Summary of thesis

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The stochastic fluid-fluid model (SFFM) is a Markov process $\{(X_t, Y_t, \varphi_t)\}_{t\geq 0}$, where $\{\varphi_t\}_{t\geq 0}$ is a continuous-time Markov chain, the first fluid, $\{X_t\}_{t\geq 0}$, is a classical stochastic fluid process driven by $\{\varphi_t\}_{t\geq 0}$, and the second fluid, $\{Y_t\}_{t\geq 0}$, is driven by the pair $\{(X_t, \varphi_t)\}_{t\geq 0}$. Operator-analytic expressions for performance measures of fluid-fluid queues, such as the stationary distribution and first return distribution of the SFFM, have been derived [3]. These performance measures are in terms of the infinitesimal generator of the process $\{(X_t, \varphi_t)\}_{t\geq 0}$, which is a differential operator. Thus, the operator-analytic expressions do not lend themselves to direct computation.

In this thesis we investigate approximations to the infinitesimal generator of the process $\{(X_t, \varphi_t)\}_{t\geq 0}$ so that one may approximate the performance measures derived in [3]. As a first step we first introduce modified versions of the operators derived in [3] which are partitioned so that we can directly correspond elements of the true and approximated operators.

Perhaps the simplest approximation is to approximate the continuous fluid process $\{(X_t, \varphi_t)\}_t$ by a discrete-time Markov chain. This was considered in [2] where they use a quasi-birth-and-death process (QBD) to approximate $\{(X_t, \varphi_t)\}_t$. Another approach is to use a finite-element method to approximate the infinitesimal generator of $\{(X_t, \varphi_t)\}_t$. In Chapter 2 we show how we can use the discontinuous Galerkin (DG) method to derive approximations to the infinitesimal generator of $\{(X_t, \varphi_t)\}_t$, and then use these to approximate the operator-analytic expressions from [3]. When discontinuities or steep gradients are present in the distribution of $\{(X_t, \varphi_t)\}_t$, the DG method can result in approximations to the cumulative distribution function (CDF) which oscillate and, in the worst cases, are non-monotonic and can take values outside the interval $[0, 1]$. This is clearly undesirable. In contrast, the QBD approximation of [2] does not suffer the same problems due to its interpretation as a stochastic process, which ensures that the resulting approximations to the CDF are well-behaved. However, the rate of convergence of the QBD approximation is, in general, much slower than the DG approximation.

Motivated by this, the thesis proceeds to derive a new approximation to a fluid queue. The approximation is inspired by the observation that the QBD approximation of [2] effectively uses an Erlang distribution to model the sojourn time of $\{(X_t, \varphi_t)\}_t$ in a given interval on the event that the phase of the fluid is constant. The aforementioned sojourn time is a deterministic event, and it is known that the Erlang distribution is the least-variable Phase-type distribution so, in this sense, the Erlang distribution is the best approximation to the distribution of this deterministic sojourn time. Thus, we argue that the approximation of [2] is the best-possible Markov chain approximation to the fluid queue $\{(X_t, \varphi_t)\}_t$. In light of this, we look to a broader class of models known as quasi-birth-and-death-processes with rational-arrival-process components (QBD-RAPs) [1] to construct our approximation. QBDs have Phase-type distributed inter-event times. QBD-RAPs extend QBDs to allow *matrix-exponentially* distributed inter-event times. Matrix exponential distributions have the same functional form as Phase-type distributions without the restriction that the distribution has an interpretation in terms of the absorption time of a continuous-time Markov chain. Recently, there has been much work on a class of *concentrated matrix exponential distributions* [4] which are postulated to be the least-variable matrix exponential distribution. Thus, by using concentrated matrix exponential distributions, Chapter 3 construct a QBD-RAP which better captures the dynamics of the fluid queue than the QBD approximation in [2]. As the QBD-RAP has a stochastic interpretation then the approximations of CDFs it produces are guaranteed to be monotonic, non-decreasing and take values in $[0, 1]$.

The thesis then moves on to mathematical analysis of the constructed QBD-RAP process. Chapter 4 proves a convergence of the QBD-RAP process to the fluid queue up to the time that the fluid queue leaves a given interval. Chapter 5 then uses the results of Chapter 4 to prove a global convergence results of the QBD-RAP approximation scheme to the fluid queue. Chapter 4 uses more matrix-exponential-specific arguments to show its convergence result, while Chapter 5 uses more traditional Markov process arguments, such as partitioning on the position of the process at convenient stopping times, then exploiting the strong Markov property and time homogeneity.

Chapter 6 numerically investigates the performance of the DG, QBD-RAP and QBD approximations. The numerics demonstrate that, if the solution is smooth enough, then the discontinuous Galerkin method is highly-effective and displays rapid convergence to the solution as the number of basis functions is increased. However, when the solution is sufficiently non-smooth, oscillations and negative approximations may occur when using the discontinuous Galerkin method, leading to nonsense solutions. In these cases, the QBD-RAP approximation performs relatively well compared to the other positivity preserving schemes investigated.

References

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[2] Bean, N. G. and O’Reilly, M. M. (2013). ‘Spatially-coherent uniformisation of a stochastic fluid model to a quasi-birth-and-death-process.’ *Performance Evaluation*, 70(9):578 – 592.

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