NIP compressible types

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Workshop?

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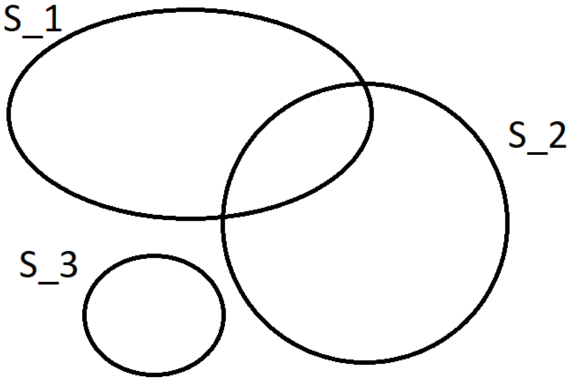

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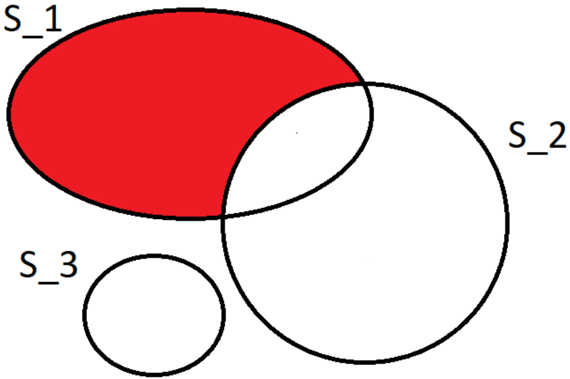

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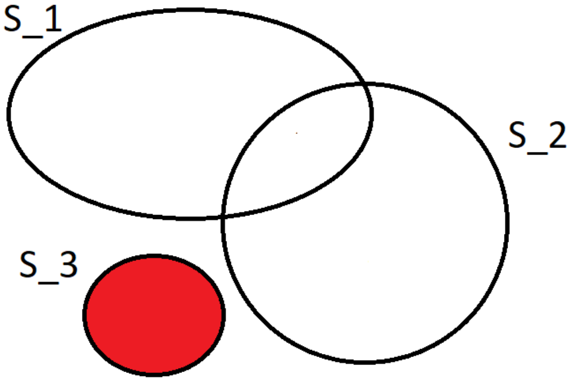

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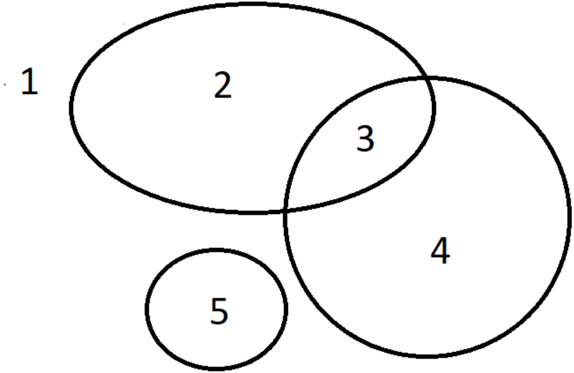

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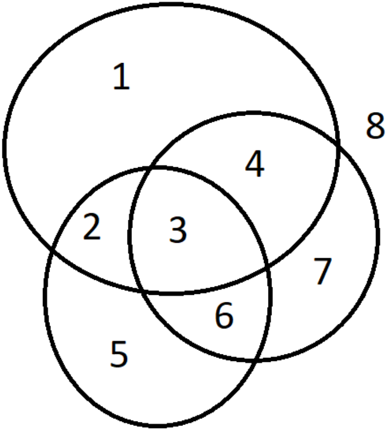

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Definition

Let $\mathcal{S}$ be a family of sets. The *dual shatter function of* $\mathcal{S}$ is the function

$$\begin{aligned} \pi^*_{\mathcal{S}} :& \omega \to \omega \\ & n \mapsto \max\{BA(\mathcal{F}) : \mathcal{F} \subseteq \mathcal{S},\ |\mathcal{F}| = n\} \end{aligned}$$

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E.g. If $\mathcal{S} = \{\textbf{intervals in } \mathbb{R}\}$ then $\pi^*_{\mathcal{S}}(n) = 2n$ and $\mathrm{vc}^*(\mathcal{S}) = 1$.

Definition

A family of sets $\mathcal{S}$ is *(downward) directed* if, for any **finite** subfamily $\mathcal{F} \subseteq \mathcal{S}$, there exists some $S \in \mathcal{S}$ such that $S \subseteq \cap\mathcal{F}$.

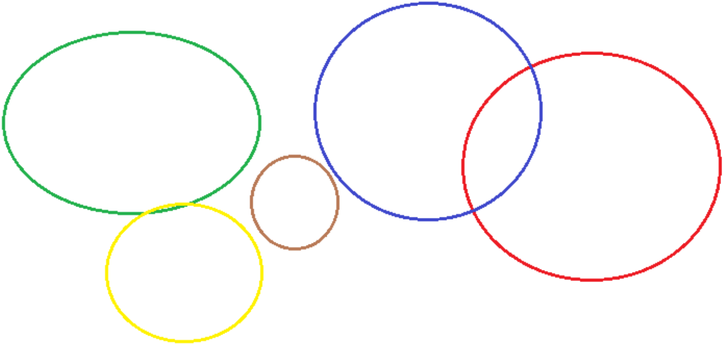
Equivalently, $\mathcal{S}$ is directed if, for any $S_1, S_2 \in \mathcal{S}$ there exists $S \in \mathcal{S}$ with $S \subseteq S_1 \cap S_2$.

E.g.

- In a poset $(P, \leq)$ the sets $\{a \in P : a \leq b\}$ for $b \in P$.
- The neighborhoods of a given point in a topological spaces.
- Any filter basis.

Definition

A family of sets $\mathcal{S}$ has a finite transversal $T = \{a_1, \ldots, a_n\}$ if every $S \in \mathcal{S}$ satisfies that $S \cap T \neq \emptyset$.



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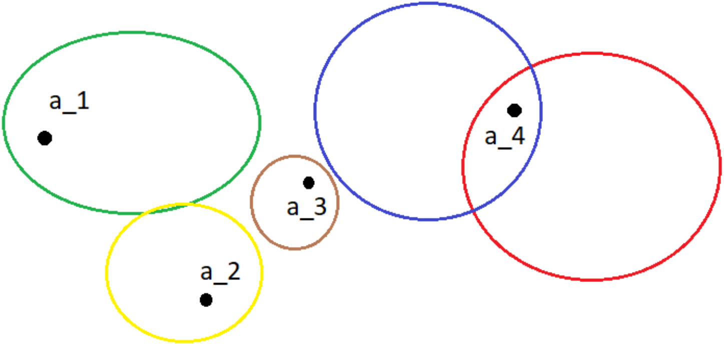

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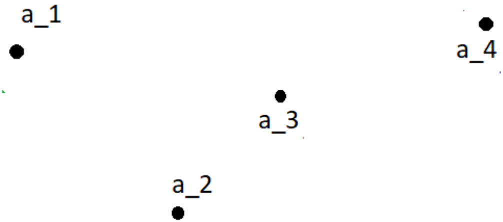

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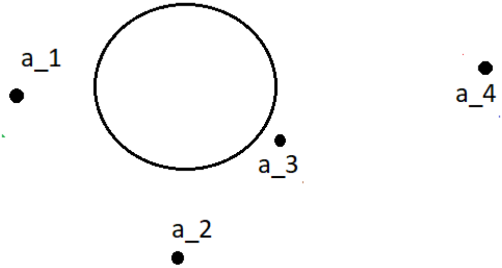

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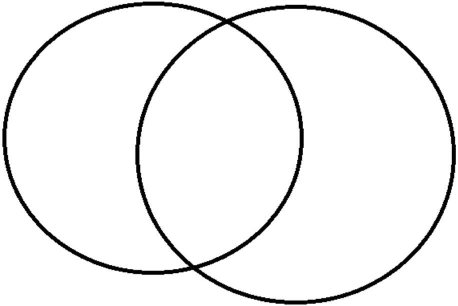

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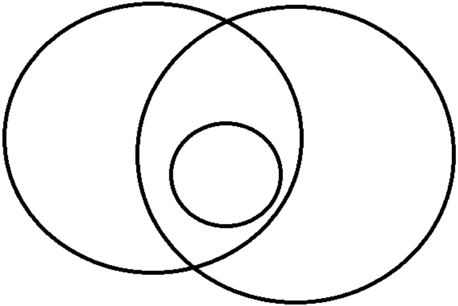

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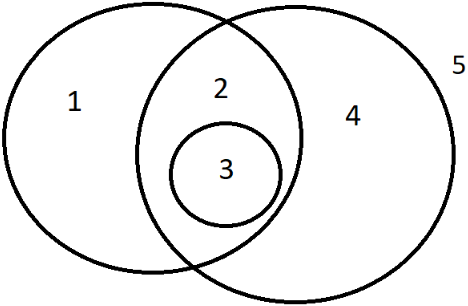

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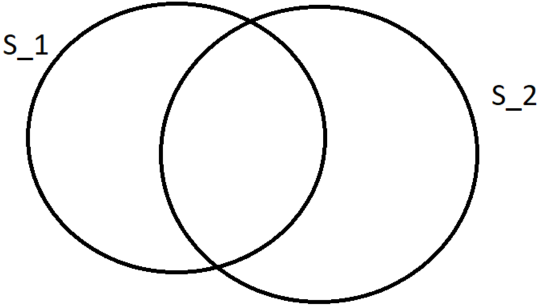

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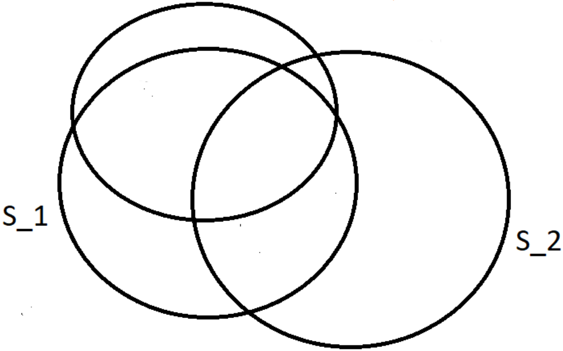

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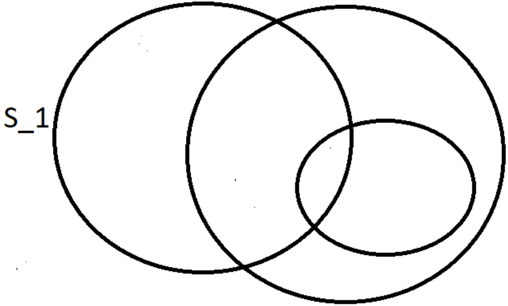

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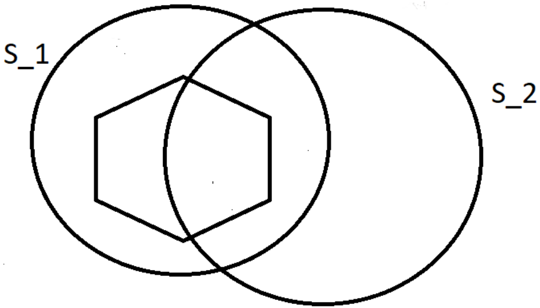

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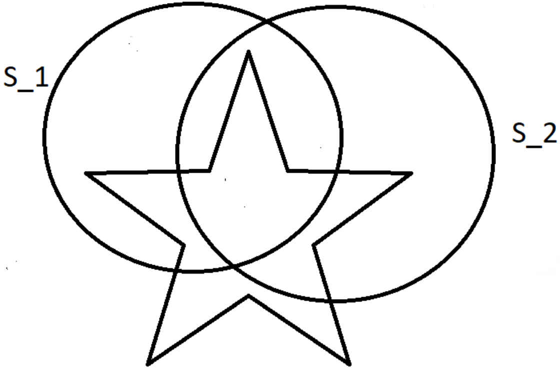

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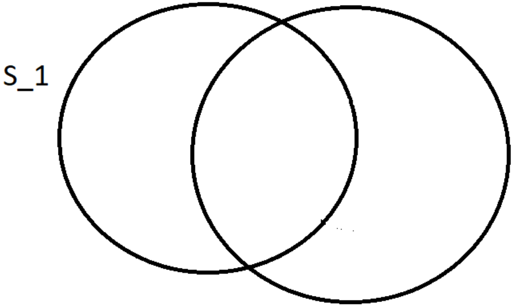

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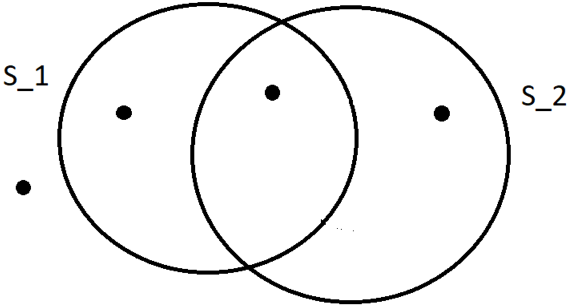

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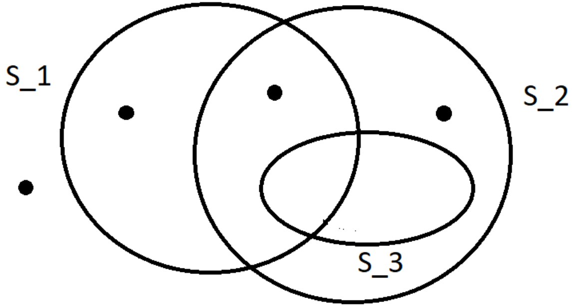

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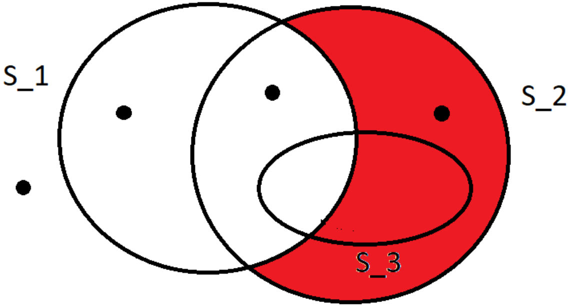

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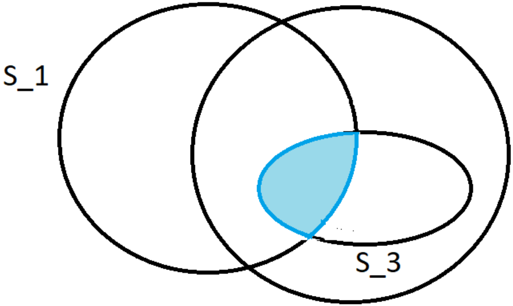

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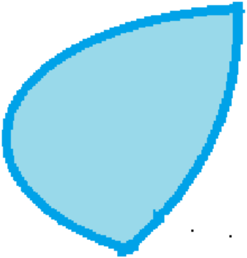

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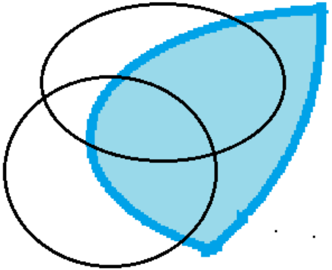

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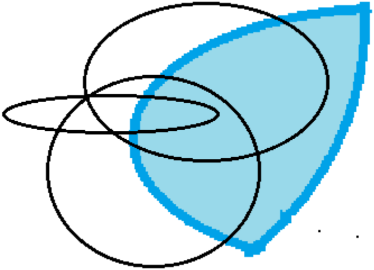

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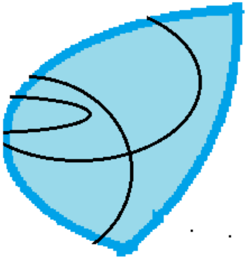

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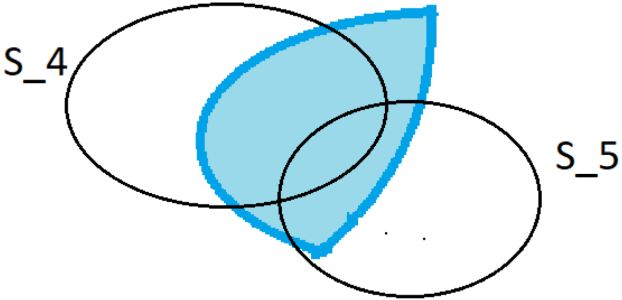

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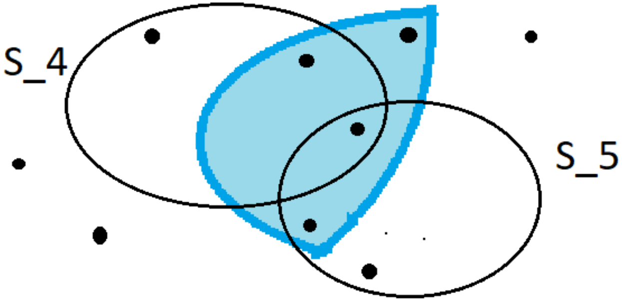

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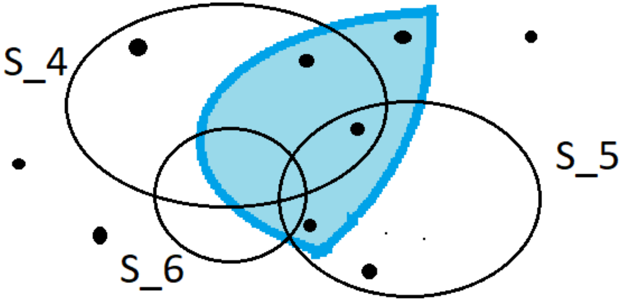

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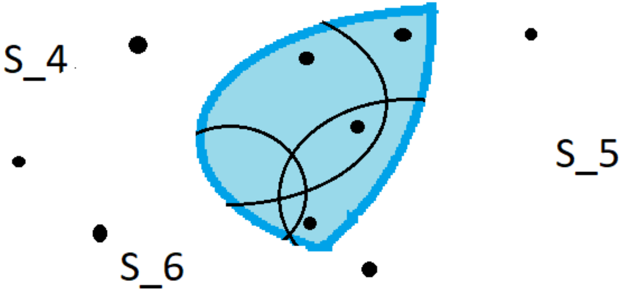

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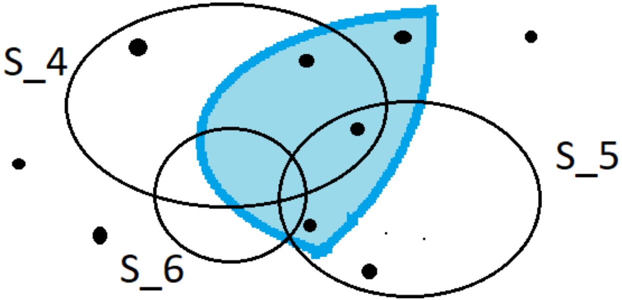

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We continue this process, and one of two things can happen:

- We find a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S}$ is directed.
- There is a sequence $(S_i)_i$ of sets in $\mathcal{S}$ such that

$$BA(B_i : i \leq 3n) \geq \sum_{j=1}^{n} j.$$

  Meaning that

$$\pi^*_{\mathcal{S}}(3n) \geq \frac{n^2+n}{2}.$$

Lemma A (A.G. '23)

Let $\mathcal{S}$ be a family of sets with $\mathrm{vc}^*(\mathcal{S}) < 2$. Then at least one holds:

(a) There exists a finite intersection F of sets in $\mathcal{S}$ such that $F \cap \mathcal{S} = \{F \cap S : S \in \mathcal{S}\}$ is directed.

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(b) There exists $F_1, \ldots, F_m$, finite non-empty Boolean combinations of sets in $\mathcal{S}$ such that, for every $S \in \mathcal{S}$ there is some $i \leq m$ such that $F_i \subseteq S$.

Now let us talk compressibility.
Our two main references are:

- Martin Bays, Itay Kaplan and Pierre Simon. *Density of compressible types and some consequences.* ArXiv: 2107.05197, 2022.
- Xi Chen, Yu Cheng, and Bo Tang. *On the recursive teaching dimension of VC classes.* Advances in Neural Information Processing Systems 29: Annual Conference on Neural Information Processing Systems, 2016, Barcelona, Spain, p 2164–2171.

Model-theoretic framework

Let T be a theory and and $\varphi(x, y)$ be a (partitioned) formula. For any $M \models T$ and $b \in M^y$ let $\varphi(M, b) = \{a \in M^x : M \models \varphi(a, b)\}$. We identify $\varphi(x, y)$ with the family of sets $\{\varphi(M, b) : b \in M^y\}$:

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- It has dual shatter function π^*_φ.
- It has VC-codensity ...

(All these are independent of M.)

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A theory T is *NIP* if every formula φ is NIP.

Conjecture: in any dp-minimal theory (a subclass of NIP theories) any formula $\varphi(x, y)$ with $|x| = 1$ has VC-codensity at most one.

Theorem (Baldwin-Saxl '76)

Let G be a group definable in an NIP structure. Let $\mathcal{H}$ be a (uniformly) definable family of subgroups of G. Then there is $k < \omega$ such that, for any finite family $\mathcal{F} \subseteq \mathcal{H}$, there is a subfamily $\mathcal{C} \subseteq \mathcal{F}$ with $|\mathcal{C}| \leq k$ such that $\cap\mathcal{C} = \cap\mathcal{F}$.

Definition

Let $k < \omega$. A family of sets $\mathcal{S}$ is *k-compressible* if, for any **finite** family $\mathcal{F} \subseteq \mathcal{S}$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

Equivalently, $\mathcal{S}$ is k-compressible if, for any family $\mathcal{F} \subseteq \mathcal{S}$ of size $k+1$, there exists a subfamily $\mathcal{G} \subseteq \mathcal{S}$ of size **at most** k such that $\cap\mathcal{G} \subseteq \cap\mathcal{F}$.

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Note: Directed = 1-compressible.

"This terminology is inspired by the terminology around compression schemes in the statistical learning literature."

- The set of real intervals in 0^+ is 1-compressible.
- The set of unbounded real intervals in 0^+ is 2-compressible.
- In $\mathbb{R}^k$, the set of "stripes" of the form $\{(x_1, \ldots, x_k) : a < x_i < b\}$ for $a \leq 0 < b$ and $i \leq n$, is k-compressible.

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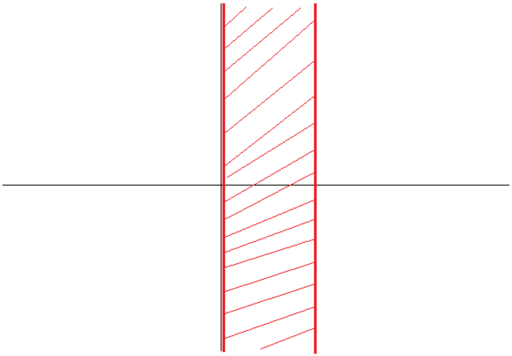

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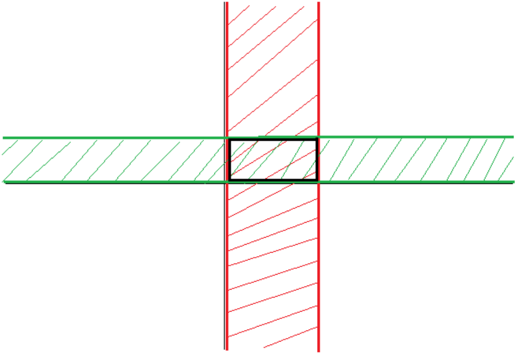

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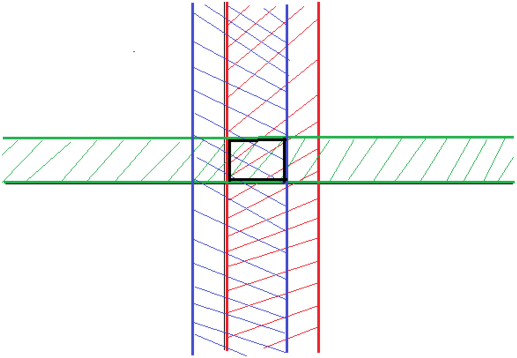

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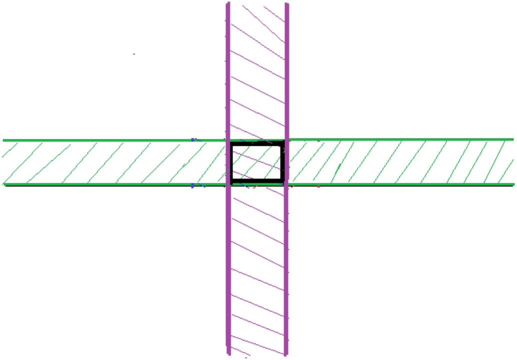

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A k-compressible family of sets is consistent iff it is k-consistent (every k sets have non-empty intersection).

So for k-compressible **definable** families of sets consistency is expressible with a single first order sentence.

Let T be a theory and $M \models T$.

For a formula $\varphi(x, y)$ and $i \in \{0, 1\}$ let $\varphi^1(x, y) = \varphi(x, y)$ and $\varphi^0(x, y) = \neg\varphi(x, y)$.

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Definition

A φ-type $p(x)$ is k-compressible if for any **finite** partial type $q(x) \subseteq p(x)$ there exists $r(x) \subseteq p(x)$ with $|r(x)| \leq k$ such that

$$r(x) \vdash q(x) \quad \text{(i.e. } r(M) \subseteq q(M)\text{)}.$$

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x, y)$ be an NIP formula in a theory T and $M \models T$.

There exists $k, n < \omega$ such that, for every $B \subseteq M^y$ and φ-type $p(x)$ over B, there exist n many k-compressible φ-types over B,

$$q_1(x), \ldots, q_n(x),$$

such that any formula $\varphi(x, b)$, $b \in B$, belongs in $p(x)$ if and only if it belongs in more than $n/2$ of the types $q_i(x)$:

$$\begin{aligned} p(x) = &\{\varphi^i(x, b) : b \in B,\ i \in \{0,1\},\ \exists I \subseteq \{1, \ldots, n\}, \\ &|I| > n/2,\ \varphi^i(x, b) \in q_i(x)\, \forall i \in I\}. \end{aligned}$$

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The n and k depend only on the dual VC-dimension of φ (the maximum d such that $\pi^*_\varphi(d) = 2^d$).

Theorem B (Bays-Kaplan-Simon '22)

Let $\varphi(x,y)$ be an NIP formula in a theory T and $M \models T$. Fix $1/2 \le \alpha < 1$.

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Theorem B was used to prove existence of uniform honest definitions for NIP formulas.

Two considerations around Theorem B

- Does VC-codensity play a role in refining Theorem B?
- If the type $p(x)$ is definable, can the k-compressible types $q_1(x), \ldots, q_n(x)$ be chosen definable too?

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Proposition C (A.G. '23) - *Uniform Better Lemma A*

Let $\varphi(x,y)$ be a formula in a theory T with $\mathbf{vc}^*(\varphi) < 2$. Let $M \models T$. Then there exist $k, n < \omega$ such, for every φ-type $p(x)$ over any $B \subseteq M^y$, at least one holds:

(a) $p(x)$ is k-compressible.

(b) There are n many partial φ-types $q_1(x), \ldots q_n(x)$, with $\mathbf{vc}^*(q_i) < 1$ for every $i \leq n$, such that every formula in $p(x)$ belongs in at least one of the $q_i(x)$.

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Does Proposition C have a generalization to $\mathrm{vc}^*(\varphi) < n$?

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Corollary D (of Theorem B)

Let M be a distal structure, $\varphi(x, y)$ a formula, and $p(x)$ a φ-type. Then there exists another formula $\psi(x, z)$ and a directed (i.e. 1-compressible) ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

($q(x)$ can be chosen over the same parameters as $p(x)$)

This corollary is a strong "uniform" density result for k-compressible types.

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Question E

In Corollary D, if the φ-type $p(x)$ extends to a definable type over M (in the full language), can the directed ψ-type $q(x)$ be chosen definable?

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A positive answer to our question about Theorem B, would yield a positive answer to Question E.

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$\uparrow$
This one.

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If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

A theory T is *of VC-density one* if every formula $\varphi(x, y)$ with $|x| = 1$ has VC-codensity less than 2.

Using Proposition C we can show that Question E has a positive answer in distal theories of VC-density one.

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Using Proposition C we can show that Question E has a positive answer in distal theories of VC-density one.

Theorem (A.G. '23)

Let M be a distal structure of VC-density one, $\varphi(x, y)$ be a formula, and $p(x)$ a φ-type. If $p(x)$ extends to a **definable** type in $S_x(M)$, then there exists another formula $\psi(x, z)$ and a directed **definable** ψ-type $q(x)$ such that

$$q(x) \vdash p(x).$$

If the type $p(x)$ is definable, can the k-compressible types $q_i(x)$ be chosen definable too?

Itay Kaplan <[redacted]>

Para: Pablo Andujar Guerrero

Mar 07/03/2023 18:57

I talked about it with Martin and Pierre.
It seems that there might be a counterexample but maybe it's not so easy to verify.
The idea is to take a meet-tree with a height function. So let's take M to be (omega^<omega, omega, h) where h goes from w^<w to w and gives the level. The language has the tree order, the omega order, h (and the meet). Now if you take a point s in the tree, you can look at the type saying x>=s and x is incomparable to any other point >=s (the generic type of a "closed ball") and h(x) > M (height = infinity).
Then this type is the average of compressible types by taking 3 different branches > s, say B1, B2, B3, and looking at the types saying x>B_i, h(x)>M, but these branches won't be definable.
In order to verify that this actually works, one would need to understand what are definable types, etc.

Itay

In the 2-sorted structure $(\omega^{<\omega} \cup \omega, <, <_\omega, h)$, with $h : \omega^{<\omega} \to \omega$ the height function, consider the formulas

$$\varphi_1(x, y_1) := \text{“}(x = y_1) \vee (x < y_1) \vee (x > y_1)\text{” for } y_1 \in \omega^{<\omega},$$
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Consider the $\{\varphi_1, \varphi_2\}$-type $p(x)$ given by

$$\begin{aligned} &\varphi_1(x, ()), \\ \neg&\varphi_1(x, b_1) \text{ for every } b_1 \in \omega^{<\omega} \setminus \{()\}, \\ &b_2 <_\omega h(x) \text{ for every } b_2 < \omega. \end{aligned}$$

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Pick 3 disjoint branches B_1, B_2, B_3. For each $i \in \{1, 2, 3\}$ let $q_i(x)$ be the $\{\varphi_1, \varphi_2\}$-type:

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The type $p(x)$ is the rounded average of the types $q_1(x)$, $q_2(x)$, $q_3(x)$. Moreover these are 2-compressible. However they are probably not definable.

Email (Bays-Kaplan-Simon)

The $\{\varphi_1, \varphi_2\}$-type $p(x)$ is definable but might not be the rounded average of finitely many k-compressible (for any k) **definable** $\{\varphi_1, \varphi_2\}$-types.

In particular $p(x)$ is the rounded average of the 2-compressible types $q_1(x)$, $q_2(x)$ and $q_3(x)$, but these are probably not definable.

The big picture: what do we know about NIP types?

1. Few types over finite sets.
2. Indiscernible sequences converge to types.
3. Equivalence of invariance, non-dividing and non-forking over models.
4. Existence of honest definitions.
5. Described by **compressible types** (Theorem B).

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Opinion: VC-(co)density is conspicuously absent.