Graph Theory: Isomorphism, Eulerian and Hamiltonian Walks

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Adjacency and Incidence Matrices

Adjacency Matrix: The adjacency matrix A of a graph with n vertices is an $n \times n$ matrix where $A[i][j] = 1$ if there is an edge between vertex i and vertex j, otherwise $A[i][j] = 0$.

Example: For the graph G_1:

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Incidence Matrix: The incidence matrix I of a graph with n vertices and m edges is an $n \times m$ matrix where $I[i][j] = 1$ if vertex i is incident to edge j, otherwise $I[i][j] = 0$.

Example of Incidence Matrix

Consider the graph G_1 with edges $E = \{e_1, e_2, e_3, e_4, e_5\}$:

$$I = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

This matrix shows how each vertex connects to the edges.

Understanding Incidence Matrices

Definition: An incidence matrix represents the relationship between vertices and edges in a graph.

Structure:

- Rows represent vertices.
- Columns represent edges.
- Entry (i, j) is 1 if vertex i is incident to edge j, otherwise 0.

Example: Consider a graph with $V = \{A, B, C, D\}$ and edges $E = \{e_1, e_2, e_3, e_4, e_5\}$, its incidence matrix is:

$$I = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Isomorphism and Incidence Matrices

Definition: Two graphs are isomorphic if there exists a one-to-one mapping between their vertices and edges that preserves adjacency.

Properties:

- If two graphs are isomorphic, their incidence matrices differ only by permutations of rows (vertices) and columns (edges).
- Permuting rows corresponds to relabeling vertices.
- Permuting columns corresponds to relabeling edges.

Steps to Prove Isomorphism

Step 1: Construct Incidence Matrices
Create the incidence matrices for both graphs.
Step 2: Check for Permutation
Determine if one matrix can be transformed into the other by permuting its rows and columns.
Step 3: Conclusion

- If a permutation exists, the graphs are isomorphic.
- If no such permutation exists, the graphs are not isomorphic.

Example: Graph Isomorphism Using Incidence Matrices

Given two graphs:

- Graph G_1 with vertices $V_1 = \{A, B, C, D\}$ and edges $E_1 = \{e_1, e_2, e_3, e_4, e_5\}$.
- Graph G_2 with vertices $V_2 = \{1, 2, 3, 4\}$ and edges $E_2 = \{f_1, f_2, f_3, f_4, f_5\}$.

Their incidence matrices are:

$$I_1 = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}, \quad I_2 = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Conclusion: Since I_2 is a row and column permutation of I_1, the graphs are isomorphic.

Conclusion

- Incidence matrices help in understanding the structure of graphs.
- Isomorphic graphs have incidence matrices that can be obtained through row and column permutations.
- Verifying isomorphism using incidence matrices is a useful method in graph theory.

Definition: Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be *isomorphic* if there exists a bijection $f : V_1 \to V_2$ such that $(u, v) \in E_1$ if and only if $(f(u), f(v)) \in E_2$.

Example: The following two graphs are isomorphic:

- G_1: $V_1 = \{A, B, C, D\}, E_1 = \{(A, B), (B, C), (C, D), (D, A), (A, C)\}$
- G_2: $V_2 = \{1, 2, 3, 4\}, E_2 = \{(1, 2), (2, 3), (3, 4), (4, 1), (1, 3)\}$

Eulerian Walks

Definition: An *Eulerian walk* in a graph is a walk that visits every edge exactly once.

Eulerian Circuit: If an Eulerian walk starts and ends at the same vertex, it is called an *Eulerian circuit*.

Theorem: A connected graph has an Eulerian circuit if and only if every vertex has an even degree.

Example of Eulerian Walk

Consider the graph:

- $V = \{A, B, C, D, E\}$
- $E = \{(A, B), (B, C), (C, D), (D, E), (E, A), (A, C), (C, E)\}$

The Eulerian walk for this graph is:
$A \to B \to C \to A \to E \to C \to D \to E$.

Hamiltonian Walks

Definition: A *Hamiltonian walk* in a graph is a walk that visits every vertex exactly once.

Hamiltonian Cycle: If a Hamiltonian walk starts and ends at the same vertex, it is called a *Hamiltonian cycle*.

Example: The complete graph K_4 has a Hamiltonian cycle: $A \to B \to C \to D \to A$.

Problems

Problem 1: Determine whether the following graphs are isomorphic:

- G_1: $V_1 = \{A, B, C, D\}, E_1 = \{(A, B), (B, C), (C, D), (D, A), (A, C)\}$
- G_2: $V_2 = \{1, 2, 3, 4\}, E_2 = \{(1, 2), (2, 3), (3, 4), (4, 1), (1, 3)\}$

Solution: The graphs have the same number of vertices and edges and the same connectivity pattern. Therefore, they are isomorphic.

Eulerian Path Problem

Problem 2: Does the following graph have an Eulerian path?

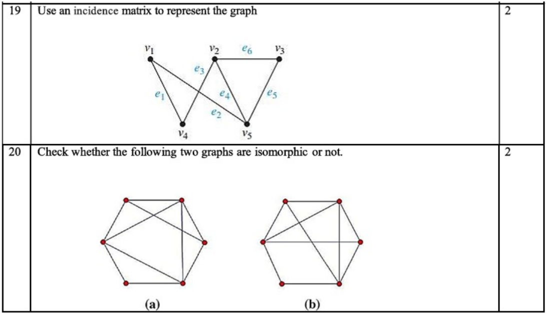
- Graph with degrees:
  $deg(A) = 2, deg(B) = 3, deg(C) = 3, deg(D) = 2$

Solution: A graph has an Eulerian path if and only if it has exactly 0 or 2 vertices of odd degree. Here, B and C have odd degrees, so the graph has an Eulerian path.

Conclusion

- Graph isomorphism helps identify structurally similar graphs.
- Eulerian walks visit every edge once, while Hamiltonian walks visit every vertex once.
- Graph theory has many real-world applications, including network routing and circuit design.

Problem: Use an incidence matrix to represent the given graph.



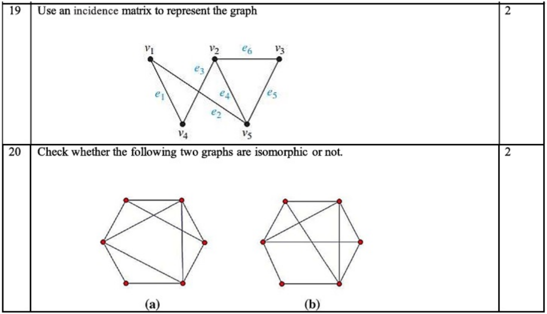
19	Use an incidence matrix to represent the graph	2
20	Check whether the following two graphs are isomorphic or not.	2

Solution: The incidence matrix for the graph with vertices $\{v_1, v_2, v_3, v_4, v_5\}$ and edges $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ is:

$$I = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Checking Graph Isomorphism

Problem: Check whether the following two graphs are isomorphic.



19	Use an incidence matrix to represent the graph	2
20	Check whether the following two graphs are isomorphic or not.	2

Solution:

- Count the number of vertices and edges in both graphs.
- Compare their degree sequences.
- Construct adjacency matrices and check for similarity through row/column permutations.

Since both graphs have the same degree sequences and adjacency matrix up to permutations, they are isomorphic.

Conclusion

- Incidence matrices help in representing graphs in a structured manner.
- Graph isomorphism can be checked using degree sequences and adjacency matrices.
- This approach is useful in graph theory applications such as network analysis.

Question 13: Prove that C_{2n+1} is not Bipartite

Definition: A graph is bipartite if and only if it does not contain an odd cycle.

Proof:

- C_{2n+1} is an odd cycle.
- Attempt to 2-color the vertices alternately.
- The last vertex must be different from the first, leading to a contradiction.

Conclusion: Since C_{2n+1} contains an odd cycle, it is not bipartite.

Question 14: Graph Isomorphism

Definition: Two graphs are isomorphic if there exists a bijection between vertex sets preserving adjacency.

- Both graphs have the same number of vertices and edges.
- Degree sequences are identical.
- A mapping between corresponding vertices preserves adjacency.

Conclusion: The two graphs are isomorphic.

Question 15: Euler Graph Theorem

Theorem: A connected graph is Eulerian if and only if every vertex has an even degree.

- ($\Rightarrow$) If a graph has an Eulerian circuit, every vertex must have an even degree (entering and leaving balance).
- ($\Leftarrow$) If every vertex has an even degree, we can construct an Eulerian circuit using Euler's theorem.

Conclusion: A connected graph is Eulerian if and only if all vertices have even degrees.

Cyclic Graph: Definition and Example

Definition: A cyclic graph is a graph that consists of at least one cycle, where a vertex is reachable from itself.

Cycle Graph C_n: A special case where n vertices are connected in a closed chain, and each vertex has exactly two neighbors.

Example: Cycle Graph C_5

- Vertices: $V = \{v_1, v_2, v_3, v_4, v_5\}$
- Edges: $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_5), (v_5, v_1)\}$

Properties:

- Each vertex in C_n has degree 2.
- A cycle of even length is bipartite, but a cycle of odd length is not.
- The smallest cycle graph is C_3, also called a triangle.

Hamiltonian Path

Definition: A Hamiltonian path is a path in a graph that visits each vertex exactly once.

Properties:

- A Hamiltonian path does not necessarily form a cycle.
- A graph may have multiple Hamiltonian paths or none.
- Finding a Hamiltonian path is NP-complete.

Question 3: Checking for a Hamiltonian Path

Given Graph:

- Vertices: a, b, c, d, e.
- Edges: $(a, b), (a, e), (e, d), (d, c), (b, c), (b, d)$.

Finding a Hamiltonian Path:

- Possible path: $a \to b \to d \to c \to e$.
- Visits all vertices exactly once.

Conclusion: The graph has a Hamiltonian path.

Question 4: Checking for a Hamiltonian Path

Given Graph:

- Vertices: a, b, c, d, e, f.
- Edges: $(a, c), (b, c), (c, f), (f, d), (f, e)$.

Checking for a Hamiltonian Path:

- Trying different paths leads to either repetition or missing vertices.
- The presence of bottleneck vertices (c and f) disrupts a Hamiltonian path.

Conclusion: The graph does not have a Hamiltonian path.