Approximations to generators of Stochastic Fluid Models

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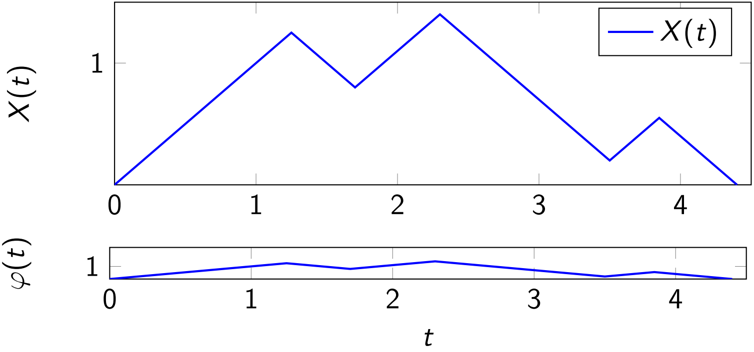
Stochastic Fluid Models

- Two-dimensional Markov process

$$\{\varphi(t), X(t)\}_{t\geq 0}.$$

- $\{\varphi(t)\}_{t\geq 0}$ a CTMC with generator T, on finite state space $\mathcal{S}$.
- $X(t) = \int\limits_{s=0}^{t} c_{\varphi(s)} ds$

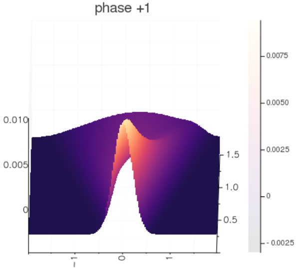


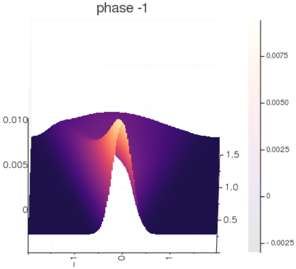
The PDE

$$\frac{\partial}{\partial t} f_i(x,t) = \sum_{j \in \mathcal{S}} f_j(x,t) T_{ji} - c_i \frac{\partial}{\partial x} f_i(x,t)$$

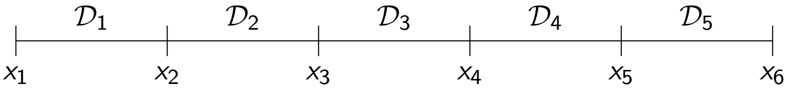
where

$$f_i(x,t)dx = P(\varphi(t) = i, X(t) \in dx)$$





The Galerkin method – The Stencil and Meshes



We also have (for example)

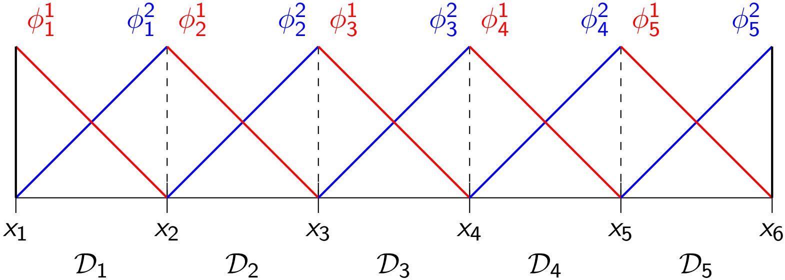
$$\phi_k^1, \phi_k^2$$

compactly supported basis functions on the k-th mesh. On each $\mathcal{D}_k$ we approximate

$$f_i(x,t) \approx \sum_{\ell=1}^{N} a_{i,k}^{\ell}(t)\phi_k^{\ell}(x).$$

The Galerkin method – The Mesh

Consider a simple mesh



Basis functions:

$$\phi_k^1(x) = \frac{x_{k+1} - x}{\Delta_k}, \qquad \phi_k^2(x) = \frac{x - x_k}{\Delta_k}.$$

The Galerkin method – The weak form

Equivalent to the PDE with right choice of basis functions.
For each basis function on each mesh $k = 1, 2, ..., K$,

$$\frac{\partial}{\partial t} \int_{\mathcal{D}_k} f_i(x,t)\phi_k^m(x)dx$$
$$= \sum_{j \in \mathcal{S}} \int_{\mathcal{D}_k} f_j(x,t) T_{ji} \phi_k^m(x)dx - c_i \int_{\mathcal{D}_k} \left[\frac{\partial}{\partial x} f_i(x,t) \right] \phi_k^m(x)dx.$$

Integrate by parts to shift derivative, create *flux* term.

$$= \sum_{j \in \mathcal{S}} \int_{\mathcal{D}_k} f_j(x,t) T_{ji} \phi_k^m(x)dx + c_i \int_{\mathcal{D}_k} f_i(x,t) \frac{\partial}{\partial x} \phi_k^m(x)dx$$
$$- c_i \left[f_i(x,t)\phi_k^m(x) \right]_{x_k}^{x_{k+1}}.$$

The Galerkin method – Approximation

Recall: on each $\mathcal{D}_k$, approximate

$$f_i(x,t) \approx \sum_{\ell=1}^{N} a_{i,k}^{\ell}(t)\phi_k^{\ell}(x).$$

On substitution into PDE

$$\frac{\partial}{\partial t}\sum_{\ell=1}^{N} a_{i,k}^{\ell}(t)\overbrace{\int_{\mathcal{D}_k}\phi_k^{\ell}(x)\phi_k^{m}(x)dx}^{\left[M_k\right]_{\ell m}}$$

$$= \sum_{j\in\mathcal{S}} T_{ji}\sum_{\ell=1}^{N} a_{j,k}^{\ell}(t)\int_{\mathcal{D}_k}\phi_k^{\ell}(x)\phi_k^{m}(x)dx$$

$$+ c_i\sum_{\ell=1}^{N} a_{i,k}^{\ell}(t)\underbrace{\int_{\mathcal{D}_k}\phi_k^{\ell}(x)\frac{\partial}{\partial x}\phi_k^{m}(x)dx}_{\left[G_k\right]_{\ell m}} - c_i\left[f_i(x,t)\phi_k^{m}(x)\right]_{x_k}^{x_{k+1}}.$$

The Galerkin method – Matrix form

Let

$$\boldsymbol{a}_{i,k}(t) = (a^1_{i,k}(t), \ldots, a^N_{i,k}(t))$$

then

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{a}_{i,k}(t) M_k = \sum_{j\in\mathcal{S}} \boldsymbol{a}_{j,k}(t) M_k T_{ji} + c_i \boldsymbol{a}_{i,k}(t) G_k \\ - c_i \left[f_i(x,t)\phi^m_k(x)\right]^{x_{k+1}}_{x_k}.\end{aligned}$$

$$[M_k]_{\ell m} = \int_{D_k} \phi^\ell_k(x)\phi^k_\ell(x)dx, \quad [G_k]_{\ell m} = \int_{D_k} \phi^\ell_k(x)\frac{\partial}{\partial x}\phi^k_\ell(x)dx$$

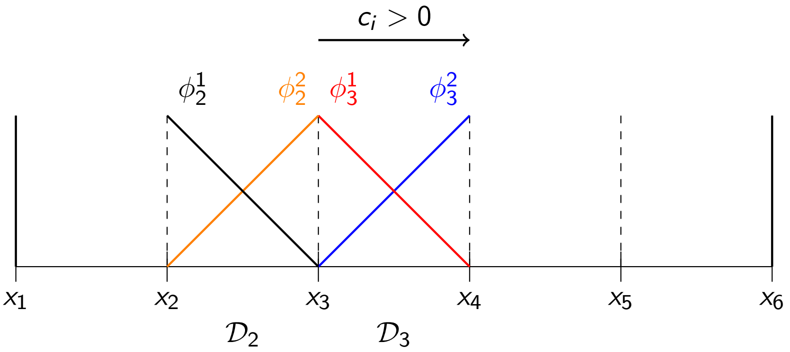
The Galerkin method – The Flux

The Flux is $-c_i\left[f_i(x,t)\phi_k^m(x)\right]_{x_k}^{x_{k+1}}$. Flow for $\mathcal{D}_3$ is

$$-c_i\left[f_i(x,t)\phi_3^1(x)\right]_{x_3}^{x_4} \approx -c_i\left[0 - a_{i,2}^2(t)\right],$$

$$-c_i\left[f_i(x,t)\phi_3^2(x)\right]_{x_3}^{x_4} \approx -c_i\left[a_{i,3}^2(t) - 0\right]$$

links intervals together (boundary condition).



The Galerkin method – Matrix form

$$M = diag(M_k), \quad G = diag(G_k), \quad F_i = flux$$

$$\boldsymbol{a}_i(t) = (\boldsymbol{a}_{i,1}(t), \ldots, \boldsymbol{a}_{i,k}(t), \ldots, \boldsymbol{a}_{i,K}(t))$$

$$\frac{\partial}{\partial t}\boldsymbol{a}_i(t) = \sum_{j \in \mathcal{S}} \boldsymbol{a}_j(t) T_{ji} + \boldsymbol{a}_i(t) c_i (G - F_i) M^{-1}.$$

With $\boldsymbol{a}(t) = (\boldsymbol{a}_1(t), \ldots, \boldsymbol{a}_p(t))$ the solution is

$$\boldsymbol{a}(t) = \boldsymbol{a}(0) \exp((T \otimes I + diag(c_i(G - F_i)M^{-1}))t).$$

Density to probability

We are actually interested in

$$
\begin{aligned}
&\mathbb{P}(X(t) \in \mathcal{D}_k, \varphi(t) = i) \\
&= \int_{\mathcal{D}_k} f_i(x,t)dx \\
&\approx \sum_{\ell} a^{\ell}_{i,k}(t) \int_{\mathcal{D}_k} \phi^{\ell}_k(x)dx \\
&= \boldsymbol{a}_{i,k}(t) \underbrace{\begin{bmatrix} \int_{\mathcal{D}_k} \phi^1_k(x)dx & & \\ & \ddots & \\ & & \int_{\mathcal{D}_k} \phi^N_k(x)dx \end{bmatrix}}_{D_k} \mathbf{1} \\
&=: \boldsymbol{\alpha}_{i,k}(t)\mathbf{1}.
\end{aligned}
$$

$\alpha^{\ell}_{i,k}(t)$ is the probability *mass* captured by basis function $\phi^{\ell}_k(x)$.

Density to probability

Also define $D = diag(D_k)$. Since

$$\boldsymbol{\alpha}_i(t)D^{-1} = \boldsymbol{a}_i(t),$$

then

$$\frac{\partial}{\partial t}\boldsymbol{a}(t) = \boldsymbol{a}(t)T \otimes I + \boldsymbol{a}(t)diag(c_i(G - F_i)M^{-1}),$$

is equivalent to

$$\frac{\partial}{\partial t}\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(t)T \otimes I + \boldsymbol{\alpha}(t)diag(c_i(G - D^{-1}F_iD)M^{-1}),$$

since M, G, $T \otimes I$ commute with D.

Approximation to the generator

Solution is

$$\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(0)\exp\Big(\overbrace{\big[T \otimes I + diag(c_i(G - D^{-1}F_iD)M^{-1})\big]}^{\text{Generator, } B}\, t\Big),$$

and

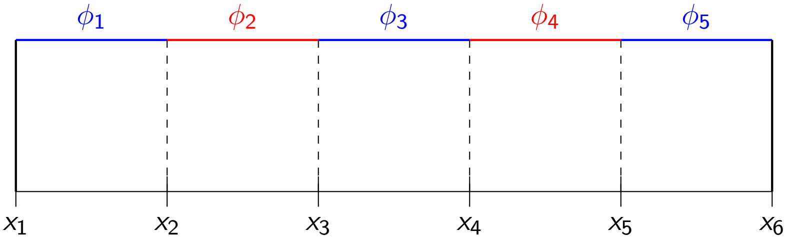
$$\big[T \otimes I + diag(c_i(G - D^{-1}F_iD)M^{-1})\big]$$

is the approximation to the generator of the SFM.

- $T \otimes I$ stochastic jumps of $\varphi(t)$.
- G transfer of density *within* mesh.
- F_i transfer of density *between* meshes.
- M and D account for mesh/basis function 'size'.

Example

Consider the simplest mesh,



with $x_{k+1} - x_k = \Delta$ and basis functions:

$$\phi_k(x) = 1.$$

Example – Order 0

Re-order elements of $\boldsymbol{\alpha}(t)$ according to

1. phase, $\varphi(t)$, then
2. basis function, then
3. mesh.

$\widehat{\boldsymbol{\alpha}}(t) = (\alpha^1_{1,1}(t), ..., \alpha^1_{p,1}(t), \alpha^2_{1,1}(t), ..., \alpha^2_{p,1}(t), ..., \alpha^n_{p,K}(t), ..., \alpha^n_{p,K}(t))$.

Example – Order 0

Solution

$$\widehat{\boldsymbol{\alpha}}(t) = \widehat{\boldsymbol{\alpha}}(0)\exp(Bt)$$

$$B = \begin{bmatrix} T - C_+ & C_+ & & & & \\ C_- & T - C & C_+ & & & \\ & & \ddots & & & \\ & & & C_- & T - C & C_+ \\ & & & & C_- & T - C_- \end{bmatrix}$$

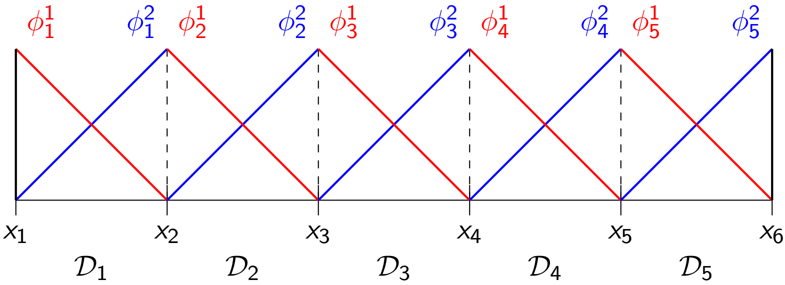
where

$$C_- = diag(|c_i|1(c_i < 0)/\Delta), \; C_+ = diag(|c_i|1(c_i > 0)/\Delta).$$

$$C = C_- + C_+.$$

- The QBD approximation of Bean & O'Reilly (2013).
- Conserves positivity, total probability.

Example – Order 1



Basis functions:

$$\phi_k^1(x) = \frac{x_{k+1} - x}{\Delta_k}, \qquad \phi_k^2(x) = \frac{x - x_k}{\Delta_k}.$$

Example – Order 1

$$\mathcal{S} = \{1, 2\}, \quad c_1 > 0, \quad c_2 < 0.$$

$$B = I \otimes T$$

$$+\frac{1}{\Delta}\left[\begin{array}{c|c|c|c|c|c|c}
\begin{matrix} B_1^+ & \\ & B_1^- \end{matrix} & \begin{matrix} A_2^+ & \\ & \end{matrix} & & & & & \\ \hline
\begin{matrix} & \\ & A_0^- \end{matrix} & \begin{matrix} A_1^+ & \\ & A_1^- \end{matrix} & \begin{matrix} A_2^+ & \\ & \end{matrix} & & & & \\ \hline
 & & & \ddots & & & \\ \hline
 & & & & \begin{matrix} & \\ & A_0^- \end{matrix} & \begin{matrix} A_1^+ & \\ & A_1^- \end{matrix} & \begin{matrix} A_2^+ & \\ & \end{matrix} \\ \hline
 & & & & & \begin{matrix} & \\ & A_0^- \end{matrix} & \begin{matrix} D_1^+ & \\ & D_1^- \end{matrix}
\end{array}\right]$$

$$B_1^+ = \begin{bmatrix} -3c_1 & 3c_1 \\ -1c_1 & -1c_1 \end{bmatrix}, \quad A_2^+ = \begin{bmatrix} 0 & 0 \\ 4c_1 & -2c_1 \end{bmatrix}, \quad D_1^+ = \begin{bmatrix} -3c_1 & 3c_1 \\ -3c_1 & 3c_1 \end{bmatrix},$$

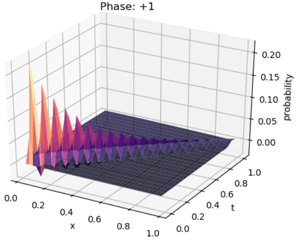
$$B_1^- = \begin{bmatrix} 3|c_2| & -3|c_2| \\ 3|c_2| & -3|c_2| \end{bmatrix}, \quad A_0^- = \begin{bmatrix} -2|c_2| & 4|c_2| \\ 0 & 0 \end{bmatrix}, \quad B_1^- = \begin{bmatrix} -1|c_2| & -1|c_2| \\ 3|c_2| & -3|c_2| \end{bmatrix}$$

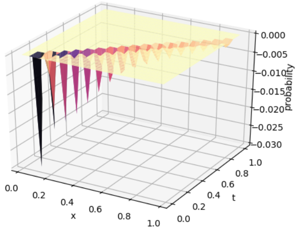
- Conserves *total* probability.
- Does not necessarily conserve positivity.

Example – Order 1

$$\mathcal{S} = \{1, 2\}, \qquad \Delta = 0.025 \text{ (40 intervals)}, \qquad c_1 = 1,$$
$$T = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}, \qquad \text{Interval: } [0, 1], \qquad c_2 = -1.$$

Initial distribution : point mass at 0 in phase 1.

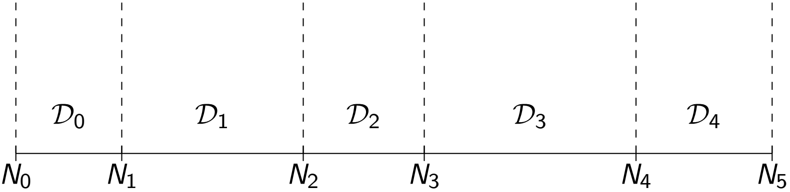




Future work

- Use ideas from probability to build approximations with probabilistic properties.
- i.e. stochastic generator matrices.
- probability
- functional analysis

Update!



$$\{N_k\} \sim \text{ Poisson process} \left(\frac{1}{\Delta}\right).$$

Update!

The generator of the probabilities $P(\varphi(t) = i, X(t) \in \mathcal{D}_k)$ is

$$B = \begin{bmatrix} T - C_+ & C_+ & & \\ C_- & T - C & C_+ & \\ & & \ddots & \end{bmatrix}$$

where

$$C_- = diag(|c_i| 1(c_i < 0)/\Delta), \; C_+ = diag(|c_i| 1(c_i > 0)/\Delta).$$

$$C = C_- + C_+.$$

- The QBD approximation of Bean & O'Reilly (2013), again!
- Can we generalise to MAP arrivals?