MfP2 Lecture 4

Previously, we defined the Vector Cross product in 3D as follows,

$$\underline{a} \times \underline{b} = \hat{\underline{e}}_1 (a_2 b_3 - a_3 b_2) + \hat{\underline{e}}_2 (a_3 b_1 - a_1 b_3) + \hat{\underline{e}}_3 (a_1 b_2 - a_2 b_1)$$

The RHS of this equation can be written in shorthand by introducing the symbol $|\,|$,

i.e $$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{\underline{e}}_1 & \hat{\underline{e}}_2 & \hat{\underline{e}}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{\underline{e}}_1 (a_2 b_3 - a_3 b_2) + \hat{\underline{e}}_2 (a_3 b_1 - a_1 b_3) + \hat{\underline{e}}_3 (a_1 b_2 - a_2 b_1)$$

and we call this symbol for DETERMINANT.

Although the determinant is very useful for calculating the vector cross product, its meaning and use goes beyond vector algebra.

For instance, we can define a 2×2 determinant: $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$

i.e. $\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - b_1 a_2$, and this is how to calculate a determinant.
In fact, this is the definition of a 2×2 determinant.

We already have seen an example of a 3×3 determinant via the vector cross product. If we were to generalise this result to any 3×3 determinant, it would look like this:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 - a_1 b_3 c_2$$

Since a lot of applications are mainly concerned about 3D space, it would be quite useful to have a reliable method for calculating 3×3 determinants.
We will show here two methods.

Method 1: evaluating 3×3 determinants by Sarrus' Rule (Pierre Frederic Sarrus, 1798-1861)
Note: this method only applies on 2×2 and 3×3 determinants.

$$\left|\begin{array}{ccc|cc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}\right. = a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_3 b_2 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3$$

We recognize the structure of even and odd permutations:
+ 123, 231, 312 i.e adding the products of terms along each NW to SE diagonals;
− 213, 132, 321 i.e subtracting the — " — " — " — NE to SW — " —.

Method 2: A general method for evaluating determinants is by Laplace's expansion (Pierre-Simon Laplace 1749-1827)

Consider first the 3x3 example:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + (-1)^{1+2} a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + (-1)^{1+3} a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$\begin{aligned} = \; & a_1b_2c_3 - a_1b_3c_2 \\ & + a_2b_3c_1 - a_2b_1c_3 \\ & + a_3b_1c_2 - a_3b_2c_1 \end{aligned}$$

In this example we have expanded the determinant by the first row and thus reduced this to calculating a sum of 2x2 determinants.

Note: recognize the even and odd permutations in the answer,

In detail this is what we did:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = (-1)^{1+1} a_1 \underbrace{\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}}_{M_{11}} + \dots$$

row ← column of a_1 (element in 1st row and 1st column) often labeled as a_{11}

the sign of M_{11}

M_{11}, the MINOR of a_1 (or a_{11} to give the full address of a_1)

The SIGNED MINOR $(-1)^{i+j} M_{ij}$, where i = row and j = column, is called the COFACTOR of a_{ij} (i.e the element in row i and column j)

$$C_{ij} \equiv (-1)^{i+j} M_{ij}$$

If $|A|$ denotes an $n \times n$ determinant, $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & & a_{2n} \\ \vdots & & \ddots & & \\ a_{n1} & a_{n2} & & \dots & a_{nn} \end{vmatrix}$

then $\boxed{|A| = \sum_{i=1}^{n} a_{ij} C_{ij}}$

Note: the sign of the cofactors can be easily remembered as a "check board":

$$\begin{vmatrix} + & - & + & - & + & \dots \\ - & + & - & + & - & \dots \\ + & - & + & - & + & \dots \\ \vdots & \vdots & & & & \end{vmatrix} \text{ etc.}$$

Thus, by Laplace expansion method, we expand by a row (or a column) of $n \times n$ determinant, and obtain a sum of order $(n-1)$ determinants.
This process is then repeated until we are down to a sum of 2×2 determinants which are calculated by their definition.

> **Laplace expansion of determinants:**
> multiply each element of a row (or column) by its cofactor and add the results.

Properties of Determinants

(i) If each element of ONE row (or one column) of a determinant is multiplied by a number λ, then the value of the determinant is multiplied by λ:

$$\text{If } |A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$\text{then } \begin{vmatrix} \lambda a_1 & \lambda a_2 & \lambda a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \lambda a_1(b_2c_3 - b_3c_2) + \lambda a_2(b_3c_1 - b_1c_3) + \lambda a_3(b_1c_2 - b_2c_1)$$
$$= \lambda |A| \text{ as required;}$$

(ii) The value of a determinant is zero if

(a) all elements of ONE row (or ONE column) are zero; or if
(b) two ROWS (or two COLUMNS) are identical, or if
(c) two ROWS (or two COLUMNS) are proportional.

$$|A| = \begin{vmatrix} a & b & c \\ 0 & 0 & 0 \\ k & l & m \end{vmatrix} = a\,0 + b\,0 + c\,0 = 0 \;;$$

$$|B| = \begin{vmatrix} a & b & c \\ a & b & c \\ k & l & m \end{vmatrix} = \cancel{a}(\cancel{bm} - \cancel{cl}) - \cancel{b}(\cancel{am} - \cancel{ck}) + \cancel{c}(\cancel{al} - \cancel{bk}) = 0 \;;$$

$$|C| = \begin{vmatrix} a & b & c \\ \lambda a & \lambda b & \lambda c \\ k & l & m \end{vmatrix} = \underbrace{\lambda |B|}_{\text{by (i)}} = 0 \;;$$

(iii) If two ROWS (or two COLUMNS) are interchanged, then the sign of the determinant is reversed.

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ then $|A'| = \begin{vmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = b_1 a_2 c_3 - b_1 a_3 c_2 + b_2 a_3 c_1 - b_2 a_1 c_3 + b_3 a_1 c_2 - b_3 a_2 c_1$

$= -|A|$;

(iv) The value of a determinant is UNCHANGED if

(a) the ROWS are written as COLUMNS, or if

(b) we add to each element of one row, λ times the corresponding element of another row, where λ is any number (and a similar statement for columns).

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$

then interchanging rows and columns gives

$$|B| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) = |A|$$ as required;

If $|C| = \begin{vmatrix} a_1 + \lambda b_1 & b_1 & c_1 \\ a_2 + \lambda b_2 & b_2 & c_2 \\ a_3 + \lambda b_3 & b_3 & c_3 \end{vmatrix} = \underbrace{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}_{= |B| \text{ as in above}} + \underbrace{\begin{vmatrix} \lambda b_1 & b_1 & c_1 \\ \lambda b_2 & b_2 & c_2 \\ \lambda b_3 & b_3 & c_3 \end{vmatrix}}_{= 0 \text{ by (ii) c.}}$

$\therefore$ $|C| = |B|$ as required;

Here we have stated these facts with an illustrative example but without proofs. For proofs we refer you to books on algebra.