Axiomatization of Some Contact Logics with a Qualitative Measure

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1 Introduction

The aim of this work is to explore the axiomatization and decidability of the quantifier-free theories of a structure, which arises from a certain kind of geometric objects on the real line. Three relations between these objects are considered: parthood, contact and qualitative measure.

The objects are referred to as polytopes, though they are in fact unions of what may usually be understood by the term. A key property by which they are chosen is that they have a true interior.

The parthood relationship between these objects gives rise to a Boolean algebra and two further relations are considered: contact and qualitative measure.

2 Notation and Notions

Listed below are some well-established notions and the notation for them in this text.

- $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^+$ the positive real numbers and $\mathbb{R}_0^+$ the non-negative ones.
- $-\infty$ and ∞ are the least and greatest elements of $\mathbb{R} \cup \{-\infty, \infty\}$.
- For the purposes of measure, the $+$ operation over $\mathbb{R}_0^+$ is extended in the usual way:

$$r + \infty \stackrel{\text{def}}{=} \infty + r \stackrel{\text{def}}{=} \infty, \text{ for any } r \in \mathbb{R}_0^+ \cup \{\infty\}$$

- $\mathcal{P}(X)$ denotes the set of all subsets of X.

- TODO: Closure and interior
- TODO: Graph, Cycle
- TODO: Tree, Rooted Tree, Children(T, u), SubTree(T, v)

3 Language

This section describes the language, whose semantics will be considered in a couple of contexts.

Definition (Language)**.** *$\mathcal{L}$ denotes the first-order language with equality of only quantifier-free formulas, containing the following non-logical symbols:*

- *predicate symbols:*
  - *binary infix $\sqsubseteq$: parthood*
  - *binary infix $\preceq$: measure comparison*
  - *binary prefix C: contact*
- *function symbols:*
  - *binary infix function $\sqcup$: union*
  - *binary infix function $\sqcap$: intersection*
  - *unary postfix function *: complement*
- *constant symbols:*
  - 0*: empty polytope*
  - 1*: universe*

The logical symbols $\wedge$, $\vee$, $\neg$, $\Leftrightarrow$, $\Rightarrow$, $\top$, $\bot$ are used in the usual way and the set of individual variables is denoted Vars. The symbol for formal equality is $\doteq$.

Further, for any terms τ_1 and τ_2, the following abbreviations will be used:

- *$\tau_1 \asymp \tau_2$ for $\tau_1 \preceq \tau_2 \wedge \tau_2 \preceq \tau_1$ and*
- *$\tau_1 \prec \tau_2$ for $\neg(\tau_2 \preceq \tau_1)$.*

4 Semantics

Though the aim is to interpret the language on the real line, other models will also be needed. It is useful fix their common semantics, which are defined below in the expected way.

Let $\mathcal{B}$ be a Boolean algebra with carrier B and $\mathcal{C}$ and $\mathcal{M}$ be relations over B. Let $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$.

Definition (Value of a Term in $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$)**.** *Let $v : Vars \to B$. Then, v^S denotes the extension of v to the terms of $\mathcal{L}$ in the following structurally recursive way:*

- $v^S(0)$ *is the zero of* $\mathcal{B}$
- $v^S(1)$ *is the unit of* $\mathcal{B}$
- $v^S(\tau_1^*)$ *is the complement of* $v(\tau_1)$ *in* $\mathcal{B}$*,*
- $v^S(\tau_1 \sqcup \tau_2)$ *is the join of* $v(\tau_1)$ *and* $v(\tau_2)$ *in* $\mathcal{B}$*,*
- $v^S(\tau_1 \sqcap \tau_2)$ *is the meet of* $v(\tau_1)$ *and* $v(\tau_2)$ *in* $\mathcal{B}$*,*

for any terms τ_1 and τ_2 of $\mathcal{L}$.

Definition (Validity of a Formula in $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$)**.** *Again, let $v : Vars \to B$. Validity of a formula ϕ in S with valuation v is denoted $\langle S, v \rangle \models \phi$ and defined over atomic formulas like so:*

- $\langle S, v \rangle \models \tau_1 \sqsubseteq \tau_2 \leftrightarrow v^S(\tau_1)$ *is less than or equal to* $v^S(\tau_2)$ *in* $\mathcal{B}$*,*
- $\langle S, v \rangle \models C(\tau_1, \tau_2) \leftrightarrow \mathcal{C}(v^S(\tau_1), v^S(\tau_2))$*,*
- $\langle S, v \rangle \models \tau_1 \preceq \tau_2 \leftrightarrow \mathcal{M}(v^S(\tau_1), v^S(\tau_2))$*,*

for any terms τ_1 and τ_2 of $\mathcal{L}$. For complex formulas, the extension is done in the usual way:

- $\langle S, v \rangle \models \top$ *and* $\langle S, v \rangle \not\models \bot$*,*
- $\langle S, v \rangle \models \neg\phi \leftrightarrow \langle S, v \rangle \not\models \phi$*,*
- $\langle S, v \rangle \models \phi \wedge \psi \leftrightarrow \langle S, v \rangle \models \phi$ *and* $\langle S, v \rangle \models \psi$*,*
- $\langle S, v \rangle \models \phi \vee \psi \leftrightarrow$ *at least one of* $\langle S, v \rangle \models \phi$ *and* $\langle S, v \rangle \models \psi$ *holds,*

where ϕ and ψ are (quantifier-free) formulas of $\mathcal{L}$.

If $\langle S, v \rangle \models \phi$ for all $v : Vars \to B$, then ϕ is valid in S, denoted by $S \models \phi$.

If there exists $v : Vars \to B$ such that $\langle S, v \rangle \models \phi$, then ϕ is satisfiable.

4.1 Polytopes on the Real Line

A specific kind of objects will be considered: finite unions of closed, potentially infinite, intervals on the real line. These are defined below, along with the interpretation operations and properties with which the language is concerned.

Definition (Basis Polytope)**.** *For any m, $n \in \mathbb{R}$ such that $m < n$, the intervals $[m, n]$, $(-\infty, m]$, $[m, \infty)$ and $(-\infty, \infty)$ are called* basis polytopes*. The set of all basis polytopes is $Bas(\mathbb{R})$.*

Definition (Polytope)**.** *For any finite set of basis polytopes $C \subseteq Bas(\mathbb{R})$, $\bigcup C$ is called a* polytope*. The set of all polytopes is denoted $Pol(\mathbb{R})$.*

Remark that for $C = \emptyset$, the empty set is also a polytope.

Proposition. *Any polytope p can be uniquely represented as the union of a finite set C_p of non-intersecting basis polytopes.*

Proof. Let $A \overset{\text{def}}{=} \{C \mid (\forall b \in C)(b \in Bas(\mathbb{R}))$ and $\bigcup C = p\}$. Since p is a polytope, $A \neq \emptyset$. Let C' be any element of A, having the least number of elements, i.e. $(\forall C \in A)(\overline{\overline{C'}} \leq \overline{\overline{C}})$.

Let $b_1 \in C'$, $b_2 \in C'$ and $b_1 \neq b_2$. Suppose for the sake of contradiction that $b_1 \cap b_2 \neq \emptyset$. Then, $b_1 \cup b_2 \in Bas(\mathbb{R})$. Let $D \overset{\text{def}}{=} C' \cup \{b_1 \cup b_2\} \setminus \{b_1, b_2\}$. $\bigcup D = \bigcup C'$, so $D \in A$ and $\overline{\overline{D}} < \overline{\overline{C'}}$, a contradiction. Thus, the elements of C' are non-intersecting.

Suppose $C'' \in A$ and $\overline{\overline{C''}} = \overline{\overline{C'}}$. By the same argument as for C', the elements of C'' are also non-intersecting.

Let $b' \in C'$. $b' \subseteq \bigcup C'$, so $b' \subseteq \bigcup C''$. Let $x \in b'$. Then $x \in \bigcup C''$, so let $b'' \in C''$, such that $x \in b''$.

Suppose for the sake of contradiction that $b'' \setminus b' \neq \emptyset$. Let $y \in b'' \setminus b'$. □

Definition. *(Standard Representation) The set C_p from the above proposition is called the* standard representation *of the polytope p.*

Definition (Polytope Operations)**.** *For any polytopes p and q, we define the following operations as modifications of intersection and complement:*

- $p \Cap q \overset{def}{=} Cl(Int(p \cap q))$;
- $p^{\circledast} \overset{def}{=} Cl(\mathbb{R} \setminus p)$.

The union operation $p \cup q$ will be considered in the same context, though no modification is needed.

The modification of intersection ensures that there are no isolated points and the modification of complement ensures that results of the operations remain a union of *closed* intervals.

Proposition. *$Pol(\mathbb{R})$ forms a Boolean algebra with*

- $\subseteq$ *for Boolean inequality,*
- $\Cap$ *for meet,*
- $\cup$ *for join,*
- ${}^{\circledast}$ *for complement,*
- $\emptyset$ *for the zero and*
- $\mathbb{R}$ *for the unit.*

This algebra will be denoted $\mathcal{B}^{\mathbb{R}}$.

Proof. It will be shown that the axioms for Boolean algebra given in the Axiomatization section are met. These are as in (TODO: Cite Halmos). □

Proposition. *$\mathcal{B}^{\mathbb{R}}$ is not closed under supremum of countable sets (countable join operations).*

Proof. Consider the countably many polytopes $[2n, 2n+1]$, for each natural number n. Their join is the set $[0,1] \cup [2,3] \cup [4,5] \cup \ldots$.

Suppose for the sake of contradiction that this set is a polytope p. Then p is a finite union of a basis polytopes C, $p = \bigcup C$. Clearly, $C \neq \emptyset$.

If $[m, \infty) \in C$ for some m, then let $o > m$ be the least odd natural number, greater than m. $o + \frac{1}{2} \in [m, \infty)$, but $o + \frac{1}{2} \notin [0,1] \cup [2,3] \cup [4,5] \cup \ldots$. Therefore, $[m, \infty) \notin C$ for any m. Similarly, $(-\infty, \infty) \notin C$, since e.g. $1\frac{1}{2} \notin p$.

Let

$$R \stackrel{\text{def}}{=} \{r \mid (\exists l \in \mathbb{R})([l, r] \in C) \text{ or } (-\infty, r] \in C\}$$

Since C is finite, R is finite and there is a greatest element in R. Let o' be the least natural number, greater than $\max R$. $o' \notin \bigcup C$, yet $o' \in [0,1] \cup [2,3] \cup [4,5] \cup \ldots$, a contradiction. □

Definition (Line Contact)**.** *Two polytopes p and q are* in contact *if $p \cap q \neq \emptyset$. This is denoted $\mathcal{C}^{\mathbb{R}}(p, q)$.*

Definition (Polytope Measure)**.** *The measure of a basis polytope of the kind $[m, n]$, for $m, n \in \mathbb{R}$ is $n - m$. The measure of a basis polytope with an infinite bound is ∞.*

The measure *of a polytope p, denoted $\mu^{\mathbb{R}}(p)$, is the sum of the measures of the basis polytopes of its standard representation.*

The qualitative measure relation induced by $\mu^{\mathbb{R}}$ *is defined*

$$p \leq^{\mathbb{R}}_{\mu} q \leftrightarrow \mu^{\mathbb{R}}(p) \leq \mu^{\mathbb{R}}(q),$$

for any $p, q \in Pol(\mathbb{R})$.

Proposition. *$\mu^{\mathbb{R}}$ is a countably additive measure.*

Proof. TODO: write proof □

$\mu^{\mathbb{R}}$ is a restriction of the usual measure on $\mathbb{R}$.

Proposition. *The only polytope with measure 0 is $\emptyset$. Further, every non-zero polytope has an arbitrary small non-zero sub-polytope. Consequently, $\mathcal{B}^{\mathbb{R}}$ is atomless.*

Proof. TODO: write proof □

Given the definition of $\mathcal{B}^{\mathbb{R}}$, measure and contact above, the model on the real line is defined directly:

Definition (Real Line Model)**.**

$$S^{\mathbb{R}} \stackrel{def}{=} \langle \mathcal{B}^{\mathbb{R}}, \mathcal{C}^{\mathbb{R}}, \leq^{\mathbb{R}}_{\mu} \rangle$$

4.2 Relational Models

In order to find appropriate value functions for a well chosen kind of formulas, an abstraction model will be needed. It is quite generic, yet it turns out to be easily transformed into an equivalent real line model, given some constraints.

Let W be a finite set and $\mathcal{B}^W$ be the Boolean algebra of all subsets of W. Let c be an arbitrary symmetric and reflexive relation over W and $m : W \to \mathbb{R}^+ \cup \{\infty\}$.

Let $\mathcal{C}^c$ be the relation over $\mathcal{P}(W)$ defined as

$$\mathcal{C}^c(a, b) \leftrightarrow (\exists i \in a)(\exists j \in b)(\langle i, j\rangle \in c)$$

Let $\mu^m : \mathcal{P}(W) \to \mathbb{R}_0^+ \cup \{\infty\}$ such that:

$$\mu^m(a) \overset{\text{def}}{=} \sum_{i \in a} m(i)$$

And finally, let $\leq_m$ be the relation over $\mathcal{P}(W)$:

$$a \leq_m b \leftrightarrow \mu^m(a) \leq \mu^m(b),$$

all for a, $b \subseteq W$.

Definition (Relational Model)**.** $\langle \mathcal{B}^W, \mathcal{C}^c, \leq_m\rangle$ *is called the* relational model for W, c and m.

4.2.1 Converting to a Real Line Model

Relational models are useful because they can be converted to real line models under certain conditions.

Definition (Contact Graph)**.** *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m\rangle$ be a relational model. Let E denote the set of pairs of different vertices in c, but unordered: $E \overset{def}{=} \{\, \{i, j\} \mid \langle i, j\rangle \in c \;\&\; i \neq j \,\}$. Note that since c is reflexive and symmetric, no information is lost when obtaining E from c. The graph $\langle W, E\rangle$ is called the* contact graph *of S and denoted $Gr(S)$.*

Definition (Convertible Relational Model)**.** *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m\rangle$ be a relational model and suppose that the following constraints hold:*

- *$Gr(S)$ is connected and*
- *there are exactly two elements of W with infinite values for m, i.e. there exist $i \in W$ and $j \in W$, $i \neq j$ such that:*

$$m(i) = m(j) = \infty \text{ and } (\forall k \in W \setminus \{i, j\})(m(k) \neq \infty).$$

Then, S is called a convertible relational model.

Definition (Disjoint Valuation)**.** *Suppose* $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ *is a convertible relational model and* $v : \mathit{Vars} \to \mathcal{P}(W)$ *such that*

$$(\forall x \in \mathit{Vars})(\forall y \in \mathit{Vars})(v(x) \neq v(y) \to v(x) \cap v(y) = \emptyset).$$

Then, v *is called a* disjoint valuation*.*

Although a valuation is an infinite object, any given formula contains a finite number of variables. Therefore, only a finite part of the valuation is relevant to the truth value of that formula. The effective construction of a valuation as discussed below is possible because the valuation will always be considered in the context of a formula and will therefore be encodable by a finite object.

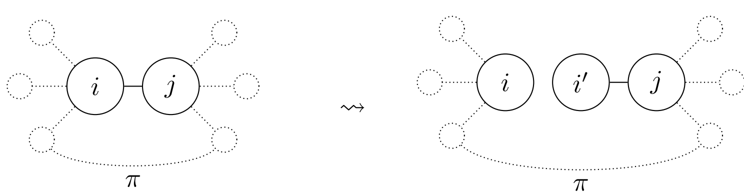
Lemma (Untying)**.** *Let* $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ *be a convertible relational model and* v *be a disjoint valuation. Suppose* $Gr(S)$ *is not a tree. Then, there is a procedure to effectively construct a convertible relational model* S' *and a disjoint valuation* v' *for* S' *such that:*

- $Gr(S')$ *has one vertex more and the same number of edges and*
- *for any formula* ϕ *in* $\mathcal{L}$*,* $\langle S, v \rangle \models \phi \leftrightarrow \langle S', v' \rangle \models \phi$*.*

Proof. Given that $Gr(S)$ is not a tree, yet is connected, it must contain at least one cycle. A cycle can be effectively found using a depth-first search. Let π be such a cycle.

Let i and j be any two consecutive vertices in π such that $m(i) \neq \infty$. This requirement is always achievable, since there are at least three vertices in a cycle and exactly two vertices with infinite values for m in a convertible relational model. Therefore, there will be at least one vertex with a finite value for m in any cycle.

The aim is to disconnect π by removing the edge between i and j. In order to achieve that, intuitively, the portion of i which is in contact only with j will be separated out into a separate atomic object, i', having half the measure.



Let $i' \notin W$ and let:

$$W' \stackrel{\text{def}}{=} W \cup \{i'\}$$
$$c' \stackrel{\text{def}}{=} (c \setminus \{\langle i, j\rangle, \langle j, i\rangle\}) \cup \{\langle i', j\rangle, \langle j, i'\rangle, \langle i', i'\rangle\}$$
$$m'(k) \stackrel{\text{def}}{=} \begin{cases} m(k) & \text{if } k \notin \{i, i'\} \\ m(i)/2 & \text{otherwise} \end{cases}, \text{ for any } k \in W'.$$
$$S' \stackrel{\text{def}}{=} \langle \mathcal{B}^{W'}, \mathcal{C}^{c'}, \leq_{m'} \rangle$$
$$v'(x) \stackrel{\text{def}}{=} \begin{cases} v(x) \cup \{i'\} & \text{if } i \in v(x) \\ v(x) & \text{otherwise} \end{cases}, \text{ for any } x \in \text{Vars}.$$

Note that S' and v' are defined in a constructive way. The procedure consists of copying S and v, except for a small number of changes, not depending on the size of S.

It is directly clear that S' is a relational model. Compared to S, there is one more vertex in $Gr(S')$ and the same number of edges ($\{i, j\}$ is removed, $\{i', j\}$ is added).

Any path between two vertices in $Gr(S)$ corresponds to a path in $Gr(S')$, by potentially substituting the edge $\{i, j\}$ for the detour path along π. Therefore, all vertices in W are connected in $Gr(S')$. i' is connected to i and from there to the rest of W as well, so $Gr(S')$ is connected.

m' has the same values as m over all elements of W', except i and i', where it's values are finite. Therefore, the same two elements of W' that have infinite values for m, also have infinite values for m' and no others.

Thus, S' is a convertible relational model.

To demonstrate by contraposition that v' is a disjoint valuation, let $x \in \text{Vars}$, $y \in \text{Vars}$ and assume that $v'(x) \cap v'(y) \neq \emptyset$. Let $k \in v'(x) \cap v'(y)$, for some $k \in W'$. If $k = i'$, $i' \in v'(x)$, so $i \in v(x)$ by the definition of v'. Analogously, $i \in v(y)$ and $v(x) \cap v(y) \neq \emptyset$. On the other hand, if $k \neq i'$, again by the definition of v', $k \in v(x) \cap v(y)$. In both cases, since v is a disjoint valuation and $v(x) \cap v(y) \neq \emptyset$, $v(x) = v(y)$. From here, $v'(x) = v'(y)$ and v' is a disjoint valuation.

Now follows a proof by induction on terms in $\mathcal{L}$, that

$$v'^{S'}(\tau) = \begin{cases} v^S(\tau) \cup \{i'\} & \text{if } i \in v^S(\tau) \\ v^S(\tau) & \text{otherwise} \end{cases}, \text{ for any term } \tau \text{ of } \mathcal{L}.$$

- For the two constants 0 and 1,

$$v'^{S'}(0) = \emptyset = v^S(0) \text{ and } i \notin v^S(0)$$

$$v'^{S'}(1) = W' = W \cup \{i'\} = v^S(1) \cup \{i'\} \text{ and } i \in v^S(1)$$

- For terms consisting of individual variables, the statement holds by the definition of v'.

- Suppose, as induction hypothesis, that the statement holds for τ_1 and τ_2.
  - To show that the statement holds for $\tau_1 \sqcup \tau_2$,
    - suppose first that $i \in v^S(\tau_1 \sqcup \tau_2)$. Then, $i \in v^S(\tau_1) \cup v^S(\tau_2)$ and $i \in v^S(\tau_1)$ or $i \in v^S(\tau_2)$. Without loss of generality, assume $i \in v^S(\tau_1)$. From here,

$$v'^{S'}(\tau_1) \overset{\text{i.h.}}{=} v^S(\tau_1) \cup \{i'\} \text{ and } v'^{S'}(\tau_2) \overset{\text{i.h.}}{\subseteq} v^S(\tau_2) \cup \{i'\}.$$

      Using this,

$$v'^{S'}(\tau_1) \cup v'^{S'}(\tau_2) = v^S(\tau_1) \cup v^S(\tau_2) \cup \{i'\}.$$

      By applying the definitions of v^S and $v'^{S'}$ to the above equality,

$$v'^{S'}(\tau_1 \sqcup \tau_2) = v^S(\tau_1 \cup \tau_2) \cup \{i'\}.$$

    - Alternatively, suppose $i \notin v^S(\tau_1 \sqcup \tau_2)$. Then

$$v'^{S'}(\tau_1 \sqcup \tau_2) \overset{\text{def}}{=} v'^{S'}(\tau_1) \cup v'^{S'}(\tau_2) \overset{\text{i.h.}}{=} v^S(\tau_1) \cup v^S(\tau_2) \overset{\text{def}}{=} v^S(\tau_1 \sqcup \tau_2).$$

  - To show that the statement holds for $\tau_1 \sqcap \tau_2$,
    - suppose $i \in v^S(\tau_1 \sqcap \tau_2)$. Then, $i \in v^S(\tau_1) \cap v^S(\tau_2)$, so $i \in v^S(\tau_1)$ and $i \in v^S(\tau_2)$. Thus,

$$v'^{S'}(\tau_1 \sqcap \tau_2) \overset{\text{def}}{=} v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2) \overset{\text{i.h.}}{=} (v^S(\tau_1) \cup \{i'\}) \cap (v^S(\tau_2) \cup \{i'\}) = (v^S(\tau_1) \cap v^S(\tau_2)) \cup \{i'\} \overset{\text{def}}{=} v^S(\tau_1 \sqcup \tau_2) \cup \{i'\}.$$

    - Alternatively, if $i \notin v^S(\tau_1 \sqcap \tau_2)$, then $i' \notin v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2)$ and

$$v'^{S'}(\tau_1 \sqcap \tau_2) \overset{\text{def}}{=} v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2) \overset{\text{i.h.}}{=} v^S(\tau_1) \cap v^S(\tau_2) \overset{\text{def}}{=} v^S(\tau_1 \sqcap \tau_2).$$

  - To show the statement holds for τ_1^*, consider that

$$v'^{S'}(\tau_1^*) \overset{\text{def}}{=} W' \setminus v'^{S'}(\tau_1) \overset{\text{def}}{=} (W \cup \{i'\}) \setminus v'^{S'}(\tau_1).$$

    - If $i \notin v^S(\tau_1^*)$, then $i \in v^S(\tau_1)$ and

$$(W \cup \{i'\}) \setminus v'^{S'}(\tau_1) \overset{\text{i.h.}}{=} (W \cup \{i'\}) \setminus (v^S(\tau_1) \cup \{i'\}) = W \setminus v^S(\tau_1) \overset{\text{def}}{=} v^S(\tau_1^*).$$

* If $i \in v^S(\tau_1^*)$, then $i \notin v^S(\tau_1)$ and

$$(W \cup \{i'\}) \setminus v'^{S'}(\tau_1) = (W \cup \{i'\}) \setminus v^S(\tau_1) =$$
$$(W \setminus v^S(\tau_1)) \cup \{i'\} \stackrel{\text{def}}{=} v^S(\tau_1^*) \cup \{i'\}$$

Now to demonstrate that for any formula ϕ in $\mathcal{L}$, $\langle S, v\rangle \models \phi \leftrightarrow \langle S', v'\rangle \models \phi$ by induction on the construction of ϕ, let τ_1 and τ_2 be terms of $\mathcal{L}$.

- For $\bot$ and $\top$, the statement is trivial.
- For parthood atomic formulas:
  - Suppose $i \in v^S(\tau_2)$. By definition,

    $$\langle S, v\rangle \models \tau_1 \sqsubseteq \tau_2 \leftrightarrow v^S(\tau_1) \subseteq v^S(\tau_2)$$

    Since $v^S(\tau_{1,2}) \subseteq W$ and $i' \notin W$,

    $$v^S(\tau_1) \subseteq v^S(\tau_2) \leftrightarrow v^S(\tau_1) \cup \{i'\} \subseteq v^S(\tau_2) \cup \{i'\}$$

    and given that $v^S(\tau_2) \cup \{i'\} = v'^{S'}(\tau_2)$,

    $$\begin{aligned} v^S(\tau_1) \cup \{i'\} \subseteq v^S(\tau_2) \cup \{i'\} &\leftrightarrow v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \\ &\leftrightarrow \langle S', v'\rangle \models \tau_1 \sqsubseteq \tau_2. \end{aligned}$$

  - conversely, if $i \notin v^S(\tau_2)$,

    $$\begin{aligned} &\langle S, v\rangle \models \tau_1 \sqsubseteq \tau_2 \\ \leftrightarrow\, & v^S(\tau_1) \subseteq v^s(\tau_2) \\ \leftrightarrow\, & v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \\ \leftrightarrow\, & \langle S', v'\rangle \models \tau_1 \sqsubseteq \tau_2 \end{aligned}$$

- To prove the statement for contact atomic formulas in one direction, assume that $\langle S, v\rangle \models C(\tau_1, \tau_2)$. From here, there exist $k \in v^S(\tau_1)$ and $l \in v^S(\tau_2)$ such that $\langle k, l\rangle \in c$.
  - Suppose $\{k, l\} = \{i, j\}$. Without loss of generality, $k = i$ and $l = j$. Then, since $i \in v^S(\tau_1)$, $i' \in v'^{S'}(\tau_1)$ must hold. Further, $j \in v'^{S'}(\tau_2)$ and $\langle i', j\rangle \in c'$, so $\langle S', v'\rangle \models C(\tau_1, \tau_2)$.
  - Alternatively, if $\{k, l\} \neq \{i, j\}$, then $\langle k, l\rangle \in c'$ (because $\langle k, l\rangle \in c$), $k \in v'^{S'}(\tau_1)$ and $l \in v'^{S'}(\tau_2)$, so $\langle S', v'\rangle \models C(\tau_1, \tau_2)$.

  In the opposite direction, assume $\langle S', v'\rangle \models C(\tau_1, \tau_2)$. Again, there must exist $k \in v'^{S'}(\tau_1)$ and $l \in v'^{S'}(\tau_2)$ such that $\langle k, l\rangle \in c'$.
  - Suppose $\{k, l\} = \{i', j\}$. Just as before, without loss of generality, $k = i'$ and $l = j$. Then, since $i' \in v'^{S'}(\tau_1)$, $i \in v^S(\tau_1)$ must hold. Further, $j \in v^S(\tau_2)$ and $\langle i, j\rangle \in c$, so $\langle S, v\rangle \models C(\tau_1, \tau_2)$.

  - Alternatively, suppose $\{k,l\} \neq \{i',j\}$. The only ordered pairs in c' that contain i' are $\{\langle i',j\rangle, \langle j,i'\rangle, \langle i',i'\rangle\}$ and $\langle k,l\rangle \in c'$. Thus, if $k = i'$, then $l = i'$ must hold, since $l \neq j$. Analogously, if $k = i'$, then $l = i'$. In both cases, $i \in v^S(\tau_1)$ and $i \in v^S(\tau_2)$ and $\langle i,i\rangle \in c$. Thus $\langle S,v\rangle \models C(\tau_1,\tau_2)$.

    If both $k \neq i'$ and $l \neq i'$, then $k \in W$, $l \in W$, $k \in v^S(\tau_1)$, $l \in v^S(\tau_2)$ and $\langle k,l\rangle \in c$, so $\langle S,v\rangle \models C(\tau_1,\tau_2)$.

- For atomic formulas with qualitative measure, first observe that for any term τ, it holds that $\mu^m(v^S(\tau)) = \mu^{m'}(v'^{S'}\tau)$. To prove this, consider again two cases:

  - If $i \in v^S(\tau)$

$$\begin{aligned}
&\mu^m(v^S(\tau)) \stackrel{\text{def}}{=} \sum_{k\in v^S(\tau)} m(k) = m(i) + \sum_{k\in v^S(\tau)\setminus\{i\}} m(k) = \\
&m(i)/2 + m(i)/2 + \sum_{k\in v^S(\tau)\setminus\{i\}} m(k) \stackrel{\text{def}}{=} \\
&m'(i) + m'(i') + \sum_{k\in v^S(\tau)\setminus\{i\}} m'(k) = \sum_{k\in v^S(\tau)\cup\{i'\}} m'(k) = \\
&\sum_{k\in v'^{S'}(\tau)} m'(k) \stackrel{\text{def}}{=} \mu^{m'}(v'^{S'}(\tau))
\end{aligned}$$

  - If $i \notin v^S(\tau)$, then

$$\mu^m(v^S(\tau)) \stackrel{\text{def}}{=} \sum_{k\in v^S(\tau)} m(k) = \sum_{k\in v'^{S'}(\tau)} m'(k) \stackrel{\text{def}}{=} \mu^{m'}(v'^{S'}(\tau))$$

  Given this, $\langle S',v'\rangle \models \tau_1 \preceq \tau_2 \leftrightarrow \langle S,v\rangle \models \tau_1 \preceq \tau_2$ trivially holds.

- Assuming that the statement holds for ϕ and ψ, the proof that it holds for $\phi \vee \psi$, $\phi \wedge \psi$, $\neg\phi$, $\phi \Rightarrow \psi$ is direct.

□

The untying lemma allows for the gradual increase in the number of vertices in a model's contact graph until all cycles have been eliminated, resulting in a tree. Intuitively, the information stored in the model is being moved to the valuation.

Corollary. *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m\rangle$ be a convertible relational model and v be a disjoint valuation. Then, there is a procedure to effectively construct a convertible relational model S' and a disjoint valuation v' for S' such that $Gr(S')$ is a tree and S' models the same formulas.*

Proof. Suppose $Gr(S)$ has n vertices and m edges. Let $S_0 = S$, $w_0 = v$ and S_{k+1} and w_{k+1} be the convertible relational model and disjoint valuation obtained by applying the untying lemma to S_k and w_k, for $k = 0, \dots, m-n$. After each application of the lemma, the number of vertices increases by one and the number of edges remains the same. Thus, S_{m-n+1} has $n+m-n+1 = m+1$ vertices and m edges and therefore $Gr(S_{m-n+1})$ is a tree. $\square$

The rest of this section deals with converting this tree to a real line model.

Definition (Preserving Tree Mapping)**.** *Let $T = \langle V, E\rangle$ be a tree and $m : V \to \mathbb{R}^+$. A function $f : V \to Pol(\mathbb{R})$ is called a* preserving tree mapping *of $\langle T, m\rangle$ if the following hold:*

- $(\forall u \in V)(\forall w \in V)(u \neq w \to f(u) \Cap f(w) = \emptyset)$ *(parthood),*
- $(\forall u \in V)(\forall w \in V)(u \neq w \to (\{u, w\} \in E \leftrightarrow \mathcal{C}^{\mathbb{R}}(f(u), f(v))))$ *(contact),*
- $(\forall u \in V)(m(u) = \mu^{\mathbb{R}}(f(u)))$ *(measure).*

For notational convenience, throughout the rest of the section m and functions that play an analogous role of giving weight to vertices are implicitly extended to any set of vertices $X \subseteq V$ in the following way: $m(X) \overset{\text{def}}{=} \sum_{u\in X} m(u)$.

Lemma (Tree Mapping to Interval, Root at Both Ends)**.** *Let $T = \langle V, E\rangle$ be a rooted tree with root r. For any $m : V \to \mathbb{R}^+$, $a \in \mathbb{R}$ and $b = a + \sum_{u\in V} m(u)$ there is an effective procedure to construct a preserving tree mapping f of $\langle T, m\rangle$ such that:*

- $\bigcup_{u\in V} f(u) = [a, b]$,
- $a \in f(r)$ *and* $b \in f(r)$.

Proof. Let $w \in V$, $Children(T, w) = \{c_1, c_2, \dots c_n\}$.

Consider the assumption that for all $i \in \{1, 2, \dots n\}$ the statement holds for $SubTree(T, c_i)$. If, given this assumption, the statement can be proven for $SubTree(T, w)$, then the statement will have been proven for T by induction.

Thus, suppose as induction hypothesis that the statement holds for all $SubTree(T, c_i)$ where $i \in \{1, 2, \dots n\}$.

Let $SubTree(T, w) = \langle V_w, E_w\rangle$ and $SubTree(T, c_i) = \langle V_{c_i}, E_{c_i}\rangle$, for $i \in \{1, 2, \dots, n\}$.

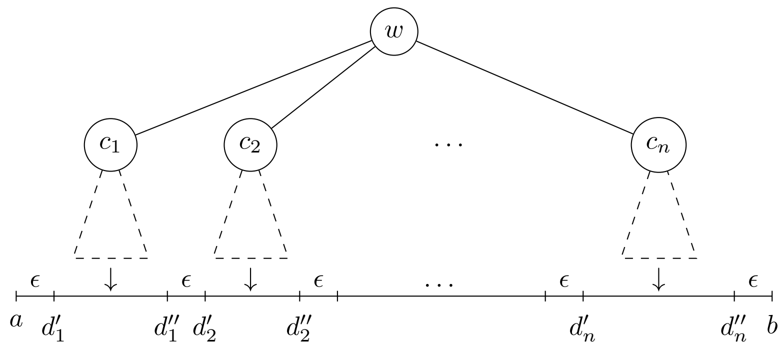
Let $m : V_w \to \mathbb{R}^+$. Let $a \in \mathbb{R}$ be arbitrary, $b = a + m(V_w)$. Define:

$$\epsilon \overset{\text{def}}{=} \frac{m(w)}{n+1}$$

$$d'_i \overset{\text{def}}{=} \epsilon + \sum_{1\leq k<i} (\epsilon + m(V_{c_k})), \text{ for } 1 \leq i \leq n$$

$$d''_i \overset{\text{def}}{=} d'_i + m(V_{c_i}), \text{ for } 1 \leq i \leq n$$

$$p \overset{\text{def}}{=} \begin{cases} [a, b] & \text{if } n = 0 \\ [a, d'_1] \cup [d''_n, b] \cup \bigcup_{1\leq i<n} [d''_i, d'_{i+1}] & \text{otherwise} \end{cases}$$



Let f_i be the preserving tree mapping from the induction hypothesis for $SubTree(T, c_i)$, $m \restriction V_{c_i}$ and d'_{c_i} (the interval $[d'_{c_i}, d''_{c_i}]$). Now, we can define f:

$$f \stackrel{\text{def}}{=} \langle w, p \rangle \cup \bigcup_{1 \leq i \leq n} f_i$$

The domains of these functions are disjoint, so f is a function from V_w to $\mathrm{Pol}(\mathbb{R})$. Clearly $a < d'_i < d''_i < d'_j < d''_j < b$ for $1 \leq i < j \leq n$, so

$$(\forall u \in V_w)(f(u) \subseteq [a, b]).$$

Further,

$$\bigcup_{u \in V_w} f(u) = p \cup \bigcup_{1 \leq i \leq n} [d'_i, d''_i] = [a, b]$$

and $a \in p = f(w)$, $b \in p = f(w)$.

Now it remains to prove that f is a preserving tree mapping.

- To show the parthood property, suppose $u_1 \in V_w$, $u_2 \in V_w$ and $u_1 \neq u_2$.
  - If u_1 and u_2 are both in V_{c_i} for some $i \in \{1, \ldots n\}$, given that f_i is a preserving tree mapping, $f(u_1) \Cap f(u_2) = f_i(u_1) \Cap f_i(u_2) = \emptyset$.
  - If $u_1 \in V_{c_i}$ and $u_2 \in V_{c_j}$, for some $i, j \in \{1, \ldots, n\}$ and $i \neq j$, then $f(u_1) = f_i(u_1) \subseteq [d'_i, d''_i]$ and $f(u_2) = f_j(u_2) \subseteq [d'_j, d''_j]$. $[d'_i, d''_i] \cap [d'_j, d''_j] = \emptyset$, so $f(u_1) \Cap f(u_2) \subseteq f(u_1) \cap f(u_2) = \emptyset$.
  - If $u_1 = w$ or $u_2 = w$, without loss of generality, assume $u_1 = w$. Then $u_2 \in V_i$ for some $i \in \{1, \ldots n\}$. Thus, $f(u_1) = p$ and $f(u_2) \subseteq [d'_i, d''_i]$.

    $$f(u_1) \Cap f(u_2) \subseteq f(u_1) \cap f(u_2) \subseteq p \cap [d'_i, d''_i] \subseteq \{d'_i, d''_i\}$$

    and so $f(u_1) \Cap f(u_2) = \emptyset$.
- To show the contact property, suppose $u_1 \in V_w$, $u_2 \in V_w$ and $u_1 \neq u_2$.

– If u_1 and u_2 are both in V_{c_i} for some $i \in \{1, \dots n\}$, $f(u_1) = f_i(u_1)$, $f(u_2) = f_i(u_2)$

$$\mathcal{C}^{\mathbb{R}}(f(u_1), f(u_1)) \leftrightarrow \mathcal{C}^{\mathbb{R}}(f_i(u_1), f_i(u_1)) \overset{\text{i.h.}}{\leftrightarrow}$$
$$\{u_1, u_2\} \in E_{c_i} \leftrightarrow \{u_1, u_2\} \in E_w.$$

– If $u_1 \in V_{c_i}$ and $u_2 \in V_{c_j}$, for some $i, j \in \{1, \dots, n\}$ and $i \neq j$, then $\{u_1, u_1\} \notin E_w$. $f(u_1) = f_i(u_1) \subseteq [d'_i, d''_i]$, $f(u_2) = f_j(u_2) \subseteq [d'_j, d''_j]$. $[d'_i, d''_i] \cap [d'_j, d''_j] = \emptyset$, so $f(u_1) \cap f(u_2) = \emptyset$ and $\mathcal{C}^{\mathbb{R}}(f(u_1), f(u_2))$ does not hold.

– If $u_1 = w$ or $u_2 = w$, without loss of generality, assume $u_1 = w$. Then $u_2 \in V_i$ for some $i \in \{1, \dots n\}$.

$$\begin{aligned}
\{u_1, u_2\} \in E_W &\leftrightarrow \\
u_2 = c_i &\overset{\text{i.h.}}{\leftrightarrow} && \text{(looking at the Boolean} \\
& && \text{disjoint polytopes in } [d'_i, d''_i]) \\
d'_i \in f_i(c_i) &\leftrightarrow && \text{(considering the definition of p)} \\
p \cap f_i(c_i) \neq \emptyset &\leftrightarrow \\
f(u_1) \cap f(u_2) \neq \emptyset &\leftrightarrow \\
\mathcal{C}^{\mathbb{R}}(f(u_1), f(u_2)&)
\end{aligned}$$

- For the measure property, simply note that it holds for f_i and $u \in V_{c_i}$, where $i = 1, \dots, n$, by induction hypothesis and that

$$\mu^{\mathbb{R}}(f(w)) = \mu^{\mathbb{R}}(p) \overset{\text{def}}{=} \epsilon + \epsilon + (n-1)\epsilon = (n+1)\frac{m(w)}{n+1} = m(w).$$

Finally, note that the given definition of f is easily implemented in a procedure, f is effectively constructable from $f_1, f_2, \dots, f_n$. □

Remark that $a \in f(r)$ in the lemma above means also that for all $u \in V \setminus \{r\}$, $a \notin f(u)$. This is because if $a \in f(u)$, for some u, then an interval in the standard representation of $f(r)$ and an interval in the standard representation of $f(u)$ would both contain a. Since both intervals are of non-zero length, they would have a regular, closed intersection and therefore $f(r) \Cap f(u) \neq \emptyset$, which is not allowed by the definition of preserving tree mapping. This reasoning will also be implicitly used on other occasions in the rest of the text.

The result of the lemma is slightly generalized below to allow a different vertex in the tree to touch the right end of the interval.

Lemma (Tree Mapping to Interval, Arbitrary Right End). *Let $T = \langle V, E\rangle$ be a rooted tree with root r. For any $m : V \to \mathbb{R}^+$, $s \in V$, $a \in \mathbb{R}$ and $b = a + \sum_{u \in V} m(u)$ there is an effective procedure to construct a preserving tree mapping f of $\langle T, m\rangle$ such that:*

- $\bigcup_{u \in V} f(u) = [a, b]$,
- $a \in f(r)$ *and* $b \in f(s)$.

Proof. If $r = s$, the previous lemma is directly applied.

Suppose $r \neq s$ and $v_1, v_2, \ldots, v_n = s$ are the vertices along the path from r to s. Let

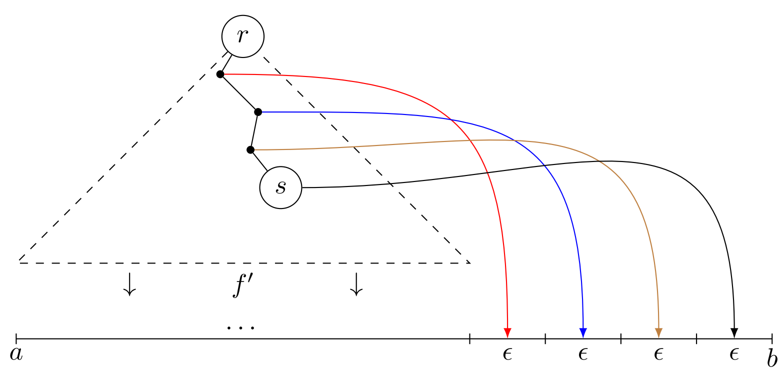
$$\epsilon \stackrel{\text{def}}{=} \frac{\min\{v_1, v_2, \ldots, v_n\}}{2}.$$

Consider the modification to m:

$$m'(u) \stackrel{\text{def}}{=} \begin{cases} m(u) - \epsilon & \text{if } u \in \{v_1, v_2, \ldots, v_n\} \\ m(u) & \text{otherwise.} \end{cases}$$

Let f' be the result of the application of the previous lemma to the same tree, but with the function m' and for the interval $[a, a + m'(V)] = [a, b - n\epsilon]$.

$$f(u) \stackrel{\text{def}}{=} \begin{cases} f'(u) \cup [b - (n - i + 1)\epsilon, b - (n - i)\epsilon] & \text{if } u = v_i, \text{ for } i \in \{1, 2, \ldots, n\} \\ f'(u) & \text{otherwise} \end{cases}$$



Clearly, for $1 \leq i < j \leq n$,

$$\begin{gathered} a < b - n\epsilon \leq \\ b - (n - i + 1)\epsilon < b - (n - i)\epsilon \leq \\ b - (n - j + 1)\epsilon < b - (n - j)\epsilon \leq b. \end{gathered}$$

It can be seen by direct inspection that

$$\bigcup_{u \in V} f(u) = [a, b],$$
$$a \in f(r) \text{ and } b \in f(s).$$

- To show the parthood and contact properties of f simultaneously, consider $u_1, u_2 \in V$, $u_1 \neq u_2$.

- If both u_1 and u_2 are not in $\{r, v_1, \ldots, v_n\}$, then $f(u_1) = f'(u_1)$ and $f(u_2) = f'(u_2)$ and the properties hold since f' is a preserving tree mapping.
- If $u_1 \in \{r, v_1, \ldots, v_n\}$, $u_2 \notin \{r, v_1, \ldots, v_n\}$ or vice-versa, then

  $$f(u_2) = f'(u_2) \subsetneq [a, b - n\epsilon], \text{ so } f(u_1) \cap f(u_2) \subseteq [a, b - n\epsilon].$$

  Therefore,

  $$\begin{aligned} f(u_1) \cap f(u_2) &= f'(u_1) \cap f'(u_2) \\ f(u_1) \Cap f(u_2) &= f'(u_1) \Cap f'(u_2) \end{aligned}$$

  and again the fact that f' is a preserving tree mapping is used.
- If $u_1 = v_i$ and $u_2 = v_j$, without loss of generality, $i < j$.
  * If $\{u_1, u_2\} \in E$, then $i + 1 = j$.

    $$\begin{aligned} &f(u_1) \Cap f(u_2) \subseteq f(u_1) \cap f(u_2) = \\ &(f'(u_1) \cap f'(u_2)) \cup \{b - (n - i)\epsilon\}. \end{aligned}$$

    $f'(u_1) \cap f'(u_2)$ is finite, so $f(u_1) \Cap f(u_2) = \emptyset$. Also,

    $$b - (n - i)\epsilon \in f(u_1) \cap f(u_2), \text{ i.e. } \mathcal{C}^{\mathbb{R}}(f(u_1), f(u_2)).$$

  * If $\{u_1, u_2\} \notin E$, $f'(u_1) \cap f'(u_2) = \emptyset$, because f' is a preserving tree mapping and

    $$(f(u_1) \cap f(u_2)) \cap [b - n\epsilon, b] = \emptyset,$$

    so $f(u_1) \cap f(u_2) = \emptyset$ and of course $f(u_1) \Cap f(u_2) = \emptyset$.
- To show the measure property for $u \in V$, consider two simple cases:
  * if $u \in \{v_1, \ldots, v_n\}$, then

    $$\begin{aligned} \mu^{\mathbb{R}}(f(u)) &= \mu^{\mathbb{R}}(f'(u)) + \epsilon = \\ m'(u) + \epsilon &= (m(u) - \epsilon) + \epsilon = \\ &= m(u); \end{aligned}$$

  * otherwise, $\mu^{\mathbb{R}}(f(u)) = \mu^{\mathbb{R}}(f'(u)) = m'(u) = m(u)$.

□

The same idea is easily generalized to trees with an infinite root and half the real line.

Lemma (Tree Mapping to Half Line)**.** *Let $T = \langle V, E \rangle$ be a rooted tree with root r. For any $m : V \to \mathbb{R}^+ \cup \{\infty\}$ such that $(\forall v \in V)(m(v) = \infty \leftrightarrow v = r)$ and any $a \in \mathbb{R}$ there is an effective procedure to construct a preserving tree mapping f of $\langle T, m \rangle$ such that:*

- $\bigcup_{u \in V} f(u) = [a, \infty)$,
- $a \in f(r)$.

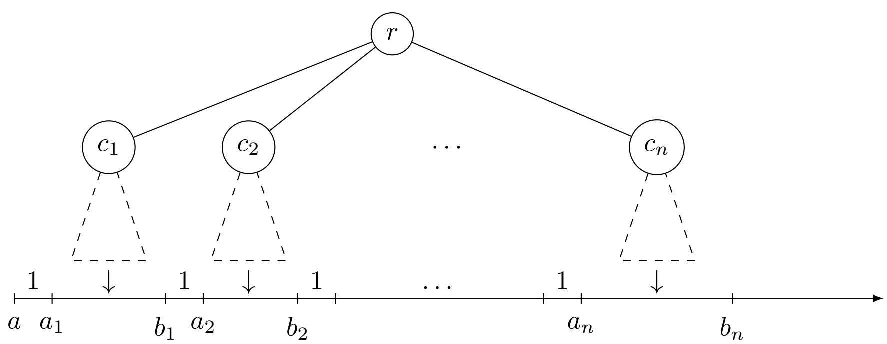
Proof. Let $c_1, c_2, \ldots, c_n$ be the children of r. If $n = 0$, the entire mapping is $\{\langle r, [a, \infty)\rangle\}$. If not, let

$$a_i \stackrel{\text{def}}{=} a + \sum_{j<i} (1 + m(\mathit{SubTree}(T, c_j)))$$

$$b_i \stackrel{\text{def}}{=} a_i + m(\mathit{SubTree}(T, c_i)),$$

for $i \in \{1, \ldots, n\}$. Let

$$p \stackrel{\text{def}}{=} [a, a_1] \cup \bigcup_{1 \leq i \leq n} [b_i, a_{i+1}]$$



We apply the (Tree Mapping to Interval, Root at Both Ends) lemma to each child c_i ($i \in \{1, \ldots, n\}$) and interval $[a_i, b_i]$ to get $f_1, f_2, \ldots, f_n$ and define f:

$$f(u) \stackrel{\text{def}}{=} \begin{cases} f_i(u) & \text{if } u \in \mathit{SubTree}(T, c_i), \text{ for some } i \in \{1, \ldots, n\} \\ p & \text{if } u = r \end{cases}$$

The rest of the proof is analogous to the induction step of the proof of the (Tree Mapping to Interval, Root at Both Ends) lemma, except for the simplification of the measure of $f(r)$ being ∞. □

Corollary. *The same can be done on the interval $(-\infty, a]$ by mirror symmetry.*

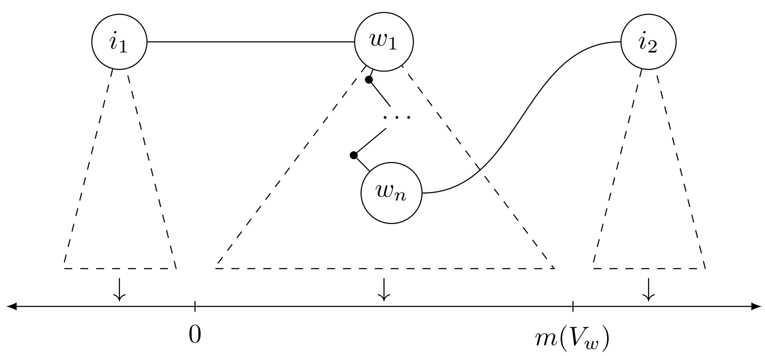
Lemma (Relational Model Conversion)**.** *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ be a convertible relational model and v be a disjoint valuation. If ϕ is a formula from $\mathcal{L}$ and $\langle S, v\rangle \models \phi$, then a valuation $v_{\mathbb{R}} : \mathit{Vars} \to \mathit{Pol}(\mathbb{R})$ such that $\langle S^{\mathbb{R}}, v_{\mathbb{R}}\rangle \models \phi$ can be effectively constructed.*

Proof. S is a convertible relational model, so by the corollary of the (Untying) lemma, let $S' = \langle \mathcal{B}^{W'}, \mathcal{C}^{c'}, \leq_{m'} \rangle$ and v' be a convertible relational model and a valuation which model the same formulas and it holds that $Gr(S') = \langle V, E\rangle$ is a tree.

Since S' is convertible, there are exactly two vertices with infinite values for m.Let i_1 and i_2 be those vertices, $m(i_1) = m(i_2) = \infty$, and $i_1, w_1, w_2, \ldots, w_n, i_2$ be the path from i_1 to i_2. The graph $\langle V, E \setminus \{\{i_1, w_1\}, \{w_n, i_2\}\}\rangle$ has three connected components: one containing i_1, one containing $w_1, \ldots, w_n$ and one containing i_2. Let $\langle V_{i_1}, E_{i_1}\rangle$, $\langle V_w, E_v\rangle$ and $\langle V_{i_2}, E_{i_2}\rangle$ respectively be the trees induced by those components.

Consider $\langle V_{i_1}, E_{i_1}\rangle$ and $\langle V_{i_2}, E_{i_2}\rangle$ as rooted trees with roots i_1 and i_2 respectively. They satisfy the conditions for the (Tree Mapping to Half Line) lemma and corollary (with $m \restriction V_{i_1}$ and $m \restriction V_{i_2}$). Thus, let f_{i_1} be the mapping from the corollary for $\langle V_{i_1}, E_{i_1}\rangle$ and interval $(-\infty, 0]$ and f_{i_2} be the mapping from the lemma for $\langle V_{i_1}, E_{i_1}\rangle$ and interval $[m(V_w), \infty)$.

Now, consider $\langle V_w, E_v\rangle$ as a rooted tree with root w_1. The (Tree Mapping to Interval, Arbitrary Right End) lemma is applied to this rooted tree with w_n in the role of the vertex s (and $m \restriction V_w$), resulting in a preserving tree mapping f_w onto the interval $[0, m(V_w)]$.



Of course, it should be noted that if $n = 0$, i.e. i_1 and i_2 are directly connected by an edge, then there would only be two connected components. In that case, $\langle V_w, E_v\rangle$ would be an empty graph. This poses no problem to the rest of the proof, as then $m(V_w) = 0$ and this empty graph would be mapped to a zero-length interval.

Let $f = f_{i_1} \cup f_w \cup f_{i_2}$. First, it is useful to note that

$$\bigcup_{i \in V} f(i) = \bigcup_{i \in V_{i_1}} f_{i_1}(i) \cup \bigcup_{i \in V_w} f_w(i) \cup \bigcup_{i \in V_{i_2}} f_{i_2}(i) =$$
$$(-\infty, 0] \cup [0, m(V_w)] \cup [m(V_w), \infty) = \mathbb{R}.$$

Now follows a proof that f is a preserving tree mapping of $\langle\langle V, E\rangle, m\rangle$.

- Let $u_1 \in V$ and $u_2 \in V$, $u_1 \neq u_2$.
  - If u_1 and u_2 are in the same connected component of the disconnected graph, then the parthood and contact properties are satisfied for f since f_{i_1}, f_w, and f_{i_2} are preserving tree mappings.

- If u_1 is in one of V_{i_1}, V_{i_2} and u_2 in the other, then $f(u_1) \subseteq (-\infty, 0]$ and $f(u_2) \subseteq [m(V_w), \infty)$ or vice-versa. Thus, $f(u_1) \cap f(u_2) = \emptyset$ is only possible if V_w is empty ($n = 0$) and $\{u_1, u_2\} = \{i_1, i_2\}$, in which case $\{u_1, u_2\} \in E$. In any case, $f(u_1) \Cap f(u_2) = \emptyset$.
- Suppose u_1 is in V_w and u_2 in V_{i_1}. Then $f(u_1) \subseteq (-\infty, 0]$ and $f(u_2) \subseteq [0, m(V_w)]$. Of course, $f(u_1) \Cap f(u_2) = \emptyset$. $f(u_1) \cap f(u_2)$ holds by construction of the lemma exactly if $u_1 = i_1$ and $u_2 = w_1$, in which case $\{u_1, u_2\} \in E$.

  A similar argument is made if u_1 and u_2 switch places above or if u_1 is in V_{i_2} and u_2 is in V_w or the other way around.

* For any $u \in V$, u is in one of V_{i_1}, V_w or V_{i_1}. In all three cases, $\mu^{\mathbb{R}}(f(u)) = m(u)$ since f_{i_1}, f_w and f_{i_2} are preserving tree mappings.

Let $v_{\mathbb{R}}(x) \stackrel{\text{def}}{=} \bigcup\{f(i) \mid i \in v'(x)\}$. The proof that for any term τ of $\mathcal{L}$, $v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau) = \bigcup\{f(i) \mid i \in v'^{S'}(\tau)\}$ follows.

For variables the statement is true by definition. For the constants:

$$v_{\mathbb{R}}^{S^{\mathbb{R}}}(0) \stackrel{\text{def}}{=} \emptyset = \bigcup\{f(i) \mid i \in \emptyset\} \stackrel{\text{def}}{=} \bigcup\{f(i) \mid i \in v'^{S'}(0)\}$$

and

$$\begin{aligned}
&v_{\mathbb{R}}^{S^{\mathbb{R}}}(1) \stackrel{\text{def}}{=} \mathbb{R} = (-\infty, 0] \cup [0, m(V_w)] \cup [m(V_w), \infty) = \\
&\bigcup\{f(i) \mid i \in V_{i_1}\} \cup \bigcup\{f(i) \mid i \in V_w\} \cup \bigcup\{f(i) \mid i \in V_{i_2}\} = \\
&\bigcup\{f(i) \mid i \in V\} \stackrel{\text{def}}{=} \bigcup\{f(i) \mid i \in v'^{S'}(1)\}
\end{aligned}$$

Suppose that the statement is true for τ_1 and τ_2, terms of $\mathcal{L}$. For the union operation:

$$\begin{aligned}
&v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1 \sqcup \tau_2) \stackrel{\text{def}}{=} v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1) \cup v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_2) \stackrel{\text{i.h.}}{=} \\
&\bigcup\{f(i) \mid i \in v'^{S'}(\tau_1)\} \cup \bigcup\{f(i) \mid i \in v'^{S'}(\tau_2)\} = \\
&\bigcup\{f(i) \mid i \in v'^{S'}(\tau_1 \sqcup \tau_2)\}
\end{aligned}$$

and for intersection:

$$\begin{aligned}
&v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1 \sqcap \tau_2) \stackrel{\text{def}}{=} v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1) \Cap v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_2) \stackrel{\text{i.h.}}{=} \\
&\bigcup\left\{f(i) \mid i \in v'^{S'}(\tau_1)\right\} \Cap \bigcup\left\{f(j) \mid j \in v'^{S'}(\tau_2)\right\} = && /\mathcal{B}^{\mathbb{R}} \text{ is Boolean algebra} \\
&\bigcup\left\{f(i) \Cap \bigcup\left\{f(j) \mid j \in v'^{S'}(\tau_2)\right\} \mid i \in v'^{S'}(\tau_1)\right\} = && /\mathcal{B}^{\mathbb{R}} \text{ is Boolean algebra} \\
&\bigcup\left\{\bigcup\left\{f(i) \Cap f(j) \mid j \in v'^{S'}(\tau_2)\right\} \mid i \in v'^{S'}(\tau_1)\right\} = && /B^{\mathbb{R}} \text{ is Boolean algebra} \\
&\bigcup\left\{f(i) \Cap f(j) \mid j \in v'^{S'}(\tau_2), i \in v'^{S'}(\tau_1)\right\} = && /i \neq j \to f(i) \Cap f(j) = \emptyset \\
&\bigcup\left\{f(i) \mid i \in v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2)\right\} \stackrel{\text{def}}{=} \\
&\bigcup\left\{f(i) \mid i \in v'^{S'}(\tau_1 \sqcap \tau_2)\right\}
\end{aligned}$$

For complement:

$$v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1^*) \stackrel{\text{def}}{=} Cl\left(\mathbb{R} \setminus v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1)\right) \stackrel{\text{i.h.}}{=} Cl\left(\mathbb{R} \setminus \bigcup\left\{f(i) \mid i \in v'^{S'}(\tau_1)\right\}\right) =$$
$$Cl\left(\bigcup\left\{f(i) \mid i \in \left(v'^{S'}(\tau_1) \cup V \setminus v'^{S'}(\tau_1)\right)\right\} \setminus \bigcup\left\{f(i) \mid i \in v'^{S'}(\tau_1)\right\}\right) =$$
$$Cl\left(\bigcup\left\{f(i) \mid i \in V \setminus v'^{S'}(\tau_1)\right\} \setminus \bigcup\left\{f(j) \mid j \in v'^{S'}(\tau_1)\right\}\right) =$$
$$Cl\left(\bigcup\left\{f(i) \setminus \bigcup\left\{f(j) \mid j \in v'^{S'}(\tau_1)\right\} \mid i \in V \setminus v'^{S'}(\tau_1)\right\}\right) =$$
$$Cl\left(\bigcup\left\{\bigcap\left\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\right\} \mid i \in V \setminus v'^{S'}(\tau_1)\right\}\right) =$$
$$\bigcup\left\{Cl\left(\bigcap\left\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\right\}\right) \mid i \in V \setminus v'^{S'}(\tau_1)\right\}$$

Consider $f(i) \setminus f(j)$ in the above expression. $i \in V \setminus v'^{S'}(\tau_1)$ and $j \in v'^{S'}(\tau_1)$, so $i \neq j$. This means that $\emptyset = f(i) \Cap f(j) = Cl(Int(f(i) \cap f(j)))$, so $Int(f(i) \cap f(j)) = \emptyset$. Thus, $Int(f(i)) \subseteq f(i) \setminus f(j)$. From here, $Int(f(i)) \subseteq \bigcap\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\}$ and applying Cl to both sides, we get $Cl(Int(f(i))) \subseteq Cl\left(\bigcap\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\}\right)$. Polytopes are regular closed sets, so $f(i) \subseteq Cl\left(\bigcap\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\}\right)$. On the other hand, $Cl\left(\bigcap\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\}\right) \subseteq Cl\left(\bigcap\{f(i) \mid j \in v'^{S'}(\tau_1)\}\right) = Cl(f(i)) = f(i)$. Thus, $f(i) = Cl\left(\bigcap\{f(i) \setminus f(j) \mid j \in v'^{S'}(\tau_1)\}\right)$ and by substituting in the last of the series of equations above, $v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1 \sqcap \tau_2) = \bigcup\{f(i) \mid i \in V \setminus v'^{S'}(\tau_1)\} \stackrel{\text{def}}{=} \bigcup\{f(i) \mid i \in v'^{S'}(\tau_1^*)\}$.

To finish the proof of the lemma, a proof by induction on the construction of ϕ that $\langle S^{\mathbb{R}}, v_{\mathbb{R}}\rangle \models \phi \leftrightarrow \langle S', v'\rangle \models \phi$ follows.

- For $\bot$ and $\top$, the statement is trivial.
- For $\tau_1 \sqsubseteq \tau_2$,

$$\begin{aligned} \langle \mathcal{B}^{\mathbb{R}}, v_{\mathbb{R}}\rangle \models \tau_1 \sqsubseteq \tau_2 \leftrightarrow \\ v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1) \subseteq v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_2) \leftrightarrow \\ \bigcup\{f(i) \mid i \in v'^{S'}(\tau_1)\} \subseteq \bigcup\{f(i) \mid i \in v'^{S'}(\tau_2)\} \leftrightarrow \qquad & (*) \\ v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \leftrightarrow \\ \langle S', v'\rangle \models \tau_1 \sqsubseteq \tau_2 \end{aligned}$$

  The equivalence at $(*)$ from right to left is direct.

  From left to right, first observe that for all $i, j \in V$, if $i \neq j$, then $Int(f(i)) \cap f(j) = \emptyset$. To show this, suppose the opposite: that $x \in Int(f(i)) \cap f(j)$. x is an interior point for $f(i)$, so there must be a neighborhood X of x, contained in $f(i)$. x is in $f(j)$, a regular closed set, so X contains an interior point y of $f(j)$. Let Y be a neighborhood of y, contained in $f(j)$. $y \in X \cap Y$, so $X \cap Y$ is a non-empty open set. $X \cap Y \subseteq f(i) \cap f(j)$, which is a contradiction with $Cl(Int(f(i) \cap f(j))) = \emptyset$. Bearing this in mind, let $i \in v'^{S'}(\tau_1)$. $Int(f(i)) \subseteq \bigcup\{f(i) \mid i \in v'^{S'}(\tau_1)\} \subseteq \bigcup\{f(i) \mid i \in v'^{S'}(\tau_2)\}$, so $i \in v'^{S'}(\tau_2)$ and $v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2)$.

- For $C(\tau_1, \tau_2)$,

$$\begin{aligned}
\langle \mathcal{B}^{\mathbb{R}}, v_{\mathbb{R}} \rangle \models C(\tau_1, \tau_2) \leftrightarrow \\
v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1) \cap v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_2) \neq \emptyset \leftrightarrow \\
\bigcup\{f(i) \mid i \in v'^{S'}(\tau_1)\} \cap \bigcup\{f(i) \mid i \in v'^{S'}(\tau_2)\} \neq \emptyset \leftrightarrow \\
(\exists i \in v'^{S'}(\tau_1))(\exists j \in v'^{S'}(\tau_2))(f(i) \cap f(j) \neq \emptyset) \leftrightarrow \\
(\exists i \in v'^{S'}(\tau_1))(\exists j \in v'^{S'}(\tau_2))(\mathcal{C}^{\mathbb{R}}(f(i), f(j))) \leftrightarrow \\
(\exists i \in v'^{S'}(\tau_1))(\exists j \in v'^{S'}(\tau_2))(\{i, j\} \in E) \leftrightarrow \\
v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \leftrightarrow \\
\langle S', v' \rangle \models C(\tau_1, \tau_2)
\end{aligned}$$

- For $\tau_1 \preceq \tau_2$,

$$\begin{aligned}
\langle \mathcal{B}^{\mathbb{R}}, v_{\mathbb{R}} \rangle \models \tau_1 \preceq \tau_2 \leftrightarrow \\
\mu^{\mathbb{R}}(v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_1)) \leq \mu^{\mathbb{R}}(v_{\mathbb{R}}^{S^{\mathbb{R}}}(\tau_2)) \leftrightarrow \\
\mu^{\mathbb{R}}\left(\bigcup\{f(i) \mid i \in v'^{S'}(\tau_1)\}\right) \leq \mu^{\mathbb{R}}\left(\bigcup\{f(i) \mid i \in v'^{S'}(\tau_2)\}\right) \leftrightarrow \text{TODO} \\
\sum_{i \in v'^{S'}(\tau_1)} \mu^{\mathbb{R}}(f(i)) \leq \sum_{i \in v'^{S'}(\tau_2)} \mu^{\mathbb{R}}(f(i)) \leftrightarrow \\
\sum_{i \in v'^{S'}(\tau_1)} m(i) \leq \sum_{i \in v'^{S'}(\tau_2)} m(i) \leftrightarrow \\
\langle S', v' \rangle \models \tau_1 \preceq \tau_2
\end{aligned}$$

□

5 Axiomatization

This section presents a decidable set of formulas of $\mathcal{L}$ which axiomatize $S^{\mathbb{R}}$. That is to say they are valid in $S^{\mathbb{R}}$ and all formulas of $\mathcal{L}$, valid in $S^{\mathbb{R}}$, can be formally derived from them.

Formal derivation is meant in the usual proof-theoretic sense. The logical axioms and rules of inference can for example be in Hilbert-style for a first order language with equality. In the case of $\mathcal{L}$, where no quantifiers are present, logical axioms or rules relating to quantifiers are superfluous and could be left out.

The axioms that pertain to $\sqcup$, $\sqcap$ and * are those of a Boolean algebra and are given below. The particular set chosen here is as in (TODO: cite Halmos).

$$x \sqcap 1 \doteq x \qquad\qquad x \sqcup 0 \doteq x$$

$$x \sqcap x^* \doteq 0 \qquad\qquad x \sqcup x^* \doteq 1$$

$$x \sqcap y \doteq y \sqcap x \qquad\qquad x \sqcup y \doteq y \sqcup x$$

$$x \sqcap (y \sqcup z) \doteq (x \sqcap y) \sqcup (x \sqcap z) \qquad\qquad x \sqcup (y \sqcap z) \doteq (x \sqcup y) \sqcap (x \sqcup z)$$

It will be assumed without further mention that basic statements about Boolean algebras can be formally derived from these and that these formal proofs can be effectively generated. The parthood relation is the partial ordering of the Boolean algebra. Formal equality occurs when two objects are parts of each other.

$$(x \sqsubseteq y) \Leftrightarrow (x \sqcap y \doteq x) \qquad\qquad (x \doteq y) \Leftrightarrow (x \sqsubseteq y \wedge y \sqsubseteq x)$$

The axioms for contact are:

$$(x \not\doteq 0) \Leftrightarrow C(x,x) \qquad\qquad C(x, y \sqcup z) \Leftrightarrow C(x,y) \vee C(x,z)$$

$$C(x,y) \Rightarrow C(y,x) \qquad\qquad (x \not\doteq 0) \wedge (x \not\doteq 1) \Rightarrow C(x, x^*)$$

Below are are some of the axioms for measure.

$$(x \preceq y) \wedge (y \sqcap z \doteq 0) \Rightarrow (x \sqcap z) \preceq (y \sqcap z)$$

$$(x \sqcap z \doteq 0) \wedge (y \sqcap z \doteq 0) \wedge (z \prec 1) \Rightarrow (x \preceq y \Leftrightarrow (x \sqcup z) \preceq (y \sqcup z))$$

$$(x \sqcap z \doteq 0) \wedge (y \sqcap z \doteq 0) \wedge (z \prec 1) \Rightarrow (x \prec y \Leftrightarrow (x \sqcup z) \prec (y \sqcup z))$$

$$(x \asymp 1) \vee (x^* \asymp 1)$$

$$\neg(x \asymp 1 \wedge y \asymp 1 \wedge z \asymp 1 \wedge (x \sqcap y \doteq 0) \wedge (x \sqcap z \doteq 0) \wedge (y \sqcap z \doteq 0))$$

$$(x \asymp 0) \Leftrightarrow (x \doteq 0)$$

The rest of the measure axioms form an infinite, but decidable set and are presented below.

Definition ($(n, \leq)$-type and $(n, <)$-type inequalities)**.** *Let $r_1, r_2, \ldots, r_n$ be variables for real numbers and $I' \subseteq \{1, 2, \ldots, n\}$, $I'' \subseteq \{1, 2, \ldots, n\}$. Then*

$$\sum_{i \in I'} r_i \leq \sum_{i \in I''} r_i$$

is an $(n, \leq)$-type inequality *and*

$$\sum_{i \in I'} r_i < \sum_{i \in I''} r_i$$

is an $(n, <)$-type inequality*.*

Definition (M_n System of Inequalities)**.** *Any system of $(n, \leq)$- and $(n, <)$-type inequalities over the same variables $r_1, r_2, \ldots, r_n$, which also contains the inequalities $0 \leq r_i$, for all $i \in 1, 2, \ldots, n$, is called an M_n* system.

Definition (Formula ϕ_M, Matched to M_n System)**.** *Let M be an M_n system. To any $(n, \leq)$-type inequality*

$$\sum_{i \in I'} r_i \leq \sum_{i \in I''} r_i,$$

the formula of $\mathcal{L}$

$$\bigsqcup_{i \in I'} x_i \preceq \bigsqcup_{i \in I''} x_i$$

is matched. Analogously, to any $(n, <)$-type inequality

$$\sum_{i \in I'} r_i < \sum_{i \in I''} r_i,$$

the formula of $\mathcal{L}$

$$\bigsqcup_{i \in I'} x_i \prec \bigsqcup_{i \in I''} x_i$$

is matched.

The conjunction of all formulas matched to $(n, \leq)$- and $(n, <)$-type inequalities of M is denoted ϕ_M, the formula matched to M.

Given these definitions, the rest of the axioms that concern the qualitative measure can be defined. For every natural number n, for every M_n system M, if M does not have a solution in which exactly one or exactly two variables are assigned values of ∞, then

$$\left(\bigwedge_{1 \leq i < j \leq n} (x_i \sqcap x_j \doteq 0) \wedge \bigsqcup_{1 \leq i \leq n} x_i \doteq 1 \right) \Rightarrow \neg \phi_M$$

is an axiom.

5.1 Decidability

The axioms described above are infinitely many. This raises the question of their decidability, particularly the set of axioms which correspond to M_n systems, which form the infinite part.

Proposition. *There exists an effective procedure to determine if any system with n real variables (not allowing ∞), containing non-strict and strict linear* *inequalities with real coefficients, i.e.*

$$
\left|
\begin{array}{ll}
a_{1,1}r_1 + a_{1,2}r_2 + \cdots + a_{1,n}r_n & \leq b_1 \\
a_{2,1}r_1 + a_{2,2}r_2 + \cdots + a_{2,n}r_n & \leq b_2 \\
\ldots & \ldots \\
a_{m,1}r_1 + a_{m,2}r_2 + \cdots + a_{m,n}r_n & \leq b_m \\
a_{m+1,1}r_1 + a_{m+1,2}r_2 + \cdots + a_{m+1,n}r_n & < b_{m+1} \\
a_{m+2,1}r_1 + a_{m+2,2}r_2 + \cdots + a_{m+2,n}r_n & < b_{m+2} \\
\ldots & \ldots \\
a_{m+k,1}r_1 + a_{m+k,2}r_2 + \cdots + a_{m+k,n}r_n & < b_{m+k}
\end{array}
\right.
$$

has a solution and if so, to produce that solution.

Proof. After rearrangement of the inequalities to leave r_n on the left side,

$$
\left|
\begin{array}{ll}
r_n \underline{\triangle} & -\frac{a_{1,1}}{a_{1,n}}r_1 - \frac{a_{1,2}}{a_{1,n}}r_2 - \cdots - \frac{a_{1,n-1}}{a_{1,n}}r_{n-1} + \frac{b_1}{a_{1,n}} \\
r_n \underline{\triangle} & -\frac{a_{2,1}}{a_{2,n}}r_1 - \frac{a_{2,2}}{a_{2,n}}r_2 - \cdots - \frac{a_{2,n-1}}{a_{2,n}}r_{n-1} + \frac{b_2}{a_{2,n}} \\
\ldots & \ldots \\
r_n \underline{\triangle} & -\frac{a_{m,1}}{a_{m,n}}r_1 - \frac{a_{m,2}}{a_{m,n}}r_2 - \cdots - \frac{a_{m,n-1}}{a_{m,n}}r_{n-1} + \frac{b_m}{a_{m,n}} \\
r_n \triangle & -\frac{a_{m+1,1}}{a_{m+1,n}}r_1 - \frac{a_{m+1,2}}{a_{m+1,n}}r_2 - \cdots - \frac{a_{m+1,n-1}}{a_{m+1,n}}r_{n-1} + \frac{b_{m+1}}{a_{m+1,n}} \\
r_n \triangle & -\frac{a_{m+2,1}}{a_{m+2,n}}r_1 - \frac{a_{m+2,2}}{a_{m+2,n}}r_2 - \cdots - \frac{a_{m+2,n-1}}{a_{m+2,n}}r_{n-1} + \frac{b_{m+2}}{a_{m+2,n}} \\
\ldots & \ldots \\
r_n \triangle & -\frac{a_{m+k,1}}{a_{m+k,n}}r_1 - \frac{a_{m+k,2}}{a_{m+k,n}}r_2 - \cdots - \frac{a_{m+k,n-1}}{a_{m+k,n}}r_{n-1} + \frac{b_{m+k}}{a_{m+k,n}}
\end{array}
\right.
$$

is obtained, where $\underline{\triangle}$ stands for $\leq$ or $\geq$ and $\triangle$ stands for $<$ or $>$, depending on the sign of $a_{i,n}$, for $i = 1, \ldots, m+k$. Thus, for any assignment to $r_1, r_2, \ldots, r_{n-1}$, a series of strict and non-strict upper and lower bounds on r_n are achieved.

Given this, the rest of the proof will be by induction on the number of variables.

Suppose $n = 1$. The strict and non-strict upper and lower bounds on the single variable r_1 are fully determined and do not depend on any further variable assignments. If two lower bounds or two upper bounds are equal, one strict and one non-strict, then the strict one implies the non-strict one, so the non-strict is superfluous and is ignored. Let L be the set of lower bounds and U be the set of upper bounds. If $\max L = \min U$, and these two bounds are non-strict, then $\max L$ is a solution for r_1. Additionally, if $\max L < \min U$, then any value in (L_g, U_l) is a solution, for example $(L_g + U_l)/2$. In the other cases, there is no solution.

Suppose $n > 1$. For each right-hand side (linear expression) of the transformed system that forms a lower bound l and each each right-hand side that forms an upper bound u, consider the inequality $U \leq L$, if both bounds are non-strict and $U < L$, if at least one of them is strict. Thus, a new system of these linear inequalities is constructed for the $n-1$ variables $r_1, r_2, \ldots, r_{n-1}$. By induction hypothesis, there is a procedure to find a solution to this system or to conclude that there is none.

This new system is a consequence of the first, so if it has no solution, then the initial system also has no solution. On the other hand, if $\rho_1, \rho_2, \ldots, \rho_{n-1}$ is a solution for $r_1, r_2, \ldots, r_{n-1}$ in the new system, let L_ρ and U_ρ be the sets of upper and lower bounds on r_n, achieved by substituting r_i for ρ_i in the previous system, where $i = 1, 2, \ldots, n-1$. Again, as in the base case, superfluous non-strict bounds are ignored.

If $\max L_\rho = \min U_\rho$, then both bounds are non-strict (if they were strict, $\rho_1, \rho_2, \ldots, \rho_{n-1}$ wouldn't be a solution) and assigning $\max L_\rho$ to r_n yields a solution for the original system. Otherwise, assigning any value in the interval $(\max L_\rho, \min U_\rho)$ to r_n yields a solution, for example $(\max L_\rho + \min U_\rho)/2$. □

Corollary. *There exists an effective procedure to determine if any M_n system M has a solution where exactly one or exactly two variables are assigned ∞ and if so, to produce that solution.*

Proof. There are finitely many possible choices for exactly one or exactly two variables of those in M to be assigned ∞. For any such assignment, a new system is obtained. If a solution is found for any these new systems, it yields a solution to M and otherwise M has no solution (in which exactly one or exactly two variables are assigned ∞).

For any inequality of this new system, the following cases are considered:

- when the inequality is strict,
  - if an infinite assignment features on the left side, the system has no solution;
  - if an infinite assignment features on the right side, but not on the left, the inequality can be ignored;
  - if no infinite assignment features in the inequality, no change is made;
- when the inequality is non-strict,
  - if an infinite assignment features on the right side, the inequality can be ignored;
  - if an infinite assignment features on the left side, but not on the right, the system has no solution;
  - if no infinite assignment features in the inequality, no change is made.

If after this consideration, it has not been established that the system has no solution, then the inequalities which feature ∞ have been eliminated and the obtained system is of linear strict and non-strict inequalities. Therefore, the statement from the proposition can be applied. □

Proposition. *There is an effective procedure to decide if any formula ϕ of $\mathcal{L}$ is an axiom.*

Proof. First, check if ϕ is any of the finitely many axioms for Boolean algebra, contact or measure. If not, what remains is to check whether it corresponds to an appropriate M_n system.

If there are k variables in ϕ, then ϕ can correspond only to an M_k system. There are finitely many M_k systems so it remains only to exhaustively check for each one of them if:

- it has a solution with exactly one or exactly two variables assigned to ∞,
- ϕ corresponds to that system.

If both are true for some M_k system, then ϕ is an axiom. Otherwise it isn't. □

5.2 Correctness

5.3 Completeness