An algorithm for inference of a class of Markovian Regime-Switching Models

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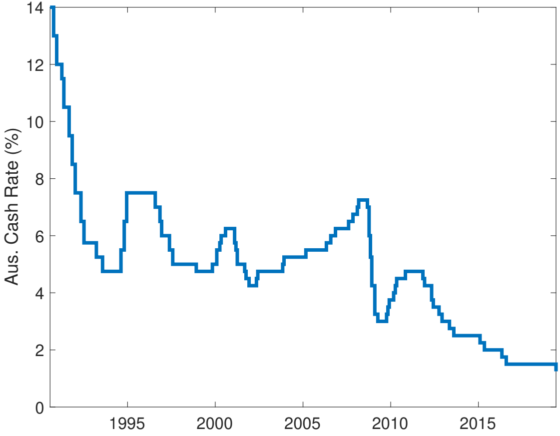
Markovian-Regime-Switching (MRS) models



- Hamilton, 1988 *Rational expectations econometric analysis of changes in regimes: An investigation of the term structure of interest rates*

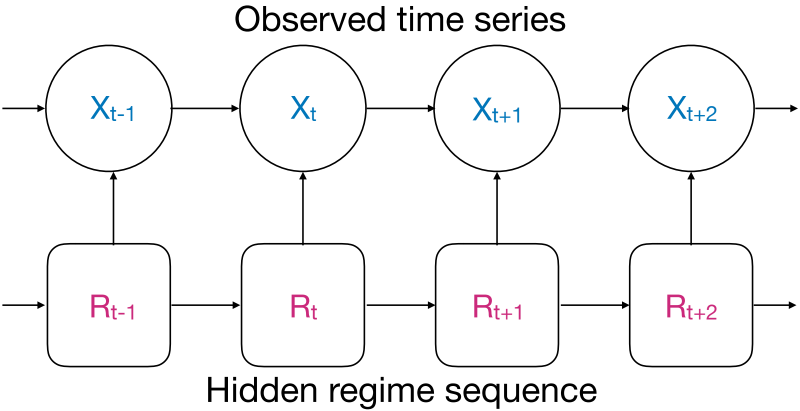
Markovian-Regime-Switching (MRS) models

Typically specified as

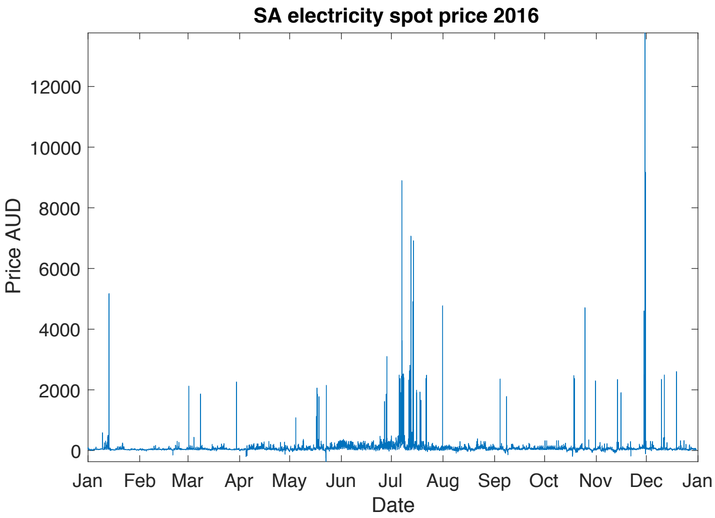
$$X_t = \alpha^{(i)} + \phi_1^{(i)} X_{t-1} + \cdots + \phi_p^{(i)} X_{t-p} + \sigma^{(i)} \varepsilon_t \text{ when } R_t = i$$

where $\{R_t\}_{t \in \mathbb{N}}$ is a Markov chain.

Dependent regime MRS models



A new model – motivation



Independent regime MRS models

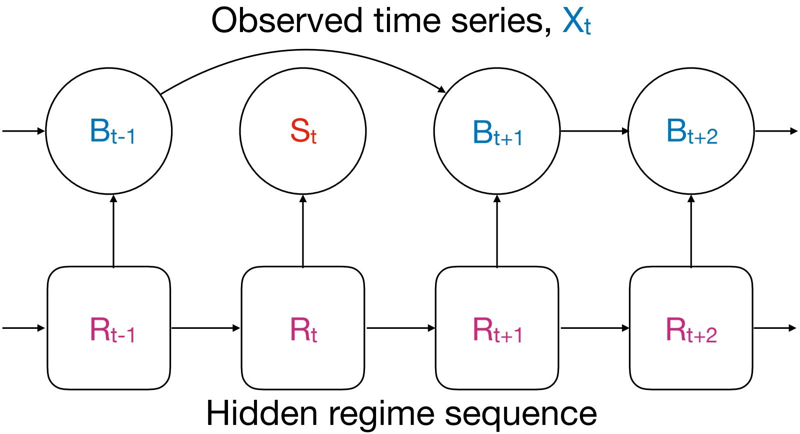
$$X_t = \begin{cases} B_t^1, & R_t = 1 \\ S_t^2, & R_t = 2 \end{cases}$$

where

$$\begin{aligned} B_t^1 &= \phi_1 B_{t-1}^1 + \sigma_1 \varepsilon_t^1, \quad \text{is AR(1)}, \\ \varepsilon_t^1 &\sim N(0,1) \\ S_t^2 &\sim \text{i.i.d. Log-Normal}(\mu_2, \sigma_2^2). \end{aligned}$$

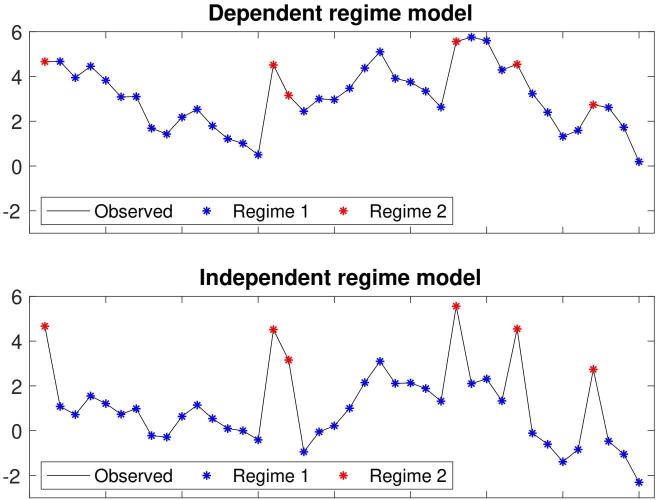
The sets $A_i := \{X_t \mid t \geq 0, R_t = i\}$, $i = 1, 2$, are independent.

Independent regime MRS models

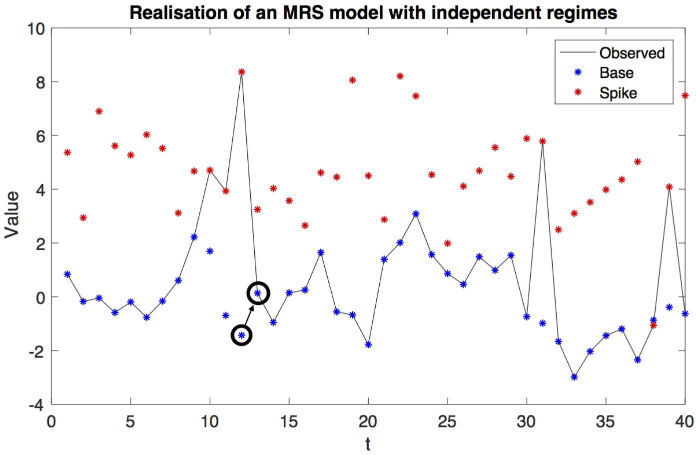


MRS models of electricity prices

- *Independent regime* MRS models allow 'jumpy' behaviour

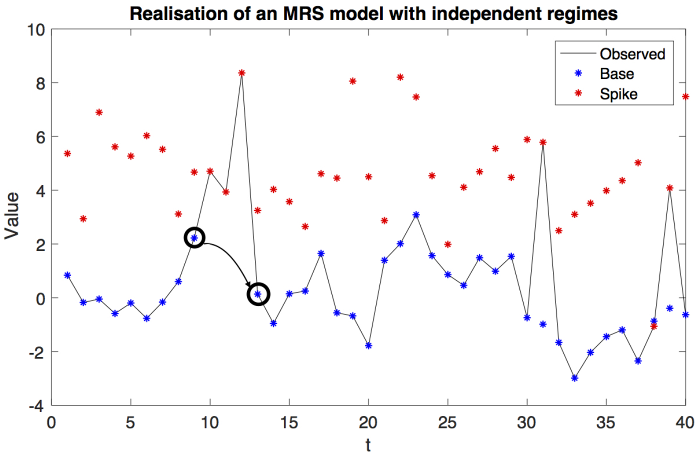


Independent regimes – Example



- B_t depends on B_{t-1}, but it might be unobserved

Independent regimes – Example



- Can integrate unobserved prices away
- B_t depends on a random lagged observation

Evaluating likelihoods

- Hamilton's work on dependent regime models relies on computing the densities

$$f(x_t \mid R_t = i, \boldsymbol{x}_{0:t-1}).$$

  where $\boldsymbol{x}_{0:t-1} = (x_0, ..., x_{t-1})$.
- For independent regime models we need to know about the history, $R_0, ..., R_{t-1}$.

$$f(x_t \mid R_0, ..., R_t, \boldsymbol{x}_{0:t-1})$$

  exponential complexity – too much history!

Problem: How to efficiently capture the history of $R_0, \dots, R_t$?

Idea: augment $\{R_t\}$ with time-since-last-visit counters

- Let $N_{t,i} = \ell$, for $i = 1, \dots, k$, denote the event that the last visit to state i before time t was ℓ transitions ago.
- Define an augmented process

$$\{\boldsymbol{H}_t\} = \{(\boldsymbol{N}_t, R_t)\} = \{(N_{t,1}, \dots, N_{t,k}, R_t)\}$$

  which is Markovian.
- Borrow ideas from hidden semi-Markov models
  - Counters for time since last transition

$\boldsymbol{H}_t = (\boldsymbol{N}_t, R_t)$

- $\{\boldsymbol{H}_t\}$ is a Markov chain.
- Transition probabilities

$$P_\theta(\boldsymbol{H}_{t+1} = (\boldsymbol{N}_{t+1}, j) \mid \boldsymbol{H}_t = (\boldsymbol{N}_t, i))$$

$$= \begin{cases} p_{ij} & \text{for } i \in \{k+1, ..., M\}, j \in \mathcal{S}, \boldsymbol{N}_{t+1} = \boldsymbol{N}_t + \boldsymbol{1}, \\ p_{ij} & \text{for } i \in \{1, ..., k\}, j \in \mathcal{S}, \boldsymbol{N}_{t+1} = \boldsymbol{N}_t^{(-i)} + \boldsymbol{1}, \\ 0 & \text{otherwise,} \end{cases}$$

where $\boldsymbol{N}_t^{(-i)} = (N_{t,1}, ..., N_{t,i-1}, 0, N_{t,i+1}, ..., N_{t,k})$.

- The possible values of $\boldsymbol{N}_t$ are ‘relatively few’.
- Number of accessible states of $\{\boldsymbol{N}_t\}$ by time t is

$$\sum_{m=0}^{\min(t,k)} \binom{t}{m} \binom{k}{m} m! \approx \mathcal{O}(t^k k^k).$$

New algorithms

We developed new algorithms for inference of independent regime MRS models.

1. Forward algorithm: evaluate likelihoods
2. Backward algorithm: infer hidden states
3. EM algorithm: find maximum likelihood estimates

Evaluating likelihoods

Independent regime models

Algorithm #1: A new forward algorithm[1]

For $t = 1, ..., T$

$$f_\theta(R_t, \boldsymbol{x}_{0:t}) = f_\theta(x_t \mid R_t, \boldsymbol{x}_{0:t-1}) f_\theta(R_t, \boldsymbol{x}_{0:t-1})$$

[1] A computationally stable (normalised) algorithm can also be derived

Evaluating likelihoods

Independent regime models

Algorithm #1: A new forward algorithm[1]

For $t = 1, ..., T$

$$f_\theta((\boldsymbol{N}_t, R_t), \boldsymbol{x}_{0:t}) = f_\theta(x_t \mid (\boldsymbol{N}_t, R_t), \boldsymbol{x}_{0:t-1}) f_\theta((\boldsymbol{N}_t, R_t), \boldsymbol{x}_{0:t-1})$$

where

$$f_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}_t, j), \boldsymbol{x}_{0:t-1})$$
$$= \begin{cases} \sum_{i \in \mathcal{S}} p_{ij} f_\theta((\boldsymbol{N}_{t-1}, R_{t-1}) = (\boldsymbol{n}_t - \boldsymbol{1}, i), \boldsymbol{x}_{0:t-1}), \\ \qquad \text{if the last regime did not have a counter} \\ \sum_{m=1}^{t} p_{ij} f_\theta((\boldsymbol{N}_{t-1}, R_{t-1}) = (\boldsymbol{n}_t - \boldsymbol{1} + m\boldsymbol{e}_i, i), \boldsymbol{x}_{0:t-1}), \\ \qquad \text{if the last regime had a counter} \end{cases}$$

[1] A computationally stable (normalised) algorithm can also be derived

Evaluating likelihoods

Independent regime models

A new forward algorithm – continued

$$L(\theta) = \sum_{j \in \mathcal{S}} \sum_{\boldsymbol{n}} f_\theta((\boldsymbol{N}_T, R_T) = (\boldsymbol{n}, j), \boldsymbol{x}_{0:T})$$

Forward algorithm – More than just likelihoods

$$\overbrace{P_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}_t, i) \mid \boldsymbol{x}_{0:t})}^{\text{filtered probabilities}} = \frac{f_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}_t, i), \boldsymbol{x}_{0:t})}{\sum_{j \in \mathcal{S}} \sum_{\boldsymbol{n}} f_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}, j), \boldsymbol{x}_{0:t})}$$

Similarly,

$$\overbrace{P_\theta((\boldsymbol{N}_{t+1}, R_{t+1}) = (\boldsymbol{n}, i) \mid \boldsymbol{x}_{0:t})}^{\text{prediction probabilities}}$$

Smoothed probabilities

Independent regime models

Algorithm #2: Backward algorithm

For $t = T-1, ..., 0$

$$\overbrace{P_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}_t, i) \mid \boldsymbol{x}_{0:T})}^{\text{smoothed probabilities}}$$
$$= \overbrace{P_\theta((\boldsymbol{N}_t, R_t) = (\boldsymbol{n}_t, i) \mid \boldsymbol{x}_{0:t})}^{\text{filtered probabilities}}$$
$$\times \sum_{j \in \mathcal{S}} p_{ij} \frac{P_\theta((\boldsymbol{N}_{t+1}, R_{t+1}) = (\boldsymbol{n}_{t+1}, j) \mid \boldsymbol{x}_{0:T})}{\underbrace{P_\theta((\boldsymbol{N}_{t+1}, R_{t+1}) = (\boldsymbol{n}_{t+1}, j) \mid \boldsymbol{x}_{\boldsymbol{0:t}})}_{\text{prediction probabilities}}}.$$

Where $\boldsymbol{n}_{t+1}$ is known when we have $\boldsymbol{n}_t$ and $R_t = i$.

Algorithm #3: EM algorithm

Iterate between E- and M-steps

E. Construct

$$
\begin{aligned}
Q(\theta, \theta_n) &= E_{\theta_n}\left[\log f_\theta(\boldsymbol{H}_0, \ldots, \boldsymbol{H}_T, \boldsymbol{x}_{0:T}) \mid \boldsymbol{x}_{0:T}\right] \\
&= E_{\theta_n}\left[\log f_\theta(\boldsymbol{x}_{0:T} \mid \boldsymbol{H}_0, \ldots, \boldsymbol{H}_T) \mid \boldsymbol{x}_{0:T}\right] \\
&\quad + E_{\theta_n}\left[\log f_\theta(\boldsymbol{H}_0, \ldots, \boldsymbol{H}_T) \mid \boldsymbol{x}_{0:T}\right]
\end{aligned}
$$

M. Set $\theta_{n+1} = \arg\max_\theta Q(\theta, \theta_n)$

Maximising likelihoods

HMMs and dependent regime models

Algorithm #3: EM algorithm

Iterate

E. Construct

$$Q(\theta,\theta_n) = \sum_{t=1}^{T}\sum_{i\in\mathcal{S}}\sum_{\boldsymbol{n}_t}\Big[\overbrace{P_{\theta_n}((\boldsymbol{N}_t,R_t)=(\boldsymbol{n}_t,i)\mid \boldsymbol{x}_{0:T})}^{\text{smoothed probabilities}}$$

$$\times \log f_\theta(x_t \mid (\boldsymbol{N}_t,R_t)=(\boldsymbol{n}_t,i),\boldsymbol{x}_{0:t-1})\Big]$$

$$+\sum_{i,j\in\mathcal{S}}\log(p_{ij})\sum_{t=1}^{T}P_{\theta_n}(R_t=j,R_{t-1}=i|\boldsymbol{x}_{0:T})$$

M. Set $\theta_{n+1} = \arg\max_\theta Q(\theta,\theta_n)$

Complexity

- Naive complexity $\mathcal{O}(M^T)$
- Complexity is $\mathcal{O}(M^2 T^{k+1} k^k)$
    - M – the number of regimes
    - T – the number of observations
    - k – the number of AR(1) processes/counters

- $\mathcal{O}(M^2 \color{red}{T^{k+1}} k^k)$ may be large
    - Truncate

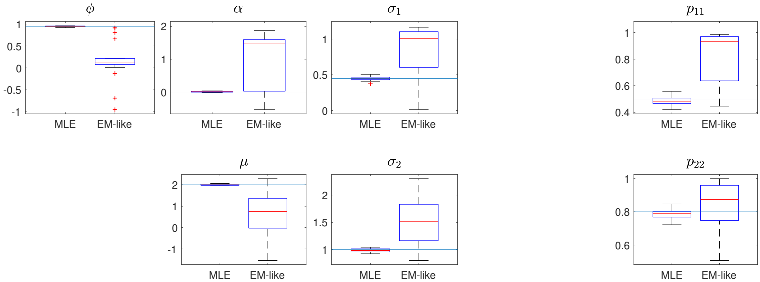
$$N_{t,i} \in \{1, 2, ..., D-1, D\}, \text{ where } D \ll T$$

  and enforce transitions $D \to D$
    - Complexity is now $\mathcal{O}(M^2 D^k T k^k)$
    - Naive complexity: $\mathcal{O}(M^D T)$

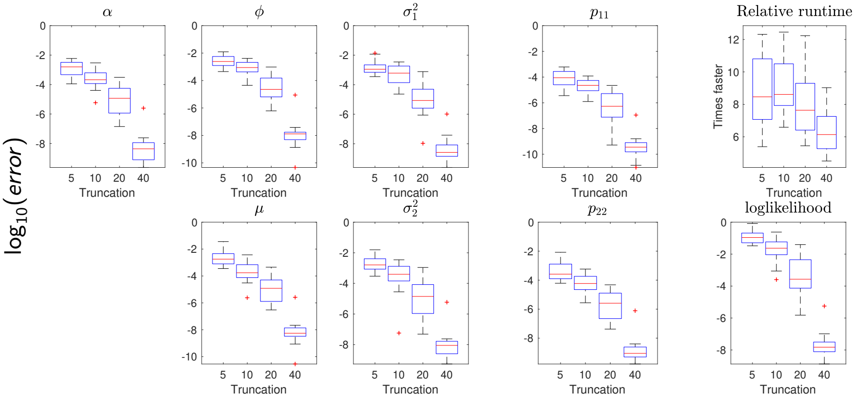
Comparison with state of the art (EM-like)

$$X_t = \begin{cases} B_t, & \text{if } R_t = 1, \\ S_t, & \text{if } R_t = 2, \end{cases}$$

- $B_t = 0.95B_{t-1} + 0.2\varepsilon_t$,
- $S_t \sim N(2, 1)$
- $p_{11} = 0.5,\ p_{22} = 0.8$



An example – Truncation, $D = 5, 10, 20, 40$



Summary

- Developed novel algorithms for independent regime MRS models
  - Forward, backward, EM
  - Efficient approximations to all of the above
  - Key idea: the augmented hidden chain $\{\boldsymbol{H}_t\}$
- These ideas can be extended to more general MRS processes
  - more general AR(p) processes
- *Estimation of Markovian-Regime-Switching models with independent regimes*, Lewis, Nguyen, Bean, Submitted, `https://arxiv.org/abs/1906.07957`
- MATLAB code: `https://github.com/angus-lewis/IRMRS`

An example – Consistency