Day 1

Discrete Mathematics

Sets, Relations, and Functions

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Reference book for this material is

Rosen, K. H., & Krithivasan, K. (1999). *Discrete mathematics and its applications* (Vol. 6). New York: McGraw-hill.

Set theory

A set is an unordered collection of objects, called elements or members of the set. A set is said to contain its elements. We write $a \in A$ to denote that a is an element of the set A. The notation $a \in A$ denotes that a is not an element of the set A.

EXAMPLE 1 The set V of all vowels in the English alphabet can be written as $V = \{a, e, i, o, u\}$.

EXAMPLE 2 The set O of odd positive integers less than 10 can be expressed by $O = \{1, 3, 5, 7, 9\}$.

Set theory

$N = \{0, 1, 2, 3,...\}$, the set of natural numbers

$Z = \{..., -2, -1, 0, 1, 2,...\}$, the set of integers

$Z+ = \{1, 2, 3,...\}$, the set of positive integers

$Q = \{p/q \mid p \in Z, q \in Z, \text{and } q = 0\}$, the set of rational numbers

R, the set of real numbers

R+, the set of positive real numbers

C, the set of complex numbers.

Set theory

Two sets are equal if and only if they have the same elements. Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$. We write $A = B$ if A and B are equal sets.

The sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal, because they have the same elements. Note that the order in which the elements of a set are listed does not matter.

THE EMPTY SET There is a special set that has no elements. This set is called the empty set, or null set, and is denoted by $\emptyset$. The empty set can also be denoted by $\{\ \}$.

Set theory

The set A is a subset of B if and only if every element of A is also an element of B. We use the notation $A \subseteq B$ to indicate that A is a subset of the set B.

Note that to show that A is not a subset of B we need only find one element $x \in A$ with $x \notin B$. Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.

The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10, the set of rational numbers is a subset of the set of real numbers, the set of all computer science majors at your school is a subset of the set of all students at your school, and the set of all people

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Set theory

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

Let S be a set. If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is a finite set and that n is the cardinality of S. The cardinality of S is denoted by $|S|$.

Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by P(S).

What is the power set of the set {0, 1, 2}?

Solution: The power set P({0, 1, 2}) is the set of all subsets of {0, 1, 2}. Hence, $P(\{0, 1, 2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$. Note that the empty set and the set itself are members of this set of subsets.

Set operations

Union

Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

$$A \cup B = \{x \mid x \in A \lor x \in B\}.$$

The union of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 2, 3, 5\}$; that is, $\{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$.

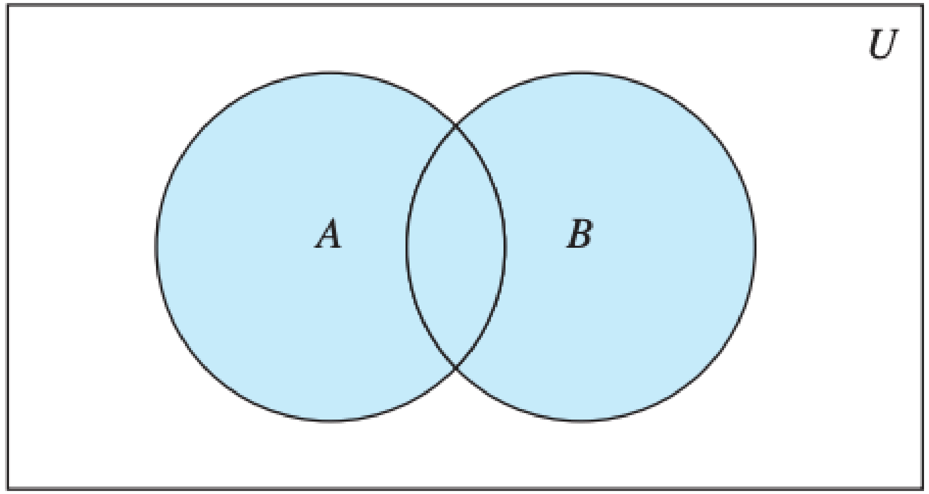
Intersection

Let A and B be sets. The intersection of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.

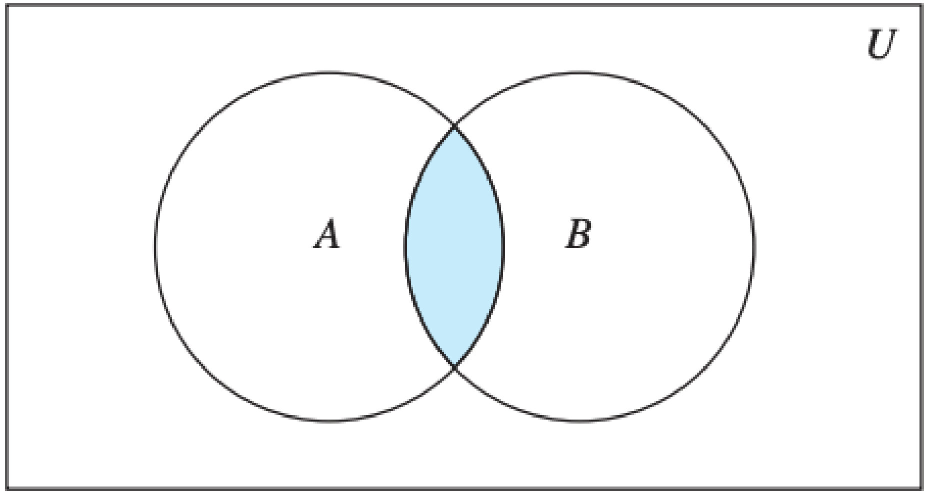
$$A \cap B = \{x \mid x \in A \land x \in B\}.$$

The intersection of the sets $\{1, 3, 5\}$ and $\{1, 2, 3\}$ is the set $\{1, 3\}$; that is, $\{1, 3, 5\} \cap \{1, 2, 3\} = \{1, 3\}$.

Set operations



$A \cup B$ is shaded.



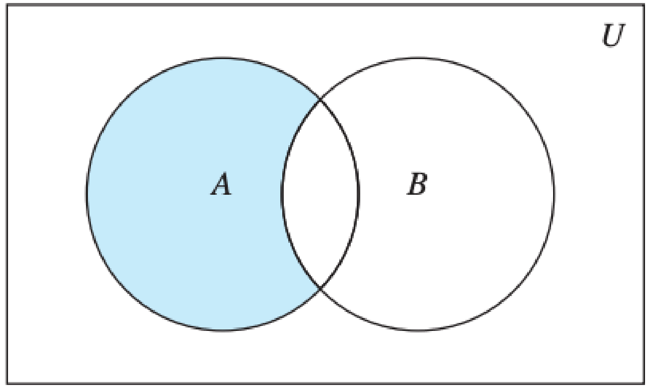
$A \cap B$ is shaded.

Set operations

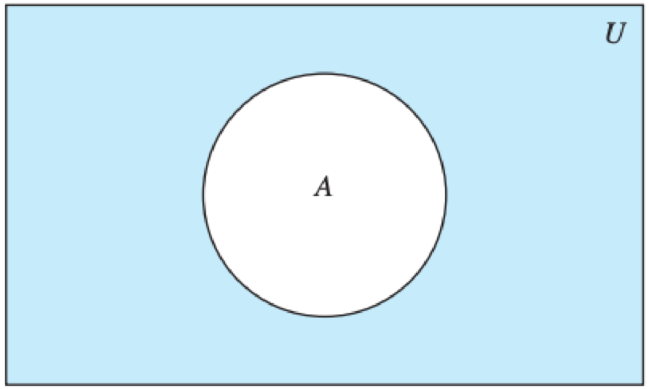
Let A and B be sets. The difference of A and B, denoted by A − B, is the set containing those elements that are in A but not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \ \land x \notin B\}$$

The difference of {1, 3, 5} and {1, 2, 3} is the set {5}; that is, {1, 3, 5}−{1, 2, 3}={5}. This is different from the difference of {1, 2, 3} and {1, 3, 5}, which is the set {2}.



$A - B$ is shaded.



$\bar{A}$ is shaded.

Note: The difference of sets A and B is sometimes denoted by A\B.

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Set operations

Let U be the universal set. The complement of the set A, denoted by A, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

An element belongs to A if and only if x /∈ A. This tells us that A = {x ∈ U | x ∉ A}.

Let A = {a, e, i, o, u} (where the universal set is the set of letters of the English alphabet). Then A = {b, c, d, f, g, h, j, k, l, m, n, p, q, r, s, t, v, w, x, y, z}.

Set identities

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set identities

Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

We can prove this identity with the following steps.

$\overline{A \cap B} = \{x \mid x \notin A \cap B\}$	by definition of complement
$= \{x \mid \neg(x \in (A \cap B))\}$	by definition of does not belong symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by the first De Morgan law for logical equivalences
$= \{x \mid x \notin A \vee x \notin B\}$	by definition of does not belong symbol
$= \{x \mid x \in \overline{A} \vee x \in \overline{B}\}$.	by definition of complement
$= \{x \mid x \in \overline{A} \cup \overline{B}\}$	by definition of union
$= \overline{A} \cup \overline{B}$	by meaning of set builder notation

Cartesian product

The ordered n-tuple (a1, a2,...,an) is the ordered collection that has a1 as its first element, a2 as its second element,..., and an as its nth element.

- We say that two ordered n-tuples are equal if and only if each corresponding pair of their elements is equal.

- In other words, (a1, a2,...,an) = (b1, b2,...,bn) if and only if ai = bi, for i = 1, 2,...,n. In particular, ordered 2-tuples are called ordered pairs. The ordered pairs (a, b) and (c, d) are equal if and only if a = c and b = d.

- Note that (a, b) and (b, a) are not equal unless a = b.

Cartesian product

Let A and B be sets. The Cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, $A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.

What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

Cartesian product

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What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?

Solution: The Cartesian product $A \times B$ is $A \times B = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

$B \times A = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}$.

This is not equal to $A \times B$

Cartesian product

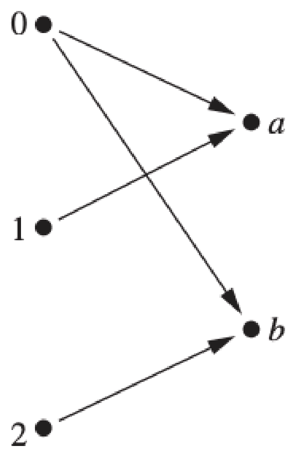
The Cartesian product of the sets A1, A2,...,An, denoted by $A1 \times A2 \times \cdots \times An$, is the set of ordered n-tuples (a1, a2,...,an), where ai belongs to Ai for i = 1, 2,...,n. In other words, $A1 \times A2 \times \cdots \times An = \{(a1, a2,...,an) \mid ai \in Ai \text{ for } i = 1, 2,...,n\}$.

Relation

Let A and B be sets. A binary relation from A to B is a subset of A × B.

We use the notation aRb to denote that (a, b) ∈ R. Moreover, when (a, b) belongs to R, a is said to be related to b by R.

Let A = {0, 1, 2} and B = {a, b}. Then {(0, a), (0, b), (1, a), (2, b)} is a relation from A to B. This means, for instance, that 0 R a, but that 1 not R b.



R	*a*	*b*
0	×	×
1	×	
2		×

Relation

A relation on a set A is a relation from A to A.

Let A be the set {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a divides b}?

Solution: Because (a, b) is in R if and only if a and b are positive integers not exceeding 4 such that a divides b, we see that R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4)}.

Relation properties

A relation R on a set A is called reflexive if (a, a) ∈ R for every element a ∈ A.

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1, 1), (1, 2), (2, 1), (2, 2), (3, 4), (4, 1), (4, 4)\},$$
$$R_2 = \{(1, 1), (1, 2), (2, 1)\},$$
$$R_3 = \{(1, 1), (1, 2), (1, 4), (2, 1), (2, 2), (3, 3), (4, 1), (4, 4)\},$$
$$R_4 = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\},$$
$$R_5 = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 3), (3, 4), (4, 4)\},$$
$$R_6 = \{(3, 4)\}.$$

Which of these relations are reflexive?

Solution: The relations R3 and R5 are reflexive because they both contain all pairs of the form (a, a), namely, (1, 1), (2, 2), (3, 3), and (4, 4). The other relations are not reflexive because they do not contain all of these ordered pairs. In particular, R1, R2, R4, and R6 are not reflexive because (3, 3) is not in any of these relations.

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Relation properties

A relation R on a set A is called symmetric if(b, a) ∈ R whenever(a, b) ∈ R, for all a, b ∈ A. A relation R on a set A such that for all a, b ∈ A, if (a, b) ∈ R and (b, a) ∈ R, then a = b is called antisymmetric.

Consider these relations on the set of integers:

$$R_1 = \{(a, b) \mid a \le b\},$$
$$R_2 = \{(a, b) \mid a > b\},$$
$$R_3 = \{(a, b) \mid a = b \text{ or } a = -b\},$$
$$R_4 = \{(a, b) \mid a = b\},$$
$$R_5 = \{(a, b) \mid a = b + 1\},$$
$$R_6 = \{(a, b) \mid a + b \le 3\}.$$

Which of the relations from are symmetric and which are antisymmetric?

The relations R3, R4, and R6 are symmetric.

R1, R2, R4, and R5 are antisymmetric.

Relation properties

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Which of the relations in Example of reflexive property are transitive?

Solution: R_4, R_5, and R_6 are transitive.

A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Relation properties

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$, for all $a, b, c \in A$.

Which of the relations in Example of reflexive property are transitive?

Solution: R4, R5, and R6 are transitive.

A relation on a set A is called an equivalence relation if it is reflexive, symmetric, and transitive.

Relation properties

Let R be the relation on the set of real numbers such that aRb if and only if $a - b$ is an integer. Is R an equivalence relation?

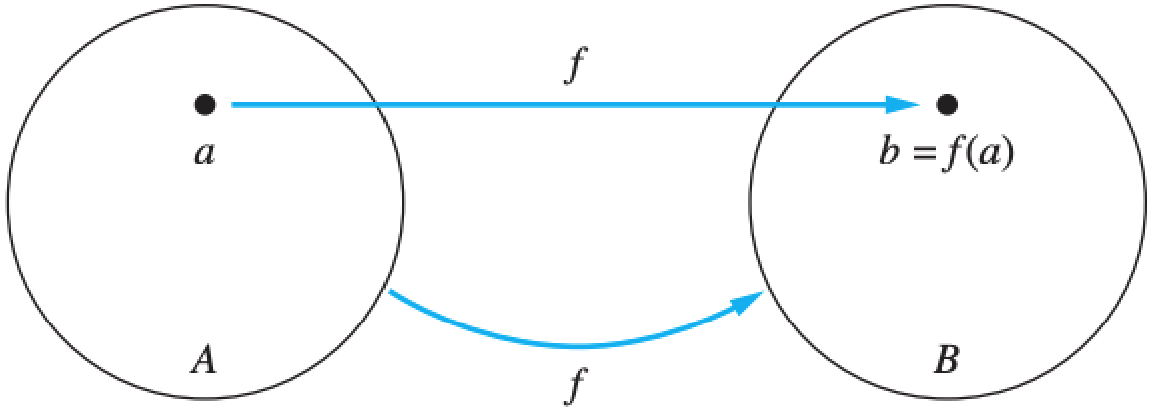
Solution:

- ✓ Because $a - a = 0$ is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reflexive.
- ✓ Now suppose that aRb. Then $a - b$ is an integer, so $b - a$ is also an integer. Hence, bRa. It follows that R is symmetric.
- ✓ If aRb and bRc, then $a - b$ and $b - c$ are integers. Therefore, $a - c = (a - b) + (b - c)$ is also an integer. Hence, aRc. Thus, R is transitive.
- ✓ Consequently, R is an equivalence relation.

Function

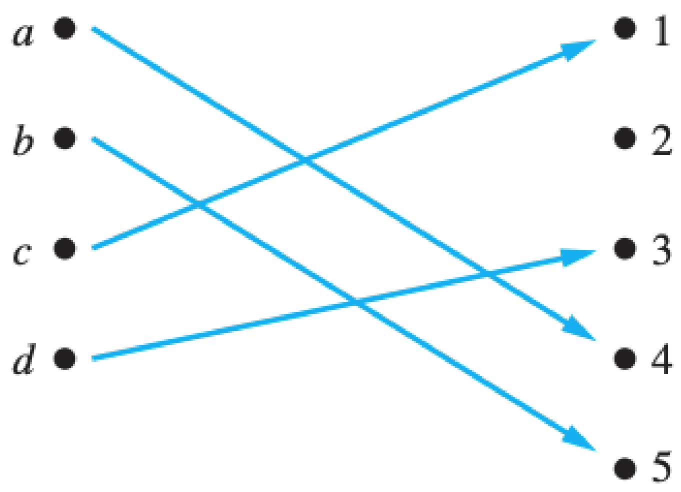
Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A. We write f (a) = b if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write f : A → B.

If f is a function from A to B, we say that A is the domain of f and B is the codomain of f. If f (a) = b, we say that b is the image of a and a is a preimage of b. The range, or image, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f maps A to B.



Function

A function f is said to be one-to-one, or an injunction, if and only if f (a) = f (b)implies that a = b for all a and b in the domain of f. A function is said to be injective if it is one-to-one.



Function

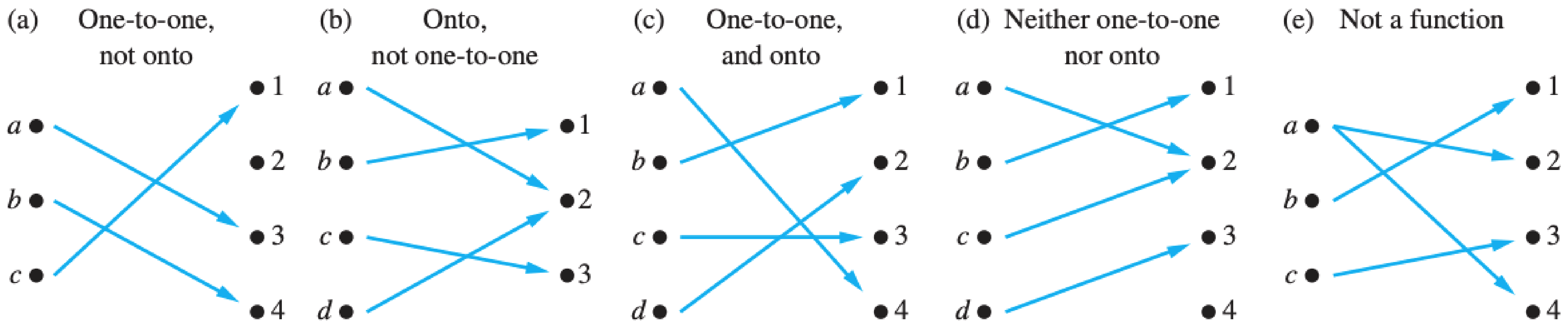
A function f from A to B is called onto, or a surjection, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

Let f be the function from {a, b, c, d} to {1, 2, 3} defined by f (a) = 3, f (b) = 2, f (c) = 1, and f (d) = 3. Is f an onto function?

Solution: Because all three elements of the codomain are images of elements in the domain, we see that f is onto.

Note that if the codomain were {1, 2, 3, 4}, then f would not be onto.

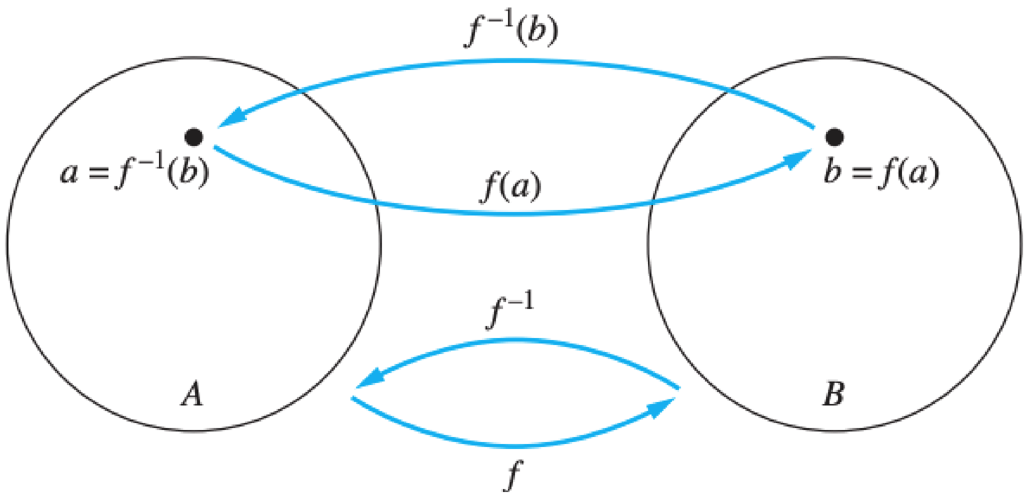
Function



The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto. We also say that such a function is bijective.

Function

Let f be a one-to-one correspondence from the set A to the set B. The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that f (a) = b. The inverse function of f is denoted by f^{-1}. Hence, f^{-1}(b) = a when f (a) = b.



A one-to-one correspondence is called invertible because we can define an inverse of this function. A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist.

Function

Let f : Z → Z be such that f (x) = x + 1. Is f invertible, and if it is, what is its inverse?

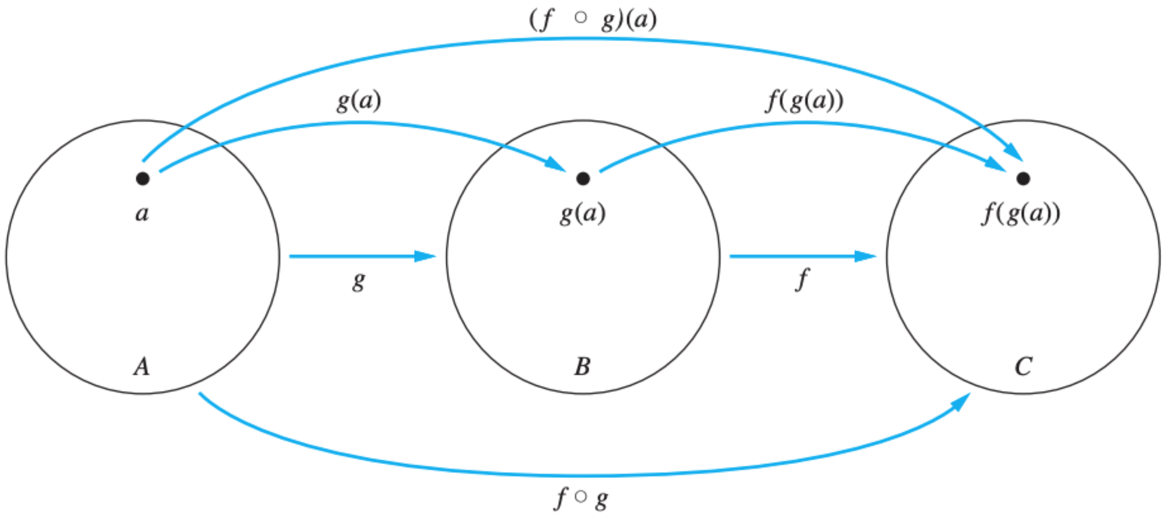
Solution: The function f has an inverse because it is a one-to-one correspondence, as follows from Examples 10 and 14. To reverse the correspondence, suppose that y is the image of x, so that y = x + 1. Then x = y − 1. This means that y − 1 is the unique element of Z that is sent to y by f . Consequently, f^{-1}(y) = y − 1.

Let f be the function from R to R with f (x) = x^2. Is f invertible?

Solution: Because f (−2) = f (2) = 4, f is not one-to-one. If an inverse function were defined, it would have to assign two elements to 4. Hence, f is not invertible.
(Note we can also show that f is not invertible because it is not onto.)

Function

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The composition of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.



❖ Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f ?

❖ Solution: Both the compositions $f \circ g$ and $g \circ f$ are defined. Moreover, $(f \circ g)(x) = f(g(x)) = f(3x + 2) = 2(3x + 2) + 3 = 6x + 7$ and $(g \circ f)(x) = g(f(x)) = g(2x + 3) = 3(2x + 3) + 2 = 6x + 11$.

Thank you

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