Introduction

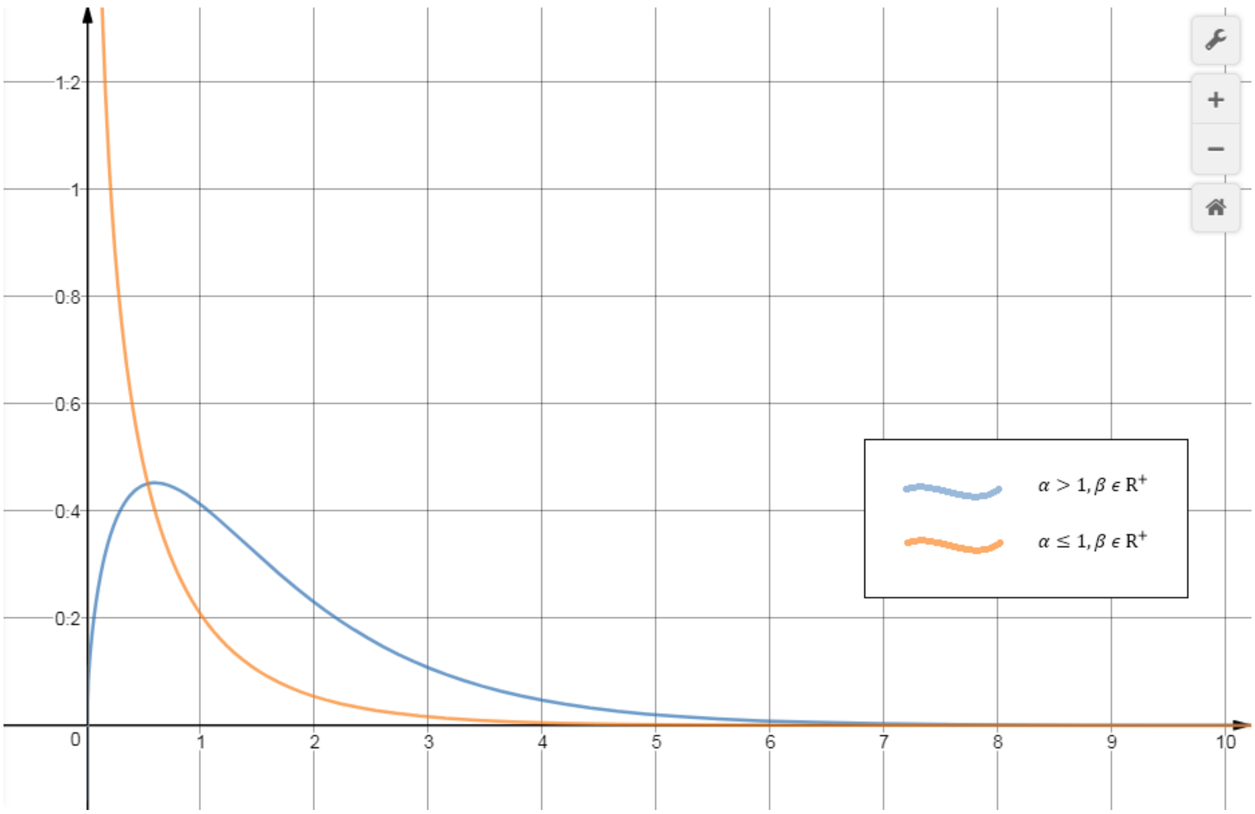
We need to generate a random sample from the gamma distribution.

› Characteristics of Gamma distribution

Let X be a random variable such that $X \sim Gamma(\alpha, \beta)$. The probability density function of the gamma distribution is as follows:

$$f_X(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} I_{\{x \geq 0\}}$$

where $\alpha \in \mathfrak{R}^+$ is the shape parameter and $\beta \in \mathfrak{R}^+$ is the scale parameter. The density when plotted looks like the following.



The cumulative distribution function of the gamma distribution is

$$F_X(x; \alpha, \beta) = P(X \leq x) = \frac{1}{\Gamma(\alpha)} \gamma(\alpha, \beta x) I_{\{x \geq 0\}}$$

where $\gamma(s, x) = \int_0^x t^{s-1} e^{-t} dt$.

› Problems with random sample generation

The simplest method for generation of random sample from a random variable $X \sim F_X(x)$ is by using the probability inverse transform which uses the fact that for any distribution, the random variable $Z = F_X(X)$ follows a uniform(0,1) distribution.

Proof: $G_Z(z) = P[Z \leq z] = P[F_X(X) \leq z] = P[X \leq F_X^{-1}(z)] = F_X\left(F_X^{-1}(z)\right) = z$

which is the cdf of uniform(0,1).

Therefore it is easy to generate a random sample u from uniform(0,1) and the transformation $x = F_X^{-1}(u)$ gives a random sample from F_X.

The function $F_X^{-1}(.)$ does not always have a closed form such as in case of a gamma distribution and thus the probability inverse transform method cannot be used to directly generate a random sample from the gamma distribution.

› Overview

In this report, a number of different methods presently used to generate samples from gamma distribution are presented and in the end a new method for generation is proposed.

Methods

We provide a few commonly used algorithms for generation of a random sample from distributions with references in case the reader is interested in further theory and explanations.

Acceptance-Rejection Method (Ross) [1]

Suppose we have $f(x)$ and the associated cdf $F(x)$. We try to find a pdf $g(x)$ from which we can generate easily and $f(x) \le cg(x),\ \forall x, c \ge 1$.

Steps:

1. Repeat:
   - Generate $Y \sim g(.)$
   - Generate U uniformly from [0,1]

   Until: $U \le \frac{f(Y)}{cg(Y)}$
2. Return Y.

Ratio of Uniform Method (Kinderman and Manmohan, 1977) [2]

Let $A = \left\{(u, v): 0 \le u \le \sqrt{f\left(\frac{u}{v}\right)}\right\}$, where $f \ge 0$ is an integrable function. If (U, V) is a random vector uniformly distributed over A, then $\frac{U}{V}$ has density $\frac{1}{c}f$ where $c = \int A$.

We can generate easily unform random number of U and V. So, it is sufficient to enclose area A by a simple set such as a rectangle. If and only if $f(x)$ and $x^2 f(x)$ are bounded, A can be closed in a rectangle. Assume that $A \subseteq [0, b] \times [a_-, a_+]$ for some finite constants $b \ge 0$, $a_- \le 0$ and $a_+ \ge 0$.

Steps:

1. Set up
   Compute b, a_- and a_+ for an enclosing rectangle, where

   $b \ge sup\sqrt{f(x)}$
   $a_- \le inf\ x\sqrt{f(x)}$ and
   $a_+ \ge sup\ x\sqrt{f(x)}$

2. Generator
   Repeat: Generate U uniformly on $[0, b]$, and V uniformly on $[a_-, a_+]$.

   $$X = \frac{V}{U}$$

   Until: $U^2 \le f(X)$

3. Return X.

D. Kundu and R. D. Gupta's Method using the Generalized Exponential Distribution (Kundu and Gupta) [3]

Set $d = 1.0334 - 0.0766e^{2.2942\alpha}$

$a = 2^{\alpha}\left(1 - e^{-\frac{d}{2}}\right)^{\alpha}$

$b = \alpha d^{\alpha-1}e^{-d}$

$c = a + b.$

Steps:

1. Generate U from uniform $(0, 1)$.
2. If U $\leq \frac{a}{a+b}$, then $= -2\ln\left[1 - \frac{(cU)^{1/\alpha}}{2}\right]$, otherwise $X = -\ln\left[\frac{c(1 - U)}{\alpha d^{\alpha-1}}\right]$.
3. Generate V from uniform $(0, 1)$.

   If $X \leq d$, check whether $V \leq \frac{X^{\alpha-1}e^{-X/2}}{2^{\alpha-1}\left(1-e^{-X/2}\right)^{\alpha-1}}$

   If true return X, otherwise go back to 1.

   If $X > d$, check whether $V \leq \left(\frac{d}{X}\right)^{1-\alpha}$.

   If true return X, otherwise go back to 1.

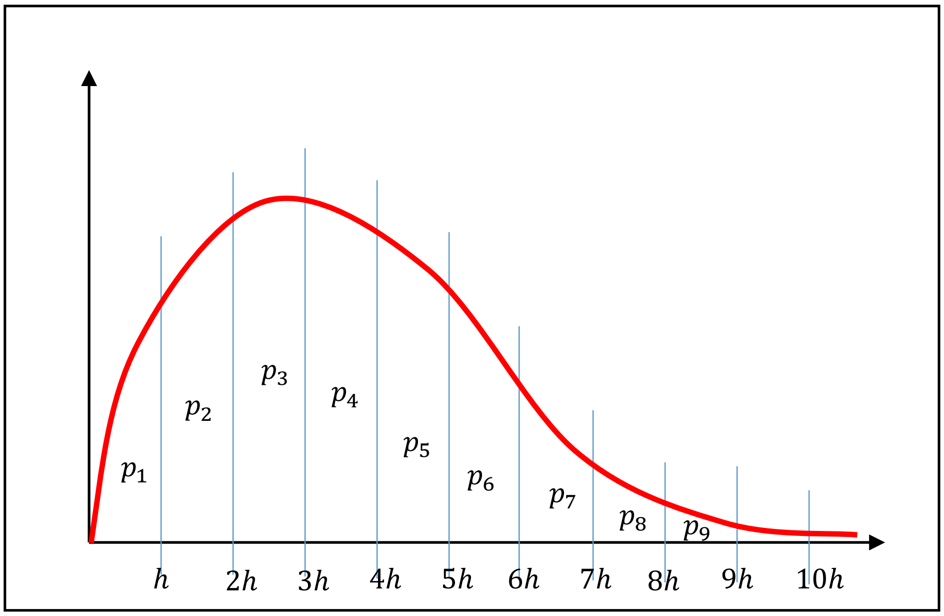
Proposed method

Overview

We partition the support $\mathfrak{X} \epsilon \mathfrak{R}$ into a number k of intervals of equal length $h > 0$. Smaller the value of h, better the results should be. It should be noted that the whole of the support $\mathfrak{X} = [0, \infty)$ cannot be partitioned since it is infinite. However, we can certainly choose k such that the probability of an observation to come from beyond the k intervals is less than ε for any $\varepsilon > 0$, however small. That is, mathematically

$$P(X > kh) < \varepsilon \text{ for any } \varepsilon > 0$$

since $\lim_{\delta \to \infty} P(|X| \leq \delta) = 1$ and X is a positive random variable.
Now, let the probability of an observation to lie in the i^{th} partition be p_i.



Clearly, from the diagram, we can see that $p_1 = F(h) - F(0), p_2 = F(2h) - F(h)$ and so on.

In general, $p_i = F(ih) - F(\{i-1\}h)$.

Now, for each interval, say $(x_1, x_2]$, we substitute $f(x)$ with a linear function $l(x) = \alpha + \beta x$ such that the area under both the curves $f(x)$ and $l(x)$ remains the same in the interval $(x_1, x_2]$. That is, mathematically

$$\int_{x_1}^{x_2} f(x)dx = \int_{x_1}^{x_2} l(x)dx = \int_{x_1}^{x_2} (\alpha + \beta x)\, dx$$

Note that $l(x)$ is the probability density-mass function of a mixed type probability distribution.

$$\int_0^\infty l(x)dx = \int_0^{x_1} l(x)dx + \int_{x_1}^{x_2} l(x)dx + \cdots = \int_0^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots = \int_0^\infty f(x)dx = 1$$

How to choose α and β?

We would like to choose α and β such that they minimize the overall squared (or absolute) deviation of $l(x)$ from $f(x)$ for each interval $(x_1, x_2]$. That is,

$$\frac{\partial}{\partial \alpha}\int_{x_1}^{x_2}\{f(x)-(\alpha+\beta x)\}^2 dx = 0$$

$$\frac{\partial}{\partial \beta}\int_{x_1}^{x_2}\{f(x)-(\alpha+\beta x)\}^2 dx = 0$$

$$\Rightarrow \begin{cases} \displaystyle\int_{x_1}^{x_2} f(x)dx = \alpha(x_2-x_1)+\frac{\beta}{2}(x_2^2-x_1^2) \\ \displaystyle\int_{x_1}^{x_2} xf(x)dx = \frac{\alpha}{2}(x_2^2-x_1^2)+\frac{\beta}{3}(x_2^3-x_1^3) \end{cases}$$

The values of α and β, respectively $\hat{\alpha}$ and $\hat{\beta}$, that satisfy the above system, minimizes the deviation.

Let us set $\boldsymbol{\hat{\beta}} = \frac{\boldsymbol{f(x_2)-f(x_1)}}{\boldsymbol{x_2-x_1}}$. Then,

$$\int_{x_1}^{x_2} f(x)dx = \alpha(x_2-x_1)+\frac{\beta}{2}(x_2^2-x_1^2)$$

$$\Rightarrow \frac{F(x_2)-F(x_1)}{x_2-x_1} = \alpha+\frac{\beta}{2}(x_2+x_1)$$

$$\Rightarrow \hat{\alpha} = \frac{F(x_2)-F(x_1)}{x_2-x_1}-\frac{\hat{\beta}}{2}(x_2+x_1)$$

$$\Rightarrow \boldsymbol{\hat{\alpha}} = \boldsymbol{\frac{F(x_2)-F(x_1)}{x_2-x_1}-\frac{1}{2}\frac{f(x_2)-f(x_1)}{x_2-x_1}(x_2+x_1)}$$

So, we obtain the mixed probability density-mass function $g(x)$

$$g(x) = \begin{cases} \alpha_0+\beta_0 x; & x\epsilon(x_0,x_1) \\ \alpha_1+\beta_1 x; & x\epsilon(x_1,x_2) \\ \dots \end{cases}$$

and the mixed distribution function $G(x)$

$$G(x) = \begin{cases} \alpha_0(x-x_0)+\frac{\beta_0}{2}(x^2-x_0^2); & x\epsilon(x_0,x_1) \\ G(x_1)+\alpha_1(x-x_1)+\frac{\beta_0}{2}(x^2-x_1^2); & x\epsilon(x_1,x_2) \\ \dots \end{cases}$$

Note that, at the edges x_i of each interval, $G(x_i) = F(x_i)$, $\forall i = 0,1,2 \dots$ by construction.

How to find (or approximate) the gamma cdf $F(x)$?

Now, since $F(x)$ has a lower incomplete gamma functional form, we cannot compute the value of the integral directly. Here, we provide a few methods of approximating the value of the integral.

The first method is using simulation [1]. We have to find the value of

$$F(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^x u^{\alpha-1} e^{-\beta u} du$$

$$= \frac{(\beta x)^\alpha}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} e^{-\beta x t} dt \text{, substituting } t = \frac{u}{x}$$

Now, $\int_0^1 t^{\alpha-1} e^{-\beta x t} dt = E[g(Y)]$ where $Y \sim U(0,1)$ and $g(Y) = y^{\alpha-1} e^{-\beta x y}$.

If we have a random sample of size n viz $Y_{1,}\ Y_2 \dots Y_n$, then by the SLLN we have,

$$\frac{1}{n} \sum_{i=1}^{n} Y_i^{\alpha-1} e^{-\beta x Y_i} \ \rightarrow\ E[g(y)]$$

The larger is n, the closer we can get to the actual value of the integral.

A series expansion of $\gamma(s,x)$ can be obtained by plugging the Taylor series of the exponential function e^{-t} and integrating each term separately (Abramowitz and Stegun, 1970) [4].

$$\gamma(s,x) = x^s \sum_{i=0}^{\infty} (-1)^i \frac{x^i}{i!\,(i+s)}$$

Another representation can be obtained from the above by separating the series representation of e^{-t} and solving for the series coefficients (Press et al., 1993) [5]

$$\gamma(s,x) = e^{-x} x^s \sum_{i=0}^{\infty} \frac{x^i}{(s+i)(s+i-1)\dots s}$$

It should be noted that we can sum upto n terms such that n is large enough since $x\ \in (0,1)$ and $\lim_{i\to\infty} x^i = 0$.

How to generate a random sample?

Now, we draw a random number u_1 from $U(0,1)$. And then we find an integer m such that

$$\sum_{i=1}^{m-1} p_i < u_1 \leq \sum_{i=1}^{m} p_i$$

We assume $\sum_1^0 p_i = 0$.

We choose the partition with the interval $(\{m-1\}h, mh]$. Let us denote it by $(x_{m-1}, x_m]$.

Essentially, $u_1 = G(x)$ for some $x \,\epsilon\, \boldsymbol{\mathfrak{X}}$. Our aim is to find x.

We have, $F(x_{m-1}) < u_1 < F(x_m)$

$$\Rightarrow u_1 = F(x_{m-1}) + \alpha_{m-1}(x - x_{m-1}) + \frac{\beta_{m-1}}{2}(x^2 - x_{m-1}^2)$$

$$\Rightarrow \frac{\beta_{m-1}}{2}x^2 + \alpha_{m-1}x + \left\{F(x_{m-1}) - \alpha_{m-1}x_{m-1} - \frac{\beta_{m-1}}{2}x_{m-1}^2 - u_1\right\} = 0$$

$$\Rightarrow x = \frac{-\alpha_{m-1} \pm \sqrt{\alpha_{m-1}^2 - 4 \times \left\{\frac{\beta_{m-1}}{2}\right\} \times \{F(x_{m-1}) - \alpha_{m-1}x_{m-1} - \frac{\beta_{m-1}}{2}x_{m-1}^2 - u_1\}}}{\beta_{m-1}} \qquad (*)$$

Note that $x > 0$. Therefore, we will consider the positive or negative square root in the formula above accordingly.

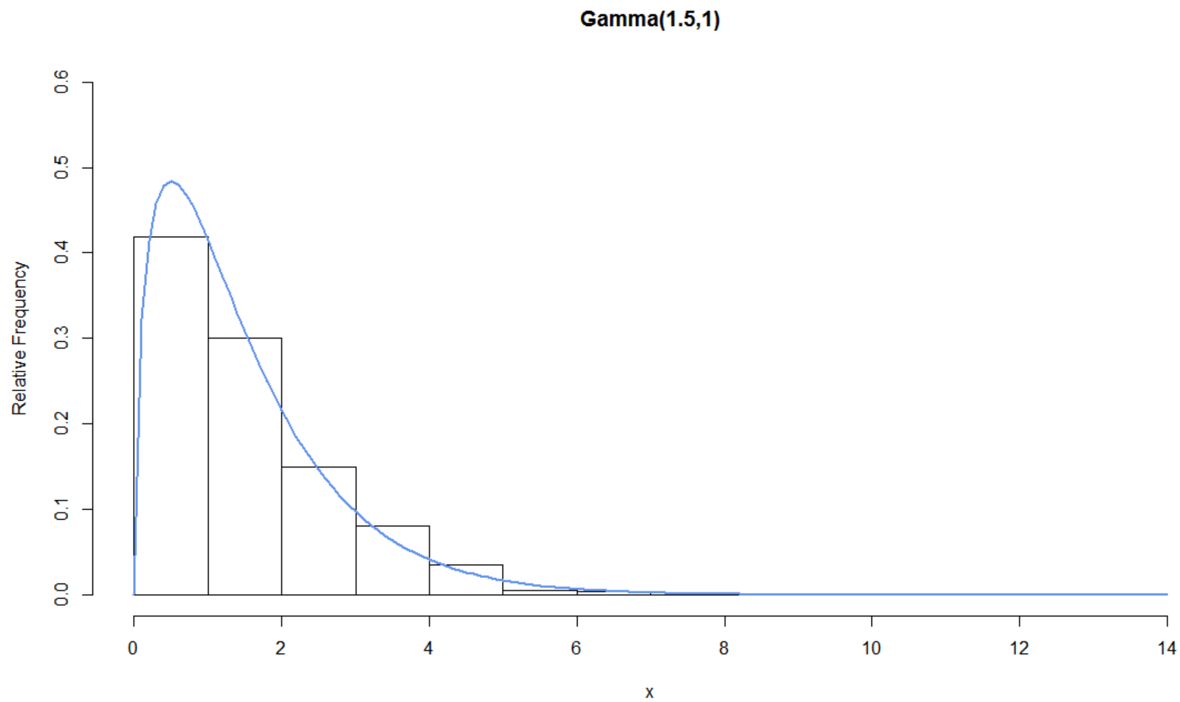
Remark. If $X \sim F_X, Y \sim G_Y$ (G is as defined earlier.) Then by construction, $Y \to X$ as $h \to 0$.

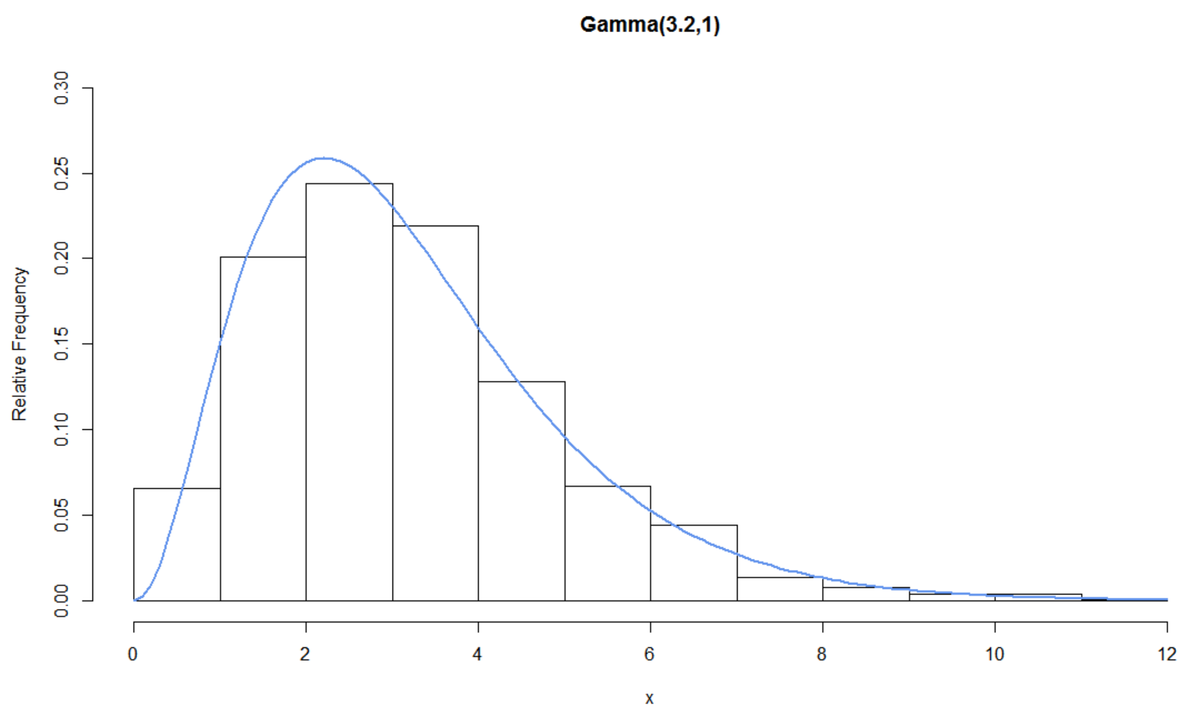
Steps:

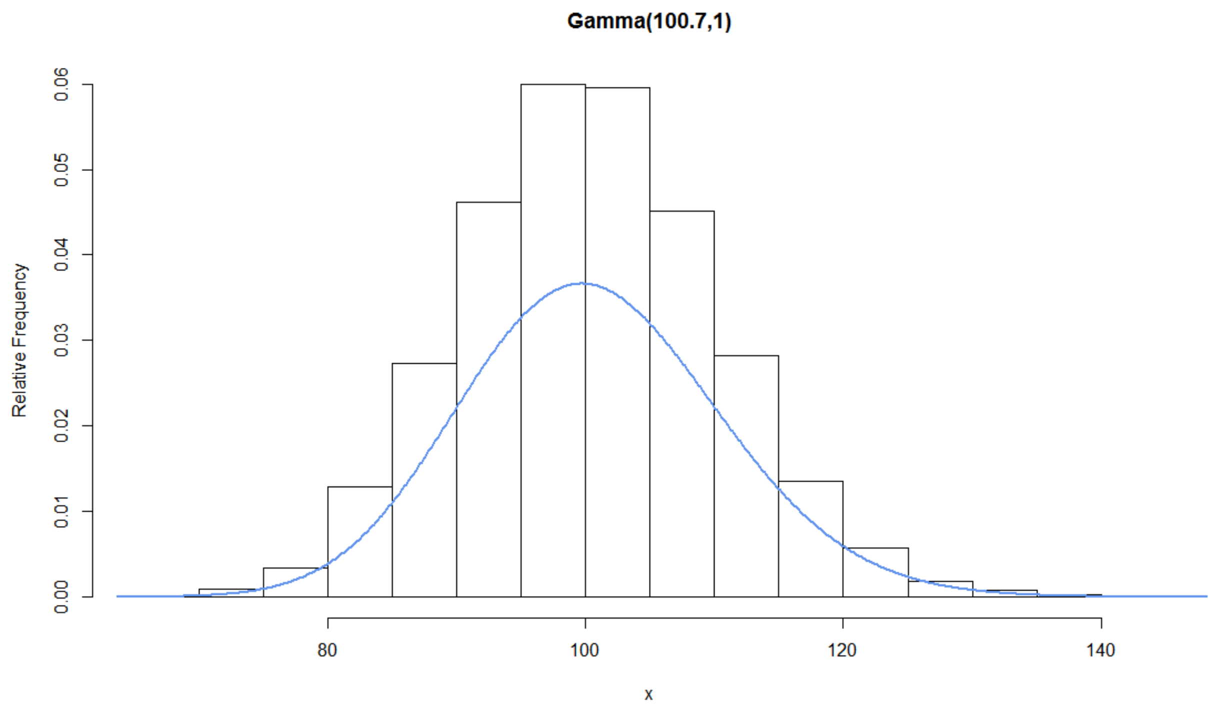
1. Generate u_1 from $U(0,1)$
2. Find m such that $\sum_{i=1}^{m-1} p_i < u_1 \leq \sum_{i=1}^{m} p_i$
3. Obtain α_{m-1} and β_{m-1} based on the chosen interval $(x_{m-1}, x_m]$
4. Return x as obtained from $(*)$.

Numerical Results

We provide histograms and the probability density function on top of it for the proposed method.






Gamma(100.7,1)
Relative Frequency
0.00
0.01
0.02
0.03
0.04
0.05
0.06
80
100
120
140
x

Conclusion

The proposed method generates a sample that fits the distribution seemingly well for small values of α.

But, we have in case of α being large (viz. 100.7) the random sample shows a higher frequency than expected.

This difference tends to reduce as we generate samples of larger sizes (say, n=10000 instead of 1000). However, there is an immense scope for further improvement of the method so as to make it more efficient. One such example of a scope would be that we might consider $l(x)$ being a polynomial function instead of a linear function to substitute the pdf $f(x)$.

References

[1] Sheldon M. Ross (2006). "*Simulation", 4th edition*, Elsevier Academic Press, UK

[2] Kinderman and Manmohan (1977)

[3] Kundu, D. and Gupta, R.D: "*A Convenient Way of Generating Gamma Random Variables using Generalised Exponential Distribution*"

[4] Abramowitz, M. and Stegun, I. (1970). "*Handbook of Mathematical Functions", 9-Revised edition*, Dover Publications, USA

[5] Press, W.H, Flannery, B.P, Teukolsky, S.A & Vetterling, W.T (1989). *Numerical Recipes in Pascal: The Art of Scientific Computing*. Cambridge University Press, New York.