Axiomatization of Some Contact Logics with a Qualitative Measure

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1 Introduction

The aim of this work is to explore the axiomatization and decidability of the quantifier-free theories of a structure, which arises from a certain kind of geometric objects on the real line. Three relations between these objects are considered: parthood, contact and qualitative measure.

The objects are referred to as polytopes, though they are in fact unions of what may usually be understood by the term. A key property by which they are chosen is that they have a true interior.

The parthood relationship between these objects gives rise to a Boolean algebra and two further relations are considered: contact and qualitative measure.

2 Notation and Notions

Listed below are some established notions and their notation in this text.

- $\mathbb{R}$ denotes the real numbers, $\mathbb{R}^+$ the positive real numbers and $\mathbb{R}_0^+$ the non-negative ones.
- $-\infty$ and ∞ are the least and greatest elements of $\mathbb{R} \cup \{-\infty, \infty\}$.
- For the purposes of measure, the $+$ operation over $\mathbb{R}_0^+$ is extended in the usual way:

$$r + \infty \stackrel{\text{def}}{=} \infty + r \stackrel{\text{def}}{=} \infty, \text{ for any } r \in \mathbb{R}_0^+ \cup \{\infty\}$$

- $\mathcal{P}(X)$ denotes the set of all subsets of X.

3 Language

This section describes the language, whose semantics we will consider in a couple of contexts.

Definition (Language)**.** *$\mathcal{L}$ denotes the first-order language of only quantifier-free formulas, which contains the following non-logical symbols:*

- *predicate symbols:*
  - *binary infix $\sqsubseteq$: parthood*
  - *binary infix $\preceq$: measure comparison*
  - *binary prefix C: contact*
- *functional symbols:*
  - *binary infix function $\sqcup$: union*
  - *binary infix function $\sqcap$: intersection*
  - *unary postfix function *: complement*
- *constant symbols:*
  - 0*: empty polytope*
  - 1*: universe*

The logical symbols $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\top$, $\bot$ are used in the usual way and the set of individual variables is denoted Vars.

4 Semantics

Though the aim is to interpret the language on the real line, other models will also be needed. It is beneficial fix their common semantics, which are defined below in the expected way.

Let $\mathcal{B}$ be a Boolean algebra with carrier B and $\mathcal{C}$ and $\mathcal{M}$ be relations over B. Further, $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$.

Definition (Value of a Term in $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$)**.** *Let $v : Vars \to B$. Then, v^S denotes the extension of v to the terms of $\mathcal{L}$ in the following structurally recursive way:*

- *$v^S(0)$ is the zero of $\mathcal{B}$*
- *$v^S(1)$ is the unit of $\mathcal{B}$*
- *$v^S(\tau_1^*)$ is the complement of $v(\tau_1)$ in $\mathcal{B}$,*
- *$v^S(\tau_1 \sqcup \tau_2)$ is the join of $v(\tau_1)$ and $v(\tau_2)$ in $\mathcal{B}$,*
- *$v^S(\tau_1 \sqcap \tau_2)$ is the meet of $v(\tau_1)$ and $v(\tau_2)$ in $\mathcal{B}$,*

for any terms τ_1 and τ_2 of $\mathcal{L}$.

Definition (Validity of a Formula in $S = \langle \mathcal{B}, \mathcal{C}, \mathcal{M} \rangle$)**.** *Again, let $v : Vars \to B$. Validity of a formula ϕ in S with valuation v is denoted $\langle S, v \rangle \models \phi$ and defined over elementary formulas like so:*

- $\langle S, v \rangle \models \tau_1 \sqsubseteq \tau_2 \longleftrightarrow v^S(\tau_1)$ *is less than or equal to* $v^S(\tau_2)$ *in* $\mathcal{B}$,
- $\langle S, v \rangle \models C(\tau_1, \tau_2) \longleftrightarrow \mathcal{C}(v^S(\tau_1), v^S(\tau_2))$,
- $\langle S, v \rangle \models \tau_1 \preceq \tau_2 \longleftrightarrow \mathcal{M}(v^S(\tau_1), v^S(\tau_2))$,

for any terms τ_1 and τ_2 of $\mathcal{L}$. For complex formulas, the extension is done in the usual way:

- $\langle S, v \rangle \models \top$ *and* $\langle S, v \rangle \not\models \bot$,
- $\langle S, v \rangle \models \neg\phi \longleftrightarrow \langle S, v \rangle \not\models \phi$,
- $\langle S, v \rangle \models \phi \wedge \psi \longleftrightarrow \langle S, v \rangle \models \phi$ *and* $\langle S, v \rangle \models \psi$,
- $\langle S, v \rangle \models \phi \vee \psi \longleftrightarrow$ *at least one of* $\langle S, v \rangle \models \phi$ *and* $\langle S, v \rangle \models \psi$ *holds,*

where ϕ and ψ are (quantifier-free) formulas of $\mathcal{L}$.
If $\langle S, v \rangle \models \phi$ for all $v : Vars \to B$, then $S \models \phi$.

4.1 Polytopes on the Real Line

A specific kind of objects will be considered: finite unions of closed, potentially infinite, intervals on the real line. These are defined below, along with the operations and properties with which the language will be concerned.

Definition (Basis Polytope)**.** *For any m, $n \in \mathbb{R}$ such that $m < n$, the intervals $[m, n]$, $(-\infty, m]$, $[m, \infty)$ and $(-\infty, \infty)$ are called* basis polytopes*.*

Definition (Polytope)**.** *For any finite set of basis polytopes B, $\bigcup B$ is called a* polytope*. The set of all polytopes is denoted $Pol(\mathbb{R})$.*

Remark that for $B = \emptyset$, the empty set is also a polytope.

Proposition. *Any non-empty polytope can be uniquely represented as the union of a finite set of non-intersecting basic polytopes.*

Proof. TODO: write proof □

Definition. *(Standard Representation) The set from the above proposition is called the* standard representation *of a polytope.*

Definition (Polytope Operations)**.** *For any polytopes p and q, we define the following operations as modifications of intersection and complement:*

- $p \Cap q \overset{def}{=} Cl(Int((p \cup q)))$;

- $p^{\circledast} \overset{def}{=} Cl(\mathbb{R} \setminus p)$.

The union operation $p \cup q$ will be considered in the same context, though no modification is needed.

The modification of union ensures that there are no isolated points and the modification of complement ensures that results of the operations remain a union of *closed* intervals.

Proposition. *$Pol(\mathbb{R})$ forms a Boolean algebra with*

- *$\subseteq$ for Boolean inequality,*
- *$\Cap$ for meet,*
- *$\cup$ for join,*
- *${}^{\circledast}$ for complement,*
- *$\emptyset$ for the zero and*
- *$\mathbb{R}$ for the unit.*

This algebra will be denoted $\mathcal{B}^{\mathbb{R}}$.

Proof. TODO: write proof ☐

Proposition. *$\mathcal{B}^{\mathbb{R}}$ is incomplete.*

Proof. TODO: write proof ☐

Definition (Line Contact)**.** *Two polytopes p and q are* in contact *if $p \cap q \neq \emptyset$. This is denoted $\mathcal{C}^{\mathbb{R}}(p, q)$.*

Definition (Polytope Measure)**.** *The measure of a basic polytope of the kind $[m, n]$, for $m, n \in \mathbb{R}$ is $n - m$. The measure of a basic polytope with an infinite bound is ∞.*

The measure *of a polytope p, denoted $\mu^{\mathbb{R}}(p)$, is the sum of the measures of the basic polytopes of its standard representation.*

The qualitative measure relation induced by $\mu^{\mathbb{R}}$ *is defined*

$$p \leq^{\mathbb{R}}_{\mu} q \longleftrightarrow \mu^{\mathbb{R}}(p) \leq \mu^{\mathbb{R}}(q),$$

for any $p, q \in Pol(\mathbb{R})$.

Proposition. *$\mu^{\mathbb{R}}$ is a measure.*

Proof. TODO: write proof ☐

$\mu^{\mathbb{R}}$ is a restriction of the usual measure on $\mathbb{R}$.

Proposition. *The only polytope with measure* 0 *is $\emptyset$. Further, every non-zero polytope has an arbitrary small non-zero sub-polytope. Consequently, $\mathcal{B}^{\mathbb{R}}$ is atomless.*

Proof. TODO: write proof □

Given the definition of $\mathcal{B}^{\mathbb{R}}$, measure and contact above, the model on the real line is defined directly:

Definition (Real Line Model)**.**

$$S^{\mathbb{R}} \stackrel{def}{=} \langle \mathcal{B}^{\mathbb{R}}, \mathcal{C}^{\mathbb{R}}, \leq_{\mu}^{\mathbb{R}} \rangle$$

4.2 Relational Models

In order to find appropriate value functions for a well chosen kind of formulas, an abstraction model will be needed. It is quite generic, yet it turns out to be easily transformed into an equivalent real line model, given some constraints.

Let W be a finite set and $\mathcal{B}^W$ be the Boolean algebra of all subsets of W. Let c be an arbitrary symmetric and reflexive relation over W and $m : W \to \mathbb{R}^+ \cup \{\infty\}$.

Let $\mathcal{C}^c$ be the relation over $\mathcal{P}(W)$ defined as

$$\mathcal{C}^c(a, b) \longleftrightarrow (\exists i \in a)(\exists j \in b)(\langle i, j \rangle \in c)$$

Let $\mu^m : \mathcal{P}(W) \to \mathbb{R}_0^+ \cup \{\infty\}$ such that:

$$\mu^m(a) \stackrel{\text{def}}{=} \sum_{i \in a} m(i)$$

And finally, let $\leq_m$ be the relation over $\mathcal{P}(W)$:

$$a \leq_m b \longleftrightarrow \mu^m(a) \leq \mu^m(b),$$

all for a, $b \subseteq W$.

Definition (Relational Model)**.** $\langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ *is called the* relational model for W, c and m.

4.2.1 Converting to a Real Line Model

Relational models are useful because they can be converted to real line models under certain conditions.

Definition (Contact Graph)**.** *Let* $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ *be a relational model. Let* E *denote the set of pairs of different vertices in* c*, but unordered:* $E \stackrel{def}{=} \{\, \{i, j\} \mid \langle i, j \rangle \in c \;\&\; i \neq j \,\}$*. Note that since* c *is reflexive and symmetric, no information is lost when obtaining* E *from* c*. The graph* $\langle W, E \rangle$ *is called the* contact graph of S *and denoted* $Gr(S)$*.*

Definition (Convertable Relational Model)**.** *Let* $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ *be a relational model and suppose that the following constraints hold:*

- *$Gr(S)$ is connected and*
- *there are exactly two elements of W with infinite values for m, i.e. there exist $i \in W$ and $j \in W$, $i \neq j$ such that:*

$$m(i) = m(j) = \infty \text{ and } (\forall k \in W \setminus \{i, j\})(m(k) \neq \infty).$$

Then, S is called a convertible relational model*.*

Definition (Disjoint Valuation)**.** *Suppose $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ is a convertible relational model and $v : \mathit{Vars} \to \mathcal{P}(W)$ such that*

$$(\forall x \in \mathit{Vars})(\forall y \in \mathit{Vars})(v(x) \neq v(y) \to v(x) \cap v(y) = \emptyset).$$

Then, v is called a disjoint valuation*.*

Although a valuation is an infinite object, any given formula contains a finite number of variables. Therefore, only a finite part of the valuation is relevant to the truth value of that formula. The effective construction of a valuation as discussed below is possible because the valuation will always be considered in the context of a formula and will therefore be encodable by a finite object.

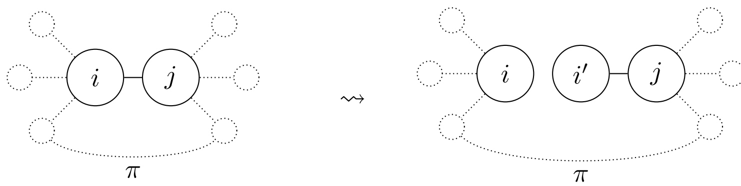
Lemma (Untying)**.** *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m \rangle$ be a convertible relational model and v be a disjoint valuation. Suppose $Gr(S)$ is not a tree. Then, there is a procedure to effectively construct a convertible relational model S' and a disjoint valuation v' for S' such that:*

- *$Gr(S')$ has one vertex more and the same number of edges and*
- *for any formula ϕ in $\mathcal{L}$, $\langle S, v \rangle \models \phi \longleftrightarrow \langle S', v' \rangle \models \phi$.*

Proof. Given that $Gr(S)$ is not a tree, yet is connected, it must contain at least one cycle. A cycle can be effectively found using a depth-first search. Let π be such a cycle.

Let i and j be any two consecutive vertices in π such that $m(i) \neq \infty$. This requirement is always achievable, since there are at least three vertices in a cycle and exactly two vertices with infinite values for m in a convertible relational model. Therefore, there will be at least one vertex with a finite value for m in any cycle.

The aim is to disconnect π by removing the edge between i and j. In order to achieve that, intuitively, the portion of i which is in contact only with j will be separated out into a separate atomic object, i', having half the measure.



Let $i' \notin W$ and let:

$$W' \stackrel{\text{def}}{=} W \cup \{i'\}$$
$$c' \stackrel{\text{def}}{=} (c \setminus \{\langle i, j\rangle, \langle j, i\rangle\}) \cup \{\langle i', j\rangle, \langle j, i'\rangle, \langle i', i'\rangle\}$$
$$m'(k) \stackrel{\text{def}}{=} \begin{cases} m(k) & \text{if } k \notin \{i, i'\} \\ m(i)/2 & \text{otherwise} \end{cases}, \text{ for any } k \in W'.$$
$$S' \stackrel{\text{def}}{=} \langle \mathcal{B}^{W'}, \mathcal{C}^{c'}, \leq_{m'} \rangle$$
$$v'(x) \stackrel{\text{def}}{=} \begin{cases} v(x) \cup \{i'\} & \text{if } i \in v(x) \\ v(x) & \text{otherwise} \end{cases}, \text{ for any } x \in \text{Vars}.$$

Note that S' and v' are defined in a constructive way. The procedure consists of copying S and v, except for a small number of changes, not depending on the size of S.

It is directly clear that S' is a relational model. Compared to S, there is one more vertex in $Gr(S')$ and the same number of edges ($\{i, j\}$ is removed, $\{i', j\}$ is added).

Any path between two vertices in $Gr(S)$ corresponds to a path in $Gr(S')$, by potentially substituting the edge $\{i, j\}$ for the detour path along π. Therefore, all vertices in W are connected in $Gr(S')$. i' is connected to i and from there to the rest of W as well, so $Gr(S')$ is connected.

m' has the same values as m over all elements of W', except i and i', where it's values are finite. Therefore, the same two elements of W' that have infinite values for m, also have infinite values for m' and no others.

Thus, S' is a convertible relational model.

To demonstrate by contraposition that v' is a disjoint valuation, let $x \in$ Vars, $y \in$ Vars and assume that $v'(x) \cap v'(y) \neq \emptyset$. Let $k \in v'(x) \cap v'(y)$, for some $k \in W'$. If $k = i'$, $i' \in v'(x)$, so $i \in v(x)$ by the definition of v'. Analogously, $i \in v(y)$ and $v(x) \cap v(y) \neq \emptyset$. On the other hand, if $k \neq i'$, again by the definition of v', $k \in v(x) \cap v(y)$. In both cases, since v is a disjoint valuation and $v(x) \cap v(y) \neq \emptyset$, $v(x) = v(y)$. From here, $v'(x) = v'(y)$ and v' is a disjoint valuation.

Now follows a proof by induction on terms in $\mathcal{L}$, that

$$v'^{S'}(\tau) = \begin{cases} v^S(\tau) \cup \{i'\} & \text{if } i \in v^S(\tau) \\ v^S(\tau) & \text{otherwise} \end{cases}, \text{ for any term } \tau \text{ of } \mathcal{L}.$$

- For the two constants 0 and 1,

$$v'^{S'}(0) = \emptyset = v^S(0) \text{ and } i \notin v^S(0)$$

$$v'^{S'}(1) = W' = W \cup \{i'\} = v^S(1) \cup \{i'\} \text{ and } i \in v^S(1)$$

- For terms consisting of individual variables, the statement holds by the definition of v'.

- Suppose, as induction hypothesis, that the statement holds for τ_1 and τ_2.

  - To show that the statement holds for $\tau_1 \sqcup \tau_2$,

    * suppose first that $i \in v^S(\tau_1 \sqcup \tau_2)$. Then, $i \in v^S(\tau_1) \cup v^S(\tau_2)$ and $i \in v^S(\tau_1)$ or $i \in v^S(\tau_2)$. Without loss of generality, assume $i \in v^S(\tau_1)$. From here,

    $$v'^{S'}(\tau_1) \overset{\text{i.h.}}{=} v^S(\tau_1) \cup \{i'\} \text{ and } v'^{S'}(\tau_2) \overset{\text{i.h.}}{\subseteq} v^S(\tau_2) \cup \{i'\}.$$

    Using this,

    $$v'^{S'}(\tau_1) \cup v'^{S'}(\tau_2) = v^S(\tau_1) \cup v^S(\tau_2) \cup \{i'\}.$$

    By applying the definitions of v^S and $v'^{S'}$ to the above equality,

    $$v'^{S'}(\tau_1 \sqcup \tau_2) = v^S(\tau_1 \cup \tau_2) \cup \{i'\}.$$

    * Alternatively, suppose $i \notin v^S(\tau_1 \sqcup \tau_2)$. Then

    $$\begin{aligned} v'^{S'}(\tau_1 \sqcup \tau_2) &\overset{\text{def}}{=} \\ v'^{S'}(\tau_1) \cup v'^{S'}(\tau_2) &\overset{\text{i.h.}}{=} v^S(\tau_1) \cup v^S(\tau_2) \\ &\overset{\text{def}}{=} v^S(\tau_1 \sqcup \tau_2). \end{aligned}$$

  - To show that the statement holds for $\tau_1 \sqcap \tau_2$,

    * suppose $i \in v^S(\tau_1 \sqcap \tau_2)$. Then, $i \in v^S(\tau_1) \cap v^S(\tau_2)$, so $i \in v^S(\tau_1)$ and $i \in v^S(\tau_2)$. Thus,

    $$v'^{S'}(\tau_1 \sqcap \tau_2) \overset{\text{def}}{=} v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2) \overset{\text{i.h.}}{=}$$
    $$(v^S(\tau_1) \cup \{i'\}) \cap (v^S(\tau_2) \cup \{i'\}) = (v^S(\tau_1) \cap v^S(\tau_2)) \cup \{i'\} \overset{\text{def}}{=}$$
    $$v^S(\tau_1 \sqcup \tau_2) \cup \{i'\}.$$

    * Alternatively, if $i \notin v^S(\tau_1 \sqcap \tau_2)$, then $i' \notin v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2)$ and

    $$v'^{S'}(\tau_1 \sqcap \tau_2) \overset{\text{def}}{=} v'^{S'}(\tau_1) \cap v'^{S'}(\tau_2) \overset{\text{i.h.}}{=} v^S(\tau_1) \cap v^S(\tau_2) \overset{\text{def}}{=} v^S(\tau_1 \sqcap \tau_2).$$

  - To show the statement holds for τ_1^*, consider that

    $$v'^{S'}(\tau_1^*) \overset{\text{def}}{=} W' \setminus v'^{S'}(\tau_1) \overset{\text{def}}{=} (W \cup \{i'\}) \setminus v'^{S'}(\tau_1).$$

* If $i \notin v^S(\tau_1^*)$, then $i \in v^S(\tau_1)$ and

$$(W \cup \{i'\}) \setminus v'^{S'}(\tau_1) \overset{\text{i.h.}}{=} (W \cup \{i'\}) \setminus (v^S(\tau_1) \cup \{i'\}) = W \setminus v^S(\tau_1) \overset{\text{def}}{=} v^S(\tau_1^*).$$

* If $i \in v^S(\tau_1^*)$, then $i \notin v^S(\tau_1)$ and

$$(W \cup \{i'\}) \setminus v'^{S'}(\tau_1) = (W \cup \{i'\}) \setminus v^S(\tau_1) = (W \setminus v^S(\tau_1)) \cup \{i'\} \overset{\text{def}}{=} v^S(\tau_1^*) \cup \{i'\}$$

Now to demonstrate that for any formula ϕ in $\mathcal{L}$, $\langle S, v\rangle \models \phi \longleftrightarrow \langle S', v'\rangle \models \phi$ by induction on the construction of ϕ, let τ_1 and τ_2 be terms of $\mathcal{L}$.

- For $\bot$ and $\top$, the statement is trivial.
- For parthood atomic formulas:
  - Suppose $i \in v^S(\tau_2)$. By definition,

    $$\langle S, v\rangle \models \tau_1 \sqsubseteq \tau_2 \longleftrightarrow v^S(\tau_1) \subseteq v^S(\tau_2)$$

    Since $v^S(\tau_{1,2}) \subseteq W$ and $i' \notin W$,

    $$v^S(\tau_1) \subseteq v^S(\tau_2) \longleftrightarrow v^S(\tau_1) \cup \{i'\} \subseteq v^S(\tau_2) \cup \{i'\}$$

    and given that $v^S(\tau_2) \cup \{i'\} = v'^{S'}(\tau_2)$,

    $$\begin{aligned} v^S(\tau_1) \cup \{i'\} \subseteq v^S(\tau_2) \cup \{i'\} &\longleftrightarrow v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \\ &\longleftrightarrow \langle S', v'\rangle \models \tau_1 \sqsubseteq \tau_2. \end{aligned}$$

  - conversely, if $i \notin v^S(\tau_2)$,

    $$\begin{aligned} &\langle S, v\rangle \models \tau_1 \sqsubseteq \tau_2 \\ \longleftrightarrow\ & v^S(\tau_1) \subseteq v^s(\tau_2) \\ \longleftrightarrow\ & v'^{S'}(\tau_1) \subseteq v'^{S'}(\tau_2) \\ \longleftrightarrow\ & \langle S', v'\rangle \models \tau_1 \sqsubseteq \tau_2 \end{aligned}$$

- To prove the statement for contact atomic formulas in one direction, assume that $\langle S, v\rangle \models C(\tau_1, \tau_2)$. From here, there exist $k \in v^S(\tau_1)$ and $l \in v^S(\tau_2)$ such that $\langle k, l\rangle \in c$.
  - Suppose $\{k, l\} = \{i, j\}$. Without loss of generality, $k = i$ and $l = j$. Then, since $i \in v^S(\tau_1)$, $i' \in v'^{S'}(\tau_1)$ must hold. Further, $j \in v'^{S'}(\tau_2)$ and $\langle i', j\rangle \in c'$, so $\langle S', v'\rangle \models C(\tau_1, \tau_2)$.
  - Alternatively, if $\{k, l\} \neq \{i, j\}$, then $\langle k, l\rangle \in c'$ (because $\langle k, l\rangle \in c$), $k \in v'^{S'}(\tau_1)$ and $l \in v'^{S'}(\tau_2)$, so $\langle S', v'\rangle \models C(\tau_1, \tau_2)$.

In the opposite direction, assume $\langle S', v'\rangle \models C(\tau_1, \tau_2)$. Again, there must exist $k \in v'^{S'}(\tau_1)$ and $l \in v'^{S'}(\tau_2)$ such that $\langle k, l\rangle \in c'$.

- Suppose $\{k, l\} = \{i', j\}$. Just as before, without loss of generality, $k = i'$ and $l = j$. Then, since $i' \in v'^{S'}(\tau_1)$, $i \in v^S(\tau_1)$ must hold. Further, $j \in v^S(\tau_2)$ and $\langle i, j\rangle \in c$, so $\langle S, v\rangle \models C(\tau_1, \tau_2)$.
- Alternatively, suppose $\{k, l\} \neq \{i', j\}$. The only pairs in c' that contain i are $\{\langle i', j\rangle, \langle j, i'\rangle, \langle i', i'\rangle\}$ and $\langle k, l\rangle \in c'$. If $k = i'$, then $l = i'$ must hold, since $l \neq j$. Analogously, if $k = i'$, then $l = i'$. In both cases, $i \in v^S(\tau_1)$ and $i \in v^S(\tau_2)$ and $\langle i, i\rangle \in c$. Thus $\langle S, v\rangle \models C(\tau_1, \tau_2)$.

  If both $k \neq i'$ and $l \neq i'$, then $k \in W$, $l \in W$, $k \in v^S(\tau_1)$, $l \in v^S(\tau_2)$ and $\langle k, l\rangle \in c$, so $\langle S, v\rangle \models C(\tau_1, \tau_2)$.

• For atomic formulas with qualitative measure, first observe that for any term τ, it holds that $\mu^m(v^S(\tau)) = \mu^{m'}(v'^{S'}\tau)$. To prove this, consider again two cases:

- If $i \in v^S(\tau)$

$$\mu^m(v^S(\tau)) \overset{\text{def}}{=} \sum_{k \in v^S(\tau)} m(k) = m(i) + \sum_{k \in v^S(\tau)\setminus\{i\}} m(k) =$$
$$m(i)/2 + m(i)/2 + \sum_{k \in v^S(\tau)\setminus\{i\}} m(k) \overset{\text{def}}{=}$$
$$m'(i) + m'(i') + \sum_{k \in v^S(\tau)\setminus\{i\}} m'(k) = \sum_{k \in v^S(\tau)\cup\{i'\}} m'(k) =$$
$$\sum_{k \in v'^{S'}(\tau)} m'(k) \overset{\text{def}}{=} \mu^{m'}(v'^{S'}(\tau))$$

- If $i \notin v^S(\tau)$, then

$$\mu^m(v^S(\tau)) \overset{\text{def}}{=} \sum_{k \in v^S(\tau)} m(k) = \sum_{k \in v'^{S'}(\tau)} m'(k) \overset{\text{def}}{=} \mu^{m'}(v'^{S'}(\tau))$$

Given this, $\langle S', v'\rangle \models \tau_1 \preceq \tau_2 \longleftrightarrow \langle S, v\rangle \models \tau_1 \preceq \tau_2$ trivially holds.

• Assuming that the statement holds for ϕ and ψ, the proof that it holds for $\phi \vee \psi$, $\phi \wedge \psi$, $\neg\phi$, $\phi \Rightarrow \psi$ is direct.

□

Lemma (Relational Model Conversion)**.** *Let $S = \langle \mathcal{B}^W, \mathcal{C}^c, \leq_m\rangle$ be a convertible relational model and v be a disjoint valuation. If ϕ is a formula from $\mathcal{L}$ and $\langle S, v\rangle \models \phi$, then there exists $v^{\mathbb{R}} : Vars \to Pol(\mathbb{R})$ such that $\langle S^{\mathbb{R}}, v^{\mathbb{R}}\rangle \models \phi$.*

Proof. TODO: write proof. □

5 Axiomatization

5.1 Correctness

5.2 Completeness

5.3 Finite Axiomatization