Directed sets and topological spaces definable in o-minimal structures

Pablo Andújar Guerrero

Universität Konstanz

pa377@cantab.net

Conventions

- Throughout $\mathcal{R} = (R, 0, 1, +, \cdot, <, \ldots)$ denotes an o-minimal expansion of a field and definable means definable in $\mathcal{R}$ (possibly with parameters).
- A curve in $X \subseteq R^n$ is a map $\gamma : (0, \infty) \to X$. If τ is a topology on X we say that γ τ-converges to $x \in X$ if, for every open $A \ni x$, $\gamma(t) \in A$ for all $t > 0$ small enough.

Definition 1

A **definable topological space** (X, τ) is a topological space such that $X \subseteq R^n$ is definable and there exists a definable basis $\mathcal{B} = \{A_u : u \in R^m\} \subseteq \mathcal{P}(X)$ for the topology τ.

E.g. Definable groups and fields [3], definable metric spaces [4], quotient spaces of definable open equivalence relations [1].

Definition 2

A **definable (downward) directed set** is a tuple $(\Omega, \preccurlyeq)$ such that $\Omega \subseteq R^n$ is definable and $\preccurlyeq$ is a definable preorder (i.e. a reflexive and transitive relation) on Ω such that, for every finite subset $F \subseteq \Omega$,

$$\exists u \in \Omega \text{ such that } u \preccurlyeq v \, \forall v \in F.$$

Theorem 1

Let $(\Omega, \preccurlyeq)$ be a definable directed set. There exists a definable curve $\gamma : (0, \infty) \to \Omega$ such that, for every $u \in \Omega$, $\gamma(t) \preccurlyeq u$ for all $t > 0$ small enough.

Given (X, τ) a definable topological space with definable basis $\mathcal{B} = \{A_u : u \in R^m\}$ and $x \in X$ consider $\Omega = \{u \in R^m : x \in A_u\}$. The family $\{A_u : u \in \Omega\}$ clearly induces a definable directed set $(\Omega, \preccurlyeq)$ where $u \preccurlyeq v \Leftrightarrow A_u \subseteq A_v$. Hence the following theorem follows from Theorem 1.

Theorem 2 (Definable first countability)

Let (X, τ) be a definable topological space. For every $x \in X$ there exists a definable basis of open neighborhoods of x of the form $\{A_t : t > 0\}$.

There are uniform versions of theorems 1 and 2. In particular in Theorem 2 the definable basis can be taken uniformly on x.

Corollary 1 (Curve Selection)

Let (X, τ) be a definable topological space and $Y \subseteq X$ be definable. Then $x \in X$ belongs in the τ-closure of Y iff there exists a definable curve in Y τ-converging to x.

Corollary 2

If the ambient structure $\mathcal{R}$ is separable (e.g. if $\mathcal{R}$ expands $\bar{\mathbb{R}}$) then any definable topological space is first countable. In general the character of a definable topological space is bounded by the density of the ambient structure.

Definition 3

- A definable topological space (X, τ) is **Definably compact 1 (DC1)** if every definable curve in X τ-converges [2].
- A definable topological space (X, τ) is **Definably compact 2 (DC2)** if every definable (downward) directed family of nonempty closed sets has nonempty intersection [1].

Corollary 3

A definable topological space (X, τ) is **DC1** if and only if it is **DC2**.

By Corollary 3 we may define **definable compactness** to be either DC1 or DC2. If $\mathbb{R}$ expands the field of reals, then we know that a definable topological space is definably compact iff it is compact.

If $\mathbb{R}$ is an arbitrary o-minimal structure (not necessarily expanding a field) we may always prove that DC2 $\Rightarrow$ DC1 but it is no longer necessarily true that DC1 $\Rightarrow$ DC2.

Question: What should be the definition for definable compactness in the general setting of $\mathcal{R}$ being an arbitrary o-minimal structure?

References

[1] W. Johnson, *Topologizing interpretable sets in O-minimal Structures*, preprint 2014.
[2] Y. Peterzil and C. Steinhorn, *Definable compactness and definable subgroups of o-minimal groups*, J. London Math. Soc., Vol. 59, 3, p. 769-786, June 1999.
[3] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra, Vol. 53, p. 239-255, 1988.
[4] E. Walsberg, *On the Topology of Metric Spaces Definable in o-minimal expansions of fields*, preprint 2015.