Solution for Assignment 1
PCCCS401

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Problem 1

Explain the physical meaning of the cartesian product $A \times B \times C$, where A is the set of all airlines and B and C are both the set of all cities in the United States? Give an example of how this Cartesian product can be used. (5)

Solution: The Cartesian product $A \times B \times C$ represents all possible ordered triples where the first element is an airline, the second element is a city of departure, and the third element is a city of arrival.

- **Physical Meaning:** It represents all possible flight routes operated by airlines in the United States, where an airline flies from one city to another.
- **Example:** If $A = \{\text{Delta}, \text{American}\}$, $B = \{\text{New York}, \text{Los Angeles}\}$, and $C = \{\text{Chicago}, \text{Miami}\}$, then $A \times B \times C$ would include triples like (Delta, New York, Chicago), (American, Los Angeles, Miami), etc. This can be used to model flight schedules or routes.

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Problem 2

What does the cartesian product $A \times B$ indicate? where, A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university? Give an example of how this Cartesian product can be used. (5)

Solution: The Cartesian product $A \times B$ represents all possible ordered pairs where the first element is a course and the second element is a professor.

- **Physical Meaning:** It represents all possible assignments of courses to professors in the mathematics department.

- **Example:** If $A = \{\text{Calculus}, \text{Algebra}\}$ and $B = \{\text{Dr. Smith}, \text{Dr. Jones}\}$, then $A \times B$ would include pairs like (Calculus, Dr. Smith), (Algebra, Dr. Jones), etc. This can be used to assign professors to courses.

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Problem 3

Let R be an equivalence relation on a set of positive integers defined as xRy if and only if $x \equiv y \pmod 3$. Then, find the equivalence class of 2 and also find the partition generated by the equivalence relation. (5)

Solution:

- **Equivalence Class of 2:** The equivalence class of 2 consists of all integers that are congruent to 2 modulo 3. Thus, the equivalence class of 2 is:

$$[2] = \{2, 5, 8, 11, \dots\}$$

- **Partition Generated by R:** The equivalence relation partitions the set of positive integers into three equivalence classes based on their remainder when divided by 3:

$$\{[0], [1], [2]\}$$

  where:

$$[0] = \{3, 6, 9, \dots\}, \quad [1] = \{1, 4, 7, \dots\}, \quad [2] = \{2, 5, 8, \dots\}$$

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Problem 4

If m and n are two positive integers such that $m = qn + r$, where $0 \leq r < n$. Then prove that $\gcd(m, n) = \gcd(n, r)$. (5)

Solution:

- **Proof:** Let $d = \gcd(m, n)$. Since d divides both m and n, it must also divide $r = m - qn$. Therefore, d is a common divisor of n and r, so $d \leq \gcd(n, r)$.

  Conversely, let $d' = \gcd(n, r)$. Since d' divides both n and r, it must also divide $m = qn + r$. Therefore, d' is a common divisor of m and n, so $d' \leq \gcd(m, n)$.

  Combining both results, we conclude that $\gcd(m, n) = \gcd(n, r)$.

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Problem 5

State and prove division algorithm. (10)
Solution:

- **Division Algorithm:** For any integers a and b with $b > 0$, there exist unique integers q and r such that:

$$a = bq + r \quad \text{where} \quad 0 \leq r < b$$

- **Proof:**

1. **Existence:** Consider the set $S = \{a - bk \mid k \in \mathbb{Z}, a - bk \geq 0\}$. By the well-ordering principle, S has a smallest element, say $r = a - bq$. Clearly, $r \geq 0$. If $r \geq b$, then $r - b = a - b(q + 1)$ would be a smaller non-negative element in S, contradicting the minimality of r. Thus, $0 \leq r < b$.
2. **Uniqueness:** Suppose there are two pairs (q, r) and (q', r') such that $a = bq + r$ and $a = bq' + r'$ with $0 \leq r, r' < b$. Then $b(q - q') = r' - r$. Since $|r' - r| < b$, the only solution is $q = q'$ and $r = r'$.

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Problem 6

Let $A = \{1, 2, 3, 4\}$. Let R be a relation on A defined as $R = \{(a, b) : a + b > 4\}$. Draw the graph of the relation R. (5)
Solution:

- **Relation R:** The relation R consists of all pairs (a, b) where $a + b > 4$. For $A = \{1, 2, 3, 4\}$, the relation R is:

$$R = \{(1, 4), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)\}$$

- **Graph:** The graph can be drawn with elements of A as nodes and directed edges from a to b for each $(a, b) \in R$.

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Problem 7

Define an equivalence relation. Determine whether the relation R on a set of integers $\mathbb{Z}$ defined as: xRy if and only if $x - y = 0$, is equivalence or not. (5)
Solution:

- **Equivalence Relation:** A relation R on a set A is an equivalence relation if it is reflexive, symmetric, and transitive.
- **Check for R:**
  1. **Reflexive:** $x - x = 0$, so xRx holds for all $x \in \mathbb{Z}$.
  2. **Symmetric:** If xRy, then $x - y = 0$, which implies $y - x = 0$, so yRx.
  3. **Transitive:** If xRy and yRz, then $x - y = 0$ and $y - z = 0$, so $x - z = 0$, which implies xRz.

  Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

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Problem 8

When a number is divided by 36, it leaves a remainder of 19. What will be the remainder when the number is divided by 12? (2)
Solution:

- Let the number be N. Given $N \equiv 19 \pmod{36}$, we can write $N = 36k+19$ for some integer k.
- When N is divided by 12:

$$N = 36k + 19 \equiv 0 + 19 \pmod{12} \equiv 7 \pmod{12}$$

- Thus, the remainder when N is divided by 12 is **7**.

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Problem 9

Using Euclidean algorithm, find gcd of 315 and 4235 and also find integers x and y such that $\gcd(315, 4235) = x \cdot 315 + y \cdot 4235$. (10)
Solution:

- **Step 1:** Apply the Euclidean algorithm:

$$4235 = 13 \times 315 + 140$$

$$315 = 2 \times 140 + 35$$

$$140 = 4 \times 35 + 0$$

  The gcd is **35**.

- **Step 2:** Express 35 as a linear combination of 315 and 4235:

$$35 = 315 - 2 \times 140$$

$$140 = 4235 - 13 \times 315$$

Substituting back:

$$35 = 315 - 2 \times (4235 - 13 \times 315) = 27 \times 315 - 2 \times 4235$$

Thus, $x = 27$ and $y = -2$.

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Problem 10

In a survey of 60 people, it is found that 25 read Newsweek magazine, 26 read Time, 26 read Fortune, 9 read both Newsweek and Fortune, 11 read both Newsweek and Time, 8 read both Time and Fortune, 3 read all three. Find the number of people reading (a) At least one of the three newspapers (b) Only one newspaper. [5+5]

Solution:

- **Given:**
  - Total people = 60
  - Newsweek (N) = 25
  - Time (T) = 26
  - Fortune (F) = 26
  - N ∩ F = 9
  - N ∩ T = 11
  - T ∩ F = 8
  - N ∩T ∩ F = 3
- **Part (a): At least one of the three newspapers** Using the principle of inclusion-exclusion:

$$|N \cup T \cup F| = |N| + |T| + |F| - |N \cap T| - |N \cap F| - |T \cap F| + |N \cap T \cap F|$$

$$|N \cup T \cup F| = 25 + 26 + 26 - 11 - 9 - 8 + 3 = 52$$

So, **52** people read at least one of the three newspapers.

- **Part (b): Only one newspaper**
  - Only Newsweek: $|N| - |N \cap T| - |N \cap F| + |N \cap T \cap F| = 25 - 11 - 9 + 3 = 8$

- Only Time: $|T|-|N\cap T|-|T\cap F|+|N\cap T\cap F| = 26-11-8+3 = 10$
- Only Fortune: $|F|-|N\cap F|-|T\cap F|+|N\cap T\cap F| = 26-9-8+3 = 12$
- Total: $8+10+12 = 30$

So, **30** people read only one newspaper.

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