Fundamental Bounds on Quantum State Tomography

Angus Lowe

In the beginning...

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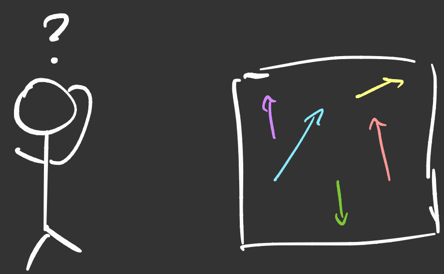
$$\nearrow, \uparrow, \dots \in V^{\otimes m}, \quad \dim(V^{\otimes m}) = \dim(V)^m$$



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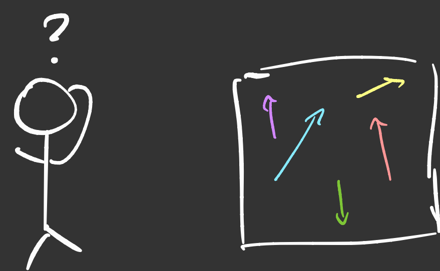

$\in (V^{\otimes m})^{\otimes n}$

Coin Flipping

n correlated coins

Coin Flipping

m correlated coins

$$\Big\} \; p = \nearrow \in V^{\otimes m}, \qquad V \to \mathbb{R}^2$$

$$= \left(p_{HHH\ldots H},\; p_{THH\ldots H},\; \ldots,\; p_{TTT\ldots T}\right)$$

$$\|p\|_1 = \sum_j |p_j| = 1.$$

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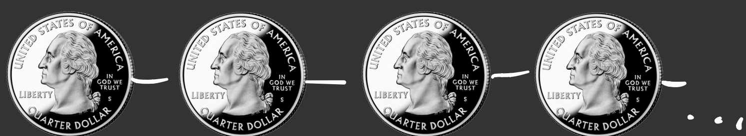

$$\Big\} \; q$$

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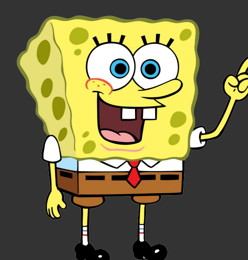


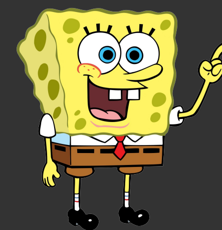
Prepare p w/prob. $1/2$

Prepare q w/prob. $1/2$

Guess p or q

Coin Flipping

$\boxed{?}$



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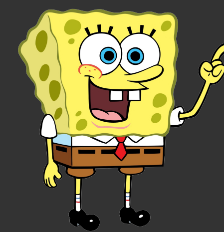

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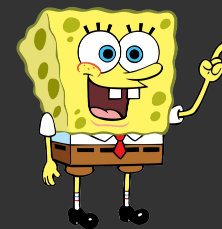
Fact (Bayes' rule): Optimal prob. of success is

$$p_{succ} = \frac{1}{2} + \frac{1}{4}\|p - q\|_1$$

Coin Flipping



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Fact (Bayes' rule): Optimal prob. of success is

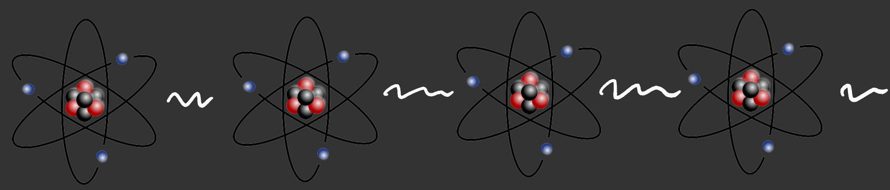
$$P_{succ} = \frac{1}{2} + \frac{1}{4}\|p - q\|_1$$

Fact: if $\frac{1}{2}\|p-q\|_1 \geq c$, then

$$\frac{1}{2}\|p^{\otimes n} - q^{\otimes n}\|_1 \geq 1 - (1 - c^2/2)^n$$

$$\Longrightarrow \quad P_{succ} \xrightarrow{n\to\infty} 1$$

Quantum State Distinguishability

 ,

n entangled qubits

$\rho \in V^{\otimes n}$

$\rightsquigarrow \mathbb{C}^{2\times 2}$

$\sigma \in V^{\otimes n}$

$\|\rho\|_1 = \sum_j |\lambda_j| = 1.$

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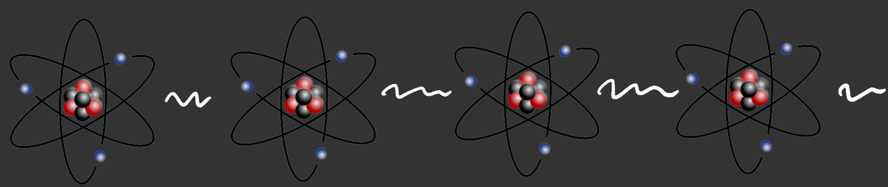

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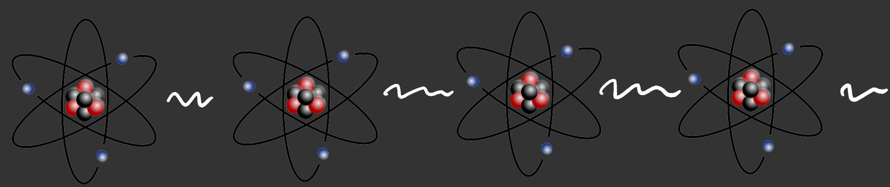

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Thm (Holevo-Helstrom): the optimal meas. $\{E_0, E_1\}$ succeeds w/ prob.

$$P_{succ} = \mathrm{Tr}(E_0 \rho) + \mathrm{Tr}(E_1 \sigma) = \frac{1}{2} + \frac{1}{4}\|\rho - \sigma\|_1$$

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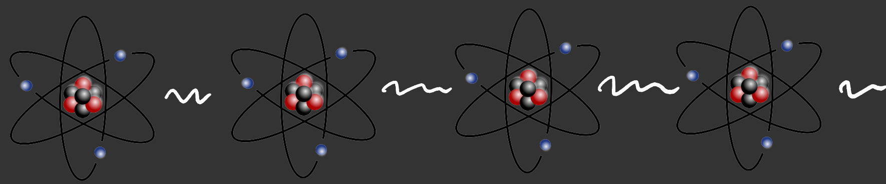

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Say, $\rho = |\psi\rangle\langle\psi|$

$\sigma = |\phi\rangle\langle\phi|$,

then $\frac{1}{2}\|\rho - \sigma\|_1 = \sqrt{1 - |\langle\phi|\psi\rangle|}$

$\frac{1}{2}\|\rho^{\otimes n} - \sigma^{\otimes n}\|_1 = \sqrt{1 - |\langle\phi|\psi\rangle|^n}$

$\xrightarrow{n \to \infty} 1$

$\Rightarrow P_{succ} \xrightarrow{n \to \infty} 1$.

Distribution Learning

If we can distinguish, there is hope to learn.

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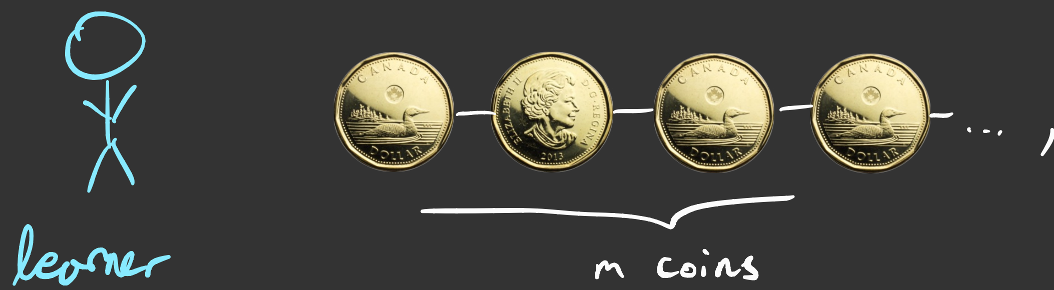

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$d = 2^m$

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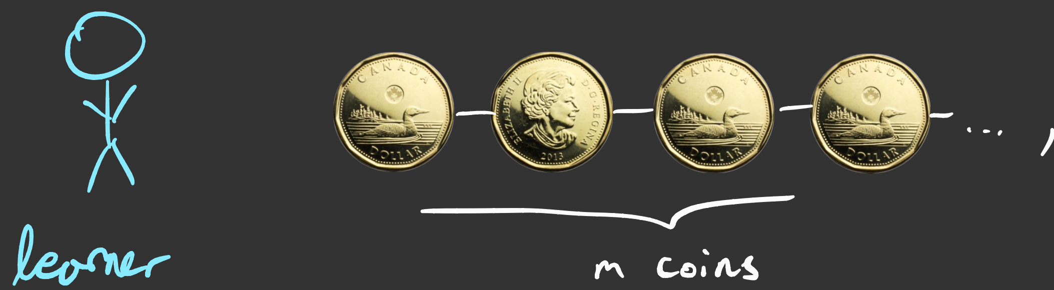

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Distribution Learning Problem

Given: $(X_1, \ldots, X_n) \sim p^{\otimes n}$.

Output: $\hat{p}_n$ s.t. $\|\hat{p}_n - p\|_1 \leq \epsilon$.

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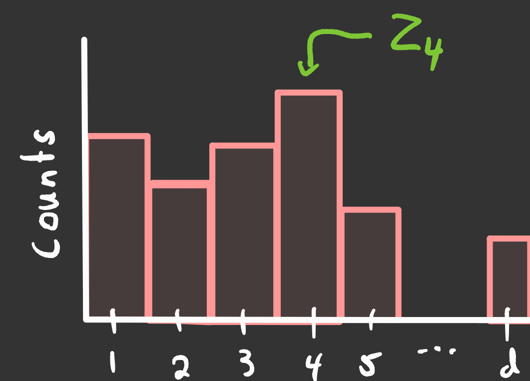

Sol^n:

Let $\hat{p}_n(i) = Z_i/n$, $\quad \vec{Z} = (Z_1, \dots, Z_d) \sim \text{Multi}(n, d)$

$\rightarrow$ can show $\mathbb{E}\|\hat{p}_n - p\|_1 \le \sqrt{d/n}$, so $n \sim d/\epsilon^2$ suffices.

Distribution Learning

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$n \sim d/\epsilon^2$ suffices.

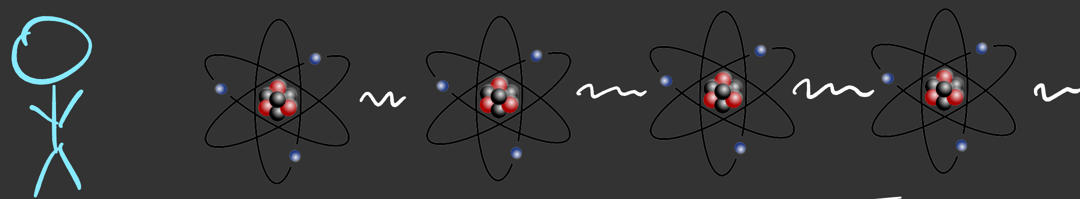
Note: CLT not good enough!

$$\left(\frac{Z_i - np_i}{\sqrt{n}}\right)_{i=1}^{d} \xrightarrow{n \to \infty} N(0, \Sigma)$$

$\iff \forall \epsilon > 0 \ \exists n_0$ s.t. $|\mathbb{E} Z_i - np_i| \le \epsilon\sqrt{n} \quad \forall n \ge n_0.$

What if $n_0 \sim d^2, d^3, e^d$?

Quantum State Tomography



learner

m spin-$\frac{1}{2}$ particles

$\rho \in \mathbb{C}^{d \times d}$

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Given: $\rho^{\otimes n}$

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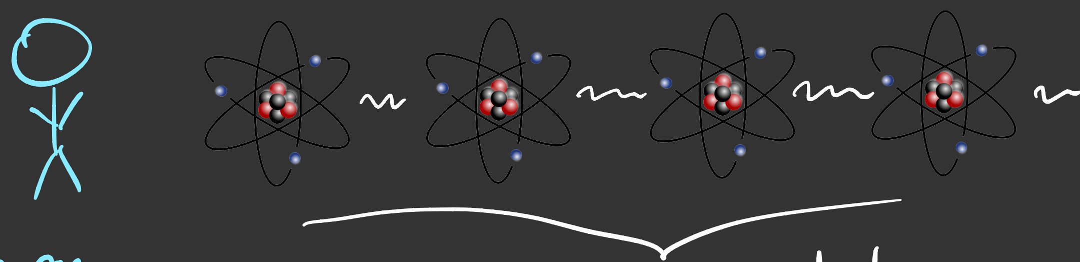

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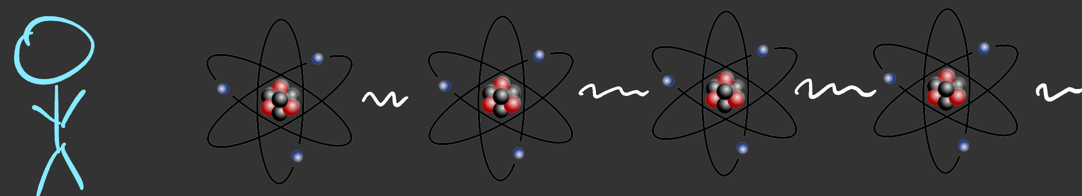

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Quantum Tomography Problem

Given:	$\rho^{\otimes n} \rightarrow$ measurement?
Output:	$\hat{\rho}_n$ s.t. $\lVert \hat{\rho}_n - \rho \rVert_1 \le \epsilon$

- Unlike the classical case, we get different answers depending on the allowed measurements.

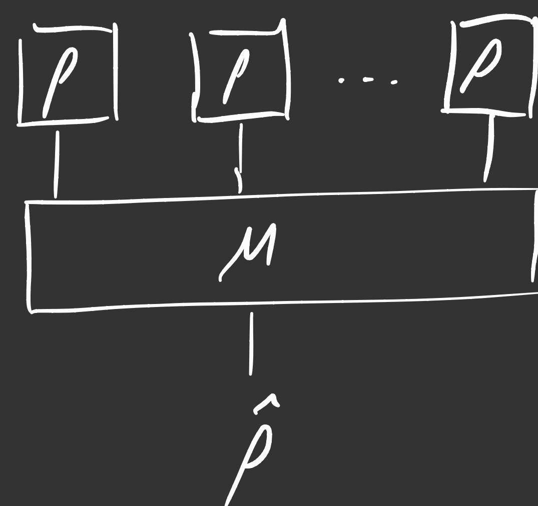
Quantum State Tomography

Quantum Tomography Problem

$$\boxed{\begin{array}{l} \text{Given: } \rho^{\otimes n} \longrightarrow \text{measurement?} \\[2em] \text{Output: } \hat{\rho}_n \text{ s.t. } \|\hat{\rho}_n - \rho\|_1 \leq \epsilon \end{array}}$$

Entangled	$\sim d^2/\epsilon^2$
Nonadaptive	$\sim d^3/\epsilon^2$
Adaptive	$\sim d^3/\epsilon^2$
Binary	$\sim d^4/\epsilon^2$

cf. classical distr. learning



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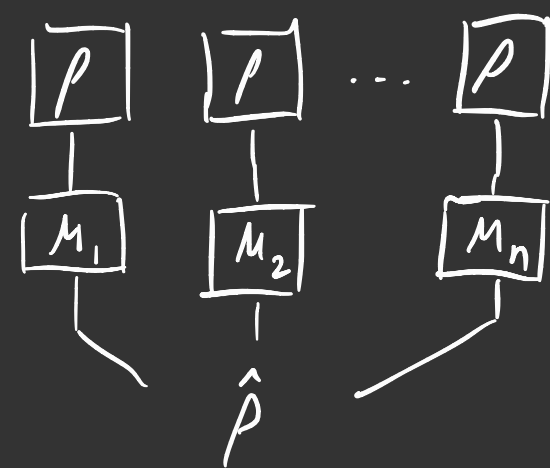

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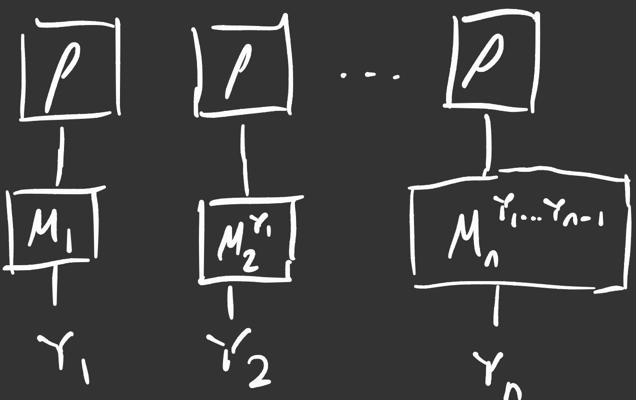

Quantum State Tomography

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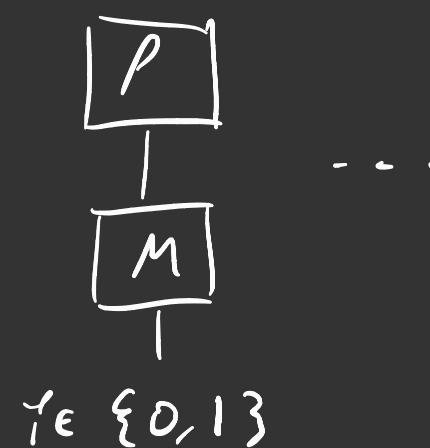
Entangled	$\sim d^2/\epsilon^2$ → [HHJWY '17, OW '17] cf. classical distr. learning
Nonadaptive	$\sim d^3/\epsilon^2$ → [KRT '15, HHJWY '17]
Adaptive	$\sim d^3/\epsilon^2$ → [LN '22, CHLLS '23]
Binary	$\sim d^4/\epsilon^2$ → [LN '22]



Lower Bound

$(*)$

Want to show : $\boxed{n \ll d^2/\epsilon^2 \text{ impossible for quantum tomography}}$

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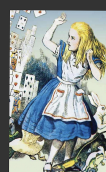
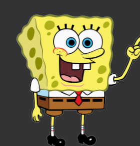


$X \in \{1, \dots, 2^N + 1\}$

$\rho \in (\mathbb{C}^{2\times 2})^{\otimes N} \longrightarrow$



Measure ρ, outcome Y

Decode $\hat{X}(Y) = X$.

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(*)

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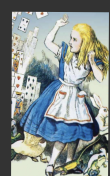


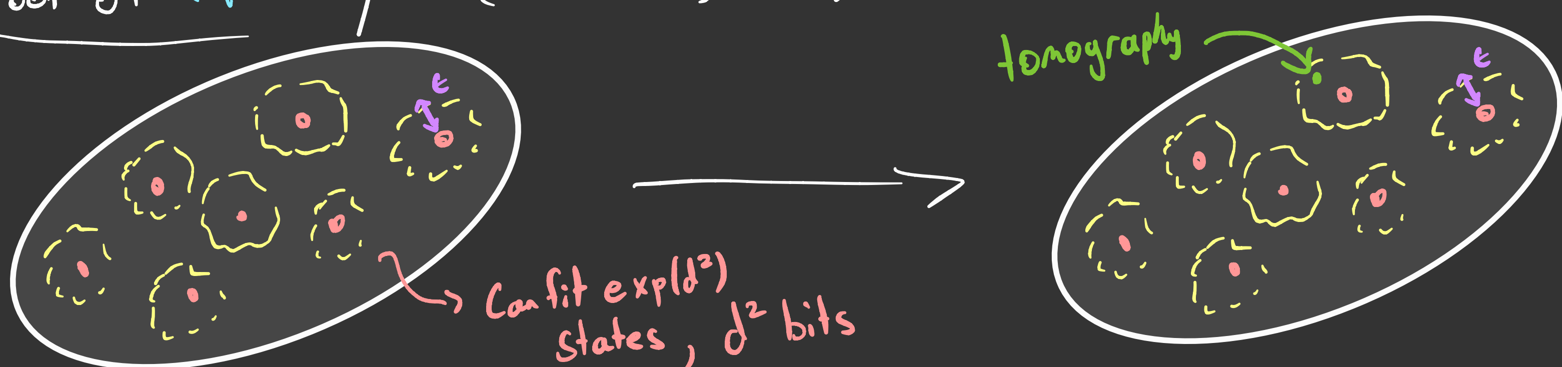
$X \in \{1, \ldots, 2^N + 1\}$

$\rho \in (\mathbb{C}^{2\times 2})^{\otimes N} \longrightarrow$

Measure ρ, outcome Y

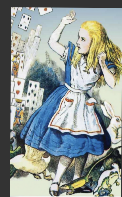
Decode $\hat{X}(Y) = X$.

Proof of (*): $\rho^{\otimes n} \in (\mathbb{C}^{d\times d})^{\otimes n}$, $\#\text{qubits} \equiv \log \dim. = n \log d$.



Upper Bound

Recall state discrimination



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Meas. w/ $\{E_0, E_1\}$

Guess ρ or σ

$\downarrow$

$\boxed{?}$

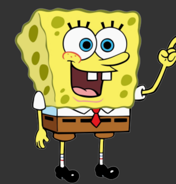
Prepare ρ_j w/prob. $1/M$

$j = 1, \ldots, M$

Meas. w/ $\{E_1, \ldots, E_M\}$

Guess j.

Upper Bound



Prepare ρ_j w/prob. $1/M$

$j=1,\dots,M$

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Guess j.

Thm (Pretty Good Measurement): Let $\rho := \frac{1}{M}\sum_{j=1}^{M} \rho_j$,

$E_j = \rho^{-\frac{1}{2}}\rho_j\rho^{-\frac{1}{2}}/M$. Then

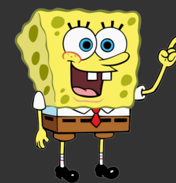
$$P_{succ} \geq \big(\underbrace{\text{optimal prob. of success}}_{P_{opt}}\big)^2$$

Proof:

$$
\begin{aligned}
P_{opt} &= \frac{1}{M}\sum_{j=1}^{M} \mathrm{Tr}(F_j \rho_j) \\
&= \frac{1}{M}\sum_{j=1}^{M} \mathrm{Tr}\left(\rho^{\frac{1}{4}} F_j \rho^{\frac{1}{4}} \rho^{-\frac{1}{4}} \rho_j \rho^{-\frac{1}{4}}\right) \\
\text{(Cauchy-Schwarz)} \quad &\leq \frac{1}{M}\sum_{j=1}^{M} \left\|\rho^{\frac{1}{4}} F_j \rho^{\frac{1}{4}}\right\|_2 \left\|\rho^{-\frac{1}{4}} \rho_j \rho^{-\frac{1}{4}}\right\|_2 \\
\text{(Cauchy-Schwarz)} \quad &\leq \frac{1}{M} \underbrace{\sqrt{\sum_{j=1}^{M} \left\|\rho^{\frac{1}{4}} F_j \rho^{\frac{1}{4}}\right\|_2^2}}_{\leq 1} \sqrt{\sum_{j'=1}^{M} \left\|\rho^{-\frac{1}{4}} \rho_{j'} \rho^{-\frac{1}{4}}\right\|_2^2} \\
\left(\sum_j F_j = \mathbb{1}\right) \quad &\leq \sqrt{\sum_{j=1}^{M} \left\|\rho^{-\frac{1}{4}} \rho_j / M \, \rho^{-\frac{1}{4}}\right\|_2^2} \\
&= \sqrt{P_{succ}}
\end{aligned}
$$

Upper Bound

$\boxed{?} \longrightarrow$



Prepare ρ_j w/ prob. $1/M$

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Meas. w/ $\{E_1, \dots, E_M\}$

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- PGM works well. Silver bullet?

Upper Bound

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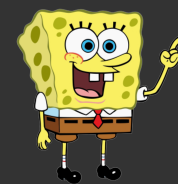

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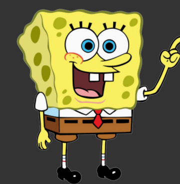
- PGM works well. Silver bullet?
- Idea: $\rho = \frac{1}{M} \sum_{j=1}^{M} \rho_j \longrightarrow \int d\rho \, \rho^{\otimes n}$

what measure?

Upper Bound

$\boxed{?}$ $\longrightarrow$



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- Idea: $$\rho = \frac{1}{M}\sum_{j=1}^{M}\rho_j \longrightarrow \int d\rho\, \rho^{\otimes n}$$

  what measure?
- Symmetries to exploit: $\left[\int d\rho\, \rho^{\otimes n}, U^{\otimes n}\right] = 0$.

- Schur-Weyl duality $\int d\rho \, \rho^{\otimes n} = \bigoplus_{\lambda \vdash n} c_\lambda \underbrace{\mathbb{1}_{Q_\lambda}}_{\text{irrep of } GL(d)} \otimes \underbrace{\mathbb{1}_{P_\lambda}}_{\text{irrep of } S_n}$

$$= \sum_\lambda c_\lambda \Pi_\lambda$$

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- Skipping details about how to compute c_λ ... (Young-Tableaus, Schur polynomials)

- Schur-Weyl duality $\int d\rho\, \rho^{\otimes n} = \bigoplus_{\lambda \vdash n} c_\lambda \underbrace{\mathbb{1}_{Q_\lambda}}_{\text{irrep of } GL(d)} \otimes \underbrace{\mathbb{1}_{P_\lambda}}_{\text{irrep of } S_n}$

$$= \sum_\lambda c_\lambda \Pi_\lambda$$

- c_λ related to characters of irreps of $GL(d)$.
- Skipping details about how to compute c_λ ... (Young-Tableaus, Schur polynomials)
- The PGM becomes

$$M_\sigma d\sigma = \left(\int d\rho\, \rho^{\otimes n}\right)^{-\frac{1}{2}} \sigma \left(\int d\rho\, \rho^{\otimes n}\right)^{-\frac{1}{2}}$$

$$M_\sigma \, d\sigma = \left(\int d\rho \, \rho^{\otimes n} \right)^{-\frac{1}{2}} \sigma \left(\int d\rho \, \rho^{\otimes n} \right)^{-\frac{1}{2}} d\sigma$$

$$= \sum_\lambda \frac{1}{c_\lambda} \Pi_\lambda \, \sigma^{\otimes n} \, \Pi_\lambda \, d\sigma$$

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- If we believe in the PGM, and we believe tomography doable w/ $\sim d^2/\epsilon^2$ copies, this will do it!

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$\Longrightarrow n \sim d^2 / \|\rho - \sigma\|_1^2$ suffices.

Final Thoughts

- We showed $\sim d^2/\epsilon^2$ necessary and sufficient for tomography in most generous setting.
  - ↳ tight bounds recently shown in more realistic ones
- "Shadow tomography" → learning full state not necessary.
  - ↳ what is the copy complexity?
- Other learning / testing problems.

→ Quantum ≈ classical often, but showing is hard. Is this true for computational problems as well?