Tarski Fixed Point Computation and the Arrival Problem

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Overview

1. Basic Definitions and Algorithmic Results
   - The integer lattice, monotone functions, Tarski's theorem
   - The problem, Lower bounds
   - Known Upper Bounds
2. The Arrival Problem
   - Definitions
   - Reduction to TARSKI
3. Progress, Next Steps, Open Questions
   - Testing the Algorithms
   - Progress, Next Steps, Open Problems

The integer lattice, monotone functions, Tarski's theorem

Definition (Bounded Integer Lattice)

The *bounded d-dimensional integer lattice* $[N]^d = \{1, ..., N\}^d$ is equipped with a lattice ordering where for $(x_1, ..., x_d), (x'_1, ..., x'_d) \in [N]^d$, $(x_1, ..., x_d) \leq (x'_1, ..., x'_d)$ if $x_i \leq x'_i$ for each $i \in [d]$.

Definition (Monotone function)

A function $f : [N]^d \to [N]^d$ is monotone if whenever $x, x' \in [N]^d$ with $x \leq x'$, $f(x) \leq f(x')$.

Theorem (Tarski)

If $f : [N]^d \to [N]^d$ is monotone, then there is a point $x \in [N]^d$ with $f(x) = x$.

The problem, Lower Bounds

Definition (Tarski)

The problem $\textsc{Tarski}(N, d)$ is, given oracle access to a monotone function $f : [N]^d \to [N]^d$, find a point $x \in [N]^d$ such that $f(x) = x$.

Theorem (Etessami, Papadimitriou, Rubinstein, Yannakakis)

The query complexity of $\textsc{Tarski}(N, 2)$ *is* $\Theta(\log^2 N)$.

Corollary

The query complexity of $\textsc{Tarski}(N, d)$ *is* $\Omega(\log^2 N)$ *for* $d \geq 2$.

Known Upper Bounds

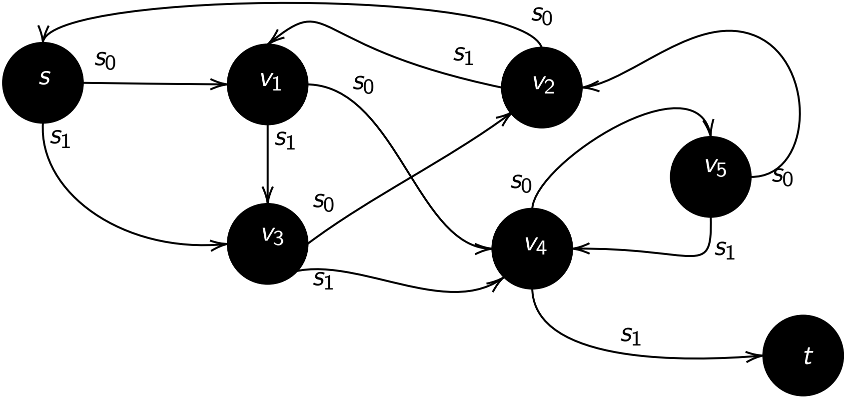
Theorem (Fearnley, Pálvölgyi, Savani)

The query complexity of $\textsc{Tarski}\ (N, 3)$ *is* $\Theta(\log^2 N)$.

Theorem (Chen, Li)

The query complexity of $\textsc{Tarski}\ (N, d)$ *is* $O(\log^{\lceil (d+1)/2 \rceil} N)$.

The Arrival Problem

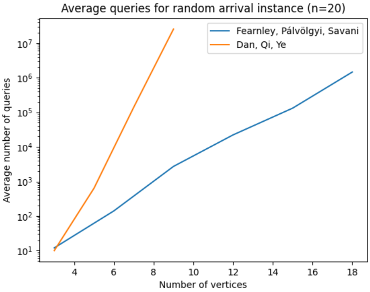


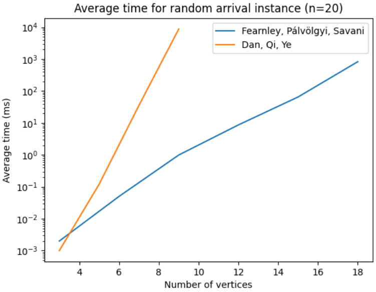
Reduction to TARSKI

Theorem (Gärtner, Haslebacher, Hoang)

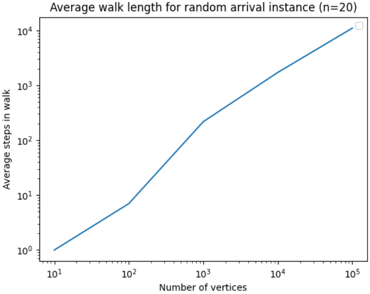
ARRIVAL *is polynomial-time reducible to* TARSKI.

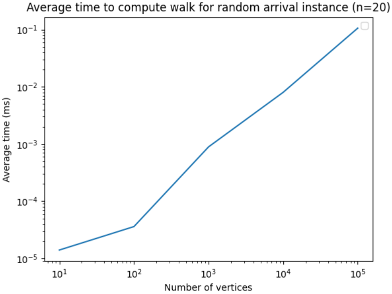
Testing the Algorithms





Reality


Average walk length for random arrival instance (n=20)
Average steps in walk
10^0 10^1 10^2 10^3 10^4
10^1 10^2 10^3 10^4 10^5
Number of vertices


Average time to compute walk for random arrival instance (n=20)
Average time (ms)
10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1}
10^1 10^2 10^3 10^4 10^5
Number of vertices

Progress and Next Steps

- Progress
  - Explored solving the 4-dimensional TARSKI problem,
  - Implemented Fearnley, Pálvölgyi, Savani algorithm,
  - Applied the algorithm to randomly generated instances of ARRIVAL.
- Next Steps
  - Explore solving the 4-dimensional TARSKI problem in the special case of monotone functions corresponding to arrival instances.
- Open Problems
  - Is TARSKI fixed-parameter tractable? That is, is the query complexity of $\text{TARSKI}(N, d)$ $\Theta(\log^2 n)$ for fixed d?
  - Is TARSKI fixed-parameter tractable in the special case of monotone functions from ARRIVAL?