Introduction

My research interests are in mathematical logic, in particular in model theory, which is the study of the relationship between collections of sentences in a formal language (theories) and the structures that satisfy them (models). More specifically I am interested in o-minimal and other "tame" theories, where "tame" references the geometric nature of sets that are definable in models using the language (e.g. in the reals in the language with addition and multiplication the graph of any polynomial is definable). O-minimal structures are linearly ordered structures where every unary definable set is a finite union of points and intervals. The study of mathematical objects definable in o-minimal structures has been a fertile area of research in the last decades, motivated by the evidence that o-minimality provides a setting that is both tame and rich enough to allow a variety of applications both within and outside mathematics (more on this in the next paragraph). My doctoral research centered on topologies that are definable in o-minimal theories (in the sense of Pillay [Pil87]), as well as in the connected notions of o-minimal types, combinatorics, and forking. My goal is to advance and unify previous research in this area, and develop an overarching tame framework of general topology with ample applicability. While I continue to pursue this aim within o-minimality, I presently have an interest in studying definable topologies in other tame theories, such as dp-minimal valuation topologies. In broad terms, my topological interests lie on the intersections of model theory with general topology and functional analysis. On the other hand, I have developed an independent interest in the study of types and combinatorics in o-minimal and more general NIP theories.

O-minimality has established itself as the realization of the "topologie modérée" of Grothendieck, a setting for analysis that avoids the pathologies of general set-theoretic topology. It generalizes semialgebraic and semianalytic geometry, as well as the real field with the exponential function, and has found applications in a wide range of areas including number theory [PW06], machine learning [KM95] and economics [RW00]. The seminal work of Knight, Pillay and Steinhorn [KPS86] established the topological tameness of definable sets with the canonical "euclidean topology". Pillay [Pil88] proved that definable groups in o-minimal structures have a natural definable manifold topology that makes them topological groups. This motivated the study of definable topologies within o-minimality beyond the euclidean one. Van den Dries [vdD98] applied techniques from semialgebraic topology to classify these manifold spaces. Recently, Johnson [Joh18] proved that interpretable sets have a similiar piecewise euclidean topological structure. On the other hand, Thomas [Tho12] was motivated by the proof of the Reparametrization Theorem of Pila and Wilkie, an application of o-minimality to diophantine geometry, to study the topology of definable families of functions with C^r-norms and norms involving a Lipschitz constant. She proposed the development of a theory of definable normed spaces in this setting. This was generalized by Walsberg [Wal15], who introduced and studied definable metric spaces. Finally, definable orders in o-minimal structures, which encapsule definable order topologies, were studied by Ramakrishnan and Steinhorn ([Ram13], [RS14]), with applications to economics.

My research into o-minimal definable topologies seeks to unify all the above areas. I used methods from o-minimality, general topology and functional analysis to reach classification results and establish a broad tame setting of definable topology. A central goal, which is motivated by previous similar work by van den Dries [vdD98] and Walsberg [Wal15], is the characterization of those spaces that are definably homeomorphic to a euclidean space (affine spaces) in o-minimal expansions of ordered fields. In joint work with Thomas and Walsberg [ATW] we extensively study and answer this question for one-dimensional spaces. We moreover prove a decomposition result for one-dimensional spaces in terms of the lexicographic order topology and the Alexandrov double circle topology, and use it to establish universality results. We show that the Cantor space is not definable, as well as a strong version of the Gruenhage conjecture of set theoretic topology within o-minimality. Finally, I also proved an affineness theorem for spaces of all dimensions. I describe these results in more detail in the next section. We are able to conclude that the setting of definable o-minimal topology admits an ample variety of classical topological spaces, while mantaining powerful classification results and avoiding some pathological spaces in general topology.

We work with suitable analogues of classical topological properties, such as compactness and separability. This invites the investigation of which results and proof techniques from general topology and functional analysis can be imported to our setting, and which ones are beyond the tools of model theory or o-minimality. This process also informs our conjectures. For example we work with definable spaces of functionals to adapt metrization theorems on compact Hausdorff spaces.

A definition of definable compactness in the o-minimal setting introduced by Peterzil and Steinhorn [PS99] has been crucial both in the study of o-minimal groups, where it was used in the formulation in Pillay's conjecture [PP07] that a certain quotient of a definably compact group is a compact Lie group, and in the work in normed space theory and metric topology of Thomas and Walsberg respectively. I studied a different modern definition of definable compactness of current interest both in o-minimality and general model theory. This work led me to a more general result in connection with combinatorics, forking and types in model theory. It is in particular related to the Alon-Kleitman-Matousek (AKM) (p, q)-theorem, that applies to VC set systems, which are objects of study in machine learning. In model theory they play a central role in the definition of NIP theories, a class extending o-minimal theories that has seen renewed interest in the last two decades, partly in the context of proving Pillay's conjecture.

Finally, my work has led me to new density results for o-minimal types and new proofs of known combinatorial and forking results. These have motivated ongoing work on a novel classification of o-minimal types, based partially on the work of Dolich on forking in o-minimality [Dol04], as well as given rise to my interest in some combinatorial conjectures in dp-minimal and NIP theories, which I address in the next section.

I aspire to continue developing the theory of definable topology in o-minimal and other tame structures, studying in particular the connections with classical functional analysis and proving universality and classification results. Moreover, I am intrigued to continue the research of types and combinatorics in NIP settings, and their connections to topological notions such as definable compactness.

I now move on to describe in more detail some of the aforementioned topics and results.

Classifying definable topologies and affine spaces

Much of this endeavour was pursued jointly with Erik Walsberg and Margaret Thomas.

In [AGTW21] we prove, by analyzing definable downward directed families of sets, that spaces definable in o-minimal expansions of ordered groups and fields display properties akin to first countability. In [vdD98] van den Dries argues that definable curves take the role of sequences in the o-minimal setting. We support this thesis by proving that, given any definable topology in an o-minimal expansion of an ordered field, a point is in the closure of a definable set if and only if it is the limit of a definable curve in the set (definable curve selection).

In [ATW] we undertake an in-depth analysis of one-dimensional definable topological spaces in o-minimal structures. We prove the following.

Theorem 1 ([ATW])**.** *Let $\mathcal{M} = (M, < \dots)$ be an o-minimal structure and (X, τ), $X \subseteq M$, a Hausdorff topological space definable in $\mathcal{M}$. There exists a finite partition $\mathcal{X}$ of X into points and intervals such that, for every interval $I \in \mathcal{X}$, the subspace topology $I|_\tau$ is either the euclidean, discrete, or the right or left half-open interval topology.*

This gives a positive answer in the o-minimal setting to a generalized form of the Gruenhage conjecture of set theoretic topology [Gru88], which states that every uncountable, first countable, regular Hausdorff topological space has a subspace of cardinality $\aleph_1$ that embeds into $\mathbb{R}$ with the euclidean, discrete, or right half-open interval topology.

In pursuing universality results we prove that any one-dimensional Hausdorff regular space can be partitioned into a finite set and two definable open sets, where the open sets are definably homeomorphic to a space with either the lexicographic order topology or a generalization of the Alexandrov double circle topology. We derive in particular that any one-dimensional space definable in an o-minimal expansion of $(\mathbb{R}, <)$ that is Hausdorff, regular and separable has a cofinite subset definably homeomorphic to a set with the lexicographic order topology.

A definable topological space is affine if it is definably homeomorphic to a euclidean space. The study of affine spaces in o-minimal expansions of ordered fields was started by van den Dries [vdD98], who proved, using techniques from semialgebraic topology, that spaces with a manifold structure are affine if and only if they are regular. Walsberg [Wal15] introduced definable metric spaces and proved that they are affine if and only if they are definably separable, a suitable analogue of separability that is equivalent to its classical counterpart in o-minimal expansions of the reals. We continued this line of research by proving the following.

Theorem 2 ([ATW])**.** *A one-dimensional definable topological space in an o-minimal expansion of an ordered field is affine if and only if it is Hausdorff and can be partitioned into finitely many points and intervals where the subspace topology is euclidean.*

This result was also proved independently by Peterzil and Rosel [PR20].

Classically, the study of the existence of certain continuous maps on topological spaces based on their separation properties goes back to the Urysohn metrization theorem, as well as the Urysohn Lemma and equivalent Tietze extension theorem. These results yield the existence of a “rich” space of scalars (real-valued continuous functions) on a space, in the sense that the scalars completely describe the space topology (i.e. their induced weak topology equals the space topology). Nevertheless, the proofs of these results cannot be imported to the definable model theoretic setting. To prove his affineness theorem Walsberg [Wal15] used the definability of the metric map, as well as the definable Michael Selection theorem (although I refined the proof to avoid the use of the latter). On the other hand, in the general definable topological setting, it is not clear whether "nice" spaces (e.g. compact Hausdorff) admit rich definable continuous functions. Advancing this line of research requires circumventing this obstacle. A straightforward way of doing it is by simply assuming that a space (definable in some structure $\mathcal{M}$) admits a rich definable family of (M-valued) scalars. When the topology is also Hausdorff I call this, in parallelism with classical topology, being definably Tychonoff. It applies for example to locally affine spaces. Working with definably Tychonoff spaces allows the adoption proof schemes from functional analysis, such as seeing the space topology as a pointwise convergence topology.

Another tool that has proven to be useful in classifying o-minimal definable topologies is the condition that the topology behaves nicely with respect to o-minimal dimension. Specifically, we consider the frontier dimension inequality: the dimension of the frontier of any definable set is strictly less than the dimension of the set. Notably this is a property of the euclidean topology and of any definable metric topology (but not of every definable topology).

The following is my main affineness result for definable topological spaces of all dimensions.

Theorem 3 ([AG21a]). *Let (X, τ) be a definably compact definable topological space (of any dimension) in an o-minimal expansion of an ordered field. The following are equivalent.*

(i) (X, τ) *is definably Tychonoff and has the frontier dimension inequality.*

(ii) (X, τ) *is affine.*

Future directions

While Theorem 3 characterizes affine o-minimal definable topologies that are definably compact, the question of a general characterization remains open. I believe that the missing piece is a suitable notion of definable second countability which, in parallel to Urysohn's metrization theorem, should yield a characterization of affine definable topologies to be those that are Hausdorff, regular, definably second countable and have the frontier dimension inequality.

More generally, I am interested in investigating definable topologies in other tame (in particular NIP) structures, such as the valuation topology in the field of p-adic numbers. I would inform this research with the framework I helped develop within o-minimality, contrasting the tools, definitions and results that are available in the distinct settings. I am generally interested in the study and development of definable functional analysis in tame fields.

Definable compactness and combinatorial results

I have extensively studied the various notions of definable topological compactness within o-minimality. The precise property I label "definable compactness" is the following: every definable downward directed family of closed nonempty sets has nonempty intersection. This modern notion was approached by Fornasiero [For], who presents it as a powerful condition that mimics its classical counterpart in the definable model theoretic setting. For example the continuous definable image of a definably compact space is definably compact, and a set definable in an o-minimal structure is definably compact in the euclidean topology if and only if it is closed and bounded. The previous definition introduced in the o-minimal setting by Peterzil and Steinhorn [PS99] was that every definable curve converges. This was used by Walsberg in proving his affineness result [Wal15]. Both notions are known to be equivalent for the eucidean topology in an o-minimal structure [Joh18]. Another notion, studied in [HL16], is the property that for every definable (complete) type in a space there is a point contained in every closed set in the type.

I characterize definable compactness within o-minimality, showing in particular equivalences between the different notions, as follows.

Theorem 4 ([AG21b]). *Let $\mathcal{M}$ be an o-minimal structure. Let (X, τ) be a definable topological space in $\mathcal{M}$. The following are equivalent.*

(1) (X, τ) is definably compact, i.e. every downward directed definable family of closed sets has nonempty intersection.

(2) Every definable type in X has a limit, i.e. there is a point contained in every closed set in the type.

(3) Any definable family of closed sets that extends to a definable type has nonempty intersection.

(4) Any definable family of closed sets with the finite intersection property has a finite transversal, i.e. there exists a finite set that intersects every set in the family.

(5) Any definable family $\mathcal{C}$ of nonempty closed sets with the (p, q)-property, where $p \geq q > \dim \cup\mathcal{C}$, has a finite transversal.

(6) Any definable family $\mathcal{C}$ of nonempty closed sets with the (p, q)-property, where $p \geq q$ and q is greater than the VC-codensity of $\mathcal{C}$, has a finite transversal.

Moreover all the above imply and, if τ is Hausdorff or $\mathcal{M}$ has definable choice, are equivalent to:

(7) Every definable curve in (X, τ) converges.

As part of their work in definably compact groups Peterzil and Pillay [PP07] proved that, in an o-minimal theory with definable choice functions, every definable family of closed and bounded sets with the finite intersection property has a finite transversal. I generalize this in three ways: by considering closed sets in any definably compact topology, by weakening the intersection property, and by dropping the assumption of having choice functions. From the above theorem I derived, using the Marker-Steinhorn

Theorem [MS94], that definable compactness is equivalent to compactness in o-minimal expansions of $(\mathbb{R}, <)$.

The main ingredient in the proof of Theorem 4 was the following more general result.

Theorem 5 ([AG21b]). *Let $\mathcal{M} = (M, <, \dots)$ be an o-minimal structure and $\mathcal{S}$ be a definable family of nonempty subsets of M^n. Suppose either of the following two conditions hold.*

(a) $\mathcal{S}$ has the (p, q)-property for some $p \geq q > \dim \cup \mathcal{S}$.

(b) $\mathcal{S}$ has the (p, q)-property for some $p \geq q$ with q greater than the VC-codensity of $\mathcal{S}$.

Then $\mathcal{S}$ can be partitioned into finitely many subfamilies, each of which extends to a definable type.

The proof of the above theorem given condition (b) uses the AKM (p, q)-theorem. This theorem implies that an arbitrary family with the described intersection property can be partitioned into finitely many subfamilies, each having the finite intersection property. Condition (a) can be derived from condition (b) using the work of Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko [ADH$^+$16] on bounding the VC-density of definable families of sets in o-minimal and other tame theories. The proof I produce assuming condition (a) is elementary in that it circumvents these results and relies solely in o-minimal cell decomposition.

The proof of the AKM (p, q)-theorem used a method developed for the solution to the long-standing (p, q)-conjecture for convex sets. This area of research can be traced back to the classical Helly theorem for convex sets, which states that, if a finite family of convex subsets of $\mathbb{R}^d$ satisfies that any $d + 1$ sets intersect, then the whole family intersects. By a straightforward compactness argument it also holds for infinite families of compact convex sets. The definable Helly theorem of Aschenbrenner and Fischer [AF11] proved this for definable families of closed and bounded convex sets in any definably complete real closed field. I generalize this result within o-minimality from closed and bounded sets to sets definably compact in any definable topology. Moreover I observe that, if a definable family of convex subsets of R^d, where $\mathcal{R}$ is an o-minimal expansion of an ordered field, satisfies that every $d+1$ sets intersect, then the whole family extends to a definable type.

Theorem 5 is deeply interconnected with the work of Dolich, Starchenko, Simon and others ([SS14], [Dol04]) on forking in the o-minimal and more general NIP context. As of today the more general theorem [SS14] in this respect states that, in a large class of dp-minimal theories, a formula does not fork (equivalently does not divide) over a model M if and only if it extends to a M-definable type. I observe that, in an o-minimal theory, this holds given forking over any set (not just a model), and derive from Theorem 5 the equivalence given an intersection property weaker than not dividing.

Finally, motivated by developing the bridge between results involving definable compactness and more general results in o-minimal combinatorics, I studied types. This includes in particular types that have a basis given by their restriction to a formula. I proved in particular that any definable family of sets in an o-minimal structure that extends to a definable type also extends to one with one such basis.

Future directions

Simon conjectured [Sim15, Conjecture 5.2] that Theorem 5 (version (b)) holds in all dp-minimal theories, as well as a weaker similar conjecture (the definable (p, q)-conjecture) for all NIP theories [Sim15, Conjecture 5.1]. I am interested in these conjectures, and currently investigating whether my proof of Theorem 5 can be generalized to offer any new insight towards their solution.

Johnson proved [AG21b, Appendix B] that, even though every definable family of sets that extends to a definable type also extends to one with a basis given by its restriction to a formula, this basis might not be definable over the same parameters as the family. This has motivated my ongoing research on density results among types, building on the notion of "goodness" for definable sets introduced by Dolich [Dol04]. I aspire to extend this research to the study of more general NIP theories, and explore deeper the connection between these results, NIP combinatorics, and notions of definable compactness.

References

[ADH+16] Matthias Aschenbrenner, Alf Dolich, Deirdre Haskell, Dugald Macpherson, and Sergei Starchenko. Vapnik-Chervonenkis density in some theories without the independence property, I. *Trans. Amer. Math. Soc.*, 368(8):5889–5949, 2016.

[AF11] Matthias Aschenbrenner and Andreas Fischer. Definable versions of theorems by Kirszbraun and Helly. *Proc. Lond. Math. Soc. (3)*, 102(3):468–502, 2011.

[AG21a] Pablo Andújar Guerrero. *Definable Topological Spaces in O-minimal Structures*. PhD thesis, Purdue University, 2021.

[AG21b] Pablo Andújar Guerrero. Types, transversals and definable compactness in o-minimal structures, 2021. arXiv:2111.03802.

[AGTW21] Pablo Andújar Guerrero, Margaret Thomas, and Erik Walsberg. Directed sets and topological spaces definable in o-minimal structures. *J. London Math. Soc. (2)*, 104(3):989–1010, 2021.

[ATW] Pablo Andújar, Margaret Thomas, and Erik Walsberg. One dimensional definable topological spaces in o-minimal structures. In preparation.

[Dol04] Alfred Dolich. Forking and independence in o-minimal theories. *J. Symbolic Logic*, 69(1):215–240, 2004.

[vdD98] Lou van den Dries. *Tame Topology and O-minimal Structures*, volume 248 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1998.

[For] Antongiulio Fornasiero. Definable compactness for topological structures. In preparation.

[Gru88] G. Gruenhage. On the existence of metrizable or Sorgenfrey subspaces. In *General topology and its relations to modern analysis and algebra, VI (Prague, 1986)*, volume 16 of *Res. Exp. Math.*, pages 223–230. Heldermann, Berlin, 1988.

[HL16] Ehud Hrushovski and François Loeser. *Non-archimedean tame topology and stably dominated types*, volume 192 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2016.

[Joh18] Will Johnson. Interpretable sets in dense o-minimal structures. *J. Symb. Log.*, 83(4):1477–1500, 2018.

[KM95] Marek Karpinski and Angus Macintyre. Bounding VC-dimension for neural networks: progress and prospects. In *Computational learning theory (Barcelona, 1995)*, volume 904 of *Lecture Notes in Comput. Sci.*, pages 337–341. Springer, Berlin, 1995.

[KPS86] Julia F. Knight, Anand Pillay, and Charles Steinhorn. Definable sets in ordered structures. II. *Trans. Amer. Math. Soc.*, 295(2):593–605, 1986.

[MS94] David Marker and Charles I. Steinhorn. Definable types in O-minimal theories. *J. Symbolic Logic*, 59(1):185–198, 1994.

[Pil87] Anand Pillay. First order topological structures and theories. *J. Symbolic Logic*, 52(3):763–778, 1987.

[Pil88] Anand Pillay. On groups and fields definable in o-minimal structures. *J. Pure Appl. Algebra*, 53(3):239–255, 1988.

[PP07] Ya'acov Peterzil and Anand Pillay. Generic sets in definably compact groups. *Fund. Math.*, 193(2):153–170, 2007.

[PR20] Ya'acov Peterzil and Ayala Rosel. Definable one-dimensional topologies in o-minimal structures. *Arch. Math. Logic*, 59(1-2):103–125, 2020.

[PS99] Ya'acov Peterzil and Charles Steinhorn. Definable compactness and definable subgroups of o-minimal groups. *J. London Math. Soc. (2)*, 59(3):769–786, 1999.

[PW06] J. Pila and A. J. Wilkie. The rational points of a definable set. *Duke Math. J.*, 133(3):591–616, 2006.

[Ram13] Janak Ramakrishnan. Definable linear orders definably embed into lexicographic orders in o-minimal structures. *Proc. Amer. Math. Soc.*, 141(5):1809–1819, 2013.

[RS14] Janak Ramakrishnan and Charles Steinhorn. Definably extending partial orders in totally ordered structures. *MLQ Math. Log. Q.*, 60(3):205–210, 2014.

[RW00] Marcel K. Richter and Kam-Chau Wong. Definable utility in o-minimal structures. *J. Math. Econom.*, 34(2):159–172, 2000.

[Sim15] Pierre Simon. Invariant types in NIP theories. *J. Math. Log.*, 15(2):1550006, 26, 2015.

[SS14] Pierre Simon and Sergei Starchenko. On forking and definability of types in some DP-minimal theories. *J. Symb. Log.*, 79(4):1020–1024, 2014.

[Tho12] Margaret E. M. Thomas. Convergence results for function spaces over o-minimal structures. *J. Log. Anal.*, 4(1), 2012.

[Wal15] Erik Walsberg. On the topology of metric spaces definable in o-minimal expansions of fields. *arXiv e-prints*, 2015. arXiv:1510.07291.