

Tarski Fixed Point Computation and the Arrival Problem

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January 31, 2024

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- The integer lattice, monotone functions, Tarski's theorem
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2 The Arrival Problem

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- Progress, Next Steps, Open Problems

The integer lattice, monotone functions, Tarski's theorem

Definition (Bounded Integer Lattice)

The *bounded d -dimensional integer lattice* $[N]^d = \{1, \dots, N\}^d$ is equipped with a lattice ordering where for $(x_1, \dots, x_d), (x'_1, \dots, x'_d) \in [N]^d$, $(x_1, \dots, x_d) \leq (x'_1, \dots, x'_d)$ if $x_i \leq x'_i$ for each $i \in [d]$.

Definition (Monotone function)

A function $f : [N]^d \rightarrow [N]^d$ is monotone if whenever $x, x' \in [N]^d$ with $x \leq x'$, $f(x) \leq f(x')$.

Theorem (Tarski)

If $f : [N]^d \rightarrow [N]^d$ is monotone, then there is a point $x \in [N]^d$ with $f(x) = x$.

The problem, Lower Bounds

Definition (TARSKI)

The problem $\text{TARSKI}(N, d)$ is, given oracle access to a monotone function $f : [N]^d \rightarrow [N]^d$, find a point $x \in [N]^d$ such that $f(x) = x$.

Theorem (Etesami, Papadimitriou, Rubinfeld, Yannakakis)

The query complexity of $\text{TARSKI}(N, 2)$ is $\Theta(\log^2 N)$.

Corollary

The query complexity of $\text{TARSKI}(N, d)$ is $\Omega(\log^2 N)$ for $d \geq 2$.

Known Upper Bounds

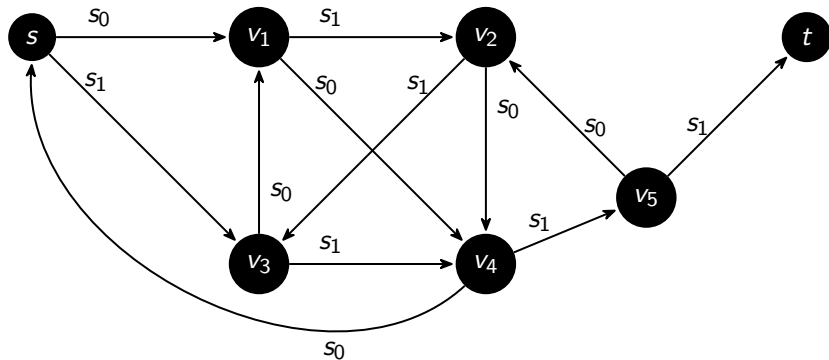
Theorem (Fearnley, Pálvölgyi, Savani)

The query complexity of Tarski $(N, 3)$ is $\Theta(\log^2 N)$.

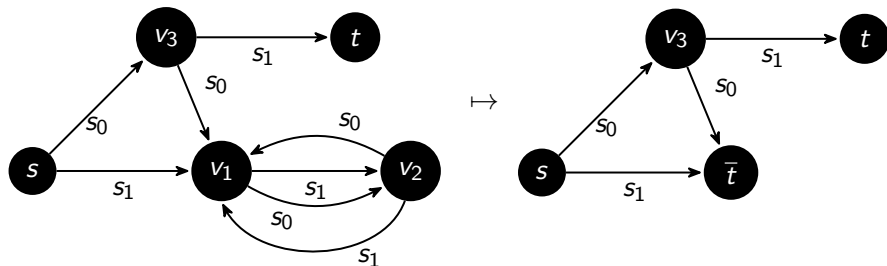
Theorem (Chen, Li)

The query complexity of Tarski (N, d) is $O(\log^{\lceil (d+1)/2 \rceil} N)$.

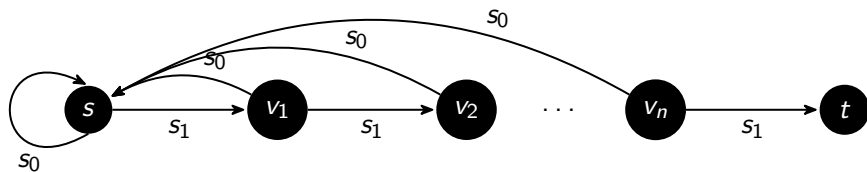
The Arrival Problem



Finiteness



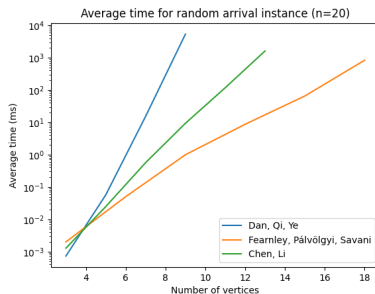
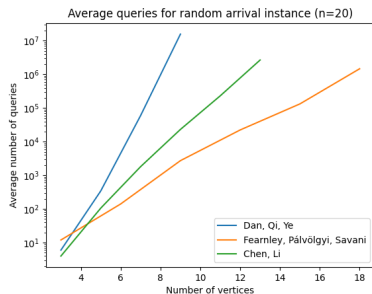
Long Walks

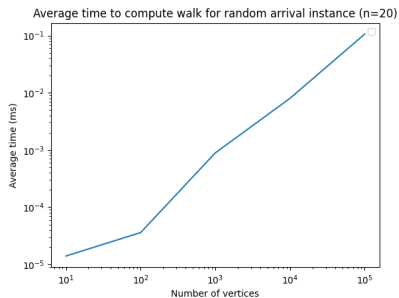
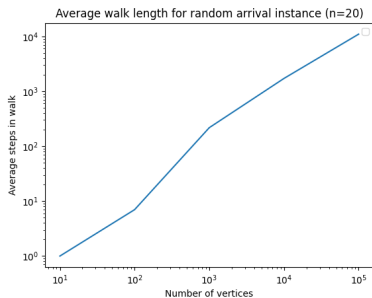


Theorem (Gärtner, Haslebacher, Hoang)

ARRIVAL is polynomial-time reducible to TARSKI.

Testing the Algorithms





Progress and Next Steps

- Progress
 - Explored solving the 4-dimensional TARSKI problem,
 - Implemented Fearnley, Pálvölgyi, Savani algorithm,
 - Applied the algorithm to randomly generated instances of ARRIVAL.
- Next Steps
 - Explore solving the 4-dimensional TARSKI problem in the special case of monotone functions corresponding to arrival instances.
- Open Problems
 - Is TARSKI fixed-parameter tractable? That is, is the query complexity of $\text{TARSKI}(N, d)$ $\Theta(\log^2 n)$ for fixed d ?
 - Is TARSKI fixed-parameter tractable in the special case of monotone functions from ARRIVAL?