Tarski Fixed Point Computation and the Arrival Problem

Angus Joshi

University of Edinburgh

s1712180@ed.ac.uk

February 1, 2024

Overview

1. Basic Definitions and Algorithmic Results
   - The integer lattice, monotone functions, Tarski's theorem
   - The problem, Lower bounds
   - Known Upper Bounds
2. The Arrival Problem
   - Definitions
   - Reduction to TARSKI
3. Progress, Next Steps, Open Problems
   - Testing the Algorithms
   - Progress, Next Steps, Open Problems

The integer lattice, monotone functions, Tarski's theorem

Definition (Bounded Integer Lattice)

The *bounded d-dimensional integer lattice* $[N]^d = \{1, ..., N\}^d$ is equipped with a lattice ordering where for $(x_1, ..., x_d), (x'_1, ..., x'_d) \in [N]^d$, $(x_1, ..., x_d) \leq (x'_1, ..., x'_d)$ if $x_i \leq x'_i$ for each $i \in [d]$.

Definition (Monotone function)

A function $f : [N]^d \to [N]^d$ is monotone if whenever $x, x' \in [N]^d$ with $x \leq x'$, $f(x) \leq f(x')$.

Theorem (Tarski, '55)

If $f : [N]^d \to [N]^d$ is monotone, then there is a point $x \in [N]^d$ with $f(x) = x$.

The problem, Lower Bounds

Definition ($\textsc{Tarski}$)

The problem $\textsc{Tarski}(N, d)$ is, given oracle access to a monotone function $f : [N]^d \to [N]^d$, find a point $x \in [N]^d$ such that $f(x) = x$.

Theorem (Etessami, Papadimitriou, Rubinstein, Yannakakis, '20)

The query complexity of $\textsc{Tarski}(N, 2)$ *is* $\Theta(\log^2 N)$.

Corollary

The query complexity of $\textsc{Tarski}(N, d)$ *is* $\Omega(\log^2 N)$ *for* $d \geq 2$.

Known Upper Bounds

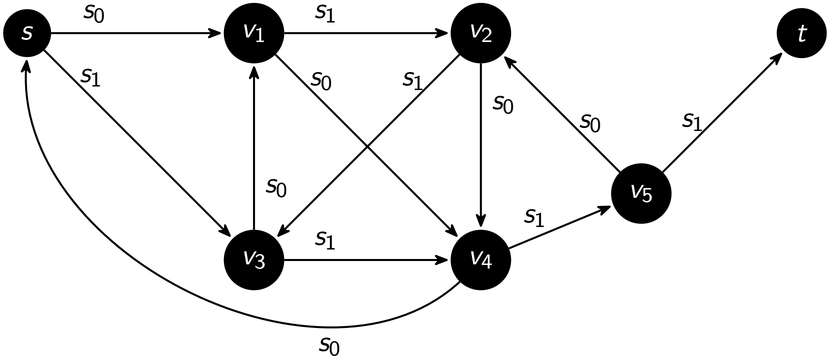
Theorem (Fearnley, Pálvölgyi, Savani, '21)

The query complexity of $\textsc{Tarski}(N, 3)$ *is* $\Theta(\log^2 N)$.

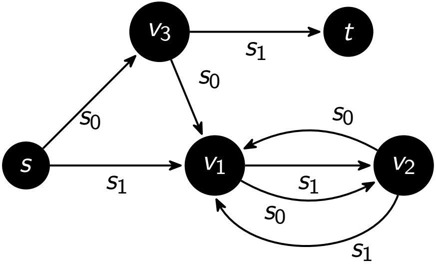
Theorem (Chen, Li, '22)

The query complexity of $\textsc{Tarski}(N, d)$ *is* $O(\log^{\lceil (d+1)/2 \rceil} N)$.

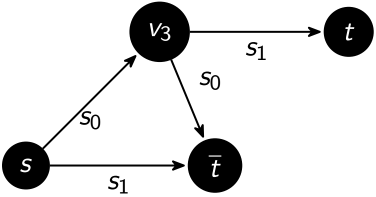
The Arrival Problem



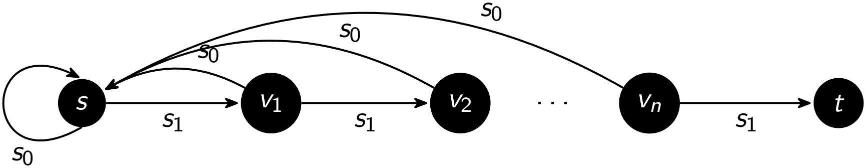
Finiteness



$\mapsto$



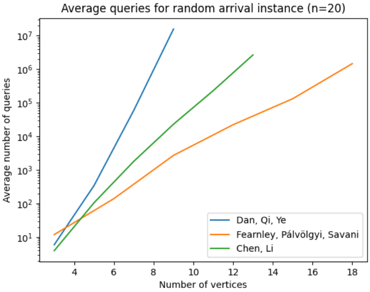
Long Walks

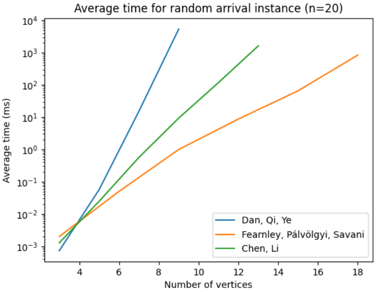


Theorem (Gärtner, Haslebacher, Hoang, '21)

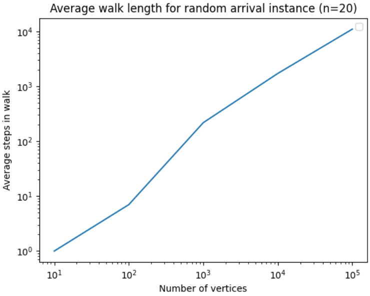
ARRIVAL *is polynomial-time reducible to* TARSKI.

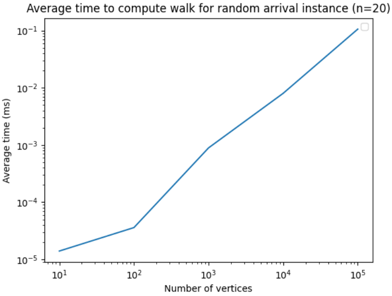
Testing the Algorithms





Reality


Average walk length for random arrival instance (n=20)
Average steps in walk
Number of vertices
10^0
10^1
10^2
10^3
10^4
10^5


Average time to compute walk for random arrival instance (n=20)
Average time (ms)
Number of vertices
10^{-5}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^1
10^2
10^3
10^4
10^5

Progress and Next Steps

- Progress
  - Explored solving the 4-dimensional TARSKI problem,
  - Implemented all of the currently known algorithms,
  - Tested the performance of the algorithms on randomly generated instances of ARRIVAL.
- Next Steps
  - Test the algorithms on pathological instances of the arrival problem,
  - Explore the TARSKI problem in the special case of monotone functions from ARRIVAL.
- Open Problems
  - Is TARSKI fixed-parameter tractable? That is, is the query complexity of TARSKI(N, d) $\Theta(\log^2 n)$ for fixed d?
  - Is the query complexity of TARSKI$(N, 4)$ $\Theta(\log^2 N)$?
  - Is there an $\Omega(\log^3 N)$ lower bound to TARSKI(N, d) for some d?
  - Is there something to be said about TARSKI in the special case of monotone functions corresponding to arrival instances?