Indian Institute of Technology, Guwahati
Department of Mechanical Engineering
Engineering Computing Lab
ME 502 (September-Dec, 2020)
Assignment 4: Ordinary Differential Equation (Initial value problems)
FullMarks: 90

A. Solving Initial value problem (Ordinary differential equation with initial condition):

We want to solve following ODE (Ordinary differential equation)

$$y' = f(x, y) \text{ where } y(x_0) = y_0;$$

i.e. we want to get values of $y(x)$ at different x points ($0 < x < L$) such that it satisfy the above ODE.

<u>Different Numerical Methods:</u>

1. **Euler Method:**

$$y_{n+1} = y(x_{n+1}) = y_n + hf(x_n, y_n),$$

where $h = x_{n+1} - x_n$.

2. **Mid Point Method:**

$$y_{n+1} = y(x_{n+1}) = y_{n-1} + 2hf(x_n, y_n),$$

where $h = x_{n+1} - x_n$.

Detailed Algorithm for Mid Point Method:
Input:

(a) Initial value (x_0, y_0),
(b) End point L,
(c) No. of steps N,
(d) Function $f(x, y)$: Should be provided as different function,
(e) Analytical Solution (if available), $y = g(x)$: Should be provided as another function for error calculation

Algorithm:

(i) $h = \frac{L - x_0}{N}$;
(ii) Initialize a one dimensional array $y_{\text{val}}[N]$ with $y_{\text{val}}[0] = y_0$;
(iii) esum = 0, (To calculate L_2 norm of error);
(iv) $y_1 = y_0 + hf(x_0, y_0)$; (First step by Euler method)
(v) $\text{esum} = \text{esum} + (y_1 - g(x_0 + h))^2$;
(vi) $y_{n-1} = y_0; x_{n-1} = x_0$;
(vii) $y_n = y_1; x_n = x_0 + h$;

(viii) *For* $i = 2 : N$

$$
\begin{aligned}
&y_{n+1} = y_{n-1} + hf(x_n, y_n);\\
&y_{\text{val}}[i] = y_{n+1};\\
&\text{esum} = \text{esum} + (y_{n+1} - g(x_n + h))^2;\\
&x_n = x_n + h;\\
&y_{n-1} = y_n, y_n = y_{n+1};
\end{aligned}
$$

End For(i)

(ix) $eL_2 = \frac{1}{N}\sqrt{\text{esum}}$;

3. **Runge-Kutta method of second order (RK2):**

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{2}\left(k_1 + k_2\right),\\
k_1 &= hf(x_n, y_n),\\
k_2 &= hf(x_{n+1}, y_n + k_1),
\end{aligned}
$$

where $h = x_{n+1} - x_n$.

4. **Runge-Kutta method of fourth order (RK4):**

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right),\\
k_1 &= hf(x_n, y_n),\\
k_2 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}),\\
k_3 &= hf(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}),\\
k_4 &= hf(x_{n+1}, y_n + k_3),
\end{aligned}
$$

where $h = x_{n+1} - x_n$.

Detailed Algorithm for RK4 Method:

Input:

(a) Initial value (x_0, y_0),

(b) End point L,

(c) No. of steps N,

(d) Function $f(x, y)$: Should be provided as different function,

(e) Analytical Solution (if available), $y = g(x)$: Should be provided as another function for error calculation

Algorithm:

(i) $h = \frac{L - x_0}{N}$;

(ii) Initialize a one dimensional array $y_{\text{val}}[N]$ with $y_{\text{val}}[0] = y_0$;

(iii) esum $= 0$, (To calculate L_2 norm of error);

(iv) $y_n = y_0; x_n = x_0$;

(v) *For* $i = 1 : N$

$$k_1 = hf(x_n, y_n);$$
$$k_2 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_1}{2});$$
$$k_3 = hf(x_n + \tfrac{h}{2}, y_n + \tfrac{k_2}{2});$$
$$k_4 = hf(x_{n+1}, y_n + k_3);$$
$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4);$$
$$y_{\text{val}}[i] = y_{n+1};$$
$$\text{esum} = \text{esum} + (y_{n+1} - g(x_n + h))^2;$$
$$x_n = x_n + h;$$
$$y_n = y_{n+1};$$

End For(i)

(vi) $eL_2 = \frac{1}{N}\sqrt{\text{esum}};$

Problem 1: Let us consider following initial value problem

$$y' = y + 2x - x^2, \text{ where } y(0) = 1.$$

We want to find distribution of y in $0 < x < 5$ by above four numerical methods. Analytical solution of above ODE is given as

$$y(x) = x^2 + e^x.$$

- Write a function which calculate $f(x, y)$ in $y' = f(x, y)$.
- Write another function which calculate analytical solution $g(x) = x^2 + e^x$.
- Write four different functions *Euler, Midpt, RK2, RK4* where for each function INPUTS: x_0, y_0, L, N and OUTPUTS: $y_{\text{val}}[N], eL_2$.
  In each of these functions, functions $f(x, y)$ and $g(x)$ will be called for required calculation.

(1) Find $y_{\text{val}}[N]$ for $N = 5, 10, 20$ for all four methods. You should represent your output in tabular form. Table 1 represent one such table for $N = 5$. There will be similar

x	Analytical, $y(x)$	y_n (Euler)	y_n (MidPoint)	y_n (RK2)	y_n (RK4)
0.0					
1.0					
2.0					
3.0					
4.0					
5.0					

Table 1: Values of y_n for different methods for $N = 5$.

tables for $N = 10$ and $N = 20$. **[10+4+4=18]**

(2) Find eL_2 for $N = 2, 5, 10, 15, 20, 25$ for all four methods. Represent your results in tabular form as shown in Table 2 **[12]**

N	Euler	MidPoint	RK2	RK4
2				
5				
10				
15				
20				
25				

Table 2: L_2 error norms for different methods.

B. Solving system of initial value problems:

Consider following system of initial value problems

$$\begin{aligned} u'(x) &= f(x,u,v), & u(a) &= u_0, \\ v'(x) &= g(x,u,v), & v(a) &= v_0, \end{aligned}$$

where we want to find $u(x)$ and $v(x)$ in $a < x < b$ such that both u and v satisfy above coupled ODEs.

Runge-Kutta of Fourth order (RK4) for system of ODEs:

$$\begin{aligned} u_{n+1} &= u_n + \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right), & v_{n+1} &= v_n + \frac{1}{6}\left(l_1 + 2l_2 + 2l_3 + l_4\right), \\ k_1 &= hf(x_n, u_n, v_n), & l_1 &= hg(x_n, u_n, v_n), \\ k_2 &= hf(x_n + \frac{h}{2}, u_n + \frac{k_1}{2}, v_n + \frac{l_1}{2}), & l_2 &= hg(x_n + \frac{h}{2}, u_n + \frac{k_1}{2}, v_n + \frac{l_1}{2}), \\ k_3 &= hf(x_n + \frac{h}{2}, u_n + \frac{k_2}{2}, v_n + \frac{l_2}{2}), & l_3 &= hg(x_n + \frac{h}{2}, u_n + \frac{k_2}{2}, v_n + \frac{l_2}{2}), \\ k_4 &= hf(x_{n+1}, u_n + k_3, v_n + l_3), & l_4 &= hg(x_{n+1}, u_n + k_3, v_n + l_3), \end{aligned}$$

Detailed Algorithm for RK4 Method for system of ODEs:
Input:

1. Initial value (x_0, u_0, v_0),
2. End point L,
3. No. of steps N,
4. Function $f(x,u,v)$ and $g(x,u,v)$: Should be provided as different function,
5. Analytical Solution (if available), $u = l(x), v = m(x)$: Provided for error calculation.

Algorithm:

(i) $h = \frac{L - x_0}{N}$;

(ii) Initialize two one dimensional arrays $u_{\text{val}}[N]$ with $u_{\text{val}}[0] = u_0$ and $v_{\text{val}}[N]$ with $v_{\text{val}}[0] = v_0$;

(iii) esumu = 0, esumv = 0 (To calculate L_2 norm of errors for both u and v);

(iv) $u_n = u_0; v_n = v_0; x_n = x_0;$

(v) *For* $i = 1 : N$

$$
\begin{aligned}
&k_1 = hf(x_n, u_n, v_n);\\
&l_1 = hg(x_n, u_n, v_n);\\
&k_2 = hf(x_n + \tfrac{h}{2}, u_n + \tfrac{k_1}{2}, v_n + \tfrac{l_1}{2});\\
&l_2 = hg(x_n + \tfrac{h}{2}, u_n + \tfrac{k_1}{2}, v_n + \tfrac{l_1}{2});\\
&k_3 = hf(x_n + \tfrac{h}{2}, u_n + \tfrac{k_2}{2}, v_n + \tfrac{l_2}{2});\\
&l_3 = hg(x_n + \tfrac{h}{2}, u_n + \tfrac{k_2}{2}, v_n + \tfrac{l_2}{2});\\
&k_4 = hf(x_{n+1}, u_n + k_3, v_n + l_3);\\
&l_4 = hg(x_{n+1}, u_n + k_3, v_n + l_3);\\
&u_{n+1} = u_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4);\\
&v_{n+1} = v_n + \tfrac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4);\\
&u_{\text{val}}[i] = u_{n+1};\\
&v_{\text{val}}[i] = v_{n+1};\\
&\text{esumu} = \text{esumu} + (u_{n+1} - l(x_n + h))^2;\\
&\text{esumv} = \text{esumv} + (v_{n+1} - m(x_n + h))^2;\\
&x_n = x_n + h;\\
&u_n = u_{n+1};\\
&v_n = v_{n+1};
\end{aligned}
$$

End For(i)

(vi) $e_u L_2 = \frac{1}{N}\sqrt{\text{esumu}};$

(vii) $e_v L_2 = \frac{1}{N}\sqrt{\text{esumv}};$

Problem 2: Let us consider following system of initial value problem

$$
\begin{aligned}
y' &= 2xz^2/y, && \text{where } y(1) = 1,\\
z' &= y/z^2, && \text{where } z(1) = 1.
\end{aligned}
$$

We want to find distribution of y and z in $1 < x < 5$ by RK4 method. Analytical solution of above set of ODEs is given as

$$y(x) = x^2, \qquad z(x) = x.$$

- Write two functions which calculate $f(x, y, z)$ in $y' = f(x, y, z)$ and $g(x, y, z)$ in $z' = g(x, y, z)$.
- Write two functions $l(x)$ and $m(x)$ which calculate two analytical solutions.
- Write the function *RK4system* where
  INPUTS: x_0, y_0, z_0, L, N and
  OUTPUTS: $y_{\text{val}}[N], z_{\text{val}}[N], e_y L_2, e_z L_2$.
  In *RK4system*, functions $f(x, y, z), g(x, y, z), l(x), m(x)$ will be called for required calculation.

(1) Find $y_{\text{val}}[N]$ and $z_{\text{val}}[N]$ for $N = 5, 10, 20$ with RK4 method. You should represent your output in tabular form. Table 3 represent one such table for $N = 5$. There will be similar tables for $N = 10$ and $N = 20$. **[10+4+4=18]**

x	Analytical, $y(x)$	y_n (RK4)	Analytical, $z(x)$	z_n (RK4)
1.0				
1.8				
2.6				
3.4				
4.2				
5.0				

Table 3: Values of y_n and z_n for RK4 method for $N = 5$.

(2) Find $e_y L_2$ and $e_z L_2$ for $N = 2, 5, 10, 15, 20, 25$. Represent your results in tabular form as shown in Table 4 **[12]**

N	$e_y L_2$	$e_z L_2$
2		
5		
10		
15		
20		
25		

Table 4: Error norms ($e_y L_2$ and $e_z L_2$).

C. Expressing higher order ODE as system of first order ODEs:

Let us consider following higher order ODE

$$y''' - xy'' + y' - 8y^4 = y \tan x, \text{ where } y(1) = 5, y'(1) = 0, y''(1) = 10$$

We can express above equation as system of 3 first order ODEs as below

$$\begin{aligned} y' &= u, & y(1) &= 5 \\ u' &= v, & u(1) &= 0 \\ v' &= y \tan x + 8y^4 - u + xv & v(1) &= 10 \end{aligned}$$

Problem 3: Let us consider following higher order ODE

$$y'' + y = 6\cos x + 2 \text{ subjected to } y(0) = 4, y'(0) = -\pi.$$

We want to find distribution of y and y' in $0 < x < \pi$ by RK4 method. Analytical solution of above ODE is given as

$$y = 2\cos x - \pi \sin x + 3x \sin x + 2$$

(1) Find $y_{\text{val}}[N]$ and $y'_{\text{val}}[N]$ for $N = 5, 10, 20$ with RK4 method. You should represent your output in tabular form. Table 5 represent one such table for $N = 5$. There will be similar tables for $N = 10$ and $N = 20$. **[10+4+4=18]**

(2) Find $e_y L_2$ and $e_{y'} L_2$ for $N = 2, 5, 10, 15, 20, 25$. Represent your results in tabular form as shown in Table 6 **[12]**

x	Analytical, $y(x)$	y_n (RK4)	Analytical, $y'(x)$	y'_n (RK4)
0.0				
0.6283				
1.2566				
1.8850				
2.5133				
3.1416				

Table 5: Values of y_n and y'_n for RK4 method for $N = 5$.

N	$e_y L_2$	$e_{y'} L_2$
2		
5		
10		
15		
20		
25		

Table 6: Error norms ($e_y L_2$ and $e_{y'} L_2$).