Exercice 5.3 *On souhaite déterminer les expressions de u_p, v_p, et w_p définies par le système*

$$\left\{\begin{array}{lcl} u_{p+1} & = & \underline{3u_p - v_p + w_p} \\ v_{p+1} & = & \underline{2v_p} \\ w_{p+1} & = & \underline{u_p - v_p + 3w_p} \end{array}\right., \qquad \left\{\begin{array}{lcl} u_0 & = & 1 \\ v_0 & = & 1 \\ w_0 & = & 2 \end{array}\right. . \qquad U_p = \begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix}$$

(i) Écrire le système sous la forme $U_{p+1} = AU_p$.

$$\begin{pmatrix} u_{p+1} \\ v_{p+1} \\ w_{p+1} \end{pmatrix} = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}\begin{pmatrix} u_p \\ v_p \\ w_p \end{pmatrix} \qquad A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

(ii) Diagonalisons A :

* Poly. caractéristiq. : $P_A(X) = \det(XI - A) = \begin{vmatrix} X-3 & 1 & -1 \\ 0 & X-2 & 0 \\ -1 & 1 & X-3 \end{vmatrix}$

on développe sur la 2e ligne

$$= +(X-2)\begin{vmatrix} X-3 & -1 \\ -1 & X-3 \end{vmatrix}$$

$$\underset{C_1+C_2}{=} (X-2)\begin{vmatrix} X-4 & -1 \\ X-4 & X-3 \end{vmatrix}$$

$$\underset{L_2-L_1}{=} (X-2)\begin{vmatrix} X-4 & -1 \\ 0 & X-2 \end{vmatrix} \quad \text{triangulaire}$$

$$= (X-2)(X-4)(X-2) = (X-4)(X-2)^2$$

Valeurs propres : 2 et 4.

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

• Vecteur propre $v_4 \in \operatorname{Ker}(A - 4I)$: $\quad v_4 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$(A-4I)v_4 = 0 \iff \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff \left\{\begin{array}{l} -x - y + z = 0 \\ -2y = 0 \\ x - y - z = 0 \end{array}\right. \iff \left\{\begin{array}{l} -x + z = 0 \\ y = 0 \end{array}\right.$$

$$v_4 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Vecteurs propres $u_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \operatorname{Ker}(A - 2I)$ $\quad A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$

$$(A - 2I) u_2 = 0 \iff \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff x - y + z = 0$$

On a $\dim \operatorname{Ker}(A - 2I) = 2$
car 1 éq. cartésienne.

$$u_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ et } u_3 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

On a alors

$$A = P D P^{-1} \qquad \text{avec} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\underbrace{\text{colonnes } 1, 2}_{\operatorname{Ker}(A-2I)} \qquad \underbrace{\text{colonne } 3}_{\operatorname{Ker}(A-4I)}$$

Calcul de P^{-1} :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \underset{L_2 - L_1}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\underset{L_3 - L_2}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & 1 \end{array}\right) \underset{\substack{L_1 - 0{,}5L_3 \\ L_2 + 0{,}5L_3 \\ 0{,}5L_3}}{\sim} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 1/2 \\ 0 & 0 & 1 & 1/2 & -1/2 & 1/2 \end{array}\right)$$

$$P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

$A^p = PD\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}}\,\cancel{P}D\cancel{P^{-1}} \cdots \cancel{P}DP^{-1}$

(iv) Calculer la matrice A^p. $A^p = PD^pP^{-1}$

(v) En déduire la forme explicite de la suite U_p. $U_{p+1} = AU_p$

donc $U_p = A^p U_0$ or $\begin{cases} u_0 = 1 \\ v_0 = 1 \\ w_0 = 2 \end{cases}$.

$$U_p = PD^pP^{-1}U_0 = P\begin{pmatrix} 2^p & 0 & 0 \\ 0 & 2^p & 0 \\ 0 & 0 & 4^p \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2}P\begin{pmatrix} 2^p & 0 & 0 \\ 0 & 2^p & 0 \\ 0 & 0 & 4^p \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}$$

$$= \frac{1}{2}P\begin{pmatrix} 0 \\ 2^{p+1} \\ 2\times 4^p \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2^{p+1} \\ 2\times 4^p \end{pmatrix}$$

$$U_p = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2^p \\ 4^p \end{pmatrix} = \begin{pmatrix} 4^p \\ 2^p \\ 2^p + 4^p \end{pmatrix}$$