The Accept-Reject Method: Anthony Gusman, Nicholas Sullivan

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Introduction

The process of generating pseudo random numbers using a given density function of a random variable is easy if the inverse of the CDF is known or quick to find. However, the general method breaks down when there does not exist a happy way to find the inverse of the CDF. This is why we will be exploring something called the *Acceptance-Rejection Method*.

Overview of Method

The Acceptance-Rejection Method is actually quite simple. We implement the method with our `acceptReject` function as described in plain English below:

1. Given a probability density function (**PDF**) f_X of a random variable X with compact support find its maximum over its range (call it c).
2. Generate a pair of random numbers (X, Y) such that $X \sim U[support\ of f_X]$ and $Y \sim U[0, c]$.
3. For each point (x_i, y_i) if $0 \leq y_i \leq f_X(x_i)$ then it is "accepted" otherwise it is "rejected."

As you can see this is essentially the same process used in Monte-Carlo integration.

The Functions

We provide three functions in this paper, `acceptReject.m`, `acceptRejectPlot.m` and `randPDF.m` with the following outputs and arguments:

`[A,varargout] = acceptReject(fun,a,b,n,varargin)`

- **Output:** numerical array A containing the accepted X-values (i.e., the values of the random variable with density function f).
- **Input:** fun, PDF function f_X
- a, lower bound of fun's domain
- b, upper bound of fun's domain
- n, number of (x_i, y_i) pairs (increase this for higher accuracy)
- The optional input string 'plot' returns a histogram comparing the produced values with the actual user-specified density.

`acceptRejectPlot(fun,a,b,Xc,Yc)`

- **Output:** plots, one of the pdf with a histogram of the accepted values generated and one of the pdf along with accepted and rejected points.
- **Input:** fun, PDF function f_X

- a, lower bound of fun's domain
- b, upper bound of fun's domain
- X, the X returned by `acceptReject`
- Y, the Y returned by `acceptReject` **Note:** This will have little use aside from its call by `acceptReject.m`

`[x] = randPDF( fun, a, b, n)`

- **Output:** a 1xn vector of random values with density provided by fun
- **Input:** fun, PDF function fx
- a, lower bound of fun's domain
- b, upper bound of fun's domain

Accept Reject and Plotter function code

here is the full code of all functions with comments and syntax high lighting:

acceptReject.m

```
function [A,varargout] = acceptReject(fun,a,b,n,varargin)
% ACCEPTREJECT  Produces random variables with accept/reject method.
%
%   Examples:
%       f = @(x) 6 * x.^2 .* (1-x).^2;
%       A = acceptReject(f, -1, 1, 1e4);
%
%       This takes function f, left and right endpoints a = -1, b = 1
%       with n = 1e4 trials; returns numerical array A containing the
%       accepted X-values (i.e., the values of the random variable with
%       density function f).
%   ------------------------------------------------------------------
%
%       f = @(x) 6 * x.^2 .* (1-x).^2;
%       [A,Xc,Yc] = acceptReject(f, -1, 1, 1e4);
%
%       The optional output arguments Xc and Yc return cell arrays:
%       The first rows Xc{1,:} and Yc{1,:} contain the accepted x- and
%       y-values, respectively. The second rows Xc{2,:} and Yc{2,:} contain
%       the rejected values.
%   ------------------------------------------------------------------
%
%       A = acceptReject(f, -1, 1, 1e4,'plot');
%
%       The optional input string 'plot' returns a histogram comparing the
%       produced values with the actual user-specified density.

% Scale density function to appropriate size.
I = integral(fun,a,b);
f = @(x) (1/I)*fun(x);
c = max([f(a),f(b),f(fminbnd(@(x) -f(x),a,b))]);     % locate maximum
M = @(x) c + (x-x);

% Initialize random variables for n trials.
T = (b-a)*rand(1,n)+a;          % x-value uniform distribution on S
```

```
U = rand(1,n);                    % y-scale factor uniform distribution

% Test criteria.
TEST = [ U .* M(T) <= f(T) ];      % logic array (0 if reject, 1 if accept)

Xc = {T(TEST); ...          % row 1 <- accepted X values
      T(~TEST)};            % row 2 <- rejected X values

Yc = {U(TEST).*c; ...       % row 1 <- accepted Y values
      U(~TEST).*c};         % row 2 <- rejected Y values

A = Xc{1,:};                % accepted X values as num array

% Report accepted/rejected data if requested.
varargout{1} = Xc;   varargout{2} = Yc;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Optional Routines
if ~isempty(varargin)
    switch varargin{1}
        case 'plot'      % Plotting routine
            acceptRejectPlot(fun,a,b,Xc,Yc)
    end
end
```

acceptRejectPlot.m

```
function acceptRejectPlot(fun,a,b,Xc,Yc)
% ACCEPTREJECTPLOT  Plots results from acceptReject.
%
%   Example:
%       f = @(x) 6 * x.^2 .* (1-x).^2;
%       [A,Xc,Yc] = acceptReject(f,-1,1,1e4);
%
%       acceptRejectPlot(f, -1, 1, Xc, Yc);
%
%       This plots the approximate density using the histogram of Xc and
%       the actual density f. A second plot contains some sample
%       accept/reject points.

% Recover data
I = integral(fun,a,b);
f = @(x) (1/I)*fun(x);

X = Xc{1,:};    notX = Xc{2,:};
n = length(X) + length(notX);

nACCEPT = length(X);              % count how many we accepted (stats)
pACCEPT = nACCEPT/n;              % percentage accepted (stats)

% Plotting
x = linspace(a,b,200);                % domain for actual density

% Plot some sample accept/reject points.
figure(1)
```

```
hold on

nEnd = min([.5e4,length(Xc{1,:})]);
nEnd2 = min([.5e4,length(Xc{2,:})]);

plot(Xc{1,:}(1:nEnd),Yc{1,:}(1:nEnd),'b+')
plot(Xc{2,:}(1:nEnd2),Yc{2,:}(1:nEnd2),'rx')

plot(x,f(x),'k--','LineWidth',2);    % plot actual density
xlabel('x'); ylabel('f(x)'); title('Sample Points');

% Plot the approximate and actual density.
figure(2)

m = 50;                                      % number of bins
EDGES = linspace(a,b,m);                     % define bins for histogram
FREQ = histc(X,EDGES);                       % report frequency in each bin
A = nACCEPT*(b-a)/(m-1);                     % 'area' of histogram
approx = bar(EDGES,FREQ/A,'histc');          % plot approximate density

hold on

plot(x,f(x),'r--','LineWidth',2);    % plot actual density

xlabel('x'); ylabel('f(x)'); title('Accept/Reject Method');
set(approx,'FaceColor',[1 1 1],'LineWidth',2);
```

randPDF.m

```
function [x] = randPDF( fun, a, b, n)
% RANDPDF Generate random numbers from user-specified distribution.
%
% Produces a 1 x n vector of random numbers following the distribution of a
% given PDF.
%
%   Output:
%   x: 1 x n vector of random numbers following the distribution of fun
%
%   Input:
%   fun: a PDF function with compact support over [a,b]
%   a: the lower bound for the support of fun
%   b: the upper bound for the support of fun
%   n: the desired number of random values

x = zeros(1,n); % placeholder for generated numbers
filled = 0;     % keeping track of the accepted values

% Find ratio of area beneath pdf to box size.
I = integral(fun,a,b);
f = @(x) (1/I)*fun(x);
c = max([f(a),f(b),f(fminbnd(@(x) -f(x),a,b))]);     % locate maximum
boxsize = ceil(c*(b-a));

% Fill the output array with the desired amount of random numbers.
filled = 1;
oldfill = 1;
```

```
x = zeros(1,n);
while filled <= n
    A = acceptReject(f,a,b, ((n - filled)*boxsize));
    oldfill = filled;
    filled = filled + length(A);
    if filled <= n
        x(oldfill:filled-1) = A;
    else
        x(oldfill:n) = A(1:n-oldfill+1);
    end
end
```

Example of the inside work with the Beta Density

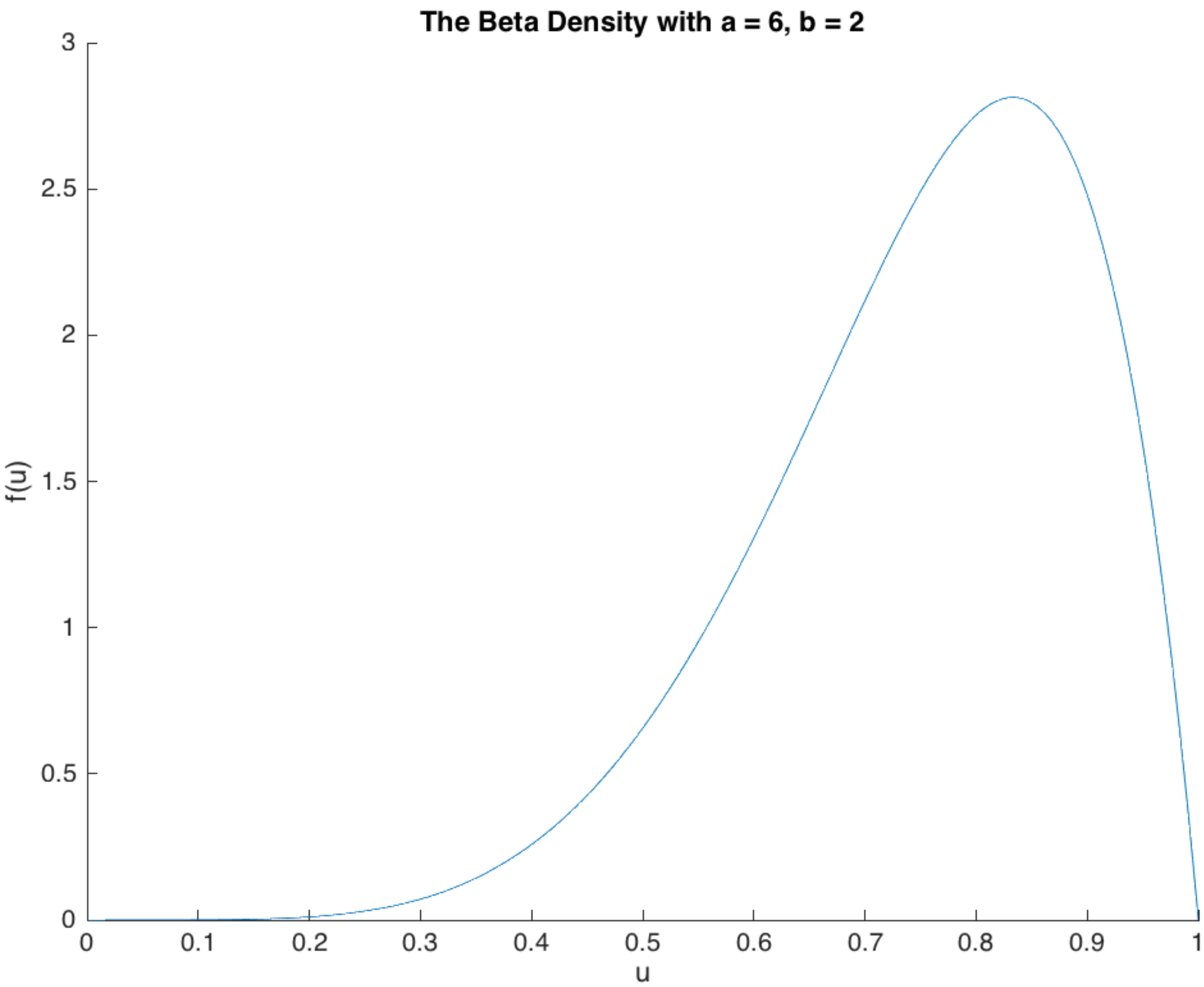
Lets take a look at how these functions work using the beta density with a = 6 and b = 2

setting up a, b and beta(u)

```
a = 6;
b = 2;
beta = @(u) gamma(a+b)/(gamma(a)*gamma(b)) * u.^(a-1) .* (1-u).^(b-1);
```

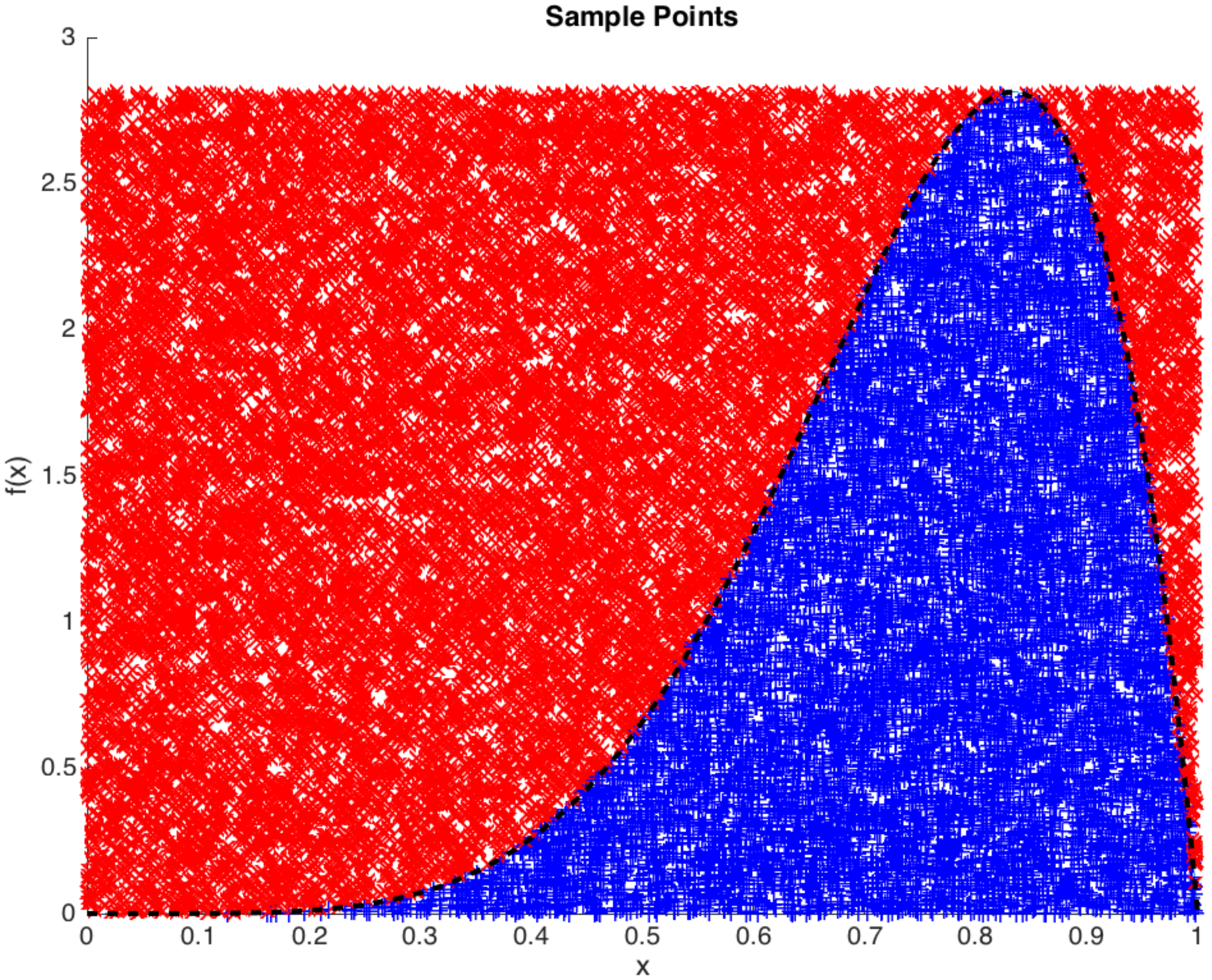
Plotting to see the pdf

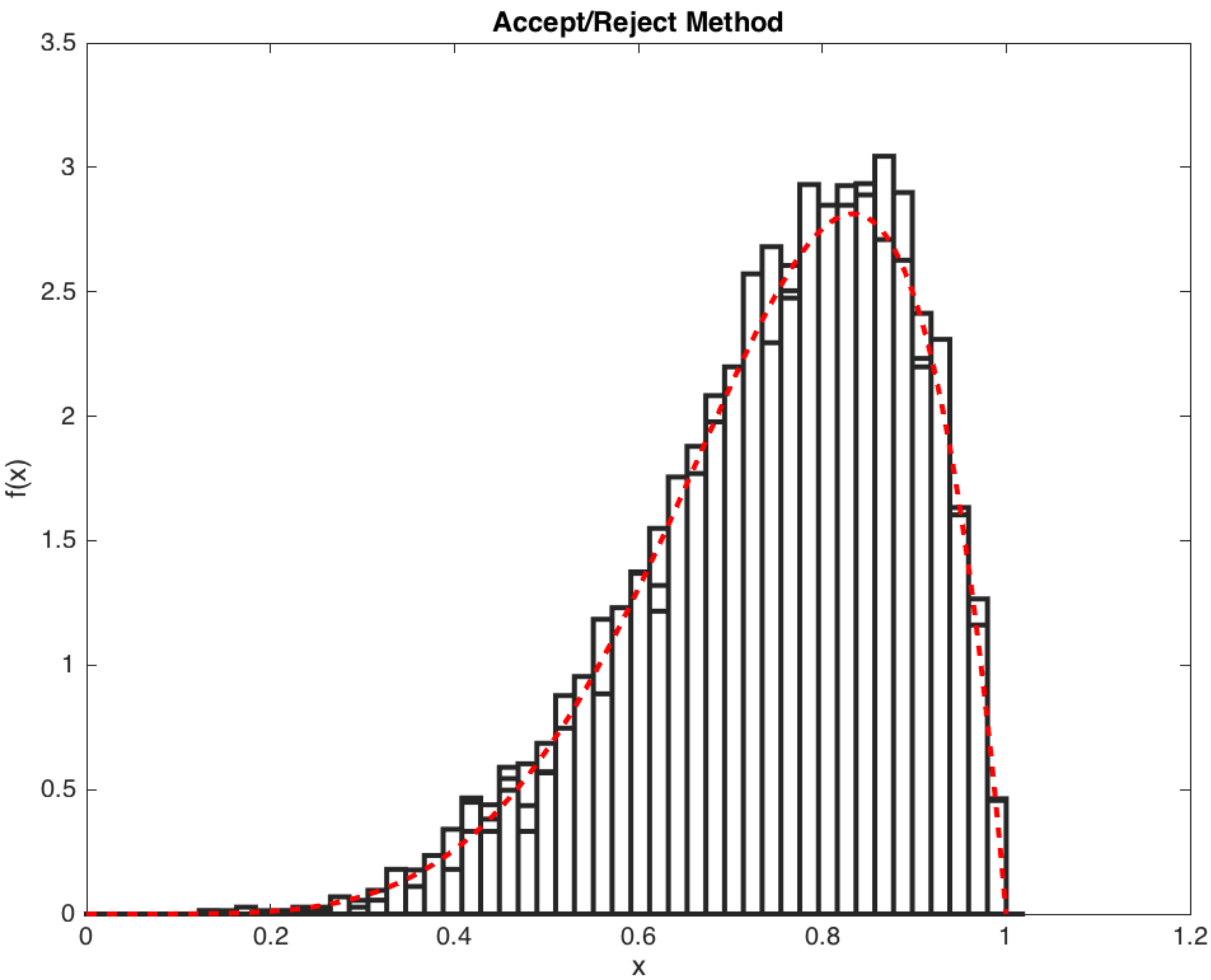
```
x = linspace(0,1,1000);

figure
hold on
title('The Beta Density with a = 6, b = 2')
plot(x,beta(x))
xlabel('u')
ylabel('f(u)')
hold off
```



Now lets test out the functions for 10,000 points

```
[A] = acceptReject(beta,0,1,10000,'plot');
```





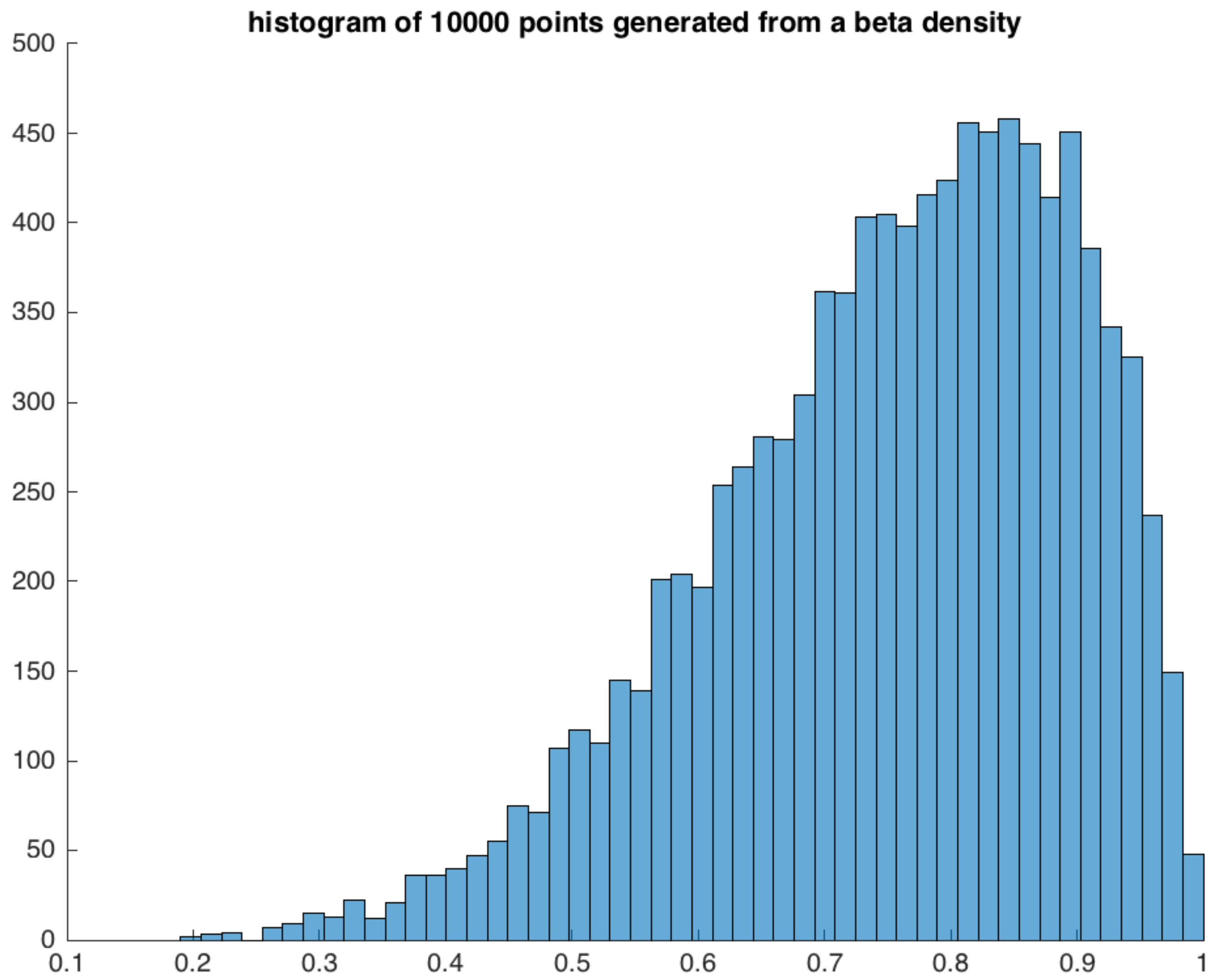
Example 10000 points with density of the Beta distribution

All random number generators out there work in the following manner, a type of random number is chosen via which rand function is used (`rand`, `randi`, ...) and then the user specifies how they want that data returned, a vector, a single random number or a matrix.

randPDF works much in the same manner, though with slightly limmited functionality at this time. given a function a range and a number of points to be returned randPDF generates that number of points required from the density given.

lets try it out using the beta density above and grabbing 10000 points.

```
n = 10000;

x = randPDF(beta, 0, 1, n);

figure
hold on
title('histogram of 10000 points generated from a beta density')
histogram(x,50)
hold off
```



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