Summary of thesis

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The stochastic fluid-fluid model (SFFM) is a Markov process $\{(X_t, Y_t, \varphi_t)\}_{t\geq 0}$, where $\{\varphi_t\}_{t\geq 0}$ is a continuous-time Markov chain, the first fluid, $\{X_t\}_{t\geq 0}$, is a classical stochastic fluid process driven by $\{\varphi_t\}_{t\geq 0}$, and the second fluid, $\{Y_t\}_{t\geq 0}$, is driven by the pair $\{(X_t, \varphi_t)\}_{t\geq 0}$. Operator-analytic expressions for the stationary distribution of the SFFM, in terms of the infinitesimal generator of the process $\{(X_t, \varphi_t)\}_{t\geq 0}$, are known [3]. However, these operator-analytic expressions do not lend themselves to direct computation.

In this thesis we investigate approximations to the infinitesimal generator of the process $\{(X_t, \varphi_t)\}_{t\geq 0}$ so that one may approximate the fluid-fluid performance measures derived in [3]. As a first step we first introduce modified versions of the operators derived in [3] which are partitioned so that we can directly correspond elements of the true and approximated operators.

Perhaps the simplest approximation is to approximate the continuous fluid process by a discrete-time Markov chain. This was considered in [2] and results in a quasi-birth-and-death process (QBD). Another approach is to use the finite-element method – in Chapter 2 we apply the discontinuous Galerkin (DG) scheme. It turns out that the simplest DG method is equivalent to the Markov chain approximation of [2]. In some circumstances the DG method can result in approximations to the cumulative distribution function (CDF) which oscillate and are non-monotonic. This is clearly undesirable as we know that CDFs must be monotonic and non-decreasing. Due to its interpretation as a stochastic process, the Markov chain approximation of [2] ensures that the resulting approximations to the CDF are monotonic and non-decreasing. However, the rate of convergence of the Markov chain approximation of [2] is, in general, much slower than the DG scheme.

Motivated by this, the thesis proceeds to derive a new approximation to a fluid queue. The approximation is inspired by the observation that the Markov chain approximation of [2] effectively uses Erlang distributions to model the sojourn time in a given interval on the event that the phase of the fluid is constant. The sojourn time in a given interval on the event that the phase of the fluid is constant is a deterministic event, and it is known that the Erlang distribution is the least-variable Phase-type distribution so, in this sense, the best approximation to this deterministic sojourn time. Thus, it appears that the approximation of [2] is the best-possible Markov chain approximation. Recently, there has been much work on a class of concentrated matrix exponential distributions [4] which are postulated to be the least-variable matrix exponential distribution. Matrix exponential distributions have the same functional form as Phase-type distributions without the restriction that the distribution has an interpretation in terms of the absorption time of a continuous-time Markov chain. A class of stochastic processes, known as quasi-birth-and-death-processes with rational-arrival-process components (QBD-RAPs) [1], extend QBDs, which have Phase-type inter-event times, to allow matrix-exponentially distributed inter-event times. Thus, by using matrix exponential distributions, Chapter 3 attempts to construct a QBD-RAP which better captures the dynamics of the fluid queue than the QBD approximation in [2]. As the QBD-RAP has a stochastic interpretation then the approximations of to CDFs it produces are guaranteed to be monotonic and non-decreasing.

The thesis then moves on to mathematical analysis of the QBD-RAP process. Chapter 4 proves a convergence of the QBD-RAP process to the fluid queue up to the time that the fluid queue leaves a given interval. Chapter 5 then uses the results of Chapter 4 to prove a global convergence results of the QBD-RAP approximation scheme to the fluid queue. Chapter 4 relies more on matrix-exponential specific arguments, while Chapter 5 uses more traditional Markov process arguments, such as partitioning on the position of the process at convenient stopping times, then exploiting the strong Markov property and time homogeneity.

Chapter 6 numerically investigates the performance of the DG, QBD-RAP and Markov chain approximations. The numerics demonstrate that, if the solution is smooth enough, then the discontinuous Galerkin method is highly-effective and displays rapid convergence to the solution as the number of basis functions is increased. However, when the solution is sufficiently non-smooth, oscillations and negative approximations may occur when using the discontinuous Galerkin method, leading to nonsense solutions. In these cases, the QBD-RAP approximation performs relatively well compared to the other positivity preserving schemes investigated.

References

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