NIP compressible types

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anguepa.github.io/handout.pdf

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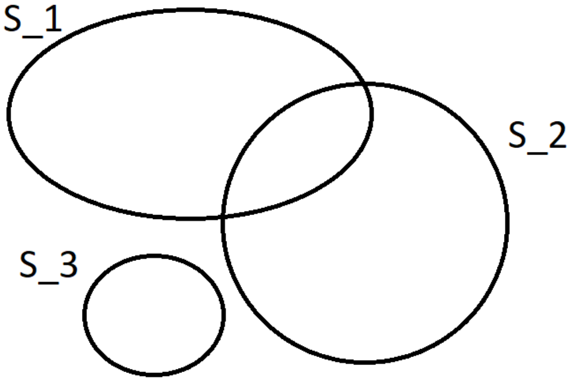

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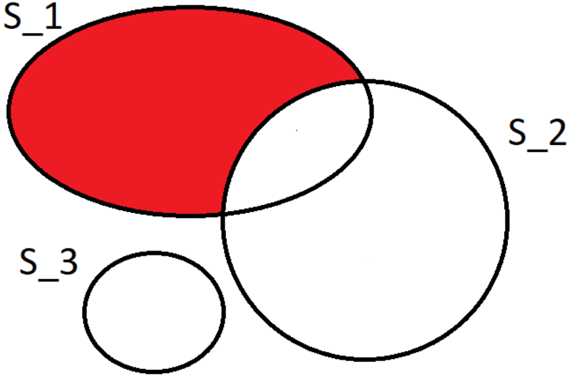

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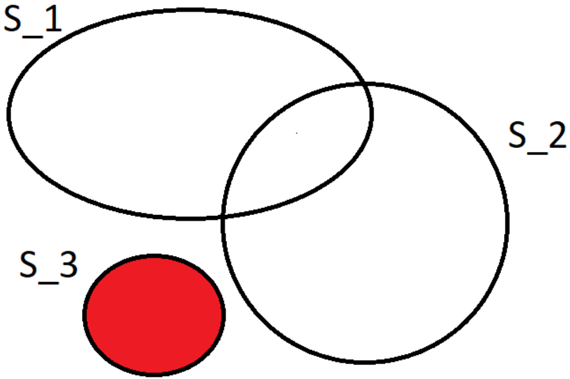

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(In model-theoretic terminology a Boolean atom can be called a type.)

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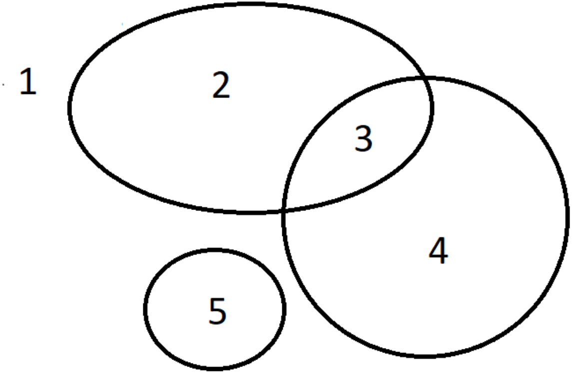
$BA(\mathcal{F}) = 5$



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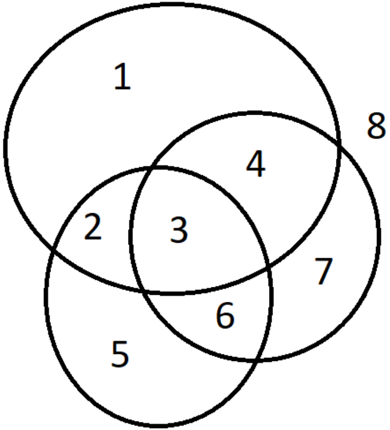
$$BA(\mathcal{F}) = 2^3 = 8$$



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Definition

Let $\mathcal{S}$ be a family of sets. The *dual shatter function of* $\mathcal{S}$ is the function

$$\begin{aligned} \pi^*_{\mathcal{S}} : & \omega \to \omega \\ & n \mapsto \max\{BA(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{S}, \, |\mathcal{F}| = n\} \end{aligned}$$

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E.g. If $\mathcal{S} = \{\textbf{intervals in } \mathbb{R}\}$ then $\pi^*_{\mathcal{S}}(n) = 2n$ and $\mathrm{vc}^*(\mathcal{S}) = 1$.

Definition

A family of sets $\mathcal{S}$ is *(downward) directed* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists some $S \in \mathcal{S}$ such that $S \subseteq \cap\mathcal{F}$.

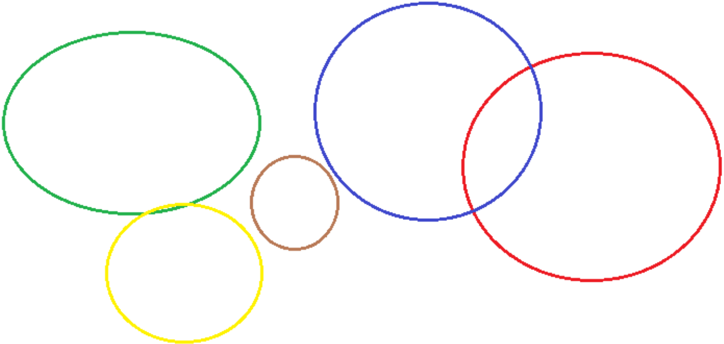
Equivalently, $\mathcal{S}$ is directed if, for any $S_1, S_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ with $S \subseteq S_1 \cap S_2$.

E.g.

- In a poset $(P, \leq)$ the sets $\{a \in P : a \leq b\}$ for $b \in P$.
- The neighborhoods of a given point in a topological spaces.
- Any filter basis.

Definition

A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.



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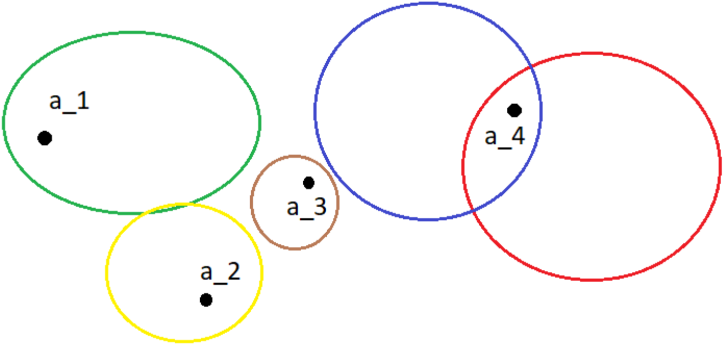

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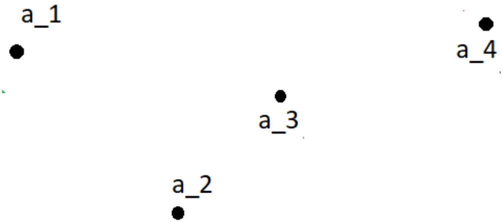

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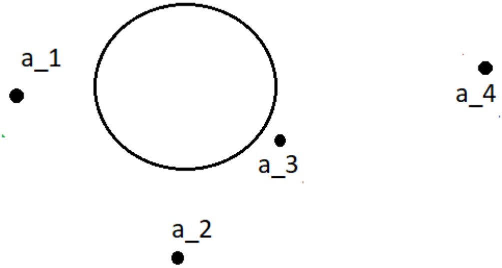

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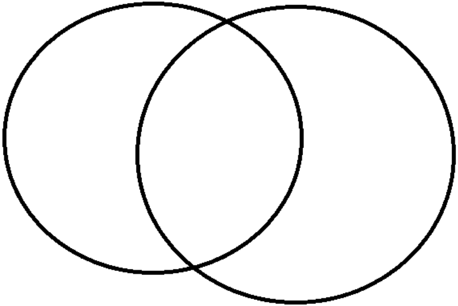

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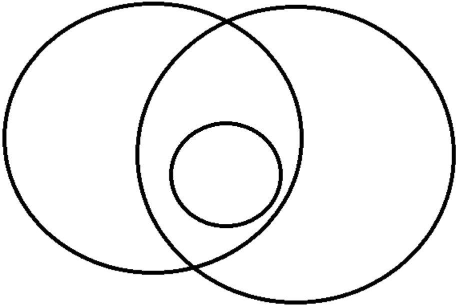

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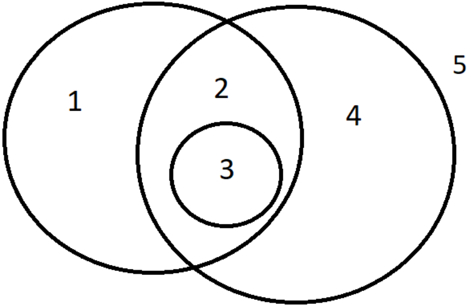

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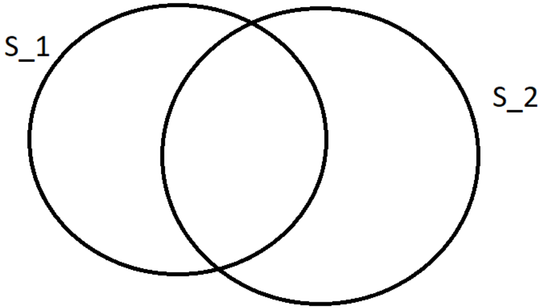

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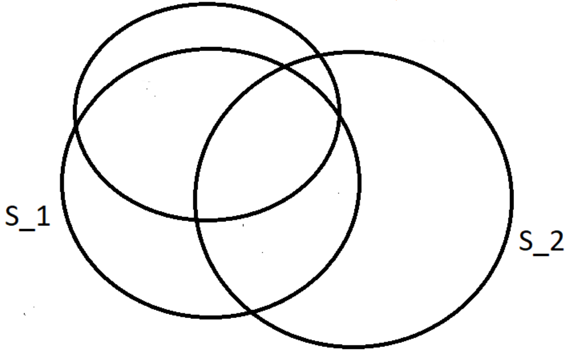

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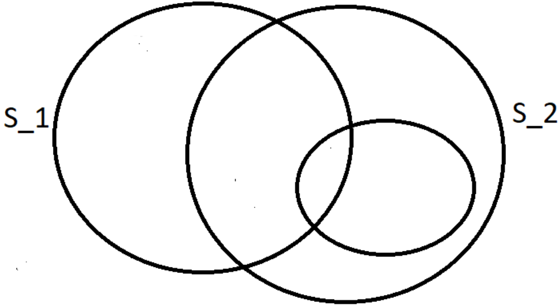

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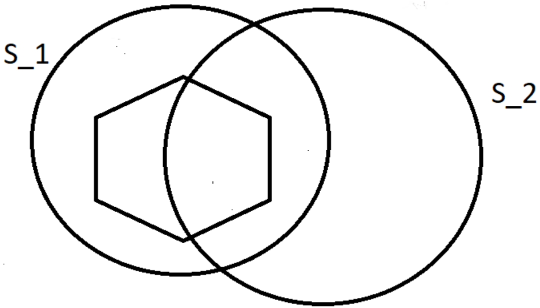

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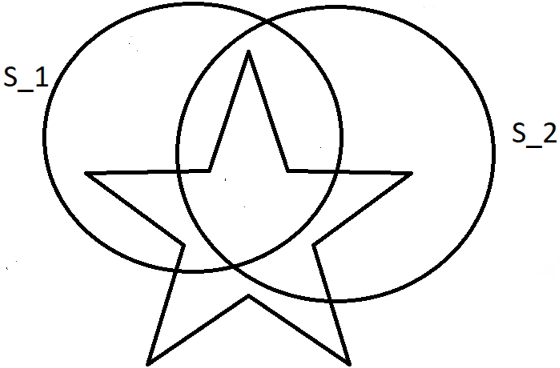

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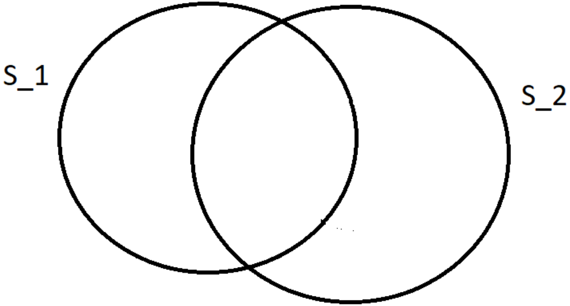
$$BA(S_1, S_2) \geq 3$$



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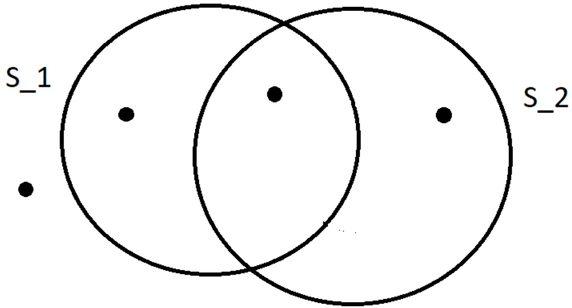

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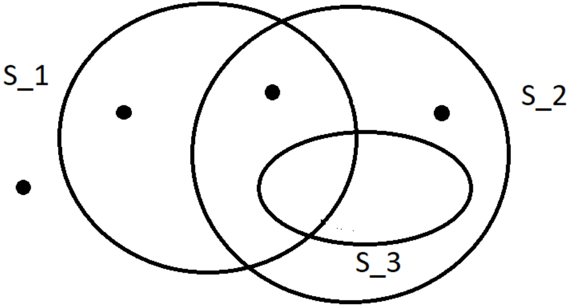
$$BA(S_1, S_2, S_3) \geq 3 + 2 = 5$$



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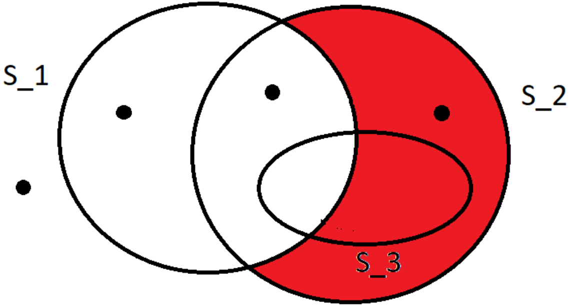

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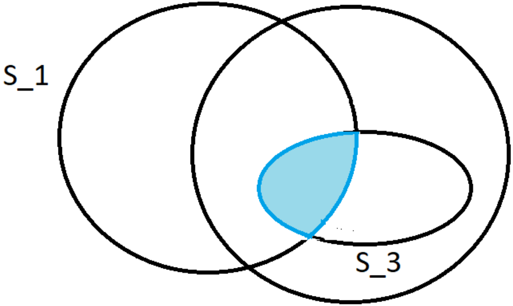

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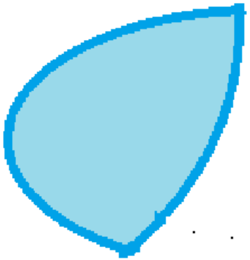

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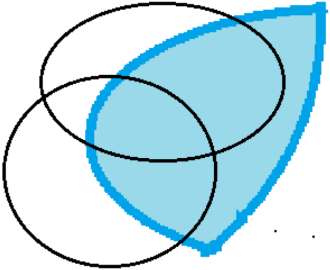

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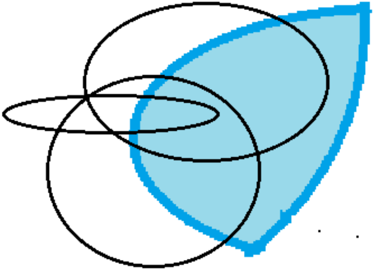

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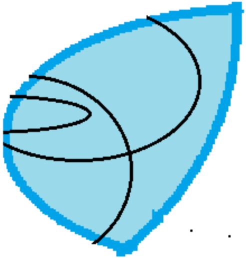

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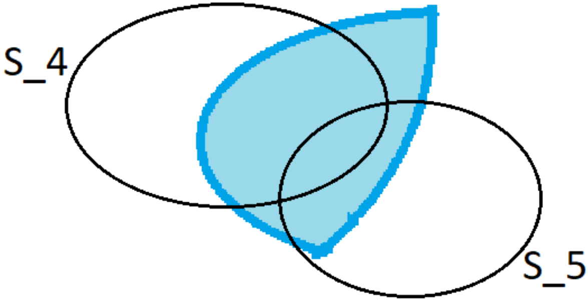
$$BA(S_1, S_2, S_3, S_4, S_5) \geq BA(S_1, S_2, S_3) + 3$$



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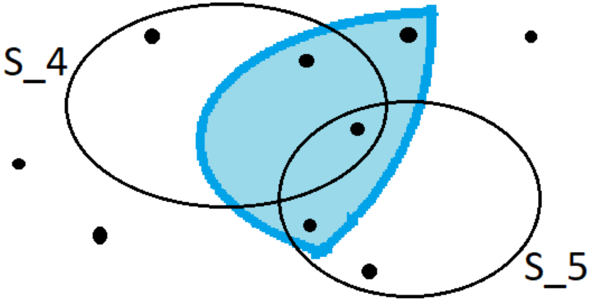

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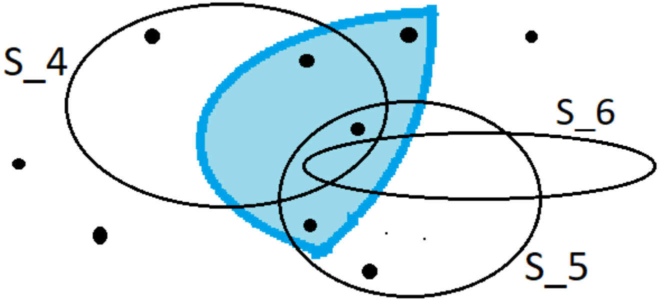

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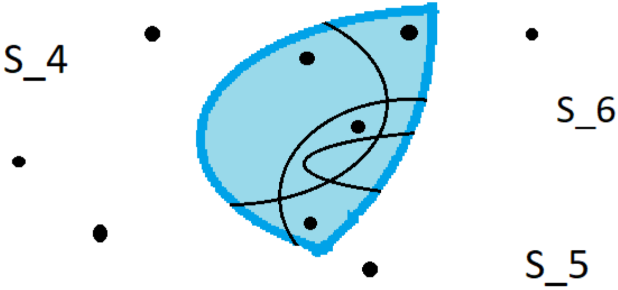
$$BA(S_1, \ldots, S_6) \geq BA(S_1, S_2, S_3) + 3 + 2.$$



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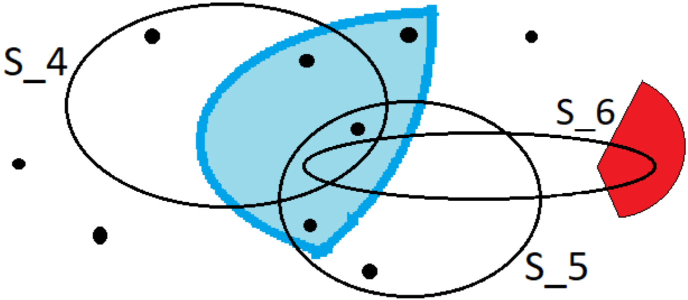

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We continue this process, and one of two things can happen:

- We find a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S}$ is directed.
- There is a sequence $(S_i)_i$ of sets in $\mathcal{S}$ such that

  $$BA(B_i : i \leq 3n) \geq \sum_{j=1}^{n} j.$$

  Meaning that

  $$\pi^*_{\mathcal{S}}(3n) \geq \frac{n^2+n}{2}.$$

Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

(b) $\mathcal{S}$ has a finite transversal.

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(b) There exists $F_1, \ldots, F_m$, finite non-empty Boolean combinations of sets in $\mathcal{S}$ such that, for every $S \in \mathcal{S}$ there is some $i \leq m$ such that $F_i \subseteq S$.

Now let us talk compressibility.
Our two main references are:

- Martin Bays, Itay Kaplan and Pierre Simon. *Density of compressible types and some consequences.* ArXiv: 2107.05197, 2022.
- Xi Chen, Yu Cheng, and Bo Tang. *On the recursive teaching dimension of VC classes.* Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems, 2016, Barcelona, Spain, p 2164–2171.

Model-theoretic framework

Let T be a theory and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

- $\varphi(x, y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

Model-theoretic framework

Let T be a theory and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

- $\varphi(x, y)$ may be directed.
- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

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A theory T is *of VC-density one* if every formula $\varphi(x, y)$ with $|x| = 1$ has VC-codensity less than 2.

$$\text{VC-density one} \subseteq \text{dp-minimal} \subsetneq \text{NIP}$$

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Conjecture

$$\boxed{\text{VC-density one} = \text{dp-minimal}} \subsetneq \text{NIP}$$

Theorem (Baldwin-Saxl '76). *Let G be a group definable in an NIP structure. Let $\mathcal{H}$ be a (uniformly) definable family of subgroups of G. Then there is $k < \omega$ such that, for any finite family $\mathcal{F} \subseteq \mathcal{H}$, there is a subfamily $\mathcal{C} \subseteq \mathcal{F}$ with $|\mathcal{C}| \leq k$ such that $\cap\mathcal{C} = \cap\mathcal{F}$.*

Definition

Let $k < \omega$. A family of sets $\mathcal{S}$ is *k-compressible* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Equivalently, $\mathcal{S}$ is k-compressible if, for any subfamily $\mathcal{F} \subseteq \mathcal{S}$ of size $k+1$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

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Note: Directed = 1-compressible.

"This terminology is inspired by the terminology around compression schemes in the statistical learning literature."
— Bays-Kaplan-Simon

- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.

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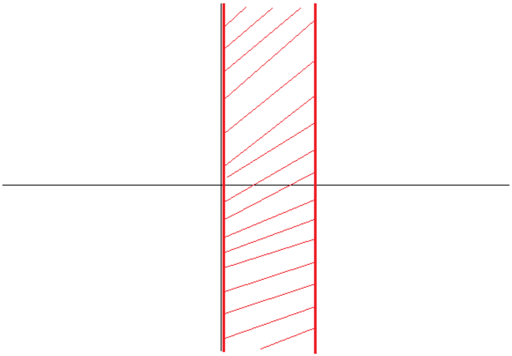

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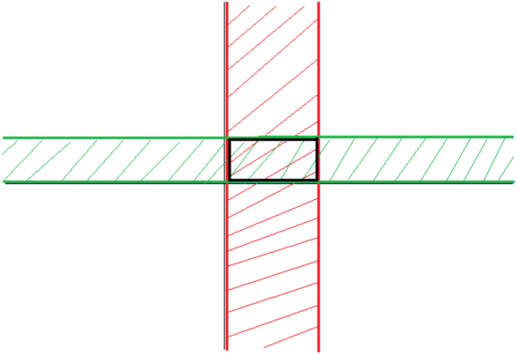

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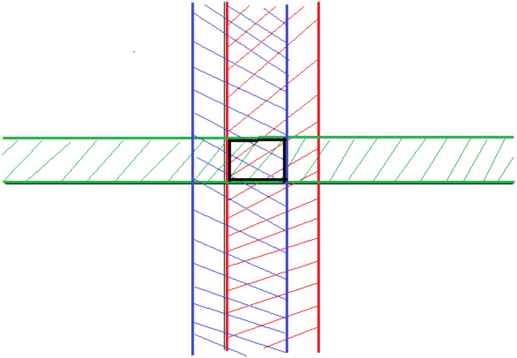

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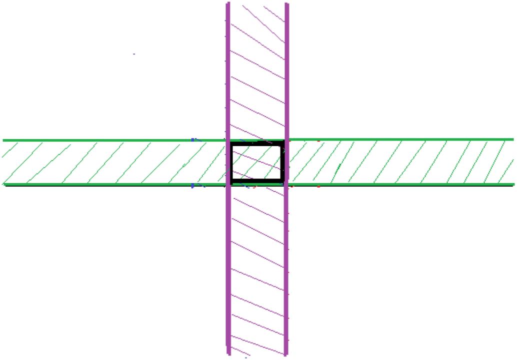

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A k-compressible family of sets is consistent iff it is k-consistent (every k sets have non-empty intersection).

So for k-compressible **definable** families of sets consistency is expressible with a single first order sentence.

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Definition

A φ-type $p(x)$ is k-compressible if for any **finite** partial type $q(x) \subseteq p(x)$ there exists $r(x) \subseteq p(x)$ with $|r(x)| \leq k$ such that

$$r(x) \vdash q(x) \quad \text{(i.e. } r(M) \subseteq q(M)\text{)}.$$

Better Lemma A (A.G. '23)

Let $\varphi(x, y)$ be a formula in a theory T and $M \models T$. Let $B \subseteq M^y$ and $p(x)$ be a φ-type over B with $\mathrm{vc}^*(p(x)) < 2$. Then at least one holds:

(a) There exists a finite conjunction $\psi(x)$ of formulas in $p(x)$ such that $\psi(x) \wedge p(x) = \{\psi(x) \wedge \sigma(x) : \sigma(x) \in p(x)\}$ is directed (i.e. 1-compressible).
In particular $p(x)$ is k-compressible for some $k < \omega$.

(b) There exist $q_1(x), \dots q_m(x)$ finite (partial) φ-types such that, for every $\sigma(x) \in p(x)$ there exists some $i \leq m$ such that $q_i(x) \vdash \sigma(x)$.

Better Lemma A* (A.G. and Johnson '23)

Let $\varphi(x, y)$ be a formula with $|x| = 1$ in a dp-minimal theory T and $M \models T$. Let $B \subseteq M^y$ and $p(x)$ be a φ-type over B ~~with $\mathrm{vc}^*(p(x)) < 2$~~. Then at least one holds:

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So the type $p(x)$ is either k-compressible (some $k = k(p)$) or it can be partitioned into finitely many types that are finitely axiomatisable (in particular k-compressible).

Can we say something similar to this for all formulas in NIP theories?

Can we say something similar to this for all formulas in NIP theories?

Yes (thanks to Bays, Kaplan and Simon).

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that any formula $\varphi(x, b)$, $b \in B$, belongs in $p(x)$ if and only if it belongs in more than $n/2$ of the types $q_i(x)$:

$$\begin{aligned} p(x) = &\{\varphi^i(x, b) : b \in B,\ i \in \{0, 1\},\ \exists I \subseteq \{1, \ldots, n\}, \\ &|I| > n/2,\ \varphi^i(x, b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

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The n and k depend only on the dual VC-dimension of φ (the maximum d such that $\pi^*_\varphi(d) = 2^d$).

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Let $\varphi(x,y)$ be an NIP formula in a theory T and $M \models T$. Fix $1/2 \leq \alpha < 1$.

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Theorem B was used to prove existence of uniform honest definitions for NIP formulas.

Two considerations around Theorem B

- Does VC-codensity play a role in refining Theorem B?
- If the type $p(x)$ is definable, can the k-compressible types $q_1(x), \ldots, q_n(x)$ be chosen definable too?

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Proposition C (A.G. '23) - *Uniform Better Lemma A*

Let $\varphi(x, y)$ be a formula with $\mathbf{vc}^*(\varphi) < 2$ in a theory T. Let $M \models T$. Then there exist $k, n < \omega$ such, for every φ-type $p(x)$ over any $B \subseteq M^y$, at least one holds:

(a) $p(x)$ is k-compressible.

(b) There are n many partial φ-types $q_1(x), \ldots q_n(x)$, with $\mathbf{vc}^*(q_i) = 0$ for every $i \leq n$, such that every formula in $p(x)$ is implied by **at least one** of the $q_i(x)$.

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Does Proposition C have a generalization to $\mathrm{vc}^*(\varphi) < n$?

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Corollary D (of Theorem B)

Let M be a distal structure, $\varphi(x, y)$ a formula, and $p(x)$ a φ-type. Then there exists another formula $\psi(x, z)$ and a directed (i.e. 1-compressible) ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

($q(x)$ can be chosen over the same parameters as $p(x)$)

This corollary is a strong "uniform" density result for k-compressible types.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ satisfying $q(x) \vdash p(x)$ be chosen definable?

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A positive answer to our question about Theorem B, would yield a positive answer to Question E.

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Using Proposition C (uniform Better Lemma A) we can show that Question E has a positive answer in distal theories of VC-density one.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Theorem (A.G. '23)

Let M be a distal structure of VC-density one, $\varphi(x, y)$ be a formula, and $p(x)$ a φ-type. If $p(x)$ extends to a **definable** type in $S_x(M)$, then there exists another formula $\psi(x, z)$ and a directed **definable** ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

(dp-minimal version to come)

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But what about our question?

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Itay Kaplan <[redacted]>
Para: Pablo Andujar Guerrero
Mar 07/03/2023 18:57

I talked about it with Martin and Pierre.
It seems that there might be a counterexample but maybe it's not so easy to verify.
The idea is to take a meet-tree with a height function. So let's take M to be (omega^<omega, omega, h) where h goes from w^<w to w and gives the level. The language has the tree order, the omega order, h (and the meet). Now if you take a point s in the tree, you can look at the type saying x>=s and x is incomparable to any other point >=s (the generic type of a "closed ball") and h(x) > M (height = infinity).
Then this type is the average of compressible types by taking 3 different branches > s, say B1, B2, B3, and looking at the types saying x>B_i, h(x)>M, but these branches won't be definable.
In order to verify that this actually works, one would need to understand what are definable types, etc.

Itay

In the 2-sorted structure $(\omega^{<\omega}, \omega; <, <_\omega, h)$, with $h : \omega^{<\omega} \to \omega$ the level function, consider the formulas

$$\varphi_1(x, y_1) := \text{“}(x = y_1) \vee (x < y_1) \vee (x > y_1)\text{” for } y_1 \in \omega^{<\omega},$$
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Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{aligned} &\varphi_1(x, ()), \\ \neg&\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

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Pick 3 disjoint branches B_1, B_2, B_3. For each $i \in \{1, 2, 3\}$ let $q_i(x)$ be the $\{\varphi_1, \varphi_2\}$-type:

$$\begin{aligned} &\varphi_1(x, b_1) \text{ iff } b_1 \in B_i, \\ &b_2 <_\omega h(x) \text{ for every } b_2 \in \omega. \end{aligned}$$

Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{gathered}\varphi_1(x, ()), \\ \neg\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ b_2 <_\omega h(x) \text{ for every } b_2 \in \omega.\end{gathered}$$

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The type $p(x)$ is the rounded average of the types $q_1(x)$, $q_2(x)$, $q_3(x)$. Moreover these are 2-compressible. However they are *probably* not definable.

Email (Bays-Kaplan-Simon)

The $\{\varphi_1, \varphi_2\}$-type $p(x)$ is definable but might not be the rounded average of finitely many k-compressible (for any k) **definable** $\{\varphi_1, \varphi_2\}$-types.

In particular $p(x)$ is the rounded average of the 2-compressible types $q_1(x)$, $q_2(x)$ and $q_3(x)$, but these are probably not definable.

The big picture: what do we know about NIP types?

1. Few types over finite sets.
2. Indiscernible sequences converge to types.
3. Equivalence of invariance, non-dividing and non-forking over models.
4. Existence of uniform honest definitions.
5. Described by **compressible types** (Theorem B).

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Opinion: VC-(co)density is conspicuously absent.

The bigger picture: VC meets NIP.

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statistics

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Erdős question

classification theory

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We see fruitful outcomes (every 20+ years) from the communication between VC machine learning and NIP model theory.