds (note that shifting the value of c_β by a constant is the same as shifting the value of the r. h. s. of $(2.38)_\beta$ by the same constant), still being consistent with (2.34). □

5.2 Construction of Π_x via integration

The construction of $\Pi_{x\beta}$ given $\Pi^-_{x\beta}$ is provided by formula $(4.11)_\beta$, the convergence of which is part of Proposition 4.3. It follows from the representation $(4.11)_\beta$ that $(5.3)_\beta$ and $(5.4)_\beta$ transmit to $(5.1)_\beta$ and $(5.2)_\beta$.

5.3 Construction of Γ^*_{xy} via re-centering encoded through $\pi^{(\mathbf{n})}_{xy}$

Via the exponential formula (2.44), Γ^*_{xy} is determined by $\{\pi^{(\mathbf{n})}_{xy}\}_{\mathbf{n}}$, so that the task is to construct the latter. For the induction step β, this means constructing $\{\pi^{(\mathbf{n})}_{xy\beta}\}_{\mathbf{n}:|\mathbf{n}|<|\beta|}$. It follows from (8.1) and (8.2) that

$$(\Gamma^*_{xy}P)_\beta \quad \text{depends on } \pi^{(\mathbf{n})}_{xy\beta'} \text{ only for } \beta'\prec\beta, \tag{5.8}$$

[79] for any c constrained by (2.34).

[80] the notation $\tilde\Pi^-_x$ differs from the one used in Sect. 2.4.

$$(\Gamma^*_{xy})_\beta \quad \text{depends on } \pi^{(\mathbf{n})}_{xy\beta'} \text{ only for } \beta' \preccurlyeq \beta. \tag{5.9}$$

In particular, at the beginning of the induction step β, the β-th row of $\Gamma^*_{xy}P$ and all rows $\beta' \prec \beta$ of Γ^*_{xy} have already "stabilized".

The task is to construct $\pi^{(\mathbf{n})}_{xy\beta}$ such that re-centering $(2.61)_{\preccurlyeq\beta}$ & $(2.62)_{\preccurlyeq\beta}$ and transitivity $(2.31)_{\preccurlyeq\beta}$ hold. By (5.9), it is enough to verify the current component $(2.62)_\beta$ & $(2.61)_\beta$ for re-centering. By the triangularity (8.9) of Γ^*_{xy} w. r. t. $\prec$, we likewise see that it is enough to establish the current component $(2.31)_\beta$ for transitivity.

By the composition rule (2.52), transitivity (2.31) is a consequence of

$$\pi^{(\mathbf{n})}_{xy} - \pi^{(\mathbf{n})}_{xz} - \Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy} = 0 \quad \text{and} \quad \pi^{(\mathbf{n})}_{xx} = 0. \tag{5.10}$$

More precisely, in view of (5.9), transitivity $(2.31)_\beta$ is a consequence of $(5.10)_{\preccurlyeq\beta}$. Hence we include $(5.10)_\beta$ into the induction step β, and note that by (5.8), the induction hypothesis $(5.10)_{\prec\beta}$ implies

$$\Gamma^*_{xy}P = \Gamma^*_{xz}\Gamma^*_{zy}P \quad \text{and} \quad \Gamma^*_{xx}P = P. \tag{5.11}$$

It follows from (2.32) and the binomial formula that (5.10) holds for purely polynomial β. The argument for the base case $\beta = 0$ is covered by the induction step below.

We proceed in three steps:

- Algebraic argument. Proposition 5.2 passes from the Π-statement $(2.61)_{\prec\beta}$ to the Π^--statement $(2.62)_\beta$. It is based on the definition (2.57) of Π^- in terms of Π and on the multiplicativity (2.49) of Γ^*. The renormalization term transforms as desired because c is independent of x and $\mathsf{z}_\mathbf{n}$. A difficulty in appealing to multiplicativity for (2.18) lies in the presence of the projection P in formula (2.57).
- Integration. Proposition 5.3 passes from the Π^--statement $(2.62)_\beta$, or rather the weaker $(2.60)_\beta$, to the Π-statement $(2.61)_\beta$. It is based on a Liouville argument for the PDE (2.35), which implies that under the growth condition (2.36), $(\Gamma^*_{xy}P\Pi_y - \Pi_x)_\beta$ is a polynomial of degree $< |\beta|$, so that we may define

$$(\Gamma^*_{xy}P\Pi_y - \Pi_x)_\beta = -\sum_{\mathbf{n}:|\mathbf{n}|<|\beta|} \pi^{(\mathbf{n})}_{xy\beta}(\cdot - y)^\mathbf{n}. \tag{5.12}$$

- Three-point argument. Proposition 5.4 uses $(5.11)_\beta$ and $(5.12)_\beta$ to establish $(5.10)_\beta$, and thus $(2.31)_\beta$.

Before we address Proposition 5.2, we need to argue that the r. h. s. term in $(2.62)_\beta$ is well-defined at this stage of the induction step (regular case 2c or singular case 2a). Indeed, because of $\Pi^- \in \tilde{\mathsf{T}}^*$ and by the triangular structure (8.9) of Γ^* w. r. t. $\prec$, the first r. h. s. term $(\Gamma^*_{xy}\Pi^-_y)_\beta$ involves $(\Gamma^*_{xy})^\gamma_\beta$ only for γ not purely polynomial, which has stabilized by (5.8), and $\Pi^-_{y\beta'}$ for $\beta' \preccurlyeq \beta$, which has been constructed. Because of the triangular structure (8.6) of $\sum_{k\geq 0}\mathsf{z}_k\pi^k\pi'$, the second r. h. s. term of $(2.62)_\beta$, namely the product $\big(\mathsf{z}_k(\Gamma^*_{xy}\Pi_y + \pi^{(\mathbf{0})}_{xy})^k\partial_1^2(\Gamma^*_{xy}\Pi_y + \pi^{(\mathbf{0})}_{xy})\big)_\beta$, depends on its factors only in terms of $(\Gamma^*_{xy}\Pi_y + \pi^{(\mathbf{0})}_{xy})_{\beta'\prec\beta}$. We note that $(\Gamma^*_{xy})^\gamma_{\beta'\prec\beta}$ has stabilized for all

γ by (5.9). Again, by the triangular structure (8.9) of Γ^*_{xy}, $\Pi_{y\beta'}$ is involved only for $\beta' \preccurlyeq \beta$, which has been constructed.

Proposition 5.2 *Suppose* $(2.61)_{\prec\beta}$ *holds. Then* $(2.62)_\beta$ *holds. Moreover,* $(2.62)_\beta$ *is of the form* $(2.60)_\beta$.

We highlight that this proposition is independent of the specific value of c and therefore can be obtained before the BPHZ choice of renormalization $(2.38)_\beta$.

Proof We deal with the three terms on the r. h. s. of (2.57) one by one. In preparation of the first one, we claim that (2.49) implies for any $\pi, \pi' \in \mathsf{T}^*$

$$\begin{aligned} &P\sum_{k\geq 0}\mathsf{z}_k\big(\Gamma^*\pi+\pi^{(\mathbf{0})}\big)^k(\Gamma^*\pi') \\ &\quad=\Gamma^* P\sum_{k\geq 0}\mathsf{z}_k\pi^k\pi' + P\sum_{k\geq 0}\mathsf{z}_k\big(\Gamma^*(\mathrm{id}-P)\pi+\pi^{(\mathbf{0})}\big)^k(\Gamma^*(\mathrm{id}-P)\pi'), \end{aligned} \tag{5.13}$$

where $\pi^{(\mathbf{0})}$ is related to Γ^* by (2.44). Note that P plays the role of the projection of $\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$ onto $\tilde{\mathsf{T}}^*$, and $\mathrm{id}-P$ the one of T^* onto $\bar{\mathsf{T}}^*$. Here comes the argument for (5.13): The $(k+1)$-fold iteration of (2.49) component-wise reads[81]

$$(\Gamma^*)_\beta^\gamma = \sum_{\beta_0+\cdots+\beta_{k+1}=\beta}(\Gamma^*)_{\beta_0}^{\gamma_0}\cdots(\Gamma^*)_{\beta_{k+1}}^{\gamma_{k+1}} \quad \text{provided } \gamma=\gamma_0+\cdots+\gamma_{k+1}.$$

This allows to characterize the commutator of Γ^* and P on a product of arbitrary $\pi^{(0)} \in \tilde{\mathsf{T}}^*$, $\pi^{(1)}, \ldots, \pi^{(k+1)} \in \mathsf{T}^*$ (so that below, P acts like a projection from $\mathbb{R}[[\mathsf{z}_k, \mathsf{z}_\mathbf{n}]]$ onto T^*):

$$\begin{aligned} &\big(P(\Gamma^*\pi^{(0)})\cdots(\Gamma^*\pi^{(k+1)})\big)_\beta \\ &\quad=\big(\Gamma^* P\pi^{(0)}\cdots\pi^{(k+1)}\big)_\beta \\ &\qquad+\sum_{\beta_0+\cdots+\beta_{k+1}=\beta}\ \sum_{\gamma_0+\cdots+\gamma_{k+1}\text{ not populated}}(\Gamma^*)_{\beta_0}^{\gamma_0}\pi_{\gamma_0}^{(0)}\cdots(\Gamma^*)_{\beta_{k+1}}^{\gamma_{k+1}}\pi_{\gamma_{k+1}}^{(k+1)}. \end{aligned} \tag{5.14}$$

We use (5.14) for $\pi^{(0)}=\mathsf{z}_k$, $\pi^{(1)}=\cdots=\pi^{(k)}=\pi$, and $\pi^{(k+1)}=\pi'$, and combine it with the following consequence[82] of the definitions (2.19) & (2.23)

$$e_k+\gamma_1+\cdots+\gamma_{k+1}\ \text{not populated} \quad\Longleftrightarrow\quad \gamma_1,\ldots,\gamma_{k+1}\ \text{purely polynomial}.$$

Hence (5.14) assumes the form of

$$P(\Gamma^*\mathsf{z}_k)(\Gamma^*\pi)^k(\Gamma^*\pi') = \Gamma^* P\mathsf{z}_k\pi^k\pi' + P(\Gamma^*\mathsf{z}_k)(\Gamma^*(\mathrm{id}-P)\pi)^k(\Gamma^*(\mathrm{id}-P)\pi'),$$

[81] with the understanding that all multi-indices are populated.

[82] recalling our implicit assumption that $\gamma_1, \ldots, \gamma_{k+1}$ are populated.

which we sum in k,

$$
\begin{aligned}
&P\sum_{k\geq 0}(\Gamma^*\mathsf{z}_k)(\Gamma^*\pi)^k(\Gamma^*\pi')\\
&\quad=\Gamma^*P\sum_{k\geq 0}\mathsf{z}_k\pi^k\pi'+P\sum_{k\geq 0}(\Gamma^*\mathsf{z}_k)(\Gamma^*(\mathrm{id}-P)\pi)^k(\Gamma^*(\mathrm{id}-P)\pi').
\end{aligned}
\tag{5.15}
$$

By (2.50) followed by the binomial formula, we obtain

$$
\sum_{k\geq 0}(\Gamma^*\mathsf{z}_k)(\Gamma^*\pi)^k(\Gamma^*\pi')=\sum_{k\geq 0}\mathsf{z}_k(\Gamma^*\pi+\pi^{(\mathbf{0})})^k(\Gamma^*\pi')
$$

and the same identity with π,π' replaced by $(\mathrm{id}-P)\pi$, $(\mathrm{id}-P)\pi'$. Inserting these two identities in (5.15) yields (5.13). We apply (5.13) with $\Gamma^*=\Gamma^*_{xy}$ (noting that the corresponding $\pi^{(\mathbf{0})}_{xy}$ is a constant in space-time), $\pi=\Pi_y$, and $\pi'=\partial_1^2\Pi_y$, which results in

$$
\begin{aligned}
&P\sum_{k\geq 0}\mathsf{z}_k(\Gamma^*_{xy}\Pi_y+\pi^{(\mathbf{0})}_{xy})^k\partial_1^2(\Gamma^*_{xy}\Pi_y+\pi^{(\mathbf{0})}_{xy})\\
&=\Gamma^*_{xy}P\sum_{k\geq 0}\mathsf{z}_k\Pi_y^k\partial_1^2\Pi_y\\
&\quad+P\sum_{k\geq 0}\mathsf{z}_k(\Gamma^*_{xy}(\mathrm{id}-P)\Pi_y+\pi^{(\mathbf{0})}_{xy})^k\partial_1^2(\Gamma^*_{xy}(\mathrm{id}-P)\Pi_y+\pi^{(\mathbf{0})}_{xy}).
\end{aligned}
\tag{5.16}
$$

We now turn to the second r. h. s. contribution to (2.57), and claim that for any $\pi\in\mathsf{T}^*$ and $c\in\tilde{\mathsf{T}}^*$ satisfying the population condition (2.34), we have

$$
\sum_{k\geq 0}\frac{1}{k!}(\Gamma^*\pi+\pi^{(\mathbf{0})})^k(D^{(\mathbf{0})})^kc=\Gamma^*\sum_{k\geq 0}\frac{1}{k!}\pi^k(D^{(\mathbf{0})})^kc. \tag{5.17}
$$

Because of (2.49), it remains to show

$$
\sum_{k\geq 0}\frac{1}{k!}(\Gamma^*\pi+\pi^{(\mathbf{0})})^k(D^{(\mathbf{0})})^kc=\sum_{k\geq 0}\frac{1}{k!}(\Gamma^*\pi)^k(\Gamma^*(D^{(\mathbf{0})})^kc). \tag{5.18}
$$

Note that the second item in the population condition (2.34) on c can be re-expressed as $D^{(\mathbf{n})}c=0$ for $\mathbf{n}\neq\mathbf{0}$, cf. (2.45). We now appeal to (4.64), which implies (5.18), once more by the binomial formula. We apply (5.17) to $\Gamma^*=\Gamma^*_{xy}$ and $\pi=\Pi_y$, to the effect of

$$
\sum_{k\geq 0}\frac{1}{k!}(\Gamma^*_{xy}\Pi_y+\pi^{(\mathbf{0})}_{xy})^k(D^{(\mathbf{0})})^kc=\Gamma^*_{xy}\sum_{k\geq 0}\frac{1}{k!}\Pi_y^k(D^{(\mathbf{0})})^kc. \tag{5.19}
$$

We finally turn to the third r. h. s. contribution to (2.57) and note that by the second item in (2.49) we have

$$
\xi\mathsf{1}=\Gamma^*_{xy}\xi\mathsf{1}.
$$

The sum of (5.16), (5.19) and the last identity yields (2.62) by definition (2.18).

Finally, in order to pass from (2.62) to (2.60), we need to argue that the β-component of the second r. h. s. term in (2.62) is a polynomial of degree $\le |\beta|-2$. In view of the definition (2.58) of P and the definition (2.21) of Π_y, the second r. h. s. term in (2.62) is (componentwise) a polynomial. In view of the triangularity (8.9) of Γ^* w. r. t. $|\cdot|$ and the structure (8.7) of $\sum_{k\ge0} \mathsf{z}_k\pi^k\pi'$ w. r. t. $|\cdot|$, combined with (2.28), the degree of the β-component of the polynomial is $\le |\beta|-2$. Hence (2.62) is indeed a stronger version of (2.60). □

Before addressing Proposition 5.3, we need to argue that $(\Gamma^*_{xy}P\Pi_y)_\beta$ appearing in (5.12) is well-defined at this stage of the induction step (regular case 5a or singular case 6c): By (5.8), $(\Gamma^*_{xy}P)_\beta$ has stabilized. By the triangularity of (8.9) of Γ^*_{xy} w. r. t. $\prec$, $(\Gamma^*_{xy}P\Pi_y)_\beta$ only involves $\Pi_{y\gamma}$'s with $\gamma \preccurlyeq \beta$, and thus already constructed. As always in an integration step, both for the estimates in Proposition 4.3 and for uniqueness here, we have to stay away from integers, cf. (2.29).

Proposition 5.3 (Liouville) *Suppose* $(2.36)_{\preccurlyeq\beta}$ *and* $(2.60)_\beta$ *hold, and that* $(2.37)^\gamma_\beta$ *holds for all* γ *not purely polynomial. There exist* $\{\pi^{(\mathbf{n})}_{xy\beta}\}_{|\mathbf{n}|<|\beta|}$ *such that* $(2.61)_\beta$ *holds.*

Proof Applying $\partial_2-\partial_1^2$ to $(\Gamma^*_{xy}P\Pi_y)_\beta$ results in

$$
\begin{aligned}
(\partial_2-\partial_1^2)(\Gamma^*_{xy}P\Pi_y)_\beta &= (\Gamma^*_{xy}(\partial_2-\partial_1^2)P\Pi_y)_\beta \\
&\overset{(2.35)_{\preccurlyeq\beta}}{=} (\Gamma^*_{xy}\Pi^-_y)_\beta \\
&\overset{(2.60)_\beta}{=} \Pi^-_{x\beta} + \text{polynomial of degree} \le |\beta|-2,
\end{aligned}
\tag{5.20}
$$

where we used the triangularity w. r. t. both $\prec$ and homogeneity $|\cdot|$, the latter in order to control the degree of the polynomial. Hence $(\Gamma^*_{xy}P\Pi_y)_\beta$ solves the same PDE (2.35) as $\Pi_{x\beta}$ modulo a polynomial of degree $|\beta|-2$, so that

$$
(\partial_2-\partial_1^2)\partial^{\mathbf{n}}(\Gamma^*_{xy}P\Pi_y-\Pi_x)_\beta = 0 \quad \text{provided } |\mathbf{n}|>|\beta|-2. \tag{5.21}
$$

In view of the kernel estimate (4.3), the quantitative $(2.36)_{\preccurlyeq\beta}$ yields the qualitative $(5.22)_{\preccurlyeq\beta}$, where

$$
\lim_{t\uparrow\infty}\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\Pi_{x\beta t}(\cdot)|^p = 0 \quad \text{provided } |\mathbf{n}|>|\beta|. \tag{5.22}
$$

Hence by the triangularity of Γ^*_{xy} w. r. t. both $\prec$ and $|\cdot|$, and using $(2.37)^{\gamma\neq\text{p.p.}}_\beta$, this implies

$$
\lim_{t\uparrow\infty}\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}(\Gamma^*_{xy}P\Pi_y-\Pi_x)_{\beta t}(\cdot)|^p = 0 \quad \text{provided } |\mathbf{n}|>|\beta|. \tag{5.23}
$$

We now argue that (5.21) and (5.23) imply

$$\partial^{\mathbf{n}}(\Gamma^*_{xy}P\Pi_y - \Pi_x)_\beta = 0 \quad \text{provided } |\mathbf{n}| > |\beta|. \tag{5.24}$$

Indeed, according to the factorization (4.4) and the definition of $(\cdot)_t$, (5.21) implies

$$\partial_t\partial^{\mathbf{n}}(\Gamma^*_{xy}P\Pi_y - \Pi_x)_{\beta t} = 0 \quad \text{provided } |\mathbf{n}| > |\beta|.$$

This permits to pass from (5.23) for $t \uparrow \infty$ to (5.24) for $t \downarrow 0$.

As a consequence of (5.24), $(\Gamma^*_{xy}P\Pi_y - \Pi_x)_\beta$ is a polynomial of degree $\le |\beta|$, which by $|\beta| \notin \mathbb{N}$, cf. (2.29), allows to define $\pi^{(\mathbf{n})}_{xy\beta}$ by (5.12). By definition (2.21), and because of $\Gamma^*\mathsf{z}_{\mathbf{n}} \overset{(2.44)}{=} \mathsf{z}_{\mathbf{n}} + \pi^{(\mathbf{n})}$ in form of $(\Gamma^*)^{e_{\mathbf{n}}}_\beta = \pi^{(\mathbf{n})}_\beta$, this turns into the desired $(2.61)_\beta$. □

Proposition 5.4 $(5.10)_{\prec\beta}$ *and* $(2.61)_{\preccurlyeq\beta}$ *imply* $(2.31)_\beta$ *and* $(5.10)_\beta$.

Proof Recall that $(2.31)_\beta$ is a consequence of $(5.10)_{\preccurlyeq\beta}$ by the composition rule (2.52). Furthermore, the second item in $(5.10)_\beta$ is a consequence of the first, and the induction hypothesis $(5.10)_{\prec\beta}$: Indeed, the first item in $(5.10)_\beta$ for $x = y = z$ assumes the form of $\pi^{(\mathbf{n})}_{xx\beta} = ((\mathrm{id} - \Gamma^*_{xx})\pi^{(\mathbf{n})}_{xx})_\beta$, so that it remains to appeal to the strict triangularity of $\Gamma^*_{xx} - \mathrm{id}$ w. r. t. $\prec$.

Akin to the three-point argument, we will establish the first item in $(5.10)_\beta$ in form of

$$\sum_{\mathbf{n}:|\mathbf{n}|<|\beta|}(\pi^{(\mathbf{n})}_{xy} - \pi^{(\mathbf{n})}_{xz} - \Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})_\beta(\cdot - y)^{\mathbf{n}} = 0. \tag{5.25}$$

The representation $(\mathrm{id} - P)\pi = \sum_{\mathbf{m}\neq\mathbf{0}}\pi_{e_{\mathbf{m}}}\mathsf{z}_{\mathbf{m}}$ allows us to rewrite (2.54) as $\pi^{(\mathbf{n})}_{zy} = P\pi^{(\mathbf{n})}_{zy} + \sum_{\mathbf{m}>\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}(y-z)^{\mathbf{m}-\mathbf{n}}\mathsf{z}_{\mathbf{m}}$. Hence by (2.51) we obtain $\Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy} = \Gamma^*_{xz}P\pi^{(\mathbf{n})}_{zy} + \sum_{\mathbf{m}>\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}(y-z)^{\mathbf{m}-\mathbf{n}}(\mathsf{z}_{\mathbf{m}} + \pi^{(\mathbf{m})}_{xz})$. Since β is not purely polynomial, this implies

$$(\pi^{(\mathbf{n})}_{xz} + \Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})_\beta = (\Gamma^*_{xz}P\pi^{(\mathbf{n})}_{zy})_\beta + \sum_{\mathbf{m}\geq\mathbf{n}}\binom{\mathbf{m}}{\mathbf{n}}(y-z)^{\mathbf{m}-\mathbf{n}}\pi^{(\mathbf{m})}_{xz\beta}.$$

Since by (2.46) the sum is restricted to $|\mathbf{m}| < |\beta|$ this yields by the binomial formula

$$\begin{aligned}&\sum_{\mathbf{n}:|\mathbf{n}|<|\beta|}(\pi^{(\mathbf{n})}_{xz} + \Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})_\beta(\cdot - y)^{\mathbf{n}}\\ &= \sum_{\mathbf{n}:|\mathbf{n}|<|\beta|}(\Gamma^*_{xz}P\pi^{(\mathbf{n})}_{zy})_\beta(\cdot - y)^{\mathbf{n}} + \sum_{\mathbf{m}:|\mathbf{m}|<|\beta|}\pi^{(\mathbf{m})}_{xz\beta}(\cdot - z)^{\mathbf{m}}.\end{aligned}$$

In view of the triangularity (8.9) of Γ^*_{xz} w. r. t. $|\cdot|$ and once more (2.46), we may rewrite the first r. h. s. term to obtain

$$\begin{aligned}
&\sum_{\mathbf{n}:|\mathbf{n}|<|\beta|}(\pi^{(\mathbf{n})}_{xz}+\Gamma^*_{xz}\pi^{(\mathbf{n})}_{zy})_\beta(\cdot-y)^{\mathbf{n}}\\
&=\sum_{\gamma\neq\text{p.p.}}(\Gamma^*_{xz})^\gamma_\beta\sum_{\mathbf{n}:|\mathbf{n}|<|\gamma|}\pi^{(\mathbf{n})}_{zy\gamma}(\cdot-y)^{\mathbf{n}}+\sum_{\mathbf{m}:|\mathbf{m}|<|\beta|}\pi^{(\mathbf{m})}_{xz\beta}(\cdot-z)^{\mathbf{m}}.
\end{aligned}\tag{5.26}$$

By the triangularity (8.9) of Γ^*_{xz} w. r. t. $\prec$, we may appeal to $(5.12)_{\preccurlyeq\beta}$ (which we recall is a consequence of $(2.61)_{\preccurlyeq\beta}$) to see that (5.25), into which we insert (5.26), reduces to

$$(\Gamma^*_{xy}P\Pi_y-\Pi_x)_\beta=(\Gamma^*_{xz}P(\Gamma^*_{zy}P\Pi_y-\Pi_z))_\beta+(\Gamma^*_{xz}P\Pi_z-\Pi_x)_\beta.$$

This identity follows from (4.79) and the first item in $(5.11)_\beta$, which as explained is a consequence of $(5.10)_{\prec\beta}$. □

5.4 Construction of $\mathrm{d}\Gamma^*_{xz}$ through $\mathrm{d}\pi^{(\mathbf{n})}_{xz}$

Recall the Ansatz (4.40) for $\mathrm{d}\Gamma^*_{xz}$. We seek $\mathrm{d}\pi^{(1,0)}_{xz}\in Q\tilde{\mathsf{T}}^*$ such that (4.42) holds, which since $\partial_1\Pi_z$ and $\partial_1\delta\Pi_x$ are well-defined[83] amounts to

$$Q\big(\partial_1\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}\partial_1\Pi_z(z)\big)=0.\tag{5.27}$$

Evaluating (4.40) on $\mathsf{z}_{(1,0)}$, and using the definitions (2.10) & (2.45) of $D^{(\mathbf{n})}$ (together with the last item in (2.49)), we obtain $\mathrm{d}\Gamma^*_{xz}\mathsf{z}_{(1,0)}=\mathrm{d}\pi^{(1,0)}_{xz}$. Applying Q and feeding in the postulate $\mathrm{d}\pi^{(1,0)}_{xz}\in Q\tilde{\mathsf{T}}^*$, we have

$$\mathrm{d}\pi^{(1,0)}_{xz}=Q\mathrm{d}\Gamma^*_{xz}\mathsf{z}_{(1,0)}.\tag{5.28}$$

Hence in view of $(\mathrm{id}-P)\partial_1\Pi_z(z)=\mathsf{z}_{(1,0)}$, which follows from (2.21), (5.27) can be re-arranged to

$$\mathrm{d}\pi^{(1,0)}_{xz}=Q\big(\partial_1\delta\Pi_x(z)-\mathrm{d}\Gamma^*_{xz}P\partial_1\Pi_z(z)\big).\tag{5.29}$$

According to the order within an induction step stated in Sect. 3.4 (singular case 9a), $(\Gamma^*_{xz}P)_{\preccurlyeq\beta}$ has been constructed (Proposition 4.1), and based on this and the induction hypothesis, $(\mathrm{d}\Gamma^*_{xz}P)_\beta$ has been constructed (Proposition 4.16). Also $(Q\Pi_x)_\beta$ and thus $(Q\delta\Pi_x)_\beta$ have been constructed (Proposition 4.3). In view of the strict triangularity (8.11) of $\mathrm{d}\Gamma^*_{xz}$ w. r. t. $\prec$, (5.29) involves only $(P\Pi_z)_{\prec\beta}$, which has been constructed by induction hypothesis. Hence $\mathrm{d}\pi^{(1,0)}_{xz\beta}$ is well-defined through (5.29).

[83] see the discussions before Proposition 2.3 and after Proposition 4.13, respectively.

6 Divergent bounds and analyticity

6.1 Divergent bounds: Proof of Propositions 2.3 and 4.13

Proof of Proposition 2.3 We now embark on the proof of $(2.40)_\beta$, $(2.41)_\beta$, $(2.42)_\beta$, and $(2.43)_\beta$. In view of (2.34) and (2.21), it is enough to consider (populated) β that are not purely polynomial. We shall frequently appeal to $\tilde\Pi^-_x$ as defined in (5.5).

Step 1. We first quantify the continuity of ξ_τ, which amount to the base case $\beta = 0$ in $(2.43)_\beta$. By a scaling argument and the fact that $\psi_{t=1}$ is a Schwartz function, we get $\|\partial^{\mathbf{n}}\psi_\tau\|_* \lesssim (\sqrt[4]{\tau})^{-2-|\mathbf{n}|+\alpha}$, see (2.4) for the definition of $\|\cdot\|_*$. By the mean-value theorem and translation invariance of the norm, this implies $\|\psi_\tau(y-\cdot)-\psi_\tau(x-\cdot)\|_* \lesssim (\sqrt[4]{\tau})^{-2}|y-x|^\alpha$. Since $\psi_\tau(y-\cdot)-\psi_\tau(x-\cdot)$ is the Malliavin derivative of the cylindrical random variable $\xi_\tau(y)-\xi_\tau(x)$, we obtain by the $\mathbb{L}^p$-version (4.17) of the SG inequality (and since by stationarity, $\mathbb{E}\xi_\tau(y)$ does not depend on y)

$$\mathbb{E}^{\frac{1}{p}}|\xi_\tau(y)-\xi_\tau(x)|^p \lesssim (\sqrt[4]{\tau})^{-2}|y-x|^\alpha. \tag{6.1}$$

Step 2. $(2.36)_{\prec\beta}$ & $(2.38)_\beta$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ & $(4.18)_\beta$ $\Longrightarrow$ $(2.40)_\beta$. In fact, we also need the following estimate on $\tilde\Pi^-_{x\beta}$,

$$\mathbb{E}^{\frac{1}{p}}|\tilde\Pi^-_{x\beta}(y)|^p \lesssim (\sqrt[4]{\tau})^{\alpha-2}(\sqrt[4]{\tau}+|y-x|)^{|\beta|-2\alpha}|y-x|^\alpha. \tag{6.2}$$

which is meant to include the statement that $\tilde\Pi^-_{x\beta}$ vanishes unless $|\beta|\geq 2\alpha$. Note that the presence of the bare factor $|y-x|^\alpha$ will be important for integration below. The proof of $(6.2)_\beta$ is similar to the one of $(6.4)_\beta$ below, using only the estimates $(4.87)_{\preccurlyeq\beta}$, $(2.41)_{\prec\beta}$, $(4.87)_{e_k+\beta_1}$ and $(2.40)_{\prec\beta}$.

We now turn to the estimate $(2.40)_\beta$. For this we note that $(2.38)_\beta$ implies

$$c_\beta = \lim_{t\uparrow\infty}\mathbb{E}(\partial_1^2\Pi_{x\beta-e_0}+\tilde\Pi^-_{x\beta})_t(x). \tag{6.3}$$

Recall now the representation (5.7). Using that $\beta-e_0\prec\beta$, we learn from $(4.7)_{\prec\beta}$ (which we recall is a consequence of $(2.36)_{\prec\beta}$) that the first term on the r. h. s. is estimated by $\sqrt[4]{\tau}^{|\beta|-2}$, as desired. Rewriting the second r. h. s. term as $\int dy\psi_\tau(x-y)\mathbb{E}\tilde\Pi^-_{x\beta}(y)$, we get the same estimate from $(6.2)_\beta$ with help of the moment bounds (4.3). For the third term, the estimate follows from $(4.18)_\beta$ (for $T=\tau$ and $y=x$).

Step 3. $(2.36)_{\prec\beta}$ & $(2.37)_{\prec\beta}$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ & $(2.42)_{\prec\beta}$ & $(2.61)_{\prec\beta}$ $\Longrightarrow$ $(2.43)_\beta$. By (5.6), noting that $\beta-e_0\prec\beta$, the required estimate follows by $(2.42)_{\prec\beta}$ and the following continuity estimate on $\tilde\Pi^-_{x\beta}$,

$$\mathbb{E}^{\frac{1}{p}}|\tilde\Pi^-_{x\beta}(y)-\tilde\Pi^-_{x\beta}(z)|^p \lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau}+|y-z|+|z-x|)^{|\beta|-\alpha}|y-z|^\alpha. \tag{6.4}$$

In order to prove $(6.4)_\beta$ we consider (5.5) componentwise and prove the corresponding continuity estimate for each summand separately. Note that each summand has the product structure $(\pi\pi')_\beta = \sum_{\beta_1+\beta_2=\beta}\pi_{\beta_1}\pi'_{\beta_2}$. In particular, summands in the first term on the r. h. s. of (5.5) correspond to $\pi = \mathsf{z}_k\Pi^k_x$ and $\pi' = \partial_1^2\Pi_x$, where purely polynomial β_1 as well as $\beta_1 = 0$ do not contribute, to the effect of $\beta_2 \prec \beta$ by (3.6). Similarly, since $\beta_2 = e_{(1,0)}$ does not contribute, a careful inspection of (3.2) shows that $\beta_1 \preccurlyeq \beta$. Therefore, the desired estimate for these terms follows via Hölder's inequality in probability from the boundedness $(4.87)_{\preccurlyeq\beta}$ (which we recall is a consequence of $(2.36)_{\prec\beta}$) & $(2.41)_{\prec\beta}$ and continuity $(4.84)_{\preccurlyeq\beta}$ (which we recall is a consequence of $(2.36)_{\prec\beta}$ & $(2.37)_{\prec\beta}$ & $(2.61)_{\prec\beta}$) & $(2.42)_{\prec\beta}$ of $\mathsf{z}_k\Pi_x^k$ and $\partial_1^2\Pi_x$. The summands in the second term on the r. h. s. of (5.5) correspond to $\pi = \Pi_x^k$ and $\pi' = (D^{(\mathbf{0})})^k c$, where π'_{β_2} depends only on c_γ for $\gamma \prec \beta$ by (8.4). Moreover, from (2.13) and (2.34) we infer that $\pi'_{\beta_2} \neq 0$ only for $k \leq [\beta_2]$, which by (3.2) implies $e_k \preccurlyeq \beta_2$ and hence $e_k + \beta_1 \preccurlyeq \beta$. Therefore, the desired estimate follows by continuity[84] $(4.84)_{e_k+\beta_1\preccurlyeq\beta}$ of Π_x^k and the estimate $|((D^{(\mathbf{0})})^k c)_{\beta_2}| = \sum_\gamma |((D^{(\mathbf{0})})^k)_{\beta_2}^\gamma||c_\gamma| \lesssim (\sqrt[4]{\tau})^{|\beta_2|-\alpha k-2}$ which follows from (2.13) and $(2.40)_{\prec\beta}$.

Step 4. $(2.36)_{\preccurlyeq\beta}$ & $(2.37)_\beta$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ & $(2.61)_\beta$ & $(2.64)_\beta$ & $(6.5)_{\prec\beta}$ $\Longrightarrow$ $(2.42)_\beta$. We start by recalling a version of Campanato's argument, namely that the pointwise $C^{2+\alpha}$-estimate $(2.42)_\beta$ may be encoded in terms of averages. More precisely, we argue that $(6.5)_\beta$ implies the pointwise control $(2.42)_\beta$, where

$$\begin{aligned}&\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\partial^{\mathbf{m}}\Pi_{x\beta t}(y)|^p \lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|)^{|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}\\ &\text{for } \mathbf{n}\neq\mathbf{0},\ \mathbf{m}\in\{(2,0),(0,1)\},\end{aligned} \tag{6.5}$$

which is part of the induction hypothesis. For simplicity, let us restrict to the case $\mathbf{m} = (2,0)$. To this purpose, we split the first term on the l. h. s. of $(2.42)_\beta$ into $(\partial_1^2\Pi_{x\beta} - \partial_1^2\Pi_{x\beta t})(y) + (\partial_1^2\Pi_{x\beta t}(y) - \partial_1^2\Pi_{x\beta t}(z)) + (\partial_1^2\Pi_{x\beta t} - \partial_1^2\Pi_{x\beta})(z)$, where the choice $t = |y-z|^4$ will turn out to be natural. Based on (4.2) we rewrite the first and last contribution as

$$\partial_1^2\Pi_{x\beta} - \partial_1^2\Pi_{x\beta t} = \int_0^t ds(-\partial_2^2 + \partial_1^4)\partial_1^2\Pi_{x\beta s}. \tag{6.6}$$

Hence on these contributions, we may use $(6.5)_\beta$ with $|\mathbf{n}| = 4$ and t replaced by s, yielding control by $(\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|+|z-x|)^{|\beta|-\alpha}\int_0^t ds(\sqrt[4]{s})^{\alpha-4}$. Note that thanks to $\alpha > 0$, the last integral converges; by the choice of $t = |y-z|^4$, and by $|y-x| \leq |y-z| + |z-x|$, this gives rise to the r. h. s. of $(2.42)_\beta$. We rewrite the remaining middle term $\partial_1^2\Pi_{x\beta t}(y) - \partial_1^2\Pi_{x\beta t}(z)$ as an integral of the derivative along the connecting segment. Since this involves $\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta t}$ for $\mathbf{n} \in \{(1,0),(0,1)\}$ we obtain from $(6.5)_\beta$ an estimate by

$$(\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|+|z-x|)^{|\beta|-\alpha}\big((\sqrt[4]{t})^{\alpha-1}|y-z| + (\sqrt[4]{t})^{\alpha-2}|y-z|^2\big).$$

By our choice of $t = |y-z|^4$ also this term can be subsumed into to the r. h. s. of $(2.42)_\beta$.

We now turn to integration proper and argue that $(6.2)_\beta$ (which we recall is a consequence of $(2.36)_{\prec\beta}$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$) and $(6.5)_{\prec\beta}$ imply $(6.5)_\beta$. Again,

[84] note that $(4.84)_{e_k+\beta_1}$ yields an estimate on $(\Pi_x^k)_{\beta_1}$.

for simplicity we restrict ourselves to the case $\mathbf{m} = (2,0)$. We first argue that it is sufficient to establish (6.5)$_\beta$ for $y = x$, which assumes the form

$$\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t}(x)|^p \lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t})^{|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|} \quad \text{for } \mathbf{n}\neq\mathbf{0}. \tag{6.7}$$

Indeed, in order to pass from the generic point y to the base point x, we appeal again to (2.61) in form of $\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta} = \sum_\gamma (\Gamma^*_{xy})_\beta^\gamma \partial^{\mathbf{n}}\partial_1^2\Pi_{y\gamma}$ and to the triangularity (8.9). Hence by (2.37)$_\beta$, (6.5)$_\beta$ is a consequence of (6.7)$_{\preccurlyeq\beta}$:

$$\begin{aligned}\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t}(y)|^p &\lesssim \sum_{|\gamma|\in\mathsf{A}\cap(-\infty,|\beta|]} |y-x|^{|\beta|-|\gamma|}(\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t})^{|\gamma|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}\\ &\lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|)^{|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}.\end{aligned}$$

We also note that for (6.7)$_\beta$, it is enough to deal with the case of $t \le \tau$. Indeed, we obtain from (4.7)$_\beta$ (with $\mathbf{n}$ replaced by $\mathbf{n}+(2,0)$, which we recall is a consequence of (2.36)$_\beta$) that $\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t}(x)|^p \lesssim (\sqrt[4]{t})^{|\beta|-2-|\mathbf{n}|}$, which for $t \ge \tau$ is dominated by the r. h. s. of (6.7)$_\beta$. Therefore, it remains to prove (6.7)$_\beta$ in the reduced form of

$$\mathbb{E}^{\frac{1}{p}}|\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t}(x)|^p \lesssim (\sqrt[4]{\tau})^{-2+|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|} \quad \text{for } \mathbf{n}\neq\mathbf{0} \text{ and } t\le\tau. \tag{6.8}$$

In establishing (6.8)$_\beta$, we distinguish the cases $|\beta| < 3$ and $|\beta| \ge 3$, starting with the former. Upon applying $\partial^{\mathbf{n}}\partial_1^2$ to the integral representation (4.11), the Taylor polynomial drops out because of $|\beta| < 3$ and $\mathbf{n}\neq\mathbf{0}$. Also applying $(\cdot)_t$, we obtain with help of (4.1)

$$\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t} = -\int_t^\infty ds\,\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\Pi^-_{x\beta\,s}.$$

We split the integral at $s = \tau \ge t$, substitute (5.6) in the first part, and we evaluate at x:

$$\begin{aligned}\partial^{\mathbf{n}}\partial_1^2\Pi_{x\beta\,t}(x) = &-\int_t^\tau ds\,\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)(\partial_1^2\Pi_{x\beta-e_0}+\tilde{\Pi}^-_{x\beta}+\xi_\tau\delta_\beta^0)_s(x)\\ &-\int_\tau^\infty ds\,\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\Pi^-_{x\beta\,s}(x).\end{aligned} \tag{6.9}$$

We start with the first r. h. s. contribution to (6.9), $\int_t^\tau ds\ \partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\times \partial_1^2\Pi_{x\beta-e_0\,s}(x)$. Our ordering $\beta - e_0 \prec \beta$ allows us to appeal to (6.8)$_{\prec\beta}$, which we use with $\mathbf{n}$ replaced by $(2,1)+\mathbf{n}$ and $(4,0)+\mathbf{n}$. Because of $|\beta-e_0| = |\beta|$, this yields control by $(\sqrt[4]{\tau})^{|\beta|-2-\alpha}\ \int_t^\tau ds\ (\sqrt[4]{s})^{\alpha-4-|\mathbf{n}|}$, which due to $\alpha < 1 \le |\mathbf{n}|$ behaves as the term on the r. h. s. (6.8)$_\beta$. We rewrite the second r. h. s. contribution to (6.9) as $\int_t^\tau ds\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\tilde{\Pi}^-_{x\beta\,s}(x) = \int_t^\tau ds\int dy\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\psi_s(x-y)\tilde{\Pi}^-_{x\beta}(y)$, we appeal to (6.2)$_\beta$ (which we recall is a consequence of (2.36)$_{\prec\beta}$ & (2.40)$_{\prec\beta}$ & (2.41)$_{\prec\beta}$) and the moment bounds (4.3) to obtain control by $(\sqrt[4]{\tau})^{\alpha-2}\int_t^\tau ds(\sqrt[4]{s})^{\alpha-|\mathbf{n}|-4}(\sqrt[4]{\tau+s})^{|\beta|-2\alpha}$. Because of $s\le\tau$, the last factor is

$\sim(\sqrt[4]{\tau})^{|\beta|-2\alpha}$, so that by $\alpha>0$ we again arrive at $(\sqrt[4]{\tau})^{|\beta|-\alpha-2}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}$. For the third r. h. s. contribution to (6.9), $\int_t^\tau ds\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\xi_{\tau+s}(x)$, we appeal to (6.1) to obtain an estimate by $(\sqrt[4]{\tau})^{-2}\int_t^\tau ds(\sqrt[4]{s})^{\alpha-|\mathbf{n}|-4}$, which acts as the previous term because of $|\beta|=\alpha$ for $\beta=0$.

We now turn to the last contribution in (6.9), namely $\int_\tau^\infty ds\partial^{\mathbf{n}}\partial_1^2(\partial_2+\partial_1^2)\Pi^-_{x\beta\, s}(x)$, and appeal to the standard estimates $(2.64)_\beta$ in their upgraded form $(4.12)_\beta$. Equipped with $(4.12)_\beta$, which we use for $y=x$ and $\mathbf{n}$ replaced by $\mathbf{n}+(2,1)$ and $\mathbf{n}+(4,0)$, we obtain control by $\int_\tau^\infty ds(\sqrt[4]{s})^{|\beta|-6-|\mathbf{n}|}$. Since $|\beta|<3$, therefore $-2-|\mathbf{n}|+|\beta|<0$ for $\mathbf{n}\neq\mathbf{0}$, this integral is $\sim(\sqrt[4]{\tau})^{-2-|\mathbf{n}|+|\beta|}$, which due to $\alpha<|\mathbf{n}|$ for $\mathbf{n}\neq\mathbf{0}$ and $t\le\tau$ is dominated by the r. h. s. of $(6.8)_\beta$.

We conclude integration proper by dealing with the case of $|\beta|\ge 3$. In this case, we may directly infer $(6.8)_\beta$ from $(4.7)_\beta$ (with $\mathbf{n}$ replaced by $\mathbf{n}+(2,0)$, which we recall is a consequence of $(2.36)_\beta$), gaining control by $(\sqrt[4]{t})^{-|\mathbf{n}|-2+|\beta|}$, which is dominated by the r. h. s. of $(6.8)_\beta$ because $|\beta|\ge 3\ge 2+\alpha$ and $t\le\tau$.

Step 5. $(2.36)_\beta$ & $(2.42)_\beta$ $\Longrightarrow$ $(2.41)_\beta$. We pass from the homogeneous $C^{2+\alpha}$-estimate $(2.42)_\beta$ to the C^2-estimate $(2.41)_\beta$, namely by interpolation with the $C^{\alpha-2}$-estimate $(4.7)_\beta$ (which we recall is a consequence of $(2.36)_\beta$). To this purpose, we write $\partial^{\mathbf{n}}\Pi_{x\beta}(y)=\partial^{\mathbf{n}}\Pi_{x\beta\,\tau}(y)+\int dz\psi_\tau(y-z)\big(\partial^{\mathbf{n}}\Pi_{x\beta}(y)-\partial^{\mathbf{n}}\Pi_{x\beta}(z)\big)$ for $\mathbf{n}\in\{(2,0),(0,1)\}$. On the first term, we use $(4.7)_\beta$ with $t=\tau$. On the second term, we use $(2.42)_\beta$ and then the moment bound (4.3) for $t=\tau$. $\square$

Proof of Proposition 4.13 The arguments follow those of the proof of Proposition 2.3, and should be read in parallel.

Step 1. $(2.36)_{\prec\beta}$ & $(2.37)_{\prec\beta}$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ & $(2.42)_{\prec\beta}$ & $(2.61)_{\prec\beta}$ & $(4.27)_{\prec\beta}$ & $(4.34)_{\prec\beta}$ & $(4.53)_{\prec\beta}$ & $(4.54)_{\prec\beta}$ $\Longrightarrow$ $(4.55)_\beta$. In order to establish (4.55), we take the Malliavin derivative of (5.6):

$$\delta\Pi^-_{x\beta}=\partial_1^2\delta\Pi_{x\beta-e_0}+\delta\tilde{\Pi}^-_{x\beta}+\delta\xi_\tau\delta_0^\beta.$$

The desired estimate on $\delta\xi_\tau$ follows analogously to (6.1). The estimate on the first r. h. s. follows from the induction hypothesis $(4.54)_{\prec\beta}$. In order to estimate the middle r. h. s. term we take the Malliavin derivative of (5.5) which, similarly to (4.57), gives

$$\begin{aligned}\delta\Pi^-_x=P\Big(&\sum_{k\ge1}\mathsf{z}_k\Pi_x^k\partial_1^2\delta\Pi_x+\sum_{k\ge0}(k+1)\mathsf{z}_{k+1}\Pi_x^k\delta\Pi_x\partial_1^2\Pi_x\Big)\\&-\sum_{k\ge1}\frac{1}{k!}\Pi_x^k\delta\Pi_x(D^{(\mathbf{0})})^{k+1}c.\end{aligned}$$

Therefore, the additional ingredients w. r. t. the proof of $(6.4)_\beta$ are $(4.34)_{\prec\beta}$, $(4.53)_{\prec\beta}$, $(4.54)_{\prec\beta}$, and $(6.10)_{\prec\beta}$, where (6.10) is given by

$$\mathbb{E}^{\frac{1}{q'}}|\delta\Pi_{x\beta}(y)-\delta\Pi_{x\beta}(z)|^{q'}\lesssim|y-z|^\alpha(|y-x|+|z-x|)^{|\beta|-\alpha}\bar{w}. \tag{6.10}$$

In order to prove $(6.10)_\beta$ we take the Malliavin derivative of $(2.61)_\beta$ and evaluate at z, which leads to the identity $\delta\Pi_{x\beta}(y)-\delta\Pi_{x\beta}(z)=-\sum_\gamma(\delta\Gamma^*_{xy})_\beta^\gamma\Pi_{y\gamma}(z)-$

$\sum_\gamma (\Gamma^*_{xy})^\gamma_\beta \delta\Pi_{y\gamma}(z)$, where due to (8.9) and (8.10) the sums are restricted to $\gamma \preccurlyeq \beta$. Therefore, $(6.10)_\beta$ follows from $(4.27)_\beta$ & $(2.36)_{\preccurlyeq\beta}$ and $(2.37)_\beta$ & $(4.34)_{\preccurlyeq\beta}$.

Step 2. $(2.36)_{\prec\beta}$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ & $(4.34)_{\prec\beta}$ & $(4.53)_{\prec\beta}$ & $(6.11)_{\prec\beta}$ $\Longrightarrow$ $(4.54)_\beta$. We proceed as for (2.42) in Step 4 of the proof of Proposition 2.3: We derive the pointwise $(4.54)_\beta$ from the analogue of the weak $(6.5)_\beta$ for $\delta\Pi_{x\beta}$, as given by

$$\begin{aligned}&\mathbb{E}^{\frac{1}{q'}}|\partial^{\mathbf{n}}\partial^{\mathbf{m}}\delta\Pi_{x\beta\,t}(y)|^{q'}\\ &\lesssim (\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|)^{|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|}\bar{w} \quad \text{for } \mathbf{n}\neq\mathbf{0},\ \mathbf{m}\in\{(2,0),(0,1)\},\end{aligned} \tag{6.11}$$

which we state here since it is part of the induction hypothesis. The analogue of $(6.2)_\beta$ (which we recall is a consequence of $(2.36)_{\prec\beta}$ & $(2.40)_{\prec\beta}$ & $(2.41)_{\prec\beta}$) for $\delta\tilde{\Pi}^-_{x\beta}$, from which we derive (6.11) by integration, relies on the additional ingredients $(4.34)_{\prec\beta}$ and $(4.53)_{\prec\beta}$.

Step 3. $(4.34)_\beta$ & $(4.54)_\beta$ $\Longrightarrow$ $(4.53)_\beta$. We can pass from $(4.54)_\beta$ to $(4.53)_\beta$ using the analogue of $(4.7)_\beta$ for $\partial^{\mathbf{n}}\delta\Pi_{x\beta}$, which follows from $(4.34)_\beta$ and (4.3). ☐

6.2 Analyticity: Proof of Proposition 2.7

Proof of Proposition 2.7 The first statement is easy to check: The most substantial change is the representation (4.5) of the solution operator in terms of the convolution semi-group, see Sect. 4.1. This requires changing the generator of the latter to $(\partial_2 - a_0\partial_1^2)^*(\partial_2 - a_0\partial_1^2) = -\partial_2^2 + |a_0|^2\partial_1^4$. Hence the new convolution kernel is a simple spatial rescaling of the standard one:

$$\psi_t(a_0, x) = \psi_t\Big(\frac{x_1}{\sqrt{|a_0|}}, x_2\Big), \tag{6.12}$$

and thus satisfies the relevant moment bounds (4.3) locally uniformly in a_0.

For convenience of the reader, we list the annealed Hölder-type norms with respect to which we will establish complex differentiability and thus analyticity: the norm on random functions given by (2.36) of Theorem 2.2,

$$\sup_y |y-x|^{-|\beta|}\mathbb{E}^{\frac{1}{p}}|\Pi_{x\beta}(y)|^p, \tag{6.13}$$

the norm on random distributions given by (2.64) of Proposition 2.6

$$\sup_{y,t}(\sqrt[4]{t})^{2-\alpha}(\sqrt[4]{t}+|y-x|)^{\alpha-|\beta|}\mathbb{E}^{\frac{1}{p}}|\Pi^-_{x\beta\,t}(y)|^p, \tag{6.14}$$

and finally the stronger norm given by (2.41) in the proof of Proposition 2.3

$$\sup_y(\sqrt[4]{\tau}+|y-x|)^{\alpha-|\beta|}\mathbb{E}^{\frac{1}{p}}|\partial_1^2\Pi_{x\beta}(y)|^p. \tag{6.15}$$

Note that in view of the argument for (2.64), we may revert from (6.12) to the standard convolution kernel for (6.14).

Over the next several paragraphs, we establish (2.71), (2.72), and (2.74) for $\pi \in \{c, \Pi_x\}$ by induction in β with respect to the ordering $\prec$. In fact, we rephrase these statements as

$$\partial_{a_0} c_\beta = (\beta(0)+1) c_{\beta+e_0}, \tag{6.16}$$

$$\partial_{a_0} \Pi_{x\beta} = (\beta(0)+1) \Pi_{x\beta+e_0} \quad \text{w. r. t. (6.13)}, \tag{6.17}$$

with the understanding that these imply that the functions are complex differentiable – and thus analytic – in a_0.

We start with a reconstruction argument showing that $(6.16)_{\prec\beta}$ and $(6.17)_{\prec\beta}$ imply $(6.16)_\beta$ and $(6.18)_\beta$, where

$$\partial_{a_0} \Pi^-_{x\beta} + \partial_1^2 \Pi_{x\beta} = (\beta(0)+1) \Pi^-_{x\beta+e_0} \quad \text{w. r. t. (6.14)}. \tag{6.18}$$

In preparation for this, we note that in view of the locally uniform estimate (2.41), the analyticity expressed by $(6.17)_\beta$ transmits from $\Pi_{x\beta}$ to $\partial_1^2 \Pi_{x\beta}$ in form of $(6.19)_\beta$, where

$$\partial_{a_0} \partial_1^2 \Pi_{x\beta} = (\beta(0)+1) \partial_1^2 \Pi_{x\beta+e_0} \quad \text{w. r. t. (6.15)}. \tag{6.19}$$

Like in the proof of Proposition 2.3 we work with $\tilde{\Pi}^-_{x\beta}$, see (5.5), which component-wise takes the form

$$\begin{aligned}\tilde{\Pi}^-_{x\beta} = &\sum_{k\geq 1} \sum_{e_k+\beta_1+\cdots+\beta_{k+1}=\beta} \Pi_{x\beta_1} \cdots \Pi_{x\beta_k} \partial_1^2 \Pi_{x\beta_{k+1}} \\ &- \sum_{k\geq 1} \frac{1}{k!} \sum_{\beta_1+\cdots+\beta_{k+1}=\beta} \Pi_{x\beta_1} \cdots \Pi_{x\beta_k} \sum_\gamma ((D^{(\mathbf{0})})^k)^\gamma_{\beta_{k+1}} c_\gamma.\end{aligned} \tag{6.20}$$

By (8.6) and (8.4) the two sums are restricted to $\beta_1, \ldots, \beta_{k+1}, \gamma \prec \beta$. Therefore we obtain from the analyticity expressed in $(6.16)_{\prec\beta}$, $(6.17)_{\prec\beta}$, and $(6.19)_{\prec\beta}$ via Leibniz' rule

$$\begin{aligned}\partial_{a_0} \tilde{\Pi}^-_{x\beta} = &\sum_{k\geq 1} \sum_{\substack{e_k+\beta_1+\cdots+\beta_{k+1}=\beta \\ l_1+\cdots+l_{k+1}=1}} \partial_{a_0}^{l_1} \Pi_{x\beta_1} \cdots \partial_{a_0}^{l_k} \Pi_{x\beta_k} \partial_1^2 \partial_{a_0}^{l_{k+1}} \Pi_{x\beta_{k+1}} \\ &- \sum_{k\geq 1} \frac{1}{k!} \sum_{\substack{\beta_1+\cdots+\beta_{k+1}=\beta \\ l_1+\cdots+l_{k+1}=1}} \partial_{a_0}^{l_1} \Pi_{x\beta_1} \cdots \partial_{a_0}^{l_k} \Pi_{x\beta_k} \sum_\gamma ((D^{(\mathbf{0})})^k)^\gamma_{\beta_{k+1}} \partial_{a_0}^{l_{k+1}} c_\gamma.\end{aligned} \tag{6.21}$$

Inserting the corresponding formulas, we obtain

$$\partial_{a_0} \tilde{\Pi}^-_{x\beta} = (\beta(0)+1) \tilde{\Pi}^-_{x\beta+e_0} \quad \text{w. r. t. (6.15)}. \tag{6.22}$$

We now argue that $(6.22)_\beta$ implies $(6.16)_\beta$, where w. l. o. g. we restrict to $|\beta| < 2$. Rewriting the first r. h. s. term of (6.3) as $\int dy \partial_1^2 \psi_t(x-y) \mathbb{E}\Pi_{x\beta-e_0}(y)$, we learn from (2.36) that its limit vanishes. This leaves us with

$$c_\beta = \lim_{t\uparrow\infty} \mathbb{E}\tilde{\Pi}^-_{x\beta\, t}(x). \tag{6.23}$$

Since (6.22) implies that $\mathbb{E}\tilde{\Pi}^-_{x\beta\, t}(x)$ is analytic, this transmits to c_β via the locally uniform convergence (6.23).

We now may turn to (6.18), based on the identity (5.6) to which we apply ∂_{a_0}, noting that there is no contribution from ξ. According to $(6.19)_{\prec\beta}$ (recall that $\beta - e_0 \prec \beta$), to $(6.22)_\beta$, and to $(6.16)_\beta$, we obtain $(6.18)_\beta$, at first w. r. t. to the stronger topology (6.15) and then to the weaker one (6.14).

We now turn to integration, arguing that $(6.18)_\beta$ implies $(6.17)_\beta$. We consider the (what would be) first-order Taylor terms

$$\begin{aligned}
\Delta\Pi_{x\beta} &:= \Pi_{x\beta}(a_0') - \Pi_{x\beta}(a_0) - (a_0' - a_0)(\beta(0)+1)\Pi_{x\beta+e_0}(a_0),\\
\Delta\Pi^-_{x\beta} &:= \Pi^-_{x\beta}(a_0') - \Pi^-_{x\beta}(a_0) - (a_0' - a_0)\partial_{a_0}\Pi^-_{x\beta}(a_0).
\end{aligned}$$

We now appeal to (2.70), which we use for β, also with a_0 replaced by a_0', and for $\beta + e_0$. We then obtain from (6.18)

$$(\partial_2 - a_0\partial_1^2)\Delta\Pi_{x\beta} = \Delta\Pi^-_{x\beta} + (a_0' - a_0)\partial_1^2(\Pi_{x\beta}(a_0') - \Pi_{x\beta}(a_0)). \tag{6.24}$$

We note that $\Delta\Pi_{x\beta}$ (qualitatively) inherits the growth and anchoring estimate (2.36) from $\Pi_{x\beta}$ and $\Pi_{x\beta+e_0}$. Hence by the Liouville argument from the proof of Proposition 5.3 we obtain from (6.24) an integral representation of $\Delta\Pi_{x\beta}$ in terms of $\Delta\Pi^-_{x\beta} + (a_0' - a_0)\partial_1^2(\Pi_{x\beta}(a_0') - \Pi_{x\beta}(a_0))$ in form of (4.11). The argument of Proposition 4.3 then shows that

$$\begin{aligned}
&\text{norm (6.13) of } \Delta\Pi_{x\beta}(y) \lesssim \text{norm (6.14) of } \Delta\Pi^-_{x\beta\, t}\\
&\quad + |a_0' - a_0| \times \quad \text{norm (6.13) of } (\Pi_{x\beta}(a_0') - \Pi_{x\beta}(a_0)).
\end{aligned}$$

Writing $\Pi_{x\beta}(a_0') - \Pi_{x\beta}(a_0) = \Delta\Pi_{x\beta} + (a_0' - a_0)(\beta(0)+1)\Pi_{x\beta+e_0}(a_0)$, we absorb the second r. h. s. term for $|a_0' - a_0| \ll 1$ to obtain

$$\begin{aligned}
&\text{norm (6.13) of } \Delta\Pi_{x\beta}(y)\\
&\lesssim \text{norm (6.14) of } \Delta\Pi^-_{x\beta\, t} \;+\; |a_0' - a_0|^2 \times \text{ norm (6.13) of } \Pi_{x\beta+e_0}(a_0).
\end{aligned}$$

This estimate shows that (6.18) transmits to (6.17).

Let us now treat the base case $(6.16)_{\beta=0}$ and $(6.17)_{\beta=0}$. For the former we note that $\Pi^-_{x\,0} = \xi_\tau - c_0$ and $\Pi^-_{x\, e_0} = \partial_1^2\Pi_{x\,0} - c_{e_0}$; since ξ is centered by Assumption 2.1 and $\mathbb{E}|\partial_1^2\Pi_{x\,0\,t}(x)| \lesssim (\sqrt[4]{t})^{\alpha-2}$ by $(2.36)_{\beta=0}$, the BPHZ-choice (2.38) yields $c_0 = 0 = c_{e_0}$. For the latter we note that $(6.18)_{\beta=0}$ holds by the just said, and thus the integration argument established above shows $(6.17)_{\beta=0}$.

We now turn to (2.73). By (2.47), there is nothing to show for purely polynomial β, so that we turn to not purely polynomial β, which we treat by induction in $|\beta|$. According to (2.21) and (2.51), the β-component of (2.61) can be rewritten as

$$\sum_{\mathbf{n}}(z-y)^{\mathbf{n}}\pi^{(\mathbf{n})}_{xy\beta} + \sum_{\gamma\neq\text{p.p.}}(\Gamma^*_{xy})^\gamma_\beta\Pi_{y\gamma}(z) = \Pi_{x\beta}(z). \tag{6.25}$$

According to (8.1), for not purely polynomial γ, $(\Gamma^*_{xy})^\gamma_\beta$ depends on $\pi^{(\mathbf{n})}_{xy\beta'}$ via (2.44) only through $|\beta'| < |\beta|$. Hence by induction hypothesis, $(\Gamma^*_{xy})^\gamma_\beta$ is analytic in a_0 w. r. t. $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$. Together with the analyticity of $\Pi_{y\gamma}$ and $\Pi_{x\beta}$ w. r. t. (6.13), see (6.17), we obtain analyticity of $\sum_{\mathbf{n}}(\cdot - y)^{\mathbf{n}}\pi^{(\mathbf{n})}_{xy\beta}$ w. r. t. (6.13). Since by (2.46), the sum is effectively constrained to $|\mathbf{n}| < |\beta|$, the invertibility of the Vandermonde matrix yields analyticity of the coefficients $\pi^{(\mathbf{n})}_{xy\beta}$.

It remains to establish (2.74) for $\pi^{(\mathbf{n})}_{xy\beta}$, which again we do by induction in $|\beta|$. With help of (2.44), we rewrite (6.25) as

$$
\begin{aligned}
\Pi_{x\beta}(z) = \sum_{\mathbf{n}}(z-y)^{\mathbf{n}}\pi^{(\mathbf{n})}_{xy\beta} + \sum_{\gamma\neq\text{p.p.}}\sum_{k\geq 0}\frac{1}{k!}\sum_{\mathbf{n}_1,\ldots,\mathbf{n}_k}\sum_{\beta_1+\cdots+\beta_{k+1}=\beta}\\
\pi^{(\mathbf{n}_1)}_{xy\beta_1}\cdots\pi^{(\mathbf{n}_k)}_{xy\beta_k}(D^{(\mathbf{n}_k)}\cdots D^{(\mathbf{n}_1)})^\gamma_{\beta_{k+1}}\Pi_{y\gamma}(z).
\end{aligned}
$$

We now apply ∂_{a_0}, as in (6.21), use once more the triangular structure (8.1) and (8.9), to feed in the induction hypothesis of (2.74).

The same argumentation applies to establish the base case $(2.73)_{\beta=0}$ and $(2.74)_{\beta=0}$ for $\pi^{(\mathbf{n})}_{xy}$. □

7 Malliavin differentiability

In this section we prove that the quantities of interest, namely Π_x, Π^-_x, Γ^*_{xy} and $\pi^{(\mathbf{n})}_{xy}$, are Malliavin differentiable, so that Leibniz rule can indeed be applied. For the definitions and basic properties of the Malliavin–Sobolev spaces $\mathbb{H}^p$ for random variables (Γ^*_{xy} and $\pi^{(\mathbf{n})}_{xy}$) and $C^0(\mathbb{H}^p)$ for random fields (Π_x and Π^-_x) we refer to Appendix A. As before, $p < \infty$ denotes a generic exponent and its value may change from line to line, which is convenient when applying Leibniz rule.

We proceed in line with the logical order of the induction, see Sect. 3.4.

1. In Sect. 7.1 we establish [85] $\xi_\tau \in C^0(\mathbb{H}^p)$.
2. Then, for any populated multi-index β with $|\beta| < 2$, in Sect. 7.2 we establish $\Pi^-_{x\beta} \in C^0(\mathbb{H}^p)$.
3. In Sect. 7.3 we prove $\Pi_{x\beta}$, $\partial_1^2\Pi_{x\beta}$, $\partial_2\Pi_{x\beta}$, $\in C^0(\mathbb{H}^p)$ via integration and relate $\delta\Pi_{x\beta}$ to $\delta\Pi^-_{x\beta}$ through the representation (4.11).
4. In Sect. 7.4 we establish $\Gamma^*_{xy\beta}, \pi^{(\mathbf{n})}_{xy\beta} \in \mathbb{H}^p$ by the algebraic and three-point argument.

The identities (4.24), (4.35), (4.57) and (4.116) then follow by a now fully justified application of Leibniz' rule. Statements 3 and 4 are "officially" included into the induction since we need them in form of an induction hypothesis. Statement 2 is established inside the induction step.

[85] always with the understanding that this holds for all $p < \infty$.

7.1 The base case: Malliavin differentiability of $\Pi^-_{x0} = \xi_\tau$

From the argument used to derive (6.1) we learn that for every y, the cylindrical random variable $\xi_\tau(y)$ is in $\mathbb{H}^p$, with (deterministic) Malliavin derivative $\psi_\tau(y-\cdot)$. In order to argue that the random field ξ_τ is in $C^0(\mathbb{H}^p)$, we need to show that $\xi_\tau(y)$ can be approximated w. r. t. $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$ by functionals of the form (A.4) that form a Cauchy sequence w. r. t. the family of semi-norms (A.1). To this purpose, we note that for fixed $\tau > 0$, $\psi_\tau(y-\cdot)$ can be approximated, locally uniformly in y and w. r. t. to the Schwartz topology (in the active variable), by functions $\psi_N(y,\cdot)$ that are finite sums of functions of the form $\psi(y)\psi'(\cdot)$, where ψ is smooth and ψ' a Schwartz function. This structure implies that $\big(\xi, \psi_N(y,\cdot)\big)$ is of the form (A.4). On the one hand, the convergence implies that $\xi_\tau(y)$ is even the pathwise limit of $\big(\xi, \psi_N(y,\cdot)\big)$, locally uniformly in y. Since convergence in the Schwartz topology implies convergence w. r. t. $\|\cdot\|_*$, $\|\psi_\tau(y-\cdot) - \psi_N(y,\cdot)\|_*$ is small locally uniformly in y. Since $\psi_N(y,\cdot)$ is the Malliavin derivative of $\big(\xi, \psi_N(y,\cdot)\big)$, we obtain from the $\mathbb{L}^p$-version (4.17) of the SG inequality that $\big(\xi, \psi_N(y,\cdot)\big)$ forms a Cauchy sequence in $C^0(\mathbb{H}^p)$, locally uniformly in y.

7.2 Reconstruction

In this subsection, we establish $\Pi^-_{x\beta} \in C^0(\mathbb{H}^p)$. It follows from (8.6) and (8.4) that $\Pi^-_{x\beta}$ depends on $\Pi_{\beta'}$ only for $\beta' \prec \beta$ and on $c_{\beta'}$ only for $\beta' \preccurlyeq \beta$. At this stage of the induction step (singular case 4a), we already know that the deterministic constant c_β is finite. Hence by the induction hypothesis and (A.5) we obtain $\Pi^-_{x\beta} \in C^0(\mathbb{H}^p)$. In more quantitative terms, we obtain from $(2.36)_{\prec\beta}$ & $(2.41)_{\prec\beta}$ on the $\mathbb{L}^p$-norms and $(4.34)_{\prec\beta}$ & $(4.53)_{\prec\beta}$ on the $\mathbb{L}^p(H^*)$-norms that

$$\sup_y \frac{1}{1+|y|^{|\beta|}} \|\Pi^-_{x\beta}(y)\|_{\mathbb{H}^p} < \infty. \tag{7.1}$$

7.3 Integration

We now turn to integration. By (7.1), the first part of Lemma A.2 implies that $(\partial_1^2 + \partial_2)\Pi^-_{x\beta\,t}(y) \in \mathbb{H}^p$ is continuous in $(t,y) \in (0,\infty)\times\mathbb{R}^2$. In order to deduce that $\Pi_{x\beta} \in C^0(\mathbb{H}^p)$ from the integral representation (4.11), we shall apply the second part of Lemma A.2 to $F(t,y) = (1-\mathrm{T}_x^{|\beta|})(\partial_1^2+\partial_2)\Pi^-_{x\beta\,t}(y)$. In order to check assumption (A.7), we revisit the proof of Proposition 4.3, this time starting from (2.64) and the un-dualized version of (4.22), which combine to

$$\|\Pi^-_{x\beta\,t}(y)\|_{\mathbb{H}^p} \lesssim (\sqrt[4]{t})^{\alpha-2}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}.$$

As for (4.12), this can be upgraded to

$$\|\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}(y)\|_{\mathbb{H}^p} \lesssim (\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}. \tag{7.2}$$

This allows us to derive, in analogy to (4.13) and (4.14),

$$\|\mathrm{T}_x^{|\beta|}\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}(y)\|_{\mathbb{H}^p}\lesssim\sum_{|\mathbf{m}|<|\beta|}(\sqrt[4]{t})^{|\beta|-2-|\mathbf{n}|-|\mathbf{m}|}|y-x|^{|\mathbf{m}|}\tag{7.3}$$

and

$$\begin{aligned}&\|(1-\mathrm{T}_x^{|\beta|})\partial^{\mathbf{n}}\Pi^-_{x\beta\,t}(y)\|_{\mathbb{H}^p}\\&\quad\lesssim\sum_{\substack{|\mathbf{m}|\geq|\beta|\\ m_1+m_2<|\beta|+1}}(\sqrt[4]{t})^{\alpha-2-|\mathbf{n}|-|\mathbf{m}|}(\sqrt[4]{t}+|y-x|)^{|\beta|-\alpha}|y-x|^{|\mathbf{m}|}.\end{aligned}\tag{7.4}$$

The far-field part of (A.7), i. e. the integral over (N,∞), now follows directly from (7.4) with $\mathbf{n}\in\{(2,0),(0,1)\}$ and thus $|\mathbf{n}|=2$, because thanks to (2.29) we have $|\mathbf{m}|>|\beta|$, so that all exponents on $\sqrt[4]{t}$ are strictly less than -4. Hence the far-field integral decays as some negative power of N, uniformly for bounded $|y-x|$. For the near-field part of (A.7), i. e. the integral over $(0,N^{-1})$, we combine (7.2) and (7.3) by the triangle inequality and note that all exponents on $\sqrt[4]{t}$ are strictly larger than -4.

We now turn to the argument for $\partial^{\mathbf{m}}\Pi_{x\beta}\in C^0(\mathbb{H}^p)$ for $\mathbf{m}\in\{(2,0),(0,1)\}$. We appeal to the integral representation (6.6) in form of

$$\partial^{\mathbf{m}}\Pi_{x\beta}=\partial^{\mathbf{m}}\Pi_{x\beta\,\tau}+\int_0^\tau dt(\partial_1^4-\partial_2^2)\partial^{\mathbf{m}}\Pi_{x\beta\,t}.\tag{7.5}$$

By the first part of Lemma A.2 we have $\partial^{\mathbf{m}}\Pi_{x\beta\,\tau}\in C^0(\mathbb{H}^p)$, and that $(\partial_1^4-\partial_2^2)\partial^{\mathbf{m}}\Pi_{x\beta\,t}(y)\in\mathbb{H}^p$ is continuous in $(t,y)\in(0,\infty)\times\mathbb{R}^2$. It remains to appeal to the second part of Lemma A.2 with $F(t,y)=(\partial_1^4-\partial_2^2)\partial^{\mathbf{m}}\Pi_{x\beta\,t}$ for $t<\tau$ (and vanishing for $t\geq\tau$) in order to infer that the integral on the r. h. s. of (7.5) is also in $C^0(\mathbb{H}^p)$. For this, we need to verify the near-field part of (A.7). We therefore combine (6.5) and (6.11) to

$$\|\partial^{\mathbf{n}}\partial^{\mathbf{m}}\Pi_{x\beta\,t}(y)\|_{\mathbb{H}^p}\lesssim(\sqrt[4]{\tau})^{-2}(\sqrt[4]{\tau+t}+|y-x|)^{|\beta|-\alpha}(\sqrt[4]{t})^{\alpha-|\mathbf{n}|},$$

which we use for $\mathbf{n}\in\{(4,0),(0,2)\}$. Hence the exponent on $\sqrt[4]{t}$ is $\alpha-4>-4$, so that (A.7) decays as $(N^{-1})^\alpha$ uniformly for bounded $|y-x|$, as desired.

7.4 Algebraic and three-point arguments

For $(\Gamma^*_{xy}P)_\beta\in\mathbb{H}^p$ we appeal to the exponential formula (2.44), and the fact that it involves $\pi^{(\mathbf{n})}_{x\beta'}$ only for $\beta'\prec\beta$, see (8.1). Hence we may appeal to the induction hypothesis Statement 4, and use the Leibniz rule (A.5).

For $\pi^{(\mathbf{n})}_{xy\beta}\in\mathbb{H}^p$ we appeal to the three-point identity (4.16). By the just established $(\Gamma^*_{xy}P)_\beta\in\mathbb{H}^p$ and the previously established $\Pi_x,\Pi_y\in C^0(\mathbb{H}^p)$ we obtain with help of Leibniz rule (A.5) that $\sum_{\mathbf{n}}\pi^{(\mathbf{n})}_{xy}(\cdot-y)^{\mathbf{n}}\in C^0(\mathbb{H}^p)$. Since the latter is a polynomial of order $<|\beta|$, this implies the desired statement on its coefficients.

The Malliavin differentiability of the full row $\Gamma^*_{xy\beta}$ now follows from the identity (2.51) in form of $(\Gamma^*_{xy})^{e_{\mathbf{n}}}_\beta = \pi^{(\mathbf{n})}_{xy\beta}$.

8 Triangular structures and dependencies

We start this section by providing the (elementary) arguments from Sect. 2.2. We first address (2.10). First of all, since $\mathbb{R}[[u]] \ni a \mapsto a(\cdot + v) \in \mathbb{R}[[u]]$ is linear, $v \mapsto \pi[a(\cdot + v)]$ is polynomial for $\pi \in \mathbb{R}[\mathsf{z}_k]$, so that indeed, $\frac{d}{dv}_{|v=0}\pi[a(\cdot + v)]$ is well-defined as a linear form on the algebra $\mathbb{R}[\mathsf{z}_k]$ that satisfies Leibniz' rule. The same is true for the l. h. s. of (2.10), when applied to π and evaluated at a. Hence it is enough to check (2.10) on the coordinates (2.8). Since z_k is a linear functional we learn from (2.8) that $\frac{d}{dv}_{|v=0}\mathsf{z}_k[a(\cdot + v) = \mathsf{z}_k[\frac{da}{du}] = (k+1)\mathsf{z}_{k+1}[a]$, in agreement with the r. h. s. of (2.10).

We now turn to the argument for (2.24). The r. h. s. inclusion is automatically satisfied for the last contribution to (2.18). By (8.3), the β-component of the middle term in (2.18) is only non-vanishing for $[\beta] \geq 0$. Only for the first r. h. s. term in (2.18) we need the additional component in (2.24): It is the linear combination of terms of the form $\mathsf{z}_k \Pi_x^k \partial_1^2 \Pi_x$ where $k \geq 0$. In view of (8.8), its β-component is non-vanishing if either $[\beta] \geq 0$, or if all its constituents are purely polynomial. By (2.21), the latter case corresponds to a space-time polynomial.

We now address (2.32). By the binomial formula, the postulate $(\cdot - y)^{\mathbf{n}}\Gamma_{yx} = (\cdot - x)^{\mathbf{n}}$ translates into $(\cdot)^{\mathbf{n}}\Gamma_{yx} = \sum_{\mathbf{m}} \binom{\mathbf{n}}{\mathbf{m}}\,(y-x)^{\mathbf{n}-\mathbf{m}}(\cdot)^{\mathbf{m}}$, with the understanding that the binomial coefficient vanishes unless $\mathbf{m} \leq \mathbf{n}$ componentwise. By definition (2.15), this implies $\mathsf{z}_{\mathbf{m}}.\Gamma_{yx}(\cdot)^{\mathbf{n}} = \binom{\mathbf{n}}{\mathbf{m}}(y-x)^{\mathbf{n}-\mathbf{m}}$. By duality[86] this implies on the component-wise level $(\Gamma^*_{xy})^{e_{\mathbf{m}}}_{e_{\mathbf{n}}} = \binom{\mathbf{n}}{\mathbf{m}}(y-x)^{\mathbf{n}-\mathbf{m}}$. This yields (2.32).

Finally, we turn to the (strict) triangular structure of $\Gamma^* - \mathrm{id}$ and $\mathrm{d}\Gamma^*$ w. r. t. $|\cdot|$ and $|\cdot|_\prec$, on which the entire induction argument relies on. Equally important are the strict triangular dependencies of Γ^* and $\mathrm{d}\Gamma^*$ on $\pi^{(\mathbf{n})}$ and $\mathrm{d}\pi^{(\mathbf{n})}$, respectively. The same applies to the dependencies of the expressions $\mathsf{z}_k\pi^k\pi'$ and $\pi^k(D^{(\mathbf{0})})^k c$ on π, π', c.

Lemma 8.1 (Triangular dependencies) *For γ not purely polynomial,*

$$(\Gamma^*)^\gamma_\beta \quad \textit{does not depend on} \quad \pi^{(\mathbf{n})}_{\beta'} \quad \textit{unless} \quad \beta' \prec \beta, \tag{8.1}$$

and, for arbitrary γ,

$$(\Gamma^*)^\gamma_\beta \quad \textit{does not depend on} \quad \pi^{(\mathbf{n})}_{\beta'} \quad \textit{unless} \quad \beta' \preccurlyeq \beta. \tag{8.2}$$

For $k \geq 1$, $\pi \in \mathsf{T}^*$ and $c \in \tilde{\mathsf{T}}^*$,

$$\pi^k(D^{(\mathbf{0})})^k c \in \tilde{\mathsf{T}}^*, \tag{8.3}$$

[86] with our abuse of notation.

$$(\pi^k (D^{(\mathbf{0})})^k c)_\beta \quad \textit{does not depend on} \quad \pi_{\beta'},\, c_{\beta'} \quad \textit{unless} \quad \beta' \prec \beta. \tag{8.4}$$

For $k \geq 0$ *and* $\pi^{(1)}, \dots, \pi^{(k+1)} \in \mathsf{T}^*$,

$$(\mathsf{z}_k \pi^{(1)} \cdots \pi^{(k+1)})_\beta = \sum_{e_k + \beta_1 + \cdots + \beta_{k+1} = \beta} \pi^{(1)}_{\beta_1} \cdots \pi^{(k+1)}_{\beta_{k+1}} \tag{8.5}$$

involves only multi-indices $\beta_1, \dots, \beta_{k+1}$ *satisfying*

$$\beta_1, \dots, \beta_{k+1} \prec \beta, \tag{8.6}$$

$$|\beta_1| + \cdots + |\beta_{k+1}| = |\beta|, \tag{8.7}$$

and for $\pi^{(k+1)} \in \tilde{\mathsf{T}}^*$,

$$\mathsf{z}_k \pi^{(1)} \cdots \pi^{(k+1)} \in \tilde{\mathsf{T}}^*. \tag{8.8}$$

Lemma 8.2 (Triangular structures) *For any* $\Gamma^* \in \mathsf{G}^*$,

$$(\Gamma^* - \mathrm{id})_\beta^\gamma \neq 0 \quad \Longrightarrow \quad \gamma \prec \beta \quad \textit{and} \quad |\gamma| < |\beta|, \tag{8.9}$$

$$(\delta\Gamma^*)_\beta^\gamma \neq 0 \quad \Longrightarrow \quad \gamma \prec \beta \quad \textit{and} \quad |\gamma| < |\beta|, \tag{8.10}$$

$$(\mathrm{d}\Gamma^*)_\beta^\gamma \neq 0 \quad \Longrightarrow \quad \gamma \prec \beta \quad \textit{and} \quad |\gamma| \leq |\beta| + 1 - \alpha. \tag{8.11}$$

In particular,

$$(\Gamma^*)_0^\gamma \neq 0 \quad \Longrightarrow \quad \gamma = 0, \tag{8.12}$$

$$(\mathrm{d}\Gamma^*_{xy} - \mathrm{d}\Gamma^*_{xz}\Gamma^*_{zy})_\beta^\gamma \neq 0 \quad \Longrightarrow \quad \gamma \prec \beta. \tag{8.13}$$

We state (and instantly prove) one further property:

$$(\mathrm{d}\Gamma^*_{xz})_\beta^\gamma = 0 \quad \text{for } \beta \in \mathbb{N}_0 e_0 \text{ and } \gamma \text{ not purely polynomial.} \tag{8.14}$$

By definition (4.40), for (8.14) it suffices to show that $(\Gamma^* D^{(\mathbf{n})})_\beta^\gamma$ vanishes for such β, γ. Since by (2.27) and $\alpha \in (0, \frac{1}{2}]$,

$$\beta \in \mathbb{N}_0 e_0 \quad \Longleftrightarrow \quad |\beta| = \min \mathsf{A} \quad \Longleftrightarrow \quad |\beta| < 2\alpha, \tag{8.15}$$

the latter follows from the triangularity of Γ^* w. r. t. $|\cdot|$ and the fact that $(D^{(\mathbf{n})})_\beta^\gamma$ vanishes for such β, γ, see (2.10) and (2.45).

Proof of Lemma 8.1 Proof of (8.1) & (8.2). Indeed, (8.2) is an immediate consequence of (8.1) by (2.51). By (2.44) in its component-wise version, we see that for (8.1) we have to show

$$\pi^{(\mathbf{n}_1)}_{\beta_1} \cdots \pi^{(\mathbf{n}_k)}_{\beta_k} \big(D^{(\mathbf{n}_1)} \cdots D^{(\mathbf{n}_k)}\big)_{\beta_{k+1}}^\gamma \neq 0 \quad \Longrightarrow \quad \beta_1, \dots, \beta_k \prec \beta, \tag{8.16}$$

where $k \geq 1$ and $\beta_1, \ldots, \beta_{k+1}$ satisfy

$$\sum_{k'=1}^{k+1} \beta_{k'} = \beta \quad \text{and} \quad |\mathbf{n}_{k'}| < |\beta_{k'}| \quad \text{for} \quad k' = 1, \ldots, k, \tag{8.17}$$

where the second part comes from (2.46). From the definitions (2.10) and (2.45) we infer

$$(D^{(\mathbf{0})})_\beta^\gamma \neq 0 \implies \Big([\gamma] = [\beta] - 1 \text{ and } |\gamma|_p = |\beta|_p \text{ and } \gamma(0) \leq \beta(0) + 1\Big), \tag{8.18}$$

and for $\mathbf{n} \neq \mathbf{0}$

$$(D^{(\mathbf{n})})_\beta^\gamma \neq 0 \implies \Big([\gamma] = [\beta] - 1 \text{ and } |\gamma|_p = |\beta|_p + |\mathbf{n}| \text{ and } \gamma(0) = \beta(0)\Big). \tag{8.19}$$

This yields by iteration

$$\begin{aligned} &\big(D^{(\mathbf{n}_1)} \cdots D^{(\mathbf{n}_k)}\big)_{\beta_{k+1}}^\gamma \neq 0 \quad \implies \\ &[\gamma] = [\beta_{k+1}] - k \quad \text{and} \quad |\gamma|_p = |\beta_{k+1}|_p + \sum_{k'=1}^{k} |\mathbf{n}_{k'}| \\ &\quad \text{and} \quad \gamma(0) \leq \beta_{k+1}(0) + \sum_{k'=1}^{k} \delta_{\mathbf{0}}^{\mathbf{n}_{k'}}, \end{aligned} \tag{8.20}$$

which by definition (3.2) of $|\cdot|_\prec$ implies

$$\begin{aligned} &\big(D^{(\mathbf{n}_1)} \cdots D^{(\mathbf{n}_k)}\big)_{\beta_{k+1}}^\gamma \neq 0 \\ &\quad \implies \quad |\gamma|_\prec \leq |\beta_{k+1}|_\prec + \sum_{k'=1}^{k} \big(\tfrac{1}{2}|\mathbf{n}_{k'}| + \tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}} - 1\big). \end{aligned} \tag{8.21}$$

Hence by the first item in (8.17) and the additivity of $|\cdot|_\prec$, the l. h. s. of (8.16) yields

$$|\beta|_\prec = \sum_{k'=1}^{k+1} |\beta_{k'}|_\prec \geq |\gamma|_\prec + \sum_{k'=1}^{k} \Big(|\beta_{k'}|_\prec - \big(\tfrac{1}{2}|\mathbf{n}_{k'}| + \tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}} - 1\big)\Big). \tag{8.22}$$

We now argue that each summand in the last term of (8.22) is positive

$$\tfrac{1}{2}|\mathbf{n}_{k'}| + \tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}} - 1 < |\beta_{k'}|_\prec \quad \text{for} \quad k' = 1, \ldots, k. \tag{8.23}$$

Indeed, for $\mathbf{n}_{k'} = \mathbf{0}$ we have as desired

$$\tfrac{1}{2}|\mathbf{n}_{k'}| + \tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}} - 1 = \tfrac{1}{4} - 1 < -\tfrac{1}{2} \overset{(3.5)}{\leq} |\beta_{k'}|_\prec.$$

For $\mathbf{n}_{k'} \neq \mathbf{0}$, we have

$$\tfrac{1}{2}|\mathbf{n}_{k'}| + \tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}} - 1 = \tfrac{1}{2}|\mathbf{n}_{k'}| - 1 \overset{(8.17)}{<} \tfrac{1}{2}|\beta_{k'}| - 1,$$

and conclude by

$$\tfrac{1}{2}|\beta_{k'}|-1 \overset{(2.27)}{=} \tfrac{1}{2}\big(\alpha([\beta_{k'}]+1)+|\beta_{k'}|_p\big)-1$$
$$\overset{(3.5)}{\leq} [\beta_{k'}]+\tfrac{1}{2}|\beta_{k'}|_p+\tfrac{1}{4}\beta_{k'}(0) \overset{(3.2)}{=} |\beta_{k'}|_{\prec}.$$

Using (8.23), we obtain from (8.22)

$$|\beta|_{\prec} \geq |\beta_{k'}|_{\prec}+|\gamma|_{\prec}-(\tfrac{1}{2}|\mathbf{n}_{k'}|+\tfrac{1}{4}\delta_{\mathbf{0}}^{\mathbf{n}_{k'}}-1) \quad \text{for any} \quad k'=1,\dots,k. \tag{8.24}$$

In case $\mathbf{n}_{k'}=\mathbf{0}$, we obtain from (8.24) as desired

$$|\beta|_{\prec} \geq |\beta_{k'}|_{\prec}+|\gamma|_{\prec}-\tfrac{1}{4}+1 \overset{(3.5)}{>} |\beta_{k'}|_{\prec}.$$

In case $\mathbf{n}_{k'}\neq\mathbf{0}$ we obtain from (8.24)

$$|\beta|_{\prec} \geq |\beta_{k'}|_{\prec}+|\gamma|_{\prec}-\tfrac{1}{2}|\mathbf{n}_{k'}|+1 \overset{(3.2)}{=} |\beta_{k'}|_{\prec}+[\gamma]+\tfrac{1}{2}(|\gamma|_p-|\mathbf{n}_{k'}|)+\tfrac{1}{4}\gamma(0)+1.$$

Since $|\gamma|_p-|\mathbf{n}_{k'}|\geq 0$ by (8.20), $[\gamma]\geq 0$ by the assumption that γ is not purely polynomial, and $\gamma(0)\geq 0$, this yields again $|\beta|_{\prec}>|\beta_{k'}|_{\prec}$, which finishes the argument for (8.1).

Proof of (8.3). The β-component of (8.3) equals

$$\sum_{\substack{\beta_1+\dots+\beta_{k+1}=\beta \\ \gamma}} \pi_{\beta_1}\cdots\pi_{\beta_k}((D^{(\mathbf{0})})^k)_{\beta_{k+1}}^{\gamma}\, c_\gamma, \tag{8.25}$$

where by additivity of $[\cdot]$ and the first item of (8.20) the multi-indices are restricted to

$$[\beta]=[\beta_1]+\dots+[\beta_{k+1}]=[\beta_1]+\dots+[\beta_k]+[\gamma]+k.$$

Since $[\cdot]\geq -1$ we obtain $[\beta]\geq[\gamma]$, and by $c\in\tilde{\mathsf{T}}^*$ we have $[\gamma]\geq 0$ which yields as desired $[\beta]\geq 0$.

Proof of (8.4). We use again that the β-component of (8.3) equals (8.25). From (8.20) we obtain $|\beta_{k+1}|_{\prec}\geq|\gamma|_{\prec}+\frac{3}{4}k$, and hence by additivity of $|\cdot|_{\prec}$

$$|\beta|_{\prec}=|\beta_1|_{\prec}+\dots+|\beta_{k+1}|_{\prec}\geq|\beta_1|_{\prec}+\dots+|\beta_k|_{\prec}+|\gamma|_{\prec}+\tfrac{3}{4}k.$$

Since $|\cdot|_{\prec}\geq -1/2$, we obtain for any $\beta'\in\{\beta_1,\dots,\beta_k,\gamma\}$

$$|\beta|_{\prec}\geq|\beta'|_{\prec}+\tfrac{1}{4}k>|\beta'|_{\prec},$$

which finishes the proof of (8.4).

Proof of (8.6) and (8.7). We have to show that

$$e_k+\beta_1+\dots+\beta_{k+1}=\beta \quad \Longrightarrow \quad \begin{cases} \beta_1,\dots,\beta_{k+1}\prec\beta, \\ |\beta_1|+\dots+|\beta_{k+1}|=|\beta|. \end{cases}$$

For the upper item we distinguish $k=0$ from $k\neq 0$. In the first case we have $|\beta|_\prec = |\beta_1|_\prec + \frac{1}{4} > |\beta_1|_\prec$. In the latter one, we have by additivity $|\beta|_\prec = k + |\beta_1|_\prec + \cdots + |\beta_{k+1}|_\prec$, and since $|\cdot|_\prec \geq -1/2$, we obtain $|\beta|_\prec \geq k/2 + |\beta_i|_\prec > |\beta_i|_\prec$ for all $i = 1,\ldots,k+1$. The lower item is an immediate consequence of the definition of $|\cdot|$, cf. (2.27).

Proof of (8.8). The β-component of (8.8) equals (8.5), hence $[\beta] = [e_k] + [\beta_1] + \cdots + [\beta_{k+1}]$. Since $[e_k] = k$, $[\cdot] \geq -1$ and $[\beta_{k+1}] \geq 0$ by $\pi_{k+1} \in \tilde{\mathsf{T}}^*$, see (2.23), we obtain as desired $[\beta] \geq 0$. $\square$

Proof of Lemma 8.2 Proof of (8.9). Recall that $(\Gamma^* - \mathrm{id})_\beta^\gamma$ is a linear combination of terms of the form of the l. h. s. of (8.16), involving multi-indices $\beta_1,\ldots,\beta_{k+1}$ for $k\geq 1$ subject to (8.17). Putting together (8.22) and (8.23), we obtain $\gamma \prec \beta$. From (8.20) and

$$\sum_{k'=1}^{k+1}[\beta_{k'}] = [\beta], \qquad \sum_{k'=1}^{k+1}|\beta_{k'}|_p = |\beta|_p,$$

which is an immediate consequence of (8.17), we see that the l. h. s. of (8.16) implies

$$[\gamma] = [\beta] - \sum_{k'=1}^{k}[\beta_{k'}] - k \quad \text{and} \quad |\gamma|_p < |\beta|_p + \sum_{k'=1}^{k}(|\beta_{k'}| - |\beta_{k'}|_p).$$

This yields

$$|\gamma| < |\beta| + \sum_{k'=1}^{k}\big(-\alpha(1+[\beta_{k'}]) + |\beta_{k'}| - |\beta_{k'}|_p\big) = |\beta|,$$

establishing the last item in (8.9).

Proof of (8.10). This is an immediate consequence of (8.9).

Proof of (8.11). We first note that

$$D_{\gamma'}^{\gamma} \neq 0 \quad \Longrightarrow \quad \gamma \prec \gamma' \quad \text{and} \quad |\gamma| \leq |\gamma'| + 1 - \alpha \quad \text{for} \quad D \in \{D^{(\mathbf{0})}, D^{(1,0)}\},$$

which is an immediate consequence of (8.18) and (8.19). By (8.9) we therefore obtain

$$(\Gamma^* D)_{\beta''}^{\gamma} \neq 0 \quad \Longrightarrow \quad \gamma \prec \beta'' \quad \text{and} \quad |\gamma| \leq |\beta''| + 1 - \alpha \quad \text{for } D \in \{D^{(\mathbf{0})}, D^{(1,0)}\}.$$

To establish the last item in (8.11), we also note that by $\beta' + \beta'' = \beta$ and $|\cdot| \geq \alpha$, we have $|\beta''| = |\beta| - |\beta'| + \alpha \leq |\beta|$, which yields as desired $|\gamma| \leq |\beta| + 1 - \alpha$. The first item in (8.11) follows from $\beta'' \preccurlyeq \beta$, which we shall establish now. From the definition (4.40) of $\mathrm{d}\Gamma_{xz}^*$, we see that $(\mathrm{d}\Gamma_{xz}^*)_\beta^\gamma$ is a linear combination of terms of the form $\mathrm{d}\pi_{xz\beta'}^{(\mathbf{n})}(\Gamma_{xz}^* D^{(\mathbf{n})})_{\beta''}^{\gamma}$ with $\beta' + \beta'' = \beta$ and $\mathbf{n} = \mathbf{0}, (1,0)$. By (4.26) and the first item in (4.41) we see that purely polynomial β' do not contribute, hence $\beta'' \preccurlyeq \beta$ by (3.6). $\square$

Appendix A: Malliavin–Sobolev spaces for random variables and random fields

In this section we recall the definitions of the classical Malliavin–Sobolev spaces and extend it to the case of random fields. We also show that these definitions are stable under the operations used to construct the model, which allows us to prove Malliavin differentiability in Sect. 7.

We start with some notation. We denote by H^* the Hilbert space with norm $\|\cdot\|_*$ given by (2.4). The space of p-integrable random variables is denoted by $\mathbb{L}^p$ and the space of p-integrable random variables with values in H^* by $\mathbb{L}^p(H^*)$. The natural norms on these spaces are given by $\mathbb{E}^{\frac{1}{p}}|\cdot|^p$ and $\mathbb{E}^{\frac{1}{p}}\|\cdot\|_*^p$ respectively.

The classical Malliavin–Sobolev space[87] $\mathbb{H}^p$ is given by the completion of cylindrical functionals (2.5) w. r. t. the stochastic norm

$$\|F\|_{\mathbb{H}^p} := \mathbb{E}^{\frac{1}{p}}\left(|F|^p + \|\frac{\partial F}{\partial \xi}\|_*^p\right).$$

Since by Assumption 2.1 the Malliavin derivate $\frac{\partial}{\partial \xi}$ is closable from $\mathbb{L}^2$ to $\mathbb{L}^2(H^*)$, it is also closable from $\mathbb{L}^p$ to $\mathbb{L}^2(H^*)$ for $p \geq 2$, and naturally extends to a bounded linear operator from $\mathbb{L}^p$ to $\mathbb{L}^p(H^*)$. Therefore, if a sequence of cylindrical functionals $\{F_N\}_N$ of the form (2.5) converges to F in $\mathbb{L}^p$ and it is Cauchy w. r. t. $\|\cdot\|_{\mathbb{H}^p}$, then $F \in \mathbb{H}^p$ and $\frac{\partial F}{\partial \xi} = \lim_{N\uparrow\infty} \frac{\partial F_N}{\partial \xi}$ in $\mathbb{L}^p(H^*)$.

Since we deal with random functionals of the noise which are continuous in an annealed sense, cf. (2.42), it is convenient to work with the space $C^0(\mathbb{H}^p)$. We stress that by $C^0(\mathbb{H}^p)$ we do not mean the Banach space endowed with the norm $\sup_{y\in\mathbb{R}^2}\|F(y)\|_{\mathbb{H}^p}$, but the linear space of continuous (and possibly unbounded) functions from $\mathbb{R}^2$ with values in $\mathbb{H}^p$, endowed with the topology given by the family of semi-norms

$$\sup_{y\in K}\|F(y)\|_{\mathbb{H}^p} \quad \text{for all } K \subset \mathbb{R}^2 \text{ compact.} \tag{A.1}$$

Clearly $C^0(\mathbb{H}^p)$ has the property

$$C^0(\mathbb{H}^p) \ni F \mapsto F(y) \in \mathbb{H}^p \quad \text{is bounded for all } y \in \mathbb{R}^2, \tag{A.2}$$

and for a compactly supported continuous function ψ,[88]

$$C^0(\mathbb{H}^p) \ni F \mapsto \psi * F \in C^0(\mathbb{H}^p) \quad \text{is continuous,} \tag{A.3}$$

see Lemma A.2 for a refinement. The latter statement follows from the fact that functionals F of the form

$$F[\xi](y) = \bar{F}\big(y; (\xi, \zeta_1), \ldots, (\xi, \zeta_N)\big), \tag{A.4}$$

[87] see [54, Sect. 1.2] for the Gaussian case.

[88] with the understanding that this linear map is well-defined.

where $N \in \mathbb{N}$, $\bar{F}$ is a smooth function on $\mathbb{R}^2 \times \mathbb{R}^N$ and $\zeta_1, \ldots, \zeta_N$ are Schwartz functions, are dense in $C^0(\mathbb{H}^p)$ w. r. t. the semi-norms (A.1) and (A.3) preserves their form and is bounded under (A.1). Here comes the argument for density: By Cantor's diagonal sequence argument, it is enough to establish the approximation for fixed K in (A.1). Because K is compact and $F \in C^0(K; \mathbb{H}^p)$, given $\delta > 0$, there exist finitely many open sets $U_1, \ldots, U_M \subset \mathbb{R}^2$ covering K such that the oscillation of F on each U_m w. r. t. $\| \cdot \|_{\mathbb{H}^p}$ is $\leq \delta$. For $m = 1, \ldots, M$, pick an $x_m \in U_m$. Since $F(x_m) \in \mathbb{H}^p$ there exists a cylindrical functional F_m of the form (2.6), such that $\|F(x_m) - F_m\|_{\mathbb{H}^p} \leq \delta$. Pick a smooth partition of unity $\eta_1, \ldots, \eta_M \geq 0$ subordinate to the covering $\{U_i\}_{i=1}^M$. Then $\tilde{F}(x) = \sum_{m=1}^M \eta_m(x) F_m$ is of the form (A.4) and we have $\sup_{x \in K} \|F(x) - \tilde{F}(x)\|_{\mathbb{H}^p} \leq 2\delta$.

Appealing to closability we have

Remark A.1 If a sequence $\{F_N\}_N$ in $C^0(\mathbb{H}^p)$ converges to a random field F, pointwise in y w. r. t. $\mathbb{E}^{\frac{1}{p}} |\cdot|^p$, and is a Cauchy sequence w. r. t. the family of semi-norms (A.1), then $F \in C^0(\mathbb{H}^p)$ with $\frac{\partial F(y)}{\partial \xi} = \lim_{N \uparrow \infty} \frac{\partial F_N(y)}{\partial \xi}$. Furthermore, convergence of $\{F_N\}_N$ to F takes place w. r. t. these semi-norms.

For reconstruction, and also the algebraic and the three-point argument, we need Leibniz rule, meaning that the two bilinear maps

$$\left\{ \begin{array}{c} \mathbb{H}^{p_1} \times \mathbb{H}^{p_2} \ni (F_1, F_2) \mapsto F_1 F_2 \in \mathbb{H}^p \\ C^0(\mathbb{H}^{p_1}) \times C^0(\mathbb{H}^{p_2}) \ni (F_1, F_2) \mapsto F_1 F_2 \in C^0(\mathbb{H}^p) \end{array} \right\} \text{ are continuous}$$
$$\text{provided } \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \tag{A.5}$$

where the relation between the exponents is dictated by Hölder's inequality (in probability). This follows from the fact that the maps (A.5) preserve (2.6) and (A.4), respectively, and that the latter is bounded as a bilinear map under (A.1).

The refinement of (A.3) we need in the integration step is given by the following lemma,

Lemma A.2 *Suppose for some $\theta > 0$ that $F \in C^0(\mathbb{H}^p)$ satisfies*

$$\sup_y \frac{1}{1 + |y|^\theta} \|F(y)\|_{\mathbb{H}^p} < \infty. \tag{A.6}$$

Then we have $\psi * F \in C^0(\mathbb{H}^p)$ for any Schwartz kernel ψ, with the natural formula for the Malliavin derivative.

Moreover, suppose that $(0, \infty) \times \mathbb{R}^2 \ni (t, y) \mapsto F(t, y) \in \mathbb{H}^p$ is continuous and satisfies

$$\lim_{N \uparrow \infty} \int_{(0, N^{-1}) \cup (N, \infty)} dt \, \|F(t, y)\|_{\mathbb{H}^p} = 0 \quad \textit{locally uniformly in } y. \tag{A.7}$$

Then $\int_0^\infty dt \, F(t, \cdot) \in C^0(\mathbb{H}^p)$ with the natural formula for the Malliavin derivative.

Proof We start by addressing the first statement in the lemma. To this purpose, we select a sequence ψ_N of compactly supported kernels that approximate ψ in Schwartz space. According to (A.3), we have $\psi_N * F \in C^0(\mathbb{H}^p)$. By Remark A.1 it suffices to argue that $\psi_N * F$ is a Cauchy sequence in $C^0(\mathbb{H}^p)$. Indeed, using the triangle inequality in $C^0(\mathbb{H}^p)$ in form of

$$\begin{aligned}&\|(\psi_N * F - \psi_{N'} * F)(y)\|_{\mathbb{H}^p}\\ &\quad\le \sup_{z\in\mathbb{R}^2} \frac{1}{1+|z|^\theta}\|F(z)\|_{\mathbb{H}^p} \int dz\,|(\psi_N - \psi_{N'})(y-z)|(1+|z|^\theta),\end{aligned}$$

we see that (A.6) is sufficient to convert the convergence of the kernels in Schwartz space to the Cauchy criterion w. r. t. the family of semi-norms (A.1).

We now turn to the second statement in this lemma. By Remark A.1, it is enough to establish that $\int_{N-1}^{N} dt\, F(t,\cdot)$, which automatically is an element of $C^0(\mathbb{H}^p)$, is in fact a Cauchy sequence in $C^0(\mathbb{H}^p)$. This is precisely what assumption (A.7) ensures. □

Appendix B: Summary of the logical order of the proof

To assist the reader, here we include a detailed description of the inductive proof according to the steps outlined in Sect. 3.4, both for regular (Table 4) and singular (Table 5) multi-indices. We include all the statements, their inputs and outputs.

Table 4 Logic of an induction step for regular β. The attentive reader may have noticed that (2.40)$_{\prec\beta}$ is contained in several hypotheses, but (2.40)$_\beta$ is not part of any output; the reason is that (2.40)$_\beta$ trivially holds by the population constraint (2.34) for these multi-indices

Step	Statement	Input	Output
1	Proposition 4.1	(2.55)$_{\prec\beta}$	(2.37)$_\beta^{\gamma\neq\text{p.p.}}$
2. a)	Proposition 2.3	(2.36)$_{\prec\beta}$, (2.37)$_{\prec\beta}$, (2.40)$_{\prec\beta}$, (2.41)$_{\prec\beta}$, (2.42)$_{\prec\beta}$, (2.61)$_{\prec\beta}$	(2.43)$_\beta$
2. b)	Sect. 5.1	(5.1)$_{\prec\beta}$, (5.2)$_{\prec\beta}$	(5.3)$_\beta$, (5.4)$_\beta$
2. c)	Proposition 5.2	(2.61)$_{\prec\beta}$	(2.60)$_\beta$, (2.62)$_\beta$
3	Proposition 4.2	(2.36)$_{\prec\beta}$, (2.37)$_\beta^{\gamma\neq\text{p.p.}}$, (2.42)$_{\prec\beta}$, (2.43)$_\beta$, (2.55)$_{\prec\beta}$, (2.62)$_\beta$, (2.64)$_{\prec\beta}$	(2.64)$_\beta$
4. a)	Proposition 4.3	(2.64)$_\beta$	(2.36)$_\beta$, (4.11)$_\beta$
4. b)	Sect. 5.2	(4.11)$_\beta$, (5.3)$_\beta$, (5.4)$_\beta$	(5.1)$_\beta$, (5.2)$_\beta$
5. a)	Proposition 5.3	(2.36)$_{\preccurlyeq\beta}$, (2.37)$_\beta^{\gamma\neq\text{p.p.}}$, (2.60)$_\beta$	(2.61)$_\beta$
5. b)	Proposition 5.4	(2.61)$_{\preccurlyeq\beta}$, (5.10)$_{\prec\beta}$	(2.31)$_\beta$, (5.10)$_\beta$
5. c)	Proposition 4.4	(2.36)$_{\preccurlyeq\beta}$, (2.37)$_\beta^{\gamma\neq\text{p.p.}}$, (2.61)$_\beta$	(2.55)$_\beta$
	Proposition 4.5	(2.55)$_{\preccurlyeq\beta}$	(2.37)$_\beta$
6	Proposition 2.3	(2.36)$_{\preccurlyeq\beta}$, (2.37)$_\beta$, (2.40)$_{\prec\beta}$, (2.41)$_{\prec\beta}$, (2.61)$_\beta$, (2.64)$_\beta$, (6.5)$_{\prec\beta}$	(2.41)$_\beta$, (2.42)$_\beta$, (6.5)$_\beta$

Table 5 Logic of an induction step for singular β. Steps 1b, 4a, 7a, 7c and 9a, involving only constructions or Malliavin differentiability, have been omitted

Step	Statement	Input	Output
1. a)	Proposition 4.1	$(2.55)_{\prec\beta}$	$(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$
1. c)	Proposition 4.8	$(2.37)_{\preccurlyeq\beta}^{\gamma\neq\text{p.p.}}$, $(4.25)_{\prec\beta}$	$(4.27)_{\beta}^{\gamma\neq\text{p.p.}}$
1. d)	Proposition 4.16	$(2.37)_{\preccurlyeq\beta}^{\gamma\neq\text{p.p.}}$, $(4.34)_{\prec\beta}$, $(4.107)_{\prec\beta}$	$(4.108)_{\beta}^{\gamma\neq\text{p.p.}}$
1. e)	Proposition 4.11	$(2.37)_{\preccurlyeq\beta}^{\gamma\neq\text{p.p.}}$, $(4.44)_{\prec\beta}$, $(5.10)_{\prec\beta}$	$(4.47)_{\beta}^{\gamma\neq\text{p.p.}}$
2. a)	Proposition 5.2	$(2.61)_{\prec\beta}$	$(2.60)_{\beta}$, $(2.62)_{\beta}$
2. b)	Sect. 5.1	$(5.1)_{\prec\beta}$, $(5.2)_{\prec\beta}$	$(5.3)_{\beta}$, $(5.4)_{\beta}$
2. c)	Proposition 4.6	$(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.64)_{\prec\beta}$, $(5.3)_{\beta}$	$(4.18)_{\beta}$
3. a)	Proposition 5.1	$(4.18)_{\beta}$, $(5.3)_{\beta}$, $(5.4)_{\beta}$	$(2.38)_{\beta}$
3. b)	Proposition 2.3	$(2.36)_{\prec\beta}$, $(2.38)_{\beta}$, $(2.40)_{\prec\beta}$, $(2.41)_{\prec\beta}$, $(4.18)_{\beta}$	$(2.40)_{\beta}$
3. c)	Proposition 4.7	$(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.38)_{\beta}$, $(2.60)_{\beta}$, $(2.64)_{\prec\beta}$, $(4.18)_{\beta}$	$(4.20)_{\beta}$
4. b)	Proposition 4.13	$(2.36)_{\prec\beta}$, $(2.37)_{\prec\beta}$, $(2.40)_{\prec\beta}$, $(2.41)_{\prec\beta}$, $(2.42)_{\prec\beta}$, $(2.61)_{\prec\beta}$, $(4.27)_{\prec\beta}$, $(4.34)_{\prec\beta}$, $(4.53)_{\prec\beta}$, $(4.54)_{\prec\beta}$	$(4.55)_{\beta}$
4. c)	Proposition 4.12	$(2.36)_{\prec\beta}$, $(2.37)_{\prec\beta}$, $(2.42)_{\prec\beta}$, $(2.43)_{\prec\beta}$, $(2.60)_{\prec\beta}$, $(2.61)_{\prec\beta}$, $(2.64)_{\prec\beta}$, $(4.47)_{\preccurlyeq\beta}^{\gamma\neq\text{p.p.}}$, $(4.54)_{\prec\beta}$, $(4.55)_{\beta}$, $(4.89)_{\prec\beta}$, $(4.108)_{\preccurlyeq\beta}^{\gamma\neq\text{p.p.}}$	$(4.52)_{\beta}$
4. d)	Proposition 4.18	$(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.60)_{\beta}$, $(2.64)_{\prec\beta}$, $(4.22)_{\prec\beta}$, $(4.27)_{\beta}^{\gamma\neq\text{p.p.}}$, $(4.52)_{\beta}$, $(4.108)_{\beta}^{\gamma\neq\text{p.p.}}$	$(4.22)_{\beta}$
5	SG inequality	$(4.20)_{\beta}$, $(4.22)_{\beta}$	$(2.64)_{\beta}$
6. a)	Proposition 4.3	$(2.64)_{\beta}$	$(2.36)_{\beta}$, $(4.11)_{\beta}$
6. b)	Sect. 5.2	$(4.11)_{\beta}$, $(5.3)_{\beta}$, $(5.4)_{\beta}$	$(5.1)_{\beta}$, $(5.2)_{\beta}$
6. c)	Proposition 5.3	$(2.36)_{\preccurlyeq\beta}$, $(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.60)_{\beta}$	$(2.61)_{\beta}$
6. d)	Proposition 5.4	$(2.61)_{\preccurlyeq\beta}$, $(5.10)_{\prec\beta}$	$(2.31)_{\beta}$, $(5.10)_{\beta}$
6. e)	Proposition 4.4 Proposition 4.5	$(2.36)_{\preccurlyeq\beta}$, $(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.61)_{\beta}$ $(2.55)_{\preccurlyeq\beta}$	$(2.55)_{\beta}$ $(2.37)_{\beta}$
7. b)	Proposition 4.9	$(4.22)_{\beta}$	$(4.34)_{\beta}$
7. d)	Proposition 4.10	$(2.36)_{\prec\beta}$, $(2.37)_{\beta}^{\gamma\neq\text{p.p.}}$, $(2.61)_{\beta}$, $(4.27)_{\beta}^{\gamma\neq\text{p.p.}}$, $(4.34)_{\preccurlyeq\beta}$	$(4.25)_{\beta}$, $(4.27)_{\beta}$
8. a)	Proposition 2.3	$(2.36)_{\prec\beta}$, $(2.37)_{\prec\beta}$, $(2.40)_{\prec\beta}$, $(2.41)_{\prec\beta}$, $(2.42)_{\prec\beta}$, $(2.61)_{\prec\beta}$	$(2.43)_{\beta}$
8. b)	Proposition 2.3	$(2.36)_{\preccurlyeq\beta}$, $(2.37)_{\beta}$, $(2.40)_{\prec\beta}$, $(2.41)_{\prec\beta}$, $(2.61)_{\beta}$, $(2.64)_{\beta}$ $(6.5)_{\prec\beta}$	$(2.41)_{\beta}$, $(2.42)_{\beta}$, $(6.5)_{\beta}$
8. c)	Proposition 4.13	$(2.36)_{\prec\beta}$, $(2.40)_{\prec\beta}$, $(2.41)_{\prec\beta}$, $(2.61)_{\beta}$, $(4.34)_{\preccurlyeq\beta}$, $(4.53)_{\prec\beta}$, $(6.11)_{\prec\beta}$	$(4.53)_{\beta}$, $(4.54)_{\beta}$, $(6.11)_{\beta}$
9. b)	Proposition 4.14	$(2.36)_{\prec\beta}$, $(2.64)_{\prec\beta}$, $(4.22)_{\beta}$, $(4.34)_{\beta}$, $(4.52)_{\beta}$, $(4.108)_{\beta}^{\gamma\neq\text{p.p.}}$	$(4.88)_{\beta}$, $(4.89)_{\beta}$
9. c)	Proposition 4.15	$(2.36)_{\prec\beta}$, $(2.37)_{\prec\beta}^{\gamma\neq\text{p.p.}}$, $(2.55)_{\prec\beta}$, $(2.61)_{\prec\beta}$, $(4.47)_{\beta}^{\gamma\neq\text{p.p.}}$, $(4.89)_{\beta}$, $(4.108)_{\beta}^{\gamma\neq\text{p.p.}}$	$(4.44)_{\beta}$
9. d)	Proposition 4.17	$(2.36)_{\prec\beta}$, $(4.34)_{\beta}$, $(4.89)_{\beta}$, $(4.108)_{\beta}^{\gamma\neq\text{p.p.}}$	$(4.107)_{\beta}$, $(4.108)_{\beta}$

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