My beautiful dissertation

My Name

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Acknowledgements

Dedication

Abstract

Chapter 1 introduces

Chapter 1

Introduction

A fluid queue is a two-dimensional stochastic process $\{(X(t), \varphi(t))\}_{t\geq 0}$. The phase process, also known as the driving process, $\{\varphi(t)\}_{t\geq 0}$, is a continuous-time Markov chain (CTMC). The level process, $\{X(t)\}_{t\geq 0}$, is a real-valued, continuous, and piecewise linear.

Stochastic fluid queues have found a variety of applications such as telecommunications (see Anick et al. (1982) as a canonical application in this area), power systems (Bean et al. 2010), risk processes (Badescu et al. 2005) and environmental modelling (Wurm 2020). Fluid queues are relatively well studied. Largely, the analysis of fluid queues falls into two categories, matrix-analytic methods (Ahn et al. 20035, Ahn & Ramaswami 2003, 2004, Bean et al. 2005*a*,*b*, 2009*a*,*b*, da Silva Soares 2005, Latouche & Nguyen 2019), and differential equation-based methods (Anick et al. 1982, Karandikar & Kulkarni 1995, Bean et al. 2021).

More recently, Bean & O'Reilly (2014) extended fluid queues to *so-called* stochastic fluid-fluid queues. In a fluid-fluid queue there is a second level process, $\{Y(t)\}_{t\geq 0}$ which is itself driven by a fluid queue, $\{(X(t), \varphi(t))\}_{t\geq 0}$. The analysis, (Bean & O'Reilly 2014), is in principal similar to the matrix-analytic methods of (Bean et al. 2005*b*), and derives results about the second level process $\{Y(t)\}_{t\geq 0}$ in terms of the infinitesimal generator (a differential operator) of the fluid queue, $\{(X(t), \varphi(t))\}_{t\geq 0}$. For practical computation of the results of Bean & O'Reilly (2014), a matrix-discretisation of the infinitesimal generator of the fluid queue can be used. To this end, to date, two possible discretisation have been suggested. Taking a differential equations-based approach, Bean et al. (2021) use the discontinuous Galerkin (DG) method to discretise this operator, while Bean & O'Reilly (2013) take a stochastic modelling and matrix-analytic methods approach to approximate the fluid-queue by a quasi-birth-and-death (QBD) process. Both approaches are insightful and offer different tools and perspectives with which to analyse the resulting approximations. It turns out that the latter approach is a sub-class of the former; the QBD can be viewed as the simplest DG scheme where the operator is projected onto a basis of piecewise constant functions.

In the context of approximating fluid queues, one advantage of the QBD discretisation and, equivalently, DG schemes with constant basis functions, is that they guarantee probabilities computed from the approximation are positive (Koltai 2011, Section 3.3), see also (Hesthaven & Warburton 2007) and references therein. One justification for the positivity preserving property is from the interpretation of the discretisation as a stochastic process thereby ensuring positivity. For higher order DG schemes there is no such interpretation. Moreover, higher-order DG approximation schemes may produce negative, or highly oscillatory solutions, particularly when discontinuities or steep gradients are present. Methods to navigate the problem of negative or highly oscillatory solutions have been developed, such as slope limiters, and filtering (see (Hesthaven & Warburton 2007, Section 6.5) and references therein). Slope limiting effectively alters the discretised operator in regions where oscillations are detected and lowers the order of the approximation in theses regions. Filtering is a post-hoc method which looks to recover an accurate solution, given an oscillatory approximation.

1.1 Fluid queues

A fluid queue is a two-dimensional stochastic process $\{(X(t),\varphi(t))\}_{t\geq 0}$ where $\{\varphi(t)\}_{t\geq 0}$ is known as the phase or driving process, and $\{X(t)\}_{t\geq 0}$ is known as the level process or buffer. The phase process $\{\varphi(t)\}_{t\geq 0}$, is an irreducible continuous-time Markov chain (CTMC) with finite state space, which we we assume to be $\mathcal{S}=\{1,2,\dots,N\}$ without loss of generality, and infinitesimal generator $\boldsymbol{T}=[T_{ij}]_{i,j\in\mathcal{S}}$. We assume that $\boldsymbol{T}$ is *conservative.* Associated with states $i\in\mathcal{S}$ are real-valued *rates* $c_i\in\mathbb{R}$.

Partition the state space $\mathcal{S}$ into $\mathcal{S}_+=\{i\in\mathcal{S}\mid c_i>0\}$, $\mathcal{S}_-=\{\imath\in\mathcal{S}\mid c_i<0\}$ and $\mathcal{S}_0=\{i\in\mathcal{S}\mid c_i=0\}$. We assume, without loss of generality, that the generator $\boldsymbol{T}$ is partitioned into sub-matrices

$$\boldsymbol{T}=\left[\begin{array}{ccc}\boldsymbol{T}_{++} & \boldsymbol{T}_{+-} & \boldsymbol{T}_{+0}\\ \boldsymbol{T}_{-+} & \boldsymbol{T}_{--} & \boldsymbol{T}_{-0}\\ \boldsymbol{T}_{0+} & \boldsymbol{T}_{0-} & \boldsymbol{T}_{00}\end{array}\right],$$

where $\boldsymbol{T}_{mn}=[T_{ij}]_{i\in\mathcal{S}_m,j\in\mathcal{S}_n}$, $m,n\in\{+,-,0\}$. Also define the diagonal matrices

$$\boldsymbol{C}=\left[\begin{array}{ccc}\boldsymbol{C}_+ & & \\ & \boldsymbol{C}_- & \\ & & \boldsymbol{0}\end{array}\right],\qquad \boldsymbol{C}_+=diag(c_i,i\in\mathcal{S}_+),\qquad \boldsymbol{C}_-=diag(|c_i|,i\in\mathcal{S}_-),$$

where $diag(a_i,i\in\mathcal{I})$ denotes a diagonal matrix with entries a_i down the diagonal.

When no boundary conditions are imposed, the level process is given by

$$X(t)=X(0)+\int_{s=0}^{t}c_{\varphi(s)}\,\mathrm{d}s.$$

Sample paths of $\{X(t)\}$ are continuous and piecewise linear, with $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = c_\varphi(t)$, when $X(t)$ is differentiable. Given sample paths of $\{\varphi(t)\}$, then $\{X(t)\}$ is deterministic, and in this sense, $\{\varphi(t)\}$ is the only stochastic element of the fluid queue.

Often, boundary conditions are imposed. Here, we consider a mixture of *regulated* and *reflecting* boundary conditions. Upon hitting a boundary we suppose that, with probability p_{ij}, $i, j \in \mathcal{S}$, the phase process instantaneously transitions from phase i to phase j (note that we might have $i = j$ i.e. no transition). At a lower boundary, if $j \in \mathcal{S}_0 \cup \mathcal{S}_-$, then $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = 0$, and the phase process continues to evolve according to the sub-generator

$$\begin{bmatrix} \boldsymbol{T}_{--} & \boldsymbol{T}_{-0} \\ \boldsymbol{T}_{0-} & \boldsymbol{T}_{00} \end{bmatrix},$$

until such a time that $\varphi(t)$ transitions to a phase $k \in \mathcal{S}_+$, at which time $X(t)$ leaves the boundary. Similarly, at an upper boundary if $j \in \mathcal{S}_0 \cup \mathcal{S}_+$, then $\frac{\mathrm{d}}{\mathrm{d}t}X(t) = 0$ and the phase process continues to evolve according to the sub-generator

$$\begin{bmatrix} \boldsymbol{T}_{++} & \boldsymbol{T}_{+0} \\ \boldsymbol{T}_{0+} & \boldsymbol{T}_{00} \end{bmatrix},$$

until such a time that $\varphi(t)$ transitions to a phase $k \in \mathcal{S}_-$ at which time $X(t)$ leaves the boundary. Without loss of generality, we assume the lower and upper boundaries (when present) are at $x = 0$ and $x = M > 0$, respectively.

In summary, the evolution of the level can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t}X(t) = \begin{cases} c_{\varphi(t)}, & \text{if } X(t) > 0, \\ \max\{0, c_{\varphi(t)}\}, & \text{if } X(t) = 0, \\ \min\{0, c_{\varphi(t)}\}, & \text{if } X(t) = M. \end{cases}$$

Let $\boldsymbol{f}(x,t) = \left[f_i(x,t)\right]_{i\in\mathcal{S}}$ be a row-vector function where $f_i(x,t)$ is the density of $\mathbb{P}(X(t) \leq x, \varphi(t) = i)$, assuming it exists. When a differentiable density exists, the system of partial differential equation which describes the evolution of the densities $\boldsymbol{f}(x,t)$ is

$$\frac{\partial}{\partial t}\boldsymbol{f}(x,t) = \boldsymbol{f}(x,t)\boldsymbol{T} - \frac{\partial}{\partial x}\boldsymbol{f}(x,t)\boldsymbol{C}. \tag{1.1}$$

The initial condition is the initial distribution of the fluid queue, $f_i(x,0)$. Often a differentiable density function does not exist and therefore the partial differential equation (1.1) is not well-defined. For example, for a fluid queue with a regulated boundary, if the initial distribution of the fluid queue is a point mass at any point $x_0 \geq 0$ and in phase $i \in \mathcal{S}_+ \cup \mathcal{S}_0$, then a density function $f_i(x,t)$ will not exist for any finite t. Specifically, a point mass will persist along the ray $x_0 + c_i t$, $t \geq 0$. In such situations, it is the *weak solution* to (1.1) that we seek.

Boundary conditions may also be imposed on (1.1).

Discretisation methods approximate the operator on the right-hand side of (1.1) by a matrix, in our case, by the generator of a QBD-RAP.

1.2 Matrix-exponential distributions

Here we recount some facts about matrix exponential distributions. See (Bladt & Nielsen 2017) for a more detailed exposition. A random variable, Z, is said to have a matrix-exponential distribution if it has a distribution function of the form $1 - \boldsymbol{\alpha} e^{\boldsymbol{S}x}(-\boldsymbol{S})^{-1}\boldsymbol{s}$, where $\boldsymbol{\alpha}$ is a $1 \times p$ *initial vector*, $\boldsymbol{S}$ a $p \times p$ matrix, and $\boldsymbol{s}$ a $p \times 1$ *closing vector*, and $e^{\boldsymbol{S}x} := \sum_{n=0}^{\infty} \dfrac{(\boldsymbol{S}x)^n}{n!}$ is the matrix exponential. The density function of Z is given by $f_Z(x) = \boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s}$. The only restrictions on the parameters $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ are that $\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s}$ be a valid density function, i.e. $\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s} \geq 0$, for all $x \geq 0$ and $\lim_{x\to\infty} 1 - \boldsymbol{\alpha} e^{\boldsymbol{S}x}(-\boldsymbol{S})^{-1}\boldsymbol{s} = 1$. There is the possibility of an *atom* (a point mass) at 0, but here we do not consider this possibility. These condition that $\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s}$ be a valid density imposes some properties on representations $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$. However, in general there is no way to determine whether, given a triplet $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ whether it is a representation of a matrix-exponential distribution, or not. Nonetheless, some properties of a triplet $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ are known, such as the following, which is used in the characterisation of QBD-RAPs.

Theorem 1.2.1 (Theorem 4.1.3, Bladt & Nielsen (2017)). *The density function of a matrix-exponential distribution with representation $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ can be expressed in terms of real-valued constants as*

$$\begin{aligned}\psi(x) = &\sum_{j=1}^{m_1}\sum_{k=1}^{p_j} c_{jk}\frac{x^{k-1}}{(k-1)!}e^{\mu_j x} + \sum_{j=1}^{m+2}\sum_{k=1}^{q_j} d_{jk}\frac{x^{k-1}}{(k-1)!}e^{\eta_j x}\cos(\sigma_j x) \\ &+ \sum_{j=1}^{m_2}\sum_{k=1}^{q_j} e_{jk}\frac{x^{k-1}}{(k-1)!}e^{\eta_j x}\sin(\sigma_j x), \end{aligned} \tag{1.2}$$

for integers m_1, m_2, p_j, and q_j and some real constants c_{jk}, d_{jk}, e_{jk}, μ_j, η_j, and σ_j. Here μ_j, $j = 1, \dots, m_1$ are the real eigenvalues of $\boldsymbol{S}$, while $\eta_j + i\sigma_j$, $\eta_j - i\sigma_j$, $j = 1, \dots, m_2$ denote its complex eigenvalues, which come in conjugate pairs. Thus $m_1 + 2m_2$ is the total number of eigenvalues, while the dimension of the representation is given by $p = \sum_{j=1}^{m+1} p_j + 2\sum_{j=1}^{m_2} q_j$.

Theorem 1.2.2 (Theorem 4.1.4, Bladt & Nielsen (2017)). *Consider the nonvanishing terms of the matrix exponential density (1.2),* i.e., *the terms for which $c_{jk} = 0$, $d_{jk} = 0$,* *or* $e_{jk} = 0$. *Among the corresponding eigenvalues* λ_j*, there is a real dominating eigenvalue* κ*, say. That is,* κ *is real,* $\kappa \geq Re(\lambda_j)$ *for all* j*, and the multiplicity of* κ *is at least the multiplicity of every other eigenvalue with real part* κ*.*

Corollary 1.2.3 (Corollary 4.1.5, Bladt & Nielsen (2017))**.** *If* $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ *is a representation for a matrix-exponential distribution, then* $\boldsymbol{S}$ *has a real dominating eigenvalues.*

Theorem 1.2.4 (Theorem 4.1.6, Bladt & Nielsen (2017))**.** *Let* Z *be a matrix-exponentially distributed random variable with density (1.2). Then the dominant real eigenvalue* κ *of Theorem 1.2.2 is strictly negative.*

We define $dev(\boldsymbol{S})$ to be the real dominating eigenvalue of $\boldsymbol{S}$, that is $dev(\boldsymbol{S}) = \kappa$ in Theorem 1.2.2.

The class of matrix-exponential distribution is characterised as the class of probability distributions which have a rational Laplace transform. That is, $\displaystyle\int_{x=0}^{\infty} e^{-\lambda x}\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x$ is a ratio of two polynomial functions in λ. Matrix exponential distributions are an extension of Phase-type distributions, where for the latter, $\boldsymbol{S}$ must be a sub-generator matrix of a CTMC, $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e}$ where $\boldsymbol{e}$ is a $1 \times p$ vector of ones, and $\boldsymbol{\alpha}$ is a discrete probability distribution.

A *representation* of a matrix exponential distribution is a triplet $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$, and we write $Z \sim ME(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ to denote that Z has a matrix-exponential distribution with this representation. The order of the representation is the dimension of the square matrix $\boldsymbol{S}$, i.e. if $\boldsymbol{S}$ is $p \times p$, then the matrix exponential distribution is said to be of order p. Representations of matrix-exponential distributions are not unique (Bladt & Nielsen 2017). A representation is called *minimal* when $\boldsymbol{S}$ has the smallest possible dimension. Throughout this work, we assume that the representation of any matrix exponential distribution is minimal. Let $\boldsymbol{e}_i$ be a vector with a 1 in the ith position and zeros elsewhere. We assume that $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e}$, and that $(\boldsymbol{e}_i, \boldsymbol{S}, \boldsymbol{s})$ for $i = 1, \ldots, p$ are representations of matrix exponential distributions. It is always possible to find such a representation (Bladt & Nielsen 2017, Theorem 4.5.17, Corollary 4.5.18). As such, we abbreviate our notation $Z \sim ME(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ to $Z \sim ME(\boldsymbol{\alpha}, \boldsymbol{S})$. Further, given $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e}$ then observe that $\displaystyle\int_{x=0}^{\infty} e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x = (-\boldsymbol{S})^{-1}\boldsymbol{s} = \boldsymbol{e}$.

For a given $p \times p$ matrix $\boldsymbol{S}$, denote by $\mathcal{A} \subset \mathbb{R}^p$ the space of all possible vectors $\boldsymbol{a}$ such that $(\boldsymbol{a}, \boldsymbol{S})$ is a valid representation of a matrix exponential distribution.

1.3 QBD-RAPs

As RAPs are an extension of Markovian arrival processes to include matrix-exponential inter-arrival times, QBD-RAPs are extensions of QBDs to include matrix-exponential times between level changes. To define a QBD-RAP we first need a Batch (Marked) RAP.

Let $\mathscr{K} \subset \mathbb{Z}$ be a set of *marks*. Let N be a point process, and $Y_0 = 0 < Y_1 < Y_2 \cdots$ be event times of N. Let $\{N(t)\}$ be the counting process associated with N such that $N(t)$ returns the number of events by time t. Associated with the nth event is a mark M_n. For $i \in \mathscr{K}$, let N_i be simple point processes associated with events with marks of type i only, and let $\{N_i(t)\}_{t\geq 0}$ be the associated counting processes of events of mark i. For a matrix $\boldsymbol{B}$ let $dev(\boldsymbol{B})$ denote the real part of the dominant eigenvalue of $\boldsymbol{B}$ (the one with maximal real part). Denote by $f_{N,n}(y_1, m_1, y_2, m_2, \dots, y_n, m_n)$ the joint density, probability mass function of the first n inter-arrival times, $Y_1, Y_2 - Y_1, \dots, Y_n - Y_{n-1}$, and the associated marks M_n. From Bean and Nielsen (Bean & Nielsen 2010, Theorem 1) we have the following.

Theorem 1.3.1. *A process N is a Marked RAP if there exist matrices $\boldsymbol{S}$, $\boldsymbol{D}_i$, $i \in \mathscr{K}$, and a row vector $\boldsymbol{\alpha}$ such that $dev(\boldsymbol{S}) < 0$, $dev(\boldsymbol{S}+\boldsymbol{D}) = 0$, $(\boldsymbol{S}+\boldsymbol{D})\boldsymbol{e} = 0$, $\boldsymbol{D} = \sum_{i\in\mathscr{K}} \boldsymbol{D}_i$, and*

$$f_{N,n}(y_1, m_1, y_2, m_2, \dots, y_n, m_n) = \boldsymbol{\alpha} e^{\boldsymbol{S}y_1} \boldsymbol{D}_{m_1} e^{\boldsymbol{S}y_2} \boldsymbol{D}_{m_2} \dots e^{\boldsymbol{S}y_n} \boldsymbol{D}_{m_n} \boldsymbol{e}. \tag{1.3}$$

Conversely, if a point process has the property (1.3) then it is a Marked RAP.

Denote such a process $N \sim BRAP(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{D}_i,\, i \in \mathscr{K})$.

Also from Bean & Nielsen (2010), associated with a Marked RAP is a row-vector-valued *orbit* process, $\{\boldsymbol{A}(t)\}_{t\geq 0}$,

$$\boldsymbol{A}(t) = \frac{\boldsymbol{\alpha}\left(\prod_{i=1}^{N(t)} e^{\boldsymbol{S}(Y_i - Y_{i-1})} \boldsymbol{D}_{M_i}\right) e^{\boldsymbol{S}(t - Y_{N(t)})}}{\boldsymbol{\alpha}\left(\prod_{i=1}^{N(t)} e^{\boldsymbol{S}(Y_i - Y_{i-1})} \boldsymbol{D}_{M_i}\right) e^{\boldsymbol{S}(t - Y_{N(t)})} \boldsymbol{e}}.$$

Thus, $\{\boldsymbol{A}(t)\}$ is a piecewise-deterministic Markov process where, in between events $\{\boldsymbol{A}(t)\}$ evolves deterministically according to

$$\boldsymbol{A}(t) = \frac{\boldsymbol{A}(Y_{N(t)}^-) e^{\boldsymbol{S}(t - Y_{N(t)})}}{\boldsymbol{A}(Y_{N(t)}^-) e^{\boldsymbol{S}(t - Y_{N(t)})} \boldsymbol{e}},$$

where $\boldsymbol{A}(Y_{N(t)}^-) = \lim_{u\to 0^+} \boldsymbol{A}(Y_{N(t)} - u)$. The process $\{\boldsymbol{A}(t)\}$ "jumps" at event times of N (the process may not always jump at these times, but typically the dynamics change discontinuously at this point). At time t the intensity with which $\{\boldsymbol{A}(t)\}$ has a jump is $\boldsymbol{A}(t)\boldsymbol{D}\boldsymbol{e}$. Upon an event the event is associated with mark i with probability $\boldsymbol{A}(t)\boldsymbol{D}_i\boldsymbol{e}/\boldsymbol{A}(t)\boldsymbol{D}\boldsymbol{e}$. Upon on an event at time t with mark i, the new position of the orbit is $\boldsymbol{A}(t) = \boldsymbol{A}(t^-)\boldsymbol{D}_i/\boldsymbol{A}(t^-)\boldsymbol{D}_i\boldsymbol{e}$.

Marked RAPs are an extension of Marked Markovian arrival processes to include matrix-exponential inter-arrival times. For Marked MAPs, the vector $\boldsymbol{A}(t)$ is a vector of posterior probabilities of a continuous-time Markov chain.

Intuitively, $\boldsymbol{A}(t)$ encodes all of the information about the event times of the Marked RAP and associated marks up to time t that is needed to determine the future behaviour of the point process. Let $\mathcal{F}_t$ be the σ-algebra generated by $N(u), u \in [0,t]$. Then $N \mid \mathcal{F}_t \equiv N \mid \boldsymbol{A}(t) \sim BRAP(\boldsymbol{A}(t), \boldsymbol{S}, \boldsymbol{D}_i, i \in \mathscr{K})$. In words, the future of the point process after time t given all of the information about the process up to and including time t, is distributed as a Marked RAP with initial vector $\boldsymbol{A}(t)$.

Now consider a Marked RAP, $N \sim BRAP(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{D}_i,\ i \in \{-1, 0, +1\})$. The process $\{(L(t), \boldsymbol{A}(t))\}_{t\geq 0}$ formed by letting $L(t) = N_{+1}(t) - N_{-1}(t)$ is a QBD-RAP.

Chapter 2

A stochastic modelling approach to approximating fluid queues

In this chapter we develop a new discretisation of a fluid queue using a *quasi-birth-and-death process with rational arrival process components* (QBD-RAP). In the following chapter, Chapter 3 we prove the scheme converges weakly. To improve upon the discretisation of Bean & O'Reilly (2013), we argue that the shift from QBD to QBD-RAP is necessary. The approximation can be made arbitrarily accurate. Due to its stochastic interpretation, the QBD-RAP discretisation ensures solutions are positive, even when discontinuities are present. Unlike slope limiters and post-hoc filtering, for the QBD-RAP the positivity preserving nature is a property of the discretised operator.

The structure of this chapter is as follows. Section 2.1 motivates the idea using the QBD approximation of Bean & O'Reilly (2014). Sections 2.2 and 2.3 describe the modelling of certain events of the fluid queue with matrix exponential distributions to ultimately construct the behaviour of the QBD-RAP on the event that the QBD-RAP remains in a given level. Section 2.4 describes the dynamics of the construction and also constructs the level process of the QBD-RAP. Section 2.5 deals with modelling boundary behaviour. Section 2.6 describes how to model initial conditions of the fluid queue. Section 2.7 introduces the concept of a *closing operator* which maps the state space of the QBD-RAP to estimates of the density of the fluid queue.

2.1 Inspiration and motivation

The inspiration for our approach stems from the QBD-based discretisation of Bean & O'Reilly (2013). A QBD can be seen as a two dimensional CTMC, $\{(L(t), \phi(t))\}_{t\geq 0}$, where $\{L(t)\}$ is the discrete level, and $\{\phi(t)\}$ is the phase. Bean & O'Reilly (2013) discretise the state space of the fluid into small intervals of width Δ; denote these by $\mathcal{D}_k = [k\Delta, (k+1)\Delta]$. Their approximation captures the dynamics of the phase process exactly, but discretises the level process of the fluid queue. Their approximation supposes that when $L(t) = k$, $\phi(t) = i$, then $X(t) \approx k\Delta$, $\phi(t) = i$.

When in level k and phase i, the approximation sees events at rate $|T_{ii}| + |c_i|\Delta$. Upon an event, with probability $T_{ij}/(|T_{ii}| + |c_i|\Delta)$ a change of phase from i to j is observed and the approximation remains in level k; with probability $|c_i|\Delta/(|T_{ii}| + |c_i|\Delta)$ a change of level occurs, to $k+1$ if $c_i > 0$ and $k-1$ if $c_i < 0$. The generator of their QBD approximation (for an unbounded fluid queue) is

$$\boldsymbol{B} = \begin{bmatrix} \ddots & \ddots & \ddots & \\ \boldsymbol{B}_{-1}(\Delta) & \boldsymbol{B}_0(\Delta) & \boldsymbol{B}_{+1}(\Delta) & \\ & \boldsymbol{B}_{-1}(\Delta) & \boldsymbol{B}_0(\Delta) & \boldsymbol{B}_{+1}(\Delta) \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{B}_0 = \boldsymbol{T} - \boldsymbol{C}\Delta, \qquad \boldsymbol{B}_{-1} = \begin{bmatrix} \boldsymbol{0} & & \\ & \boldsymbol{C}_-\Delta & \\ & & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B}_{+1} = \begin{bmatrix} \boldsymbol{C}_+\Delta & & \\ & \boldsymbol{0} & \\ & & \boldsymbol{0} \end{bmatrix}. \tag{2.1}$$

Bean and O'Reilly then take $\Delta \to 0$ and show that the approximation converges weakly to the fluid queue. It seems that this is the best we can do if we want to keep the interpretation of the approximation as a QBD. We now elaborate on this point.

For a given Δ, the discretisation of Bean and O'Reilly supposes that, when the phase is i and on the event of no change of phase, the sojourn time in a given level has an exponential distribution with rate $|c_i|\Delta$. However, the corresponding sojourn time of the fluid queue is deterministic: given $X(t) = x \in \mathcal{D}_k$, then $\{X(t)\}$ will leave $\mathcal{D}_k$ in exactly $((k+1)\Delta - x)/|c_i|$ units of time if $c_i > 0$, and $(x - k\Delta)/|c_i|$ units of time if $c_i < 0$, on the event that the phase does not change before this time. We can extend the QBD model of Bean and O'Reilly to model this determinism more accurately by supposing that the sojourn times in each interval have Erlang distributions (rather than exponential). However, we effectively realise the same QBD approximation as the original, but on a finer discretisation. For example, using Erlang distributions of order 2 we would get a generator of the QBD approximation

$$\boldsymbol{B} = \begin{bmatrix} \ddots & \ddots & \ddots & \\ \boldsymbol{B}_{-1}(\Delta) & \boldsymbol{B}_0(\Delta) & \boldsymbol{B}_{+1}(\Delta) & \\ & \boldsymbol{B}_{-1}(\Delta) & \boldsymbol{B}_0(\Delta) & \boldsymbol{B}_{+1}(\Delta) \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\boldsymbol{B}_0 = \boldsymbol{T} \otimes I_2 - \begin{bmatrix} \begin{bmatrix} -\Delta/2 & \Delta/2 \\ 0 & -\Delta/2 \end{bmatrix} \otimes \boldsymbol{C}_+ & & \\ & \begin{bmatrix} -\Delta/2 & 0 \\ \Delta/2 & -\Delta/2 \end{bmatrix} \otimes C_- & \\ & & \boldsymbol{0} \end{bmatrix}, \tag{2.2}$$

$$\boldsymbol{B}_{-1} = \left[\begin{array}{ccc} \boldsymbol{0} & & \\ & \boldsymbol{C}_{-} \otimes \left[\begin{array}{cc} 0 & \Delta/2 \\ 0 & 0 \end{array} \right] & \\ & & \boldsymbol{0} \end{array} \right], \quad \boldsymbol{B}_{+1} = \left[\begin{array}{ccc} \boldsymbol{C}_{+} \otimes \left[\begin{array}{cc} 0 & 0 \\ \Delta/2 & 0 \end{array} \right] & & \\ & \boldsymbol{0} & \\ & & \boldsymbol{0} \end{array} \right]. \quad (2.3)$$

But this is the same discretisation we would get if we were to use intervals of width $\Delta/2$ in the QBD approximation of Bean & O'Reilly (2013), with the rows/columns reordered. Indeed, this appears to be the best we can do with a QBD-MAP approximation as the Erlang distribution is known to be the least variable Phase-type distribution for a given order (David & Larry 1987) and therefore gives the closest approximation to the deterministic behaviour we are trying to approximate. This necessitates the extension to QBD-RAPs, whereby we can find more concentrated distributions than any Phase-type, but retain (some of) the stochastic interpretation and matrix-analytic tools.

Here, the construction of the approximating QBD-RAP is developed intuitively, from a stochastic modelling perspective. The key observation is that, given $\{\varphi(t)\}$ is constant, then $\{X(t)\}$ moves deterministically. Upon discretising the state space of the level process into intervals of width Δ, then, given $X(t)$, the distribution of time it takes for $\{X(t)\}$ to leave a given interval on the event that $\{\varphi(t)\}$ remains in the same phase, is deterministic. We model this deterministic behaviour approximately by concentrated matrix exponential distributions (matrix exponential distributions with low-variance). As in Bean & O'Reilly (2013), the discretisation of the state space of $\{X(t)\}$ corresponds to the level process of a QBD-RAP. Unlike Bean & O'Reilly (2013), however, here we do not take $\Delta \to 0$. Rather, we suppose that the variance of the concentrated matrix exponential distributions we use to approximate deterministic events gets small. This necessarily means that the order of the matrix exponential increases.

It turns out that the corresponding orbit process $\{\boldsymbol{A}(t)\}$ can be seen to "track", approximately, how far $X(t)$ is from the left of a given interval when the phase is in $\mathcal{S}_+$, or how far from the right of the interval $X(t)$ is when the phase is in $\mathcal{S}_-$. This leaves us with two problems. 1) on a transition from $\mathcal{S}_+$ to $\mathcal{S}_-$ or $\mathcal{S}_-$ to $\mathcal{S}_+$ how must $\boldsymbol{A}(t)$ jump to retain this information about where $X(t)$ is within a given interval. 2) how to use the orbit position at time t, $\boldsymbol{A}(t)$, to obtain an approximation for the density of $X(t)$. Answers to both problems can be derived from the *residual time* of a matrix exponential distribution. Let us now formalise these concepts some more.

2.2 Time to exit an interval

Consider partition of the state space of the level of a fluid queue, $[0, M]$ into K intervals of width $\Delta = M/(K+1)$, specifically $\mathcal{D}_{k,i} = [k\Delta, (k+1)\Delta)$ if $i \in \mathcal{S}_+$ and $\mathcal{D}_{k,i} = (k\Delta, (k+1)\Delta]$ if $i \in \mathcal{S}_-$, $k = 0, \dots, K+1$. The distinction between $\mathcal{D}_{k,i}$ for $i \in \mathcal{S}_+$ and $\mathcal{S}_-$ is a technical one and will be discussed later. For now, one reason we might want to specify the discretisation in this way, is that it ensures the stochastic process

$$\left\{\sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{S}} k1(X(t)\in\mathcal{D}_{k,i})\right\}_{t\geq 0} \tag{2.4}$$

is cádlág. The expression (2.4) is the discrete process of the fluid-queue which the level process of the QBD-RAP will approximate. Let $y_k = k\Delta$, $k = 0,\dots,K+1$ and $\mathcal{D}_k = [y_k, y_{k+1}]$. We now look to model the behaviour of $\{X(t)\}$ on an interval $\mathcal{D}_k$ over the time $[t, t+u]$ given information about the phase process over this time.

Modelling the residual time to exit on no change of phase We first consider no change of phase in $[t, t+u]$. Suppose that $\varphi(t) = i \in \mathcal{S}_+$ and $X(t) = x \in \mathcal{D}_{k,i}$. Observe that on no change of phase, $\varphi(t+s) = i$, $s \in [0,u]$, and at time $t+s$ the level of the fluid queue is given by $X(t+s) = x + c_i s$, $s \in [0,u]$. Further, for $u > (y_{k+1}-x)/c_i$ the level process $\{X(t)\}$ leaves the interval $\mathcal{D}_{k,i}$ at exactly time $t + (y_{k+1}-x)/c_i$. Similarly, for $i \in \mathcal{S}_-$, $\{X(t)\}$ leaves the interval $\mathcal{D}_{k,i}$ at exactly time $t + (x-y_k)/|c_i|$ on the event that there is no change of phase on $[t, t+u]$ for $u > (x-y_k)/|c_i|$.

To approximate the deterministic behaviour observed above, consider a matrix-exponential distribution $Z \sim ME(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$ with mean Δ and low variance. As we shall see later, the assumption that Z has low variance will allow us to claim that Z can be used to approximate deterministic behaviour. Let $Z_i \sim ME(\boldsymbol{\alpha}, \boldsymbol{S}_i, \boldsymbol{s}_i)$ where $\boldsymbol{S}_i = |c_i|\boldsymbol{S}$, $\boldsymbol{s}_i = -\boldsymbol{S}_i\boldsymbol{e}$, $i \in \mathcal{S}$. The random variable Z_i has mean $\Delta/|c_i|$. Further, let $R_i(u)$ be the residual time, $R_i(u) = Z_i - u \mid \{Z_i - u > 0\}$, $i \in \mathcal{S}$. Suppose first that $X(t) = y_k$ and $\varphi(t) = i \in \mathcal{S}_+$, so that $X(t)$ will leave $\mathcal{D}_{k,i}$ in exactly $\Delta/|c_i|$ units of time on the event that the phase process remains in i for at least this amount of time.

After $u \in [0, \Delta/|c_i|)$ amount of time has elapsed, on the event that $\varphi(t+s) = i$, for all $s \in [0,u)$, then $X(t+u) = y_k + c_i u$. If the phase remains i for a further $\Delta/|c_i| - u$ amount of time, then $\{X(t)\}$ will leave $\mathcal{D}_{k,i}$ at exactly this time. Also at time $t+u$, given $Z_i > u$, the distribution of the residual time, $R_i(u)$, has density

$$f_{R_i(u)}(r) = \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i(u+r)}\boldsymbol{s}_i}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}} = \boldsymbol{A}(t+u)e^{\boldsymbol{S}_i r}\boldsymbol{s}_i.$$

Here, the event $Z_i > u$ approximates the event $X(t) \in \mathcal{D}_{k,i}$ and we want the residual time, $R_i(u)$, to approximate the time until $\{X(t)\}$ leaves $\mathcal{D}_{k,i}$. That is, we want $R_i(u)$ to approximate a deterministic random variable at $\Delta/|c_i| - u$. To this end, we observe that, if the variance of Z_i is sufficiently low, then for $\varepsilon > 0$ and $u < \Delta/|c_i| - \varepsilon$, then

$$\begin{aligned}
&\mathbb{P}(R_i(u) \in (\Delta/|c_i| - u - \varepsilon, \Delta/|c_i| - u + \varepsilon)) \\
&= \mathbb{P}(Z_i \in (\Delta/|c_i| - \varepsilon, \Delta/|c_i| + \varepsilon) \mid Z_i > u) \\
&= \frac{\mathbb{P}(Z_i \in (\Delta/|c_i| - \varepsilon, \Delta/|c_i| + \varepsilon))}{\mathbb{P}(Z_i > u)}
\end{aligned}$$

$$= \frac{\mathbb{P}(Z \in (\Delta - \varepsilon, \Delta + \varepsilon))}{\mathbb{P}(Z_i > u)}$$
$$\geq \frac{1 - \dfrac{\mathrm{Var}(Z)}{\varepsilon^2}}{1}$$

since, by Chebyshev's inequality, $\mathbb{P}(Z_i \in (\Delta/|c_i| - \varepsilon, \Delta/|c_i| + \varepsilon)) \geq 1 - \dfrac{\mathrm{Var}(Z_i)}{\varepsilon^2}$. Choosing $\varepsilon = \mathrm{Var}(X)^{1/3}$, then

$$\mathbb{P}(R_i(u) \in (\Delta/|c_i| - u - \varepsilon, \Delta/|c_i| - u + \varepsilon)) \geq 1 - \mathrm{Var}(Z)^{1/3} \approx 1.$$

That is, when the variance of Z_i is low, the residual time $R_i(u)$ will be concentrated around $\Delta/|c_i| - u$, as required. Figure 2.1 gives an example of a density function for a Z_i with mean $\Delta = 1$ and $c_i = 1$, as well as the density function of $R_i(0.3)$ and $R_i(0.5)$ for comparison. Observe that the density of the residual life $R_i(0.3)$ is concentrated around $\Delta - 0.3 = 0.7$ and, similarly the density of the residual life $R_i(0.6)$ is concentrated around $\Delta - 0.6 = 0.4$.

We can interpret the position of the orbit $\boldsymbol{A}(t+u) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u} \boldsymbol{e}}$ as corresponding to the fact that $X(t+u)$ is $|c_i|u$ units from the left-hand boundary of the interval $\mathcal{D}_{k,i}$. This gives a heuristic argument as to how we can model the sojourn times in a given interval $\mathcal{D}_{k,i}$ on the event that the phase does not change.

We can apply analogous arguments to heuristically develop a model for the sojourn time of the fluid queue in an interval, $\mathcal{D}_{k,i}$, $i \in \mathcal{S}_-$, on the event that the phase does not change, given the fluid is $X(t) = y_{k+1}$. In this case $\boldsymbol{A}(t+u) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u} \boldsymbol{e}}$ is interpreted as corresponding to $X(t)$ being $|c_i|u$ units from the right-hand boundary of the interval $\mathcal{D}_{k,i}$, y_{k+1}.

On a change of phase from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_+$ Consider the residual time $R_j(|c_i|u/|c_j|)$, which is the distribution of time until Z_j occurs, given $Z_j > |c_i|u/|c_j|$. The initial vector associated with the residual time $R_j(|c_i|u/|c_j|)$ is $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|} \boldsymbol{e}}$. This is the initial vector which is obtained if at time t the phase is $\varphi(t) = j$ and the orbit is $\boldsymbol{A}(t) = \alpha$, so that the QBD-RAP has just entered level k, and there is no change of phase by time $|c_i|u/|c_j|$. Above, we argued that the vector $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|} \boldsymbol{e}}$ can be seen to correspond to the event that $X(t+u)$ is $|c_j| \, (|c_i|u/|c_j|) = |c_i|u$ units greater than y_k.

Notice that we can re-write the vector $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|} \boldsymbol{e}}$ as $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u} \boldsymbol{e}}$. This is the orbit position on the event that at time t there is a change of level, so that $\boldsymbol{A}(t) = \boldsymbol{\alpha}$, and the

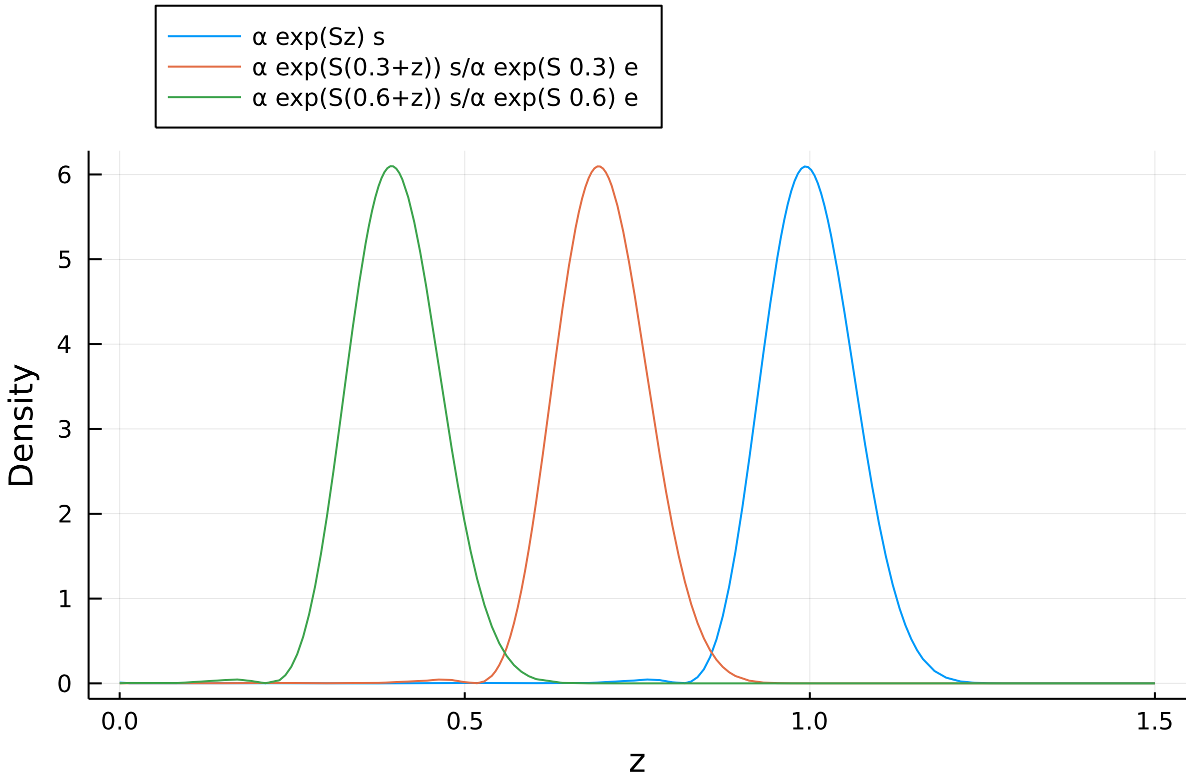

Figure 2.1: The density function for a *concentrated matrix exponential of order 21* from (Horváth et al. 2020) (blue) and corresponding density functions of the residual lives, $R_{0.3}$ (red), $R_{0.5}$ (blue). Observe how the density function of the Z_i (blue) to approximate a point mass at $\Delta = 1$, while the density functions of $R_{0.3}$ (red) and $R_{0.5}$ (blue) approximate point masses at 0.7 and 0.4, respectively.

phase is $\varphi(t) = i$, followed by no change of phase by time $t+u$. We have argued that this is the orbit position which approximates the fact that $\{X(t)\}$ is $|c_i|u$ units greater than y_k.

Therefore, at time $t + u$, if there is a change of phase from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_+$ then the orbit position should not change as in both cases the orbit position $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_j |c_i|u/|c_j|}\boldsymbol{e}}$ corresponds to an approximation to $X(t)$ being $|c_i|u$ units greater than y_k.

Further to this argument, observe that the residual time $R_j(|c_i|u/|c_j|)$ has density

$$f_{R_j(|c_i|u/|c_j|)}(r) = \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}} e^{\boldsymbol{S}_j r}\boldsymbol{s}_j,$$

and

$$\begin{aligned}
&\mathbb{P}(R_j(|c_i|u/|c_j|) \in ((\Delta - |c_i|u)/|c_j| - \varepsilon, (\Delta - |c_i|u)/|c_j| + \varepsilon)) && (2.5)\\
&= \mathbb{P}(Z_j \in (\Delta/|c_j| - \varepsilon, \Delta/|c_j| + \varepsilon) \mid Z_j > |c_i|u/|c_j|) && (2.6)\\
&= \frac{\mathbb{P}(Z_j \in (\Delta/|c_j| - \varepsilon, \Delta/|c_j| + \varepsilon))}{\mathbb{P}(Z_j > |c_i|u/|c_j|)} && (2.7)\\
&\geq 1 - \mathrm{Var}(Z)^{1/3} \approx 1. && (2.8)
\end{aligned}$$

Hence, when the variance of Z is low, the residual time, $R_j(|c_i|u/|c_j|)$, is concentrated around $(\Delta - |c_i|u)/|c_j|$ as required.

Therefore, in our model, upon the event of a change from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_+$ at time $t + u$ the orbit should be $\boldsymbol{A}(t+u) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}}$. That is, the initial condition for the next segment of the orbit process is just the current orbit position and the orbit should start to evolve according to the matrix $\boldsymbol{S}_j$.

Analogous arguments suggest that the same applies for changes of phase from $i \in \mathcal{S}_-$ to $j \in \mathcal{S}_-$.

On a change of phase from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ Now suppose $X(t) = y_k$, $\varphi(t) = i \in \mathcal{S}_+$, and the phase remains in state i until there is a change of phase phase from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ at time $t+u$, $u \in [0, \Delta/|c_i|)$. We have reasoned, intuitively, that, for $i \in \mathcal{S}_+$ and for $x = |c_i|u \in [0, \Delta)$, the orbit position $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S} x}}{\boldsymbol{\alpha} e^{\boldsymbol{S} x}\boldsymbol{e}}$ corresponds to the fluid level position $X(t) = y_k + x$, that is x units greater than y_k. Meanwhile, the residual time $R_i(u)$ *approximates* the fact that $X(t)$ is $\Delta - x$ units to the left of y_{k+1}. We have also reasoned that for $i \in \mathcal{S}_-$ the orbit position $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S} x}}{\boldsymbol{\alpha} e^{\boldsymbol{S} x}\boldsymbol{e}}$ corresponds to the fluid level position $X(t) = y_{k+1} - x$, x units less than y_{k+1}, and the residual time $R_i(u)$ *approximates* the fact that $X(t)$ is $\Delta - x$ units greater than y_k. Notice that in positive phases the orbit position describes the exact position of $X(t)$ from the left of the interval but only approximates the position of $X(t)$ from the right of the interval, and vice-versa for negative phases.

To construct the QBD-RAP we need to find a matrix $\boldsymbol{D}$, say, to take the information about the position of $X(t)$ contained in the orbit when in positive phases, and map it to information about the position of $X(t)$ in negative phases. Then, when there is a change of phase at time $t+u$ from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$, the orbit position will jump from $\boldsymbol{A}(t+u^-)$ to $\boldsymbol{A}(t+u) = \boldsymbol{A}(t+u^-)\boldsymbol{D}$. The matrix $\boldsymbol{D}$ must have the property that, for any $\boldsymbol{a} \in \mathcal{A}$, $(\boldsymbol{aD}, \boldsymbol{S})$ is a representation of a matrix exponential distribution. The matrix $\boldsymbol{D}$ should also have the property that $\boldsymbol{De} = \boldsymbol{e}$ as this will mean that we can model the phase process of the fluid exactly.

To construct such a matrix $\boldsymbol{D}$. If we were given $R_i(u) = r$, then the appropriate orbit position upon a jump from phase $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ would be $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}\boldsymbol{e}}$ as this corresponds to the fluid level position $y_{k+1} - |c_i| r$. However, at time $t+u$, $R_i(u)$ is a random variable about the future of the process and therefore not known, so instead, we take the expected initial vector

$$\mathbb{E}\left[\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i R_i(u)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i R_i(u)}\boldsymbol{e}}\right] = \int_{r=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}} e^{\boldsymbol{S}_i r}\boldsymbol{s}_i \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}\boldsymbol{e}} \mathrm{d}r = \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}} \int_{r=0}^{\infty} e^{\boldsymbol{S}_i r}\boldsymbol{s}_i \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i r}\boldsymbol{e}} \mathrm{d}r.$$

After a change of variables $x = |c_i| r$ and defining $\boldsymbol{D} = \displaystyle\int_{x=0}^{\infty} e^{\boldsymbol{S}x}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}x}}{\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{e}} \mathrm{d}x$, we get

$$\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}} \int_{x=0}^{\infty} e^{\boldsymbol{S}x}|c_i|\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}x}}{\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{e}} \mathrm{d}x/|c_i| = \boldsymbol{A}(t+u)\int_{x=0}^{\infty} e^{\boldsymbol{S}x}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}x}}{\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{e}} \mathrm{d}x = \boldsymbol{A}(t+u)\boldsymbol{D},$$

since at time $t+u$, the orbit position is $\boldsymbol{A}(t+u) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_i u}\boldsymbol{e}}$. This suggests that upon a jump from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$, the orbit position jumps according to the matrix $\boldsymbol{D}$.

Observe that $\boldsymbol{De} = \displaystyle\int_{x=0}^{\infty} e^{\boldsymbol{S}x}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}x}}{\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{e}} \mathrm{d}x\boldsymbol{e} = \int_{x=0}^{\infty} e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x = \boldsymbol{e}$. This is an important observation as it implies that events corresponding to jumps from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ occur with intensity $\boldsymbol{A}(t)T_{ij}\boldsymbol{De} = T_{ij}$ which is constant with respect to orbit position and time and this implies they are exponential. Further, since $\mathcal{A}$ is closed and convex (Bladt & Nielsen 2017), then $(\boldsymbol{aD}, \boldsymbol{S}, \boldsymbol{s})$ is a representation of a matrix exponential distribution for any $\boldsymbol{a} \in \mathcal{A}$.

Note that the matrix $\boldsymbol{D}$ is a modelling choice and other choices are possible. For example, $\boldsymbol{D} = \displaystyle\int_{x=0}^{\Delta-\varepsilon} e^{\boldsymbol{S}x}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}x}}{\boldsymbol{\alpha} e^{\boldsymbol{S}x}\boldsymbol{e}} \mathrm{d}x$. It may also be possible to construct other matrices $\boldsymbol{D}$, perhaps via geometric arguments.

The Phase-type case If $Z \sim ME(\boldsymbol{\alpha}, \boldsymbol{S})$ is chosen to be a Phase-type distribution then there is another possibility for the matrix $\boldsymbol{D}$. A Phase-type distribution is the distribution of time until absorption of a finite state absorbing Markov chain with transient states $\{1, \dots, p\}$. The sub-generator matrix describing the dynamics of the Markov chain on transient states is $\boldsymbol{S}$, and $\boldsymbol{\alpha}$ is an initial probability distribution over the transient states. Let $\{J(t)\}_{t\in(-\infty,\infty)}$ be the Markov chain to which the Phase-type distribution corresponds.

Let $\boldsymbol{\Pi} = diag\,(\boldsymbol{\pi})$, where $\left[\pi_k\right]_{k\in\{1,\dots,p\}} =: \boldsymbol{\pi} = \boldsymbol{\alpha}(-\boldsymbol{S})^{-1}/m$ and $m = \boldsymbol{\alpha}(-\boldsymbol{S})^{-1}\boldsymbol{e}$. There is a time-reverse representation of a Phase-type distribution given by $(\widetilde{\boldsymbol{\alpha}}, \widetilde{\boldsymbol{S}}, \widetilde{\boldsymbol{s}})$, where $\widetilde{\boldsymbol{\alpha}} = m\boldsymbol{s}'\boldsymbol{\Pi}$, $\widetilde{\boldsymbol{S}} = \boldsymbol{\Pi}^{-1}\boldsymbol{S}'\boldsymbol{\Pi}$ and $\widetilde{\boldsymbol{s}} = \boldsymbol{\Pi}^{-1}\boldsymbol{\alpha}'/m$ (Asmussen 2008, Page 91).

Rather than use the residual time $R_i(u)$ to estimate how far $X(t)$ is from the edge of an interval, we can use the information contained in $J(t+u)$. Suppose there is a change of phase of the fluid queue from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ at time $t+u$ at which time $J(t+u) = k$. At a random time before $t+u$ the Phase-type random variable was initialised. Let C_{t+u} be the *age* of the Phase-type distribution at time $t+u$, that is, the time since the Phase-type random variable was initialised. Suppose that $\varphi(v) = i$, $v \in [t+u-C_{t+u}, t+u)$, that is, there has been no change of phase of the fluid queue during the current life of the Phase-type random variable. Given $J(t+u) = k$, we wish to construct an approximation to the distribution of C_{t+u}.

According to Hautphenne et al. (2017), the distribution of C_{t+u} depends on the method of observation. Here, the Phase-type distribution is observed after an exponential time with rate $-T_{ii}$. Proposition 4.1, Hautphenne et al. (2017) states that the cumulative distribution function (CDF) of C_{t+u}, given $J(t+u) = k$ and the process is observed after an exponential time with rate $-T_{ii}$ is

$$\mathbb{P}(C_{t+u} \le a) = 1 - \frac{\boldsymbol{\alpha} e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I})a}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}\boldsymbol{e}_k}{\boldsymbol{\alpha}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}\boldsymbol{e}_k}.$$

Let $\widehat{\boldsymbol{S}}_i(T_{ii}) = diag(\boldsymbol{\nu})^{-1}\boldsymbol{S}_i' diag(\boldsymbol{\nu})$, where $\boldsymbol{\nu} = \boldsymbol{\alpha}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}/(\boldsymbol{\alpha}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}\boldsymbol{e})$. Algebraic manipulations show

$$1 - \frac{\boldsymbol{\alpha} e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I})a}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}\boldsymbol{e}_k}{\boldsymbol{\alpha}(-(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}))^{-1}\boldsymbol{e}_k} = 1 - \boldsymbol{e}_k' e^{(\widehat{\boldsymbol{S}}_i(T_{ii}) + T_{ii}\boldsymbol{I})a}\boldsymbol{e}, \tag{2.9}$$

which is of Phase-type. The conversion between the intensity at which events occurs when the fluid queue is in phase i compared to phase j is $|c_j|/|c_i|$. Therefore, when there is a change of phase from $i \in \mathcal{S}_+$ to $j \in \mathcal{S}_-$ at time $t+u$, if the Phase-type process which approximates the fluid level starts in phase k and evolves according to the generator $|c_j|/|c_i|(\widehat{\boldsymbol{S}}_i(T_{ii}) + T_{ii}\boldsymbol{I})$. The distribution of time until the next event in the QBD-RAP after time $t+u$ is

$$1 - \boldsymbol{e}_k e^{(|c_j|/|c_i|(\widehat{\boldsymbol{S}}_i(T_{ii}) + T_{ii}\boldsymbol{I}) + T_{jj}\boldsymbol{I})a}\boldsymbol{e}.$$

While this does approximate the desired quantity, it is not quite satisfactory for the purpose of our approximation scheme due to dependence on the sample path of $\{\varphi(t)\}$.

Specifically, the evolution of the QBD-RAP from time $t+u$ until the next event depends on the phase immediately before the change of phase at time $t+u$, $\varphi(t+u^-)=i$. This increases the size of the approximating QBD-RAP as we need a separate model for each $\varphi(t+u^-)\in\mathcal{S}_+$. Furthermore, we have not yet considered how to model further changes from $\mathcal{S}_-$ to $\mathcal{S}_+$ or beyond, which further complicates matters.

A solution is to suppose that, rather than observing C_{t+u} at an exponential time with rate $-T_{ii}$, we instead observe $\{J(t+u)\}$ uniformly randomly on the lifetime of length Z_i. In this case, the cumulative distribution function (CDF) of C_{t+u}, given $J(t+u)=k$ is (Hautphenne *et al.* (Hautphenne et al. 2017, Lemma 3.1))

$$\mathbb{P}(C_{t+u}\leq a)=1-\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}_iu}(-\boldsymbol{S}_i)^{-1}\boldsymbol{e}_k}{\boldsymbol{\alpha}(-\boldsymbol{S}_i)^{-1}\boldsymbol{e}_k}=1-\boldsymbol{e}_ke^{\widetilde{\boldsymbol{S}}_iu}\boldsymbol{e}. \tag{2.10}$$

Once again, noting that the conversion between the intensity at which events occurs when the fluid queue is in phase i compared to phase j is $|c_j|/|c_i|$then, upon a change of phase from $i\in\mathcal{S}_+$ to $j\in\mathcal{S}_-$ at time $t+u$, the Phase-type process which approximates the fluid level starts in phase k and evolves according to the reverse-time generator $\widetilde{\boldsymbol{S}}_j$. The distribution of time until the next event in the QBD-RAP after time $t+u$ is

$$1-\boldsymbol{e}_ke^{(\widetilde{\boldsymbol{S}}_j+T_{jj}\boldsymbol{I})a}\boldsymbol{e}.$$

This suggests that, at a jump from $\mathcal{S}_+$ to $\mathcal{S}_-$ or from $\mathcal{S}_-$ to $\mathcal{S}_+$, the state of $\{J(t+u)\}$ does not change, and begins to evolve according to the time-reverse generator at an appropriate speed.

With this construction, and choosing $Z\sim Erlang(p,\Delta/p)$, we recover the discretisation of Bean & O'Reilly (2014) with discretisation parameter Δ/p.

Upon exiting $\mathcal{D}_{k,i}$ Suppose that upon exiting $\mathcal{D}_{k,i}$ at time t the phase is $\varphi(t)=i\in\mathcal{S}_+$. At this time $X(t)=y_{k+1}$ which is the left-hand endpoint of $\mathcal{D}_{k+1,i}$. Hence, we restart the model of the sojourn time with the initial condition $\boldsymbol{A}(t)=\boldsymbol{\alpha}$. Similarly, upon exiting $\mathcal{D}_{k,i}$ at time t in phase $\varphi(t)=i\in\mathcal{S}_-$, then $X(t)=y_k$, which is the right-hand endpoint of $\mathcal{D}_{k,i}$, and so we restart the model of the sojourn time with the initial condition $\boldsymbol{A}(t)=\boldsymbol{\alpha}$.

2.2.1 Computing $\boldsymbol{D}$

In practice we use the class of concentrated matrix exponential distributions (CMEs) found numerically in (Horváth et al. 2020). For this class of CMEs, we take the index p to be the order of the representation. Moreover, we take p to be odd. The justification for considering representations of odd orders only is that the variance of CME representations of orders $2p$ and $2p-1$ are relatively similar and therefore have similar abilities to represent the delta function (Horváth et al. 2020). Hence, if we construct a

QBD-RAP approximation of with a representation of order $2p$ we expect it to perform similarly to an approximation constructed with representations of order $2p-1$. However, the computational cost of the latter is lower, so we opt for the order $2p-1$ representation.

For a given CME with odd order, p, and representation $(\boldsymbol{\alpha}, \boldsymbol{S})$, the matrix $\boldsymbol{S}$ has one real eigenvalue, and $p-1$ complex eigenvalues and all eigenvalues have the same real part.

We numerically evaluate the matrix $\boldsymbol{D}$ where

$$\boldsymbol{D} = \int_{t=0}^{\infty} e^{\boldsymbol{S}t}\boldsymbol{s} \cdot \frac{\boldsymbol{\alpha}e^{\boldsymbol{S}t}}{\boldsymbol{\alpha}e^{\boldsymbol{S}t}\boldsymbol{e}} \,\mathrm{d}t$$

using a trapezoidal rule as follows. For CMEs, the vector function $\boldsymbol{a}(t) := \dfrac{\boldsymbol{\alpha}e^{\boldsymbol{S}t}}{\boldsymbol{\alpha}e^{\boldsymbol{S}t}\boldsymbol{e}}$ is periodic with period $\rho = 2\pi/\omega$ where $\omega = min_i(|\Im(\lambda_i)|)$, λ_i are the eigenvalues of $\boldsymbol{S}$ and $\Im(z)$ is the imaginary component of a complex number z. Let $\boldsymbol{f}(t) = e^{\boldsymbol{S}t}\boldsymbol{s}$. Then $\boldsymbol{f}(t)e^{-\lambda t}$ where $\lambda = \Re(\lambda_i)$ is the real part of the eigenvalues of $\boldsymbol{S}$ (the all share the same real part), is also periodic with the same period. Hence we can simplify the integral to a finite one;

$$\begin{aligned}
\boldsymbol{D} &= \int_{t=0}^{\infty} \boldsymbol{f}(t) \cdot \boldsymbol{a}(t) \,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} \int_{k\rho}^{(k+1)\rho} e^{\lambda t}e^{-\lambda t}\boldsymbol{f}(t) \cdot \boldsymbol{a}(t) \,\mathrm{d}t \\
&= \sum_{k=0}^{\infty} \int_{0}^{\rho} e^{\lambda(k\rho+t)}e^{-\lambda(k\rho+t)}\boldsymbol{f}(k\rho+t) \cdot \boldsymbol{a}(k\rho+t) \,\mathrm{d}t. \qquad (2.11)
\end{aligned}$$

By periodicity, then $e^{-\lambda(k\rho+t)}\boldsymbol{f}(k\rho+t) \cdot \boldsymbol{a}(k\rho+t) = e^{-\lambda t}\boldsymbol{f}(t) \cdot \boldsymbol{a}(t)$, hence (2.11) is equal to

$$\sum_{k=0}^{\infty}(e^{\lambda\rho})^k \int_0^{\rho} e^{\lambda t}e^{-\lambda t}\boldsymbol{f}(t) \cdot \boldsymbol{a}(t) \,\mathrm{d}t = \frac{1}{1-e^{\lambda\rho}} \int_0^{\rho} \boldsymbol{f}(t) \cdot \boldsymbol{a}(t) \,\mathrm{d}t, \qquad (2.12)$$

where the sum converges as it is a geometric series and $\lambda < 0$, $\rho > 0$.

To approximate (2.12) numerically, we first partition $[0, \rho)$ into N equal-width intervals $[t_n, t_{n+1})$, where $t_n = (n-1)\rho/N$, $n = 1, 2, ..., N+1$. On $[t_n, t_{n+1})$ we approximate the orbit $\boldsymbol{a}(t)$ by a constant $\boldsymbol{a}(t) \approx \boldsymbol{a}_n := \dfrac{1}{2}\left(\boldsymbol{a}(t_n) + \boldsymbol{a}(t_{n+1})\right)$, $t \in [t_n, t_{n+1})$. Substituting this approximation into the expression for $\boldsymbol{D}$ gives

$$\begin{aligned}
\boldsymbol{D} &\approx \frac{1}{1-e^{\lambda\rho}} \sum_{n=1}^{N} \int_{t_n}^{t_{n+1}} \boldsymbol{f}(t) \cdot \boldsymbol{a}_n \,\mathrm{d}t \\
&= \frac{1}{1-e^{\lambda\rho}} \sum_{n=1}^{N} \left[e^{\boldsymbol{S}t_{n+1}} - e^{\boldsymbol{S}t_n}\right] \boldsymbol{e} \cdot \boldsymbol{a}_n.
\end{aligned}$$

This approximation preserves the property that $\boldsymbol{De} = \boldsymbol{e}$.

Computationally efficient expressions for $e^{\boldsymbol{S}t}\boldsymbol{e}$ and $\boldsymbol{a}(t)$ are provided in (Horváth et al. 2020).

2.3 The association of $j \in \mathcal{S}_0$ with $\mathcal{S}_+$ or $\mathcal{S}_-$

Let $X(0) = y_\ell$ and consider the event where $\{\varphi(t)\}$ transitions from $j_0 \to j_1 \to j_2$ where $j_0 \in \mathcal{S}_+$, $j_1 \in \mathcal{S}_0$ and $j_2 \in \mathcal{S}_-$, before there is a change of level, i.e.

$$\varphi(t) = \begin{cases} j_0 & t \in [0, t_1)\,, \\ j_1 & t \in [t_1, t_2)\,, \\ j_2 & t \in [t_2, t_3)\,, \end{cases}$$

and $X(t) \in \mathcal{D}_\ell$, $t \in [0, t_3)$. On approximating this event, the initial orbit position is $\boldsymbol{A}(0) = \boldsymbol{\alpha}$, and the condition $X(t) \in \mathcal{D}_\ell$, $t \in [0, t_3)$ is modelled by $L(t) = \ell$, $t \in [0, t_3)$. The corresponding sample path of the orbit process on $t \in [0, t_3)$, is

$$\boldsymbol{A}(t) = \begin{cases} \dfrac{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t}}{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t}\boldsymbol{e}} & t \in [0, t_1)\,, \\ \dfrac{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t_1}\boldsymbol{D}(j_0, j_1) e^{T_{j_1 j_1}(t-t_1)}}{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t_1}\boldsymbol{D}(j_0, j_1) e^{T_{j_1 j_1}(t-t_1)}\boldsymbol{e}} & t \in [t_1, t_2)\,, \\ \dfrac{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t_1}\boldsymbol{D}(j_0, j_1) e^{T_{j_1 j_1}(t_2-t_1)}\boldsymbol{D}(j_1, j_2) e^{(\boldsymbol{S}_{j_2} + T_{j_2 j_2}\boldsymbol{I})(t-t_2)}}{\boldsymbol{\alpha} e^{(\boldsymbol{S}_{j_0} + T_{j_0 j_0}\boldsymbol{I})t_1}\boldsymbol{D}(j_0, j_1) e^{T_{j_1 j_1}(t_2-t_1)}\boldsymbol{D}(j_1, j_2) e^{(\boldsymbol{S}_{j_2} + T_{j_2 j_2}\boldsymbol{I})(t-t_2)}\boldsymbol{e}} & t \in [t_2, t_3)\,, \end{cases}$$

$$= \begin{cases} \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}\boldsymbol{e}} & t \in [0, t_1)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}(j_0, j_1)}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}(j_0, j_1)\boldsymbol{e}} & t \in [t_1, t_2)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}(j_0, j_1)\boldsymbol{D}(j_1, j_2) e^{\boldsymbol{S}_{j_2}(t-t_2)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}(j_0, j_1)\boldsymbol{D}(j_1, j_2) e^{\boldsymbol{S}_{j_2}(t-t_2)}\boldsymbol{e}} & t \in [t_2, t_3)\,, \end{cases}$$

for some matrices $\boldsymbol{D}(j_0, j_1)$ and $\boldsymbol{D}(j_2, j_1)$. Notice that $\{\boldsymbol{A}(t)\}$ is constant on $t \in [t_1, t_2)$, and ultimately, $\{\varphi(t)\}$ transitions from a positive phase to a negative phase. The matrix product $\boldsymbol{D}(j_0, j_1)\boldsymbol{D}(j_1, j_2)$ must capture that there is ultimately a change of phase from $\mathcal{S}_+$ to $\mathcal{S}_-$. Hence $\boldsymbol{D}(j_0, j_1)\boldsymbol{D}(j_1, j_2)$ should be equal to $\boldsymbol{D}$. These types of sample paths are the reason we need to associate states $j_1 \in \mathcal{S}_0$ with either $\mathcal{S}_+$ or $\mathcal{S}_-$.

Augmented state-space schemes

One way to approach this problem is to augment the state space of the phase process by duplicating $\mathcal{S}_0$ and associating one copy of $\mathcal{S}_0$ with $\mathcal{S}_+$ and one copy of $\mathcal{S}_0$ with $\mathcal{S}_-$. Let

$\{\varphi^*(t)\}$ be the augmented CTMC with state space $\mathcal{S}^*$ and generator $\boldsymbol{T}^*$. Let $\mathcal{S}_+$ and $\mathcal{S}_-$ be as before and $\mathcal{S}_{m0} = \{(m,i) \mid i \in \mathcal{S}_m\}$, $m \in \{+,-\}$, then $\mathcal{S}^* = \mathcal{S}_+ \cup \mathcal{S}_- \cup \mathcal{S}_{+0} \cup \mathcal{S}_{-0}$. The generator of $\varphi^*(t)$ is

$$\boldsymbol{T}^* = \left[\begin{array}{cccc} \boldsymbol{T}_{++} & \boldsymbol{T}_{+-} & \boldsymbol{T}_{+0} & 0 \\ \boldsymbol{T}_{-+} & \boldsymbol{T}_{--} & 0 & \boldsymbol{T}_{-0} \\ \boldsymbol{T}_{0+} & \boldsymbol{T}_{0-} & \boldsymbol{T}_{00} & 0 \\ \boldsymbol{T}_{0+} & \boldsymbol{T}_{0-} & 0 & \boldsymbol{T}_{00} \end{array}\right].$$

Also define a fluid level $\{X^*(t)\}$ using $\{\varphi^*(t)\}$, with rates $c_i^* = c_i$ for $i \in \mathcal{S}_+ \cup \mathcal{S}_-$ and $c_{(m,i)}^* = 0$ for $(m,i) \in \mathcal{S}_{0+} \cup \mathcal{S}_{0-}$. The process $\{\varphi(t)\}$ is imbedded within $\{\varphi^*(t)\}$ and is recovered by marginalising over $\mathcal{S}_{0+}$ and $\mathcal{S}_{0-}$. Given $X^*(0) = X(0)$, the fluid levels $X^*(t)$ and $X(t)$ match exactly. Hence, if we approximate $\{(X^*(t), \varphi^*(t))\}$, then we can recover an approximation to $\{(X(t), \varphi(t))\}$. This construction removes the problem of having to choose how to associate states $j \in \mathcal{S}_0$ with either $\mathcal{S}_+$ or $\mathcal{S}_-$

The generator for the QBD-RAP approximation to the augmented fluid process is

$$\boldsymbol{B} = \left[\begin{array}{cccccc} \ddots & \ddots & \ddots & & & \\ & \boldsymbol{B}_{-1} & \boldsymbol{B}_0 & \boldsymbol{B}_{+1} & & \\ & & \boldsymbol{B}_{-1} & \boldsymbol{B}_0 & \boldsymbol{B}_{+1} & \\ & & & \ddots & \ddots & \ddots \end{array}\right],$$

where,

$$\boldsymbol{B}_0 = \left[\begin{array}{cccc} \boldsymbol{C}_+ \otimes \boldsymbol{S} + \boldsymbol{T}_{++} \otimes \boldsymbol{I} & \boldsymbol{T}_{+-} \otimes \boldsymbol{D} & \boldsymbol{T}_{+0} \otimes \boldsymbol{I} & 0 \\ \boldsymbol{T}_{-+} \otimes \boldsymbol{D} & \boldsymbol{C}_- \otimes \boldsymbol{S} + \boldsymbol{T}_{--} \otimes \boldsymbol{I} & 0 & \boldsymbol{T}_{-0} \otimes \boldsymbol{I} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{I} & \boldsymbol{T}_{0-} \otimes \boldsymbol{D} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} & 0 \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{D} & \boldsymbol{T}_{0-} \otimes \boldsymbol{I} & 0 & \boldsymbol{T}_{00} \otimes \boldsymbol{I} \end{array}\right],$$

$$\boldsymbol{B}_{-1} = \left[\begin{array}{cccc} \boldsymbol{0} & & & \\ & \boldsymbol{C}_- \otimes (\boldsymbol{s}\boldsymbol{\alpha}) & & \\ & & \boldsymbol{0} & \\ & & & \boldsymbol{0} \end{array}\right], \qquad \boldsymbol{B}_{+1} = \left[\begin{array}{cccc} \boldsymbol{C}_+ \otimes (\boldsymbol{s}\boldsymbol{\alpha}) & & & \\ & \boldsymbol{0} & & \\ & & \boldsymbol{0} & \\ & & & \boldsymbol{0} \end{array}\right].$$

With this construction, jumps according to the matrix $\boldsymbol{D}$ occur only on transitions from $\mathcal{S}_m \to \mathcal{S}_n$, or $\mathcal{S}_m \to \mathcal{S}_{0m} \to \mathcal{S}_n$, $m, n \in \{+.-\}$, $m \neq n$.

Other approaches

There are other possible approaches to model this behaviour which do not require duplicating states and may achieve a small computational saving, however they are less amenable to analysis and possibly lead to less accurate schemes. Instead of duplicating states in $\mathcal{S}_0$, we can choose to associate $j_1 \in \mathcal{S}_0$ with either $\mathcal{S}_+$ or $\mathcal{S}_-$. If we associate j_1 with $\mathcal{S}_+$, this amounts to choosing $\boldsymbol{D}(j_0, j_1) = \boldsymbol{I}$ and $\boldsymbol{D}(j_1, j_2) = \boldsymbol{D}$; if we associate j_1 with $\mathcal{S}_-$, this amounts to choosing $\boldsymbol{D}(j_0, j_1) = \boldsymbol{D}$ and $\boldsymbol{D}(j_1, j_2) = \boldsymbol{I}$.

There are some consequences to this choice. Let $k_2 \in \mathcal{S}_+$. Consider an event where the phase process of the fluid queue transitions from $j_0 \to j_1 \to k_2$ and there is no change of level. If j_1 is associated with $\mathcal{S}_+$, then $\boldsymbol{D}(j_0, j_1) = \boldsymbol{D}(j_1, k_2) = \boldsymbol{I}$ and the corresponding orbit process, given $\boldsymbol{A}(0) = \boldsymbol{\alpha}$, is

$$\boldsymbol{A}(t) = \begin{cases} \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}\boldsymbol{e}} & t \in [0, t_1)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{e}} & t \in [t_1, t_2)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1} e^{\boldsymbol{S}_{k_2}(t-t_2)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1} e^{\boldsymbol{S}_{k_2}(t-t_2)}\boldsymbol{e}} & t \in [t_2, t_3)\,. \end{cases}$$

Notice that there is no matrix $\boldsymbol{D}$ in this expression.

Compare this to if j_1 is associated with $\mathcal{S}_-$. In this case $\boldsymbol{D}(j_0, j_1) = \boldsymbol{D}(j_1, k_2) = \boldsymbol{D}$ and the corresponding orbit process of the approximation, given $\boldsymbol{A}(0) = \boldsymbol{\alpha}$ and there are no change of level in $[t_1, t_3)$, is

$$\boldsymbol{A}(t) = \begin{cases} \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t}\boldsymbol{e}} & t \in [0, t_1)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}\boldsymbol{e}} & t \in [t_1, t_2)\,, \\ \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}\boldsymbol{D} e^{\boldsymbol{S}_{k_2}(t-t_2)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}_{j_0} t_1}\boldsymbol{D}\boldsymbol{D} e^{\boldsymbol{S}_{k_2}(t-t_2)}\boldsymbol{e}} & t \in [t_2, t_3)\,. \end{cases}$$

Ideally $\boldsymbol{D}^2 = \boldsymbol{I}$, however this is not the case here. Recall that a jump according to $\boldsymbol{D}$ corresponds to approximating the residual life by an expectation. With this interpretation as an approximation, it suggests that we might want to minimise the number of jumps according to $\boldsymbol{D}$ which occur. Therefore, for $j_1 \in \mathcal{S}_0$, if transitions $\mathcal{S}_+ \to j_1 \to \mathcal{S}_+$ occur with high probability compared to transition $\mathcal{S}_- \to j_1 \to \mathcal{S}_-$, then this suggests we might want to associate j_1 with $\mathcal{S}_+$. Which association is chosen will depend on the parameters of the fluid queue and on which aspects of the model we wish to approximate. Although this advice is based on intuition only, numerical results in Section TBC suggest that it is reasonable.

2.4 The dynamics of the QBD-RAP approximation

We now have all the elements we need to describe the dynamics of the QBD-RAP approximation. Recall that a QBD-RAP is a stochastic process $\{(L(t), \boldsymbol{A}(t))\}_{t\geq 0}$ where $L(t)$ is a discrete, level process and $\boldsymbol{A}(t)$ is a piecewise deterministic orbit process. Since the phase dynamics are a CTMC, we choose to use an alternate notation, $\{(L(t), \boldsymbol{A}(t), \phi(t))\}_{t\geq 0}$ where $\{L(t)\}$ is the level, $\{\boldsymbol{A}(t)\}$ is the orbit process and $\{\phi(t)\}$ is the phase process. We choose this representation to make explicit how the approximation captures the phase dynamics of the fluid queue. We will show later that $\phi(t)$ captures the phase dynamics of the fluid queue exactly, provided the phase process is independent of the fluid level $\{X(t)\}$. We use the level $L(t) = \ell$ to approximate which band $\mathcal{D}_\ell$ that $X(t)$ is in at time t, and use the orbit $\boldsymbol{A}(t)$ to obtain approximations about where $X(t)$ is within the interval $\mathcal{D}_\ell$.

We now proceed to describe the evolution of the orbit and phase processes, before introducing the level variable later.

The orbit process and phase dynamics On $\phi(t) = i$, between event epochs, the process $\{\boldsymbol{A}(t)\}$ evolves deterministically according to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{A}(t) = \boldsymbol{A}(t)(\boldsymbol{S}_i - \boldsymbol{A}(t)(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}_p)\boldsymbol{e})\boldsymbol{I}_p. \tag{2.13}$$

Let $\boldsymbol{a}_0$ be a vector such that $\boldsymbol{a}_0\boldsymbol{e} = 1$ and $(\boldsymbol{a}_0, \boldsymbol{S}, \boldsymbol{s})$ is a representation of a ME distribution. Given no events occur before time $t + u$, $\boldsymbol{A}(u) = \boldsymbol{a}_0$ and $\phi(u) = i$, the solution to (2.13) states that $\boldsymbol{A}(t + u)$ evolves deterministically according to

$$\boldsymbol{A}(t+u) \mid \{\boldsymbol{A}(u) = \boldsymbol{a}_0\} = \frac{\boldsymbol{a}_0 e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}_p)t}}{\boldsymbol{a}_0 e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}_p)t}\boldsymbol{e}} = \frac{\boldsymbol{a}_0 e^{\boldsymbol{S}_i t}}{\boldsymbol{a}_0 e^{\boldsymbol{S}_i t}\boldsymbol{e}}.$$

At time t, given $\boldsymbol{A}(t)$, an event occurs at rate

$$\boldsymbol{A}(t)(\boldsymbol{S}_i - T_{ii}\boldsymbol{I}_p)\boldsymbol{e} = \boldsymbol{A}(t)\boldsymbol{s}_i - T_{ii}.$$

More precisely, an event corresponding to a change in phase for $\phi(t)$ occurs at rate $-\boldsymbol{A}(t)T_{ii}\boldsymbol{e} = -T_{ii}$ and an event corresponding to a *change of level* occurs at rate $-\boldsymbol{A}(t)\boldsymbol{S}_i\boldsymbol{e} = \boldsymbol{A}(t)\boldsymbol{s}_i$. Later, we will make clear why we say that the latter event corresponds to a change of level. Upon an event occurring at time t, with probability $\dfrac{-T_{ii}}{-T_{ii} + \boldsymbol{A}(t^-)\boldsymbol{s}_i}$ the event corresponds to a change of phase and with probability $\dfrac{\boldsymbol{A}(t^-)\boldsymbol{s}_i}{-T_{ii} + \boldsymbol{A}(t^-)\boldsymbol{s}_i}$ the event corresponds a change of level.

Upon an event corresponding to a change of level occurring at time t the process $\{\boldsymbol{A}(t)\}$ jumps to $\boldsymbol{A}(t) = \boldsymbol{\alpha}$.

Upon an event corresponding to a change of phase from i to $j \neq i$ occurring at time t, there are two possibilities; either $sign(c_i) = sign(c_j)$, or $sign(c_i) \neq sign(c_j)$. As discussed earlier, for states $i \in \mathcal{S}_0$ we must specify some association with either $\mathcal{S}_+$ or $\mathcal{S}_-$. Here we choose the augmented state space approach and duplicate $\mathcal{S}_0$ and associate one copy with $i \in \mathcal{S}_+$ and one copy with $\mathcal{S}_-$; call these $\mathcal{S}_{+0}$ and $\mathcal{S}_{-0}$, respectively. We take $sign(c_i) = +$ for $i \in \mathcal{S}_{+0}$ and $sign(c_i) = -$ for $i \in \mathcal{S}_{-0}$.

Upon an event corresponding to a change of phase from i to $j \neq i$ occurring at time t, in the case $sign(c_i) = sign(c_j)$, at the time of the event $\boldsymbol{A}(t)$ is unchanged but begins to evolves according to (2.13) with i replaced by j, while $\{\phi(t)\}$ jumps from $\phi(t^-) = i$ to $\phi(t) = j$.

Upon an event corresponding to a change of phase from i to $j \neq i$ occurring at time t, in the case $sign(c_i) \neq sign(c_j)$, the process $\{\boldsymbol{A}(t)\}$ jumps to $\boldsymbol{A}(t) = \dfrac{\boldsymbol{A}(t^-)T_{ij}\boldsymbol{D}}{\boldsymbol{A}(t^-)T_{ij}\boldsymbol{D}\boldsymbol{e}} = \dfrac{\boldsymbol{A}(t^-)T_{ij}\boldsymbol{D}}{T_{ij}} = \boldsymbol{A}(t^-)\boldsymbol{D}$ and then proceeds to evolve according to (2.13) with i replaced by j, and $\{\phi(t)\}$ jumps from $\phi(t^-) = i$ to $\phi(t) = j$.

The two scenarios, $sign(c_i) \neq sign(c_j)$ and $sign(c_i) = sign(c_j)$, can be written succinctly by stating that, at the time of an event corresponding to a change of phase from i to j, $\{\boldsymbol{A}(t)\}$ jumps to $\boldsymbol{A}(t)\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}$ and begins to evolve according to (2.13) with i replaced by j, meanwhile $\{\phi(t)\}$ jumps from $\phi(t^-) = i$ to $\phi(t) = j$. Moreover, given a change of phase event occurs at time t with $\phi(t^-) = i$ then, with probability $\dfrac{\boldsymbol{A}(t^-)T_{ij}\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}\boldsymbol{e}}{-\boldsymbol{A}(t^-)T_{ii}\boldsymbol{e}} = \dfrac{T_{ij}}{-T_{ii}}$, the event corresponds to a change of phase from i to j.

The following result states that $\{\phi(t)\}$ has the same distribution as $\{\varphi(t)\}$ when the fluid queue is unbounded, the latter being the phase process of the fluid queue.

Theorem 2.4.1. *Let Θ_i be the time at which the first jump of the phase process of the unbounded QBD-RAP, $\{\phi(t)\}$, occurs given $\phi(0) = i$. For any initial orbit $\boldsymbol{a} \in \mathcal{A}$, then Θ_i has an exponential distribution with rate parameter $|T_{ii}|$. Furthermore, given $\phi(t)$ leaves state i, it jumps to state j with probability $T_{ij}/|T_{ii}|$. Hence $\phi(t)$ and $\varphi(t)$ have the same probability law.*

Proof. Let $\{\tau_n\}_{n\geq 0}$ with $\tau_0 = 0$ and τ_n the time of the n-th change of level of the QBD-RAP. Consider partitioning $\{\Theta_i \leq t\}$ with respect to $\{\tau_{n-1} < t \leq \tau_n\}$, $n = 1, 2, \ldots$. For $n = 1$ we can write

$$\mathbb{P}(\Theta_i \leq t, \tau_0 < t \leq \tau_1 \mid \boldsymbol{A}(0) = \boldsymbol{a}) = \boldsymbol{a}e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}_p)t}\boldsymbol{e}$$

and since $T_{ii}\boldsymbol{I}_p$ commutes with $\boldsymbol{S}_i$ and $e^{T_{ii}t}$ is a scalar, then this is equal to

$$\boldsymbol{a}e^{\boldsymbol{S}_i t}e^{T_{ii}\boldsymbol{I}_p t}\boldsymbol{e} = \boldsymbol{a}e^{\boldsymbol{S}_i t}\boldsymbol{e}e^{T_{ii}t} = \mathbb{P}(\tau_0 < t \leq \tau_1)e^{T_{ii}t}.$$

For $n > 1$, by partitioning on the times of the first $n-1$ level changes, $\tau_1, \dots, \tau_n$, we get

$$\begin{aligned}
&\mathbb{P}(\Theta_i \leq t, \tau_{n-1} < t \leq \tau_n \mid \boldsymbol{A}(0) = \boldsymbol{a})\\
&= \int_{t_1=0}^{t}\int_{t_2=t_1}^{t}\dots\int_{t_{n-1}=t_{n-2}}^{t} \boldsymbol{a} e^{(\boldsymbol{S}_i + T_{ii}\boldsymbol{I}_p)t_1}\boldsymbol{s}_i\left(\prod_{k=2}^{n-1}\boldsymbol{\alpha} e^{(\boldsymbol{S}_i+T_{ii}\boldsymbol{I}_p)(t_k-t_{k-1})}\boldsymbol{s}_i\right)\\
&\quad\times\boldsymbol{\alpha} e^{(\boldsymbol{S}_i+T_{ii}\boldsymbol{I}_p)(t-t_{n-1})}\boldsymbol{e}\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\dots\,\mathrm{d}t_1
\end{aligned}$$

where t_k is the time of the k change of level. Since $T_{ii}\boldsymbol{I}_p$ commutes with $\boldsymbol{S}_i$, $e^{T_{ii}t_k}$, $k = 1, ..., n-1$ are scalars, and $t_1 + (t_2 - t_1) + \dots + (t_{n-1} - t_{n-2}) + (t - t_{n-1}) = t$, then this is equal to

$$\begin{aligned}
&\int_{t_1=0}^{t}\int_{t_2=t_1}^{t}\dots\int_{t_{n-1}=t_{n-2}}^{t} \boldsymbol{a} e^{\boldsymbol{S}_i t_1}\boldsymbol{s}_i\left(\prod_{k=2}^{n-1}\boldsymbol{\alpha} e^{\boldsymbol{S}_i(t_k-t_{k-1})}\boldsymbol{s}_i\right)\boldsymbol{\alpha} e^{\boldsymbol{S}_i(t-t_{n-1})}\boldsymbol{e}\times e^{T_{ii}t}\,\mathrm{d}t_{n-1}\,\mathrm{d}t_{n-2}\dots\,\mathrm{d}t_1\\
&= \mathbb{P}(\tau_{n-1} < t \leq \tau_n)e^{T_{ii}t}.
\end{aligned}$$

Hence, by the law of total probability,

$$\begin{aligned}
\mathbb{P}(\Theta_i \leq t) &= \sum_{n=1}^{\infty}\mathbb{P}(\Theta_i \leq t, \tau_{n-1} < t \leq \tau_n)\\
&= \sum_{n=1}^{\infty}\mathbb{P}(\tau_{n-1} < t \leq \tau_n)e^{T_{ii}t}\\
&= e^{T_{ii}t},
\end{aligned}$$

and therefore Θ_i has an exponential distribution with rate $|T_{ii}|$.

Upon leaving state i at time t, $\phi(t)$ transitions to state j with probability

$$\frac{\left(\dfrac{\boldsymbol{A}(t)\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}T_{ij}\boldsymbol{e}}{\sum_{j\in\mathcal{S}}\boldsymbol{A}(t)\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}T_{ij}\boldsymbol{e} + \boldsymbol{A}(t)\boldsymbol{s}_i}\right)}{\left(\dfrac{\sum_{j\in\mathcal{S}}\boldsymbol{A}(t)\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}T_{ij}\boldsymbol{e}}{\sum_{j\in\mathcal{S}}\boldsymbol{A}(t)\boldsymbol{D}^{1(sign(c_i)\neq sign(c_j))}T_{ij}\boldsymbol{e} + \boldsymbol{A}(t)\boldsymbol{s}_i}\right)} = \frac{\boldsymbol{A}(t)\boldsymbol{e}T_{ij}}{\sum_{j\in\mathcal{S}}\boldsymbol{A}(t)\boldsymbol{e}T_{ij}} = \frac{T_{ij}}{-T_{ii}}.$$

Therefore the process $\{\phi(t)\}$ is has the same probability law as $\{\varphi(t)\}$. $\square$

Remark 2.4.2. *The same result can be shown for a regulated boundary. For boundary conditions which interact with the phase dynamics, such as a reflecting boundary, the* *result does not hold. The cause is the fact that the phase dynamics are level dependent – we see a forced change of phase upon a boundary being hit – and the QBD-RAP can only approximate the level process of the fluid queue. However, until a boundary is hit (by either the fluid queue or QBD-RAP) then the phase processes match. We show later that, in the limit as the variance of the the matrix exponential distribution used in the construction of the QBD-RAP goes to zero, then the dynamics of the level process of the fluid queue, $X(t)$, are captured by the QBD-RAP, and boundary behaviour which interacts with the phase dynamics can be captured too.*

Since the phase processes $\{\phi(t)\}$ and $\{\varphi(t)\}$ have the same law when boundaries are not present, henceforth, we shall assume $\{\phi(t)\}$ and $\{\varphi(t)\}$ are coupled when possible (they share the same sample path). Specifically, in Section 3.1, we will analyse the QBD-RAP on the event that it remains in the same level, ℓ, say, and we compare this to the fluid queue on the event that the level remains in the band $\mathcal{D}_\ell$. No boundary behaviour is involved in this calculation, so we treat $\{\phi(t)\}$ and $\{\varphi(t)\}$ as coupled and use the latter notation. When we must distinguish the two processes, we use $\phi(t)$ for the phase of the QBD-RAP and $\varphi(t)$ for the phase of the fluid.

The level process To event epochs of $\{(\boldsymbol{A}(t), \varphi(t))\}_{t\geq 0}$ we associate marks $\{-1, 0, +1\}$ in the following way.

- To events epochs corresponding to a change of phase of $\varphi(t)$ we associate the mark 0.
- To event epochs at time t which correspond to a change in level and for which $\varphi(t^-) = i \in \mathcal{S}_-$ we associate the mark -1.
- To event epochs at time t which correspond to a change in level and for which $\varphi(t^-) = i \in \mathcal{S}_+$ we associate the mark $+1$.

Now define $N_+(t)$ ($N_-(t)$) as the simple point process which counts the number of event epochs with marks $+1$ (-1) which have occurred in the time up to and including time t. The level process of the QBD-RAP is given by $L(t) = N_+(t) - N_-(t)$. The process $\{(L(t), \boldsymbol{A}(t), \varphi(t))\}_{t\geq 0}$ forms a QBD with RAP components.

One way to specify the QBD with RAP components, $\{(L(t), \boldsymbol{A}(t), \varphi(t))\}$, is to describe its generator:

$$\boldsymbol{B} = \begin{bmatrix} \ddots & \ddots & \ddots & & & \\ & \boldsymbol{B}_{-1} & \boldsymbol{B}_0 & \boldsymbol{B}_{+1} & & \\ & & \boldsymbol{B}_{-1} & \boldsymbol{B}_0 & \boldsymbol{B}_{+1} & \\ & & & \ddots & \ddots & \ddots \end{bmatrix},$$

where,

$$\boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{C}_+ \otimes \boldsymbol{S} + \boldsymbol{T}_{++} \otimes \boldsymbol{I} & \boldsymbol{T}_{+-} \otimes \boldsymbol{D} & \boldsymbol{T}_{+0} \otimes \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T}_{-+} \otimes \boldsymbol{D} & \boldsymbol{C}_- \otimes \boldsymbol{S} + \boldsymbol{T}_{--} \otimes \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{T}_{-0} \otimes \boldsymbol{I} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{I} & \boldsymbol{T}_{0-} \otimes \boldsymbol{D} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{D} & \boldsymbol{T}_{0-} \otimes \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} \end{bmatrix},$$

$$\boldsymbol{B}_{-1} = \begin{bmatrix} \boldsymbol{0} & & & \\ & \boldsymbol{C}_- \otimes (\boldsymbol{s\alpha}) & & \\ & & \boldsymbol{0} & \\ & & & \boldsymbol{0} \end{bmatrix}, \qquad \boldsymbol{B}_{+1} = \begin{bmatrix} \boldsymbol{C}_+ \otimes (\boldsymbol{s\alpha}) & & & \\ & \boldsymbol{0} & & \\ & & \boldsymbol{0} & \\ & & & \boldsymbol{0} \end{bmatrix}.$$

2.5 Boundary conditions

In the study and practical application of fluid queues, regulated, reflecting, or a mixture of these boundary conditions may be imposed. We now present the intuition as to how one may include such boundary conditions in the QBD-RAP approximation scheme.

Without loss of generality, assume that there is a boundary for the fluid level at $y_0 = 0$. This boundary can only be hit in phases $i \in \mathcal{S}_-$. Suppose that, upon hitting the boundary in phase $i \in \mathcal{S}_-$, the phase jumps from i to $j \in \mathcal{S}$ with probability p_{ij}. If $j \in \mathcal{S}_- \cup \mathcal{S}_0$ the process remains at the boundary and the phase process evolves among states in $\mathcal{S}_- \cup \mathcal{S}_0$ until the first transition to a state in $\mathcal{S}_+$, at which point the level $\{X(t)\}$ immediately leaves the boundary. If $j \in \mathcal{S}_+$ the process immediately leaves the boundary. We collect the probabilities p_{ij} into the matrices $\boldsymbol{P}_{-+} = [p_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_+}$, $\boldsymbol{P}_{-0} = [p_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_0}$, and $\boldsymbol{P}_{--} = [p_{ij}]_{i\in\mathcal{S}_-, j\in\mathcal{S}_-}$.

Adjacent to the boundary at $y_0 = 0$ is the band $\mathcal{D}_0 = [0, \Delta]$ which corresponds to level 0 for the QBD-RAP. Modelling the behaviour of the fluid queue at boundaries can be broken down into three components. 1) Modelling the time and phase when the fluid level hits the boundary. 2) Modelling the phase whilst the fluid level remains at the boundary. 3) Modelling the fluid level and phase at the exit from the boundary.

We claim that the event $\{L(t) = \ell, \phi(t) = i\}$ models the event $\{X(t) \in \mathcal{D}_{\ell,i}, \varphi(t) = i\}$, and further that instants u with $L(u^-) \neq L(u), \phi(u) = i$ model the events $\{X(u^-) \in \mathcal{D}_{\ell,i}, X(u) \notin \mathcal{D}_{\ell,i}, \varphi(u) = i\}$. With this in mind, we suppose that when the QBD-RAP is in level 0 in phase $i \in \mathcal{S}_-$ and there is a change of level, this corresponds approximating the event that the fluid level $\{X(t)\}$ hits the boundary at 0. We refer to this as the QBD-RAP hitting the boundary also. We show later that, using matrix exponentials with sufficiently small variance in the construction of the QBD-RAP, then the distribution of time until the QBD-RAP first hits the boundary closely models the time until the fluid level hits the boundary closely. This addresses component 1).

For regulated behaviour at a lower boundary at 0, given the fluid level reaches zero at time u in phase $i \in \mathcal{S}_-$, then the phase transitions to some phase $j \in \mathcal{S}_- \cup \mathcal{S}_0$ with probability p_{ij}, and the distribution of the phase $\varphi(t)$ until the fluid queue leaves the boundary for the first time after u is given by the elements of the vector

$$p_{ij}\boldsymbol{e}_j \exp\left(\left[\begin{array}{cc} \boldsymbol{T}_{--} & \boldsymbol{T}_{-0} \\ \boldsymbol{T}_{0-} & \boldsymbol{T}_{00} \end{array}\right](t-u)\right).$$

The distribution for the time until $\{X(t)\}$ leaves $x=0$ for the first time after u is

$$1 - p_{ij}\boldsymbol{e}_j \exp\left(\left[\begin{array}{cc} \boldsymbol{T}_{--} & \boldsymbol{T}_{-0} \\ \boldsymbol{T}_{0-} & \boldsymbol{T}_{00} \end{array}\right](t-u)\right)\boldsymbol{e}.$$

Therefore, in the case of a regulated boundary, we can capture the behaviour of the process at the boundary exactly. For a reflecting boundary there is no time spent at the boundary. This addresses component 2) of the modelling problem.

For both regulated boundary behaviour and reflecting boundary behaviour, given the QBD-RAP is in the correct phase at the instant upon which it hits the boundary, then upon exiting the boundary the phase can be captured exactly too. For a regulated boundary, given the fluid/QBD-RAP hits the boundary in phase $i \in \mathcal{S}_-$ at time t, it then jumps to phase $j \in \mathcal{S}_+ \cup \mathcal{S}_0$ and exits the boundary in phase $k \in \mathcal{S}_+$ at time $(t-u)$ with density

$$p_{ij}\boldsymbol{e}_j \exp\left(\left[\begin{array}{cc} \boldsymbol{T}_{--} & \boldsymbol{T}_{-0} \\ \boldsymbol{T}_{0-} & \boldsymbol{T}_{00} \end{array}\right](t-u)\right)\left[\begin{array}{c} \boldsymbol{T}_{-+} \\ \boldsymbol{T}_{0+} \end{array}\right]\boldsymbol{e}_k.$$

For a reflecting boundary, given the boundary is hit in phase $i \in \mathcal{S}_-$, the phase upon leaving the boundary is $j \in \mathcal{S}_+$ with probability p_{ij}. Upon leaving the boundary, the appropriate orbit position for the QBD-RAP is $\boldsymbol{\alpha}$, as this corresponds to the fluid level $X=0$.

To summarise, at time t the rate at which probability mass accumulates at the regulated boundary in phase $j \in \mathcal{S}_- \cup \mathcal{S}_0$ upon hitting the boundary in phase $i \in \mathcal{S}_-$ is

$$c_i p_{ij}\boldsymbol{A}(t)\boldsymbol{s}.$$

The rate at which mass transitions from phase $i \in \mathcal{S}_-$ to $j \in \mathcal{S}_+$ upon hitting a boundary is

$$c_i p_{ij}\boldsymbol{A}(t)\boldsymbol{s},$$

and upon this transition, the orbit jumps to $\boldsymbol{\alpha}$. The rate at which mass leaves the regulated boundary from phase $i \in \mathcal{S}_-$ into phases $j \in \mathcal{S}_+$ is

$$T_{ij},$$

and upon leaving the boundary the orbit is $\boldsymbol{\alpha}$.

This information can be inscribed in the generator of the QBD-RAP. For example, using the augmented state-space scheme to account for phases $\mathcal{S}_0$, the generator of the QBD-RAP described above has boundary conditions included as

$$\left[\begin{array}{cc|cccc|c}
\boldsymbol{T}_{--} & \boldsymbol{T}_{0-} & \boldsymbol{T}_{-+}\otimes\boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \\
\boldsymbol{T}_{-0} & \boldsymbol{T}_{00} & \boldsymbol{T}_{0+}\otimes\boldsymbol{\alpha} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \\
\hline
\boldsymbol{0} & \boldsymbol{0} & & & & & \\
(\boldsymbol{C}_-\boldsymbol{P}_{--})\otimes\boldsymbol{s} & (\boldsymbol{C}_-\boldsymbol{P}_{-0})\otimes\boldsymbol{s} & \multicolumn{4}{c|}{\check{\boldsymbol{B}}_0} & \boldsymbol{B}_{+1} \\
\boldsymbol{0} & \boldsymbol{0} & & & & & \\
\boldsymbol{0} & \boldsymbol{0} & & & & & \ddots \\
\hline
 & & \multicolumn{4}{c|}{\boldsymbol{B}_{-1}} & \boldsymbol{B}_0 \\
 & & & & & \ddots & \ddots
\end{array}\right],$$

where

$$\check{\boldsymbol{B}}_0 = \begin{bmatrix}
\boldsymbol{C}_+\otimes\boldsymbol{S}+\boldsymbol{T}_{++}\otimes\boldsymbol{I} & \boldsymbol{T}_{+-}\otimes\boldsymbol{D} & \boldsymbol{T}_{+0}\otimes\boldsymbol{I} & \\
(\boldsymbol{C}_-\boldsymbol{P}_{-+})\otimes\boldsymbol{s\alpha}+\boldsymbol{T}_{-+}\otimes\boldsymbol{D} & \boldsymbol{C}_-\otimes\boldsymbol{S}+\boldsymbol{T}_{--}\otimes\boldsymbol{I} & & \boldsymbol{T}_{-0}\otimes\boldsymbol{I} \\
\boldsymbol{T}_{0+}\otimes\boldsymbol{I} & \boldsymbol{T}_{0-}\otimes\boldsymbol{D} & \boldsymbol{T}_{00}\otimes\boldsymbol{I} & \\
\boldsymbol{T}_{0+}\otimes\boldsymbol{D} & \boldsymbol{T}_{0-}\otimes\boldsymbol{I} & & \boldsymbol{T}_{00}\otimes\boldsymbol{I}
\end{bmatrix}.$$

The top-left block represents the point masses due to the regulated boundary. The bottom-left block is the transition of density into point masses due to the regulated boundary. The top-right block is the transition of point masses into density as the process leaves the boundary. The term $(\boldsymbol{C}_-\boldsymbol{P}_{-+})\otimes\boldsymbol{s\alpha}$ in $\check{\boldsymbol{B}}_0$ incorporates the reflecting boundary.

Upper boundaries can be included in an analogous manner.

The orbit process $\{\boldsymbol{A}(t)\}$ is not required to model the behaviour at the regulated boundary. However, we suppose that $\boldsymbol{A}(t) = 1$ at the boundaries. This choice is arbitrary but allows us to generalise some notation later when we are describing the evolution of the QBD-RAP. Further, for boundaries at $y_0 = 0$ and $y_{K+1} = (K+1)\Delta$, we let the level process take the value $L(t) = -1$ at the lower boundary, and $L(t) = K+1$ at the upper boundary. Denote the set of levels (including boundary levels) by $\mathcal{K} = \{-1, 0, \dots, K, K+1\}$.

Remark 2.5.1. *It may be possible to extend the QBD-RAP approximation to model jumps into and out of the boundary, provided that the jumps into/out of the boundary happen at an intensity which can be described by a matrix exponential distribution.*

Furthermore, it may be possible to model more general boundary behaviours which can be approximated by a limit of matrix exponentials. For example, a boundary condition where the fluid queue spends a deterministic time at the boundary.

2.6 Initial conditions

We argued earlier that we can think of the orbit $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}|c_i|t}}{\boldsymbol{\alpha} e^{\boldsymbol{S}|c_i|t}\boldsymbol{e}}$ as corresponding to the fluid being a distance of $|c_i|t$ from the left boundary of an interval when $i \in \mathcal{S}_+$, or from the right boundary of an interval when $i \in \mathcal{S}_-$. With this interpretation, an approximation to the initial condition $(X(0), \varphi(0)) = (x_0, i)$, $x_0 \in \mathcal{D}_{\ell,i}$, $i \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$ is the set the initial orbit to $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x_0-y_\ell)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}(x_0-y_\ell)}\boldsymbol{e}}$. Similarly, an approximation to the initial condition $(X(0), \varphi(0)) = (x_0, i)$, $x_0 \in \mathcal{D}_{\ell,i}, x_0 \neq y_{\ell+1}$, $i \in \mathcal{S}_- \cup \mathcal{S}_{-0}$ is to set the initial orbit position to $\dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}(y_{\ell+1}-x_0)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}(y_{\ell+1}-x_0)}\boldsymbol{e}}$.

We use the notation $\boldsymbol{a}_{\ell,i}(x_0) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x_0-y_\ell)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}(x_0-y_\ell)}\boldsymbol{e}}$ for the initial orbit position corresponding to x_0 when the initial phase is $i \in \mathcal{S}_+$ and $x_0 \in \mathcal{D}_{\ell,i}$. Similarly, for $i \in \mathcal{S}_-$ and $x_0 \in \mathcal{D}_{\ell,i}$ define the notation $\boldsymbol{a}_{\ell,i}(x_0) = \dfrac{\boldsymbol{\alpha} e^{\boldsymbol{S}(y_{\ell+1}-x_0)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}(y_{\ell+1}-x_0)}\boldsymbol{e}}$.

More generally, given an initial measure $\mu_i(\cdot) := \mathbb{P}(X(0) \in \cdot, \varphi(0) = i)$, an approximation to this initial condition is to set the orbit to $\displaystyle\int_{x \in \mathcal{D}_{\ell,i}} \boldsymbol{a}_{\ell,i}(x)\,\mathrm{d}\mu_i$, for $i \in \mathcal{S}$, $\ell \in \mathcal{K} \setminus \{-1, K+1\}$.

2.7 At time t – closing operators

We now describe how the orbit position $\boldsymbol{A}(t)$ can be used to approximate the density of the position of $X(t)$ in a given level.

Suppose that the QBD-RAP is in level $L(t) = \ell$, phase $\phi(t) = j \in \mathcal{S}_+$, and the orbit is $\boldsymbol{A}(t) = \boldsymbol{a}$. If the QBD-RAP remains in phase j, then QBD-RAP approximation will transition out of level ℓ in the infinitesimal time interval $t+\,\mathrm{d}u$ with density $\boldsymbol{a} e^{|c_j|\boldsymbol{S}u}|c_j|\boldsymbol{s}\,\mathrm{d}u$. At the time of the change of level we estimate the position of $X(t+u)$ by $X(t+u) \approx y_{\ell+1}$. Tracing this back to time t, we estimate the position of $X(t)$ as $X(t) \approx y_{\ell+1} - |c_j|u$. Reverse engineering this logic, the approximation to the density of $X(t) \in \mathrm{d}x$ is $\boldsymbol{a} e^{|c_j|\boldsymbol{S}(y_{\ell+1}-x)/|c_j|}|c_j|\boldsymbol{s}\,\mathrm{d}x/|c_j|$, since $\mathrm{d}x = |c_j|\,\mathrm{d}u$ where $\mathrm{d}x$ is an infinitesimal interval in space and $\mathrm{d}u$ is an infinitesimal interval with respect to time.

Similarly, for $j \in \mathcal{S}_-$, if the QBD-RAP remains in phase j then the QBD-RAP approximation will transition out of level ℓ in the infinitesimal time interval $t+\,\mathrm{d}u$ with density $\boldsymbol{a} e^{|c_j|\boldsymbol{S}u}|c_j|\boldsymbol{s}\,\mathrm{d}u$. At the time of the transition of level, we estimate the position of $X(t+u)$ by $X(t+u) \approx y_\ell$. Tracing this back to time t, we estimate the position of $X(t)$ as $X(t) \approx y_\ell + |c_j|u$. Reverse engineering this logic, the approximation to the density $X(t) \in \mathrm{d}x$ is $\boldsymbol{a} e^{|c_j|\boldsymbol{S}(x-y_\ell)/|c_j|}|c_j|\boldsymbol{s}\,\mathrm{d}x/|c_j|$, since $\mathrm{d}x = |c_j|\,\mathrm{d}u$.

For phases $j \in \mathcal{S}_0$, if phase j is associated with $\mathcal{S}_+$, we use the same approximation as if $j \in \mathcal{S}_+$, and if phase $j \in \mathcal{S}_0$ is associated with $\mathcal{S}_-$, we use the same approximation as if

$j \in \mathcal{S}_-$.

This reasoning leads to an approximation of the distribution of the fluid at time t, $\mathbb{P}(X(t) \in \mathrm{d}x, \varphi(t) = j \mid X(0) = x_0, \varphi(0) = i)$, as

$$\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i) \\ \times \boldsymbol{a}e^{\boldsymbol{S}(y_{\ell+1}-x)}\boldsymbol{s}\,\mathrm{d}x, \tag{2.14}$$

for $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$ and

$$\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i) \\ \times \boldsymbol{a}e^{\boldsymbol{S}(x-y_\ell)}\boldsymbol{s}\,\mathrm{d}x, \tag{2.15}$$

for $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$, where $x \in \mathcal{D}_{\ell,j}$, $x_0 \in \mathcal{D}_{\ell_0,i}$.

While the estimates in the right-hand side of (2.14) and (2.15) are appealing, they are, however, defective estimates – they do not integrate to 1 for any finite-dimensional matrix exponential distribution. To see this, compute

$$\begin{aligned}
&\sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}_+\cup\mathcal{S}_{+0}}\int_{x\in\mathcal{D}_{\ell,j}}\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0),\\
&\qquad\qquad \phi(0) = i)\boldsymbol{a}e^{\boldsymbol{S}(y_{\ell+1}-x)}\boldsymbol{s}\,\mathrm{d}x\\
&\qquad +\sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}_-\cup\mathcal{S}_{-0}}\int_{x\in\mathcal{D}_{\ell,j}}\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0,\\
&\qquad\qquad \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0), \phi(0) = i)\boldsymbol{a}e^{\boldsymbol{S}(x-y_\ell)}\boldsymbol{s}\,\mathrm{d}x\\
&= \sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}_+\cup\mathcal{S}_{+0}}\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0),\\
&\qquad\qquad \phi(0) = i)(1 - \boldsymbol{a}e^{\boldsymbol{S}\Delta}\boldsymbol{e})\\
&\qquad +\sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}_-\cup\mathcal{S}_{-0}}\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0),\\
&\qquad\qquad \phi(0) = i)(1 - \boldsymbol{a}e^{\boldsymbol{S}\Delta}\boldsymbol{e})\\
&< 1,
\end{aligned}$$

since $\boldsymbol{a}e^{\boldsymbol{S}u}\boldsymbol{e} > 0$ for any $u > 0$. For an orbit position $\boldsymbol{A}(t) = \boldsymbol{a}$, the amount of mass missing from each level and phase is

$$\mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0), \phi(0) = i)\boldsymbol{a}e^{\boldsymbol{S}\Delta}\boldsymbol{e}\,\mathrm{d}x.$$

To partially rectify this we can instead approximate $\mathbb{P}(X(t) \in \mathrm{d}x, \varphi(t) = j \mid X(0) = x_0, \varphi(0) = i)$ by

$$\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell,i}(x_0), \phi(0) = i)\boldsymbol{a}u_+^\ell(x)\,\mathrm{d}x, \tag{2.16}$$

where

$$\boldsymbol{a}u_+^\ell(x) = \boldsymbol{a}\left(e^{\boldsymbol{S}(y_{\ell+1}-x)}\boldsymbol{s} + e^{\boldsymbol{S}(2\Delta-(y_{\ell+1}-x))}\boldsymbol{s}\right), \tag{2.17}$$

for $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$, and

$$\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t) = \ell, \boldsymbol{A}(t) \in \mathrm{d}\boldsymbol{a}, \phi(t) = j \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)\boldsymbol{a}u_-^\ell(x)\,\mathrm{d}x, \tag{2.18}$$

where

$$\boldsymbol{a}u_-^\ell(x) = \boldsymbol{a}\left(e^{\boldsymbol{S}(x-y_\ell)}\boldsymbol{s} + e^{\boldsymbol{S}(2\Delta-(x-y_\ell))}\boldsymbol{s}\right), \tag{2.19}$$

for $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$.

Intuitively, we take the density function $\boldsymbol{a}e^{\boldsymbol{S}x}\boldsymbol{s}$ on $x \in [0, 2\Delta)$, and 'fold' it back on itself around Δ, to create a density function on $[0, \Delta)$,

$$\boldsymbol{a}e^{\boldsymbol{S}x}\boldsymbol{s} + \boldsymbol{a}e^{\boldsymbol{S}(2\Delta-x)}\boldsymbol{s},\ x \in [0, \Delta).$$

The missing mass is now proportional to

$$\boldsymbol{a}e^{\boldsymbol{S}2\Delta}\boldsymbol{e},$$

which is less than the quantity $\boldsymbol{a}e^{\boldsymbol{S}\Delta}\boldsymbol{e}$ from the approximation scheme (2.14)-(2.15).

The difference between the approximation schemes is the way in which the vector $\boldsymbol{a}$ is used. We generalise this idea with the concept of *closing operators* which is a linear operator $V(x) : \mathcal{A} \to \mathbb{R}$, for each $x \in [0, \Delta)$. For example, the closing operator in Equations (2.14)-(2.15) is the operator $V(x)$, $x \in [0, \Delta)$ such that for any $\boldsymbol{a} \in \mathcal{A}$,

$$\boldsymbol{a}V(x) = \boldsymbol{a}e^{\boldsymbol{S}x}\boldsymbol{s}. \tag{2.20}$$

Similarly, in (2.16) and (2.18)

$$\boldsymbol{a}V(x) = \boldsymbol{a}\left(e^{\boldsymbol{S}x}\boldsymbol{s} + e^{\boldsymbol{S}(2\Delta-x)}\boldsymbol{s}\right). \tag{2.21}$$

The operator $V(x)$ is a choice which forms part of the definition of the approximation scheme. As we shall see, given certain properties of $V(x)$, we can prove that the approximation scheme converges, and ensures positivity. In the cases above, all the closing operators (2.20)-(2.21) lead to an approximation which converges and, due to their interpretation as probability densities, ensure positivity.

In (2.20)-(2.21), since V is linear, then we actually have

$$\begin{aligned}&\int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(L(t)=\ell, \boldsymbol{A}(t)\in \mathrm{d}\boldsymbol{a}, \phi(t)=j \mid L(0)=\ell_0, \boldsymbol{A}(0)=\boldsymbol{a}_{\ell_0,i}(x_0), \phi(0)=i)\boldsymbol{a}V(x)\\ &= \mathbb{E}[\boldsymbol{A}(t)1(L(t)=\ell,\phi(t)=j) \mid L(0)=\ell_0, \boldsymbol{A}(0)=\boldsymbol{a}_{\ell_0,i}(x_0), \phi(0)=i]V(x). \qquad (2.22)\end{aligned}$$

Remark 2.7.1. *I considered taking the closing operators to be*

$$\boldsymbol{a}V(x) = \frac{\boldsymbol{a}\left(e^{\boldsymbol{S}x}\boldsymbol{s} + e^{\boldsymbol{S}(2\Delta - x)}\boldsymbol{s}\right)}{1-\boldsymbol{a}e^{\boldsymbol{S}2\Delta}\boldsymbol{e}}, \qquad (2.23)$$

which also ensures that the solution always preserves mass (the result of integrating (2.23) over $[0,\Delta)$ is 1). However, (2.23) is not a linear operator. The importance of this fact is that for a linear operator,

$$\mathbb{E}[\boldsymbol{A}(t)V(x)] = \mathbb{E}[\boldsymbol{A}(t)]V(x),$$

thus, the computation of $\mathbb{E}[\boldsymbol{A}(t)V(x)]$ can be achieved by first computing $\mathbb{E}[\boldsymbol{A}(t)]$ and then applying $V(x)$ to the result. In this sense, $\mathbb{E}[\boldsymbol{A}(t)]$ contains all of the information about the history of the process up to time t needed to compute $\mathbb{E}[\boldsymbol{A}(t)V(x)]$. This is much the same as in RAPs and QBD-RAPs where $\mathbb{E}[\boldsymbol{A}(t)]$ is all that is required to compute probabilities about the future of the processes from time t onwards.

When $V(x)$ is not linear then, in general,

$$\mathbb{E}[\boldsymbol{A}(t)V(x)] \neq \mathbb{E}[\boldsymbol{A}(t)]V(x).$$

For example, with (2.23),

$$\mathbb{E}[\boldsymbol{A}(t)V(x)] = \int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(\boldsymbol{A}(t)\in \mathrm{d}\boldsymbol{a})\frac{\boldsymbol{a}\left(e^{\boldsymbol{S}x}\boldsymbol{s} + e^{\boldsymbol{S}(2\Delta - x)}\boldsymbol{s}\right)}{1-\boldsymbol{a}e^{\boldsymbol{S}2\Delta}\boldsymbol{e}}. \qquad (2.24)$$

While the calculation in (2.24) is theoretically possible, practically it is not. The seemingly innocuous integral over $\boldsymbol{a}\in\mathcal{A}$ actually amounts to an integral over all possible sample paths of the QBD-RAP (i.e. all possible event times of the QBD-RAP before time t) and in general, there is no 'nice' way to do this computation.

In the linear case, computation of

$$\mathbb{E}[\boldsymbol{A}(t)] = \int_{\boldsymbol{a}\in\mathcal{A}} \mathbb{P}(\boldsymbol{A}(t)\in \mathrm{d}\boldsymbol{a})\boldsymbol{a},$$

amounts to a matrix-exponential calculation.

In practice, to ensure that the approximation integrates to 1, we do re-normalise the estimates of the density. Specifically, we use the estimates

$$\frac{\mathbb{E}[\boldsymbol{A}(t)1(L(t)=\ell,\phi(t)=j)\mid L(0)=\ell_0,\boldsymbol{A}(0)=\boldsymbol{a}_{\ell,i}(x_0),\phi(0)=i]V(x)}{\int_{x\in[0,\Delta)}\mathbb{E}[\boldsymbol{A}(t)1(L(t)=\ell,\phi(t)=j)\mid L(0)=\ell_0,\boldsymbol{A}(0)=\boldsymbol{a}_{\ell,i}(x_0),\phi(0)=i]V(x)\,\mathrm{d}x}. \tag{2.25}$$

Chapter 3

Weak convergence of the QBD-RAP scheme

This chapter details convergence of the approximation scheme constructed in Chapter 2. The QBD-RAP is constructed using *matrix-exponential distributions*. The main result shows the convergence of the approximation scheme under the assumption that the variance of the matrix-exponential distribution(s) used in the construction tends to 0. Given that there is no known simple closed form for the transient distributions of a fluid queue, and that we make minimal assumptions about the matrix-exponential distributions used in the construction of the approximation, it is somewhat remarkable that we are able to establish a convergence result. The generality of the result with respect to the assumptions on the matrix-exponential distributions used in the construction of the approximation is necessitated by the fact that we use a class of *concentrated matrix-exponential distributions* found numerically in (Horváth et al. 2020), and for which there is relatively little known about their properties.

We suppose that we have a sequence $\{Z^{(p)}\}_{p\geq 1}$ of matrix exponential distributions, $Z^{(p)} \sim ME(\boldsymbol{\alpha}^{(p)}, \boldsymbol{S}^{(p)}, \boldsymbol{s}^{(p)})$, such that $\mathrm{Var}\left(Z^{(p)}\right) \to 0$ as $p \to \infty$. Ultimately, we show weak convergence (in both space and time) of the QBD-RAP approximation scheme via convergence of various Laplace transforms with respect to time of the QBD-RAP to the corresponding Laplace transforms of the fluid queue. We use the superscript (p) to denote dependence on the underlying choice of matrix exponential distribution that is used in the construction of the QBD-RAP scheme. To simplify notation, we omit the super script (p) where possible. In the following we show error bounds for an arbitrary parameter $\varepsilon > 0$. However, keep in mind the ultimate intention is to show convergence, for which we choose this parameter to be $\varepsilon^{(p)} = \mathrm{Var}\left(Z^{(p)}\right)^{1/3}$. Other notations which have been defined so far which are defined by $Z^{(p)}$ and therefore also implicitly depend on p are $\boldsymbol{\alpha}^{(p)}$, $\boldsymbol{S}^{(p)}$, $\boldsymbol{s}^{(p)}$, $\boldsymbol{S}_i^{(p)}$, $\boldsymbol{s}_i^{(p)}$, $\boldsymbol{D}^{(p)}$, $\mathcal{A}^{(p)}$.

In the following we show various results which involve integrating a function g, or a sequence of functions $g_1, g_2, \ldots$. We make the following assumptions about such functions,

Assumptions 3.0.1. *Let g be a function $g : [0,\infty) \to [0,\infty)$ which is*
(i) non-negative,

$$g(x) \geq 0 \text{ for all } x \geq 0,$$

(ii) bounded,

$$g(x) \leq G < \infty \text{ for all } x \geq 0,$$

(iii) integrable,

$$\int_{x=0}^{\infty} g(x)\,\mathrm{d}x \leq \widehat{G} < \infty,$$

(iv) and Lipschitz continuous

$$|g(x) - g(u)| \leq L|x-u| \text{ for all } x,\, u \geq 0. \tag{3.1}$$

We also need a corresponding sequence of closing operators which we denote by $V^{(p)}$. For the convergence results, we require the following properties of the closing operators $V^{(p)}(x)$, $x \in [0.\Delta)$.

Properties 3.0.2. *Let $\{V^{(p)}(x)\}_{p\geq 1}$ be a sequence of closing operators.*
(i) For $x \in [0,\Delta), u \geq 0$,

$$\boldsymbol{\alpha}^{(p)} e^{\boldsymbol{S}^{(p)}u} (-\boldsymbol{S}^{(p)})^{-1} V^{(p)}(x) \leq \boldsymbol{\alpha}^{(p)} e^{\boldsymbol{S}^{(p)}u} \boldsymbol{e} G_V,$$

for some $0 \leq G_V < \infty$ independent of $p > p_0$ for some $p_0 < \infty$.
(ii) Let g be a function satisfying the Assumptions 3.0.1. For $u \leq \Delta - \varepsilon^{(p)}$, $v \in [0,\Delta)$, then

$$\left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha}^{(p)} e^{\boldsymbol{S}^{(p)}(u+x)}}{\boldsymbol{\alpha}^{(p)} e^{\boldsymbol{S}^{(p)}u} \boldsymbol{e}} V^{(p)}(v) g(x)\,\mathrm{d}x - g(\Delta - u - v) 1(u+v \leq \Delta - \varepsilon^{(p)}) \right| = |r_V^{(p)}(u,v)|,$$

where

$$\int_{u=0}^{\Delta} \left| r_V^{(p)}(u,v) \right| \mathrm{d}u \leq R_{V,1}^{(p)} \to 0$$

and

$$\int_{v=0}^{\Delta} \left| r_V^{(p)}(u,v) \right| \mathrm{d}v \leq R_{V,2}^{(p)} \to 0$$

as $\mathrm{Var}(Z^{(p)}) \to 0$.

For convenience, in the sequel, let us adopt the notation

$$\mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \phi^{(p)}(0) = i) \tag{3.2}$$

for an approximation to

$$\mathbb{P}(X(t) \in \mathrm{d}x, \varphi(t) = j \mid X(0) = x_0, \varphi(0) = i), \tag{3.3}$$

$x \in \mathcal{D}_{\ell,j}$.

Lemma A.2.2 and Corollary A.2.1 in Appendix A.1 show that the operator (2.20) satisfies properties 3.0.2(i) and 3.0.2(ii). Lemma A.2.4 and Corollary A.2.5 in Appendix A.1 show that the operator (2.21) satisfies properties 3.0.2(i) and 3.0.2(ii).

Ultimately, we will apply the Extended Continuity Theorem for Laplace transforms (Feller 1957 - 1971, Chapter XIII, Theorem 2a) to claim convergence. The Extended Continuity Theorem for Laplace transforms requires us to show convergence of the Laplace transform pointwise with respect to the Laplace transform parameter, λ. Therefore, we can take $\lambda > 0$ fixed the the following sections.

The structure of this chapter is as follows. First, in Sections 3.1 we analyse the behaviour of the fluid queue and QBD-RAP on the event that they stay in the same level. Then, in Section 3.2, we derive expressions for certain Laplace transforms with respect to time of the QBD-RAP and fluid queue on the event that there is no change of level. Section 3.3 provides error bounds between the Laplace transforms of the QBD-RAP and the fluid queue that were derived in Section 3.2 and also a domination conditions used so that we may apply the Dominated Convergence Theorem. At this stage we have shown weak convergence of the approximation up to the first change of level. The subsequent sections generalise this to global convergence. We start by showing weak convergence of the approximation at the nth change of level, $n \geq 1$ in Section 3.4. Section 3.5 the generalises this to show weak convergence between the nth and $n+1$th change of level. Via the Dominated Convergence Theorem (again), Section 3.6 proves the weak convergence of the QBD-RAP approximation to the fluid queue. So as to not obscure the logical flow of the presentation, technical proofs are reserved for Appendix A.

3.1 Characterisations on no change of level

3.1.1 Characterising the fluid queue

Let τ_1^X be the minimum of the time at which $\{X(t)\}$ exits $\mathcal{D}_{\ell_0}$ for the first time, where $X(0) = x_0 \in \mathcal{D}_{\ell_0}$, or $\{X(t)\}$ hits a boundary, or $\{X(t)\}$ exits a boundary. More precisely,

$$\tau_1^X = \min \left\{ \begin{array}{c} \inf\{t > 0 \mid X(t) = y_\ell, \ell \in \mathcal{K}\}, \\ \inf\{t > 0 \mid X(t) \neq 0, X(0) = 0\}, \\ \inf\{t > 0 \mid X(t) \neq y_{K+1}, X(0) = y_{K+1}\} \end{array} \right\}.$$

Consider the measures

$$\mu^{\ell_0}(t)(\mathrm{d}x, j; x_0, i) := \mathbb{P}(X(t) \in \mathrm{d}x, \tau_1^X > t, s \in [0,t], \varphi(t) = j \mid X(0) = x_0, \varphi(0) = i), \tag{3.4}$$

$\ell_0 \in \{0, \dots, K\}$, $x, x_0 \in \mathcal{D}_{\ell_0,i}, i, j \in \mathcal{S}, t \geq 0$. In words, this is the distribution of the fluid queue at time t on the event that the fluid level remains within $\mathcal{D}_{\ell_0}$ up to and including time t and is in phase j at time t, given that is started at $X(0) = x_0 \in \mathcal{D}_{\ell_0,i}$ in phase i.

The QBD-RAP approximation to (3.4) is

$$(\boldsymbol{e}_i \otimes \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0)) \exp\left(\boldsymbol{B}^{(p)} t\right) (\boldsymbol{e}_j \otimes V^{(p)}(y_{\ell_0+1} - x))\,\mathrm{d}x,\; j \in \mathcal{S}_+ \cup \mathcal{S}_{+0} \tag{3.5}$$

where

$$\boldsymbol{B}^{(p)} = \begin{bmatrix} \boldsymbol{C}_+ \otimes \boldsymbol{S}^{(p)} + \boldsymbol{T}_{++} \otimes \boldsymbol{I} & \boldsymbol{T}_{+-} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{T}_{+0} \otimes \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T}_{-+} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{C}_- \otimes \boldsymbol{S}^{(p)} + \boldsymbol{T}_{--} \otimes \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{T}_{-0} \otimes \boldsymbol{I} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{I} & \boldsymbol{T}_{0-} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} & \boldsymbol{0} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{T}_{0-} \otimes \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} \end{bmatrix}.$$

For convenience, and without loss of generality, we reorder the rows and columns of $\boldsymbol{B}^{(p)}$ and partition $\boldsymbol{B}^{(p)}$ into

$$\boldsymbol{B}^{(p)} = \begin{bmatrix} \boldsymbol{B}^{(p)}_{++} & \boldsymbol{B}^{(p)}_{+-} \\ \boldsymbol{B}^{(p)}_{-+} & \boldsymbol{B}^{(p)}_{--} \end{bmatrix} \tag{3.6}$$

where

$$\boldsymbol{B}^{(p)}_{++} = \begin{bmatrix} \boldsymbol{C}_+ \otimes \boldsymbol{S}^{(p)} + \boldsymbol{T}_{++} \otimes \boldsymbol{I} & \boldsymbol{T}_{+0} \otimes \boldsymbol{I} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{I} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} \end{bmatrix}, \boldsymbol{B}^{(p)}_{+-} = \begin{bmatrix} \boldsymbol{T}_{+-} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \\ \boldsymbol{T}_{0-} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \end{bmatrix},$$
$$\boldsymbol{B}^{(p)}_{-+} = \begin{bmatrix} \boldsymbol{T}_{-+} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \\ \boldsymbol{T}_{0+} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \end{bmatrix}, \boldsymbol{B}^{(p)}_{--} = \begin{bmatrix} \boldsymbol{C}_- \otimes \boldsymbol{S}^{(p)} - \boldsymbol{T}_{--} \otimes \boldsymbol{I} & \boldsymbol{T}_{-0} \otimes \boldsymbol{I} \\ \boldsymbol{T}_{0-} \otimes \boldsymbol{I} & \boldsymbol{T}_{00} \otimes \boldsymbol{I} \end{bmatrix}.$$

For $m \in \{+, -, 0\}$, $n \in \{+, -\}$, $m \neq n$, $j \in \mathcal{S}_n$, denote $\boldsymbol{T}_{mj} = \boldsymbol{T}_{mn}(\boldsymbol{e}_j \boldsymbol{e}'_j)$ and for $m, n \in \{+, -\}$, $m \neq n$, $j \in \mathcal{S}_n$, denote

$$\boldsymbol{B}^{(p)}_{mj} = \boldsymbol{B}^{(p)}_{mn} \boldsymbol{e}_j \boldsymbol{e}'_j = \begin{bmatrix} \boldsymbol{T}_{mj} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \\ \boldsymbol{T}_{0j} \otimes \boldsymbol{D}^{(p)} & \boldsymbol{0} \end{bmatrix}. \tag{3.7}$$

There is no simple expression for (3.4). There are expressions for the Laplace transform of (3.4) with respect to time. One is in terms of the of first return matrices $\Psi(\lambda)$ and $\Xi(\lambda)$ (Bean et al. 2009*b*). Here we opt for another expression for the Laplace transform which is obtained by partitioning as follows.

Suppose $i \in \mathcal{S}_+, j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$. The arguments for all other combinations of $i, j \in \mathcal{S}$ are analogous and notable differences will be pointed out where necessary. There are some issues pertaining to starting the QBD-RAP in phases $i \in \mathcal{S}_{+0}$, hence the restriction $\varphi(0) = i \in \mathcal{S}_+$ only. We will treat the case $i \in \mathcal{S}_0$ separately and in due course.

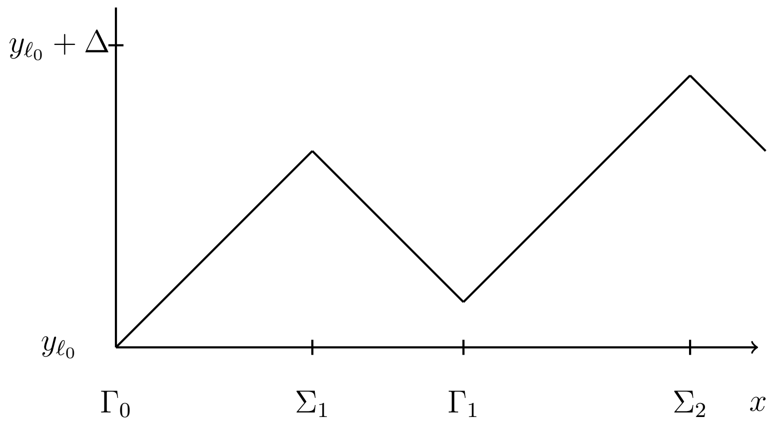

Figure 3.1: Sample paths corresponding to the summands in (3.11).

Denote by Σ_m, $m \geq 1$ the sequence of (stopping) times at which $\{\varphi(t)\}$ jumps from $\mathcal{S}_+ \cup \mathcal{S}_{+0}$ to $\mathcal{S}_-$ for the mth time. Denote by $\Gamma_m, m \geq 1$ the sequence of (stopping) times at which $\{\varphi(t)\}$ jumps from $\mathcal{S}_- \cup \mathcal{S}_{-0}$ to $\mathcal{S}_+$ for the mth time, and let $\Gamma_0 = 0$. More precisely,

$$\Sigma_m := \inf\{t > \Gamma_{m-1} \mid \varphi(t) \in \mathcal{S}_-\}, \tag{3.8}$$

$$\Gamma_m := \inf\{t > \Sigma_m \mid \varphi(t) \in \mathcal{S}_+\}. \tag{3.9}$$

$m \geq 1$. For times t such that $\Gamma_m \leq t < \Sigma_{m+1}$, then $\varphi(t) \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$. For times t such that $\Sigma_{m+1} \leq t < \Gamma_{m+1}$, then $\varphi(t) \in \mathcal{S}_- \cup \mathcal{S}_{-0}$. The events $\{\Gamma_m \leq t < \Sigma_{m+1}\}$, and $\{\Sigma_{m+1} \leq t < \Gamma_{m+1}\}$, $m \geq 0$, partition the sample paths of (3.4) into periods where the fluid is either non-decreasing or non-increasing, respectively; see Figure 3.1.

Using the law of total probability, we may write (3.4) with $i \in \mathcal{S}_+$, $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$, as

$$\sum_{m=0}^{\infty} \mathbb{P}(X(t) \in \mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Gamma_m \leq t < \Sigma_{m+1} \mid X(0) = x_0, \varphi(0) = i), \tag{3.10}$$

We can further partition by including the phases at times $\Sigma_1, \Gamma_1, \Sigma_2, \Gamma_2, \ldots$, i.e.

$$\sum_{m=0}^{\infty} \sum_{j_1 \in \mathcal{S}_-} \sum_{k_1 \in \mathcal{S}_+} \cdots \sum_{j_m \in \mathcal{S}_-} \sum_{k_m \in \mathcal{S}_+} \mathbb{P}\Big(X(t) \in \mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Gamma_m \leq t < \Sigma_{m+1}, \varphi(\Sigma_\ell) = j_\ell,$$
$$\varphi(\Gamma_\ell) = k_\ell, \ell = 1, \ldots, m \mid X(0) = x_0, \varphi(0) = i\Big). \tag{3.11}$$

Define $\mu^{\ell_0}_{m,+,+}(t)(\mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)$ by

$$\mathbb{P}\Big(X(t) \in \mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Gamma_m \leq t < \Sigma_{m+1}, \varphi(\Sigma_\ell) = j_\ell, \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \ldots, m$$

$$\mid X(0) = x_0, \varphi(0) = i\Big), \tag{3.12}$$

$i \in \mathcal{S}_+$, $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$, $j_1, j_2, \dots \in \mathcal{S}_-$, $k_1, k_2, \dots \in \mathcal{S}_+$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $x \in \mathcal{D}_{\ell_0,i}$, $t \geq 0$, $\ell_0 \in \{0, \dots, K\}$, $m \geq 0$, which are the summands in the sums (3.11). Analogously, define measures

$$\begin{aligned}&\mu^{\ell_0}_{m,+,-}(t)(\,\mathrm{d}x, j_1, k_1, \dots, j_m, k_m, j_{m+1}, j; x_0, i)\\&= \mathbb{P}\Big(X(t) \in \,\mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Sigma_{m+1} \leq t < \Gamma_{m+1}, \varphi(\Sigma_{m+1}) = j_{m+1}, \varphi(\Sigma_\ell) = j_\ell,\\&\qquad \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \dots, m \mid X(0) = x_0, \varphi(0) = i\Big),\end{aligned} \tag{3.13}$$

for $i \in \mathcal{S}_+$, $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$, $m \geq 0$;

$$\begin{aligned}&\mu^{\ell_0}_{m,-,+}(t)(\,\mathrm{d}x, k_1, j_1, \dots, j_m, k_{m+1}, j; x_0, i)\\&= \mathbb{P}\Big(X(t) \in \,\mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Gamma_{m+1} \leq t < \Sigma_{m+1}, \varphi(\Gamma_{m+1}) = k_{m+1}, \varphi(\Sigma_\ell) = j_\ell,\\&\qquad \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \dots, m \mid X(0) = x_0, \varphi(0) = i\Big),\end{aligned} \tag{3.14}$$

for $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$;

$$\begin{aligned}&\mu^{\ell_0}_{m,-,-}(t)(\,\mathrm{d}x, k_1, j_1, \dots, k_m, j_m, j; x_0, i)\\&= \mathbb{P}\Big(X(t) \in \,\mathrm{d}x, \tau_1^X > t, \varphi(t) = j, \Sigma_m \leq t < \Gamma_{m+1}, \varphi(\Sigma_\ell) = j_\ell, \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \dots, m\\&\qquad \mid X(0) = x_0, \varphi(0) = i\Big),\end{aligned} \tag{3.15}$$

for $i \in \mathcal{S}_-$, $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$, where in each case $j_1, j_2, \dots \in \mathcal{S}_-$, $k_1, k_2, \dots \in \mathcal{S}_+$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $x \in \mathcal{D}_{\ell_0,j}$, $t \geq 0$, $\ell_0 \in \{0, \dots, K\}$, $m \geq 0$. For later, also define

$$\mu^{\ell_0}_{m,+,+}(t)(\,\mathrm{d}x, j; x_0, i) = \sum_{j_1 \in \mathcal{S}_-} \sum_{k_1 \in \mathcal{S}_+} \cdots \sum_{j_m \in \mathcal{S}_-} \sum_{k_m \in \mathcal{S}_+} \mu^{\ell_0}_{m,+,+}(t)(\,\mathrm{d}x, j_1, k_1, \dots, j_m, k_m, j; x_0, i) \tag{3.16}$$

$$\begin{aligned}&\mu^{\ell_0}_{m,+,-}(t)(\,\mathrm{d}x, j; x_0, i)\\&\quad= \sum_{j_1 \in \mathcal{S}_-} \sum_{k_1 \in \mathcal{S}_+} \cdots \sum_{k_m \in \mathcal{S}_+} \sum_{j_{m+1} \in \mathcal{S}_-} \mu^{\ell_0}_{m,+,-}(t)(\,\mathrm{d}x, j_1, k_1, \dots, k_m, j_{m+1}, j; x_0, i)\end{aligned} \tag{3.17}$$

$$\begin{aligned}&\mu^{\ell_0}_{m,-,+}(t)(\,\mathrm{d}x, j; x_0, i)\\&\quad= \sum_{k_1 \in \mathcal{S}_+} \sum_{j_1 \in \mathcal{S}_-} \cdots \sum_{j_m \in \mathcal{S}_-} \sum_{k_{m+1} \in \mathcal{S}_+} \mu^{\ell_0}_{m,-,+}(t)(\,\mathrm{d}x, k_1, j_1, \dots, j_m, k_{m+1}, j; x_0, i)\end{aligned} \tag{3.18}$$

$$\mu^{\ell_0}_{m,-,-}(t)(\,\mathrm{d}x, j; x_0, i) = \sum_{k_1 \in \mathcal{S}_+} \sum_{j_1 \in \mathcal{S}_-} \cdots \sum_{k_m \in \mathcal{S}_+} \sum_{j_m \in \mathcal{S}_-} \mu^{\ell_0}_{m,-,-}(t)(\,\mathrm{d}x, k_1, j_1, \dots, k_m, j_m, j; x_0, i) \tag{3.19}$$

and

$$\mu^{\ell_0}_{+,+}(t)(\,\mathrm{d}x,j;x_0,i) := \sum_{m=0}^{\infty} \mu^{\ell_0}_{m,+,+}(t)(\,\mathrm{d}x,j;x_0,i) \qquad i\in\mathcal{S}_+,\ j\in\mathcal{S}_+\cup\mathcal{S}_{+0}, \tag{3.20}$$

$$\mu^{\ell_0}_{+,-}(t)(\,\mathrm{d}x,j;x_0,i) := \sum_{m=1}^{\infty} \mu^{\ell_0}_{m,+,-}(t)(\,\mathrm{d}x,j;x_0,i) \qquad i\in\mathcal{S}_+,\ j\in\mathcal{S}_-\cup\mathcal{S}_{-0}, \tag{3.21}$$

$$\mu^{\ell_0}_{-,+}(t)(\,\mathrm{d}x,j;x_0,i) := \sum_{m=1}^{\infty} \mu^{\ell_0}_{m,-,+}(t)(\,\mathrm{d}x,j;x_0,i) \qquad i\in\mathcal{S}_-,\ j\in\mathcal{S}_+\cup\mathcal{S}_{+0}, \tag{3.22}$$

$$\mu^{\ell_0}_{-,-}(t)(\,\mathrm{d}x,j;x_0,i) := \sum_{m=0}^{\infty} \mu^{\ell_0}_{m,-,-}(t)(\,\mathrm{d}x,j;x_0,i) \qquad i\in\mathcal{S}_-,\ j\in\mathcal{S}_-\cup\mathcal{S}_{-0}. \tag{3.23}$$

3.1.2 Characterising the QBD-RAP

Let $\tau_1^{(p)}$ be the random (stopping) time at which the QBD-RAP changes level, or hits the boundary, or exits a boundary, for the first time;

$$\tau_1^{(p)} = \inf\left\{t>0 \mid L^{(p)}(t)\neq L^{(p)}(0)\right\}.$$

According as the approximation described, the summands (3.12) are approximated by the densities

$$f^{\ell_0,(p)}_{m,+,+}(t)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i)\,\mathrm{d}x,$$

for $i\in\mathcal{S}_+$, $j\in\mathcal{S}_+\cup\mathcal{S}_{+0}$, $j_1,\dots,j_k\in\mathcal{S}_-$, $k_1,\dots,k_m\in\mathcal{S}_+$, $x_0\in\mathcal{D}_{\ell_0,i}$, $x\in\mathcal{D}_{\ell_0,j}$, $t\geq 0$, $\ell_0\in\{0,\dots,K\}$, $m\geq 0$, which are given by

$$\begin{aligned}
&\int_{\boldsymbol{a}\in\mathcal{A}^{(p)}} \mathbb{P}\Big(\boldsymbol{A}^{(p)}(t)\in\,\mathrm{d}\boldsymbol{a}, t<\tau_1^{(p)}, \varphi(t)=j, \Gamma_m\leq t<\Sigma_{m+1}, \varphi(\Sigma_\ell)=j_\ell, \varphi(\Gamma_\ell)=k_\ell,\\
&\qquad \ell=1,\dots,m \mid \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \varphi(0)=i\Big)\boldsymbol{a}V^{(p)}(y_{\ell_0+1}-x)\,\mathrm{d}x\\
&= \int_{\sigma_1=0}^{t} (\boldsymbol{e}_i\otimes\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0))e^{\boldsymbol{B}^{(p)}_{++}\sigma_1}\boldsymbol{B}^{(p)}_{+j_1}\int_{\gamma_1=\sigma_1}^{t} e^{\boldsymbol{B}^{(p)}_{--}(\gamma_1-\sigma_1)}\boldsymbol{B}^{(p)}_{-k_1}\dots\int_{\gamma_m=\sigma_m}^{t} e^{\boldsymbol{B}^{(p)}_{--}(\gamma_m-\sigma_m)}\boldsymbol{B}^{(p)}_{-k_m}\\
&\quad\times e^{\boldsymbol{B}^{(p)}_{++}(t-\gamma_m)}\left(\boldsymbol{e}_j\otimes V^{(p)}(y_{\ell_0+1}-x)\right)\,\mathrm{d}x\,\mathrm{d}\sigma_1\,\mathrm{d}\gamma_1\dots\,\mathrm{d}\sigma_m\,\mathrm{d}\gamma_m.
\end{aligned} \tag{3.24}$$

Define analogously,

$$\begin{aligned}
&f^{\ell_0,(p)}_{m,+,-}(t)(x,j_1,k_1,\dots,j_m,k_m,j_{m+1},j;x_0,i)\\
&= \int_{\boldsymbol{a}\in\mathcal{A}^{(p)}} \mathbb{P}\Big(\boldsymbol{A}^{(p)}(t)\in\,\mathrm{d}\boldsymbol{a}, t<\tau_1^{(p)}, \varphi(t)=j, \Sigma_{m+1}\leq t<\Gamma_{m+1}, \varphi(\Sigma_{m+1})=j_{m+1},\\
&\qquad \varphi(\Sigma_\ell)=j_\ell, \varphi(\Gamma_\ell)=k_\ell, \ell=1,\dots,m \mid \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \varphi(0)=i\Big)
\end{aligned}$$

$$\times \boldsymbol{a}V^{(p)}(y_{\ell_0+1} - x)\,\mathrm{d}x \tag{3.25}$$

for $i \in \mathcal{S}_+,\ j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$;

$$\begin{aligned} & f^{\ell_0,(p)}_{m,-,+}(t)(x, k_1, j_1, \dots, j_m, k_{m+1}, j; x_0, i) \\ &= \int_{\boldsymbol{a}\in\mathcal{A}^{(p)}} \mathbb{P}\Big(\boldsymbol{A}^{(p)}(t) \in \mathrm{d}\boldsymbol{a}, t < \tau_1^{(p)}, \varphi(t) = j, \Gamma_{m+1} \le t < \Sigma_{m+1}, \varphi(\Gamma_{m+1}) = k_{m+1}, \\ &\quad \varphi(\Sigma_\ell) = j_\ell, \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \dots, m \mid \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \varphi(0) = i\Big) \\ &\quad \times \boldsymbol{a}V^{(p)}(y_{\ell_0+1} - x)\,\mathrm{d}x \end{aligned} \tag{3.26}$$

for $i \in \mathcal{S}_-,\ j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$;

$$\begin{aligned} & f^{\ell_0,(p)}_{m,-,-}(t)(x, k_1, j_1, \dots, k_m, j_m, j; x_0, i) \\ &= \int_{\boldsymbol{a}\in\mathcal{A}^{(p)}} \mathbb{P}\Big(\boldsymbol{A}^{(p)}(t) \in \mathrm{d}\boldsymbol{a}, t < \tau_1^{(p)}, \varphi(t) = j, \Sigma_m \le t < \Gamma_{m+1}, \varphi(\Sigma_\ell) = j_\ell, \varphi(\Gamma_\ell) = k_\ell, \\ &\quad \ell = 1, \dots, m \mid \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \varphi(0) = i\Big) \boldsymbol{a}V^{(p)}(y_{\ell_0+1} - x)\,\mathrm{d}x \end{aligned} \tag{3.27}$$

for $i \in \mathcal{S}_-, j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$. Expressions analgous to that on the right-hand side of (3.24) can be written down for (3.25)-(3.27).

The events $\{\Gamma_m \le t < \Sigma_{m+1}\}$, and $\{\Sigma_{m+1} \le t < \Gamma_{m+1}\}$, $m \ge 0$, and the phases at these times, $\varphi(\Gamma_\ell)$ and $\varphi(\Sigma_\ell)$, also form a partition of the sample paths of the QBD-RAP. On these sample paths, the phase of the QBD-RAP changes from $\mathcal{S}_+$ to $j_m \in \mathcal{S}_-$ at times Σ_m and from $\mathcal{S}_-$ to $k_m \in \mathcal{S}_+$ at times Γ_m.

Define

$$\begin{aligned} & f^{\ell_0,(p)}_{m,+,+}(t)(x, j; x_0, i)\,\mathrm{d}x \\ &\qquad = \sum_{j_1\in\mathcal{S}_-} \sum_{k_1\in\mathcal{S}_+} \cdots \sum_{j_m\in\mathcal{S}_-} \sum_{k_m\in\mathcal{S}_+} f^{\ell_0,(p)}_{m,+,+}(t)(x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\,\mathrm{d}x, \end{aligned} \tag{3.28}$$

$$\begin{aligned} & f^{\ell_0,(p)}_{m,+,-}(t)(x, j; x_0, i)\,\mathrm{d}x \\ &= \sum_{j_1\in\mathcal{S}_-} \sum_{k_1\in\mathcal{S}_+} \cdots \sum_{k_m\in\mathcal{S}_+} \sum_{j_{m+1}\in\mathcal{S}_-} f^{\ell_0,(p)}_{m+1,+,-}(t)(x, j_1, k_1, \dots, k_m, j_{m+1}, j; x_0, i)\,\mathrm{d}x, \end{aligned} \tag{3.29}$$

$$\begin{aligned} & f^{\ell_0,(p)}_{m,-,+}(t)(x, j; x_0, i)\,\mathrm{d}x \\ &= \sum_{k_1\in\mathcal{S}_+} \sum_{j_1\in\mathcal{S}_-} \cdots \sum_{j_m\in\mathcal{S}_-} \sum_{k_{m+1}\in\mathcal{S}_+} f^{\ell_0,(p)}_{m,-,+}(t)(x, k_1, j_1, \dots, j_m, k_{m+1}, j; x_0, i)\,\mathrm{d}x, \end{aligned} \tag{3.30}$$

$$\begin{aligned} & f^{\ell_0,(p)}_{m,-,-}(t)(x, j; x_0, i)\,\mathrm{d}x \\ &\qquad = \sum_{k_1\in\mathcal{S}_+} \sum_{j_1\in\mathcal{S}_-} \cdots \sum_{k_m\in\mathcal{S}_+} \sum_{j_m\in\mathcal{S}_-} f^{\ell_0,(p)}_{m,-,-}(t)(x, k_1, j_1, \dots, k_m, j_m, j; x_0, i)\,\mathrm{d}x. \end{aligned} \tag{3.31}$$

and

$$
\begin{aligned}
f^{\ell_0,(p)}_{+,+}(t)(x,j;x_0,i)\,\mathrm{d}x &:= \sum_{m=0}^{\infty} f^{\ell_0,(p)}_{m,+,+}(t)(x,j;x_0,i)\,\mathrm{d}x && i\in\mathcal{S}_+,\ j\in\mathcal{S}_+\cup\mathcal{S}_{+0},\\
f^{\ell_0,(p)}_{+,-}(t)(x,j;x_0,i)\,\mathrm{d}x &:= \sum_{m=1}^{\infty} f^{\ell_0,(p)}_{m,+,-}(t)(x,j;x_0,i)\,\mathrm{d}x && i\in\mathcal{S}_+,\ j\in\mathcal{S}_0\cup\mathcal{S}_{-0},\\
f^{\ell_0,(p)}_{-,+}(t)(x,j;x_0,i) &:= \sum_{m=1}^{\infty} f^{\ell_0,(p)}_{m,-,+}(t)(x,j;x_0,i)\,\mathrm{d}x && i\in\mathcal{S}_-,\ j\in\mathcal{S}_+\cup\mathcal{S}_{+0},\\
f^{\ell_0,(p)}_{-,-}(t)(x,j;x_0,i)\,\mathrm{d}x &:= \sum_{m=0}^{\infty} f^{\ell_0,(p)}_{m,-,-}(t)(x,j;x_0,i)\,\mathrm{d}x && i\in\mathcal{S}_-,\ j\in\mathcal{S}_-\cup\mathcal{S}_{-0}.
\end{aligned}
$$

which are the corresponding approximations of (3.16)-(3.23). The sum (3.28) is

$$
\begin{aligned}
&\int_{\sigma_1=0}^{t} (\boldsymbol{e}_i\otimes\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0))e^{\boldsymbol{B}^{(p)}_{++}\sigma_1}\boldsymbol{B}^{(p)}_{+-}\int_{\gamma_1=\sigma_1}^{t} e^{\boldsymbol{B}^{(p)}_{--}(\gamma_1-\sigma_1)}\boldsymbol{B}^{(p)}_{-+}\dots\int_{\gamma_m=\sigma_m}^{t} e^{\boldsymbol{B}^{(p)}_{--}(\gamma_m-\sigma_m)}\\
&\quad\times\boldsymbol{B}^{(p)}_{-+}e^{\boldsymbol{B}^{(p)}_{++}(t-\gamma_m)}\big(\boldsymbol{e}_j\otimes V^{(p)}(y_{\ell_0+1}-x)\big)\,\mathrm{d}x\,\mathrm{d}\sigma_1\,\mathrm{d}\gamma_1\dots\mathrm{d}\sigma_m\,\mathrm{d}\gamma_m. \qquad (3.32)
\end{aligned}
$$

Analogous expressions can be written down for (3.28)-(3.31). For simplicity, we may drop the superscript (p).

3.2 Laplace transforms with respect to time on no change of level

For $\lambda\geq 0$, define the matrices

$$
\begin{aligned}
\boldsymbol{Q}_{+0}(\lambda) &= \boldsymbol{C}_+^{-1}\boldsymbol{T}_{+0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1},\\
\boldsymbol{Q}_{-0}(\lambda) &= \boldsymbol{C}_-^{-1}\boldsymbol{T}_{-0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1},\\
\boldsymbol{Q}_{++}(\lambda) &= \boldsymbol{C}_+^{-1}\left(\boldsymbol{T}_{++}-\lambda\boldsymbol{I}+\boldsymbol{T}_{+0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}\boldsymbol{T}_{0+}\right),\\
\boldsymbol{Q}_{+-}(\lambda) &= \boldsymbol{C}_+^{-1}\left(\boldsymbol{T}_{+-}+\boldsymbol{T}_{+0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}\boldsymbol{T}_{0-}\right),\\
\boldsymbol{Q}_{--}(\lambda) &= \boldsymbol{C}_-^{-1}\left(\boldsymbol{T}_{--}-\lambda\boldsymbol{I}+\boldsymbol{T}_{-0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}\boldsymbol{T}_{0-}\right),\\
\boldsymbol{Q}_{-+}(\lambda) &= \boldsymbol{C}_-^{-1}\left(\boldsymbol{T}_{-+}+\boldsymbol{T}_{-0}\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}\boldsymbol{T}_{0+}\right),
\end{aligned}
$$

and the functions,

$$
\boldsymbol{H}^{++}(\lambda,x) := e^{\boldsymbol{Q}_{++}(\lambda)x}\begin{bmatrix}\boldsymbol{C}_+^{-1} & \boldsymbol{Q}_{+0}(\lambda)\end{bmatrix} = \left[h^{++}_{ij}(\lambda,x)\right]_{i\in\mathcal{S}_+,j\in\mathcal{S}_+\cup\mathcal{S}_{+0}}, \qquad (3.33)
$$

$$
\boldsymbol{H}^{--}(\lambda,x) := e^{\boldsymbol{Q}_{--}(\lambda)x}\begin{bmatrix}\boldsymbol{C}_-^{-1} & \boldsymbol{Q}_{-0}(\lambda)\end{bmatrix} = \left[h^{--}_{ij}(\lambda,x)\right]_{i\in\mathcal{S}_-,j\in\mathcal{S}_-\cup\mathcal{S}_{-0}}, \qquad (3.34)
$$

$$\boldsymbol{H}^{+-}(\lambda, x) := e^{\boldsymbol{Q}_{++}(\lambda)x}\boldsymbol{Q}_{+-}(\lambda) = \left[h_{ij}^{+-}(\lambda, x)\right]_{i\in\mathcal{S}_+,\, j\in\mathcal{S}_-}, \tag{3.35}$$

$$\boldsymbol{H}^{-+}(\lambda, x) := e^{\boldsymbol{Q}_{--}(\lambda)x}\boldsymbol{Q}_{-+}(\lambda) = \left[h_{ij}^{-+}(\lambda, x)\right]_{i\in\mathcal{S}_-,\, j\in\mathcal{S}_+}, \tag{3.36}$$

for $x, \lambda \geq 0$. The function $h_{ij}^{++}(\lambda, x)$ ($h_{ij}^{--}(\lambda, x)$) is the Laplace transform with respect to time of the time taken for the fluid level to shift by an amount x whilst remaining in phases in $\mathcal{S}_+ \cup \mathcal{S}_{+0}$ ($\mathcal{S}_- \cup \mathcal{S}_{-0}$), given the phase was initially $i \in \mathcal{S}_+$ ($i \in \mathcal{S}_-$) (Bean et al. 2005*b*). The function $h_{ij}^{+-}(\lambda, x)$ ($h_{ij}^{-+}(\lambda, x)$) is the Laplace transform with respect to time of the time taken for the fluid level, $\{X(t)\}$ to shift by an amount x whilst remaining in phases in $\mathcal{S}_+ \cup \mathcal{S}_{+0}$ ($\mathcal{S}_- \cup \mathcal{S}_{-0}$), after which time the phase instantaneously changes to $j \in \mathcal{S}_-$ ($\mathcal{S}_+$), given the phase was initially $i \in \mathcal{S}_+$ ($\mathcal{S}_-$) (Bean et al. 2005*b*).

Let $\psi : [0, \Delta) \to \mathbb{R}$ be bounded and Lipschitz continuous and consider expectations with respect to measures (3.12)-(3.15). For example,

$$\int_{x=0}^{\Delta} \mu_{m,+,+}^{\ell_0}(t)(y_{\ell_0} + \,\mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)\psi(x)\,\mathrm{d}x, \tag{3.37}$$

and similarly for (3.13)-(3.15). Consider taking the Laplace transform with respect to time of (3.37);

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-\lambda t} \int_{x=0}^{\Delta} \mu_{m,+,+}^{\ell_0}(t)(y_{\ell_0} + \,\mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)\psi(x)\,\mathrm{d}t \\
&= \int_{x=0}^{\Delta} \int_{t=0}^{\infty} e^{-\lambda t} \mu_{m,+,+}^{\ell_0}(t)(y_{\ell_0} + \,\mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)\,\mathrm{d}t\psi(x) \\
&= \int_{x=0}^{\Delta} \widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(y_{\ell_0} + x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)\psi(x)\,\mathrm{d}x
\end{aligned} \tag{3.38}$$

where we use $\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(y_{\ell_0} + \mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)$ to denote the Laplace transform with respect to time of (3.12). We now proceed to an expression for the Laplace transform with respect to time, $\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(\,\mathrm{d}x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)$, from which an expression for (3.38) follows.

From the stochastic interpretations of the Laplace transforms (3.33)-(3.36) given in (Bean et al. 2005*b*) and summarised above, the Laplace transforms with respect to time, $\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)$, of (3.12) are given by

$$\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(x, j; x_0, i)\,\mathrm{d}x = h_{ij}^{++}(\lambda, x - x_0)1(x \geq x_0)\,\mathrm{d}x,$$

for $m = 0$, and

$$\int_{x_1=0}^{\Delta-(x_0-y_{\ell_0})} h_{ij_1}^{+-}(\lambda, \Delta - (x_0 - y_{\ell_0}) - x_1)$$

$$\times \left[\prod_{r=1}^{m-1} \int_{x_{2r}=0}^{\Delta - x_{2r-1}} h_{j_r k_r}^{-+}(\lambda, \Delta - x_{2r} - x_{2r-1})\,\mathrm{d}x_{2r-1} \int_{x_{2r+1}=0}^{\Delta - x_{2r}} h_{k_r j_{r+1}}^{+-}(\lambda, \Delta - x_{2r+1} - x_{2r})\,\mathrm{d}x_{2r}\right]$$
$$\times \int_{x_{2m}=0}^{\Delta - x_{2m-1}} h_{j_m k_m}^{-+}(\lambda, \Delta - x_{2m-1} - x_{2m})\,\mathrm{d}x_{2m-1} h_{k_m j}^{++}(\lambda, \Delta - x_{2m} - (y_{\ell_0+1} - x))$$
$$\times 1(\Delta - x_{2m} - (y_{\ell_0+1} - x) \geq 0)\,\mathrm{d}x_{2m}\,\mathrm{d}x \tag{3.39}$$

for $m \geq 1$. Figure 3.2 shows an example of the sample paths to which these Laplace transforms correspond. Analogously, we can write down similar expressions for the Laplace transform with respect to time of (3.13)-(3.15). We use the hat $\widehat{\ }$ notation to denote Laplace transforms with respect to time.

Observe that $\widehat{\mu}_{0,+,+}^{\ell_0}(\lambda)(x, j; x_0, i)$ is a smooth function in the variable $x > x_0$ (and $\lambda > 0$ too), even though the measure $\mu_{0,+,+}^{\ell_0}(t)(\,\mathrm{d}x, j; x_0, i)$ may have point masses. For example, given $X(0) = x_0$ and $\varphi(0) = i$, then $\mu_{0,+,+}^{\ell_0}(t)(\,\mathrm{d}x, j; x_0, i)$ has a point mass of size (at least) $e^{T_{ii}t}\delta(x - (x_0 + c_i t))1(j = i)1(x \in \mathcal{D}_{\ell_0,i})$ which arises from the event that the phase remains i until time t. Meanwhile, the Laplace transform $\widehat{\mu}_{0,+,+}^{\ell_0}(\lambda)(\,\mathrm{d}x, j; x_0, i)$ is given by $\left[e^{\boldsymbol{Q}_{++}(\lambda)(x-x_0)}1(x \geq x_0)\right]_{ij}$, which is a smooth function for all x except at $x = x_0$.

The Laplace transform with respect to time of (3.16), $\widehat{\mu}_{m,+,+}^{\ell_0}(t)(\,\mathrm{d}x, j; x_0, i)$, is the (i, j)th entry of

$$\int_{x_1=0}^{\Delta - (x_0 - y_{\ell_0})} \boldsymbol{H}^{+-}(\lambda, \Delta - (x_0 - y_{\ell_0}) - x_1) \int_{x_2=0}^{\Delta - x_1} \boldsymbol{H}^{-+}(\lambda, \Delta - x_2 - x_1)\,\mathrm{d}x_1 \dots$$
$$\times \int_{x_{2m}=0}^{\Delta - x_{2m-1}} \boldsymbol{H}^{-+}(\lambda, \Delta - x_{2m-1} - x_{2m})\,\mathrm{d}x_{2m-1} \boldsymbol{H}^{++}(\lambda, \Delta - x_{2m} - (y_{\ell_0+1} - x))$$
$$\times 1(\Delta - x_{2m} - (y_{\ell_0+1} - x) \geq 0)\,\mathrm{d}x_{2m}\,\mathrm{d}x. \tag{3.40}$$

Analogous expressions can be written down for the Laplace transforms of (3.17)-(3.19).

Now consider expectations with respect to measures (3.24)-(3.27), i.e.

$$\int_{x=0}^{\Delta} f_{m,+,+}^{\ell_0,(p)}(t)(y_{\ell_0} + \,\mathrm{d}x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\psi(x), \tag{3.41}$$

and similarly for (3.25)-(3.27). Taking the Laplace transform with respect to time of (3.41);

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{x=0}^{\Delta} f_{m,+,+}^{\ell_0,(p)}(t)(y_{\ell_0} + x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{x=0}^{\Delta} \int_{t=0}^{\infty} e^{-\lambda t} f_{m,+,+}^{\ell_0,(p)}(t)(y_{\ell_0} + x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\,\mathrm{d}t \psi(x)\,\mathrm{d}x$$
$$= \int_{x=0}^{\Delta} \widehat{f}_{m,+,+}^{\ell_0,(p)}(\lambda)(y_{\ell_0} + x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\psi(x)\,\mathrm{d}x \tag{3.42}$$

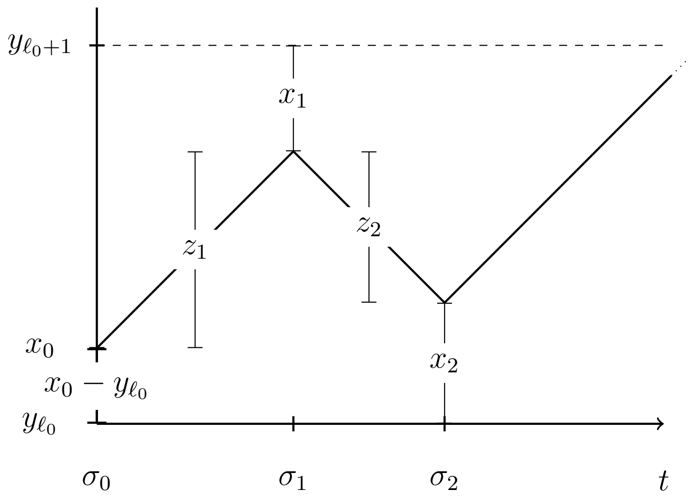

Figure 3.2: Sample paths corresponding to the Laplace transforms (3.39). $z_1 = \Delta - x_1 - (x - y_{\ell_0})$, $z_2 = \Delta - x_2 - x_1$.

where we use $\widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(y_{\ell_0} + x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)$ to denote the Laplace transform with respect to time of (3.12). We now proceed to an expression for the Laplace transform with respect to time, $\widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(\,\mathrm{d}x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)$, from which an expression for (3.42) follows.

Notice that (3.24) is a convolution. The Laplace transform with respect to time of (3.24) is

$$\begin{aligned}
&\widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x, j_1, k_1, \dots, j_m, k_m, j; x_0, i)\\
&= \int_{t=0}^{\infty} e^{-\lambda t} \int_{\boldsymbol{a}\in\mathcal{A}^{(p)}} \mathbb{P}\Big(\boldsymbol{A}^{(p)}(t) \in \,\mathrm{d}\boldsymbol{a}, t < \tau_1^{(p)}, \varphi(t) = j, \Gamma_m \le t < \Sigma_{m+1}, \varphi(\Sigma_\ell) = j_\ell,\\
&\qquad \varphi(\Gamma_\ell) = k_\ell, \ell = 1, \dots, m \mid \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \varphi(0) = i\Big) \boldsymbol{a} V^{(p)}(y_{\ell_0+1} - x)\,\mathrm{d}t\\
&= (\boldsymbol{e}_i \otimes \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0)) \int_{\sigma_1=0}^{\infty} e^{-\lambda\sigma_1} e^{\boldsymbol{B}^{(p)}_{++}\sigma_1}\,\mathrm{d}\sigma_1 \boldsymbol{B}^{(p)}_{+-} \boldsymbol{e}_{j_1} \boldsymbol{e}'_{j_1} \int_{\gamma_1=0}^{\infty} e^{-\lambda\gamma_1} e^{\boldsymbol{B}^{(p)}_{--}\gamma_1}\,\mathrm{d}\gamma_1 \boldsymbol{B}^{(p)}_{-+} \boldsymbol{e}_{k_1} \boldsymbol{e}'_{k_1}\\
&\quad \dots \int_{\gamma_m=0}^{\infty} e^{-\lambda t_m} e^{\boldsymbol{B}^{(p)}_{--}\gamma_m}\,\mathrm{d}\gamma_m \boldsymbol{B}^{(p)}_{-+} \boldsymbol{e}_{k_m} \boldsymbol{e}'_{k_m} \int_{t=0}^{\infty} e^{-\lambda t} e^{\boldsymbol{B}^{(p)}_{++}t}\,\mathrm{d}t \left(\boldsymbol{e}_j \boldsymbol{e}'_j\right) \left(\boldsymbol{I} \otimes V^{(p)}(y_{\ell_0+1} - x)\right).
\end{aligned} \tag{3.43}$$

Analogous expressions can be computed for the Laplace transforms with respect to time of (3.25)-(3.27).

In Lemma A.3.4 in Appendix A.3, we show the following relation

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix} \int_{t=0}^{\infty} e^{-\lambda t} e^{\boldsymbol{B}^{(p)}_{mm}t}\,\mathrm{d}t \boldsymbol{B}^{(p)}_{mn} = \int_{x=0}^{\infty} \left(\boldsymbol{H}^{mn}(\lambda, x) \begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0}_{n\times|\mathcal{S}_0|}\end{bmatrix}\right) \otimes e^{\boldsymbol{S}^{(p)}x} \boldsymbol{D}^{(p)}\,\mathrm{d}x, \tag{3.44}$$

for $m, n \in \{+, -\}$, $m \neq n$. Using (3.44) on all of the integrals in (3.43) gives

$$\int_{x_1=0}^{\infty} \left[\boldsymbol{H}^{+-}(\lambda, x_1)\right]_{i,j_1} \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) e^{\boldsymbol{S}^{(p)} x_1} \,\mathrm{d}x_1 \boldsymbol{D}^{(p)} \left[\prod_{r=1}^{m-1} \int_{x_{2r}=0}^{\infty} \left[\boldsymbol{H}^{-+}(\lambda, x_{2r})\right]_{j_r,k_r} e^{\boldsymbol{S}^{(p)} x_{2r}} \,\mathrm{d}x_{2r} \right.$$
$$\left. \boldsymbol{D}^{(p)} \int_{x_{2r+1}=0}^{\infty} \left[\boldsymbol{H}^{+-}(\lambda, x_{2r+1})\right]_{k_r,j_{r+1}} e^{\boldsymbol{S}^{(p)} x_{2r+1}} \,\mathrm{d}x_{2r+1} \boldsymbol{D}^{(p)} \right] \int_{x_{2m}=0}^{\infty} \left[\boldsymbol{H}^{-+}(\lambda, x_{2m})\right]_{j_m,k_m}$$
$$e^{\boldsymbol{S}^{(p)} x_{2m}} \,\mathrm{d}x_{2m} \boldsymbol{D}^{(p)} \int_{x_{2m+1}=0}^{\infty} \left[\boldsymbol{H}^{++}(\lambda, x_{2m+1})\right]_{k_m,j} e^{\boldsymbol{S}^{(p)} x_{2m+1}} \,\mathrm{d}x_{2m+1} V^{(p)}(y_{\ell_0+1} - x)$$
$$= \int_{x_1=0}^{\infty} h_{i,j_1}^{+-}(\lambda, x_1) \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) e^{\boldsymbol{S}^{(p)} x_1} \,\mathrm{d}x_1 \boldsymbol{D}^{(p)} \left[\prod_{r=1}^{m-1} \int_{x_{2r}=0}^{\infty} h_{j_r,k_r}^{-+}(\lambda, x_{2r}) e^{\boldsymbol{S}^{(p)} x_{2r}} \,\mathrm{d}x_{2r} \boldsymbol{D}^{(p)} \right.$$
$$\left. \int_{x_{2r+1}=0}^{\infty} h_{k_r j_{r+1}}^{+-}(\lambda, x_{2r+1}) e^{\boldsymbol{S}^{(p)} x_{2r+1}} \,\mathrm{d}x_{2r+1} \boldsymbol{D}^{(p)} \right] \int_{x_{2m}=0}^{\infty} h_{j_m,k_m}^{-+}(\lambda, x_{2m}) e^{\boldsymbol{S}^{(p)} x_{2m}} \,\mathrm{d}x_{2m} \boldsymbol{D}^{(p)}$$
$$\int_{x_{2m+1}=0}^{\infty} h_{k_m,j}^{++}(\lambda, x_{2m+1}) e^{\boldsymbol{S}^{(p)} x_{2m+1}} \,\mathrm{d}x_{2m+1} V^{(p)}(y_{\ell_0+1} - x)$$
$$= \int_{x_1=0}^{\infty} h_{i,j_1}^{+-}(\lambda, x_1) \int_{x_2=0}^{\infty} h_{j_1,k_1}^{-+}(\lambda, x_2) \dots \int_{x_{2m}=0}^{\infty} h_{j_m,k_m}^{-+}(\lambda, x_{2m}) \int_{x_{2m+1}=0}^{\infty} h_{k_m,j}^{++}(\lambda, x_{2m+1})$$
$$\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) e^{\boldsymbol{S}^{(p)} x_1} \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_2} \boldsymbol{D}^{(p)} \dots e^{\boldsymbol{S}^{(p)} x_{2m}} \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_{2m+1}} V^{(p)}(y_{\ell_0+1} - x)$$
$$\mathrm{d}x_{2m+1} \,\mathrm{d}x_{2m} \dots \mathrm{d}x_2 \,\mathrm{d}x_1. \tag{3.45}$$

Analogous expressions can be shown for (3.25)-(3.27). The relation (3.44) is key to our analysis. It allows us to factorise the integrand of the Laplace transform (3.45) into one factor solely related to the orbit process $\{\boldsymbol{A}^{(p)}(t)\}$ and another factor solely related to the fluid queue.

For $\varphi(0) = k \in \mathcal{S}_{-0}$ (or $\varphi(0) = k \in \mathcal{S}_{+0}$) there is a slight complexity as in this case it is possible that, upon the phase process first leaving $\mathcal{S}_{-0}$ ($\mathcal{S}_{+0}$) the phase transitions to a state in $\mathcal{S}_{+}$ ($\mathcal{S}_{-}$). Since the orbit of the QBD-RAP is constant on $\varphi(t) \in \mathcal{S}_{-0}$ ($\varphi(t) \in \mathcal{S}_{+0}$), then upon a first transition out of $\mathcal{S}_{-0}$ ($\mathcal{S}_{+0}$) and into $\mathcal{S}_{+}$ ($\mathcal{S}_{-}$) the orbit jumps to $\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0)\boldsymbol{D}^{(p)}$. For $k \in \mathcal{S}_{-0}$, $i \in \mathcal{S}_{+}$, the corresponding Laplace transform of the QBD-RAP is

$$\widehat{f}_{m,-0,+}^{\ell_0,(p)}(\lambda)(x, i, j_1, k_1, \dots, j_m, k_m, j; x_0, k)$$
$$:= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]_{ki}^{-1} \int_{x_1=0}^{\infty} h_{i,j_1}^{+-}(\lambda, x_1) \int_{x_2=0}^{\infty} h_{j_1,k_1}^{-+}(\lambda, x_2) \dots \int_{x_{2m}=0}^{\infty} h_{j_m,k_m}^{-+}(\lambda, x_{2m})$$
$$\times \int_{x_{2m+1}=0}^{\infty} h_{k_m,j}^{++}(\lambda, x_{2m+1}) \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_1} \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_2} \boldsymbol{D}^{(p)} \dots e^{\boldsymbol{S}^{(p)} x_{2m}} \boldsymbol{D}^{(p)} e^{\boldsymbol{S} x_{2m+1}}$$
$$\times V^{(p)}(y_{\ell_0+1} - x) \,\mathrm{d}x_{2m+1} \,\mathrm{d}x_{2m} \dots \mathrm{d}x_2 \,\mathrm{d}x_1. \tag{3.46}$$

The expression (3.46) is similar to (3.45), except (3.46) has a different initial condition $\boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0)\boldsymbol{D}^{(p)}$, $\varphi(0) = i$, and is multiplied by the constant $\left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}$. The Laplace transform of the fluid queue corresponding to (3.46) is Expression (3.39) multiplied by $\left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}$, i.e.

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,-0,+}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\\&:= \left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}\widehat{\mu}^{\ell_0}_{m,-0,+}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i).\end{aligned}\tag{3.47}$$

Note that there is no issue for sample paths which start in $\varphi(0) = k \in \mathcal{S}_{-0}$ (or $\varphi(0) = k \in \mathcal{S}_{+0}$), remain there for some time, then, upon leaving, transition to $\mathcal{S}_-$ ($\mathcal{S}_+$). There is no jump in the orbit at the time of this first transition out of $\mathcal{S}_{-0}$ ($\mathcal{S}_{+0}$). In these cases, the Laplace transforms for $i \in \mathcal{S}_-$ ($i \in \mathcal{S}_+$) need only to be multiplied by $\left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}$. For example, for $k \in \mathcal{S}_{-0}$, $i \in \mathcal{S}_-$, the relevant Laplace transform of the QBD-RAP is

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,-0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)\\&:= \left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}\widehat{f}^{\ell_0,(p)}_{m+1,-,+}(\lambda)(x,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,i)\end{aligned}\tag{3.48}$$

The corresponding Laplace transform of the fluid queue is

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,-0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)\\&:= \left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}\widehat{\mu}^{\ell_0}_{m+1,-,+}(\lambda)(x,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,i).\end{aligned}\tag{3.49}$$

Once we argue that expectations of $\psi(x)$ with respect to the Laplace transforms of the QBD-RAP (3.45) converge to expectations of $\psi(x)$ with respect to the Laplace transforms of the fluid queue (3.40) we also get that the necessary convergence of (3.48) to (3.49), since the only difference is the constant $\left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}$.

To be precise, we explicitly enumerate the Laplace transforms for all possible sample paths starting in $\mathcal{S}_{+0}$ or $\mathcal{S}_{-0}$. For $k \in \mathcal{S}_{-0}$, $i \in \mathcal{S}_+$,

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\\&:= \left[\lambda\boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki}\int_{x_1=0}^\infty h^{+-}_{i,j_1}(\lambda,x_1)\int_{x_2=0}^\infty h^{-+}_{j_1,k_1}(\lambda,x_2)\dots\int_{x_{2m+1}=0}^\infty h^{+-}_{k_m,j_{m+1}}(\lambda,x_{2m+1})\\&\quad\int_{x_{2m+2}=0}^\infty h^{--}_{j_{m+1},j}(\lambda,x_{2m+2})\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_1}\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_2}\boldsymbol{D}^{(p)}\dots e^{\boldsymbol{S}x_{2m+1}}\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_{2m+2}}\\&\quad V(x-y_{\ell_0})\,\mathrm{d}x_{2m+2}\,\mathrm{d}x_{2m+1}\dots\,\mathrm{d}x_2\,\mathrm{d}x_1,\end{aligned}\tag{3.50}$$

with corresponding Laplace transform of the fluid queue given by

$$\widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)$$

$$= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{\mu}^{\ell_0}_{m+1,-0,-}(\lambda)(x, j_1, k_1, \ldots, k_m, j_{m+1}, j; x_0, i). \tag{3.51}$$

For $k \in \mathcal{S}_{-0}$, $i \in \mathcal{S}_{-}$,

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x, i, k_1, j_1, \ldots, k_m, j_m, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{f}^{\ell_0,(p)}_{m,-,-}(\lambda)(x, k_1, j_1, \ldots, k_m, j_m, j; x_0, i)\end{aligned} \tag{3.52}$$

with corresponding Laplace transform of the fluid queue given by

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x, i, k_1, j_1, \ldots, k_m, j_m, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{\mu}^{\ell_0}_{m,-,-}(\lambda)(x, k_1, j_1, \ldots, k_m, j_m, j; x_0, i)\end{aligned} \tag{3.53}$$

For $k \in \mathcal{S}_{+0}$, $i \in \mathcal{S}_{+}$,

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x, i, j_1, k_1, \ldots, j_m, k_m, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i)\end{aligned} \tag{3.54}$$

with corresponding Laplace transform of the fluid queue given by

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x, i, j_1, k_1, \ldots, j_m, k_m, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x, j_1, k_1, \ldots, j_m, k_m, j; x_0, i).\end{aligned} \tag{3.55}$$

For $k \in \mathcal{S}_{+0}$, $i \in \mathcal{S}_{-}$,

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x, i, k_1, j_1, \ldots, j_m, k_{m+1}, j; x_0, k)\\ &:= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \int_{x_1=0}^{\infty} h^{-+}_{i,k_1}(\lambda, x_1) \int_{x_2=0}^{\infty} h^{+-}_{k_1,j_1}(\lambda, x_2) \ldots \int_{x_{2m+1}=0}^{\infty} h^{-+}_{j_m,k_{m+1}}(\lambda, x_{2m+1})\\ &\quad \int_{x_{2m+2}=0}^{\infty} h^{++}_{k_{m+1},j}(\lambda, x_{2m+2}) \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0) \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_1} \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_2} \boldsymbol{D}^{(p)} \ldots e^{\boldsymbol{S}^{(p)} x_{2m+1}} \boldsymbol{D}^{(p)} e^{\boldsymbol{S}^{(p)} x_{2m+2}}\\ &\quad V^{(p)}(y_{\ell_0+1} - x) \,\mathrm{d}x_{2m+2} \,\mathrm{d}x_{2m+1} \ldots \,\mathrm{d}x_2 \,\mathrm{d}x_1,\end{aligned} \tag{3.56}$$

with corresponding Laplace transform of the fluid queue given by

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x, i, k_1, j_1, \ldots, j_m, k_{m+1}, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{\mu}^{\ell_0}_{m+1,-,+}(\lambda)(x, k_1, j_1, \ldots, j_m, k_{m+1}, j; x_0, i).\end{aligned} \tag{3.57}$$

For $k \in \mathcal{S}_{+0}$, $i \in \mathcal{S}_{+}$

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x, i, j_1, k_1, \ldots, k_m, j_{m+1}, j; x_0, k)\\ &= \left[\lambda \boldsymbol{I} - \boldsymbol{T}_{00}\right]^{-1}_{ki} \widehat{f}^{\ell_0,(p)}_{m,+,-}(\lambda)(x, j_1, k_1, \ldots, k_m, j_{m+1}, j; x_0, i)\end{aligned} \tag{3.58}$$

with corresponding Laplace transform of the fluid queue given by

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\\&=\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}_{ki}\widehat{\mu}^{\ell_0}_{m+1,+,-}(\lambda)(x,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,i).\end{aligned}\tag{3.59}$$

For $k\in\mathcal{S}_{+0}$, $i\in\mathcal{S}_-$

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)\\&:=\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}_{ki}\int_{x_1=0}^{\infty}h^{-+}_{i,k_1}(\lambda,x_1)\int_{x_2=0}^{\infty}h^{+-}_{k_1,j_1}(\lambda,x_2)\dots\int_{x_{2m}=0}^{\infty}h^{+-}_{k_m,j_m}(\lambda,x_{2m})\\&\quad\int_{x_{2m+1}=0}^{\infty}h^{--}_{j_m,j}(\lambda,x_{2m+1})\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_1}\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_2}\boldsymbol{D}^{(p)}\dots e^{\boldsymbol{S}^{(p)}x_{2m}}\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_{2m+1}}\\&\quad V^{(p)}(x-y_{\ell_0})\,\mathrm{d}x_{2m+1}\,\mathrm{d}x_{2m}\dots\mathrm{d}x_2\,\mathrm{d}x_1,\end{aligned}\tag{3.60}$$

with corresponding Laplace transform of the fluid queue given by

$$\begin{aligned}&\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)\\&=\left[\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\right]^{-1}_{ki}\widehat{\mu}^{\ell_0}_{m,-,-}(\lambda)(x,k_1,j_1,\dots,k_m,j_m,j;x_0,i).\end{aligned}\tag{3.61}$$

Remark 3.2.1. *For technical reasons we should not have point masses at $x_0\in\partial\mathcal{D}_{\ell_0}$ when $\varphi(0)\in\mathcal{S}_{+0}\cup\mathcal{S}_{-0}$. Intuitively, if $\varphi(0)=k\in\mathcal{S}_{+0}$ and $x_0=y_{\ell_0}$ then, if upon exiting $\mathcal{S}_{+0}$ the phase process transitions to $\mathcal{S}_-$ then the fluid queue will instantaneously leave the interval $\mathcal{D}_{\ell_0}$ upon this transition. On the same event, the orbit of the QBD-RAP will be $\boldsymbol{\alpha}^{(p)}\boldsymbol{D}^{(p)}$ at the instant of the transition to $\mathcal{S}_-$. Roughly speaking $\boldsymbol{D}^{(p)}$ maps $\boldsymbol{\alpha}^{(p)}$ to approximately $\dfrac{\boldsymbol{\alpha}^{(p)}e^{\boldsymbol{S}^{(p)}\Delta}}{\boldsymbol{\alpha}^{(p)}e^{\boldsymbol{S}^{(p)}\Delta}\boldsymbol{e}}$ (asymptotically). Our asymptotic arguments rely on Chebyshev's inequality, in the form of a bound in terms of the distance of the random variable $Z^{(p)}\sim ME(\boldsymbol{\alpha}^{(p)},\boldsymbol{S}^{(p)})$ from its mean Δ. However, we cannot use such a technique to bound terms such as $\dfrac{\boldsymbol{\alpha}^{(p)}e^{\boldsymbol{S}^{(p)}\Delta}}{\boldsymbol{\alpha}^{(p)}e^{\boldsymbol{S}^{(p)}\Delta}\boldsymbol{e}}e^{\boldsymbol{S}^{(p)}z}\boldsymbol{s}^{(p)}$.*

In practice, it may be possible to avoid this issue by choosing the intervals $\{\mathcal{D}_\ell\}$ so that the boundaries do not align with any point masses. Another option is to append an ephemeral class of phases to the fluid queue as follows.

Suppose we wish to have a point mass at $x_0=y_\ell$, $i\in\mathcal{S}_{+0}$, and without loss of generality, assume the point mass has total mass 1. Let $\mathcal{S}^*_{+0}$ with rates $c_j=0$, $j\in\mathcal{S}^*_{+0}$. These states represent a duplicated copy of $\mathcal{S}_{+0}$. The initial orbit representing the point mass is $\boldsymbol{\alpha}^{(p)}$, $i\in\mathcal{S}^*_{+0}$. The dynamics of the QBD-RAP process regarding the ephemeral class $\mathcal{S}^*_{0+}$ is as follows. The phase process evolves according to the sub-generator matrix $\boldsymbol{T}_{00}$ whilst it remains in $\mathcal{S}^*_{0+}$. Meanwhile, the orbit and level remain constant. When in phase $k\in\mathcal{S}^*_{0+}$, at rate $[\boldsymbol{T}_{0+}]_{kj}$ the process transitions out of $\mathcal{S}^*_{0+}$ and into phase $j\in\mathcal{S}_+$.

Similarly, when in phase $k \in \mathcal{S}_{0+}^*$, at rate $[\boldsymbol{T}_{0-}]_{kj}$ the process transitions out of $\mathcal{S}_{0+}^*$ and into phase $j \in \mathcal{S}_-$. Upon a transition from $\mathcal{S}_{0+}^*$ to $j \in \mathcal{S}_+$ the orbit and level remain unchanged, and the process evolves as usual. Upon a transition from $\mathcal{S}_{0+}^*$ to $j \in \mathcal{S}_-$ the orbit remains unchanged, but the level jumps to $\ell - 1$. Thus, the process evolves as if it has just entered level $\ell - 1$ in phase $j \in \mathcal{S}_-$.

Let us now set up some notation for the following sections. Let

$$\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(x,j;x_0,i) = \sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{\mu}_{m,+,+}^{\ell_0}(\lambda)(x,j_1,k_1,\ldots,j_m,k_m,j;x_0,i) \tag{3.62}$$

$$\widehat{\mu}_{m,+,-}^{\ell_0}(\lambda)(x,j;x_0,i) = \sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{\mu}_{m+1,+,-}^{\ell_0}(\lambda)(x,j_1,k_1,\ldots,k_m,j_{m+1},j;x_0,i) \tag{3.63}$$

$$\widehat{\mu}_{m,-,+}^{\ell_0}(\lambda)(x,j;x_0,i) = \sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{\mu}_{m,-,+}^{\ell_0}(\lambda)(x,k_1,j_1,\ldots,j_m,k_{m+1},j;x_0,i) \tag{3.64}$$

$$\widehat{\mu}_{m,-,-}^{\ell_0}(\lambda)(x,j;x_0,i) = \sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{\mu}_{m,-,-}^{\ell_0}(\lambda)(x,k_1,j_1,\ldots,k_m,j_m,j;x_0,i) \tag{3.65}$$

be the Laplace transforms with respect to time of (3.16)-(3.19). Similarly, let

$$\widehat{f}_{m,+,+}^{\ell_0,(p)}(\lambda)(x,j;x_0,i) = \sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{f}_{m,+,+}^{\ell_0,(p)}(\lambda)(x,j_1,k_1,\ldots,j_m,k_m,j;x_0,i) \tag{3.66}$$

$$\widehat{f}_{m,+,-}^{\ell_0,(p)}(\lambda)(x,j;x_0,i) = \sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{f}_{m,+,-}^{\ell_0,(p)}(\lambda)(x,j_1,k_1,\ldots,k_m,j_{m+1},j;x_0,i) \tag{3.67}$$

$$\widehat{f}_{m,-,+}^{\ell_0,(p)}(\lambda)(x,j;x_0,i) = \sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{f}_{m,-,+}^{\ell_0,(p)}(\lambda)(x,k_1,j_1,\ldots,j_m,k_{m+1},j;x_0,i) \tag{3.68}$$

$$\widehat{f}_{m,-,-}^{\ell_0,(p)}(\lambda)(x,j;x_0,i) = \sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{f}_{m,-,-}^{\ell_0,(p)}(\lambda)(x,k_1,j_1,\ldots,k_m,j_m,j;x_0,i) \tag{3.69}$$

be the Laplace transforms with respect to time of (3.28)-(3.31), respectively. The right-hand sides of (3.62)-(3.69) hold by the definitions of (3.16)-(3.19) and (3.28)-(3.31) and since the Laplace transform is a linear operator. Also define

$$\begin{aligned}
&\widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\\
&\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)
\end{aligned}\tag{3.70}$$

$$\begin{aligned}
&\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\\
&\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)
\end{aligned}\tag{3.71}$$

$$\begin{aligned}
&\widehat{\mu}^{\ell_0}_{m,-0,+}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\\
&\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)
\end{aligned}\tag{3.72}$$

$$\begin{aligned}
&\widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\\
&\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)
\end{aligned}\tag{3.73}$$

and

$$\begin{aligned}
&\widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\\
&\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)
\end{aligned}\tag{3.74}$$

$$\begin{aligned}
&\widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x,j;x_0,k)\\
&=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)
\end{aligned}$$

$$+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\ldots,k_m,j_m,j;x_0,k) \quad (3.75)$$

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,-0,+}(\lambda)(x,j;x_0,k)\\ &=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+}\widehat{f}^{\ell_0,(p)}_{m,-0,+}(\lambda)(x,i,j_1,k_1,\ldots,j_m,k_m,j;x_0,k)\\ &\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_{m+1}\in\mathcal{S}_+}\widehat{f}^{\ell_0,(p)}_{m,-0,+}(\lambda)(x,i,k_1,j_1,\ldots,j_m,k_{m+1},j;x_0,k)\end{aligned} \quad (3.76)$$

$$\begin{aligned}&\widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x,j;x_0,k)\\ &=\sum_{i\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_{m+1}\in\mathcal{S}_-}\widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\ldots,k_m,j_{m+1},j;x_0,k)\\ &\quad+\sum_{i\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\sum_{j_1\in\mathcal{S}_-}\cdots\sum_{k_m\in\mathcal{S}_+}\sum_{j_m\in\mathcal{S}_-}\widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x,i,k_1,j_1,\ldots,k_m,j_m,j;x_0,k).\end{aligned} \quad (3.77)$$

3.3 Convergence on no change of level

We need the following properties of $h^{++}_{ij}(\lambda,x)$, $h^{--}_{ij}(\lambda,x)$, $h^{+-}_{ij}(\lambda,x)$, $h^{-+}_{ij}(\lambda,x)$ which follow from their interpretation as a Laplace transform of a probability distribution. Let $c_{min}=\min_{i\in\mathcal{S}_-\cup\mathcal{S}_+}|c_i|$. For all $\lambda\geq 0$, there is some $0\leq G<\infty$ such that

$$\begin{aligned}&0\leq h^{++}_{ij}(\lambda,x)\leq h^{++}_{ij}(0,x)\leq\max\{1/c_{min},1\}\leq G,\ i\in\mathcal{S}_+, j\in\mathcal{S}_+\cup\mathcal{S}_{+0},\\ &0\leq h^{--}_{ij}(\lambda,x)\leq h^{--}_{ij}(0,x)\leq\max\{1/c_{min},1\}\leq G,\ i\in\mathcal{S}_-, j\mathcal{S}_-\in\mathcal{S}_{-0},\\ &0\leq h^{+-}_{ij}(\lambda,x)\leq h^{+-}_{ij}(0,x)\leq\max_{k,\ell}\left[\boldsymbol{Q}_{+-}(0)\right]_{k,\ell}\leq G,\ i\in\mathcal{S}_+,\ j\in\mathcal{S}_-,\\ &0\leq h^{-+}_{ij}(\lambda,x)\leq h^{-+}_{ij}(0,x)\leq\max_{k,\ell}\left[\boldsymbol{Q}_{-+}(0)\right]_{k,\ell}\leq G,\ i\in\mathcal{S}_-,\ j\in\mathcal{S}_+,\end{aligned}$$

Furthermore, there exists some $0\leq\widehat{G}<\infty$ such that, for $i\in\mathcal{S}_+$, $j\in\mathcal{S}_{+0}$,

$$\int_{x=0}^{\infty}h^{++}_{ij}(\lambda,x)\,\mathrm{d}x\leq\int_{x=0}^{\infty}h^{++}_{ij}(0,x)\,\mathrm{d}x=\left[-\boldsymbol{Q}_{++}(0)^{-1}\boldsymbol{C}_+\quad-\boldsymbol{Q}_{++}(0)^{-1}\boldsymbol{Q}_{+0}(0)\right]_{ij}\leq\widehat{G},$$

for $i\in\mathcal{S}_-$, $j\in\mathcal{S}_{-0}$,

$$\int_{x=0}^{\infty}h^{--}_{ij}(\lambda,x)\,\mathrm{d}x\leq\int_{x=0}^{\infty}h^{--}_{ij}(0,x)\,\mathrm{d}x=\left[-\boldsymbol{Q}_{--}(0)^{-1}\boldsymbol{C}_-\quad-\boldsymbol{Q}_{--}(0)^{-1}\boldsymbol{Q}_{-0}(0)\right]_{ij}\leq\widehat{G},$$

for $i\in\mathcal{S}_+$, $j\in\mathcal{S}_-$,

$$\int_{x=0}^{\infty}h^{+-}_{ij}(\lambda,x)\,\mathrm{d}x\leq\int_{x=0}^{\infty}h^{+-}_{ij}(0,x)\,\mathrm{d}x=\left[-\boldsymbol{Q}_{++}(0)^{-1}\boldsymbol{Q}_{+-}(0)\right]_{ij}\leq\widehat{G},$$

for $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+$,

$$\int_{x=0}^{\infty} h_{ij}^{-+}(\lambda, x)\,\mathrm{d}x \leq \int_{x=0}^{\infty} h_{ij}^{-+}(0, x)\,\mathrm{d}x = \left[-\boldsymbol{Q}_{--}(0)^{-1}\boldsymbol{Q}_{-+}(0)\right]_{ij} \leq \widehat{G}.$$

Moreover, since $h_{ij}^{mn}(\lambda, x)$, $m, n \in \{+,-\}$, $i \in \mathcal{S}_m$, $j \in \mathcal{S}_n$, are matrix exponential functions with exponent matrix which is a sub-generator matrix, then for every $\lambda > 0$, $h_{ij}^{mn}(\lambda, x)$ is Lipschitz continuous with respect to x on $x \in [0,\infty)$. Therefore there exists some $0 < L < \infty$ such that $\left|h_{ij}^{mn}(\lambda, x) - h_{ij}^{mn}(\lambda, y)\right| \leq L|x-y|$, $m, n \in \{+,-\}$, $i \in \mathcal{S}_m$, $j \in \mathcal{S}_n \cup \mathcal{S}_{n0}$.

We now present a series of results which provide provide error bounds and/or show convergence of the Laplace transforms presented in the previous section.

Lemma 3.3.1. *Let $\psi : [0,\Delta) \to \mathbb{R}$ be bounded, $\psi(x) \leq F$, and Lipschitz. Then, for $x \in \mathcal{D}_{\ell_0,j}$, $\ell_0 \in \mathcal{K} \setminus \{-1, K+1\}$, $\lambda > 0$,*

$$\left|\int_{x=0}^{\Delta} \widehat{f}_{0,+,+}^{\ell_0,(p)}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x=0}^{\Delta} \widehat{\mu}_{0,+,+}^{\ell_0}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right| \leq R_{V,2}^{(p)}GF + \varepsilon^{(p)}GF, \tag{3.78}$$

and

$$\left|\int_{x=0}^{\Delta} \widehat{f}_{0,-,-}^{\ell_0,(p)}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x=0}^{\Delta} \widehat{\mu}_{0,-,-}^{\ell_0}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right| \leq R_{V,2}^{(p)}GF + \varepsilon^{(p)}GF. \tag{3.79}$$

Corollary 3.3.2. *Let $\psi : \mathcal{D}_{\ell_0} \to \mathbb{R}$ be bounded, $|\psi(x)| \leq F$, and Lipschitz continuous. Then, for $x_0 \in (y_{\ell_0}, y_{\ell_0+1}$, $k \in \mathcal{S}_{0+}$, $i \in \mathcal{S}_-$, $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$, there exists $r_{10}^{(p)} \to 0$ as $p \to \infty$,*

$$\left|\int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}_{0,-0,-}^{\ell_0,(p)}(x,i,j;j,x_0)\psi(x)\,\mathrm{d}x \to \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}_{0,-0,-}^{\ell_0}(x,i,j;k,x_0)\psi(x)\,\mathrm{d}x\right| \leq r_{10}^{(p)}. \tag{3.80}$$

Similarly, for $k \in \mathcal{S}_{0-}$, $i \in \mathcal{S}_+$, $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$

$$\left|\int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}_{0,+0,+}^{\ell_0,(p)}(x,i,j;j,x_0)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}_{0,+0,+}^{\ell_0}(x,i,j;k,x_0)\psi(x)\,\mathrm{d}x\right| \leq r_{10}^{(p)}, \tag{3.81}$$

where $r_{10}^{(p)} \to 0$ as $p \to \infty$.

Proof. The result follows from (A.1.5) in Appendix A. □

Define

$$w_n^{(p)}(x_0,x) = \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}^{(p)}(x_0)e^{\boldsymbol{S}^{(p)}x_1}\,\mathrm{d}x_1\boldsymbol{D}^{(p)}\left[\prod_{k=2}^{n-1}\int_{x_k=0}^{\infty} g_k(x_k)e^{\boldsymbol{S}^{(p)}x_k}\,\mathrm{d}x_k\boldsymbol{D}^{(p)}\right]$$
$$\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}^{(p)}x_n}\,\mathrm{d}x_nV^{(p)}(x), \tag{3.82}$$

where $g_1, g_2, \dots,$ are functions satisfying the Assumptions 3.0.1 and $V^{(p)}(x)$ is a closing operator with the Properties 3.0.2. Expressions such as (3.43) are specific forms of (3.82). In Appendix A.1, we prove the following results.

Corollary 3.3.3. *Let $g_1, g_2, \dots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x \in [0,\Delta)$, be a closing operator with Properties 3.0.2. Then, for $n \geq 2$, $x_0 \in [0,\Delta)$,*

$$\Bigg| \int_{x=0}^{\Delta} w_n^{(p)}(x_0,x)\psi(x)\,\mathrm{d}x - \int_{x=0}^{\Delta}\int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)$$
$$\left[\prod_{k=2}^{n-1}\int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta-u_k-u_{k-1})\,\mathrm{d}u_{k-1}\right] g_n(\Delta-x-u_{n-1})$$
$$1(\Delta-x-u_{n-1}\geq 0)\,\mathrm{d}u_{n-1}\psi(x)\,\mathrm{d}x\Bigg|$$
$$\leq \left(|r_5^{(p)}(n)| + |r_6^{(p)}(n)| + (n-1)|r_4^{(p)}(n)|\right)\Delta F, \tag{3.83}$$

where

$$|r_4^{(p)}(n)| = \left(2\varepsilon^{(p)} + \frac{\mathrm{Var}(Z^{(p)})}{\varepsilon^{(p)}}\right)\frac{1}{1-\mathrm{Var}(Z^{(p)})/(\Delta-x_0)}G\widehat{G}^{n-2}G,$$
$$|r_5(n)| = O\Bigg(\max\Bigg\{G^{n-1}\Delta^{n-2}\left(\frac{1}{2}\Delta|r_2| + 2\varepsilon^{(p)}G + \frac{1}{2}\Delta G\frac{\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}{1-\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}\right),$$
$$G^{n-1}\Delta^{n-2}R_{V,1}^{(p)}\Bigg\}\Bigg)$$
$$|r_6^{(p)}(n)| \leq \left(\varepsilon^{(p)}\right)^{n-1}G^n.$$

Corollary 3.3.4. *Let $g_1, g_2, \dots,$ be functions satisfying Assumptions 3.0.1 and let $V^{(p)}(x)$, $x \in (0,\Delta)$, be a closing operator with Properties 3.0.2. For $x_0, x \in (0,\Delta)$, $n \geq 2$*

$$\Bigg|\int_{x\in[0,\Delta)}\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}^{(p)}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_1}\,\mathrm{d}x_1\boldsymbol{D}^{(p)}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}^{(p)}x_n}\,\mathrm{d}x_n\boldsymbol{D}^{(p)}\right]$$

$$\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}^{(p)}x_n}\,\mathrm{d}x_n V^{(p)}(x)\psi(x)\,\mathrm{d}x - \int_{x\in[0,\Delta)}\int_{u_1=0}^{x_0} g_1(x_0-u_1)$$

$$\left[\prod_{k=2}^{n-1}\int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta-u_k-u_{k-1})\,\mathrm{d}u_{k-1}\right] g_n(\Delta-x-u_{n-1}) \tag{3.84}$$

$$\left.\times 1(\Delta-x-u_{n-1}\geq 0)\,\mathrm{d}u_{n-1}\psi(x)\,\mathrm{d}x\right|$$

$$\leq \left(|r_8^{(p)}(n)|+|r_5^{(p)}(n)|+|r_6^{(p)}(n)|+(n-1)|r_4^{(p)}(n)|\right)F\Delta. \tag{3.85}$$

where

$$|r_8^{(p)}(n)| \leq \left(2|r_5^{(p)}(n)|+2|r_6^{(p)}(n)|+2(n-1)|r_4^{(p)}(n)|+\varepsilon^{(p)}G^{n-1}\Delta^{n-2}(G+L\Delta)\right)$$
$$+ 2\widehat{G}^{n-2}GG_V\frac{\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}{1-\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}.$$

Upon choosing $\varepsilon^{(p)}=\mathrm{Var}(Z^{(p)})^{1/3}$, for fixed $n\geq 2$, $|r_5^{(p)}(n)|+|r_6^{(p)}(n)|+(n-1)|r_4^{(p)}(n)|\to 0$ and $|r_8^{(p)}|+|r_5^{(p)}(n)|+|r_6^{(p)}(n)|+(n-1)|r_4^{(p)}(n)|\to 0$ as $p\to\infty$.

Our first main result towards proving the convergence of the QBD-RAP scheme to the fluid queue is the following bound.

Corollary 3.3.5. *Let $\psi:\mathcal{D}_{\ell_0}\to\mathbb{R}$ be bounded, $|\psi(x)|\leq F$, and Lipschitz continuous. For $x\in\mathcal{D}_{\ell_0,j}$, $x_0\in\mathcal{D}_{\ell_0,i}$, $j_1,j_2\dots\in\mathcal{S}_-$, $k_1,k_2,\dots\in\mathcal{S}_+$, $\ell_0\in\mathcal{K}\setminus\{-1,K+1\}$, $m\geq 0$, $\lambda>0$, then*

1. *if $i\in\mathcal{S}_+$, $j\in\mathcal{S}_+\cup\mathcal{S}_{+0}$,*

$$\left|\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i)\psi(x)\,\mathrm{d}x\right.$$
$$\left.-\int_{x\in\mathcal{D}_{\ell_0}}\widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i)\psi(x)\,\mathrm{d}x\right|$$
$$\leq\begin{cases}R^{(p)}_{V,2}GF+\varepsilon^{(p)}GF, & m=0,\\ (|r_5^{(p)}(2m)|+|r_6^{(p)}(2m)|+(2m-1)|r_4^{(p)}(2m)|)\Delta F, & m\geq 1,\end{cases} \tag{3.86}$$

2. *if $i\in\mathcal{S}_-$, $j\in\mathcal{S}_-\cup\mathcal{S}_{-0}$,*

$$\left|\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}_{m,-,-}(\lambda)(x,k_1,j_1,\dots,k_m,j_m,j;x_0,i)\psi(x)\,\mathrm{d}x\right.$$

$$
\left. - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,-,-}(\lambda)(x,k_1,j_1,\dots,k_m,j_m,j;x_0,i)\psi(x)\,\mathrm{d}x \right|
$$
$$
\leq \begin{cases} R^{(p)}_{V,2}GF + \varepsilon^{(p)}GF, & m=0, \\ (|r^{(p)}_5(2m)| + |r^{(p)}_6(2m)| + (2m-1)|r^{(p)}_4(2m)|)\Delta F, & m\geq 1, \end{cases} \tag{3.87}
$$

3. *if* $i\in\mathcal{S}_+$, $j\in\mathcal{S}_-\cup\mathcal{S}_{-0}$,

$$
\left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,+,-}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j_{m+1},j;x_0,i)\psi(x)\,\mathrm{d}x \right.
$$
$$
\left. - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,-}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j_{m+1},j;x_0,i)\psi(x)\,\mathrm{d}x \right|
$$
$$
\leq (|r^{(p)}_5(2m+1)| + |r^{(p)}_6(2m+1)| + 2m|r^{(p)}_4(2m+1)|)\Delta F,\ m\geq 0, \tag{3.88}
$$

4. *and if* $i\in\mathcal{S}_-$, $j\in\mathcal{S}_+\cup\mathcal{S}_{+0}$,

$$
\left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,-,+}(\lambda)(x,k_1,j_1,\dots,k_m,j_m,k_{m+1},j;x_0,i)\psi(x)\,\mathrm{d}x \right.
$$
$$
\left. - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,-,+}(\lambda)(x,k_1,j_1,\dots,k_m,j_m,k_{m+1},j;x_0,i)\psi(x)\,\mathrm{d}x \right|
$$
$$
\leq (|r^{(p)}_5(2m+1)| + |r^{(p)}_6(2m+1)| + 2m|r^{(p)}_4(2m+1)|)\Delta F,\ m\geq 0. \tag{3.89}
$$

5. *if* $k\in\mathcal{S}_{-0}$, $i\in\mathcal{S}_+$, $j\in\mathcal{S}_+\cup\mathcal{S}_{+0}$,

$$
\Big| \int_{x=0}^{\Delta} \widehat{f}^{\ell_0}_{m,-0,+}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\psi(x)\,\mathrm{d}x
$$
$$
- \int_{x=0}^{\Delta} \widehat{\mu}^{\ell_0}_{m,-0,+}(\lambda)(x,i,j_1,k_1,\dots,j_m,k_m,j;x_0,k)\psi(x)\,\mathrm{d}x \Big|
$$
$$
\leq \begin{cases} |r^{(p)}_{10}|, & m=0 \\ (|r^{(p)}_8(2m)| + |r^{(p)}_5(2m)| + |r^{(p)}_6(2m)| + (2m-1)|r^{(p)}_4(2m)|)\Delta F & m\geq 1 \end{cases} \tag{3.90}
$$

6. *if* $k\in\mathcal{S}_{-0}$, $i\in\mathcal{S}_+$, $j\in\mathcal{S}_-\cup\mathcal{S}_{-0}$,

$$
\Big| \int_{x=0}^{\Delta} \widehat{f}^{\ell_0,(p)}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\psi(x)\,\mathrm{d}x
$$

$$- \int_{x=0}^{\Delta} \widehat{\mu}^{\ell_0}_{m,-0,-}(\lambda)(x,i,j_1,k_1,\dots,k_m,j_{m+1},j;x_0,k)\psi(x)\,\mathrm{d}x\Big|$$
$$\leq (|r_8^{(p)}(2m+1)| + |r_5^{(p)}(2m+1)| + |r_6^{(p)}(2m+1)| + 2m|r_4^{(p)}(2m+1)|)\Delta F,\ m \geq 0 \tag{3.91}$$

7. *if* $k \in \mathcal{S}_{+0}$, $i \in \mathcal{S}_-$, $j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$,

$$\Big| \int_{x=0}^{\Delta} \widehat{f}^{\ell_0,(p)}_{m,+0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)\psi(x)\,\mathrm{d}x$$
$$- \int_{x=0}^{\Delta} \widehat{\mu}^{\ell_0}_{m,+0,+}(\lambda)(x,i,k_1,j_1,\dots,j_m,k_{m+1},j;x_0,k)\psi(x)\,\mathrm{d}x\Big|$$
$$\leq (|r_8^{(p)}(2m+1)| + |r_5^{(p)}(2m+1)| + |r_6^{(p)}(2m+1)| + 2m|r_4^{(p)}(2m+1)|)\Delta F,\ m \geq 0 \tag{3.92}$$

8. *and if* $k \in \mathcal{S}_{+0}$, $i \in \mathcal{S}_-$, $j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$,

$$\Big| \int_{x=0}^{\Delta} \widehat{f}^{\ell_0,(p)}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)\psi(x)\,\mathrm{d}x$$
$$- \int_{x=0}^{\Delta} \widehat{\mu}^{\ell_0}_{m,+0,-}(\lambda)(x,i,k_1,j_1,\dots,k_m,j_m,j;x_0,k)\psi(x)\,\mathrm{d}x\Big|$$
$$\leq \begin{cases} |r_{10}^{(p)}|, & m = 0 \\ (|r_8^{(p)}(2m)| + |r_5^{(p)}(2m)| + |r_6^{(p)}(2m)| + (2m-1)|r_4^{(p)}(2m)|)\Delta F & m \geq 1. \end{cases} \tag{3.93}$$

Proof. For (3.86) and (3.87) and $m = 0$ apply Lemma 3.3.1 to the differences on the left-hand sides.

For (3.86) and (3.87) and $m \geq 1$ apply Corollary 3.3.3 to the differences on the left-hand sides.

For (3.88) and (3.89) apply Corollary 3.3.3 to the differences on the left-hand sides.

For (3.90) and (3.93) and $m = 0$ apply Corollary 3.3.2 to the differences on the left-hand sides.

For (3.90) and (3.93) and $m \geq 1$ apply Corollary 3.3.4 to the differences on the left-hand sides.

For (3.91) and (3.92) apply Corollary 3.3.4 to the differences on the left-hand sides. □

These bounds are enough to show the weak convergence (in space and time) of the QBD-RAP scheme to the fluid queue on the event on the event that there is no change of level, and on a given partition as described in (3.15)-(3.12). Since the state space of the phases, $\mathcal{S}$, is finite, then the same bounds show convergence when we sum over all possible phases at times $\{\Sigma_m\}$ and $\{\Gamma_m\}$. We formalise this with Corollary 3.3.6, below.

Corollary 3.3.6. *For $q \in \{+,-,+0,-0\}$, $r \in \{+,-\}$, $i \in \mathcal{S}_q$, $j \in \mathcal{S}_r$, $x \in \mathcal{D}_{\ell_0,j}$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $\ell_0 \in \mathcal{K}, \ell_0 \notin \{-1, K+1\}$, $m \geq 0$, $\lambda > 0$, then*

$$\left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x \right| \to 0 \tag{3.94}$$

as $p \to \infty$.

Proof. We prove the result for $q = r = +$ only, with the proofs for the other cases being analogous.

By (3.62) and (3.66) and the triangle inequality,

$$\begin{aligned}
&\left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x \right| \\
&\leq \sum_{j_1\in\mathcal{S}_-}\sum_{k_1\in\mathcal{S}_+}\cdots\sum_{j_m\in\mathcal{S}_-}\sum_{k_m\in\mathcal{S}_+} \Big| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i)\psi(x)\,\mathrm{d}x \\
&\quad - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j_1,k_1,\dots,j_m,k_m,j;x_0,i)\psi(x)\,\mathrm{d}x \Big| \\
&\leq \begin{cases} R^{(p)}_{V,2}GF + \varepsilon^{(p)}GF & m = 0 \\ (|r^{(p)}_5(2m)| + |r^{(p)}_6(2m)| + 2m|r^{(p)}_4(2m)|)\Delta F|\mathcal{S}_+|^m|\mathcal{S}_-|^m & m \geq 1. \end{cases}
\end{aligned}$$

since, by Corollary 3.3.5, each term in the sum is bounded by either $R^{(p)}_{V,2}GF + \varepsilon^{(p)}GF$ for $m = 0$ or by $(|r^{(p)}_5(2m)| + |r^{(p)}_6(2m)| + 2m|r^{(p)}_4(2m)|)|\Delta F$ for $m \geq 0$. $\square$

For $q \in \{+,-,+0,-0\}$, $r \in \{+.-\}$, consider the Laplace transforms

$$\int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x := \int_{x\in\mathcal{D}_{\ell_0}} \sum_{m=0}^{\infty} \widehat{f}^{\ell_0,(p)}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x,$$

for $i \in \mathcal{S}_q$, $j \in \mathcal{S}_r \cup \mathcal{S}_{r0}$ and

$$\int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x := \int_{x\in\mathcal{D}_{\ell_0}} \sum_{m=0}^{\infty} \widehat{\mu}^{\ell_0}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x,$$

for $i \in \mathcal{S}_q$, $j \in \mathcal{S}_r \cup \mathcal{S}_{r0}$. The difference,

$$\left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x \right|$$

is less than or equal to

$$\sum_{m=0}^{\infty}\left|\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x-\int_{x\in\mathcal{D}_{\ell_0}}\widehat{\mu}^{\ell_0}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right|,$$

which is an infinite sum of terms of the form of the left-hand side of (3.94). The issue that remains is the that the bounds in Corollary 3.3.6 are not necessarily summable (over m). In a step to resolving this issue, we have the following geometric domination condition.

Let $c_{min}=\min_{i\in\mathcal{S}_+\cup\mathcal{S}_-}|c_i|$.

Lemma 3.3.7. *For all $M\geq 0$, $x\in\mathcal{D}_{\ell_0,j}$, $x_0\in\mathcal{D}_{\ell_0,i}$, $\ell_0\in\mathcal{K},\ell_0\notin\{-1,K+1\}$, $\lambda>0$, $q\in\{+,-,+0,-0\}$, $r\in\{+,-\}$, $i\in\mathcal{S}$, $j\in\mathcal{S}_r\cup\mathcal{S}_{r0}$,*

$$\sum_{m=M+1}^{\infty}\left|\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x-\int_{x\in\mathcal{D}_{\ell_0}}\widehat{\mu}^{\ell_0}_{m,q,r}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right|\leq r_7^M \tag{3.95}$$

where

$$r_7^M=\Delta FG(1+G_V)\left(\frac{q}{q+\lambda}\right)^{2M+2}\left(1-\left(\frac{q}{q+\lambda}\right)^2\right)^{-1}.$$

Note that the bound r_7^M is independent of p.

We prove the result for $q=r=+$ only, with the proof for the other cases following analogously. Essentially, this result follows from noting the probabilistic interpretation of the Laplace transforms $\widehat{f}^{\ell_0}_{m,+,+}(t)(x,j;x_0,i)$, as the probability that,

- there are m changes from $\mathcal{S}_+$ to $\mathcal{S}_-$ and $\mathcal{S}_-$ to $\mathcal{S}_+$,
- the orbit process $\{\boldsymbol{A}(t)\}$ evolves accordingly,
- and an exponential random variable with rate λ has not yet occurred.

We obtain an upper bound by ignoring the behaviour of the orbit process $\{\boldsymbol{A}(t)\}$, then, by a uniformisation argument, we bound the probability that there are m changes from $\mathcal{S}_+$ to $\mathcal{S}_-$ and $\mathcal{S}_-$ to $\mathcal{S}_+$ before an exponential random variable with rate λ occurs, by the event that there are m independent exponential events before an exponential random variable with rate λ occurs.

Similarly, the probabilistic interpretation of the Laplace transforms $\widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)$, as the probability that,

- there are m changes from $\mathcal{S}_+$ to $\mathcal{S}_-$ and $\mathcal{S}_-$ to $\mathcal{S}_+$,
- the fluid level $X(t)$ remains in $\mathcal{D}_{\ell_0}$,

- and an exponential random variable with rate λ has not yet occurred.

We obtain an upper bound by removing the requirement that the fluid level $X(t)$ remain in $\mathcal{D}_{\ell_0}$, then applying the same uniformisation argument as we do for $\widehat{f}^{\ell_0}_{m,+,+}(t)(x,j;x_0,i)$.

Proof. The same arguments and results apply for all p, so let us drop the dependence on p.

Consider $i \in \mathcal{S}_+, j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$. By the triangle inequality,

$$\sum_{m=M+1}^{\infty} \left| \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x \right|$$
$$\leq \sum_{m=M+1}^{\infty} \int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)|\psi(x)|\,\mathrm{d}x$$
$$+ \sum_{m=M+1}^{\infty} \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)|\psi(x)|\,\mathrm{d}x,$$

since all terms are non-negative.

Consider $\widehat{f}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)$, which is given by the (i,j)th entry of

$$\int_{x_1=0}^{\infty} \boldsymbol{H}^{+-}(\lambda,x_1) \int_{x_2=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_2) \dots \int_{x_m=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_m) \int_{x_{m+1}=0}^{\infty} \boldsymbol{H}^{++}(\lambda,x_{m+1})$$
$$\boldsymbol{a}_{\ell_0,i}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1 \boldsymbol{D} e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \boldsymbol{D} \dots e^{\boldsymbol{S}x_m}\,\mathrm{d}x_m \boldsymbol{D} e^{\boldsymbol{S}x_{m+1}} V(y_{\ell_0+1}-x)\,\mathrm{d}x_{m+1}. \tag{3.96}$$

Substituting the upper bound $h^{++}_{j_m,j}(\lambda,x_{m+1}) \leq G$, then (3.96) is less than or equal to

$$\int_{x_1=0}^{\infty} \boldsymbol{H}^{+-}(\lambda,x_1) \int_{x_2=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_2) \dots \int_{x_m=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_m)\boldsymbol{e}G$$
$$\int_{x_{m+1}=0}^{\infty} \boldsymbol{a}_{\ell_0,i}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1 \boldsymbol{D} e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \boldsymbol{D} \dots e^{\boldsymbol{S}x_m}\,\mathrm{d}x_m \boldsymbol{D} e^{\boldsymbol{S}x_{m+1}} V(y_{\ell_0+1}-x)\,\mathrm{d}x_{m+1}$$
$$= \int_{x_1=0}^{\infty} \boldsymbol{H}^{+-}(\lambda,x_1) \int_{x_2=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_2) \dots \int_{x_m=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_m)\boldsymbol{e}G$$
$$\boldsymbol{a}_{\ell_0,i}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1 \boldsymbol{D} e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \boldsymbol{D} \dots e^{\boldsymbol{S}x_m}\,\mathrm{d}x_m \boldsymbol{D}(-\boldsymbol{S})^{-1} V(y_{\ell_0+1}-x)$$
$$\leq \int_{x_1=0}^{\infty} \boldsymbol{H}^{+-}(\lambda,x_1) \int_{x_2=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_2) \dots \int_{x_m=0}^{\infty} \boldsymbol{H}^{-+}(\lambda,x_m)\boldsymbol{e}G$$
$$\boldsymbol{a}_{\ell_0,i}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1 \boldsymbol{D} e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \boldsymbol{D} \dots e^{\boldsymbol{S}x_m}\,\mathrm{d}x_m \boldsymbol{e}G_V, \tag{3.97}$$

The last inequality holds since

$$\boldsymbol{D}(-\boldsymbol{S})^{-1}V(x) = \int_{u=0}^{\infty} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}(-\boldsymbol{S})^{-1}V(x)\,\mathrm{d}u \leq \int_{u=0}^{\infty} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}G_V = \boldsymbol{e}G_V,$$

which relies on Property 3.0.2(i).

Now, applying Lamma A.2.3 repeatedly gives the bound

$$\boldsymbol{a}_{\ell_0,i}(x_0)e^{\boldsymbol{S}x_1}\boldsymbol{D}e^{\boldsymbol{S}x_2}\boldsymbol{D}\dots e^{\boldsymbol{S}x_m}\boldsymbol{e} \leq 1.$$

Hence (3.97) is less than or equal to

$$\int_{x_1=0}^{\infty}\boldsymbol{H}^{+-}(\lambda,x_1)\,\mathrm{d}x_1\int_{x_2=0}^{\infty}\boldsymbol{H}^{-+}(\lambda,x_2)\,\mathrm{d}x_2\dots\int_{x_{2m}=0}^{\infty}\boldsymbol{H}^{-+}(\lambda,x_{2m})\,\mathrm{d}x_{2m}\boldsymbol{e}G_VG. \quad (3.98)$$

The stochastic interpretation of the ith element of the vector $\boldsymbol{H}^{+-}(\lambda,x)\boldsymbol{e}$ is that it is the probability density that the phase of the fluid queue changes from a positive to a negative phase at the time when the in-out fluid has increased by $\mathrm{d}x$ and before an exponential random variable with rate λ occurs, given the phase is initially i. There may be multiple changes of phase within $\mathcal{S}_+\cup\mathcal{S}_{+0}$ before the first change from $S_+\cup\mathcal{S}_{+0}$ to $\mathcal{S}_-$. The first change of phase occurs at rate (with respect to the in-out level) $-T_{ii}/|c_i|$ and this is the lowest in-out fluid level at which it may be possible for us to see a transition from $\mathcal{S}_+$ to $\mathcal{S}_-$. Consider a uniformisation of the phase process of the in-out fluid with uniformisation parameter $q=\max\limits_{i\in\mathcal{S}}-T_{ii}/|c_i|$. Then the first event of the uniformised phase process occurs at rate q and occurs at, or before, the first change of phase. Hence the first uniformisation event occurs at, or before the first change from $\mathcal{S}_+\cup\mathcal{S}_{+0}$ to $\mathcal{S}_-$. This gives the bound $\boldsymbol{H}^{+-}(\lambda,x)\boldsymbol{e}\leq qe^{-(\lambda+q)x}\boldsymbol{e}$.

Similarly for $\boldsymbol{H}^{-+}(\lambda,x)\boldsymbol{e}\leq qe^{-(\lambda+q)x}\boldsymbol{e}$.

From this stochastic interpretation, (3.98) is less than or equal to

$$\begin{aligned}
&\boldsymbol{H}^{+-}(\lambda,x_1)\,\mathrm{d}x_1\int_{x_2=0}^{\infty}\boldsymbol{H}^{-+}(\lambda,x_2)\,\mathrm{d}x_2\dots\int_{x_m=0}^{\infty}qe^{(-q-\lambda)x_m}\,\mathrm{d}x_{2m}G_VG\\
&\leq \boldsymbol{e}\int_{x_1=0}^{\infty}qe^{(-q-\lambda)x_1}\,\mathrm{d}x_1\int_{x_2=0}^{\infty}qe^{(-q-\lambda)x_2}\,\mathrm{d}x_2\dots\int_{x_{2m}=0}^{\infty}qe^{(-q-\lambda)x_{2m}}\,\mathrm{d}x_{2m}G_VG\\
&= \boldsymbol{e}\left(\frac{q}{q+\lambda}\right)^{2m}G_VG. \qquad (3.99)
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{m=M+1}^{\infty}\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)|\psi(x)|\,\mathrm{d}x\\
&\leq G_VG\sum_{m=M+1}^{\infty}\left(\frac{q}{q+\lambda}\right)^{2m}\int_{x\in\mathcal{D}_{\ell_0}}|\psi(x)|\,\mathrm{d}x\\
&\leq G_VG\left(\frac{q}{q+\lambda}\right)^{2M+2}\left(1-\left(\frac{q}{q+\lambda}\right)^2\right)^{-1}\Delta F. \qquad (3.100)
\end{aligned}$$

Now consider $\widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)$ which is given by the (i,j)th entry of

$$\begin{aligned}
&\int_{x_1=0}^{\Delta-(x_0-y_{\ell_0})} \boldsymbol{H}^{+-}(\lambda,\Delta-(x_0-y_{\ell_0})-x_1)\int_{x_2=0}^{\Delta-x_1}\boldsymbol{H}^{-+}(\lambda,\Delta-x_2-x_1)\,\mathrm{d}x_1\\
&\quad\cdots\int_{x_{2m}=0}^{\Delta-x_{m-1}}\boldsymbol{H}^{-+}(\lambda,\Delta-x_{2m-1}-x_{2m})\,\mathrm{d}x_{2m-1}\boldsymbol{H}^{++}(\lambda,\Delta-x_{2m}-(y_{\ell_0+1}-x))\,\mathrm{d}x_{2m}\\
&=\int_{x_1=(x_0-y_{\ell_0})}^{\Delta}\boldsymbol{H}^{+-}(\lambda,\Delta-x_1)\int_{x_2=x_1}^{\Delta}\boldsymbol{H}^{-+}(\lambda,\Delta-x_2)\,\mathrm{d}x_1\dots\\
&\int_{x_{2m}=x_{2m-1}}^{\Delta}\boldsymbol{H}^{-+}(\lambda,\Delta-x_{2m})\,\mathrm{d}x_{2m-1}\boldsymbol{H}^{++}(\lambda,\Delta-x_{2m}-x_{2m-1}-(y_{\ell_0+1}-x))\\
&\qquad\mathrm{d}x_{2m-1}\,\mathrm{d}x_{2m}.
\end{aligned}\tag{3.101}$$

Once again, use the bound $h^{++}_{j_m,j}(\lambda,x_{m+1})\leq G$. Hence (3.101) is less than or equal to

$$\begin{aligned}
&\int_{x_1=(x_0-y_{\ell_0})}^{\Delta}\boldsymbol{H}^{+-}(\lambda,\Delta-x_1)\int_{x_2=x_1}^{\Delta}\boldsymbol{H}^{-+}(\lambda,\Delta-x_2)\,\mathrm{d}x_1\\
&\quad\cdots\int_{x_{2m}=x_{2m-1}}^{\Delta}\boldsymbol{H}^{-+}(\lambda,\Delta-x_{2m})\,\mathrm{d}x_{2m-1}\,\mathrm{d}x_{2m-1}\,\mathrm{d}x_{2m}G.
\end{aligned}\tag{3.102}$$

The expression (3.102) differs from (3.98) only by a constant factor G_V and that the integrals in the former are finite, hence we may bound it in the same way as (3.98). Therefore,

$$\begin{aligned}
&\sum_{m=M+1}^{\infty}\int_{x\in\mathcal{D}_{\ell_0}}|\psi(x)|\,\mathrm{d}x\widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)|\psi(x)|\,\mathrm{d}x\\
&\leq G\left(\frac{q}{q+\lambda}\right)^{2M+2}\left(1-\left(\frac{q}{q+\lambda}\right)^2\right)^{-1}\Delta F.
\end{aligned}\tag{3.103}$$

Analogous arguments show the same bounds for any $i,j\in\mathcal{S}$. □

Lemma 3.3.8. *For all $x\in\mathcal{D}_{\ell_0,j}$, $x_0\in\mathcal{D}_{\ell_0,i}$, $i,j\in\mathcal{S}$, $\ell_0\in\mathcal{K},\ell_0\notin\{-1,K+1\}$, $\lambda>0$,*

$$\left|\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x-\int_{x\in\mathcal{D}_{\ell_0}}\widehat{\mu}^{\ell_0}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right|\to 0 \tag{3.104}$$

as $p\to\infty$.

Proof. Consider $i\in\mathcal{S}_+$ and $j\in\mathcal{S}_+\cup\mathcal{S}_{+0}$. By partitioning on the number of changes from $\mathcal{S}_-\to\mathcal{S}_+$, (3.104) can be written as

$$\left|\sum_{m=0}^{\infty}\int_{x\in\mathcal{D}_{\ell_0}}\widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x-\sum_{m=0}^{\infty}\int_{x\in\mathcal{D}_{\ell_0}}\widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right|$$

$$\leq \sum_{m=0}^{\infty}\left|\int_{x\in\mathcal{D}_{\ell_0}} \widehat{f}^{\ell_0,(p)}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x - \int_{x\in\mathcal{D}_{\ell_0}} \widehat{\mu}^{\ell_0}_{m,+,+}(\lambda)(x,j;x_0,i)\psi(x)\,\mathrm{d}x\right|. \tag{3.105}$$

By Corollary 3.3.6 each term in the sum (3.105) converges to 0 as $p \to \infty$. Lemma 3.3.7 gives a domination condition, so we can apply the Dominated Convergence Theorem which proves the stated convergence in (3.104).

Analogous arguments can be applied for any $i, j \in \mathcal{S}$. □

Remark 3.3.9. *For a fixed $\lambda > 0$, convergence of*

$$\left|\widehat{f}^{\ell_0,(p)}(\lambda)(x,j;x_0,i) - \widehat{\mu}^{\ell_0}(\lambda)(x,j;x_0,i)\right| \tag{3.106}$$

actually holds point-wise for each $\ell_0 \in \mathcal{K}\setminus\{-1, K+1\}$, and each $i, j \in \mathcal{S}$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $x \in \mathcal{D}_{\ell_0,j}$ except at the set of points where $x = x_0$. Specifically, the lack of point-wise convergence at this point occurs due terms with the index $m = 0$, that is, terms where there are no changes of phase from $\mathcal{S}_+ \to \mathcal{S}_-$ or $\mathcal{S}_- \to \mathcal{S}_+$. On these sample paths the relevant Laplace transforms of the fluid queue are discontinuous at this point. For example,

$$\widehat{\mu}^{\ell_0}_{0,+,+}(\lambda)(x,j;x_0,i)\,\mathrm{d}x = h^{++}_{ij}(\lambda, x-x_0)1(x \geq x_0)\,\mathrm{d}x,$$

is discontinuous at $x = x_0$. If we were to insist on point-wise convergence, then we would need to enforce the error term $r_V(u,v)^{(p)} \to 0$ point-wise for each u, v. In the cases presented here $r_V^{(p)}(u,v)$ converges point-wise to 0 everywhere except $u + v = \Delta$.

3.3.1 Convergence at the first change of level

Before moving on to global convergence we need another result about convergence of the QBD-RAP scheme and the fluid queue at changes of level.

Let E^λ be an exponential random variable with rate λ. In the following we use the stochastic interpretation of the Laplace transform of a probability distribution with non-negative support: for a random variable W with distribution function $F_W(w) = \mathbb{P}(W < w)$, then $\displaystyle\int_{w=0}^{\infty} e^{-\lambda w}\,\mathrm{d}F_W(w) = \mathbb{P}(W < E^\lambda)$. That is, the Laplace transform with parameter $\lambda > 0$ is the probability that W occurs before an exponential time with rate λ occurs.

Regarding Remark 3.3.9 we do require convergence at certain boundary points which correspond to changes of level of the QBD-RAP. To this end, we have to following result.

Corollary 3.3.10. *For $\ell, \ell_0 \in \mathcal{K}\setminus\{-1, K+1\}$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $i \in \mathcal{S}$ then, for $j \in \mathcal{S}_+$,*

$$\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_0 + 1, \varphi(\tau_1^{(p)}) = j, \tau_1^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),$$

$$
\begin{aligned}
&\varphi(0) = i) \\
&\to \mathbb{P}(X(\tau_1^X) = y_{\ell_0+1}, \varphi(\tau_1^X) = j, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = i)
\end{aligned} \tag{3.107}
$$

and for $j \in \mathcal{S}_-$

$$
\begin{aligned}
&\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_0 - 1, \varphi(\tau_1^{(p)}) = j, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \\
&\quad \varphi(0) = i) \\
&\to \mathbb{P}(X(\tau_1^X) = y_{\ell_0}, \varphi(\tau_1^X) = j, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = i).
\end{aligned}
$$

At a boundary, for $\ell_0 \in \{-1, K+1\}$,

$$
\begin{aligned}
&\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell, \varphi(\tau_1^{(p)}) = j, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = 1, \varphi(0) = i) \\
&= \mathbb{P}(X(\tau_1^X) = x, \varphi(\tau_1^X) = j, \tau_1 \le E^\lambda \mid X(0) = 0, \varphi(0) = i),
\end{aligned} \tag{3.108}
$$

where $\ell = 0$, $i \in \mathcal{S}_- \cup \mathcal{S}_{-0}$ *and* $j \in \mathcal{S}_+$ *if* $\ell_0 = -1$, *and* $\ell = K$, $i \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$ *and* $j \in \mathcal{S}_-$ *if* $\ell_0 = K+1$.

Proof. The proof follows the same structure as the proof of Lemma 3.3.8 however changes are required in all the results used in the proof, as here we do not integrate a function ψ. Here we only give and outline of the proof.

Consider first $i \in \mathcal{S}_+, j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$. Partition the probability (3.107) on the times $\{\Sigma_n\}_{n\ge1}$ and $\{\Gamma_n\}_{n\ge1}$, specifically, partition on the event that there are exactly m events $\{\Sigma_n\}_{n=1}^m$ and exactly m events $\{\Gamma_n\}_{n=1}^m$. Partition further by the phases at these times $\varphi(\Sigma_n) = j_n \in \mathcal{S}_-$, $n = 1, \dots, m$ and $\varphi(\Gamma_n) = k_n \in \mathcal{S}_+$, $n = 1, ..., m$. The resulting partitioned probabilities are

$$
\begin{aligned}
&\int_{x_1=0}^\infty h_{i,j_1}^{+-}(\lambda, x_1) \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) e^{\boldsymbol{S}^{(p)} x_1} \,\mathrm{d}x_1 \boldsymbol{D}^{(p)} \\
&\left[\prod_{r=1}^{m-1} \int_{x_{2r}=0}^\infty h_{j_r,k_r}^{-+}(\lambda, x_{2r}) e^{\boldsymbol{S}^{(p)} x_{2r}} \,\mathrm{d}x_{2r} \boldsymbol{D}^{(p)} \int_{x_{2r+1}=0}^\infty h_{k_r j_{r+1}}^{+-}(\lambda, x_{2r+1}) e^{\boldsymbol{S}^{(p)} x_{2r+1}} \,\mathrm{d}x_{2r+1} \boldsymbol{D}^{(p)} \right] \\
&\int_{x_{2m}=0}^\infty h_{j_m,k_m}^{-+}(\lambda, x_{2m}) e^{\boldsymbol{S}^{(p)} x_{2m}} \,\mathrm{d}x_{2m} \boldsymbol{D}^{(p)} \int_{x_{2m+1}=0}^\infty h_{k_m,j}^{++}(\lambda, x_{2m+1}) e^{\boldsymbol{S}^{(p)} x_{2m+1}} \,\mathrm{d}x_{2m+1} \boldsymbol{s}^{(p)}
\end{aligned} \tag{3.109}
$$

and (3.107) is obtained by summing over $j_1, \dots, j_m$, $k_1, \dots, k_m$, $m \ge 0$.

Corollary A.1.2 and Lemma A.1.4 are analogous to Lemma 3.3.1 and Corollary 3.3.1 used in the proof of Lemma 3.3.8. They provide relevant bounds for the difference between terms like (3.109) for $m = 0$ and

$$
\mathbb{P}(X(\tau_1^X) = y_{\ell_0+1}, \varphi(\tau_1^X) = j, \tau_1^X \le E^\lambda, \tau_1^X < \Sigma_1 \mid X(0) = x_0, \varphi(0) = i)
$$

$$= h_{ij}^{++}(\lambda, y_{\ell_0+1} - x_0),$$

for example.

Corollaries A.1.11 and A.1.15 are analogous to Corollaries 3.3.3 and 3.3.4 used in the proof of Lemma 3.3.8. Corollaries A.1.11 and A.1.15 give error bounds for the difference between terms such as (3.109) for $m \geq 1$ and the corresponding probability for the fluid queue,

$$\begin{aligned}\mathbb{P}(X(\tau_1^X) = y_{\ell_0+1}, \varphi(\tau_1^X) = j, \tau_1^X \leq E^\lambda, \Sigma_m \leq \tau_1^X < \Gamma_{m+1}, \varphi(\Sigma_\ell) = j_\ell, \varphi(\Gamma_\ell) = k_\ell,\\ \ell = 1, \dots, m \mid X(0) = x_0, \varphi(0) = i)\end{aligned} \tag{3.110}$$

which is given by (3.39).

The error bounds on each term in the partition imply that all the terms converge to the corresponding terms for the fluid queue. Equation (3.99) provides a domination condition. Thus we can apply the Dominated Convergence Theorem to prove that the sum of the partitioned probabilities (3.109) converges as $p \to \infty$. The sum of the limits is

$$\mathbb{P}(X(\tau_1^X) = y_{\ell_0+1}, \varphi(\tau_1^X) = j, \tau_1 \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i).$$

The result for all other cases $i, j \in \mathcal{S}$ follow analogously.

At a boundary we can model the fluid queue exactly, hence (3.108) holds. □

3.4 At the nth change of level

So far we have shown error bounds for the QBD-RAP approximation on the event that the fluid queue and QBD-RAP remain in the same band/level (Lemma 3.3.8) and also immediately upon exiting the band (Corollary 3.3.10). In the next three subsections we extend this to global convergence of the approximation. To do so, we partition the approximation and the fluid queue by the number of level changes. The local convergence on a given level, which we just proved, is then used to claim that each term in the partition converges. What remains is to argue that we can swap a limit and countable sum which we do by the Dominated Convergence Theorem.

Let $\{\tau_n^{(p)}\}_{n\geq 0}$, $\tau_0^{(p)} = 0$, and

$$\tau_n^{(p)} = \inf\left\{t \geq \tau_{n-1}^{(p)} \mid L^{(p)}(t) \neq L^{(p)}(\tau_{n-1}^{(p)})\right\},$$

be the (stopping) times at which $\{L^{(p)}(t)\}$, the level process of the QBD-RAP, changes, or the boundary is hit, or, if the process is at the boundary, the process leaves the boundary. To simplify notation, we may drop the superscript p where it is not explicitly needed.

The process $\{L^{(p)}(\tau_n^{(p)})\}_{n\geq 0}$ is the level process of the (continuous-time) QBD-RAP process observed at changes of level. When $\tau_n^{(p)}$ is a change of level, or the time when the process leaves a boundary, the value of orbit process at these times is $\boldsymbol{A}^{(p)}(\tau_n^{(p)}) = \boldsymbol{\alpha}^{(p)}$. At time $\tau_0^{(p)} = 0$, $\boldsymbol{A}^{(p)}(\tau_0^{(p)}) = \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0)$. The process $\{(L^{(p)}(\tau_n^{(p)}), \phi^{(p)}(\tau_n^{(p)}))\}_{n\geq 0}$ is a time-inhomogeneous discrete-time QBD. Further, the process $\{\phi^{(p)}(\tau_n^{(p)})\}_{n\geq 0}$ is a time-inhomogeneous discrete-time Markov chain on the state space $\mathcal{S}$. For $\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0) = \boldsymbol{\alpha}^{(p)}$, or for index $n \geq 1$, then $\{L^{(p)}(\tau_n^{(p)}), \phi^{(p)}(\tau_n^{(p)})\}_{n\geq 1}$, and $\{\phi^{(p)}(\tau_n^{(p)})\}_{n\geq 1}$ are time-homogenous.

Let $\{\tau_n^X\}_{n\geq 0}$, be the sequence of (stopping) times with $\tau_0^X = 0$, and

$$\tau_{n+1}^X = \min \left\{ \begin{array}{c} \inf\left\{t > \tau_n^X \mid X(t) = y_\ell, \ell \in \mathcal{K}\right\}, \\ \inf\left\{t > \tau_n^X \mid X(t) \neq 0, X(0) = 0\right\}, \\ \inf\left\{t > \tau_n^X \mid X(t) \neq y_{K+1}, X(0) = y_{K+1}\right\} \end{array} \right\}.$$

For $n \geq 1$, τ_n^X is the time at which $X(t)$ either changes band, or hits a boundary, or the process leaves a boundary, for the nth time.

For $n \geq 1$, consider the Laplace transform

$$\begin{aligned} &\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \in \mathrm{d}t \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \\ &\qquad \phi^{(p)}(0) = i)\,\mathrm{d}t \\ &= \mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \\ &\qquad \phi^{(p)}(0) = i), \end{aligned}$$

which is the Laplace transform of the time until the nth change of level of the QBD-RAP on the event that the level and phase at the nth change of level are ℓ and j_n, respectively, given that the initial level and phases are ℓ_0 and i, respectively and the initial orbit is $\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0)$. Partitioning on the time of the first change of level, τ_1, and the level and phase at this time gives

$$\begin{aligned} &\sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(\tau_1^{(p)}) = \ell_1, \\ &\qquad \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \leq E^\lambda) \\ &\quad \times \mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \\ &\qquad \phi^{(p)}(0) = i). \end{aligned} \tag{3.111}$$

An application of Corollary (3.3.10) to the expression on the second line of (3.111) states, for $i \in \mathcal{S}$, $j \in \mathcal{S}$, $\ell_0 \in \mathcal{K}$,

$$\begin{aligned} &\lim_{p\to\infty} \mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \\ &\qquad \phi^{(p)}(0) = i) \\ &\to \mathbb{P}(X(\tau_1^X) = y_{\ell_0+1}, \varphi(\tau_1^X) = j_1, \tau_1^X \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i) \end{aligned} \tag{3.112}$$

for $j_1 \in \mathcal{S}_+$, $\ell_1 = \ell_0 + 1$ and

$$\begin{aligned}&\lim_{p\to\infty} \mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),\\ &\qquad \phi^{(p)}(0) = i)\\ &\to \mathbb{P}(X(\tau_1^X) = y_{\ell_0}, \varphi(\tau_1^X) = j_1, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = i) \end{aligned} \tag{3.113}$$

for $j_1 \in \mathcal{S}_-$, $\ell_1 = \ell_0 - 1$.

Now, for a given j_1 and ℓ_1 consider

$$\begin{aligned}&\mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \le E^\lambda \mid L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \le E^\lambda)\\ &= \mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_1, \phi^{(p)}(0) = j_1, \tau_1^{(p)} = 0) \end{aligned} \tag{3.114}$$

by the time-homogeneous property of the QBD-RAP and the the memoryless property of the exponential distribution. Equation (3.114) appears as the first factor in the summands of (3.111). Let

$$\mathcal{P}^n(\ell_0, \ell_n) = \{(\ell_1, \dots, \ell_{n-1}) \in \mathcal{K}^{n-1} \mid |\ell_{r-1} - \ell_r| = 1, r = 1, \dots, n\}. \tag{3.115}$$

The set $\mathcal{P}^n(\ell_0, \ell)$ contains all of the possible sequences of levels which $\{L(t)\}$ or $\{X(t)\}$ may visit on a sample path which start in level ℓ_0, end in level ℓ and change level n times. By partitioning on the times τ_m, $m = 2, \dots, n-1$, and the phases and the levels at these times and using the strong Markov property of the QBD-RAP, then (3.114) is

$$\begin{aligned}&\sum_{j_2,\dots,j_{n-1}\in\mathcal{S}} \ \sum_{(\ell_2,\dots,\ell_{n-1})\in\mathcal{P}^{n-1}(\ell_1,\ell)} \prod_{m=2}^{n} \mathbb{P}(L^{(p)}(\tau_m^{(p)}) = \ell_m, \phi^{(p)}(\tau_m^{(p)}) = j_m, \tau_m^{(p)} \le E^\lambda \mid \\ &\qquad L^{(p)}(\tau_{m-1}^{(p)}) = \ell_{m-1}, \phi^{(p)}(\tau_{m-1}^{(p)}) = j_{m-1}, \tau_{m-1}^{(p)} \le E^\lambda), \end{aligned} \tag{3.116}$$

where we define $\ell_n = \ell$. Each factor in the product is

$$\begin{aligned}&\mathbb{P}(L^{(p)}(\tau_m^{(p)}) = \ell_m, \phi^{(p)}(\tau_m^{(p)}) = j_m, \tau_m^{(p)} \le E^\lambda \mid L^{(p)}(\tau_{m-1}^{(p)}) = \ell_{m-1}, \phi^{(p)}(\tau_{m-1}^{(p)}) = j_{m-1},\\ &\qquad \tau_{m-1}^{(p)} \le E^\lambda)\\ &= \mathbb{P}(L^{(p)}(\tau_m^{(p)}) = \ell_m, \phi^{(p)}(\tau_m^{(p)}) = j_m, \tau_m^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_{m-1}, \phi^{(p)}(0) = j_{m-1},\\ &\qquad \tau_{m-1^{(p)}} = 0)\\ &= \mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_m, \phi^{(p)}(\tau_1^{(p)}) = j_m, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_{m-1}, \phi^{(p)}(0) = j_{m-1}) \end{aligned} \tag{3.117}$$

by the time-homogeneous property of the QBD-RAP and the memoryless property of the exponential distribution. We can also apply Corollary (3.3.10) to the terms (3.117) and conclude

$$\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_m, \phi^{(p)}(\tau_1^{(p)}) = j_m, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_{m-1}, \phi^{(p)}(0) = j_{m-1})$$

$$\to \mathbb{P}(X(\tau_1^X) = y_{\ell_{m-1}+1}, \varphi(\tau_1^X) = j_m, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = j_{m-1}) \tag{3.118}$$

for $j_m \in \mathcal{S}_+$, $\ell_m = \ell_{m-1} + 1$ and

$$\begin{aligned}&\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_m, \phi^{(p)}(\tau_1^{(p)}) = j_m, \tau_1^{(p)} \le E^\lambda \mid L^{(p)}(0) = \ell_{m-1}, \phi^{(p)}(0) = j_{m-1})\\ &\quad\to \mathbb{P}(X(\tau_1^X) = y_{\ell_{m-1}}, \varphi(\tau_1^X) = j_m, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = j_{m-1})\end{aligned} \tag{3.119}$$

for $j_m \in \mathcal{S}_-$, $\ell_m = \ell_{m-1} - 1$.

By the time-homogeneous property of the fluid queue and the memoryless property of the exponential distribution,

$$\begin{aligned}&\mathbb{P}(X(\tau_1^X) = y_{\ell_{m-1}+1}, \varphi(\tau_1^X) = j_m, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = j_{m-1})\\ &= \mathbb{P}(X(\tau_m^X) = y_{\ell_{m-1}+1}, \varphi(\tau_m^X) = j_m, \tau_m^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = j_{m-1}, \tau_{m-1}^X = 0)\\ &= \mathbb{P}(X(\tau_m^X) = y_{\ell_{m-1}+1}, \varphi(\tau_m^X) = j_m, \tau_m^X \le E^\lambda \mid X(\tau_{m-1}^X) = x_0, \varphi(\tau_{m-1}^X) = j_{m-1},\\ &\qquad \tau_{m-1}^X \le E^\lambda).\end{aligned} \tag{3.120}$$

Similarly,

$$\begin{aligned}&\mathbb{P}(X(\tau_1^X) = y_{\ell_{m-1}}, \varphi(\tau_1^X) = j_m, \tau_1^X \le E^\lambda \mid X(0) = x_0, \varphi(0) = j_{m-1})\\ &= \mathbb{P}(X(\tau_m^X) = y_{\ell_{m-1}}, \varphi(\tau_m^X) = j_m, \tau_m^X \le E^\lambda \mid X(\tau_{m-1}^X) = x_0, \varphi(\tau_{m-1}^X) = j_{m-1}, \tau_{m-1}^X \le E^\lambda).\end{aligned} \tag{3.121}$$

Now, taking the limit of (3.116) gives

$$\begin{aligned}&\lim_{p\to\infty} \sum_{j_2,\dots,j_{n-1}\in\mathcal{S}} \sum_{(\ell_2,\dots,\ell_{n-1})\in\mathcal{P}^{n-1}(\ell_1,\ell)} \prod_{m=2}^{n} \mathbb{P}(L^{(p)}(\tau_m^{(p)}) = \ell_m, \phi^{(p)}(\tau_m^{(p)}) = j_m, \tau_m^{(p)} \le E^\lambda\\ &\qquad \mid L^{(p)}(\tau_{m-1}^{(p)}) = \ell_{m-1}, \phi^{(p)}(\tau_{m-1}^{(p)}) = j_{m-1}, \tau_{m-1}^{(p)} \le E^\lambda)\\ &= \sum_{j_2,\dots,j_{n-1}\in\mathcal{S}} \sum_{(\ell_2,\dots,\ell_{n-1})\in\mathcal{P}^{n-1}(\ell_1,\ell)} \prod_{m=2}^{n} \lim_{p\to\infty} \mathbb{P}(L^{(p)}(\tau_m^{(p)}) = \ell_m, \phi^{(p)}(\tau_m^{(p)}) = j_m, \tau_m^{(p)} \le E^\lambda\\ &\qquad \mid L^{(p)}(\tau_{m-1}^{(p)}) = \ell_{m-1}, \phi^{(p)}(\tau_{m-1}^{(p)}) = j_{m-1}, \tau_{m-1}^{(p)} \le E^\lambda),\end{aligned} \tag{3.122}$$

where we may swap the limit and the sums as they are finite, and we can swap the limit and the product since all the limits exist and the product is finite. Substituting the limits into (3.122) gives

$$\begin{aligned}&\sum_{j_2,\dots,j_{n-1}\in\mathcal{S}} \sum_{(\ell_2,\dots,\ell_{n-1})\in\mathcal{P}^{n-1}(\ell_1,\ell)} \prod_{m=2}^{n} \mathbb{P}(X(\tau_m^X) = y_{\ell_{m-1}+1(j_m\in\mathcal{S}_+)}, \varphi(\tau_m^X) = j_m, \tau_m^X \le E^\lambda\\ &\qquad \mid X(\tau_{m-1}^X) = y_{\ell_{m-2}+1(j_{m-1}\in\mathcal{S}_+}, \tau_{m-1}^X \le E^\lambda),\end{aligned}$$

$$= \mathbb{P}(X(\tau_n^X) = y_{\ell+1(j_n\in\mathcal{S}_-)}, \varphi(\tau_n^X) = j_n, \tau_n^X \leq E^\lambda \mid X(0) = y_{\ell_0+1(j_1\in\mathcal{S}_+)}, \varphi(0) = j_1, \tau_1^X \leq E^\lambda). \tag{3.123}$$

Therefore, taking the limit as $p \to \infty$ of (3.111)

$$\begin{aligned}
&\lim_{p\to\infty} \sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_1,\\
&\quad \phi^{(p)}(0) = j_1, \tau_1^{(p)} = 0)\\
&\mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0),\\
&\quad \phi^{(p)}(0) = i)\\
&= \sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \lim_{p\to\infty} \mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_1,\\
&\quad \phi^{(p)}(0) = j_1, \tau_1^{(p)} = 0)\\
&\lim_{p\to\infty} \mathbb{P}(L^{(p)}(\tau_1^{(p)}) = \ell_1, \phi^{(p)}(\tau_1^{(p)}) = j_1, \tau_1^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0),\\
&\quad \phi^{(p)}(0) = i)\\
&= \sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \mathbb{P}(X(\tau_n^X) = y_{\ell+1(j_n\in\mathcal{S}_-)}, \varphi(\tau_n^X) = j_n, \tau_n^X \leq E^\lambda \mid X(\tau_1^X) = y_{\ell_0+1(j_1\in\mathcal{S}_+)},\\
&\quad \varphi(\tau_1^X) = j_1, \tau_1^X \leq E^\lambda)\\
&\mathbb{P}(X(\tau_1^X) = y_{\ell_0+1(j_1\in\mathcal{S}_+)}, \varphi(\tau_1^X) = j_1, \tau_1^X \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i)\\
&= \mathbb{P}(X(\tau_n^X) = y_{\ell+1(j_n\in\mathcal{S}_-)}, \varphi(\tau_n^X) = j_n, \tau_n^X \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i)
\end{aligned} \tag{3.124}$$

where the swapping of limits and sums is justified as the sums are finite, and swapping limits and products is justified as the product is finite and all limits exist.

Therefore we have proved the following result

Lemma 3.4.1. *For all $\ell, \ell_0 \in \mathcal{K}$, $i, j_n \in \mathcal{S}$, $x_0 \in \mathcal{D}_{\ell_0,i}$, $n \geq 1$, then, as $p \to \infty$,*

$$\begin{aligned}
&\mathbb{P}(L^{(p)}(\tau_n^{(p)}) = \ell, \phi^{(p)}(\tau_n^{(p)}) = j_n, \tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \phi^{(p)}(0) = i),\\
&\to \mathbb{P}(X(\tau_n^X) = y_{\ell+1(j_n\in\mathcal{S}_-)}, \varphi(\tau_n^X) = j_n, \tau_n^X \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i).
\end{aligned} \tag{3.125}$$

3.5 Between the nth and $n+1$th change of level

Consider the Laplace transform

$$\int_{t=0}^\infty e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j, \tau_n^{(p)} \leq t < \tau_{n+1}^{(p)} \mid L^{(p)}(0) = \ell_0,$$

$$\boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.126}$$

Partitioning on the time of the nth change of level and the phase and level at this time gives

$$\begin{aligned}
&\int_{t=0}^{\infty} e^{-\lambda t} \int_{u_n=0}^{t} \sum_{j_n\in\mathcal{S}} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j, u_n < t \leq \tau^{(p)}_{n+1} \mid \\
&\qquad L^{(p)}(u_n) = \ell, \phi^{(p)}(u_n) = j_n, \tau^{(p)}_n = u_n)\psi(x)\,\mathrm{d}x \\
&\mathbb{P}(L^{(p)}(u_n) = \ell, \phi^{(p)}(u_n) = j_n, \tau^{(p)}_n \in \mathrm{d}u_n \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i) \\
&\qquad \mathrm{d}u_n\,\mathrm{d}t \\
&= \sum_{j_n\in\mathcal{S}} \int_{x\in\mathcal{D}_\ell} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j, \tau^{(p)}_n < t \leq \tau^{(p)}_{n+1} \mid \\
&\qquad L^{(p)}(\tau^{(p)}_n) = \ell, \phi^{(p)}(\tau^{(p)}_n) = j_n, \tau^{(p)}_n = 0)\,\mathrm{d}t\psi(x)\,\mathrm{d}x \\
&\mathbb{P}(L^{(p)}(\tau^{(p)}_n) = \ell, \phi^{(p)}(\tau^{(p)}_n) = j_n, \tau^{(p)}_n \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i)
\end{aligned} \tag{3.127}$$

by the time homogenous property of the QBD-RAP, the memoryless property of the exponential distribution, and the convolution theorem of Laplace transforms. We recognise the probability

$$\mathbb{P}(L^{(p)}(\tau^{(p)}_n) = \ell, \phi^{(p)}(\tau^{(p)}_n) = j_n, \tau^{(p)}_n \leq E^\lambda \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i) \tag{3.128}$$

as that appearing in Lemma (3.4.1). Hence (3.128) converges to

$$\mathbb{P}(X(\tau^X_n) = y_{\ell+1(j_n\in\mathcal{S}_-)}, \varphi(\tau^X_n) = j_n, \tau^X_n \leq E^\lambda \mid X(0) = x_0, \varphi(0) = i) \tag{3.129}$$

as $p \to \infty$.

Now consider the expression

$$\begin{aligned}
&\int_{x\in\mathcal{D}_\ell} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j, \tau^{(p)}_n < t \leq \tau^{(p)}_{n+1} \mid L^{(p)}(\tau^{(p)}_n) = \ell, \\
&\qquad \phi^{(p)}(\tau^{(p)}_n) = j_n, \tau^{(p)}_n = 0)\,\mathrm{d}t\psi(x)\,\mathrm{d}x
\end{aligned} \tag{3.130}$$

which appears as part of (3.127). We can rewrite (3.130) as

$$\begin{aligned}
&\int_{x\in\mathcal{D}_\ell} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \mathrm{d}x, \phi^{(p)}(t) = j, t \leq \tau^{(p)}_1 \mid L^{(p)}(0) = \ell, \\
&\qquad \phi^{(p)}(0) = j_n)\,\mathrm{d}t\psi(x)\,\mathrm{d}x
\end{aligned}$$

$$= \int_{x\in\mathcal{D}_\ell} \widehat{f}^{\ell,(p)}(\lambda)(x,j;x_0,j_n)\psi(x)\,\mathrm{d}x \tag{3.131}$$

Applying Lemma 3.3.8, then (3.131) converges to

$$\begin{aligned}&\int_{x\in\mathcal{D}_\ell} \widehat{\mu}^\ell(\lambda)(x,j;y_{\ell+1(j_n\in\mathcal{S}_-)},j_n)\psi(x)\,\mathrm{d}x\\ &= \int_{x\in\mathcal{D}_\ell}\int_{t=0}^\infty e^{-\lambda t}\mathbb{P}(X(t)\in\,\mathrm{d}x,\varphi(t)=j,t\le\tau_1^X \mid X(0)=y_{\ell+1(j_n\in\mathcal{S}_-)},\varphi(0)=j_n)\psi(x)\,\mathrm{d}x.\end{aligned} \tag{3.132}$$

Since the fluid queue is time homogeneous then we can write (3.132) as

$$\begin{aligned}\int_{x\in\mathcal{D}_\ell}\int_{t=0}^\infty & e^{-\lambda t}\mathbb{P}(X(t)\in\,\mathrm{d}x,\varphi(t)=j,\tau_n^X<t\le\tau_{n+1}^X \mid X(\tau_n^X)=y_{\ell+1(j_n\in\mathcal{S}_-)},\\ &\varphi(\tau_n^X)=j_n,\tau_n^X\le E^\lambda)\psi(x)\,\mathrm{d}x\end{aligned} \tag{3.133}$$

Returning to right-hand side of (3.127) and taking the limit as $p\to\infty$,

$$\begin{aligned}&\lim_{p\to\infty}\sum_{j_n\in\mathcal{S}}\int_{x\in\mathcal{D}_\ell}\int_{t=0}^\infty e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell,\bar{X}^{(p)}(t)\in\,\mathrm{d}x,\phi^{(p)}(t)=j,\tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid\\ &\qquad L^{(p)}(\tau_n^{(p)})=\ell,\phi^{(p)}(\tau_n^{(p)})=j_n,\tau_n^{(p)}=0)\,\mathrm{d}t\psi(x)\,\mathrm{d}x\\ &\quad\mathbb{P}(L^{(p)}(\tau_n)=\ell,\phi^{(p)}(\tau_n^{(p)})=j_n,\tau_n^{(p)}\le E^\lambda\mid L^{(p)}(0)=\ell_0,\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0),\phi^{(p)}(0)=i)\\ &=\sum_{j_n\in\mathcal{S}}\lim_{p\to\infty}\int_{x\in\mathcal{D}_\ell}\int_{t=0}^\infty e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell,\bar{X}^{(p)}(t)\in\,\mathrm{d}x,\phi^{(p)}(t)=j,\tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid\\ &\qquad L^{(p)}(\tau_n^{(p)})=\ell,\phi^{(p)}(\tau_n^{(p)})=j_n,\tau_n^{(p)}=0)\,\mathrm{d}t\psi(x)\,\mathrm{d}x\\ &\quad\lim_{p\to\infty}\mathbb{P}(L^{(p)}(\tau_n^{(p)})=\ell,\phi^{(p)}(\tau_n^{(p)})=j_n,\tau_n^{(p)}\le E^\lambda\mid L^{(p)}(0)=\ell_0,\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}_{\ell_0,i}^{(p)}(x_0),\\ &\qquad\phi^{(p)}(0)=i)\end{aligned}$$

since the sum is finite and the limits exist. Replacing the limits with their limiting values gives

$$\begin{aligned}&\sum_{j_n\in\mathcal{S}}\int_{x\in\mathcal{D}_\ell}\int_{t=0}^\infty e^{-\lambda t}\mathbb{P}(X(t)\in\,\mathrm{d}x,\varphi(t)=j_n,\tau_n^X<t\le\tau_{n+1}^X\mid X(\tau_n^X)=y_{\ell+1(j_n\in\mathcal{S}_-)},\\ &\qquad\varphi(\tau_n^X)=j_n,\tau_n^X\le E^\lambda)\psi(x)\,\mathrm{d}x\\ &\quad\mathbb{P}(X(\tau_n^X)=y_{\ell+1(j_n\in\mathcal{S}_-)},\varphi(\tau_n^X)=j_n,\tau_n^X\le E^\lambda\mid X(0)=x_0,\varphi(0)=i)\\ &=\int_{t=0}^\infty e^{-\lambda t}\int_{x\in\mathcal{D}_\ell}\mathbb{P}(X(t)\in\,\mathrm{d}x,\varphi(t)=j_n,\tau_n^X<t\le\tau_{n+1}^X\mid X(0)=x_0,\varphi(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.134}$$

Hence we have shown the following result

Lemma 3.5.1. *For $\ell, \ell_0 \in \mathcal{K}$, $i, j \in \mathcal{S}$, $n \geq 0$, then, as $p \to \infty$,*

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(L^{(p)}(t) = \ell, \bar{X}^{(p)}(t) \in \,\mathrm{d}x, \phi^{(p)}(t) = j, \tau_n^{(p)} < t \leq \tau_{n+1}^{(p)} \mid L^{(p)}(0) = \ell_0,$$
$$\boldsymbol{A}^{(p)}(0) = \boldsymbol{a}_{\ell_0,i}^{(p)}(x_0), \phi^{(p)}(0) = i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(X(t) \in \,\mathrm{d}x, \varphi(t) = j_n, \tau_n^X < t \leq \tau_{n+1}^X \mid X(0) = x_0, \varphi(0) = i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.135}$$

3.6 To global convergence

3.6.1 First, another domination condition

To extend Lemma 3.5.1 to show a global convergence result we will use the Dominated Convergence Theorem. Here we show some geometric bounds on the probability of n level changes of the QBD-RAP and fluid queue, which ultimately serve as a dominating function in the Dominated Convergence Theorem.

Lemma 3.6.1. *For all $i \in \mathcal{S}_+ \cup \mathcal{S}_-$, and $n \geq 2$,*

$$\mathbb{P}(\tau_n \leq E^\lambda \mid \phi(\tau_{n-1}) = i, \tau_{n-1} \leq E^\lambda) \leq b, \tag{3.136}$$

where

$$b = \min\left\{1 - e^{-q(\Delta+\varepsilon)}\left[1 - e^{q\varepsilon - \lambda\Delta/|c_{min}|}\right] + \frac{\mathrm{Var}(Z)}{\varepsilon^2} + |r_1|, \frac{q}{q+\lambda}\right\}$$

and

$$|r_1| \leq 2G\frac{\mathrm{Var}\,(Z)}{\varepsilon^2} + 2L\varepsilon.$$

Note that b and r_1 depend on p which has been suppressed to simplify notation. When explicitly needed, we use a superscript p to denote this dependence.

Proof. For the QBD-RAP, changes of level can only occur when $i \in \mathcal{S}_+ \cup \mathcal{S}_-$.

Suppose that the phase at time τ_{n-1} is $i \in \mathcal{S}_+$ and that at time τ_{n-1} the QBD-RAP is not at a boundary. The arguments for an initial phase $i \in \mathcal{S}_-$ are analogous. For the QBD-RAP to change level or hit a boundary or leave a boundary one of two things must happen, either;

1. the fluid remains in phase i until there is a change of level or boundary hit, or

2. the fluid changes phase before there is a change of level or a boundary is hit.

Hence, for sample paths which contribute to the Laplace transform, one of two things must happen, either;

1. the fluid remains in phase i until there is a change of level or a boundary is hit and E^λ does not occur before the change of level, or,
2. the fluid changes phase before there is a change of level or a boundary is hit and E^λ does not occur before the change of phase.

The probability of 1. is

$$\int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{s} e^{(T_{ii}-\lambda)x/|c_i|} \,\mathrm{d}x = e^{(T_{ii}-\lambda)\Delta/|c_i|} + r_1, \tag{3.137}$$

by Lemma A.1.1.

The probability of 2 is

$$\begin{aligned}\int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{e} e^{(T_{ii}-\lambda)x/|c_i|}(-T_{ii}/|c_i|) \,\mathrm{d}x = &\int_{x=0}^{\Delta+\varepsilon} \boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{e} e^{(T_{ii}-\lambda)x/|c_i|}(-T_{ii}/|c_i|) \,\mathrm{d}x \\ &+ \int_{x=\Delta+\varepsilon}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{e} e^{(T_{ii}-\lambda)x/|c_i|}(-T_{ii}/|c_i|) \,\mathrm{d}x. \end{aligned} \tag{3.138}$$

Now, since $\boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{e} \leq 1$ for $x \leq \Delta + \varepsilon$ then the first term on the right-hand side of (3.138) is less than or equal to

$$\int_{x=0}^{\Delta+\varepsilon} e^{(T_{ii}-\lambda)/|c_i|x}(-T_{ii}/|c_i|) \,\mathrm{d}x \leq \int_{x=0}^{\Delta+\varepsilon} e^{T_{ii}/|c_i|x}(-T_{ii}/|c_i|) \,\mathrm{d}x = 1 - e^{T_{ii}/|c_i|(\Delta+\varepsilon)}.$$

By Chebyshev's inequality, $\boldsymbol{\alpha} e^{\boldsymbol{S}x} \boldsymbol{e} \leq \dfrac{\mathrm{Var}(Z)}{\varepsilon^2}$ for $x > \Delta + \varepsilon$, hence the second term on the right-hand side of (3.138) is less than or equal to

$$\int_{x=\Delta+\varepsilon}^{\infty} \frac{\mathrm{Var}(Z)}{\varepsilon^2} e^{(T_{ii}-\lambda)x/|c_i|}(-T_{ii}/|c_i|) \,\mathrm{d}x \leq \frac{\mathrm{Var}(Z)}{\varepsilon^2}.$$

Putting these together, then the right-hand side of (3.138) is less than or equal to

$$1 - e^{T_{ii}(\Delta+\varepsilon)/|c_i|} + \frac{\mathrm{Var}(Z)}{\varepsilon^2}. \tag{3.139}$$

Combining (3.137) and (3.139), then $\mathbb{P}(\tau_n \leq E^\lambda \mid \phi(\tau_{n-1}) = i, \tau_{n-1} \leq E^\lambda)$ is less than or equal to

$$e^{(T_{ii}-\lambda)\Delta/|c_i|} + |r_1| + 1 - e^{T_{ii}(\Delta+\varepsilon)/|c_i|} + \frac{\mathrm{Var}(Z)}{\varepsilon^2}$$

$$= 1 - e^{T_{ii}(\Delta+\varepsilon)/|c_i|}\left[1 - e^{(-T_{ii}\varepsilon-\lambda\Delta)/|c_i|}\right] + \frac{\mathrm{Var}(Z)}{\varepsilon^2} + |r_1|$$
$$= 1 - e^{-q(\Delta+\varepsilon)}\left[1 - e^{q\varepsilon-\lambda\Delta/|c_{min}|}\right] + \frac{\mathrm{Var}(Z)}{\varepsilon^2} + |r_1|, \tag{3.140}$$

since $-T_{ii}/|c_i| \leq q$ and $\lambda\Delta/|c_i| \leq \lambda\Delta/c_{min}$ for all $i \in \mathcal{S}_+ \cup \mathcal{S}_-$.

Now consider the QBD-RAP at a boundary. To leave the boundary there must be at-least one change of phase before E^λ. By a uniformisation argument, the probability of at-least one change of phase before E^λ is $q/(q+\lambda)$. □

Lemma 3.6.2. *For $n \geq 2$, $i \in \mathcal{S}_+ \cup \mathcal{S}_-$,*

$$\mathbb{P}(\tau_n \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda) \leq b^{n-1}. \tag{3.141}$$

Proof. The proof is by induction.

For the base case, set $n = 2$ and apply Lemma 3.6.1.

Now, assume the induction hypothesis $\mathbb{P}(\tau_{n-1} \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda) \leq b^{n-2}$ for arbitrary $n \geq 3$.

Since $\{\tau_{n-1} \leq E^\lambda\}$ is a subset of $\{\tau_n \leq E^\lambda\}$, then

$$\mathbb{P}(\tau_n \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda) = \mathbb{P}(\tau_n \leq E^\lambda, \tau_{n-1} \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda). \tag{3.142}$$

Now partition (3.142) on the phase at time τ_{n-1},

$$\sum_{j_{n-1}\in\mathcal{S}} \mathbb{P}(\tau_n \leq E^\lambda, \tau_{n-1} \leq E^\lambda, \phi(\tau_{n-1}) = j_{n-1} \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda)$$
$$= \sum_{j_{n-1}\in\mathcal{S}} \mathbb{P}(\tau_n \leq E^\lambda \mid \phi(\tau_{n-1}) = j_{n-1}, \tau_{n-1} \leq E^\lambda)$$
$$\times \mathbb{P}(\phi(\tau_{n-1}) = j_{n-1}, \tau_{n-1} \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda), \tag{3.143}$$

by the strong Markov property of the QBD-RAP and the fact that $\boldsymbol{A}(\tau_{n-1}) = \boldsymbol{\alpha}$.

By Lemma 3.6.1 (3.143) is less than or equal to

$$\sum_{j_{n-1}\in\mathcal{S}} b\mathbb{P}(\phi(\tau_{n-1}) = j_{n-1}, \tau_{n-1} \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda)$$
$$= b\mathbb{P}(\tau_{n-1} \leq E^\lambda \mid \phi(\tau_1) = i, \tau_1 \leq E^\lambda)$$
$$\leq b \cdot b^{n-2}, \tag{3.144}$$

by the induction hypothesis, and this completes the proof. □

Corollary 3.6.3.

$$\left| \int_{t=0}^\infty e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathrm{d}x, \phi(t) = j, \tau_n \leq t < \tau_{n+1} \mid L(0) = \ell_0, \right.$$

$$\left.\boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t\right|$$

$$\leq \frac{Fb^{n-1}}{\lambda} \tag{3.145}$$

Proof. First, since $|\psi(x)| \leq F$, then the left-hand side of (3.145) is less than or equal to

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathrm{d}x, \phi(t) = j, \tau_n \leq t < \tau_{n+1} \mid L(0) = \ell_0,$$
$$\boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)F\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathcal{D}_{\ell,j}, \phi(t) = j, \tau_n \leq t < \tau_{n+1} \mid L(0) = \ell_0,$$
$$\boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)F\,\mathrm{d}t.$$

Partitioning on the time, τ_1, of the 1st change of level, and the phase and level at time τ_1,

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{u_1=0}^{t} \sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathcal{D}_{\ell,j}, \phi(t) = j, \tau_n \leq t < \tau_{n+1}$$
$$\mid L(\tau_1) = \ell_1, \phi(\tau_1) = j_1, \tau_1 = u_1)$$
$$\mathbb{P}(L(\tau_1) = \ell_1, \phi(\tau_1) = j_1, \tau_1 \in \mathrm{d}u_1 \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)F\,\mathrm{d}t$$
$$= \sum_{j_1\in\mathcal{S}} \sum_{\ell_1\in\{\ell_0+1,\ell_0-1\}\cap\mathcal{K}} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathcal{D}_{\ell,j}, \phi(t) = j, \tau_n \leq t < \tau_{n+1}$$
$$\mid L(0) = \ell_1, \phi(0) = j_1, \tau_1 = 0)\,\mathrm{d}t$$
$$\mathbb{P}(L(\tau_1) = \ell_1, \phi(\tau_1) = j_1, \tau_1 \leq E^\lambda \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i)F, \tag{3.146}$$

by the time-homogeneous property of the QBD-RAP and the convolution theorem for Laplace transforms.

The expression

$$\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L(t) = \ell, \bar{X}(t) \in \mathcal{D}_{\ell,j}, \phi(t) = j, \tau_n \leq t < \tau_{n+1} \mid L(0) = \ell_1, \phi(0) = j_1, \tau_1 = 0)\,\mathrm{d}t$$
$$\leq \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(\tau_n \leq t \mid L(0) = \ell_1, \phi(0) = j_1, \tau_1 = 0)\,\mathrm{d}t$$
$$\leq b^{n-1}\int_{t=0}^{\infty} e^{-\lambda t}\,\mathrm{d}t$$
$$= b^{n-1}\frac{1}{\lambda}, \tag{3.147}$$

by Lemma 3.6.2.

Using the bound(3.147) in (3.146), gives the result. □

3.6.2 Global convergence

Consider the Laplace transform of the QBD-RAP

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_{\ell,j}} \mathbb{P}(L^{(p)}(t)=\ell, \bar{X}^{(p)}(t)\in \mathrm{d}x, \phi^{(p)}(t)=j \mid L^{(p)}(0)=\ell_0,$$
$$\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.148}$$

Partition on the number of level changes by time t,

$$\int_{x\in\mathcal{D}_{\ell,j}} \int_{t=0}^{\infty} e^{-\lambda t} \sum_{n=0}^{\infty} \mathbb{P}(L^{(p)}(t)=\ell, \boldsymbol{A}^{(p)}(t)\in \mathrm{d}\boldsymbol{a}^{(p)}, \phi^{(p)}(t)=j, \tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid$$
$$L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i)\,\mathrm{d}t\psi(x)\,\mathrm{d}x$$
$$=\sum_{n=0}^{\infty} \int_{x\in\mathcal{D}_{\ell,j}} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell, \boldsymbol{A}^{(p)}(t)\in \mathrm{d}\boldsymbol{a}^{(p)}, \phi^{(p)}(t)=j, \tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid$$
$$L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i)\,\mathrm{d}t\psi(x)\,\mathrm{d}x. \tag{3.149}$$

By Lemma (3.5.1), each term in the sum (3.149) converges. Furthermore, for $n\ge 1$, each term is dominated by $\left(b^{(p)}\right)^{n-1} F/\lambda$, from Corollary 3.6.3. The dominating terms $\left(b^{(p)}\right)^{n-1} F/\lambda$ depend on p and may not be summable. However, for p sufficiently large, there exists a $p_0<\infty$ and a B with $B<1$ such that $b^{(p)}<B$ for all $p>p_0$.

Therefore we can apply the Dominated Convergence Theorem,

$$\lim_{p\to\infty}\sum_{n=0}^{\infty} \int_{x\in\mathcal{D}_{\ell,j}} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell, \boldsymbol{A}^{(p)}(t)\in \mathrm{d}\boldsymbol{a}^{(p)}, \phi^{(p)}(t)=j, \tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid$$
$$L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i)\,\mathrm{d}t\psi(x)\,\mathrm{d}x$$
$$=\sum_{n=0}^{\infty} \int_{x\in\mathcal{D}_{\ell,j}} \lim_{p\to\infty}\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell, \boldsymbol{A}^{(p)}(t)\in \mathrm{d}\boldsymbol{a}^{(p)}, \phi^{(p)}(t)=j, \tau_n^{(p)}<t\le\tau_{n+1}^{(p)} \mid$$
$$L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i)\,\mathrm{d}t\psi(x)\,\mathrm{d}x$$
$$=\sum_{n=0}^{\infty}\int_{t=0}^{\infty} e^{-\lambda t}\int_{x\in\mathcal{D}_{\ell}} \mathbb{P}(X(t)\in \mathrm{d}x, \varphi(t)=j, \tau_n<t\le\tau_{n+1} \mid X(0)=x_0, \varphi(0)=i)$$
$$\times\,\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

where the limit is given by Lemma 3.5.1. Swapping the sum and integrals and by the law of total probability,

$$\int_{t=0}^{\infty} e^{-\lambda t}\int_{x\in\mathcal{D}_{\ell}} \sum_{n=0}^{\infty}\mathbb{P}(X(t)\in \mathrm{d}x, \varphi(t)=j, \tau_n<t\le\tau_{n+1} \mid X(0)=x_0, \varphi(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

$$= \int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_\ell} \mathbb{P}(X(t)\in \,\mathrm{d}x, \varphi(t)=j \mid X(0)=x_0, \varphi(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Thus, we have shown the following result

Lemma 3.6.4. *For all $\ell_0, \ell \in \mathcal{K}$, $i,j\in\mathcal{S}$, $x_0\in\mathcal{D}_{\ell_0,i}$, as $p\to\infty$,*

$$\int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_{\ell,j}} \mathbb{P}(L^{(p)}(t)=\ell, \bar{X}^{(p)}(t)\in \,\mathrm{d}x, \phi^{(p)}(t)=j \mid L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),$$
$$\phi^{(p)}(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$\to \int_{t=0}^{\infty} e^{-\lambda t} \int_{x\in\mathcal{D}_{\ell,j}} \mathbb{P}(X(t)\in \,\mathrm{d}x, \varphi(t)=j \mid X(0)=x_0, \varphi(0)=i)\psi(x)\,\mathrm{d}x\,\mathrm{d}t.$$

Corollary 3.6.5. *Let $\psi(x,j):\mathbb{R}\times\mathcal{S}\to\mathbb{R}$ be Lipschitz continuous and bounded functions, $|\psi(x,j)|\le F$. For each $x_0\in\mathbb{R}$, $i\in\mathcal{S}$, $\ell_0\in\mathcal{K}$,*

$$\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\left[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t)) \mid L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i\right]\,\mathrm{d}t$$
$$\to \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\left[\psi(X(t),\phi(t)) \mid X(0)=x_0, \varphi(0)=i\right]\,\mathrm{d}t.$$

Proof. Consider the left-hand side

$$\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\left[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t)) \mid L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}_{\ell_0,i}(x_0)^{(p)}, \phi^{(p)}(0)=i\right]\,\mathrm{d}t$$
$$= \int_{t=0}^{\infty} e^{-\lambda t}\sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}}\mathbb{E}\Big[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t))1(L^{(p)}(t)=\ell, \bar{X}^{(p)}(t)\in\mathcal{D}_{\ell,j}, \phi^{(p)}(t)=j)$$
$$\mid L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i\Big]\,\mathrm{d}t$$
$$= \sum_{\ell\in\mathcal{K}}\sum_{j\in\mathcal{S}}\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\Big[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t))1(L^{(p)}(t)=\ell, \bar{X}^{(p)}(t)\in\mathcal{D}_{\ell,j}, \phi^{(p)}(t)=j) \mid L$$
$$^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i\Big]\,\mathrm{d}t. \tag{3.150}$$

By Lemma 3.6.4, for each $\ell\in\mathcal{K}$, $j\in\mathcal{S}$, the terms

$$\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\Big[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t))1(L^{(p)}(t)=\ell, \bar{X}^{(p)}(t)\in\mathcal{D}_{\ell,j}, \phi^{(p)}(t)=j) \mid$$
$$L^{(p)}(0)=\ell_0, \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0)=i\Big]\,\mathrm{d}t \tag{3.151}$$

converge to

$$\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\left[\psi(X(t),\phi(t))1(X(t)\in\mathcal{D}_{\ell_j}, \varphi(t)=j) \mid X(0)=x_0, \varphi(0)=i\right]\,\mathrm{d}t.$$

If $\mathcal{K}$ is finite, we are done upon taking the limit of (3.150) as $p \to \infty$ and swapping the limit and the sums.

If $\mathcal{K}$ is countably infinite, then for a given $k \in \mathcal{K}$, since ψ is bounded

$$\begin{aligned}
&\left| \sum_{j\in\mathcal{S}} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\Big[\psi(\bar{X}^{(p)}(t),\phi^{(p)}(t))1(L^{(p)}(t)=\ell,\bar{X}^{(p)}(t)\in\mathcal{D}_{\ell,j},\phi^{(p)}(t)=j)\mid L^{(p)}(0)=\ell_0,\right.\\
&\qquad \left.\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),\phi^{(p)}(0)=i\Big]\,\mathrm{d}t\right|\\
&\le F\sum_{j\in\mathcal{S}} \int_{t=0}^{\infty} e^{-\lambda t}\mathbb{E}\Big[1(L^{(p)}(t)=\ell,\bar{X}^{(p)}(t)\in\mathcal{D}_{\ell,j},\phi^{(p)}(t)=j)\mid L^{(p)}(0)=\ell_0,\\
&\qquad \boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),\phi^{(p)}(0)=i\Big]\,\mathrm{d}t\\
&\le F\int_{t=0}^{\infty} e^{-\lambda t}\mathbb{P}(L^{(p)}(t)=\ell,\bar{X}^{(p)}(t)\in\mathcal{D}_{\ell}\mid L^{(p)}(0)=\ell_0,\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),\phi^{(p)}(0)=i)\,\mathrm{d}t\\
&\le F\mathbb{P}(\tau^{(p)}_{|\ell-\ell_0|}\le E^{\lambda}\mid L^{(p)}(0)=\ell_0,\boldsymbol{A}^{(p)}(0)=\boldsymbol{a}^{(p)}_{\ell_0,i}(x_0),\phi^{(p)}(0)=i),
\end{aligned} \tag{3.152}$$

since, to be in level ℓ after starting in level ℓ_0, there must be at least $|\ell_0-\ell|$ changes of level. By Lemma 3.6.2 then (3.152) is bounded by $\left(b^{(p)}\right)^{|\ell-\ell_0|-1}$ for $|\ell-\ell_0|\ge 2$ and by 1 otherwise. Now, choose p_0 sufficiently large so that $b^{(p)}<B<1$ for all $p>p_0$. Therefore, for all $p>p_0$, the terms in (3.150) are dominated by $F\min\{B^{|\ell-\ell_0|-1},1\}$. Moreover

$$\begin{aligned}
F\sum_{\ell\in\mathcal{K}}\min\{B^{|\ell-\ell_0|-1},1\} &\le 2\sum_{n=1}^{\infty}B^{n-1}+1\\
&=\frac{2}{1-B}+1\\
&<\infty,
\end{aligned}$$

hence the dominating terms are summable. Hence we may apply the Dominated Convergence Theorem to swap the necessary limits and sums. □

The Extended Continuity Theorem for Laplace transforms (Feller 1957 - 1971, Chapter XIII, Theorem 2a) can now be use to claim that the QBD-RAP approximation scheme converges weakly to the fluid queue.

Theorem 3.6.6 (Extended Continuity Theorem)**.** *For $p=1,2,\ldots$ let U_p be a measure with Laplace transform ω_p. If $\omega_p(\lambda)\to\omega(\lambda)$ for $\lambda>a\ge 0$, then ω is the Laplace transform of a measure U and $U_p\to U$.*

Conversely, if $U_p\to U$ and the sequence $\{\omega_p(a)\}$ is bounded, then $\omega_p(\lambda)\to\omega(\lambda)$ for $\lambda>a$.

Thus, by the the Extended Continuity Theorem 3.6.6

$$\mathbb{E}\left[\psi(\bar{X}^{(p)}(t), \phi^{(p)}(t)) \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i\right]$$
$$\to \mathbb{E}\left[\psi(X(t), \phi(t)) \mid X(0) = x_0, \varphi(0) = i\right]$$

weakly in t as $p \to \infty$. Now,

$$\mathbb{E}\left[\psi(X(t), \phi(t)) \mid X(0) = x_0, \varphi(0) = i\right]$$

is a continuous function of t (it is a Feller semi-group), moreover, for $p < \infty$,

$$\mathbb{E}\left[\psi(\bar{X}^{(p)}(t), \phi^{(p)}(t)) \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i\right]$$

is also continuous in t. However, the limit

$$\lim_{p\to\infty} \mathbb{E}\left[\psi(\bar{X}^{(p)}(t), \phi^{(p)}(t)) \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i\right]$$

may not be continuous in t. We can claim that

$$\mathbb{E}\left[\psi(\bar{X}^{(p)}(t), \phi^{(p)}(t)) \mid L^{(p)}(0) = \ell_0, \boldsymbol{A}^{(p)}(0) = \boldsymbol{a}^{(p)}_{\ell_0,i}(x_0), \phi^{(p)}(0) = i\right]$$
$$\to \mathbb{E}\left[\psi(X(t), \phi(t)) \mid X(0) = x_0, \varphi(0) = i\right]$$

for almost all $t \geq 0$. At such values of t, since ψ is an arbitrary bounded, Lipschitz continuous function, then the Portmanteau Theorem states that the QBD-RAP approximation scheme converges in distribution to the fluid queue.

A sufficient condition to upgrade the convergence from weak to point-wise (in the variable t) is to show that for $t \geq 0$

$$\sup_p \mathbb{E}\left[\psi(\bar{X}(t), \phi(t)) \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i\right] \leq M(t) < \infty$$

and the sequence $\mathbb{E}\left[\psi(\bar{X}(t), \phi(t)) \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i\right]$ is eventually equicontinuous in t. That is, for every $\varepsilon > 0$ there exists a $\delta(t,\varepsilon) > 0$ and an $p_0(t,\varepsilon)$ such that $|t-u| < \delta(t,\varepsilon)$ implies that

$$\left|\mathbb{E}\left[\psi(\bar{X}(t), \phi(t)) \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i\right] - \mathbb{E}\left[\psi(\bar{X}(u), \phi(u)) \mid L(0) = \ell_0, \boldsymbol{A}(0) = \boldsymbol{a}_{\ell_0,i}(x_0), \phi(0) = i\right]\right| < \varepsilon \qquad (3.153)$$

for all $p \geq p_0(t,\varepsilon)$. This is an area for future work.

Remark 3.6.7. *While the convergence result implies a loose bound on the rate of convergence, it applies to a wide class of QBD-RAP constructions. The main cause of the loose bound on the convergence rate is due to our reliance on Chebyshev's inequality. This is necessary as we try to make minimal assumptions about the matrix-exponential distributions used in the construction. In practice, we use a class of* concentrated matrix-exponential distributions *found numerically in (Horváth et al. 2020), and for which there is relatively little known about their properties. This necessitates the generality of the convergence result.*

3.7 Extension to arbitrary (but fixed) discretisation structures

Throughout, we have assumed that all intervals are of width Δ, i.e. $|y_{\ell+1} - y_\ell| = \Delta$, and that on every interval the dynamics of the fluid queue are modelled based on the same matrix exponential representation $(\boldsymbol{\alpha}, \boldsymbol{S}, \boldsymbol{s})$. These assumptions are, in fact, not necessary, but they do serve to simplify the presentation. The convergence results can be extended to use different sequences of matrix exponential representations on each interval, provided that for each sequence of matrix exponential distributions, the variance tends to 0. Moreover, we can extend the results to intervals of arbitrary width, provided that the width of the intervals is not arbitrarily small. Here we describe how one would prove such results.

The arguments which prove Lemma 3.3.8 are independent of all other levels/intervals, i.e. the hypotheses of the Lemma depend only on the interval $\mathcal{D}_{\ell_0}$, and the sequence of matrix exponential distributions used to model the behaviour of the fluid queue on this interval and not on any other interval. Thus Lemma 3.3.8 holds independently on each interval, as does Corollary 3.3.10.

Let the width of an interval $\mathcal{D}_{\ell_0}$ be $\Delta_{\ell_0} = y_{\ell_0+1} - y_{\ell_0}$ and suppose that sequence of matrix exponential random variable used to model the dynamics of the fluid queue on the interval $\mathcal{D}_{\ell_0}$ is $Z_{\ell_0}^{(p)}$. Regarding Lemma 3.6.1, we can extend it the following version,

Lemma 3.7.1. *Assume* $\inf_{\ell_0} \Delta_{\ell_0} > 0$ *and* $\sup_{\ell_0} \mathrm{Var}(Z_{\ell_0}) < \infty$ *exist. Then, for all* $i \in \mathcal{S}_+ \cup \mathcal{S}_-$, $\ell_0 \in \mathcal{K} \setminus \{-1, K+1\}$, *and* $n \geq 2$,

$$\mathbb{P}(\tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(\tau_{n-1}^{(p)}) = \ell_0, \phi^{(p)}(\tau_{n-1}^{(p)}) = i, \tau_{n-1}^{(p)} \leq E^\lambda) \leq b_{\ell_0}^{(p)}, \tag{3.154}$$

where

$$b_{\ell_0}^{(p)} = 1 - e^{-q(\Delta_{\ell_0} + \varepsilon_{\ell_0}^{(p)})} \left[1 - e^{q\varepsilon_{\ell_0}^{(p)} - \lambda \Delta_{\ell_0}/|c_{min}|}\right] + \frac{\mathrm{Var}(Z_{\ell_0}^{(p)})}{\left(\varepsilon^{(p)}\right)^2} + |r_{1,\ell_0}^{(p)}|$$

and

$$|r_{1,\ell_0}^{(p)}| \leq 2G\frac{\mathrm{Var}\left(Z_{\ell_0}^{(p)}\right)}{\left(\varepsilon_{\ell_0}^{(p)}\right)^2} + 2L\varepsilon_{\ell_0}^{(p)}.$$

Hence, for all $i \in \mathcal{S}_+ \cup \mathcal{S}_-$, *and* $n \geq 2$,

$$\mathbb{P}(\tau_n^{(p)} \leq E^\lambda \mid \phi^{(p)}(\tau_{n-1}^{(p)}) = i, \tau_{n-1}^{(p)} \leq E^\lambda) \leq b^{(p)}, \tag{3.155}$$

where

$$b^{(p)} = \max\left\{\sup_{\ell_0} b_{\ell_0}^{(p)}, \frac{q}{\lambda+q}\right\}.$$

Proof. For the proof of (3.154) follow the same arguments as in the proof of Lemma 3.6.1. The bound in equation follows by the assumptions that $\inf_{\ell_0}\Delta_{\ell_0}$ and $\sup_{\ell_0}\mathrm{Var}(Z_{\ell_0}^{(p)})$ exist, then

$$\begin{aligned}
&\mathbb{P}(\tau_n^{(p)} \leq E^\lambda \mid \phi^{(p)}(\tau_{n-1}^{(p)}) = i, \tau_{n-1}^{(p)} \leq E^\lambda) \\
&\leq \sup_{\ell_0} \mathbb{P}(\tau_n^{(p)} \leq E^\lambda \mid L^{(p)}(\tau_{n-1}^{(p)}) = \ell_0, \phi^{(p)}(\tau_{n-1}^{(p)}) = i, \tau_{n-1}^{(p)} \leq E^\lambda) \\
&\leq \max\left\{\sup_{\ell_0} b_{\ell_0}^{(p)}, \frac{q}{\lambda+q}\right\}.
\end{aligned}$$

□

Given Lemma 3.7.1, then an equivalent of Lemma 3.6.2 remains true, the proof of which follows verbatim except with the use of Lemma 3.6.1 replaced by Lemma 3.7.1. Corollary 3.6.3 remains true without modification. Lemma 3.6.4 and Corollary 3.6.5 remain true without modification provided that $\lim_{p\to\infty}\mathrm{Var}(Z_\ell^{(p)}) \to 0$ for all ℓ.

Appendix A

Technical results for covergence

A.1 Some bounds for integrating against matrix exponential distributions.

This appendix proves some technical Lemmas about integrating functions with respect to matrix exponential distributions.

As with Section 3, we use the superscript (p) to denote dependence on the underlying choice of matrix exponential random variable $Z^{(p)}$. However, to simplify notation, we may omit the super script (p) where it is not explicitly needed. We show results for an arbitrary parameter $\varepsilon > 0$. Keep in mind the ultimate intention is to show convergence, for which we choose this parameter to be $\varepsilon^{(p)} = \mathrm{Var}\left(Z^{(p)}\right)^{1/3}$. Other notations, previously defined and which depend on p are $\boldsymbol{\alpha}^{(p)}$, $\boldsymbol{\alpha}_0^{i,\ell_0,(p)}(x_0)$, $\boldsymbol{S}^{(p)}$, $\boldsymbol{s}^{(p)}$, $\varepsilon^{(p)}$, $V^{(p)}(x)$, $R_{V,1}^{(p)}$, $R_{V,2}^{(p)}$, $\boldsymbol{D}^{(p)}$, $\mathcal{A}^{(p)}$.

A.1.1 One integral

Here we show a collection of results that show that integrating functions against a ME density function, or against the density function of a ME conditional on the ME-life-time surviving until some time $u < \Delta-\varepsilon$ where $\Delta = \mathbb{E}[Z]$ is the mean of the matrix exponential distribution, approximates integrating said function against a Kronecker delta situated at the Δ, provided the variance of the ME is sufficiently low.

Lemma A.1.1. *Let g be a function satisfying Assumptions 3.0.1, then, for $u \leq \Delta - \varepsilon$,*

$$\int_{x=0}^{\infty} g\left(x\right) \boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s} \,\mathrm{d}x = g\left(\Delta - u\right) + r_1,$$

where

$$|r_1| \leq 2G \frac{\mathrm{Var}\left(Z\right)}{\varepsilon^2} + 2L\varepsilon.$$

The proof follows closely that of (Horváth et al. 2020, Appendix A, Theorem 4).

Proof. By a change of variables,

$$\begin{aligned}
&\left|\int_{x=0}^{\infty} g\left(x\right)\boldsymbol{\alpha}e^{\boldsymbol{S}(x+u)}\boldsymbol{s}\,\mathrm{d}x - g\left(\Delta-u\right)\right| \\
&=\left|\int_{x=u}^{\infty} g\left(x-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x - g\left(\Delta-u\right)\right| \\
&=\left|\int_{x=u}^{\infty} g\left(x-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x - \int_{x=u}^{\infty} g\left(\Delta-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x - g\left(\Delta-u\right)\left(1-\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}\right)\right|.
\end{aligned}$$

By the triangle inequality this is less than or equal to

$$\begin{aligned}
&\left|\int_{x=u}^{\infty}\left(g\left(x-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| + \left|g\left(\Delta-u\right)\left(1-\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}\right)\right| \\
&=\left|\int_{x=u}^{\infty}\left(g\left(x-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| + \left|\int_{x=0}^{u} g\left(\Delta-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| \\
&\leq d_1+d_2
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \left|\int_{x=0}^{u} g\left(\Delta-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| + \left|\int_{x=u}^{\Delta-\varepsilon}\left(g\left(x-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| \\
&\quad + \left|\int_{x=\Delta+\varepsilon}^{\infty}\left(g\left(x-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}t}\boldsymbol{s}\,\mathrm{d}x\right|, \\
d_2 &= \left|\int_{x=\Delta-\varepsilon}^{\Delta+\varepsilon}\left(g\left(t-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right|.
\end{aligned}$$

Applying the triangle inequality to d_1,

$$\begin{aligned}
d_1 \leq &\int_{x=u}^{\Delta-\varepsilon}\left|g\left(x-u\right)-g\left(\Delta-u\right)\right|\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x + \int_{x=\Delta+\varepsilon}^{\infty}\left|g\left(x-u\right)-g\left(\Delta-u\right)\right|\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x \\
&+\left|\int_{x=0}^{u} g\left(\Delta-u\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right|. \qquad \text{(A.1)}
\end{aligned}$$

Since $|g\left(x\right)| \leq G$, then (A.1) is less than or equal to

$$2G\left(\int_{x=u}^{\Delta-\varepsilon}\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x + \int_{x=\Delta+\varepsilon}^{\infty}\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x + \int_{x=0}^{u}\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right) = 2G\mathbb{P}\left(|Z-\Delta|>\varepsilon\right). \qquad \text{(A.2)}$$

By Chebyshev's inequality,

$$2G\mathbb{P}\left(|Z-\Delta|>\varepsilon\right)\le 2G\frac{\mathrm{Var}\left(Z\right)}{\varepsilon^2}. \tag{A.3}$$

For the term d_2 we have

$$\begin{aligned}
d_2 &= \left|\int_{x=\Delta-\varepsilon}^{\Delta+\varepsilon}\left(g\left(x-u\right)-g\left(\Delta-u\right)\right)\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x\right| \\
&\le \int_{x=\Delta-\varepsilon}^{\Delta+\varepsilon}\left|g\left(x-u\right)-g\left(\Delta-u\right)\right|\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x \\
&\le \int_{x=\Delta-\varepsilon}^{\Delta+\varepsilon} 2L\varepsilon\boldsymbol{\alpha}e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x \\
&= 2L\varepsilon\mathbb{P}(Z\in(\Delta-\varepsilon,\Delta+\varepsilon)) \\
&\le 2L\varepsilon,
\end{aligned}$$

where the first inequality is the triangle inequality and the second inequality is from the Lipschitz property of g in Assumption 3.0.1(iv). Hence there is some r_1 such that

$$\left|\int_{x=0}^{\infty} g\left(x\right)\boldsymbol{\alpha}e^{\boldsymbol{S}(x+u)}\boldsymbol{s}\,\mathrm{d}x - g\left(\Delta-u\right)\right| = |r_1| \le 2G\frac{\mathrm{Var}(Z)}{\varepsilon^2}+2L\varepsilon,$$

and this completes the proof. □

The error term $r_1^{(p)}$ depends on p, as it is defined by $Z^{(p)}$ and $\varepsilon^{(p)}$, but we have omitted the superscript p here. Note that, upon choosing $\varepsilon=\mathrm{Var}(Z^{(p)})^{1/3}$, the error term $|r_1^{(p)}|$ is at most $O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right)$, which tends to 0 as $p\to\infty$.

Corollary A.1.2. *Let g be a function satisfying the Assumptions 3.0.1. For $u\le\Delta-\varepsilon$,*

$$\int_{x=0}^{\infty}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(x+u)}\boldsymbol{s}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}g(x)\,\mathrm{d}x = g(\Delta-u)+r_2,$$

where

$$|r_2|\le 3G\frac{\mathrm{Var}\left(Z\right)}{\varepsilon^2}+2L\varepsilon.$$

Proof. Observe that Chebyshev's inequality gives

$$\begin{aligned}
\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e} &= \mathbb{P}\left(Z>u\right) \\
&\ge \mathbb{P}\left(|Z-\Delta|\le\varepsilon\right) \\
&\ge 1-\frac{\mathrm{Var}\left(Z\right)}{\varepsilon^2}
\end{aligned}$$

$$=: 1 - \delta.$$

Now, since $1 - \delta \leq \boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e} \leq 1$, then

$$\int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s} g(x) \,\mathrm{d}x \leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x \leq \frac{1}{1-\delta} \int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s} g(x) \,\mathrm{d}x.$$

By Lemma A.1.1 we have

$$g(\Delta - u) + r_1 \leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x \leq \frac{g(\Delta - u) + r_1}{1-\delta}.$$

Multiplying by $1-\delta$, then subtracting $g(\Delta - u)$ and adding $\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x\delta$ gives

$$\begin{aligned}
&r_1(1-\delta) - g(\Delta - u)\delta + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x\delta \\
&\leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x - g(\Delta - u) \\
&\leq r_1 + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x\delta.
\end{aligned}$$

The right-hand side is bounded above as

$$r_1 + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x\delta \leq r_1 + G\delta.$$

The left-hand side is bounded below as

$$r_1(1-\delta) - g(\Delta - u)\delta + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x\delta \geq r_1(1-\delta) - g(\Delta - u)\delta.$$

Hence,

$$\left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x - g(\Delta - u) \right| \leq \max\left(r_1(1-\delta) + g(\Delta - u)\delta, r_1 + G\delta\right) \tag{A.4}$$

and therefore

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u)} \boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u} \boldsymbol{e}} g(x) \,\mathrm{d}x = g(\Delta - u) + r_2, \tag{A.5}$$

where

$$|r_2| \leq \max\left(r_1(1-\delta) + g(\Delta - u)\delta, r_1 + G\delta\right)$$

$$\begin{aligned}&\leq r_1 + G\delta \\ &= 3G\frac{\mathrm{Var}\,(Z)}{\varepsilon^2} + 2L\varepsilon. \end{aligned} \tag{A.6}$$

This completes the proof. □

The error term $r_2^{(p)}$ also depends on p, as it is defined by $Z^{(p)}$ and $\varepsilon^{(p)}$, but we have omitted the superscript p here. Choosing $\varepsilon = \mathrm{Var}(Z^{(p)})$, the error term $|r_2^{(p)}|$ is at most $O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right)$, which tends to 0 as $p \to \infty$.

Corollary A.1.3. *Let g be a function satisfying the Assumptions 3.0.1. For $u \leq \Delta - \varepsilon$, $v \geq 0$,*

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x = g(\Delta - u - v)1(u+v \leq \Delta - \varepsilon) + r_3(u+v),$$

where

$$|r_3(u+v)| \leq \begin{cases} r_2 & u+v \leq \Delta - \varepsilon, \\ G & u+v \in (\Delta - \varepsilon, \Delta + \varepsilon), \\ G\dfrac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2} & u+v \geq \Delta + \varepsilon. \end{cases}$$

Proof. For $u+v \leq \Delta - \varepsilon$ we use similar arguments those as in the proof of Corollary A.1.2. We have

$$\int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s} g(x)\,\mathrm{d}x \leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x \leq \frac{1}{1-\delta}\int_{x=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s} g(x)\,\mathrm{d}x.$$

By Lemma A.1.1

$$g(\Delta - u - v) + r_1 \leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x \leq \frac{g(\Delta - u - v) + r_1}{1-\delta}.$$

Multiplying by $1-\delta$, then subtracting $g(\Delta - u - v)$ and adding $\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x\delta$ gives

$$\begin{aligned}&r_1(1-\delta) - g(\Delta - u - v)\delta + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x\delta \\ &\leq \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x - g(\Delta - u - v) \\ &\leq r_1 + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x\delta. \end{aligned}$$

The right-hand side is bounded above by

$$r_1 + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x\delta \leq r_1 + G\delta.$$

The left-hand side is bounded below by

$$r_1(1-\delta) - g(\Delta-u-v)\delta + \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x\delta \geq r_1(1-\delta) - g(\Delta-u-v)\delta.$$

and ultimately,

$$\left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x - g(\Delta-u-v) \right| \leq 3G\frac{\mathrm{Var}\,(Z)}{\varepsilon^2} + 2L\varepsilon. \tag{A.7}$$

For $u+v \in (\Delta-\varepsilon, \Delta+\varepsilon)$,

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x \leq G\mathbb{P}(Z > u+v \mid Z > u) \leq G \tag{A.8}$$

For $u+v \geq \Delta+\varepsilon$,

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(x+u+v)}\boldsymbol{s}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} g(x)\,\mathrm{d}x \leq G\frac{\mathbb{P}(Z > u+v)}{\mathbb{P}(Z > u)} \leq G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}. \tag{A.9}$$

□

The error term $r_3^{(p)}$ depends on p, as it is defined by $Z^{(p)}$ and $\varepsilon^{(p)}$, but we have omitted the superscript p here. Choosing $\varepsilon = \mathrm{Var}(Z^{(p)})^{1/3}$ then, outside of the vanishingly small interval $u \in (\Delta - \varepsilon^{(p)}, \Delta + \varepsilon^{(p)})$, the error term $|r_3^{(p)}(u)|$ is bounded by $O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right)$, which tends to 0 as $p \to \infty$. On $u \in (\Delta - \varepsilon^{(p)}, \Delta + \varepsilon^{(p)})$ the error term $|r_3^{(p)}(u)|$ is bounded by a constant which does not tend to 0 as $p \to \infty$. However, when we integrate a bounded function against $r_3^{(p)}(u)$, then the resulting integral tends to 0, i.e. for $|\psi(x)| \leq F$, $M < \infty$, $\int_0^M f(u) r_3^{(p)}(u)\,\mathrm{d}u \leq F\Delta|r_2^{(p)}| + 2GF\varepsilon^{(p)} + (M - \Delta)GF\frac{\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}{1-\mathrm{Var}\left(Z^{(p)}\right)/\left(\varepsilon^{(p)}\right)^2} = O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right) \to 0$ as $p \to \infty$. This is the context in which we we apply Corollary A.1.3 and thus the error bound is sufficient. See, for example, Corollary A.2.1.

Proof of Lemma 3.3.1. Property 3.0.2(ii) states

$$\left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) g(x)\,\mathrm{d}x - g(\Delta-u-v)1(u+v \leq \Delta-\varepsilon) \right| = |r_V(u,v)|. \tag{A.10}$$

Setting $g(x) = h_{ij}^{++}(\lambda, x)$,

$$\left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) h_{ij}^{++}(\lambda, x)\,\mathrm{d}x - h_{ij}^{++}(\lambda, \Delta - u - v) 1(u + v \leq \Delta - \varepsilon) \right| = |r_V(u, v)|. \tag{A.11}$$

We recognise the left-most term as

$$\widehat{f}_{0,+,+}^{\ell_0,(p)}(\lambda)(y_{\ell_0+1} - v, j; y_{\ell_0} + u, i) = \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) h_{ij}^{++}(\lambda, x)\,\mathrm{d}x.$$

Now consider

$$\begin{aligned}
&\Bigg| \int_{v=0}^{\Delta} \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) h_{ij}^{++}(\lambda, x)\,\mathrm{d}x \psi(v)\,\mathrm{d}v \\
&\qquad - \int_{v=0}^{\Delta} h_{ij}^{++}(\lambda, \Delta - u - v) 1(\Delta - u - v \geq 0) \psi(v)\,\mathrm{d}v \Bigg| \\
&\leq \int_{v=0}^{\Delta} \left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) h_{ij}^{++}(\lambda, x)\,\mathrm{d}x - h_{ij}^{++}(\lambda, \Delta - u - v) 1(\Delta - u - v \geq 0) \right| |\psi(v)|\,\mathrm{d}v \\
&\leq \int_{v=0}^{\Delta} \left| \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v) h_{ij}^{++}(\lambda, x)\,\mathrm{d}x - h_{ij}^{++}(\lambda, \Delta - u - v) 1(\Delta - u - v \geq \varepsilon) \right| |\psi(v)|\,\mathrm{d}v \\
&\qquad + \int_{v=0}^{\Delta} \left| h_{ij}^{++}(\lambda, \Delta - u - v) 1(\varepsilon \geq \Delta - u - v \geq 0) \right| |\psi(v)|\,\mathrm{d}v.
\end{aligned} \tag{A.12}$$

Applying 3.0.2(ii) to the first term, then (A.12) is less than or equal to

$$\begin{aligned}
&\int_{v=0}^{\Delta} |r_V(u, v)| \, |\psi(v)|\,\mathrm{d}v + \int_{v=0}^{\Delta} \left| h_{ij}^{++}(\lambda, \Delta - u - v) 1(\varepsilon \geq \Delta - u - v \geq 0) \right| |\psi(v)|\,\mathrm{d}v \\
&\leq R_{V,2} GF + \varepsilon GF.
\end{aligned}$$

Finally, noting $h_{ij}^{++}(\lambda, \Delta - u - v) 1(\Delta - u - v \geq 0) = \widehat{\mu}_{0,+,+}^{\ell_0}(\lambda)(y_{\ell_0+1} - v, j; y_{\ell_0} + u, i)$, then we have shown (3.78).

Using analogous arguments we can show (3.79). □

Lemma A.1.4. *Let g satisfy the Assumptions 3.0.1 and $x_0 \in (2\varepsilon, \Delta - \varepsilon)$. Then*

$$\left| \int_{x=0}^{\infty} \boldsymbol{a}(x_0) \boldsymbol{D} e^{\boldsymbol{S}x} g(x) \boldsymbol{s}\,\mathrm{d}x - g(x_0) \right| \leq \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2} 4G + 3L\varepsilon + 6G \frac{\mathrm{Var}(Z)}{\varepsilon^2}. \tag{A.13}$$

Proof. First rewrite the left-hand side as

$$\left|\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x}(g(x)-g(x_0))\boldsymbol{s}\,\mathrm{d}x\right| \le \int_{x=0}^{\infty}\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x. \tag{A.14}$$

Substituting the expression for $\boldsymbol{D}$ gives,

$$\begin{aligned}
&\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=0}^{\infty}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x\\
&=\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=0}^{\Delta-\varepsilon}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x\\
&\quad+\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=\Delta-\varepsilon}^{\infty}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x
\end{aligned}\tag{A.15}$$

Since g is bounded, the second term is less than or equal to

$$\begin{aligned}
\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=\Delta-\varepsilon}^{\infty}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x2G &= \boldsymbol{a}(x_0)\int_{u=\Delta-\varepsilon}^{\infty}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}u\boldsymbol{e}2G\\
&= \boldsymbol{a}(x_0)\int_{u=\Delta-\varepsilon}^{\infty}e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u2G\\
&= \frac{\mathbb{P}(Z\ge x_0+\Delta-\varepsilon)}{\mathbb{P}(Z>x_0)}2G
\end{aligned}\tag{A.16}$$

For $x_0\in(2\varepsilon,\Delta-\varepsilon)$, then (A.16) is less than or equal to

$$\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}2G.$$

As for the first term in (A.15), it can be written as

$$\begin{aligned}
&\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=\Delta-x_0-\varepsilon}^{u=\Delta-x_0+\varepsilon}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x\\
&\quad+\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=0}^{\Delta-x_0-\varepsilon}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x\\
&\quad+\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=\Delta-x_0+\varepsilon}^{\Delta-\varepsilon}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x.
\end{aligned}\tag{A.17}$$

Since g is bounded, then the last two terms in (A.17) are

$$2G\Bigg(\int_{x=0}^{\infty}\boldsymbol{a}(x_0)\int_{u=0}^{\Delta-x_0-\varepsilon}e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\mathrm{d}ue^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x$$

$$
\begin{aligned}
&\quad + \int_{x=0}^{\infty} \boldsymbol{a}(x_0) \int_{u=\Delta-x_0+\varepsilon}^{\Delta-\varepsilon} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}} \mathrm{d}u e^{\boldsymbol{S}x}\boldsymbol{s}\,\mathrm{d}x \Bigg) \\
&= 2G\left(\boldsymbol{a}(x_0)\int_{u=0}^{\Delta-x_0-\varepsilon} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\boldsymbol{e}\,\mathrm{d}u + \boldsymbol{a}(x_0)\int_{u=\Delta-x_0+\varepsilon}^{\Delta-\varepsilon} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}\boldsymbol{e}\,\mathrm{d}u\right) \\
&= 2G\frac{\mathbb{P}(Z > x_0, Z \notin (\Delta-\varepsilon, \Delta+\varepsilon))}{\mathbb{P}(Z > x_0)} \\
&\le 2G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2},
\end{aligned} \tag{A.18}
$$

provided that $x_0 \in [0, \Delta-\varepsilon)$. Exchanging the order of integration for the first term in (A.17)

$$
\int_{u=\Delta-x_0-\varepsilon}^{u=\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\int_{x=0}^{\infty}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}}e^{\boldsymbol{S}x}|g(x)-g(x_0)|\boldsymbol{s}\,\mathrm{d}x\,\mathrm{d}u \tag{A.19}
$$

from which we see that we can apply Corollary A.1.2 to the integral over x, implying that (A.19) is less than or equal to

$$
\int_{u=\Delta-x_0-\varepsilon}^{u=\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\left(|g(\Delta-u)-g(x_0)| + 6G\frac{\mathrm{Var}(Z)}{\varepsilon^2} + 2L\varepsilon\right)\mathrm{d}u. \tag{A.20}
$$

Since g is Lipschitz, then (A.20) is less than or equal to

$$
\int_{u=\Delta-x_0-\varepsilon}^{u=\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\left(L\varepsilon + 6G\frac{\mathrm{Var}(Z)}{\varepsilon^2} + 2L\varepsilon\right)\mathrm{d}u \le L\varepsilon + 6G\frac{\mathrm{Var}(Z)}{\varepsilon^2} + 2L\varepsilon. \tag{A.21}
$$

Putting all the bounds together proves the result. □

Lemma A.1.5. *Let $\psi : \mathcal{D}_{\ell_0} \to \mathbb{R}$ be bounded $|\psi(x)| \le F$ and Lipschitz continuous function and let $\lambda > 0$. For $i \in \mathcal{S}_-, j \in \mathcal{S}_- \cup \mathcal{S}_{-0}$, $x_0 \in (0, \Delta)$,*

$$
\int_{x_1=0}^{\infty}\int_{x=0}^{\Delta} \boldsymbol{a}^{(p)}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}x_1}V^{(p)}(x)h_{ij}^{--}(\lambda, x_1)\psi(x)\,\mathrm{d}x\,\mathrm{d}x_1 \to \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0-x)\psi(x)\,\mathrm{d}x, \tag{A.22}
$$

as $p \to \infty$. Similarly, for $i \in \mathcal{S}_+, j \in \mathcal{S}_+ \cup \mathcal{S}_{+0}$

$$
\begin{aligned}
&\int_{x_1=0}^{\infty}\int_{x=0}^{\Delta} \boldsymbol{a}^{(p)}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_1}V^{(p)}(x)h_{ij}^{++}(\lambda, x_1)\psi(\Delta-x)\,\mathrm{d}x\,\mathrm{d}x_1 \\
&\to \int_{x=\Delta-x_0}^{\Delta} h_{ij}^{++}(\lambda, x-x_0)\psi(x)\,\mathrm{d}x,
\end{aligned} \tag{A.23}
$$

Proof. Assume, without loss of generality $\ell_0 = 0$ so $\mathcal{D}_{\ell_0} = [0, \Delta]$. Substituting the definition of $\boldsymbol{D}$, then the left-hand side of (A.22) is

$$\begin{aligned}
&\left| \int_{x_1=0}^{\infty} \int_{x=0}^{\Delta} \int_{u=0}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \boldsymbol{a}(u) \,\mathrm{d}u e^{\boldsymbol{S}x_1} V(x) h_{ij}^{--}(\lambda, x_1) \psi(x) \,\mathrm{d}x \,\mathrm{d}x_1 \right. \\
&\qquad \left. - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x) \psi(x) \,\mathrm{d}x \right| \\
&\leq \left| \int_{x_1=0}^{\infty} \int_{x=0}^{\Delta} \int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \boldsymbol{a}(u) \,\mathrm{d}u e^{\boldsymbol{S}x_1} V(x) h_{ij}^{--}(\lambda, x_1) \psi(x) \,\mathrm{d}x \,\mathrm{d}x_1 \right. \\
&\qquad \left. - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x) \psi(x) \,\mathrm{d}x \right| \\
&\qquad + \left| \int_{x_1=0}^{\infty} \int_{x=0}^{\Delta} \int_{u=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \boldsymbol{a}(u) \,\mathrm{d}u e^{\boldsymbol{S}x_1} V(x) h_{ij}^{--}(\lambda, x_1) \psi(x) \,\mathrm{d}x \,\mathrm{d}x_1 \right| \qquad \text{(A.24)}
\end{aligned}$$

Since $|\phi(x) \leq F$ and $|h_{ij}^{--}(\lambda, x) \leq G$, the third term in (A.24) is less than or equal to

$$\int_{x_1=0}^{\infty} \int_{x=0}^{\Delta} \int_{u=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \boldsymbol{a}(u) \,\mathrm{d}u e^{\boldsymbol{S}x_1} V(x) \,\mathrm{d}x \,\mathrm{d}x_1 GF. \qquad \text{(A.25)}$$

Computing the integral over x_1 in (A.25) gives

$$\begin{aligned}
&\int_{x=0}^{\Delta} \int_{u=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \boldsymbol{a}(u) \,\mathrm{d}u (-\boldsymbol{S})^{-1} V(x) \,\mathrm{d}x GF \\
&\leq \int_{x=0}^{\Delta} \int_{u=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \,\mathrm{d}u \,\mathrm{d}x G_V GF \\
&= \int_{u=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \,\mathrm{d}u \Delta G_V GF. \qquad \text{(A.26)}
\end{aligned}$$

since, by property 3.0.2(i), $\boldsymbol{a}(u)(-\boldsymbol{S})^{-1} V(x) \leq \boldsymbol{a}(u) \boldsymbol{e} G_V = G_V$. The bound (A.26) is equal to

$$\Delta \mathbb{P}(Z > x_0 + \Delta - \varepsilon) \,\mathrm{d}x G_V GF \leq \Delta \frac{\mathrm{Var}(Z)}{(x_0 - \varepsilon)^2} G_V GF, \qquad \text{(A.27)}$$

by Chebyshev's inequality.

The first and second terms in (A.24) are

$$\left| \int_{x=0}^{\Delta} \int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0) e^{\boldsymbol{S}u} \boldsymbol{s} \int_{x_1=0}^{\infty} \boldsymbol{a}(u) e^{\boldsymbol{S}x_1} V(x) h_{ij}^{--}(\lambda, x_1) \,\mathrm{d}x_1 \,\mathrm{d}u \psi(x) \,\mathrm{d}x \right.$$

$$
\begin{aligned}
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| \\
&= \Bigg| \int_{x=0}^{\Delta}\int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\left[h_{ij}^{--}(\lambda, \Delta - u - x)1(u + x \le \Delta - \varepsilon) + r_V(u,x)\right]\psi(x)\,\mathrm{d}u\,\mathrm{d}x \\
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| \\
&\le \Bigg| \int_{x=0}^{\Delta}\int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda, \Delta - u - x)1(u + x \le \Delta - \varepsilon)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \\
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| + \Bigg| \int_{x=0}^{\Delta}\int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}r_V(u,x)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \Bigg| \qquad \text{(A.28)}
\end{aligned}
$$

where the first equality holds from Property 3.0.2(ii). The last term in (A.28) is less than or equal to

$$
\begin{aligned}
\int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\int_{x=0}^{\Delta} \boldsymbol{s}\left|r_V(u,x)\right|\,\mathrm{d}x\,\mathrm{d}uF &\le \int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}R_{V,2}\,\mathrm{d}uF \\
&\le R_{V,2}F, \qquad \text{(A.29)}
\end{aligned}
$$

by Property 3.0.2(ii). The first two terms in (A.28) are

$$
\begin{aligned}
&\Bigg| \int_{x=0}^{\Delta}\int_{u=0}^{\Delta-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda, \Delta - u - x)1(u + x \le \Delta - \varepsilon)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \\
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| \\
&= \Bigg| \int_{x=0}^{\Delta-\varepsilon}\int_{u=0}^{\Delta-x-\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda, \Delta - u - x)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \\
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| \\
&\le \Bigg| \int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0-\varepsilon}^{\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda, \Delta - u - x)\psi(x)\,\mathrm{d}u 1(\Delta - x_0 + \varepsilon \le \Delta - x - \varepsilon)\,\mathrm{d}x \\
&\quad - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda, x_0 - x)\psi(x)\,\mathrm{d}x \Bigg| \\
&\quad + \Bigg| \int_{x=0}^{\Delta-\varepsilon}\int_{u=0}^{\min(\Delta-x_0-\varepsilon, \Delta-x-\varepsilon)} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda, \Delta - u - x)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \Bigg|
\end{aligned}
$$

$$+\left|\int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0+\varepsilon}^{\Delta-x-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda,\Delta-u-x)\psi(x)\,\mathrm{d}u\right.$$
$$\left.\times 1(\Delta-x_0+\varepsilon\le\Delta-x-\varepsilon)\,\mathrm{d}x\right|$$
$$+\left|\int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0-\varepsilon}^{\Delta-x-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda,\Delta-u-x)\psi(x)\,\mathrm{d}u\right.$$
$$\left.\times 1(\Delta-x_0-\varepsilon\le\Delta-x-\varepsilon<\Delta-x_0+\varepsilon)\,\mathrm{d}x\right| \tag{A.30}$$

Since $|\psi|\le F$, the second and third terms in (A.30) are less than or equal to

$$\begin{aligned}
&\int_{x=0}^{\Delta-\varepsilon}\int_{u=0}^{\min(\Delta-x_0-\varepsilon,\Delta-x-\varepsilon)}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u\,\mathrm{d}xGF\\
&\quad+\int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0+\varepsilon}^{\Delta-x-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u1(\Delta-x_0+\varepsilon\le\Delta-x-\varepsilon)\,\mathrm{d}xGF\\
&\le\int_{x=0}^{\Delta-\varepsilon}\int_{u=0}^{\Delta-x_0-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u\,\mathrm{d}xGF\\
&\quad+\int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0+\varepsilon}^{\infty}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u1(\Delta-x_0+\varepsilon\le\Delta-x-\varepsilon)\,\mathrm{d}xGF\\
&\le\Delta\int_{u=0}^{\Delta-x_0-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u\,\mathrm{d}xGF+\Delta\int_{u=\Delta-x_0+\varepsilon}^{\infty}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}uGF\\
&\le\Delta\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}GF.
\end{aligned}\tag{A.31}$$

Since $\displaystyle\int_{u=\Delta-x_0-\varepsilon}^{\Delta-x-\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\le 1$, the last term in (A.30) is less than or equal to

$$\int_{x=0}^{\Delta-\varepsilon}GF\,\mathrm{d}u1(\Delta-x_0-\varepsilon\le\Delta-x-\varepsilon<\Delta-x_0+\varepsilon)\,\mathrm{d}x=2\varepsilon GF. \tag{A.32}$$

Now consider the difference in the first absolute value in (A.30),

$$\left|\int_{x=0}^{\Delta-\varepsilon}\int_{u=\Delta-x_0-\varepsilon}^{\Delta-x_0+\varepsilon}\boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda,\Delta-u-x)\psi(x)\,\mathrm{d}u1(x\le x_0-2\varepsilon)\,\mathrm{d}x\right.$$
$$\left.-\int_{x=0}^{x_0}h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x\right|$$

$$\leq \left| \int_{x=0}^{x_0-2\varepsilon} \int_{u=\Delta-x_0-\varepsilon}^{\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}h_{ij}^{--}(\lambda,\Delta-u-x)\psi(x)\,\mathrm{d}u\,\mathrm{d}x \right.$$
$$\left. - \int_{x=0}^{x_0-2\varepsilon} h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x \right| + \left| \int_{x=x_0-2\varepsilon}^{x_0} h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x \right|$$
$$\leq \int_{x=0}^{x_0-2\varepsilon} \int_{u=\Delta-x_0-\varepsilon}^{\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}\left| h_{ij}^{--}(\lambda,\Delta-u-x) - h_{ij}^{--}(\lambda,x_0-x) \right| \mathrm{d}u\,|\psi(x)|\,\mathrm{d}x$$
$$+ \left| \int_{x=0}^{x_0-2\varepsilon} h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x\mathbb{P}(|Z-\Delta|>\varepsilon) \right|$$
$$+ \left| \int_{x=x_0-2\varepsilon}^{x_0} h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x \right|$$
$$\leq \int_{x=0}^{x_0-2\varepsilon} \int_{u=\Delta-x_0-\varepsilon}^{\Delta-x_0+\varepsilon} \boldsymbol{a}(x_0)e^{\boldsymbol{S}u}\boldsymbol{s}L\varepsilon\,\mathrm{d}u\,|\psi(x)|\,\mathrm{d}x + \Delta GF\frac{\mathrm{Var}(Z)}{\varepsilon^2} + 2\varepsilon GF$$
$$\leq \Delta L\varepsilon F + \Delta GF\frac{\mathrm{Var}(Z)}{\varepsilon^2} + 2\varepsilon GF. \tag{A.33}$$

In summary, we have shown

$$\left| \int_{x_1=0}^{\infty} \int_{x=0}^{\Delta} \boldsymbol{a}^{(p)}(x_0)\boldsymbol{D}^{(p)}e^{\boldsymbol{S}^{(p)}x_1}V^{(p)}(x)h_{ij}^{--}(\lambda,x_1)\psi(x)\,\mathrm{d}x\,\mathrm{d}x_1 \right.$$
$$\left. - \int_{x=0}^{x_0} h_{ij}^{--}(\lambda,x_0-x)\psi(x)\,\mathrm{d}x \right|$$
$$\leq \Delta L\varepsilon^{(p)}F + \Delta GF\frac{\mathrm{Var}(Z^{(p)})}{\left(\varepsilon^{(p)}\right)^2} + 4\varepsilon^{(p)}GF + \Delta\frac{\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}{1-\mathrm{Var}(Z^{(p)})/\left(\varepsilon^{(p)}\right)^2}GF + R_{V,2}^{(p)}F$$
$$+ \Delta\frac{\mathrm{Var}(Z^{(p)})}{(x_0-\varepsilon^{(p)})^2}G_VGF. \tag{A.34}$$

The result follows upon choosing $\varepsilon^{(p)} = \mathrm{Var}(Z^{(p)})^{1/3}$ and letting $p \to \infty$. □

A.1.2 Many integrals.

Define the column vectors

$$\mathcal{I}_{m,k}(u_k) = \left[\prod_{\ell=m}^{k-1} \int_{x_\ell=0}^{\infty} g_\ell(x_\ell)e^{\boldsymbol{S}x_\ell}\,\mathrm{d}x_\ell\boldsymbol{D}\right] \int_{x_k=0}^{\infty} g_k(x_k)e^{\boldsymbol{S}x_k}\,\mathrm{d}x_k e^{\boldsymbol{S}u_k}\boldsymbol{s} \tag{A.35}$$

for $m, k \in \{1, 2, \dots\}$, $m \le k$, where a product over an empty set is equal to 1. Also define the row vectors

$$\mathcal{J}_{k+1,k+1}(u_k, x_{k+1}) := g_{k+1}(x_{k+1}) \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_k}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e}} e^{\boldsymbol{S} x_{k+1}} \tag{A.36}$$

and

$$\mathcal{J}_{k+1,n}(u_k, x_{k+1}) := g_{k+1}(x_{k+1}) \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_k}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e}} e^{\boldsymbol{S} x_{k+1}} \boldsymbol{D} \left[\prod_{m=k+2}^{n-1} \int_{x_m=0}^{\infty} g_m(x_m) e^{\boldsymbol{S} x_m} \,\mathrm{d}x_m \boldsymbol{D} \right] \times \int_{x_n=0}^{\infty} g_n(x_n) e^{\boldsymbol{S} x_n} \,\mathrm{d}x_n \tag{A.37}$$

for $k, n \in \{0, 1, 2, \dots\}$, $k + 1 < n$. Also define the row vector function $\boldsymbol{a}(x) : [0, \infty) \to \mathcal{A} \subset \mathbb{R}^p$,

$$\boldsymbol{a}(x) = \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} x}}{\boldsymbol{\alpha} e^{\boldsymbol{S} x} \boldsymbol{e}}.$$

Lemma A.1.6. *Let $g_1, g_2, \dots,$ be functions satisfying Assumptions 3.0.1, then, for $k \in \{1, 2, \dots\}$,*

$$\boldsymbol{a}(x_0) \mathcal{I}_{1,k}(u_k) \le \frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e}. \tag{A.38}$$

Proof. Recall the definition of $\boldsymbol{D} := \int_{u=0}^{\infty} e^{\boldsymbol{S} u} \boldsymbol{s} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u} \boldsymbol{e}} \mathrm{d}u$ and substitute it into the left-hand side of (A.38),

$$\begin{aligned}
\boldsymbol{a}(x_0) \mathcal{I}_{1,k}(u_k) &= \boldsymbol{a}(x_0) \int_{x_1=0}^{\infty} g_1(x_1) e^{\boldsymbol{S} x_1} \boldsymbol{D} \mathcal{I}_{2,k}(u_k) \\
&= \boldsymbol{a}(x_0) \int_{x_1=0}^{\infty} g_1(x_1) e^{\boldsymbol{S} x_1} \int_{u_1=0}^{\infty} e^{\boldsymbol{S} u_1} \boldsymbol{s} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1} \boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k).
\end{aligned}$$

Now, since $|g_1| \le G$, then this is less than or equal to

$$\boldsymbol{a}(x_0) \int_{x_1=0}^{\infty} G e^{\boldsymbol{S} x_1} \int_{u_1=0}^{\infty} e^{\boldsymbol{S} u_1} \boldsymbol{s} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1} \boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k). \tag{A.39}$$

Computing the integral with respect to x_1 in (A.39) gives

$$\begin{aligned}
& G \boldsymbol{a}(x_0) (-\boldsymbol{S})^{-1} \int_{u_1=0}^{\infty} e^{\boldsymbol{S} u_1} \boldsymbol{s} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1} \boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k) \\
&= G \boldsymbol{a}(x_0) \int_{u_1=0}^{\infty} e^{\boldsymbol{S} u_1} \boldsymbol{e} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_1} \boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k)
\end{aligned}$$

$$= \frac{G}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}} \int_{u_1=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}(x_0+u_1)}\boldsymbol{e} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_1}\boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k), \tag{A.40}$$

since $(-\boldsymbol{S})^{-1}$ and $e^{\boldsymbol{S}t}$ commute, $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e}$ and $e^{\boldsymbol{S}(t+u)} = e^{\boldsymbol{S}t}e^{\boldsymbol{S}u}$.

Now, as $\boldsymbol{\alpha} e^{\boldsymbol{S}(x_0+u_1)}\boldsymbol{e} \leq \boldsymbol{\alpha} e^{\boldsymbol{S}u_1}\boldsymbol{e}$, then (A.40) is less than or equal to

$$G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}} \int_{u_1=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}u_1}\boldsymbol{e} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_1}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_1}\boldsymbol{e}} \mathrm{d}u_1 \mathcal{I}_{2,k}(u_k) = G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}} \int_{u_1=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}u_1} \,\mathrm{d}u_1 \mathcal{I}_{2,k}(u_k).$$

Now integrate with respect to u_1 and use the facts that $(-\boldsymbol{S})^{-1}$ and $e^{\boldsymbol{S}x}$ commute, and $\boldsymbol{s} = -\boldsymbol{S}\boldsymbol{e}$, to get

$$\begin{aligned}
&G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}}\boldsymbol{\alpha}(-\boldsymbol{S})^{-1}\mathcal{I}_{2,k}(u_k) \\
&= G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}}\boldsymbol{\alpha}(-\boldsymbol{S})^{-1} \int_{x_2=0}^{\infty} g_2(x_2) e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \int_{u_2=0}^{\infty} e^{\boldsymbol{S}u_2}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}}\mathrm{d}u_2\mathcal{I}_{3,k}(u_k) &\text{(A.41)}\\
&= G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}} \int_{x_2=0}^{\infty} g_2(x_2)\boldsymbol{\alpha} e^{\boldsymbol{S}x_2}\,\mathrm{d}x_2 \int_{u_2=0}^{\infty} e^{\boldsymbol{S}u_2}\boldsymbol{e}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}}\mathrm{d}u_2\mathcal{I}_{3,k}(u_k) &\text{(A.42)}
\end{aligned}$$

Since $\boldsymbol{\alpha} e^{\boldsymbol{S}x_2}e^{\boldsymbol{S}u_2}\boldsymbol{e} \leq \boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}$, then (A.42) is less than or equal to

$$\begin{aligned}
&G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}} \int_{x_2=0}^{\infty} g_2(x_2)\,\mathrm{d}x_2 \int_{u_2=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}}\mathrm{d}u_2\mathcal{I}_{3,k}(u_k)\\
&= G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}}\widehat{G} \int_{u_2=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\boldsymbol{e}}\mathrm{d}u_2\mathcal{I}_{3,k}(u_k)\\
&= G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}}\widehat{G} \int_{u_2=0}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S}u_2}\mathcal{I}_{3,k}(u_k)\\
&= G\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S}x_0}\boldsymbol{e}}\widehat{G}\boldsymbol{\alpha}(-\boldsymbol{S})^{-1}\mathcal{I}_{3,k}(u_k). &\text{(A.43)}
\end{aligned}$$

Repeating the arguments which got us from (A.41) to (A.43) another $k-2$ times gives the result. □

Lemma A.1.7. *Let $g_1, g_2, \ldots,$ be functions satisfying the Assumptions 3.0.1 and let $V(x)$ be a closing operator with the Properties 3.0.2, then, for $k, n \in \{1, 2, \ldots\}$, $k+1 < n$,*

$$\mathcal{J}_{k+1,n}(u_k, x_{k+1})V(x) \leq g_{k+1}(x_{k+1})\widehat{G}^{n-k-2}GG_V.$$

Proof. Starting with the left-hand side, upon substituting $\boldsymbol{D}$,

$$\mathcal{J}_{k+1,n}(u_k, x_{k+1})V(x)$$

$$
\begin{aligned}
&= \mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\boldsymbol{D}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&= \mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\int_{u_{n-1}=0}^{\infty} e^{\boldsymbol{S}u_{n-1}}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}}\,\mathrm{d}u_{n-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&\le \mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\int_{u_{n-1}=0}^{\infty} e^{\boldsymbol{S}u_{n-1}}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}}\,\mathrm{d}u_{n-1}\int_{x_n=0}^{\infty} Ge^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x). \qquad \text{(A.44)}
\end{aligned}
$$

By the Property 3.0.2(i) of $V(x)$, $\int_{x_n=0}^{\infty} \boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}e^{\boldsymbol{S}x_n}V(x)\,\mathrm{d}x_n \le \boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}G_V$. Therefore (A.44) is less than or equal to

$$
\begin{aligned}
&\mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\int_{u_{n-1}=0}^{\infty} e^{\boldsymbol{S}u_{n-1}}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}}\,\mathrm{d}u_{n-1}GG_V \\
&= \mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\int_{u_{n-1}=0}^{\infty} e^{\boldsymbol{S}u_{n-1}}\boldsymbol{s}\,\mathrm{d}u_{n-1}GG_V \\
&= \mathcal{J}_{k+1,n-1}(u_k, x_{k+1})\boldsymbol{e}GG_V \qquad \text{(A.45)} \\
&= \mathcal{J}_{k+1,n-2}(u_k, x_{k+1})\int_{u_{n-2}=0}^{\infty} e^{\boldsymbol{S}u_{n-2}}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-2}}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-2}}\boldsymbol{e}}\,\mathrm{d}u_{n-2}\int_{x_{n-1}=0}^{\infty} g_{n-1}(x_{n-1})e^{\boldsymbol{S}x_{n-1}}\,\mathrm{d}x_{n-1}\boldsymbol{e} \\
&\qquad \times GG_V. \qquad \text{(A.46)}
\end{aligned}
$$

Now, since $\boldsymbol{\alpha}e^{\boldsymbol{S}(x_{n-1}+u_{n-2})}\boldsymbol{e} \le \boldsymbol{\alpha}e^{\boldsymbol{S}(u_{n-2})}\boldsymbol{e}$, then (A.46) is less than or equal to

$$
\begin{aligned}
&\mathcal{J}_{k+1,n-2}(u_k, x_{k+1})\int_{u_{n-2}=0}^{\infty} e^{\boldsymbol{S}u_{n-2}}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-2}}\boldsymbol{e}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-2}}\boldsymbol{e}}\,\mathrm{d}u_{n-2}\int_{x_{n-1}=0}^{\infty} g_{n-1}(x_{n-1})\,\mathrm{d}x_{n-1}GG_V \\
&= \mathcal{J}_{k+1,n-2}(u_k, x_{k+1})\int_{u_{n-2}=0}^{\infty} e^{\boldsymbol{S}u_{n-2}}\boldsymbol{s}\,\mathrm{d}u_{n-2}\widehat{G}GG_V \\
&= \mathcal{J}_{k+1,n-2}(u_k, x_{k+1})\boldsymbol{e}\widehat{G}GG_V. \qquad \text{(A.47)}
\end{aligned}
$$

This is of the same form as (A.45), hence repeating the same arguments which got us from (A.45) to (A.47) another $n-k-3$ more times gives

$$
\begin{aligned}
\mathcal{J}_{k+1,k+1}(u_k, x_{k+1})\boldsymbol{e}\widehat{G}^{n-k-2}GG_V &\le g_{k+1}(x_{k+1})\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(u_k+x_{k+1})}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_k}\boldsymbol{e}}\boldsymbol{e}\widehat{G}^{n-k-2}GG_V \\
&\le g_{k+1}(x_{k+1})\widehat{G}^{n-k-2}GG_V.
\end{aligned}
$$

□

Corollary A.1.8. *Let $g_1, g_2, \dots,$ be functions satisfying the Assumptions 3.0.1 and let $V(x)$ be a closing operator with the Properties 3.0.2, then,*

$$
\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{k=2}^{n-1}\int_{x_k=0}^{\infty} g_k(x_k)e^{\boldsymbol{S}x_k}\,\mathrm{d}x_k\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_nV(x)
$$

$$\leq \widehat{G}^{n-1} G G_V \tag{A.48}$$

Proof. The left-hand side of (A.48) can be seen to be equivalent to $\mathcal{J}_{1,n+1}(x_0, x_1)$, with $g_1(x_1) = 1$, and the integrability condition on g_1 is not required to prove the bound. □

Corollary A.1.9. *Let $g_1, g_2, \ldots,$ be functions satisfying the Assumptions 3.0.1 and let $V(x)$ be a closing operator with the Properties 3.0.2, then, for $k, n \in \{1, 2, \ldots\}$, $k+1 \leq n$,*

$$\begin{aligned}&\int_{x_{k+1}=0}^{\infty} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) \mathcal{I}_{1,k}(u_k) \mathcal{J}_{k+1,n}(u_k, x_{k+1}) V(x) \\ &\leq \left(2\varepsilon + \frac{\mathrm{Var}(Z)}{\varepsilon}\right) \frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{n-2} G G_V =: |r_4(n)|. \end{aligned} \tag{A.49}$$

Proof. Consider first $k + 1 < n$. Combining Lemmas A.1.6 and A.1.7 the left-hand side of (A.49) is less than or equal to

$$\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \int_{x_{k+1}=0}^{\infty} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e} g_{k+1}(x_{k+1}) \, \mathrm{d}u_k \, \mathrm{d}x_{k+1} \widehat{G}^{n-k-2} G G_V \tag{A.50}$$

$$= \frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e} \, \mathrm{d}u_k \widehat{G}_{k+1} \widehat{G}^{n-k-2} G G_V. \tag{A.51}$$

Now

$$\begin{aligned} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e} \, \mathrm{d}u_k &= \int_{u_k=\Delta-\varepsilon}^{\Delta+\varepsilon} \mathbb{P}(Z > u_k) \, \mathrm{d}u_k + \int_{u_k=\Delta+\varepsilon}^{\infty} \mathbb{P}(Z > u_k) \, \mathrm{d}u_k \\ &\leq \int_{u_k=\Delta-\varepsilon}^{\Delta+\varepsilon} \mathrm{d}u_k + \int_{u_k=\Delta+\varepsilon}^{\infty} \frac{\mathrm{Var}(Z)}{(u_k - \Delta)^2} \mathrm{d}u_k \\ &= 2\varepsilon + \frac{\mathrm{Var}(Z)}{\varepsilon}, \end{aligned} \tag{A.52}$$

where we have used Chebyshev's inequality to bound the tail probability,

$$\mathbb{P}(Z > u_k) \leq \mathbb{P}(|Z - \Delta| > |u_k - \Delta|) \leq \frac{\mathrm{Var}(Z)}{(u_k - \Delta)^2},$$

for $u_k \geq \Delta + \varepsilon$. Hence (A.51) is less than or equal to

$$\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \left(2\varepsilon + \frac{\mathrm{Var}(Z)}{\varepsilon}\right) \widehat{G}_{k+1} \widehat{G}^{n-k-2} G G_V.$$

Now consider $k + 1 = n$. By Lemma A.1.6 we have

$$\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \int_{x_{k+1}=0}^{\infty} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e} g_{k+1}(x_{k+1}) \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u_k + x_{k+1})}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e}} V(x) \, \mathrm{d}u_k \, \mathrm{d}x_{k+1}. \tag{A.53}$$

Since $g_{k+1} \leq G$, and upon integrating over x_{k+1}, then (A.53) is less than or equal to

$$\begin{aligned}
&\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \int_{u_k=\Delta-\varepsilon}^{\infty} G \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} (-\boldsymbol{S})^{-1} V(x) \, \mathrm{d}u_k \\
&\leq \frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{k-1} \int_{u_k=\Delta-\varepsilon}^{\infty} G \boldsymbol{\alpha} e^{\boldsymbol{S} u_k} \boldsymbol{e} G_V \, \mathrm{d}u_k, \qquad \text{(A.54)}
\end{aligned}$$

where we have used Property 3.0.2(i) to get the upper bound on the right-hand side of (A.54). Using (A.52) again, then (A.54) is less than or equal to

$$\frac{1}{\boldsymbol{\alpha} e^{\boldsymbol{S} x_0} \boldsymbol{e}} G \widehat{G}^{n-2} G G_V \left(2\varepsilon + \frac{\mathrm{Var}\,(Z)}{\varepsilon} \right). \qquad \text{(A.55)}$$

This completes the proof. □

The error term $r_4(n)$ depends on p and we write $r_4^{(p)}(n)$ when we need to make this dependence explicit, otherwise it is omitted from the notation. Upon choosing $\varepsilon = \mathrm{Var}(Z^{(p)})^{1/3}$, then for fixed $n < \infty$ the error term $|r_4^{(p)}(n)| = O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right) \to 0$ as $p \to \infty$.

Define $\boldsymbol{D}(b) = \int_{u=0}^{b} e^{\boldsymbol{S} u} \boldsymbol{s} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S} u}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u} \boldsymbol{e}}$. Also define

$$\begin{aligned}
g_{2,n}^{*}(u_1, x) = &\int_{u_2=0}^{\Delta-u_1} g_2(\Delta - u_2 - u_1) \, \mathrm{d}u_1 \ldots \int_{u_{n-1}=0}^{\Delta-u_{n-2}} g_{n-1}(\Delta - u_{n-1} - u_{n-2}) \, \mathrm{d}u_{n-2} \\
&g_n(\Delta - x - u_{n-1}) 1(\Delta - x - u_{n-1} \geq 0) \, \mathrm{d}u_{n-1}, \qquad \text{(A.56)}
\end{aligned}$$

and

$$g_{1,n}^{*}(x_0, x) = \int_{u_1=0}^{\Delta-x_0} g_1(\Delta - u_1 - x_0) g_{2,n}^{*}(u_1, x) \, \mathrm{d}u_1 \qquad \text{(A.57)}$$

and

$$\begin{aligned}
g_{1,n}^{*,\varepsilon}(x_0, x) = &\int_{u_1=0}^{\Delta-\varepsilon-x_0} g_1(\Delta - u_1 - x_0) \int_{u_2=0}^{\Delta-\varepsilon-u_1} g_2(\Delta - u_2 - u_1) \, \mathrm{d}u_1 \\
&\ldots \int_{u_{n-1}=0}^{\Delta-\varepsilon-u_{n-2}} g_{n-1}(\Delta - u_{n-1} - u_{n-2}) \, \mathrm{d}u_{n-2} g_n(\Delta - x - u_{n-1}) \\
&\times 1(\Delta - x - u_{n-1} \geq \varepsilon) \qquad \text{(A.58)}
\end{aligned}$$

For later, observe that

$$g_{2,n}^{*}(u_1, x) = \int_{u_2=0}^{\Delta-u_1} g_2(\Delta - u_2 - u_1) \, \mathrm{d}u_1 \ldots \int_{u_{n-1}=0}^{\Delta-u_{n-2}} g_{n-1}(\Delta - u_{n-1} - u_{n-2}) \, \mathrm{d}u_{n-2}$$

$$
\begin{aligned}
&g_n(\Delta - x - u_{n-1})1(\Delta - x - u_{n-1} \geq 0)\,\mathrm{d}u_{n-1} \\
&\leq G^{n-1}\int_{u_2=0}^{\Delta-u_1} \mathrm{d}u_1 \ldots \int_{u_{n-1}=0}^{\Delta-u_{n-2}} \mathrm{d}u_{n-1} \\
&\leq G^{n-1}\Delta^{n-2} := G_n^*. \qquad \text{(A.59)}
\end{aligned}
$$

Lemma A.1.10. *Let $g_1, g_2, \ldots,$ be functions satisfying the Assumptions 3.0.1 and let $V(x)$ be a closing operator with the Properties 3.0.2. Then, for $n \geq 2$,*

$$
\begin{aligned}
&\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1 \boldsymbol{D}(\Delta-\varepsilon)\left[\prod_{k=2}^{n-1}\int_{x_k=0}^{\infty} g_k(x_k)e^{\boldsymbol{S}x_k}\,\mathrm{d}x_k\right]\boldsymbol{D}(\Delta-\varepsilon) \\
&\quad\times\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&= g_{1,n}^*(x_0,x) + r_5(n) + r_6(n), \qquad \text{(A.60)}
\end{aligned}
$$

where

$$
\begin{aligned}
&|r_5(n)| = O\left(\max\left\{G^{n-1}\Delta^{n-2}\left(\frac{1}{2}\Delta|r_2| + 2\varepsilon G + \frac{1}{2}\Delta G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}\right), G^{n-1}\Delta^{n-2}R_{V,1}\right\}\right), \\
&|r_6(n)| \leq \varepsilon^{n-2}G^{n-1}
\end{aligned}
$$

Proof. Rewrite the left-hand side of (A.60) as

$$
\begin{aligned}
&\int_{u_1=0}^{\Delta-\varepsilon}\int_{x_1=0}^{\infty}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(x_0+x_1+u_1)}\boldsymbol{s}}{\boldsymbol{\alpha}e^{\boldsymbol{S}x_0}\boldsymbol{e}}g_1(x_1)\,\mathrm{d}u_1\,\mathrm{d}x_1 \\
&\times\left[\prod_{\ell=2}^{n-1}\int_{u_\ell=0}^{\Delta-\varepsilon}\int_{x_\ell=0}^{\infty}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(u_{\ell-1}+x_\ell+u_\ell)}\boldsymbol{s}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{\ell-1}}\boldsymbol{e}}g_\ell(x_\ell)\,\mathrm{d}u_\ell\,\mathrm{d}x_\ell\right] \\
&\times\int_{x_n=0}^{\infty}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(u_{n-1}+x_n)}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u_{n-1}}\boldsymbol{e}}V(x)g_n(x_n)\,\mathrm{d}u_{n-1}\,\mathrm{d}x_n,
\end{aligned}
$$

then we see that we can apply Corollary A.1.3 to all of the integrals over x_k, $k = 1, \ldots, n-1$ and use Property 3.0.2(ii) of $V(x)$ to get

$$
\begin{aligned}
&\int_{u_1=0}^{\Delta-\varepsilon}[g_1(\Delta-u_1-x_0)1(u_1+x_0 \leq \Delta-\varepsilon) + r_3(u_1+x_0)] \\
&\quad\times\int_{u_2=0}^{\Delta-\varepsilon}[g_2(\Delta-u_2-u_1)1(u_2+u_1 \leq \Delta-\varepsilon) + r_3(u_2+u_1)]\,\mathrm{d}u_1 \\
&\quad\ldots\int_{u_{n-1}=0}^{\Delta-\varepsilon}[g_{n-1}(\Delta-u_{n-1}-u_{n-2})1(u_{n-1}+u_{n-2} \leq \Delta-\varepsilon) + r_3(u_{n-1}+u_{n-2})]\,\mathrm{d}u_{n-2}
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left[g_n(\Delta - u_{n-1} - x) 1(u_{n-1} + x \le \Delta - \varepsilon) + r_V(u_{n-1}, x)\right] \mathrm{d}u_{n-1} \\
&= g^{*,\varepsilon}_{1,n}(x_0, x) + r_5(n)
\end{aligned}
$$

where $r_5(n)$ is an error term. The leading terms of $r_5(n)$ are of the form

$$
\begin{aligned}
&\int_{u_1=0}^{\Delta-\varepsilon-x_0} g_1(\Delta - u_1 - x_0) \int_{u_2=0}^{\Delta-\varepsilon-u_1} g_2(\Delta - u_2 - u_1)\,\mathrm{d}u_1 \\
&\quad \ldots \int_{u_{k-1}=0}^{\Delta-\varepsilon-u_{k-2}} g_{k-1}(\Delta - u_{k-1} - u_{k-2})\,\mathrm{d}u_{k-2} \int_{u_k=0}^{\Delta-\varepsilon} r_3(u_k + u_{k-1})\,\mathrm{d}u_{k-1} \\
&\quad \times \int_{u_{k+1}=0}^{\Delta-\varepsilon-u_k} g_{k+1}(\Delta - u_{k+1} - u_k)\,\mathrm{d}u_k \ldots \int_{u_{n-1}=0}^{\Delta-\varepsilon-u_{n-2}} g_{n-1}(\Delta - u_{n-1} - u_{n-2})\,\mathrm{d}u_{n-2} \\
&\quad \times g_n(\Delta - u_{n-1} - x) 1(u_{n-1} + x \le \Delta - \varepsilon)\,\mathrm{d}u_{n-1} \\
&\le G^{k-1}\Delta^{k-2} \int_{u_{k-1}=0}^{\Delta-\varepsilon} \int_{u_k=0}^{\Delta-\varepsilon} r_3(u_k + u_{k-1})\,\mathrm{d}u_k\,\mathrm{d}u_{k-1} G^{n-k}\Delta^{n-k-1},
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{u_1=0}^{\Delta-\varepsilon-x_0} g_1(\Delta - u_1 - x_0) \int_{u_2=0}^{\Delta-\varepsilon-u_1} g_2(\Delta - u_2 - u_1)\,\mathrm{d}u_1 \\
&\ldots \int_{u_{n-1}=0}^{\Delta-\varepsilon-u_{n-2}} g_{n-1}(\Delta - u_{n-1} - u_{n-2})\,\mathrm{d}u_{n-2} r_V(u_{n-1}, x)\,\mathrm{d}u_{n-1} \\
&\le G^{n-1}\Delta^{n-2} \int_{u_{n-1}=0}^{\Delta-\varepsilon} r_V(u_{n-1}, x)\,\mathrm{d}u_{n-1}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\left| \int_{u_{k-1}=0}^{\Delta-\varepsilon} \int_{u_k=0}^{\Delta-\varepsilon} r_3(u_k + u_{k-1})\,\mathrm{d}u_k\,\mathrm{d}u_{k-1} \right| \\
&\le \int_{u_{k-1}=0}^{\Delta-\varepsilon} \Bigg[\int_{u_k=u_{k-1}}^{\Delta-\varepsilon} |r_3(u_k)|\,\mathrm{d}u_k + \int_{u_k=\Delta-\varepsilon}^{\Delta+\varepsilon} |r_3(u_k)|\,\mathrm{d}u_k \\
&\qquad + \int_{u_k=\Delta+\varepsilon}^{\Delta-\varepsilon+u_{k-1}} |r_3(u_k)|\,\mathrm{d}u_k 1(u_{k-1} > 2\varepsilon) \Bigg] \mathrm{d}u_{k-1} \\
&\le \left[\int_{u_{k-1}=0}^{\Delta-\varepsilon} (\Delta - \varepsilon - u_{k-1})|r_2| + 2\varepsilon G + G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2}(u_{k-1} - 2\varepsilon)1(u_{k-1} > 2\varepsilon) \right] \mathrm{d}u_{k-1} \\
&\le \frac{1}{2}\Delta^2|r_2| + 2\Delta\varepsilon G + \frac{1}{2}\Delta^2 G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2},
\end{aligned}
$$

and, by Property 3.0.2(ii),

$$\int_{u_{n-1}=0}^{\Delta-\varepsilon} |r_V(u_{n-1},x)| \,\mathrm{d}u_{n-1} \leq R_{V,1}$$

Therefore, the error term $|r_5(n)|$ is less than or equal to the larger of these two terms,

$$|r_5(n)| = O\left(\max\left\{G^{n-1}\Delta^{n-2}\left(\frac{1}{2}\Delta|r_2| + 2\varepsilon G + \frac{1}{2}\Delta G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}\right), G^{n-1}\Delta^{n-2}R_{V,1}\right\}\right).$$

Now,

$$\begin{aligned}
&\left|g_{1,n}^{*,\varepsilon}(x_0,x) - g_{1,n}^{*}(x_0,x)\right| \\
&= \int_{u_1=\Delta-\varepsilon-x_0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)\int_{u_2=\Delta-\varepsilon-u_1}^{\Delta-u_1} g_2(\Delta-u_2-u_1)\,\mathrm{d}u_1 \\
&\quad \dots\int_{u_{n-1}=\Delta-\varepsilon-u_{n-2}}^{\Delta-u_{n-2}} g_{n-1}(\Delta-u_{n-1}-u_{n-2})\,\mathrm{d}u_{n-2}g_n(\Delta-x-u_{n-1}) \\
&\quad \times 1(\Delta-x-u_{n-1}\geq 0)\,\mathrm{d}u_{n-1} \\
&\leq \int_{u_1=\Delta-\varepsilon-x_0}^{\Delta-x_0} G\int_{u_2=\Delta-\varepsilon-u_1}^{\Delta-u_1} G\,\mathrm{d}u_1\dots\int_{u_{n-1}=\Delta-\varepsilon-u_{n-2}}^{\Delta-u_{n-2}} G\,\mathrm{d}u_{n-2}G\,\mathrm{d}u_{n-1} \\
&= \varepsilon^{n-1}G^n
\end{aligned}$$

Therefore, the left-hand side of (A.60) is equal to

$$g_{1,n}^{*,\varepsilon}(x_0,x) + r_5(n) + r_6(n),$$

where $r_6(n) = g_{1,n}^{*}(x_0,x) - g_{1,n}^{*,\varepsilon}(x_0,x)$, and $|r_6(n)| \leq \varepsilon^{n-1}G^n$. □

The error terms $r_5(n)$ depend on p as they are functions of $r_2^{(p)}$, $\varepsilon^{(p)}$, $\mathrm{Var}\left(Z^{(p)}\right)$ and $R_{V,1}^{(p)}$. We write $r_5^{(p)}(n)$ when this dependence is explicitly needed, otherwise this dependence it omitted from the notation. Choosing $\varepsilon = \mathrm{Var}(Z^{(p)})^{1/3}$, the error term $|r_5^{(p)}(n)| = O\left(\mathrm{Var}\left(Z^{(p)}\right)^{1/3}\right) \to 0$ as $p\to\infty$. Similarly, $r_6(n)$ depends on p as it is a function of $\varepsilon^{(p)}$ and we write $r_6^{(p)}(n)$ when we need to denote this explicitly.

Corollary A.1.11. *Let $g_1, g_2, \dots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x\in[0,\Delta)$, be a closing operator with Properties 3.0.2. Then, for $n\geq 2$, $x_0\in[0,\Delta)$,*

$$\left|\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{k=2}^{n-1}\int_{x_k=0}^{\infty} g_k(x_k)e^{\boldsymbol{S}x_k}\,\mathrm{d}x_k\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_nV(x)\right.$$

$$
\begin{aligned}
&- \int_{u_1=0}^{\Delta - x_0} g_1(\Delta - u_1 - x_0) \left[\prod_{k=2}^{n-1} \int_{u_k=0}^{\Delta - u_{k-1}} g_k(\Delta - u_k - u_{k-1}) \,\mathrm{d}u_{k-1} \right] g_n(\Delta - x - u_{n-1}) \\
&\qquad 1(\Delta - x - u_{n-1} \geq 0) \,\mathrm{d}u_{n-1} \Bigg| \\
&\leq |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|, \qquad \text{(A.61)}
\end{aligned}
$$

where

$$
\begin{aligned}
|r_4(n)| &= \left(2\varepsilon + \frac{\mathrm{Var}(Z)}{\varepsilon}\right) \frac{1}{1 - \mathrm{Var}(Z)/(\Delta - x_0)} G \widehat{G}^{n-2} G, \\
|r_5(n)| &= O\left(\max\left\{ G^{n-1}\Delta^{n-2}\left(\frac{1}{2}\Delta |r_2| + 2\varepsilon G + \frac{1}{2}\Delta G \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2} \right), G^{n-1}\Delta^{n-2} R_{V,1} \right\} \right) \\
|r_6(n)| &\leq \varepsilon^{n-1} G^n.
\end{aligned}
$$

Proof. The left-most term on the left-hand side of (A.61) can be written as

$$
\begin{aligned}
&\int_{x_1=0}^{\infty} g_1(x_1) \boldsymbol{a}(x_0) e^{\boldsymbol{S} x_1} \,\mathrm{d}x_1 \boldsymbol{D}(\Delta - \varepsilon) \left[\prod_{k=2}^{n-1} \int_{x_k=0}^{\infty} g_k(x_k) e^{\boldsymbol{S} x_k} \,\mathrm{d}x_k \boldsymbol{D}(\Delta - \varepsilon) \right] \qquad \text{(A.62)} \\
&\quad \times \int_{x_n=0}^{\infty} g_n(x_n) e^{\boldsymbol{S} x_n} \,\mathrm{d}x_n V(x) + \sum_{k=1}^{n-1} \int_{x_{k+1}=0}^{\infty} \int_{u_k=\Delta-\varepsilon}^{\infty} \boldsymbol{a}(x_0) \mathcal{I}_{1,k}(u_k) \mathcal{J}_{k+1,n}(u_k, x_{k+1}) V(x).
\end{aligned}
$$

Now, substitute this into the left-hand side of (A.61), apply the triangle inequality and Lemmas A.1.9 and A.1.10 to get the result. □

Define

$$
\begin{aligned}
w_n(x_0, x) = &\int_{x_1=0}^{\infty} g_1(x_1) \boldsymbol{a}(x_0) e^{\boldsymbol{S} x_1} \,\mathrm{d}x_1 \boldsymbol{D} \left[\prod_{k=2}^{n-1} \int_{x_k=0}^{\infty} g_k(x_k) e^{\boldsymbol{S} x_k} \,\mathrm{d}x_k \boldsymbol{D} \right] \\
&\int_{x_n=0}^{\infty} g_n(x_n) e^{\boldsymbol{S} x_n} \,\mathrm{d}x_n V(x),
\end{aligned}
$$

where $g_1, g_2, \dots,$ are functions satisfying the Assumptions 3.0.1 and $V(x)$ is a closing operator with the Properties 3.0.2.

Corollary A.1.12. *Let $g_1, g_2, \dots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x \in [0, \Delta)$, be a closing operator with Properties 3.0.2. Then, for $n \geq 2$, $x_0 \in [0, \Delta)$,*

$$
\left| \int_{x=0}^{\Delta} w_n(x_0, x) \psi(x) \,\mathrm{d}x \right.
$$

$$
\begin{aligned}
& - \int_{x=0}^{\Delta} \int_{u_1=0}^{\Delta-x_0} g_1(\Delta - u_1 - x_0) \left[\prod_{k=2}^{n-1} \int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta - u_k - u_{k-1}) \,\mathrm{d}u_{k-1} \right] g_n(\Delta - x - u_{n-1}) \\
& \qquad 1(\Delta - x - u_{n-1} \geq 0) \,\mathrm{d}u_{n-1} \psi(x) \,\mathrm{d}x \Bigg| \\
& \leq (|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\Delta F. \qquad \text{(A.63)}
\end{aligned}
$$

Proof. The left-hand side of (A.63) is less than or equal to

$$
\begin{aligned}
& \int_{x=0}^{\Delta} \Bigg| w_n(x_0, x) \\
& \quad - \int_{u_1=0}^{\Delta-x_0} g_1(\Delta - u_1 - x_0) \left[\prod_{k=2}^{n-1} \int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta - u_k - u_{k-1}) \,\mathrm{d}u_{k-1} \right] g_n(\Delta - x - u_{n-1}) \\
& \qquad 1(\Delta - x - u_{n-1} \geq 0) \,\mathrm{d}u_{n-1} \Bigg| \, |\psi(x)| \,\mathrm{d}x. \qquad \text{(A.64)}
\end{aligned}
$$

Apply Corollary A.1.11 to bound the first absolute value so that (A.64) is less than or equal to

$$
\begin{aligned}
& \int_{x=0}^{\Delta} (|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|) \, |\psi(x)| \,\mathrm{d}x \\
& \leq \int_{x=0}^{\Delta} (|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|) F \,\mathrm{d}x \\
& = (|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\Delta F \qquad \text{(A.65)}
\end{aligned}
$$

□

At this point, we have all the results we need to prove weak convergence of the QBD-RAP approximation on the event that $\varphi(0) \notin \mathcal{S}_{+0} \cup \mathcal{S}_{-0}$. The difficulty that remains it to deal with sample paths of the QBD-RAP which start in $\phi(0) \in \mathcal{S}_{+0}$ (or $\mathcal{S}_{-0}$), and upon leaving this subset of phases jump to $i \in \mathcal{S}_-$ ($\mathcal{S}_+$). The orbit process at the time at which $\{\phi(t)\}$ first exits $\mathcal{S}_{+0}$ ($\mathcal{S}_{-0}$) is $\boldsymbol{a}(x_0)\boldsymbol{D}$, thus we can treat this case simply as a change in the initial condition of the QBD-RAP. We do so by showing that asymptotically, the initial conditions $\boldsymbol{a}(\Delta - x_0)$ and $\boldsymbol{a}(x_0)\boldsymbol{D}$ produce results that are arbitrarily close to each other. We first show a Lipschitz-like condition in x_0 for $w_n(x_0, x)$.

Corollary A.1.13. *For $x_0, x \in [0, \Delta)$, $n \geq 2$,*

$$
|w_n(x_0, x) - w_n(z_0, x)| \leq 2|r_5(n)| + 2|r_6(n)| + 2(n-1)|r_4(n)| + |x_0 - z_0| G_n^*(G + L\Delta), \qquad \text{(A.66)}
$$

Proof. By adding and subtracting both $\int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1$ and $\int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-z_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1$, we can write the left-hand side of (A.66) as

$$\begin{aligned}\Bigg| w_n(x_0,x) &- \int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \\ &- w_n(z_0,x) + \int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-z_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \\ &+ \int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 - \int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-z_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg|\end{aligned}$$

which, by the triangle inequality, is less than or equal to

$$\begin{aligned}\Bigg| w_n(x_0,x) &- \int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg| \\ &+ \Bigg| w_n(z_0,x) - \int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-z_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg| \\ &+ \Bigg| \int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 - \int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-z_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg|\end{aligned} \tag{A.67}$$

By Corollary A.1.11, the first two terms of (A.67) are less than or equal to $|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|$. As for the last term, adding and subtracting $\int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1$ gives

$$\begin{aligned} &= \Bigg| \int_{u_1=0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 - \int_{u_1=0}^{\Delta-z_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \\ &\qquad - \int_{u_1=0}^{\Delta-z_0} (g_1(\Delta-u_1-z_0) - g_1(\Delta-u_1-x_0))g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg| \\ &\le \Bigg| \int_{u_1=\Delta-z_0}^{\Delta-x_0} g_1(\Delta-u_1-x_0)g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \Bigg| \\ &\qquad + \int_{u_1=0}^{\Delta-z_0} |g_1(\Delta-u_1-z_0) - g_1(\Delta-u_1-x_0)|g_{2,n}^*(u_1,x)\,\mathrm{d}u_1 \\ &\le GG_n^*|x_0-z_0| + \int_{u_1=0}^{\Delta-z_0} L|x_0-z_0|G_n^*\,\mathrm{d}u_1 \end{aligned} \tag{A.68}$$

since g_1 is Lipschitz by Assumption 3.0.1(iv) and $g^*_{2,n} \leq G^*_n$. Bounding the integral over u_1 by Δ, then (A.68) is less than or equal to

$$GG^*_n|x_0 - z_0| + \Delta L|x_0 - z_0|G^*_n. \tag{A.69}$$

$\square$

Corollary A.1.14. *Let $g_1, g_2, \ldots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x \in [0, \Delta)$, be a closing operator with Properties 3.0.2. For $x_0, x \in [0, \Delta)$, $n \geq 2$,*

$$\begin{aligned}
&\Bigg| \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&\quad - w_n(\Delta - x_0, x)\Bigg| \\
&= r_8(n),
\end{aligned} \tag{A.70}$$

where

$$\begin{aligned}
|r_8(n)| \leq{}& (2|r_5(n)| + 2|r_6(n)| + 2(n-1)|r_4(n)| + \varepsilon G^*_n(G + L\Delta)) \\
&+ 2\widehat{G}^{n-2}GG_V\frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2}.
\end{aligned}$$

Proof. Observe that

$$\begin{aligned}
&\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&= \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\int_{z_0=0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}\frac{\boldsymbol{\alpha}e^{\boldsymbol{S}z_0}}{\boldsymbol{\alpha}e^{\boldsymbol{S}z_0}\boldsymbol{e}}\,\mathrm{d}z_0 e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right] \\
&\quad \times \int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) \\
&= \boldsymbol{a}(x_0)\int_{z_0=0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}w_n(z_0, x)\,\mathrm{d}z_0
\end{aligned} \tag{A.71}$$

Now

$$\begin{aligned}
&\left|\boldsymbol{a}(x_0)\int_{z_0=0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}w_n(z_0, x)\,\mathrm{d}z_0 - w_n(\Delta - x_0, x)\right| \\
&= \left|\boldsymbol{a}(x_0)\int_{z_0=0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}(w_n(z_0, x) - w_n(\Delta - x_0, x))\,\mathrm{d}z_0\right|
\end{aligned}$$

$$\begin{aligned}
&\leq \boldsymbol{a}(x_0)\int_{z_0=0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,|w_n(z_0,x)-w_n(\Delta-x_0,x)|\,\mathrm{d}z_0\\
&= \boldsymbol{a}(x_0)\int_{z_0=0}^{\Delta-\varepsilon-x_0} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,|w_n(z_0,x)-w_n(\Delta-x_0,x)|\,\mathrm{d}z_0\\
&\qquad+\boldsymbol{a}(x_0)\int_{z_0=\Delta+\varepsilon-x_0}^{\infty} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,|w_n(z_0,x)-w_n(\Delta-x_0,x)|\,\mathrm{d}z_0\\
&\qquad+\boldsymbol{a}(x_0)\int_{z_0=\Delta-\varepsilon-x_0}^{\Delta+\varepsilon-x_0} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,|w_n(z_0,x)-w_n(\Delta-x_0,x)|\,\mathrm{d}z_0. \qquad (A.72)
\end{aligned}$$

By Corollary A.1.8, for any $x,y\in[0,\Delta)$, $0\leq w_n(x,y)\leq \widehat{G}^{n-2}GG_V$, so the sum of the first two terms is less than or equal to

$$\begin{aligned}
&2\widehat{G}^{n-1}GG_V\left(\int_{z_0=0}^{\Delta-\varepsilon-x_0}\boldsymbol{a}(x_0)e^{\boldsymbol{S}z_0}\boldsymbol{s}\,\mathrm{d}z_0+\int_{z_0=\Delta+\varepsilon-x_0}^{\infty}\boldsymbol{a}(x_0)e^{\boldsymbol{S}z_0}\boldsymbol{s}\,\mathrm{d}z_0\right) \qquad (A.73)\\
&=2\widehat{G}^{n-1}GG_V\frac{\mathbb{P}(|Z-\Delta|>\varepsilon)}{\mathbb{P}(Z>x_0)}\\
&\leq 2G^n\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2} \qquad (A.74)
\end{aligned}$$

by Chebyshev's inequality. As for the last term in (A.72), we can use Corollary A.1.13 to bound the integrand so that the last term is less than or equal to

$$\begin{aligned}
&\boldsymbol{a}(x_0)\int_{z_0=\Delta-\varepsilon-x_0}^{\Delta+\varepsilon-x_0} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,\big(2|r_5(n)|+2|r_6(n)|+2(n-1)|r_4(n)|+\varepsilon G^{n-1}\Delta^{n-2}(G+L\Delta)\big)\,\mathrm{d}z_0\\
&\leq\big(2|r_5(n)|+2|r_6(n)|+2(n-1)|r_4(n)|+\varepsilon G^{n-1}\Delta^{n-2}(G+L\Delta)\big)\\
&\quad+2\widehat{G}^{n-2}GG_V\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}, \qquad (A.75)
\end{aligned}$$

since $\boldsymbol{a}(x_0)\int_{z_0=\Delta-\varepsilon-x_0}^{\Delta+\varepsilon-x_0} e^{\boldsymbol{S}z_0}\boldsymbol{s}\,\mathrm{d}z_0\leq 1$. □

Corollary A.1.15. *Let $g_1,g_2,\ldots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x\in(0,\Delta)$, be a closing operator with Properties 3.0.2. For $x_0,x\in(0,\Delta)$, $n\geq 2$*

$$\begin{aligned}
&\Bigg|\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_nV(x)\\
&\qquad-\int_{u_1=0}^{x_0} g_1(x_0-u_1)\left[\prod_{k=2}^{n-1}\int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta-u_k-u_{k-1})\,\mathrm{d}u_{k-1}\right]g_n(\Delta-x-u_{n-1})
\end{aligned}$$

$$1(\Delta - x - u_{n-1} \geq 0)\,\mathrm{d}u_{n-1}\Bigg|$$

$$\leq |r_8(n)| + |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|, \tag{A.76}$$

where

$$|r_8(n)| \leq \big(2|r_5(n)| + 2|r_6(n)| + 2(n-1)|r_4(n)| + \varepsilon G^{n-1}\Delta^{n-2}(G + L\Delta)\big) \tag{A.77}$$

$$+ 2\widehat{G}^{n-2} G G_V \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2}. \tag{A.78}$$

Proof. Adding and subtracting $w_n(\Delta - x_0, x)$ within the absolute value on the left-hand side of (A.76)

$$\Bigg| \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x)$$

$$- w_n(\Delta - x_0, x) + w_n(\Delta - x_0, x) - g^*_{1,n}(\Delta - x_0, x)\Bigg|$$

$$\leq \Bigg| \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x),$$

$$- w_n(\Delta - x_0, x)\Bigg| + \Big|w_n(\Delta - x_0, x) - g^*_{1,n}(\Delta - x_0, x)\Big|$$

where the first absolute value is less than or equal to $|r_8(n)|$ by Corollary A.1.14 and the second absolute value is less than or equal to $|r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|$ by Corollary A.1.11. □

Corollary A.1.16. *Let ψ be bounded and Lipschitz, let $g_1, g_2, \ldots,$ be functions satisfying Assumptions 3.0.1 and let $V(x)$, $x \in (0, \Delta)$, be a closing operator with Properties 3.0.2. For $x_0, x \in (0, \Delta)$, $n \geq 2$*

$$\Bigg| \int_{x\in[0,\Delta)}\int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right]$$

$$\times \int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x)\psi(x)\,\mathrm{d}x$$

$$- \int_{x\in[0,\Delta)}\int_{u_1=0}^{x_0} g_1(x_0 - u_1)\left[\prod_{k=2}^{n-1}\int_{u_k=0}^{\Delta - u_{k-1}} g_k(\Delta - u_k - u_{k-1})\,\mathrm{d}u_{k-1}\right] g_n(\Delta - x - u_{n-1})$$

$$\times 1(\Delta - x - u_{n-1} \geq 0)\,\mathrm{d}u_{n-1}\psi(x)\,\mathrm{d}x\Bigg|$$

$$\leq (|r_8(n)| + |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\, F\Delta. \tag{A.79}$$

Proof. The left-hand side of (A.79) is less than or equal to

$$\begin{aligned}
&\int_{x\in[0,\Delta)} \Bigg| \int_{x_1=0}^{\infty} g_1(x_1)\boldsymbol{a}(x_0)\boldsymbol{D}e^{\boldsymbol{S}x_1}\,\mathrm{d}x_1\boldsymbol{D}\left[\prod_{n=2}^{k-1}\int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n\boldsymbol{D}\right] \\
&\quad\times \int_{x_n=0}^{\infty} g_n(x_n)e^{\boldsymbol{S}x_n}\,\mathrm{d}x_n V(x) - \int_{u_1=0}^{x_0} g_1(x_0-u_1) \\
&\quad\times \left[\prod_{k=2}^{n-1}\int_{u_k=0}^{\Delta-u_{k-1}} g_k(\Delta-u_k-u_{k-1})\,\mathrm{d}u_{k-1}\right] g_n(\Delta-x-u_{n-1}) \\
&\quad 1(\Delta-x-u_{n-1}\geq 0)\,\mathrm{d}u_{n-1}\Bigg|\,|\psi(x)|\,\mathrm{d}x.
\end{aligned} \tag{A.80}$$

Now, apply Corollary A.1.15 to the first absolute value then (A.80) is less than or equal to

$$\begin{aligned}
&\int_{x\in[0,\Delta)} (|r_8(n)| + |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\,|\psi(x)|\,\mathrm{d}x \\
&\leq \int_{x\in[0,\Delta)} (|r_8(n)| + |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\,F\,\mathrm{d}x \\
&= (|r_8(n)| + |r_5(n)| + |r_6(n)| + (n-1)|r_4(n)|)\,\Delta F
\end{aligned} \tag{A.81}$$

□

A.2 Properties of closing operators

Corollary A.2.1. *Let g be a function satisfying the Assumptions 3.0.1 and consider the closing operator $V(x) = e^{\boldsymbol{S}x}\boldsymbol{s}$. For $u \leq \Delta - \varepsilon$, $v \geq 0$,*

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha}e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha}e^{\boldsymbol{S}u}\boldsymbol{e}} V(v)g(x)\,\mathrm{d}x = g(\Delta-u-v)1(u+v\leq\Delta-\varepsilon) + r_V(u,v),$$

where

$$\int_{u=0}^{\Delta-\varepsilon} r_V(u,v)\,\mathrm{d}u = \int_{u=0}^{\Delta-\varepsilon} r_3(u+v)\,\mathrm{d}u \leq r_2\Delta + 2\varepsilon G + \Delta G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}$$

and

$$\int_{u=0}^{\Delta} r_V(u,v)\,\mathrm{d}u \leq R_{V,1}$$

where

$$R_{V,1} = r_2\Delta + 2\varepsilon G + \Delta G\frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}.$$

Proof. By Corollary A.1.3,

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} V(v)g(x)\,\mathrm{d}x = g(\Delta - u - v)1(u + v \leq \Delta - \varepsilon) + r_3(u+v),$$

so $r_V(u,v) = r_3(u+v)$. All that remains to be shown are the bounds $R_{V,1}$ and $R_{V,2}$. To this end, observe

$$R_{V,1} = \int_{u=0}^{\Delta} r_V(u,v)\,\mathrm{d}u = \int_{u=0}^{\Delta} r_3(u+v)\,\mathrm{d}u \leq r_2\Delta + 2\varepsilon G + \Delta G \frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}.$$

$$R_{V,2} = \int_{v=0}^{\Delta} r_V(u,v)\,\mathrm{d}v = \int_{v=0}^{\Delta} r_3(u+v)\,\mathrm{d}v \leq r_2\Delta + 2\varepsilon G + \Delta G \frac{\mathrm{Var}(Z)/\varepsilon^2}{1-\mathrm{Var}(Z)/\varepsilon^2}.$$

□

Lemma A.2.2. *For any valid orbit,* $\boldsymbol{a} \in \mathcal{A}$, $x, u \geq 0$,

$$\int_{x_n=0}^{\infty} \boldsymbol{a} e^{\boldsymbol{S}(x+x_n+u)}\boldsymbol{s} = \boldsymbol{a} e^{\boldsymbol{S}(x+u)}\boldsymbol{e} \leq \boldsymbol{a} e^{\boldsymbol{S}u}\boldsymbol{e}.$$

Proof. For any valid orbit, $\boldsymbol{a} \in \mathcal{A}$,

$$\boldsymbol{a} e^{\boldsymbol{S}(x+u)}\boldsymbol{e} = \mathbb{P}(Z > x+u) \leq \mathbb{P}(X > u) = \boldsymbol{a} e^{\boldsymbol{S}u}\boldsymbol{e}.$$

□

Lemma A.2.3. *For any valid orbit,* $\boldsymbol{a} \in \mathcal{A}$, $x \geq 0$,

$$\boldsymbol{a}\boldsymbol{D} e^{\boldsymbol{S}x}\boldsymbol{e} \leq 1.$$

Proof. By the definition of $\boldsymbol{D}$ and Lemma A.2.2,

$$\begin{aligned}
\boldsymbol{a}\boldsymbol{D} e^{\boldsymbol{S}x}\boldsymbol{e} &= \boldsymbol{a}\int_{u=0}^{\infty} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}}\,\mathrm{d}u e^{\boldsymbol{S}x}\boldsymbol{e} \\
&\leq \boldsymbol{a}\int_{u=0}^{\infty} e^{\boldsymbol{S}u}\boldsymbol{s}\frac{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}}\,\mathrm{d}u \\
&= \boldsymbol{a}\int_{u=0}^{\infty} e^{\boldsymbol{S}u}\boldsymbol{s}\,\mathrm{d}u \\
&= \boldsymbol{a}\boldsymbol{e} = 1.
\end{aligned}$$

□

Let $U^{(p)}(x)$ be the closing operator such that, for $\boldsymbol{a}^{(p)} \in \mathcal{A}^{(p)}$, $x \in [0, \Delta)$,

$$\boldsymbol{a}^{(p)} U^{(p)}(x) = \boldsymbol{a}^{(p)} \left(e^{\boldsymbol{S}^{(p)} x} \boldsymbol{s}^{(p)} + e^{\boldsymbol{S}^{(p)}(2\Delta - x)} \boldsymbol{s}^{(p)} \right).$$

Lemma A.2.4. *For $x \in [0, \Delta), u \geq 0$,*

$$\boldsymbol{a} e^{\boldsymbol{S} u} (-\boldsymbol{S})^{-1} U(x) \leq 2 \boldsymbol{a} e^{\boldsymbol{S} u} \boldsymbol{e}.$$

Proof. Let $\boldsymbol{a} \in \mathcal{A}$ be arbitrary. By definition

$$\boldsymbol{a} e^{\boldsymbol{S} u} (-\boldsymbol{S})^{-1} U(x) = \boldsymbol{a} e^{\boldsymbol{S} u} (-\boldsymbol{S})^{-1} \left(e^{\boldsymbol{S} x} \boldsymbol{s} + e^{\boldsymbol{S}(2\Delta - x)} \boldsymbol{s} \right)$$

since $(-\boldsymbol{S})^{-1}$ and $e^{\boldsymbol{S} x}$ commute and $\boldsymbol{s} = -\boldsymbol{S} e$. By Lemma A.2.2 this is less than or equal to,

$$\boldsymbol{a} e^{\boldsymbol{S} u} (\boldsymbol{e} + \boldsymbol{e}) = 2 \boldsymbol{a} e^{\boldsymbol{S} u} \boldsymbol{e} \tag{A.82}$$

□

Corollary A.2.5. *Let g be a function satisfying the Assumptions 3.0.1. For $u \leq \Delta - \varepsilon$, $v \in [0, \Delta)$,*

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S} u} \boldsymbol{e}} U(v) g(x) \, \mathrm{d}x = g(\Delta - u - v) 1(u + v \leq \Delta - \varepsilon) + r_V(u, v),$$

where

$$|r_V(u, v)| \leq r_3(u + v) + r_3(u + 2\Delta - v).$$

Furthermore,

$$\int_{u=0}^{\Delta} |r_V(u, x)| \, \mathrm{d}u \leq R_{V,1},$$

and

$$\int_{v=0}^{\Delta} |r_V(u, x)| \, \mathrm{d}u \leq R_{V,2},$$

where

$$R_{V,1},\ R_{V,2} \leq 2 \left(\Delta r_2 + 2\varepsilon G + \Delta \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2} \right).$$

Proof. By the definition of the operator $U(x)$,

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{\boldsymbol{S}u}\boldsymbol{e}} U(v) g(x)\,\mathrm{d}x = \int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{Su}\boldsymbol{e}} e^{\boldsymbol{S}v}\boldsymbol{s} g(x) + \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{Su}\boldsymbol{e}} e^{\boldsymbol{S}(2\Delta - v)}\boldsymbol{s} g(x)\,\mathrm{d}x. \tag{A.83}$$

By Corollary A.1.3

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{Su}\boldsymbol{e}} e^{\boldsymbol{S}v}\boldsymbol{s} g(x)\,\mathrm{d}x = g(\Delta - u - v)1(u+v \leq \Delta - \varepsilon) + r_3(u+v), \tag{A.84}$$

$$\int_{x=0}^{\infty} \frac{\boldsymbol{\alpha} e^{\boldsymbol{S}(u+x)}}{\boldsymbol{\alpha} e^{Su}\boldsymbol{e}} e^{\boldsymbol{S}(2\Delta - v)}\boldsymbol{s} g(x)\,\mathrm{d}x = r_3(u + 2\Delta - v). \tag{A.85}$$

Therefore, (A.83) is,

$$g(\Delta - u - v)1(u+v \leq \Delta - \varepsilon) + r_3(u+v) + r_3(u + 2\Delta - v). \tag{A.86}$$

Now,

$$\begin{aligned} R_{V,1} &\leq \int_{u=0}^{\Delta} |r_V(u,v)|\,\mathrm{d}u \\ &\leq \int_{u=0}^{\Delta} |r_3(u+v)| + |r_3(u+2\Delta - v)| \\ &\leq 2\left(\Delta r_2 + 2\varepsilon G + \Delta \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2}\right). \end{aligned}$$

Similarly

$$\begin{aligned} R_{V,2} &\leq \int_{v=0}^{\Delta} |r_V(u,v)|\,\mathrm{d}v \\ &= \int_{v=0}^{\Delta} r_3(u+v) + r_3(u+2\Delta - v) \\ &\leq 2\left(\Delta r_2 + 2\varepsilon G + \Delta \frac{\mathrm{Var}(Z)/\varepsilon^2}{1 - \mathrm{Var}(Z)/\varepsilon^2}\right). \end{aligned}$$

□

The error term $r_V(u,v)$ depends on p so we should write $r_V^{(p)}(u,v)$. The error term $r_V^{(p)}(u,v)$ has similar properties to $r_3^{(p)}(u+v)$; we can prove that it converges point-wise to 0 only on some areas of its domain, however when we integrate the error against bounded functions on bounded domains, then the resulting integral tends to 0.

A.3 Kronecker properties

Here we detail some properties of Kronecker sum, products, and exponentials (see (Bladt & Nielsen 2017), Appendix A.4).

Let

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & \dots & a_{1m} \\ \dots & & \dots \\ a_{n1} & \dots & a_{nm} \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} b_{11} & \dots & b_{1m'} \\ \dots & & \dots \\ b_{n'1} & \dots & b_{n'm'} \end{bmatrix}$$

be matrices. The operator $\otimes$ is the Kronecker product of two matrices;

$$\boldsymbol{A} \otimes \boldsymbol{B} = \begin{bmatrix} a_{11}\boldsymbol{B} & \dots & a_{1m}\boldsymbol{B} \\ \dots & & \dots \\ a_{n1}\boldsymbol{B} & \dots & a_{nm}\boldsymbol{B} \end{bmatrix},$$

which is an $nn' \times mm'$ matrix.

Let $\boldsymbol{C}, \boldsymbol{D}$ be matrices with dimensions $m \times k$ and $m' \times k'$. A property of the Kronecker Product is

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = \boldsymbol{AC} \otimes \boldsymbol{BD}. \tag{Mixed Product Rule}$$

Proof. The proof follows from

$$\begin{aligned} \begin{bmatrix} a_{i1}\boldsymbol{B} & a_{i2}\boldsymbol{B} & \dots & a_{in}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} c_{1j}\boldsymbol{D} \\ c_{2j} \\ \vdots \\ c_{nj}\boldsymbol{D} \end{bmatrix} &= \left(\sum_{\ell} a_{i\ell} c_{\ell j} \right) \boldsymbol{BD} \\ &= (\boldsymbol{AC})_{ij} \, \boldsymbol{BD}. \end{aligned}$$

□

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices, then

$$(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}. \tag{A.87}$$

Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be $n \times n$ and $m \times m$ matrices, respectively. The Kronecker sum of $\boldsymbol{A}$ and $\boldsymbol{B}$ is denoted by $\oplus$ and defined as

$$\boldsymbol{A} \oplus \boldsymbol{B} := \boldsymbol{A} \otimes \boldsymbol{I}_m + \boldsymbol{I}_n \otimes \boldsymbol{B}.$$

A property of the Kronecker sum is

$$e^{\boldsymbol{A} \oplus \boldsymbol{B}} = e^{\boldsymbol{A}} \otimes e^{\boldsymbol{B}}. \tag{A.88}$$

Proof. First, the matrices $\boldsymbol{A}\otimes\boldsymbol{I}_m$ and $\boldsymbol{I}_n\otimes\boldsymbol{B}$ commute; from the mixed product rule their product is $\boldsymbol{A}\otimes\boldsymbol{B}$. Hence

$$e^{\boldsymbol{A}\oplus\boldsymbol{B}} = e^{\boldsymbol{A}\otimes\boldsymbol{I}_m} e^{\boldsymbol{I}_n\otimes\boldsymbol{B}}.$$

We now show that $e^{\boldsymbol{A}\otimes\boldsymbol{I}_m} = e^{\boldsymbol{A}}\otimes\boldsymbol{I}_m$ and $e^{\boldsymbol{I}_n\otimes\boldsymbol{B}} = \boldsymbol{I}_n\otimes e^{\boldsymbol{B}}$. The latter follows from the fact that $\boldsymbol{I}_n\otimes\boldsymbol{B}$ is a block diagonal matrix with blocks $\boldsymbol{B}$, hence its exponential is also block diagonal with blocks equal to the exponential of $\boldsymbol{B}$. The former follows from

$$\begin{aligned} e^{\boldsymbol{A}\otimes\boldsymbol{I}_m} &= \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{A}\otimes\boldsymbol{I}_m)^n \\ &= \sum_{n=0}^{\infty}\frac{1}{n!}(\boldsymbol{A}^n\otimes\boldsymbol{I}_m) \\ &= \left(\sum_{n=0}^{\infty}\frac{1}{n!}\boldsymbol{A}\otimes\boldsymbol{I}_m\right) \\ &= e^{\boldsymbol{A}}\otimes\boldsymbol{I}_m. \end{aligned} \tag{A.89}$$

Therefore

$$e^{\boldsymbol{A}\oplus\boldsymbol{B}} = \left(e^{\boldsymbol{A}}\otimes\boldsymbol{I}_m\right)\left(\boldsymbol{I}_n\otimes e^{\boldsymbol{B}}\right),$$

and the result follows by the mixed product rule. $\square$

Lemma A.3.1. *Let $\boldsymbol{T}$ and $\boldsymbol{C}$ be $n\times n$, square matrices with $\boldsymbol{C}$ diagonal and invertible; let $\boldsymbol{S}$ be a $p\times p$ matrix. Further, suppose $[\boldsymbol{T}\otimes\boldsymbol{I}+\boldsymbol{C}\otimes\boldsymbol{S}-\lambda\boldsymbol{I}]$ is invertible for $\lambda>0$. Then*

$$\int_{t=0}^{\infty} e^{-\lambda t}e^{(\boldsymbol{T}\otimes\boldsymbol{I}+\boldsymbol{C}\otimes\boldsymbol{S})t}\,\mathrm{d}t = \int_{x=0}^{\infty} e^{\boldsymbol{C}^{-1}(\boldsymbol{T}-\lambda\boldsymbol{I})x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x\,(\boldsymbol{C}\otimes\boldsymbol{I})^{-1} \tag{A.90}$$

Proof. Computing the integral on the left-hand side and then factorising the result and using the Mixed Product Rule multiple times gives

$$\begin{aligned} \int_{t=0}^{\infty} e^{-\lambda t}e^{(\boldsymbol{T}\otimes\boldsymbol{I}+\boldsymbol{C}\otimes\boldsymbol{S})t}\,\mathrm{d}t &= -\left[\boldsymbol{T}\otimes\boldsymbol{I}+\boldsymbol{C}\otimes\boldsymbol{S}-\lambda\boldsymbol{I}\right]^{-1} \\ &= -\left[\boldsymbol{T}\otimes\boldsymbol{I}+(\boldsymbol{C}\otimes\boldsymbol{I})(\boldsymbol{I}\otimes\boldsymbol{S})-\lambda\boldsymbol{I}\right]^{-1} \\ &= -\left[(\boldsymbol{C}\otimes\boldsymbol{I})\left((\boldsymbol{C}\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}\otimes\boldsymbol{I})+\boldsymbol{I}\otimes\boldsymbol{S}-(\boldsymbol{C}\otimes\boldsymbol{I})^{-1}\lambda\boldsymbol{I}\right)\right]^{-1}. \end{aligned} \tag{A.91}$$

By Equation (A.87) and since $\boldsymbol{C}$ is invertible, (A.91) is equal to

$$-\left[(\boldsymbol{C}\otimes\boldsymbol{I})\left(\left(\boldsymbol{C}^{-1}\otimes\boldsymbol{I}\right)(\boldsymbol{T}\otimes\boldsymbol{I})+\boldsymbol{I}\otimes\boldsymbol{S}-\left(\boldsymbol{C}^{-1}\otimes\boldsymbol{I}\right)\lambda\boldsymbol{I}\right)\right]^{-1}. \tag{A.92}$$

Using the Mixed Product Rule and algebraic manipulation, (A.92) is equal to

$$\begin{aligned}
&- \left[(\boldsymbol{C} \otimes \boldsymbol{I}) \left(\left(\boldsymbol{C}^{-1} \boldsymbol{T} \right) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{S} - \left(\boldsymbol{C}^{-1} \lambda \boldsymbol{I} \right) \otimes \boldsymbol{I} \right) \right]^{-1} \\
&= - \left[(\boldsymbol{C} \otimes \boldsymbol{I}) \left(\left(\boldsymbol{C}^{-1} \left(\boldsymbol{T} - \lambda \boldsymbol{I} \right) \right) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{S} \right) \right]^{-1} \\
&= - \left[\left(\boldsymbol{C}^{-1} \left(\boldsymbol{T} - \lambda \boldsymbol{I} \right) \right) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \boldsymbol{S} \right]^{-1} (\boldsymbol{C} \otimes \boldsymbol{I})^{-1} \\
&= - \left[\left(\boldsymbol{C}^{-1} \left(\boldsymbol{T} - \lambda \boldsymbol{I} \right) \right) \oplus \boldsymbol{S} \right]^{-1} (\boldsymbol{C} \otimes \boldsymbol{I})^{-1}, \qquad \text{(A.93)}
\end{aligned}$$

by definition of the Kronecker sum.

Now, for an invertible matrix $\boldsymbol{A}$ we can write $-\boldsymbol{A}^{-1} = \int_{x=0}^{\infty} e^{\boldsymbol{A}x} \, \mathrm{d}x$. Therefore (A.93) is

$$- \left[\left(\boldsymbol{C}^{-1} \left(\boldsymbol{T} - \lambda \boldsymbol{I} \right) \right) \oplus \boldsymbol{S} \right]^{-1} (\boldsymbol{C} \otimes \boldsymbol{I})^{-1} = \int_{x=0}^{\infty} e^{\left(\boldsymbol{C}^{-1} (\boldsymbol{T} - \lambda \boldsymbol{I}) x \right) \oplus \boldsymbol{S} x} \, \mathrm{d}x \, (\boldsymbol{C} \otimes \boldsymbol{I})^{-1}.$$

Using the rule in Equation (A.88) gives

$$\int_{x=0}^{\infty} e^{\left(\boldsymbol{C}^{-1} (\boldsymbol{T} - \lambda \boldsymbol{I}) \right) x} \otimes e^{\boldsymbol{S} x} \, \mathrm{d}x \, (\boldsymbol{C} \otimes \boldsymbol{I})^{-1},$$

which is the result. □

The exponential of a matrix $\boldsymbol{B}$ is

$$e^{\boldsymbol{B}} := \sum_{n=0}^{\infty} \frac{1}{n!} B^n.$$

Latouche & Nguyen (2015) show the following.

Lemma A.3.2. *Let $\boldsymbol{B}$ be the block-partitioned matrix*

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix}$$

where $\boldsymbol{B}_{11}$ and $\boldsymbol{B}_{22}$ are matrices of order m_1 and m_2, respectively. Denote by $\boldsymbol{H}_{11}(t)$ the top-left quadrant of order m_1 of $e^{\boldsymbol{B}t}$:

$$\boldsymbol{H}_{11}(t) = \begin{bmatrix} \boldsymbol{I}_{m_1 \times m_1} & \boldsymbol{0} \end{bmatrix} e^{\boldsymbol{B}t} \begin{bmatrix} \boldsymbol{I}_{m_1 \times m_1} \\ \boldsymbol{0} \end{bmatrix}.$$

The matrix $\boldsymbol{H}_{11}(t)$ is the solution of

$$\boldsymbol{H}_{11}(t) = e^{\boldsymbol{B}_{11} t} + \int_{v=0}^{t} \int_{u=v}^{t} e^{\boldsymbol{B}_{11}(t-u)} \boldsymbol{B}_{12} e^{\boldsymbol{B}_{22}(u-v)} \boldsymbol{B}_{21} \boldsymbol{H}_{11}(v) \, \mathrm{d}u \, \mathrm{d}v. \qquad \text{(A.94)}$$

Let $\boldsymbol{H}_{12}(t)$ be the top-right quadrant of $e^{\boldsymbol{B}t}$ of size $m_1 \times m_2$, i.e.

$$\boldsymbol{H}_{12}(t) = \begin{bmatrix}\boldsymbol{I}_{m_1\times m_1} & \boldsymbol{0}\end{bmatrix} e^{\boldsymbol{B}t} \begin{bmatrix}\boldsymbol{0} \\ \boldsymbol{I}_{m_2\times m_2}\end{bmatrix}. \tag{A.95}$$

Denote by $\widehat{\boldsymbol{H}}_{11}(\lambda) := \int_{t=0}^{\infty} e^{-\lambda t}\boldsymbol{H}_{11}(t)\,\mathrm{d}t$ and by $\widehat{\boldsymbol{H}}_{12}(\lambda) := \int_{t=0}^{\infty} e^{-\lambda t}\boldsymbol{H}_{12}(t)\,\mathrm{d}t$, the Laplace transforms of $\boldsymbol{H}_{11}(t)$ and $\boldsymbol{H}_{12}(t)$, respectively. Using Lemma A.3.2 we can show the following result.

Lemma A.3.3.

$$\widehat{\boldsymbol{H}}_{11}(\lambda) = \int_{x=0}^{\infty} e^{\left(\boldsymbol{B}_{11}-\lambda\boldsymbol{I}_{m_1\times m_1}+\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right)x}\,\mathrm{d}x, \tag{A.96}$$

$$\widehat{\boldsymbol{H}}_{12}(\lambda) = \int_{x=0}^{\infty} e^{\left(\boldsymbol{B}_{11}-\lambda\boldsymbol{I}_{m_1\times m_1}+\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right)x}\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\,\mathrm{d}x. \tag{A.97}$$

Proof. First we show the result for $\widehat{\boldsymbol{H}}_{11}(\lambda)$. Taking the Laplace transform of (A.94) shows that $\widehat{\boldsymbol{H}}_{11}(\lambda)$ is equal to

$$\begin{aligned}
&\int_{t=0}^{\infty}\int_{v=0}^{t}\int_{u=v}^{t} e^{-\lambda(t-u)}e^{\boldsymbol{B}_{11}(t-u)}\boldsymbol{B}_{12}e^{-\lambda(u-v)}e^{\boldsymbol{B}_{22}(u-v)}\boldsymbol{B}_{21}e^{-\lambda v}\boldsymbol{H}_{11}(v)\,\mathrm{d}u\,\mathrm{d}v \\
&\qquad + (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1} \\
&= (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1} + (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1}\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\widehat{\boldsymbol{H}}_{11}(\lambda),
\end{aligned} \tag{A.98}$$

by the convolution theorem for Laplace transforms. This implies

$$\begin{aligned}
&\left[\boldsymbol{I}_{m_1\times m_1} - (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1}\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right]\widehat{\boldsymbol{H}}_{11}(\lambda) \\
&= (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1},
\end{aligned}$$

and therefore

$$\begin{aligned}
&\widehat{\boldsymbol{H}}_{11}(\lambda) \\
&= \left[\boldsymbol{I}_{m_1\times m_1} - (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1}\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right]^{-1}(\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1} \\
&= \left[(\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})\left(\boldsymbol{I}_{m_1\times m_1} - (\lambda\boldsymbol{I}_{m_1\times m_1}-\boldsymbol{B}_{11})^{-1}\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right)\right]^{-1} \\
&= \left[\lambda\boldsymbol{I}_{m_1\times m_1} - \boldsymbol{B}_{11} - \boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right]^{-1} \\
&= \int_{t=0}^{\infty} e^{\left(\boldsymbol{B}_{11}-\lambda\boldsymbol{I}_{m_1\times m_1}+\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2}-\boldsymbol{B}_{22})^{-1}\boldsymbol{B}_{21}\right)t}\,\mathrm{d}t,
\end{aligned}$$

which is (A.96).

Now, to show (A.97), differentiate (A.95)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{H}_{12}(t) &= \begin{bmatrix}\boldsymbol{I}_{m_1\times m_1} & \boldsymbol{0}\end{bmatrix} e^{\boldsymbol{B}t}\begin{bmatrix}\boldsymbol{B}_{11} & \boldsymbol{B}_{12}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22}\end{bmatrix}\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}_{m_2\times m_2}\end{bmatrix}\\ &= \begin{bmatrix}\boldsymbol{I}_{m_1\times m_1} & \boldsymbol{0}\end{bmatrix} e^{\boldsymbol{B}t}\begin{bmatrix}\boldsymbol{B}_{12}\\ \boldsymbol{B}_{22}\end{bmatrix}\\ &= \boldsymbol{H}_{11}(t)\boldsymbol{B}_{12} + \boldsymbol{H}_{12}(t)\boldsymbol{B}_{22}. \end{aligned}\tag{A.99}$$

Now take the Laplace transform

$$\lambda\widehat{\boldsymbol{H}}_{12}(\lambda) - \boldsymbol{H}_{12}(0) = \widehat{\boldsymbol{H}}_{11}(\lambda)\boldsymbol{B}_{12} + \widehat{\boldsymbol{H}}_{12}(\lambda)\boldsymbol{B}_{22}. \tag{A.100}$$

Since $\boldsymbol{H}_{12}(0) = \boldsymbol{0}$ and after rearranging we get

$$\widehat{\boldsymbol{H}}_{12}(\lambda) = \widehat{\boldsymbol{H}}_{11}(\lambda)\boldsymbol{B}_{12}(\lambda\boldsymbol{I}_{m_2\times m_2} - \boldsymbol{B}_{22})^{-1}, \tag{A.101}$$

which gives (A.97) upon substituting (A.96). □

Corollary A.3.4. *For $m \in \{+,-\}$ the top-left quadrant of size $m_1\times m_1 = |\mathcal{S}_m|\cdot p\times|\mathcal{S}_m|\cdot p$ of $e^{\boldsymbol{B}_{mm}t}$,*

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\int_{t=0}^{\infty} e^{-\lambda t}\exp\left\{\begin{bmatrix}\boldsymbol{T}_{mm}\otimes\boldsymbol{I} + \boldsymbol{C}_m\otimes\boldsymbol{S} & \boldsymbol{T}_{m0}\otimes\boldsymbol{I}\\ \boldsymbol{T}_{0m}\otimes\boldsymbol{I} & \boldsymbol{T}_{00}\otimes\boldsymbol{I}\end{bmatrix}t\right\}\mathrm{d}t\begin{bmatrix}\boldsymbol{I}\\ \boldsymbol{0}\end{bmatrix},$$

is given by

$$\int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x(\boldsymbol{C}_m^{-1}\otimes\boldsymbol{I}). \tag{A.102}$$

For $m \in \{+,-\}$ the top-right quadrant of size $m_1\times m_2 = |\mathcal{S}_m|\cdot p\times|\mathcal{S}_0|\cdot p$ of $e^{\boldsymbol{B}_{mm}t}$,

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\int_{t=0}^{\infty} e^{-\lambda t}\exp\left\{\begin{bmatrix}\boldsymbol{T}_{mm}\otimes\boldsymbol{I} + \boldsymbol{C}_m\otimes\boldsymbol{S} & \boldsymbol{T}_{m0}\otimes\boldsymbol{I}\\ \boldsymbol{T}_{0m}\otimes\boldsymbol{I} & \boldsymbol{T}_{00}\otimes\boldsymbol{I}\end{bmatrix}t\right\}\mathrm{d}t\begin{bmatrix}\boldsymbol{0}\\ \boldsymbol{I}\end{bmatrix},$$

is given by

$$\int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x((\boldsymbol{C}_m^{-1}\boldsymbol{T}_{m0}(\lambda\boldsymbol{I} - \boldsymbol{T}_{00})^{-1})\otimes\boldsymbol{I}). \tag{A.103}$$

Also,

$$\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\int_{t=0}^{\infty} e^{-\lambda t}e^{\boldsymbol{B}_{mm}t}\,\mathrm{d}t\boldsymbol{B}_{mn} = \int_{x=0}^{\infty}\left(\boldsymbol{H}^{mn}(\lambda,x)\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0}_{n\times|\mathcal{S}_0|}\end{bmatrix}\right)\otimes e^{\boldsymbol{S}x}\boldsymbol{D}\,\mathrm{d}x, \tag{A.104}$$

for $m,n \in \{+,-\}$, $m \neq n$.

Proof. From Lemma A.3.2 the top-left quadrant of size $m_1 \times m_1 = |\mathcal{S}_m| \cdot p \times |\mathcal{S}_m| \cdot p$ of the integral with respect to t on the left-hand side of (A.102) is

$$\int_{t=0}^{\infty} e^{\left(\boldsymbol{T}_{mm}\otimes\boldsymbol{I}+\boldsymbol{C}_m\otimes\boldsymbol{S}-\lambda\boldsymbol{I}+(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{0m}\otimes\boldsymbol{I})\right)t}\,\mathrm{d}t. \tag{A.105}$$

By Lemma A.3.1, (A.105) is equal to

$$\begin{aligned}&\int_{x=0}^{\infty} e^{\boldsymbol{C}_m^{-1}\left(\boldsymbol{T}_{mm}-\lambda\boldsymbol{I}+\boldsymbol{T}_{m0}(\lambda\boldsymbol{I}-\boldsymbol{T}_{00})^{-1}\boldsymbol{T}_{0m}\right)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x(\boldsymbol{C}_m\otimes\boldsymbol{I})^{-1}\\ &=\int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x(\boldsymbol{C}_m\otimes\boldsymbol{I})^{-1},\end{aligned} \tag{A.106}$$

from the definition of $\boldsymbol{Q}_{mm}(\lambda)$. This proves (A.102).

Now, from Lemma A.3.2 the top-right quadrant of size $m_1 \times m_2 = |\mathcal{S}_m| \cdot p \times |\mathcal{S}_0| \cdot p$ of the integral with respect to t on the left-hand side of (A.103) is

$$\begin{aligned}&\int_{t=0}^{\infty} e^{\left(\boldsymbol{T}_{mm}\otimes\boldsymbol{I}+\boldsymbol{C}_m\otimes\boldsymbol{S}-\lambda\boldsymbol{I}+(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{0m}\otimes\boldsymbol{I})\right)t}(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}\\ &\qquad\times(\boldsymbol{T}_{0m}\otimes\boldsymbol{I})\,\mathrm{d}t.\end{aligned} \tag{A.107}$$

By Lemma A.3.1, (A.107) is equal to

$$\int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x(\boldsymbol{C}_m\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{0m}\otimes\boldsymbol{I}). \tag{A.108}$$

Now,

$$\begin{aligned}(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}&=\int_{u=0}^{\infty} e^{-(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})u}\,\mathrm{d}u\\ &=\int_{u=0}^{\infty} e^{-\lambda u}e^{(\boldsymbol{T}_{00}\otimes\boldsymbol{I})u}\,\mathrm{d}u\\ &=\int_{u=0}^{\infty} e^{-\lambda u}e^{\boldsymbol{T}_{00}u}\otimes\boldsymbol{I}\,\mathrm{d}u,\end{aligned}$$

by (A.89). Using this and the Mixed Product Rule we can write

$$\begin{aligned}&(\boldsymbol{C}_m\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}\otimes\boldsymbol{I})^{-1}(\boldsymbol{T}_{0m}\otimes\boldsymbol{I})\\ &=(\boldsymbol{C}_m^{-1}\otimes\boldsymbol{I})(\boldsymbol{T}_{m0}\otimes\boldsymbol{I})\int_{u=0}^{\infty} e^{-\lambda u}e^{\boldsymbol{T}_{00}u}\otimes\boldsymbol{I}\,\mathrm{d}u(\boldsymbol{T}_{0m}\otimes\boldsymbol{I})\\ &=\left(\boldsymbol{C}_m^{-1}\boldsymbol{T}_{m0}(\lambda\boldsymbol{I}-\boldsymbol{T}_{00}u)^{-1}\boldsymbol{T}_{0m}\right)\otimes\boldsymbol{I}).\end{aligned} \tag{A.109}$$

Substituting (A.109) into (A.108) completes the proof of (A.103).

Now, using (A.102) and (A.103) we can write

$$\begin{aligned}
&\begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\int_{t=0}^{\infty} e^{-\lambda t}e^{\boldsymbol{B}_{mm}t}\,\mathrm{d}t\boldsymbol{B}_{mn} = \begin{bmatrix}\boldsymbol{I} & \boldsymbol{0}\end{bmatrix}\int_{t=0}^{\infty} e^{-\lambda t}e^{\boldsymbol{B}_{mm}t}\,\mathrm{d}t\begin{bmatrix}\boldsymbol{T}_{mn}\otimes\boldsymbol{D} & \boldsymbol{0}\\ \boldsymbol{T}_{0n}\otimes\boldsymbol{D} & \boldsymbol{0}\end{bmatrix}\\
&= \int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x(\boldsymbol{C}_m^{-1}(\boldsymbol{T}_{mn}+\boldsymbol{T}_{m0}(\lambda\boldsymbol{I}-\boldsymbol{T}_{00})^{-1}\boldsymbol{T}_{0n})\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0}_{n\times|\mathcal{S}_0|}\end{bmatrix}\otimes\boldsymbol{D})\\
&= \int_{x=0}^{\infty} e^{\boldsymbol{Q}_{mm}(\lambda)x}\otimes e^{\boldsymbol{S}x}\,\mathrm{d}x\left(\left(\boldsymbol{Q}_{mn}(\lambda)\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0}_{n\times|\mathcal{S}_0|}\end{bmatrix}\right)\otimes\boldsymbol{D}\right)\\
&= \int_{x=0}^{\infty}\left(\boldsymbol{H}^{mn}(\lambda,x)\begin{bmatrix}\boldsymbol{I}_n & \boldsymbol{0}_{n\times|\mathcal{S}_0|}\end{bmatrix}\right)\otimes e^{\boldsymbol{S}x}\boldsymbol{D}\,\mathrm{d}x, \qquad \text{(A.110)}
\end{aligned}$$

for $m,n\in\{+,-\}$, $m\neq n$ which is (A.104). □

Appendix B

second appendix

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