Bounding local correlations with cardinality and symmetry constraints

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1. Introduction

Unlike Bell experiments, a measurement scenario which comprises multiple, independent sources of information admits a set of local correlations which is difficult to describe (non-convex). One approach taken by Wolfe et al. is the inflation technique[1] which provides a hierarchy of conditions converging to the local set. Additionally, Rosset et al. have shown that a model of local hidden variables (LHVs) of finite cardinality exists for any local correlation[2]. How does the local set change as one adds constraints on the cardinality of the LHVs or symmetry of the measurement scenario? We explore this question by combining the inflation technique with random sampling of models to uncover new polynomial, Bell-like inequalities which witness non-local correlations.

2. The triangle scenario

Assuming binary measurement outcomes a, b, c in the triangle scenario, a local correlation means the observed distribution can be written

$$P(abc) = \sum_{\alpha\beta\gamma} P_\alpha P_\beta P_\gamma P_A(a|\beta\gamma) P_B(b|\gamma\alpha) P_C(c|\beta\gamma)$$

One such local point is

$$P_{\neq}(abc) = \begin{cases} 0, & a = b = c \\ 1/6, & \text{otherwise} \end{cases}$$

However, it can be shown that there is no symmetric model with finite cardinality of LHVs for the point $P_{\neq}$, where symmetric means $P_\alpha = P_\beta = P_\gamma$ and $P_A = P_B = P_C$. Can we find an inequality that witnesses the non-locality of this point given finite cardinality and symmetry constraints?

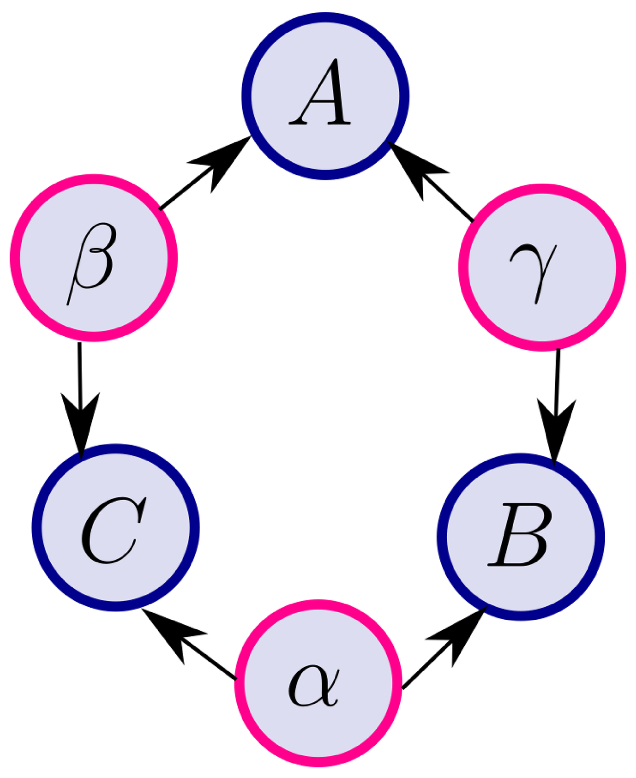


Figure 1: The triangle measurement scenario with parties A,B,C and sources of information α,β,γ

4. A linear program

Primal	Dual
$\min_x \; 0$ s.t. $Mx = b$, $x \geq 0$, $x = Sc$	$\geq$ $\max_y \; y^T b$ s.t. $M^T y \geq 0$

To impose our cardinality and symmetry constraints, we randomly sample distributions from models obeying these constraints and ask that the candidate distribution lies in the affine space of these random samples: $x = Sc$.

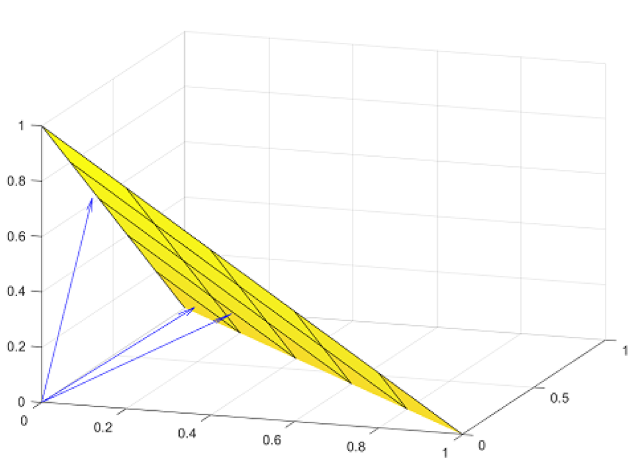


Figure 4: Three randomly sampled points are sufficient to specify a 2-D affine space.

3. The inflation technique

An inflation is a measurement scenario represented by a directed graph whose observer vertices behave exactly as in the original scenario, but with additional copies of the LHV and observer vertices in the graph. If a candidate joint probability distribution is in the local set of the original scenario, marginals of this distribution must also be compatible with a joint distribution over the inflation. For example, for the cut inflation shown in Figure 2

$$\begin{aligned} A_2 \perp B_1 \implies P^{\text{inf.}}(a_2 b_1) &= P^{\text{inf.}}(a) P^{\text{inf.}}(b) \\ \Leftrightarrow P^{\text{inf.}}(a_2 b_1) &= P^{\text{orig.}}(a) P^{\text{orig.}}(b) \\ \sum_{c_1} P^{\text{inf.}}(a_2 b_1 c_1) &= P^{\text{orig.}}(a) P^{\text{orig.}}(b) \\ &\dots \\ &\text{and so on} \\ &\dots \\ \Leftrightarrow Mx &= b[P^{\text{orig.}}] \end{aligned} \tag{1}$$

This provides us with constraints on the candidate points which can be implemented using linear programming. However, it does not give us a method to test cardinality and symmetry constraints.

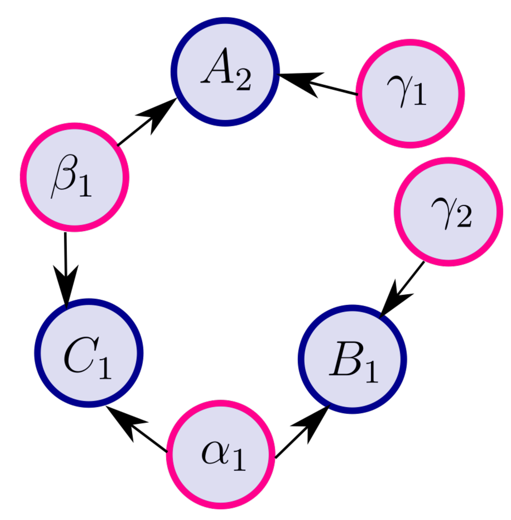


Figure 2: The cut inflation leads to constraints on the local set of correlations.

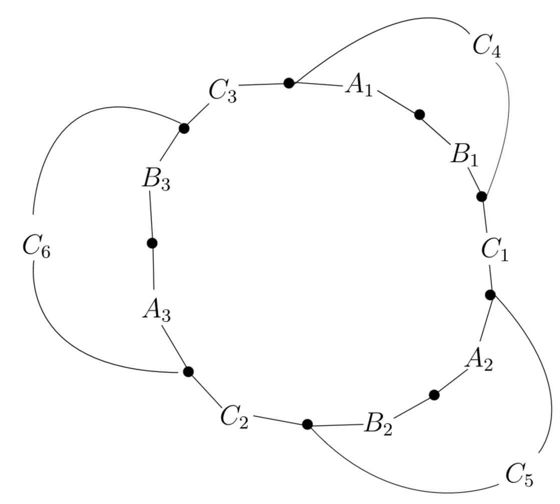


Figure 3: A larger 'ring' inflation with suppressed LHVs.

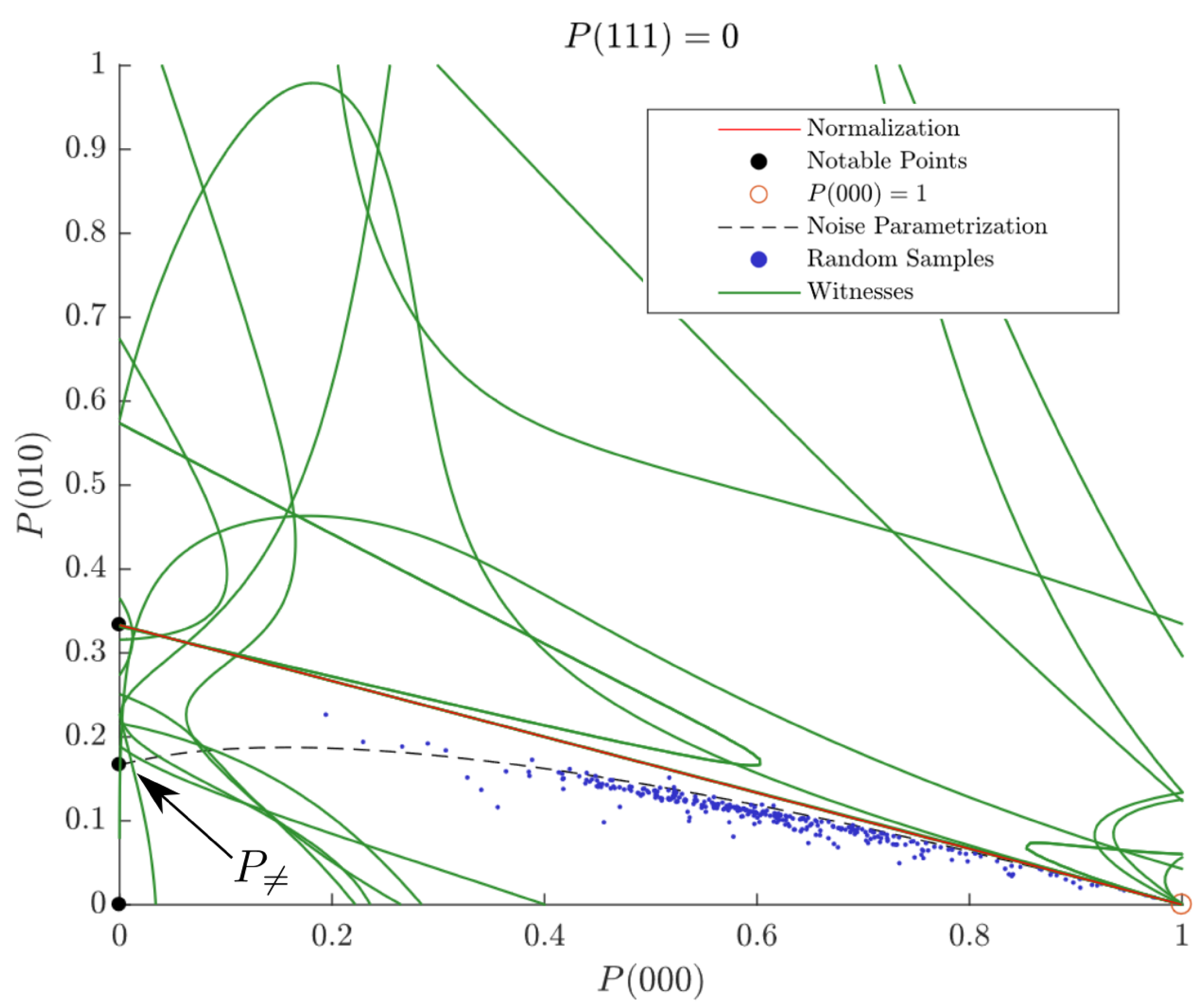
5. Results: Bell-like inequalities

Bell-like polynomial inequalities are discovered which witness the infeasibility of $P_{\neq}$. Example: using the correlator basis where the possible measurement results are ± 1, we find

$$\begin{aligned} \langle A\rangle^3 + \langle A\rangle^2 (387\langle AB\rangle - 3\langle ABC\rangle + 221) \\ + \langle A\rangle (3\langle AB\rangle^2 + 6\langle AB\rangle (31\langle ABC\rangle - 1) \\ + 3\langle ABC\rangle^2 + 134\langle ABC\rangle + 3) \\ + 289\langle AB\rangle^3 - 3\langle AB\rangle^2 (\langle ABC\rangle - 127) \\ + \langle AB\rangle \left(67\langle ABC\rangle^2 + 6\langle ABC\rangle + 291\right) \\ - \langle ABC\rangle^3 + 29\langle ABC\rangle^2 - 3\langle ABC\rangle + 63 \geq 0 \end{aligned}$$

Some open problems include:

- How does the quantum set (see [3]) compare to the local sets at different cardinalities?
- What is the lower bound on the cardinality of LHVs to reproduce the local set in the triangle scenario and more generally?



6. References

[1] Elie Wolfe, Robert W. Spekkens, and Tobias Fritz. The Inflation Technique for Causal Inference with Latent Variables. *arXiv e-prints*, page arXiv:1609.00672, Sep 2016.

[2] Denis Rosset, Nicolas Gisin, and Elie Wolfe. Universal bound on the cardinality of local hidden variables in networks. *arXiv e-prints*, page arXiv:1709.00707, Sep 2017.

[3] Tobias Fritz. Beyond bells theorem: correlation scenarios. *New Journal of Physics*, 14(10):103001, 2012.