NIP compressible types

Pablo Andújar Guerrero

University of Leeds

anguepa.github.io/handout.pdf

Theorem: For any NIP formula $\varphi(x, y)$ there exists $n, k < \omega$ such any φ-type $p(x)$ is a rounded average of n many k-compressible φ-types.

Theorem: For any NIP formula $\varphi(x, y)$ there exists $n, k < \omega$ such any φ-type $p(x)$ is a rounded average of n many k-compressible φ-types.

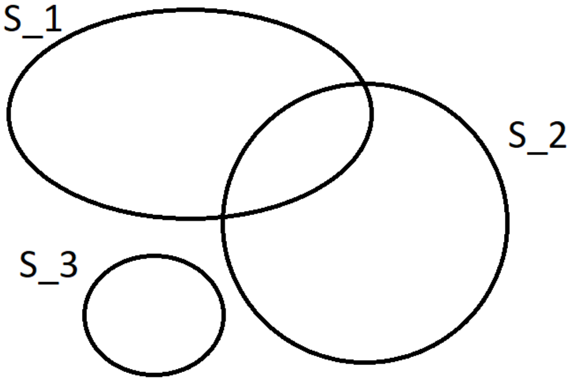
Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

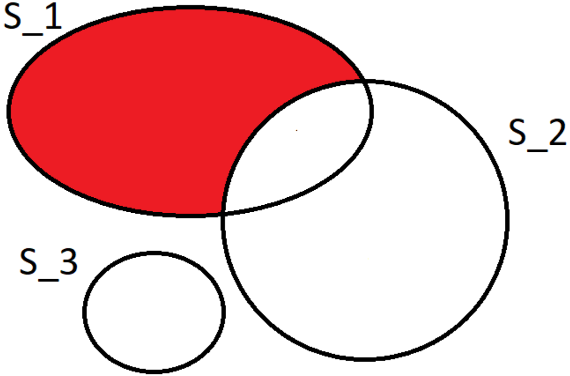
$$\mathcal{F} = \{S_1, S_2, S_3\}$$



Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

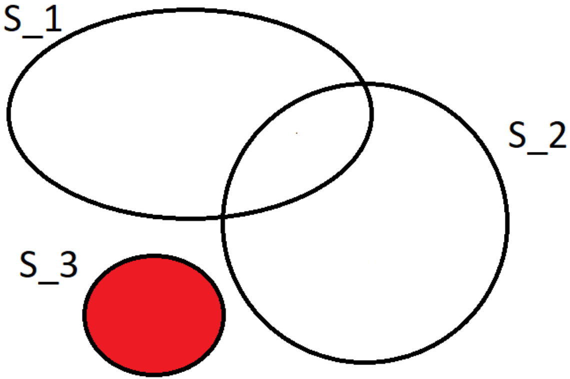
$S_1 \setminus (S_2 \cup S_3)$



Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

$S_3 \setminus (S_1 \cup S_2)$



Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

(In model-theoretic terminology a Boolean atom can be called a type.)

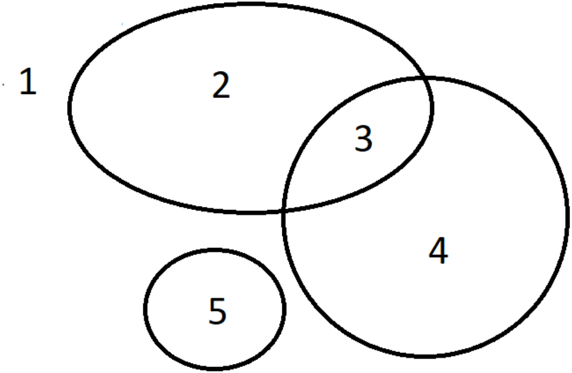
Let $BA(\mathcal{F})$ denote the number of Boolean atoms of $\mathcal{F}$.

Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

Let $BA(\mathcal{F})$ denote the number of Boolean atoms of $\mathcal{F}$.

$BA(\mathcal{F}) = 5$

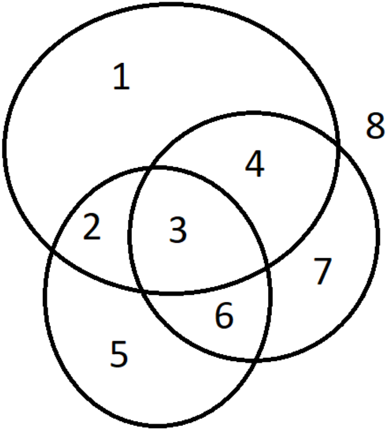


Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

Let $BA(\mathcal{F})$ denote the number of Boolean atoms of $\mathcal{F}$.

$$BA(\mathcal{F}) = 2^3 = 8$$



Counting Boolean atoms

A *Boolean atom* of a finite family of sets $\mathcal{F}$ is a maximal non-empty Boolean combination of sets in $\mathcal{F}$.

Let $BA(\mathcal{F})$ denote the number of Boolean atoms of $\mathcal{F}$.

Definition

Let $\mathcal{S}$ be a family of sets. The *dual shatter function of* $\mathcal{S}$ is the function

$$\begin{aligned} \pi^*_{\mathcal{S}} : & \omega \to \omega \\ & n \mapsto \max\{BA(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{S},\ |\mathcal{F}| = n\} \end{aligned}$$

Counting Boolean atoms

Observe that $\pi^*_{\mathcal{S}}(n) \leq 2^n$ for all n.

Counting Boolean atoms

Observe that $\pi^*_{\mathcal{S}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{S}}(d) < 2^d$ for some d then $\pi^*_{\mathcal{S}}(n) = O(n^{d-1})$ (Sauer's Lemma).

Counting Boolean atoms

Observe that $\pi^*_{\mathcal{S}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{S}}(d) < 2^d$ for some d then $\pi^*_{\mathcal{S}}(n) = O(n^{d-1})$ (Sauer's Lemma).

In this case we call $\mathcal{S}$ a *VC class* (VC=Vapnik–Chervonenkis).

Counting Boolean atoms

Observe that $\pi^*_{\mathcal{S}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{S}}(d) < 2^d$ for some d then $\pi^*_{\mathcal{S}}(n) = O(n^{d-1})$ (Sauer's Lemma).

In this case we call $\mathcal{S}$ a *VC class* (VC=Vapnik–Chervonenkis).

Definition

The VC-codensity of a VC class $\mathcal{S}$ is

$$\mathrm{vc}^*(\mathcal{S}) = \inf\{r \in \mathbb{R}^{\geq 0} : \pi^*_{\mathcal{S}}(n) = O(n^r)\}.$$

Counting Boolean atoms

Observe that $\pi^*_{\mathcal{S}}(n) \leq 2^n$ for all n.

If $\pi^*_{\mathcal{S}}(d) < 2^d$ for some d then $\pi^*_{\mathcal{S}}(n) = O(n^{d-1})$ (Sauer's Lemma).

In this case we call $\mathcal{S}$ a *VC class* (VC=Vapnik–Chervonenkis).

Definition

The VC-codensity of a VC class $\mathcal{S}$ is

$$\mathrm{vc}^*(\mathcal{S}) = \inf\{r \in \mathbb{R}^{\geq 0} : \pi^*_{\mathcal{S}}(n) = O(n^r)\}.$$

E.g. If $\mathcal{S} = \{\textbf{intervals in } \mathbb{R}\}$ then $\pi^*_{\mathcal{S}}(n) = 2n$ and $\mathrm{vc}^*(\mathcal{S}) = 1$.

Definition

A family of sets $\mathcal{S}$ is *(downward) directed* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists some $S \in \mathcal{S}$ such that $S \subseteq \cap\mathcal{F}$.

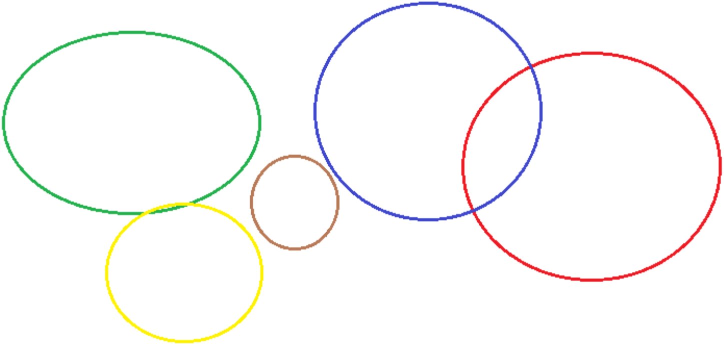
Equivalently, $\mathcal{S}$ is directed if, for any $S_1, S_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ with $S \subseteq S_1 \cap S_2$.

E.g.

- In a poset $(P, \leq)$ the sets $\{a \in P : a \leq b\}$ for $b \in P$.
- The neighborhoods of a given point in a topological spaces.
- Any filter basis.

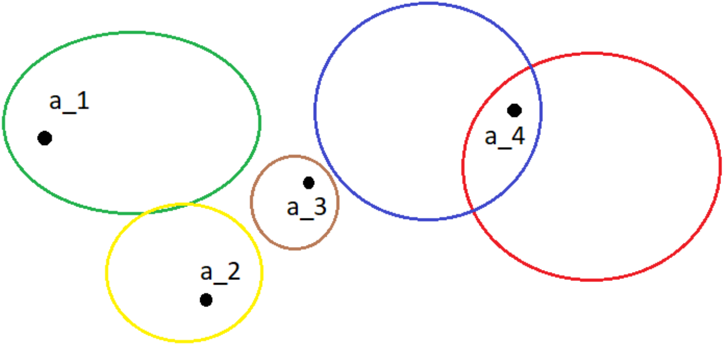
Definition

A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.



Definition

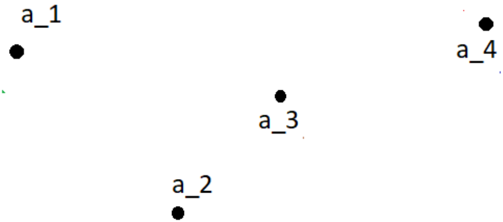
A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.



Definition

A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.

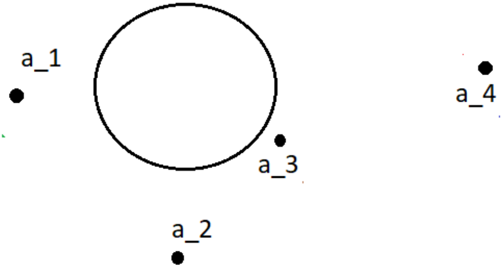
We care about the property that $\mathcal{S}$ does **not** have a finite transversal.



Definition

A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.

We care about the property that $\mathcal{S}$ does **not** have a finite transversal.



Is there a connection between directedness and the dual shatter function?

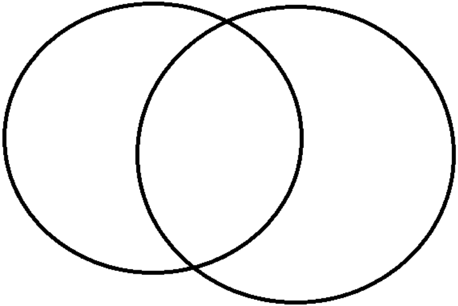
Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.
$\mathcal{S}$ could be directed.

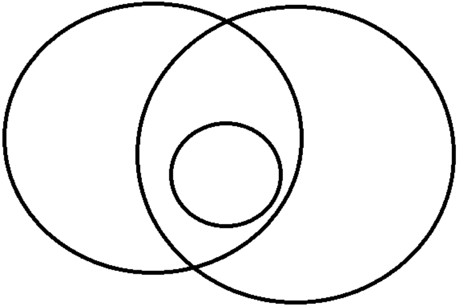
.



Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.
$\mathcal{S}$ could be directed.

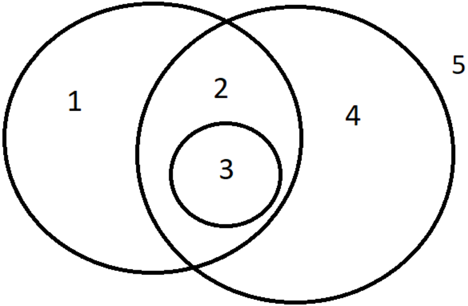
.



Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.
$\mathcal{S}$ could be directed.

$$\pi^*_{\mathcal{S}}(3) \geq 5$$

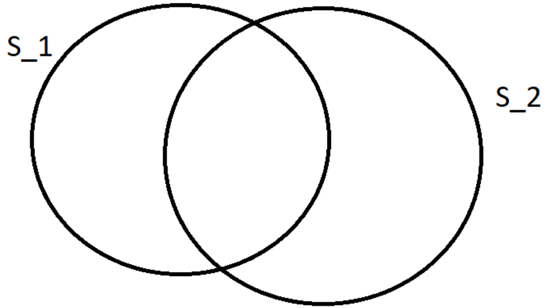


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

.

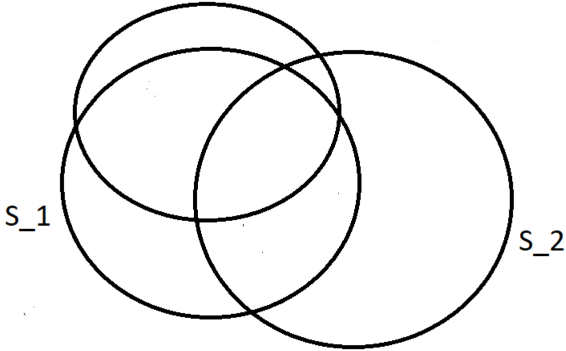


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$\pi^*_{\mathcal{S}}(3) \geq 8$

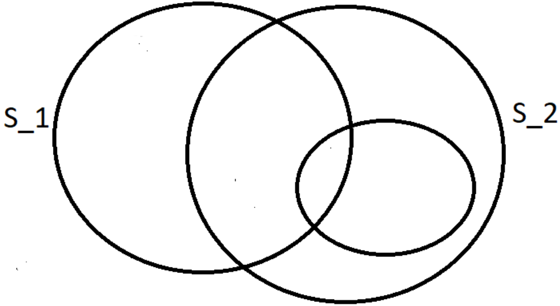


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$\pi^*_{\mathcal{S}}(3) \geq 6$$

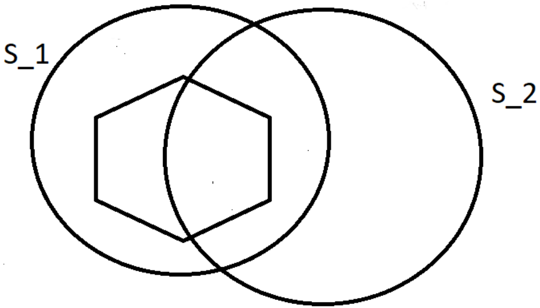


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$\pi^*_{\mathcal{S}}(3) \geq 6$$

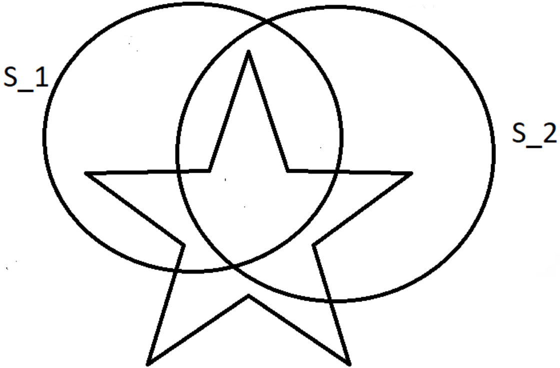


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$\pi^*_{\mathcal{S}}(3) \geq 8$$

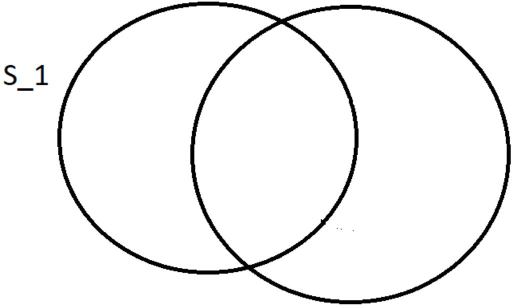


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$BA(S_1, S_2) \geq 3$$

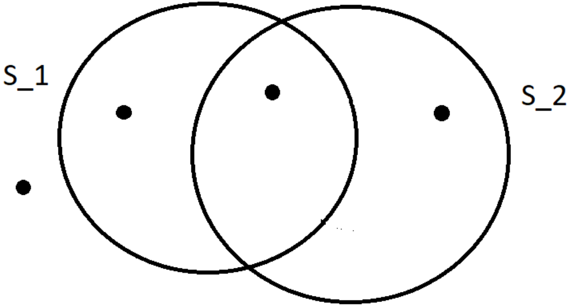


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$BA(S_1, S_2) \geq 3$$

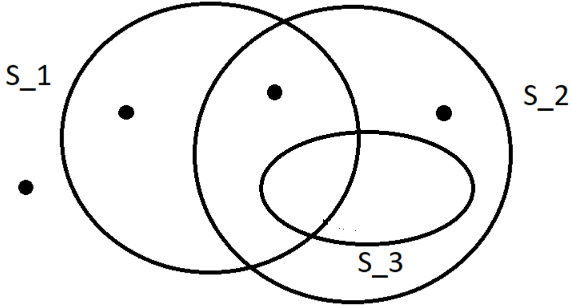


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$BA(S_1, S_2, S_3) \geq 3 + 2 = 5$$

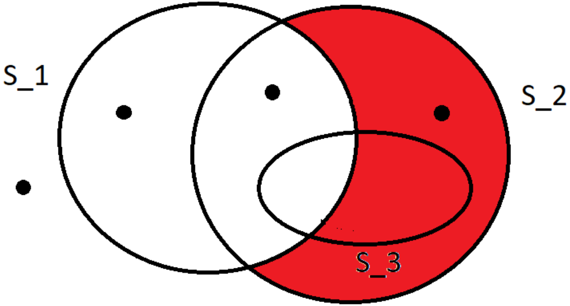


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$$BA(S_1, S_2, S_3) \geq 3 + 1 + {\color{red}1}.$$

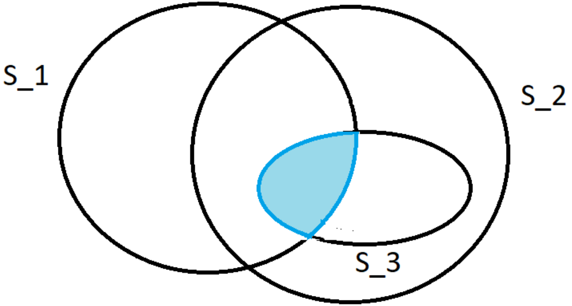


Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$F = S_1 \cap S_2 \cap S_3$



Is there a connection between directedness and the dual shatter function?

Fix a family of sets $\mathcal{S}$ that is consistent (=has the finite intersection property). Suppose it does **not** have a finite transversal.

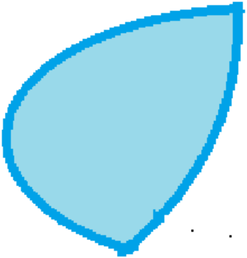
Now suppose that $\mathcal{S}$ is **not** directed, witnessed by S_1 and S_2.

$F = S_1 \cap S_2 \cap S_3$

Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

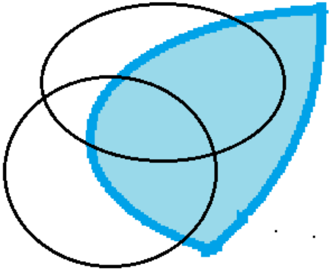
$F = S_1 \cap S_2 \cap S_3$



Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

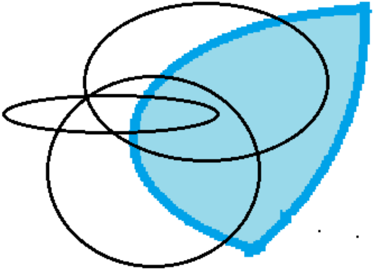
$F \cap \mathcal{S}$ could be directed.



Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

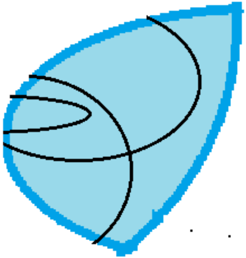
$F \cap \mathcal{S}$ could be directed.



Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

$F \cap \mathcal{S}$ could be directed.

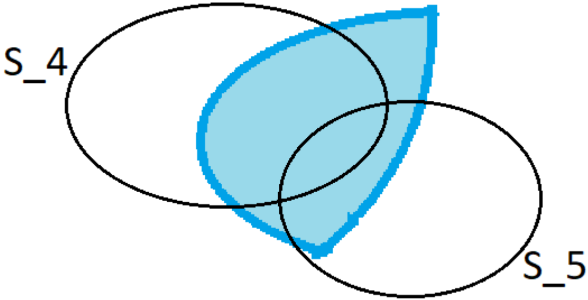


Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

Suppose that $F \cap \mathcal{S}$ is **not** directed, witnessed by S_4 and S_5.

$$BA(S_1, S_2, S_3, S_4, S_5) \geq BA(S_1, S_2, S_3) + 3$$

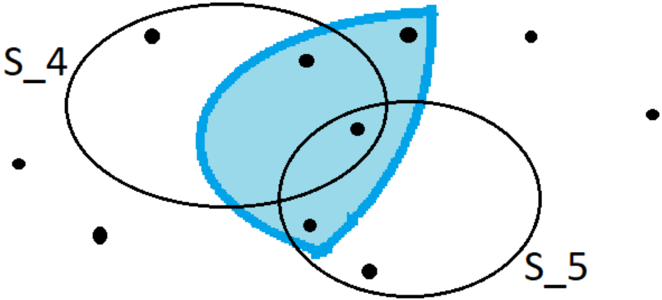


Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

Suppose that $F \cap \mathcal{S}$ is **not** directed, witnessed by S_4 and S_5.

$$BA(S_1, S_2, S_3, S_4, S_5) \geq BA(S_1, S_2, S_3) + 3$$

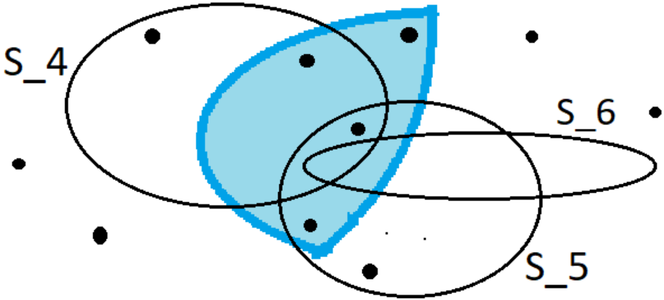


Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

Suppose that $F \cap \mathcal{S}$ is **not** directed, witnessed by S_4 and S_5.

$$BA(S_1, S_2, S_3, S_4, S_5) \geq BA(S_1, S_2, S_3) + 3$$

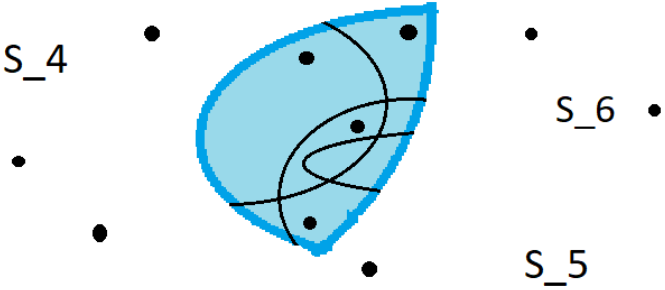


Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

Suppose that $F \cap \mathcal{S}$ is **not** directed, witnessed by S_4 and S_5.

$$BA(S_1, \ldots, S_6) \geq BA(S_1, S_2, S_3) + 3 + 2.$$

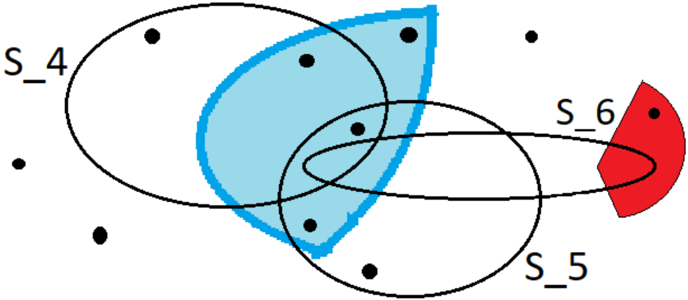


Is there a connection between directedness and the dual shatter function?

Consider the family of sets $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$.

Suppose that $F \cap \mathcal{S}$ is **not** directed, witnessed by S_4 and S_5.

$$BA(S_1, \ldots, S_6) \geq BA(S_1, S_2, S_3) + 3 {\color{blue}+2} {\color{red}+1}.$$



Is there a connection between directedness and the dual shatter function?

We continue this process, and one of two things can happen:

- We find a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S}$ is directed.
- There is a sequence $(S_i)_i$ of sets in $\mathcal{S}$ such that

$$BA(B_i : i \leq 3n) \geq \sum_{j=1}^{n} j.$$

  Meaning that

$$\pi^*_{\mathcal{S}}(3n) \geq \frac{n^2 + n}{2}.$$

Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

(b) $\mathcal{S}$ has a finite transversal.

Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

(b) $\mathcal{S}$ has a finite transversal.

In fact we can improve (b) in the above lemma.

Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

(b) $\mathcal{S}$ has a finite transversal.

In fact we can improve (b) in the above lemma.

Better Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

(b) There exists $F_1, \ldots, F_m$, finite non-empty Boolean combinations of sets in $\mathcal{S}$ such that, for every $S \in \mathcal{S}$ there is some $i \leq m$ such that $F_i \subseteq S$.

Now let us talk compressibility.
Our two main references are:

- Martin Bays, Itay Kaplan and Pierre Simon. *Density of compressible types and some consequences.* ArXiv: 2107.05197, 2022.
- Xi Chen, Yu Cheng, and Bo Tang. *On the recursive teaching dimension of VC classes.* Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems, 2016, Barcelona, Spain, p 2164–2171.

Model-theoretic framework

Let T be a theory and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

- $\varphi(x, y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

Model-theoretic framework

Let T be a theory and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

- $\varphi(x, y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

A formula φ is *NIP* if it is a VC class.
A theory T is *NIP* if every formula φ is NIP.

Model-theoretic framework

Let T be a theory and $\varphi(x,y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M,b) = \{a \in M^x : M \models \varphi(a,b)\}$. We identify $\varphi(x,y)$ with the family of sets $\{\varphi(M,b) : b \in M^y\}$:

- $\varphi(x,y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

A formula φ is *NIP* if it is a VC class.
A theory T is *NIP* if every formula φ is NIP.

A theory T is *of VC-density one* if every formula $\varphi(x,y)$ with $|x| = 1$ has VC-codensity less than 2.

$$\text{VC-density one} \subseteq \text{dp-minimal} \subsetneq \text{NIP}$$

Model-theoretic framework

Let T be a theory and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

- $\varphi(x, y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

A formula φ is *NIP* if it is a VC class.
A theory T is *NIP* if every formula φ is NIP.

A theory T is *of VC-density one* if every formula $\varphi(x, y)$ with $|x| = 1$ has VC-codensity less than 2.

Conjecture

$$\boxed{\text{VC-density one} = \text{dp-minimal}} \subsetneq \text{NIP}$$

Theorem (Baldwin-Saxl '76). *Let G be a group definable in an NIP structure. Let $\mathcal{H}$ be a definable family of subgroups of G. Then there is $k < \omega$ such that, for any finite subfamily $\mathcal{F} \subseteq \mathcal{H}$, there is a subfamily $\mathcal{C} \subseteq \mathcal{F}$ with $|\mathcal{C}| \leq k$ such that $\cap\mathcal{C} = \cap\mathcal{F}$.*

Definition

Let $k < \omega$. A family of sets $\mathcal{S}$ is *k-compressible* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Equivalently, $\mathcal{S}$ is k-compressible if, for any subfamily $\mathcal{F} \subseteq \mathcal{S}$ of size $k+1$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Definition

Let $k < \omega$. A family of sets $\mathcal{S}$ is *k-compressible* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

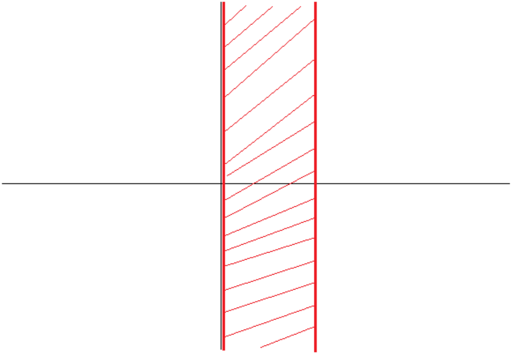
Equivalently, $\mathcal{S}$ is k-compressible if, for any subfamily $\mathcal{F} \subseteq \mathcal{S}$ of size $k+1$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Note: Directed = 1-compressible.

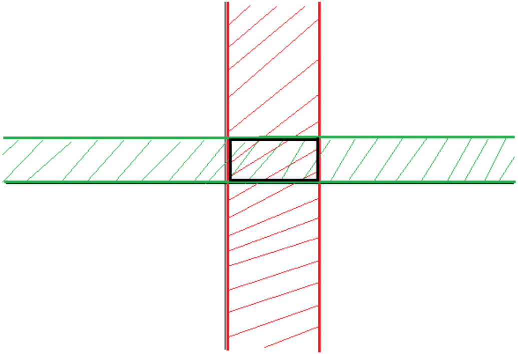
"This terminology is inspired by the terminology around compression schemes in the statistical learning literature."
— Bays-Kaplan-Simon

- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.

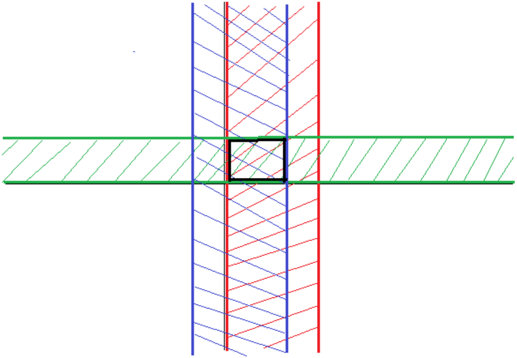
- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.



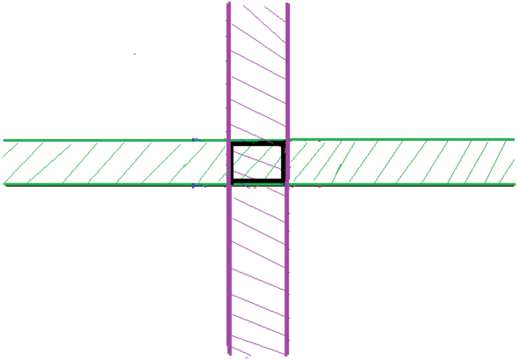
- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.



- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.



- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.



- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \dots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.

A k-compressible family of sets is consistent iff it is k-consistent (every k sets have non-empty intersection).

So for k-compressible **definable** families of sets consistency is expressible with a single first order sentence.

Let T be a theory and $M \models T$.

For a formula $\varphi(x, y)$ let $\varphi^1(x, y) = \varphi(x, y)$ and $\varphi^0(x, y) = \neg\varphi(x, y)$.

Let T be a theory and $M \models T$.

For a formula $\varphi(x, y)$ let $\varphi^1(x, y) = \varphi(x, y)$ and $\varphi^0(x, y) = \neg\varphi(x, y)$.

By a *φ-type $p(x)$ over $B \subseteq M^y$* we mean a maximal consistent collection of formulas of the form $\varphi^i(x, b)$, for $b \in B$ and $i \in \{0, 1\}$.

Let T be a theory and $M \models T$.

For a formula $\varphi(x, y)$ let $\varphi^1(x, y) = \varphi(x, y)$ and $\varphi^0(x, y) = \neg\varphi(x, y)$.

By a *φ-type $p(x)$ over $B \subseteq M^y$* we mean a maximal consistent collection of formulas of the form $\varphi^i(x, b)$, for $b \in B$ and $i \in \{0, 1\}$.

We identify types $p(x)$ with the family of sets defined in M by the formulas in it.

Let T be a theory and $M \models T$.

For a formula $\varphi(x,y)$ let $\varphi^1(x,y) = \varphi(x,y)$ and $\varphi^0(x,y) = \neg\varphi(x,y)$.

By a *φ-type $p(x)$ over $B \subseteq M^y$* we mean a maximal consistent collection of formulas of the form $\varphi^i(x,b)$, for $b \in B$ and $i \in \{0,1\}$.

We identify types $p(x)$ with the family of sets defined in M by the formulas in it.

Definition

A φ-type $p(x)$ is k-compressible if for any **finite** type $q(x) \subseteq p(x)$ there exists $r(x) \subseteq p(x)$ with $|r(x)| \leq k$ such that

$$r(x) \vdash q(x) \quad \text{(i.e. } r(M) \subseteq q(M)\text{)}.$$

Better Lemma A (A.G. '23)

Let $\varphi(x, y)$ be a formula in a theory T and $M \models T$. Let $B \subseteq M^y$ and $p(x)$ be a φ-type over B with $\mathrm{vc}^*(p(x)) < 2$. Then at least one holds:

(a) There exists a finite conjunction $\psi(x)$ of formulas in $p(x)$ such that $\psi(x) \wedge p(x) = \{\psi(x) \wedge \sigma(x) : \sigma(x) \in p(x)\}$ is directed (i.e. 1-compressible).
In particular $p(x)$ is k-compressible for some $k < \omega$.

(b) There exist $q_1(x), \dots q_m(x)$ finite φ-types such that, for every $\sigma(x) \in p(x)$ there exists some $i \leq m$ such that $q_i(x) \vdash \sigma(x)$ (i.e. $q_i(M) \subseteq \sigma(M)$).

Better Lemma A* (A.G. and Johnson '23)

Let $\varphi(x, y)$ be a formula with $|x| = 1$ in a dp-minimal theory T and $M \models T$. Let $B \subseteq M^y$ and $p(x)$ be a φ-type over B ~~with $\mathrm{vc}^*(p(x)) < 2$~~. Then at least one holds:

(a) There exists a finite conjunction $\psi(x)$ of formulas in $p(x)$ such that $\psi(x) \wedge p(x) = \{\psi(x) \wedge \sigma(x) : \sigma(x) \in p(x)\}$ is directed (i.e. 1-compressible).
In particular $p(x)$ is k-compressible for some $k < \omega$.

(b) There exist $q_1(x), \ldots q_m(x)$ finite φ-types such that, for every $\sigma(x) \in p(x)$ there exists some $i \leq m$ such that $q_i(x) \vdash \sigma(x)$ (i.e. $q_i(M) \subseteq \sigma(M)$).

Better Lemma A* (A.G. and Johnson '23)

Let $\varphi(x, y)$ be a formula with $|x| = 1$ in a dp-minimal theory T and $M \models T$. Let $B \subseteq M^y$ and $p(x)$ be a φ-type over B ~~with $\mathrm{vc}^*(p(x)) < 2$~~. Then at least one holds:

- (a) There exists a finite conjunction $\psi(x)$ of formulas in $p(x)$ such that $\psi(x) \wedge p(x) = \{\psi(x) \wedge \sigma(x) : \sigma(x) \in p(x)\}$ is directed (i.e. 1-compressible).
  In particular $p(x)$ is k-compressible for some $k < \omega$.
- (b) There exist $q_1(x), \ldots q_m(x)$ finite φ-types such that, for every $\sigma(x) \in p(x)$ there exists some $i \leq m$ such that $q_i(x) \vdash \sigma(x)$ (i.e. $q_i(M) \subseteq \sigma(M)$).

So the type $p(x)$ is either k-compressible (some $k = k(p)$) or it can be partitioned into finitely many types each of which extends to a finitely axiomatizable (in particular k-compressible) type.

Can we say something similar to this for **all** NIP formulas?

Can we say something similar to this for **all** NIP formulas?

Yes (thanks to Bays, Kaplan and Simon).

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x,y)$ be an NIP formula in a theory T and $M \models T$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that any formula $\varphi(x,b)$, $b \in B$, belongs in $p(x)$ if and only if it belongs in more than $n/2$ of the types $q_i(x)$:

$$\begin{aligned} p(x) = &\{\varphi^i(x,b) : b \in B,\, i \in \{0,1\},\, \exists I \subseteq \{1,\ldots,n\}, \\ &|I| > n/2,\, \varphi^i(x,b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that any formula $\varphi(x, b)$, $b \in B$, belongs in $p(x)$ if and only if it belongs in more than $n/2$ of the types $q_i(x)$:

$$\begin{aligned} p(x) =& \{\varphi^i(x, b) : b \in B,\, i \in \{0, 1\},\, \exists I \subseteq \{1, \ldots, n\}, \\ & |I| > n/2,\, \varphi^i(x, b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

The type $p(x)$ is the *rounded average* of the k-compressible types $q_1(x), \ldots, q_n(x)$.

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x,y)$ be an NIP formula in a theory T and $M \models T$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that any formula $\varphi(x,b)$, $b \in B$, belongs in $p(x)$ if and only if it belongs in more than $n/2$ of the types $q_i(x)$:

$$\begin{aligned} p(x) = &\{\varphi^i(x,b) : b \in B,\, i \in \{0,1\},\, \exists I \subseteq \{1,\ldots,n\}, \\ &|I| > n/2,\, \varphi^i(x,b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

The type $p(x)$ is the *rounded average* of the k-compressible types $q_1(x), \ldots, q_n(x)$.

The n and k depend only on the dual VC-dimension of φ (the maximum d such that $\pi^*_\varphi(d) = 2^d$).

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$. Fix $1/2 \leq \alpha < 1$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that

$$\begin{aligned} p(x) = &\{\varphi^i(x, b) : b \in B,\, i \in \{0,1\},\, \exists I \subseteq \{1, \ldots, n\}, \\ &|I| > \alpha n,\, \varphi^i(x, b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

The $n/2$ in the theorem can be improved to αn for any fixed $1/2 \leq \alpha < 1$.

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$. Fix $1/2 \leq \alpha < 1$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that

$$\begin{aligned} p(x) = &\{\varphi^i(x, b) : b \in B,\, i \in \{0, 1\},\, \exists I \subseteq \{1, \ldots, n\}, \\ &|I| > \alpha n,\, \varphi^i(x, b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

The $n/2$ in the theorem can be improved to αn for any fixed $1/2 \leq \alpha < 1$.

Theorem B was used to prove existence of uniform honest definitions for NIP formulas.

Two considerations around Theorem B

- Does VC-codensity play a role in refining Theorem B?
- If the type $p(x)$ is definable, can the k-compressible types $q_1(x), \ldots, q_n(x)$ be chosen definable too?

Does VC-codensity play a role in refining Theorem B?

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$. Then there exists $k, n < \omega$ such that every φ-type over any $B \subseteq M^y$ is the rounded average of n many k-compressible φ-types over B.

Does VC-codensity play a role in refining Theorem B?

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$. Then there exists $k, n < \omega$ such that every φ-type over any $B \subseteq M^y$ is the rounded average of n many k-compressible φ-types over B.

Proposition C (A.G. '23) - *Uniform Better Lemma A*

Let $\varphi(x, y)$ be a formula with $\mathbf{vc}^*(\varphi) < 2$ in a theory T. Let $M \models T$. Then there exist $k, m < \omega$ such, for every φ-type $p(x)$ over any $B \subseteq M^y$, at least one holds:

(a) $p(x)$ is k-compressible.

(b) There are m many φ-types $q_1(x), \dots q_n(x)$, with $\mathbf{vc}^*(q_i) = 0$ for every $i \leq m$, such that every formula in $p(x)$ is implied by **at least one** of the $q_i(x)$.

Does VC-codensity play a role in refining Theorem B?

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$. Then there exists $k, n < \omega$ such that every φ-type over any $B \subseteq M^y$ is the rounded average of n many k-compressible φ-types over B.

Proposition C (A.G. '23) - *Uniform Better Lemma A*

Let $\varphi(x, y)$ be a formula with $\mathbf{vc}^*(\varphi) < 2$ in a theory T. Let $M \models T$. Then there exist $k, m < \omega$ such, for every φ-type $p(x)$ over any $B \subseteq M^y$, at least one holds:

(a) $p(x)$ is k-compressible.

(b) There are m many φ-types $q_1(x), \dots q_n(x)$, with $\mathbf{vc}^*(q_i) = 0$ for every $i \leq m$, such that every formula in $p(x)$ is implied by **at least one** of the $q_i(x)$.

Does Proposition C have a generalization to $\mathrm{vc}^*(\varphi) < n$?

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Why do we care?

Because we are interested in further understanding the density of k-compressible types.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Why do we care?

Because we are interested in further understanding the density of k-compressible types.

Corollary D (of Theorem B)

Let M be a distal structure, $\varphi(x, y)$ a formula, and $p(x)$ a φ-type. Then there exists another formula $\psi(x, z)$ and a directed (i.e. 1-compressible) ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

($q(x)$ can be chosen over the same parameters as $p(x)$)

This corollary is a strong "uniform" density result for k-compressible types.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ satisfying $q(x) \vdash p(x)$ be chosen definable?

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ satisfying $q(x) \vdash p(x)$ be chosen definable?

A positive answer to our question about Theorem B, would yield a positive answer to Question E.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

$\uparrow$

This one.

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ satisfying $q(x) \vdash p(x)$ be chosen definable?

A positive answer to our question about Theorem B, would yield a positive answer to Question E.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

$\uparrow$

This one.

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ satisfying $q(x) \vdash p(x)$ be chosen definable?

A positive answer to our question about Theorem B, would yield a positive answer to Question E.

Using Proposition C (uniform Better Lemma A) we can show that Question E has a positive answer in distal theories of VC-density one.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Theorem (A.G. '23)

Let M be a distal structure of VC-density one, $\varphi(x,y)$ be a formula, and $p(x)$ a φ-type. If $p(x)$ extends to a **definable** type in $S_x(M)$, then there exists another formula $\psi(x,z)$ and a directed **definable** ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

(dp-minimal version to come)

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

$\uparrow$

This one.

But what about our question?

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Itay Kaplan <[redacted]>

Para: Pablo Andujar Guerrero

Mar 07/03/2023 18:57

I talked about it with Martin and Pierre.
It seems that there might be a counterexample but maybe it's not so easy to verify.
The idea is to take a meet-tree with a height function. So let's take M to be (omega^<omega, omega, h) where h goes from w^<w to w and gives the level. The language has the tree order, the omega order, h (and the meet). Now if you take a point s in the tree, you can look at the type saying x>=s and x is incomparable to any other point >=s (the generic type of a "closed ball") and h(x) > M (height = infinity).
Then this type is the average of compressible types by taking 3 different branches > s, say B1, B2, B3, and looking at the types saying x>B_i, h(x)>M, but these branches won't be definable.
In order to verify that this actually works, one would need to understand what are definable types, etc.

Itay

In the 2-sorted structure $(\omega^{<\omega}, \omega; <, <_\omega, h)$, with $h : \omega^{<\omega} \to \omega$ the level function, consider the formulas

$$\varphi_1(x, y_1) := \text{“}(x = y_1) \vee (x < y_1) \vee (x > y_1)\text{” for } y_1 \in \omega^{<\omega},$$
$$\varphi_2(x, y_2) := \text{“}y_2 <_\omega h(x)\text{” for } y_2 \in \omega.$$

In the 2-sorted structure $(\omega^{<\omega}, \omega; <, <_\omega, h)$, with $h : \omega^{<\omega} \to \omega$ the level function, consider the formulas

$$\varphi_1(x, y_1) := \text{“}(x = y_1) \vee (x < y_1) \vee (x > y_1)\text{” for } y_1 \in \omega^{<\omega},$$
$$\varphi_2(x, y_2) := \text{“}y_2 <_\omega h(x)\text{” for } y_2 \in \omega.$$

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{aligned} &\varphi_1(x, ()), \\ \neg &\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

The type $p(x)$ is clearly definable.

In the 2-sorted structure $(\omega^{<\omega}, \omega; <, <_\omega, h)$, with $h : \omega^{<\omega} \to \omega$ the level function, consider the formulas

$$\varphi_1(x, y_1) := \text{“}(x = y_1) \vee (x < y_1) \vee (x > y_1)\text{” for } y_1 \in \omega^{<\omega},$$
$$\varphi_2(x, y_2) := \text{“}y_2 <_\omega h(x)\text{” for } y_2 \in \omega.$$

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{gathered} \varphi_1(x, ()), \\ \neg\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{gathered}$$

The type $p(x)$ is clearly definable.
(I believe that we can apply Theorem B to $p(x)$.)

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{aligned} &\varphi_1(x, ()), \\ \neg&\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

The type $p(x)$ is clearly definable.
(I believe that we can apply Theorem B to $p(x)$.)

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{gathered}\varphi_1(x, ()),\\ \neg\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\},\\ b_2 <_\omega h(x) \text{ for every } b_2 \in \omega.\end{gathered}$$

The type $p(x)$ is clearly definable.
(I believe that we can apply Theorem B to $p(x)$.)

Pick 3 disjoint branches B_1, B_2, B_3. For each $i \in \{1, 2, 3\}$ let $q_i(x)$ be the $\{\varphi_1, \varphi_2\}$-type:

$$\begin{gathered}\varphi_1(x, b_1) \text{ iff } b_1 \in B_i,\\ b_2 <_\omega h(x) \text{ for every } b_2 \in \omega.\end{gathered}$$

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{aligned} &\varphi_1(x, ()), \\ \neg&\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

The type $p(x)$ is clearly definable.
(I believe that we can apply Theorem B to $p(x)$.)

Pick 3 disjoint branches B_1, B_2, B_3. For each $i \in \{1, 2, 3\}$ let $q_i(x)$ be the $\{\varphi_1, \varphi_2\}$-type:

$$\begin{aligned} &\varphi_1(x, b_1) \text{ iff } b_1 \in B_i, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

The type $p(x)$ is the rounded average of the types $q_1(x)$, $q_2(x)$, $q_3(x)$. Moreover these are 2-compressible. However they are *probably* not definable.

Email (Bays-Kaplan-Simon)

The $\{\varphi_1, \varphi_2\}$-type $p(x)$ is definable but might not be the rounded average of finitely many k-compressible (for any k) **definable** $\{\varphi_1, \varphi_2\}$-types.

In particular $p(x)$ is the rounded average of the 2-compressible types $q_1(x)$, $q_2(x)$ and $q_3(x)$, but these are probably not definable.

The big picture: what do we know about NIP types?

1. Few types over finite sets.
2. Indiscernible sequences converge to types.
3. Equivalence of invariance, non-dividing and non-forking over models.
4. Existence of uniform honest definitions.
5. Described by **compressible types** (Theorem B).

The big picture: what do we know about NIP types?

1. Few types over finite sets.
2. Indiscernible sequences converge to types.
3. Equivalence of invariance, non-dividing and non-forking over models.
4. Existence of uniform honest definitions.
5. Described by **compressible types** (Theorem B).

Opinion: VC-(co)density is conspicuously absent.

The bigger picture: VC meets NIP.

- 1968 – 1972: Vapnik-Chervonenkis, Shelah-Perles and Sauer all prove independently Sauer's Lemma.

The bigger picture: VC meets NIP.

statistics

- 1968 – 1972: Vapnik-Chervonenkis, Shelah-Perles and Sauer all prove independently Sauer's Lemma.

Erdős question

classification theory

The bigger picture: VC meets NIP.

statistics

- 1968 – 1972: Vapnik-Chervonenkis, Shelah-Perles and Sauer all prove independently Sauer's Lemma.

  Erdős question classification theory

- 1992: Laskowski "discovers" the connection between VC learning and NIP model theory. *He uses model theory to point out new VC classes (e.g. families of sets definable in o-minimal structures).*

The bigger picture: VC meets NIP.

statistics

- 1968 – 1972: Vapnik-Chervonenkis, Shelah-Perles and Sauer all prove independently Sauer's Lemma.

  Erdős question classification theory

- 1992: Laskowski "discovers" the connection between VC learning and NIP model theory. *He uses model theory to point out new VC classes (e.g. families of sets definable in o-minimal structures).*

- 2020: Bays, Kaplan, Simon improve a result from VC theory on recursive teaching dimension to characterize NIP types as averages of k-compressible types.

The bigger picture: VC meets NIP.

statistics

- 1968 – 1972: Vapnik-Chervonenkis, Shelah-Perles and Sauer all prove independently Sauer's Lemma.

  Erdős question classification theory

- 1992: Laskowski "discovers" the connection between VC learning and NIP model theory. *He uses model theory to point out new VC classes (e.g. families of sets definable in o-minimal structures).*

- 2020: Bays, Kaplan, Simon improve a result from VC theory on recursive teaching dimension to characterize NIP types as averages of k-compressible types.

We see fruitful outcomes (every 20+ years) from the communication between VC machine learning and NIP model theory.