1 Categories of spans

This is a rephrasing of [3].

1.1 Motivation

Given an ∞-category $\mathcal{C}$, we would like to create an ∞-category whose objects are spans in $\mathcal{C}$. That is, we would like to create a category whose objects are the objects of $\mathcal{C}$, whose morphisms are spans in $\mathcal{C}$, and whose higher simplices parametrize compositions of spans.

$$\{\ x\ \} \quad \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow} \quad \left\{ \begin{array}{ccc} & y & \\ \swarrow & & \searrow \\ x & & x' \end{array} \right\} \quad \substack{\longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow} \quad \left\{ \begin{array}{ccccccccc} & & & & z & & & & \\ & & & \swarrow & \vee & \searrow & & & \\ & & y & & & & y' & & \\ & \swarrow & & \searrow & & \swarrow & & \searrow & \\ x & & & & x' & & & & x'' \end{array} \right\} \quad \cdots$$

Our first job is to formulate this in rigorous, combinatorial language.

Definition 1. Define a functor $T\colon \mathbb{\Delta} \to \mathbb{\Delta}$ sending

$$[n] \mapsto [n] \oplus [n]^{\mathrm{op}} = \{0, \ldots, n, \bar{n}, \ldots, \bar{0}\}.$$

We can pull any simplicial set $K\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ back by T, giving a simplicial set T^*K with n-simplices

$$(T^*K)_n = K([n] \oplus [n]^{\mathrm{op}}).$$

Example 2. With $K = \Delta^1$, we have

$$(T^*\Delta^1)_0 = \{\phi\colon \{0, \bar{0}\} \mapsto \{0, 1\} \mid \phi \text{ weakly monotone}\} = \{00, 01, 11\}$$

and

$$(T^*\Delta^1)_1 = \{0000, 0001, 0011, 0111, 1111\}.$$

The map d_0 acts by pulling back by $T\partial_0$, which is given by the diagram

$$\begin{tikzcd}[row sep=small]
 & 0 \\
0 \arrow[ur, mapsto] & 1 \\
\bar{0} \arrow[dr, mapsto] & \bar{1} \\
 & \bar{0}
\end{tikzcd}.$$

That is, the source of an edge $abcd$ is ad, and the target is bc. We can draw $T^*\Delta^1$ as follows.

$$00 \xleftarrow{0001} 01 \xrightarrow{0111} 11$$

Similarly, we can draw $T^*\Delta^2$ as follows.

$$\begin{tikzcd}
 & & 11 & & \\
 & 01 \arrow[ur] \arrow[dl] & & 12 \arrow[ul] \arrow[dr] & \\
00 & & 02 \arrow[ll] \arrow[rr] \arrow[uu] \arrow[ul] \arrow[ur] & & 22
\end{tikzcd}$$

We would like the span $00 \leftarrow 02 \rightarrow 22$ to correspond to the composition of the spans $00 \leftarrow 01 \rightarrow 11$ and $11 \leftarrow 12 \rightarrow 22$, but it is not immediately clear how to impose this condition in a general way. We turn instead to a different model.

Definition 3. Denote by $\mathbb{\Sigma}^n$ the poset with objects (i, j), with $0 \leq i \leq j \leq n$, with

$$(i, j) \leq (i', j') \iff i \leq i \quad \text{and} \quad j' \leq j.$$

Since any map of totally ordered sets $\phi\colon [m] \rightarrow [n]$ induces a functor

$$\mathbb{\Sigma}^m \rightarrow \mathbb{\Sigma}^n, (i, j) \mapsto (\phi(i), \phi(j)),$$

the categories $\mathbb{\Sigma}^n$ form a simplicial object $\mathbb{\Sigma}^\bullet\colon \mathbb{\Delta} \rightarrow \mathbf{Cat}$.

Similarly, let $\mathbb{\Lambda}^n$ be the full subcategory of $\mathbb{\Sigma}^n$ on objects of the form

$$(i, j)\colon j - i \leq 1.$$

There is an obvious inclusion $i\colon \mathbb{\Lambda}^\bullet \hookrightarrow \mathbb{\Sigma}^\bullet$.

Example 4. We can draw Σ^2 as follows. The subcategory $\Lambda^2 \hookrightarrow \Sigma^2$ is in blue.

$$
\begin{tikzcd}[column sep=small]
 & & 02 \arrow[dl] \arrow[dr] & & \\
 & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22
\end{tikzcd}
$$

As simplicial sets, $N(\Sigma^n) \cong T^*\Delta^n$. The k-simplices of Σ^n are maps $[k] \mapsto \Sigma^n$, i.e. strings $(i_0, j_0) \leq \cdots \leq (i_k, j_k)$. This is a collection of integers such that

$$0 \leq i_k \leq \cdots \leq i_0 \leq j_0 \leq \cdots \leq j_k \leq n.$$

Providing such a string is equivalent to providing a map $[k]^{\mathrm{op}} \oplus [k] \to [n]$.

Definition 5 (Cartesian)**.** Let $\mathcal{C}$ be an ∞-category with finite limits. A functor $f \colon \Sigma^n \to \mathcal{C}$ is ***Cartesian*** if it is a right Kan extension of its restriction to Λ^n.

For any ∞-category $\mathcal{C}$, we might try to procede by defining a simplicial set

$$\mathrm{Hom}^{\mathrm{Cart}}(\Sigma^\bullet, \mathcal{C}) \colon \Delta^{\mathrm{op}} \to \mathbf{Set},$$

whose n-simplices are Cartesian diagrams $\Sigma^n \to \mathcal{C}$. However, it is not at all clear how to show that this is an infinity category. It turns out to be profitable to change models; we will be able to show that the functor

$$\mathbf{Fun}^{\mathrm{Cart}}(\Sigma^\bullet, \mathcal{C})^{\simeq}; \qquad \Delta^{\mathrm{op}} \to \mathcal{S}$$

is a complete Segal space.

For any ∞-category $\mathcal{C}$, we define a simplicial object

$$\mathbf{Fun}(\Sigma^\bullet, \mathbf{Cat}_\infty) \colon \Delta^{\mathrm{op}} \to \mathbf{Cat}_\infty. \tag{1.1}$$

Applying the relative nerve to Equation 1.1 gives a coCartesian fibration

$$\pi \colon \overline{\mathbf{SPAN}}^+ \to \Delta^{\mathrm{op}}.$$

The ∞-category $\overline{\mathbf{SPAN}}^+$ has

- 0-simplices given by functors $\alpha \colon \Sigma^m \to \mathcal{C}$.
- 1-simplices given by pairs (ϕ, η), where $\phi \colon [n] \to [m]$ and η is a 1-simplex

  $$\phi^*\alpha \xrightarrow{\eta} \beta$$

  in $\mathbf{Fun}(\Sigma^n, \mathcal{C})$.

- ...

Definition 6. We define $\mathbf{SPAN}^+$ to be the full subcategory of $\overline{\mathbf{SPAN}}^+$ on Cartesian functors.

Lemma 7. Let $\alpha\colon \mathbb{\Sigma}^m \to \mathcal{C}$ be a Cartesian functor. Then for any $\phi\colon [n] \to [m]$, the functor $\phi^*\alpha$ given by the composition

$$\mathbb{\Sigma}^n \xrightarrow{\phi} \mathbb{\Sigma}^m \xrightarrow{\alpha} \mathcal{C}$$

is Cartesian.

Corollary 8. The restriction

$$\pi'\colon \mathbf{SPAN}^+ \to \mathbb{\Delta}^{\mathrm{op}}$$

is a coCartesian fibration.

Proof. We need to show that we can have a sufficient supply of π'-coCartesian morphisms. A morphism f in $\overline{\mathbf{SPAN}}^+$ is π-coCartesian if and only if every horn Λ^n_i with $\Delta^{\{0,1\}} = f$ has a filler. Because $\mathbf{SPAN}^+ \hookrightarrow \overline{\mathbf{SPAN}}^+$ is a full subcategory inclusion, any □

Definition 9 (category of spans)**.** Let $\mathcal{C}$ be an ∞-category with finite limits. We define a category

$$\mathbf{Span}(\mathcal{C})$$

by

1.2 Monoidal structures

1.2.1 1-categories of spans

Let $\mathcal{C}^\otimes$ be a symmetric monoidal infinity category, which we model by a commutative monoid in $\mathbf{Cat}_\infty$, i.e. a functor

$$\mathcal{C}^\otimes\colon N(\mathcal{F}\mathrm{in}_*) \to \mathbf{Cat}_\infty$$

such that

$$\mathcal{C}^\otimes_{\langle n\rangle} \xrightarrow{\prod_i \rho_i} (\mathcal{C}^\otimes_{\langle 1\rangle})^n$$

is an equivalence for all n. Further suppose that

1. Each ∞-category $\mathcal{C}^\otimes_{\langle n\rangle}$ admits finite limits; and
2. For each morphism $\rho\colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{F}\mathrm{in}_*$, the functor $\rho_*\colon \mathcal{C}^\otimes_{\langle m\rangle} \to \mathcal{C}^\otimes_{\langle n\rangle}$ preserves finite limits.

Definition 10. We will call such a symmetric monoidal ∞-category ***pullback-compatible***.

Note 11. By presenting a symmetric monoidal ∞-category as a Cartesian fibration admitting relative finite limits, these conditions are satisfied automatically by [4, Prop. 4.3.1.10]. For example, for any ∞-category $\mathcal{C}$ with finite limits, the construction $\mathcal{C}_\times$ in [2] yields such a Cartesian fibration.

Definition 12. Define a functor $F\colon \mathbb{\Delta}^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_* \longrightarrow \mathbf{Kan}$ (where we are viewing $\mathbb{\Delta}^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*$ as a discrete simplicially enriched category) by

$$F([m], \langle n \rangle) = \mathbf{Span}(\mathcal{C}^{\otimes}_{\langle n \rangle})_m = \mathbf{Fun}^{\mathrm{Cart}}(\Sigma^m, \mathcal{C}^{\otimes}_{\langle n \rangle})^{\simeq}.$$

Taking the homotopy coherent nerve gives a functor $N(\mathbb{\Delta}^{\mathrm{op}} \times \mathcal{F}\mathrm{in}_*) \longrightarrow \mathcal{S}$. This is the adjunct to a functor

$$G\colon N(\mathcal{F}\mathrm{in}_*) \longrightarrow \mathbf{Fun}(N(\mathbb{\Delta}^{\mathrm{op}}), \mathcal{S}); \qquad \langle n \rangle \longmapsto \mathbf{Span}(\mathcal{C}^{\otimes}_{\langle n \rangle}).$$

One might worry that this construction is not well-defined. At the moment, it's seeming like one is smarter than the author.

Lemma 13. The functor F is well-defined.

Proof. The functor F is well-defined in the first slot by [3, Prop. 5.9], and well-defined in the second because if $f\colon \Sigma^n \longrightarrow \mathcal{C}^{\otimes}_{\langle m \rangle}$ is Cartesian then so is $\rho_* \circ f$. To see this, we need to show that

$$(\rho_* \circ f)(i, j) \simeq \lim \left[\mathbb{\Lambda}^m_{/(i,j)} \longrightarrow \mathbb{\Lambda}^m \xrightarrow{f} \mathcal{C}^{\otimes}_{\langle m \rangle} \xrightarrow{\rho_*} \mathcal{C}^{\otimes}_{\langle n \rangle} \right].$$

But ρ_* preserves limits, so this is equivalently

$$\lim \left[\mathbb{\Lambda}^m_{/(i,j)} \longrightarrow \mathbb{\Lambda}^m \xrightarrow{f} \mathcal{C}^{\otimes}_{\langle m \rangle} \right] \xrightarrow{\rho_*} \mathcal{C}^{\otimes}_{\langle n \rangle}$$

which is the same as $(\rho_* \circ f)(i, j)$ because f is Cartesian. □

Proposition 14. The functor G takes its values in **CSS**, the ∞-category of complete Segal spaces.

Proof. This is [3, Prop. 8.1] □

Note 15. I'm not sure what the best way of writing this next proposition is. It sort of feels like I'm missing the point here.

Proposition 16. The functor G is a commutative monoid in complete Segal spaces, i.e. for each $n \geq 0$, the maps ρ_i give an equivalence

$$G(\langle n\rangle) \simeq G(\langle 1\rangle)^n.$$

Proof. Limits in functor categories are computed pointwise. Therefore, it suffices to check this claim for fixed $[m]$. We have equivalences

$$\begin{aligned} G(\langle n\rangle)_m &\simeq \mathbf{Fun}^{\mathrm{Cart}}(\Sigma^m, \mathcal{C}^{\otimes}_{\langle n\rangle})^{\simeq} \\ &\simeq \mathbf{Fun}^{\mathrm{Cart}}(\Sigma^m, (\mathcal{C}^{\otimes}_{\langle 1\rangle})^n)^{\simeq} \\ &\simeq \left(\mathbf{Fun}^{\mathrm{Cart}}(\Sigma^m, \mathcal{C}^{\otimes}_{\langle 1\rangle})^n\right)^{\simeq} \\ &\simeq \left(\mathbf{Fun}^{\mathrm{Cart}}(\Sigma^m, \mathcal{C}^{\otimes}_{\langle 1\rangle})^{\simeq}\right)^n \\ &\simeq G(\langle 1\rangle)^n_m \end{aligned}$$

The second equivalence holds because $\mathcal{C}^{\otimes}$ is a monoid, the third holds because the hom functor commutes with limits, and the fourth holds because $(-)^{\simeq}$ is a right adjoint and hence preserves limits. □

I suspect that it will be more useful to cast everything in fibration-y language than commutative monoid in **CSS** language. However, we stop at this point to remark that our construction almost reproduces Toby's construction.

Using the equivalence $\mathbf{CSS} \to \mathbf{Cat}_\infty$ coming from forgetting higher simplices, we get a functor

$$\mathcal{F}\mathrm{in}_* \to \mathbf{CSS} \to \mathbf{Cat}_\infty; \qquad \langle n\rangle \mapsto \mathrm{Hom}^{\mathrm{Cart}}(\Sigma^\bullet, C^{\otimes}_{\langle n\rangle}),$$

where $\mathrm{Hom}^{\mathrm{Cart}}$ denotes the set of Cartesian functors. Using the relative nerve we can unstraighten this to a Cartesian fibration

$$\chi \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}.$$

The k-simplices of this are pairs of an n-simplex in $N(\mathcal{F}\mathrm{in}_*^{\mathrm{op}})$ and an n-simplex F in $\mathbf{Span}(\mathcal{C}^{\otimes}_{\langle n_1\rangle})$

$$\left(\langle n_0\rangle \xleftarrow{\phi_1} \langle n_1\rangle \xleftarrow{\phi_2} \cdots \xleftarrow{\phi_k} \langle n_k\rangle, \quad F\colon \Sigma^n \to \mathcal{C}^{\otimes}_{\langle n_0\rangle}\right),$$

where F is a Cartesian functor of the form

$$\begin{tikzcd}
 & & & & \arrow[dl, dotted] & & & \\
 & & & \delta_1 \arrow[dl] \arrow[dr] & & \arrow[dl, dotted] & & \\
 & & \gamma_1 \arrow[dl] \arrow[dr] & & \phi_1\gamma_2 \arrow[dl] \arrow[dr] & & \arrow[dl, dotted] & \\
 & \beta_1 \arrow[dl] \arrow[dr] & & \phi_1\beta_2 \arrow[dl] \arrow[dr] & & \phi_1\phi_2\beta_3 \arrow[dl] \arrow[dr] & & \arrow[dl, dotted] \\
\alpha_1 & & \phi_1\alpha_2 & & \phi_1\phi_2\alpha_3 & & \phi_1\phi_2\phi_3\alpha_4 &
\end{tikzcd}$$

and α_i, β_i, etc. are in $\mathcal{C}^{\otimes}_{\langle n_i \rangle}$. This is tantalizingly close to Toby's construction, the difference being that in my construction the tensor products are taken before we take spans.

1.2.2 Higher categories of spans

Let $\mathcal{C}^{\otimes}\colon \mathcal{F}\mathrm{in}_* \to \mathbf{Cat}_\infty$ be a pullback-compatible symmetric monoidal ∞-category. We can again define a functor

$$(\mathbb{\Delta}^{\mathrm{op}})^k \times \mathcal{F}\mathrm{in}_* \to \mathcal{S}; \qquad ([m_1], \ldots, [m_k], \langle n \rangle) \mapsto \mathbf{Fun}^{\mathrm{Cart}}(\mathbb{\Sigma}^{m_1,\ldots,m_k}, \mathcal{C}^{\otimes}_{\langle n \rangle})$$

This is well-defined in the first slot by the reasoning in [3, Cor. 5.12], and in the second by exactly the same reasoning as above.

1.3 Push/pull and the Grothendieck construction

Let X be a space. The category of presheaves over X is the ∞-category $\mathbf{Fun}(X^{\mathrm{op}}, \mathcal{S})$. However, by the Grothendieck construction, the ∞-category of functors $X^{\mathrm{op}} \to \mathcal{S}$ is equivalent to the ∞-category of spaces over X. We therefore make the following definition.

Definition 17 (category of presheaves)**.** Let X be a space. The ***category of presheaves*** on X is the category $\mathcal{P}(X)$, given by the pullback

$$\begin{tikzcd}
\mathcal{P}(X) \arrow[r] \arrow[d] & \mathbf{Fun}(\Delta^1, \mathcal{S}) \arrow[d, "\mathrm{ev}_1"] \\
\Delta^0 \arrow[r, "X"] & \mathcal{S}
\end{tikzcd}$$

Note: fix ops.

Fix once and for all $f\colon \Delta^1 \to \mathcal{S}$ be a map of spaces $f\colon X \to Y$. This gives us a functor

of ∞-categories
$$f^*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$$
between categories of presheaves via pullback. We can view this as a map $(\Delta^1)^{\mathrm{op}} \to \mathbf{Cat}_\infty$. The Grothendieck construction again tells us that this can be thought of as a Cartesian fibration $p\colon \mathcal{M} \to \Delta^1$.

Definition 18. Define a morphism of simplicial sets $p\colon \mathcal{M} \to \Delta^1$ by the following pullback.

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d, "p"'] & \mathbf{Fun}(\Delta^1, \mathcal{S}) \arrow[d, "\mathrm{ev}_1"] \\ \Delta^1 \arrow[r, "f"] & \mathcal{S} \end{tikzcd}.$$

Proposition 19. We have:

1. The map p is a Cartesian fibration, and the p-Cartesian morphisms in $\mathcal{M}$ are pullback squares in $\mathcal{S}$.
2. The map p is a coCartesian fibration, and the p-coCartesian morphisms in $\mathcal{M}$ are squares
$$\begin{tikzcd} K \arrow[r, "b"] \arrow[d] & S \arrow[d] \\ A \arrow[r, "a"] & A' \end{tikzcd},$$
in $\mathcal{S}$, where $a = \mathrm{id}_X$, f, or id_Y, and b is an equivalence.

Proof. This is wrong!!!!

1. This follows from [4, Lem. 6.1.1.1], which shows that
$$\mathbf{Fun}(\Delta^1, \mathcal{S}) \to \mathcal{S}$$
is a Cartesian fibration whose Cartesian morphisms are pullback squares
$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
in $\mathcal{S}$. Here, we expand [4, Lem. 6.1.1.1], because the proof is fucky.

Denote $\mathbf{Fun}(\Delta^1, \mathcal{S}) = \mathcal{O}_{\mathcal{S}}$. We will call an injective map $[k] \to \mathcal{P}$, for $\mathcal{P}$ a poset, a *string of length k* in $\mathcal{P}$.

For every simplicial set K, let K^+ be the full simplicial subset of $(K \star \{x\} \star \{y\}) \times \Delta^1$ spanned by all of the vertices except $(x, 0)$. Two examples, which we will use later on, $(\Delta^n)^+$ and $(\partial\Delta^n)^+$, are as follows.

- The k-simplices of $(\Delta^n)^+$ are in bijective correspondence with those maps of posets
$$[k] \to ([n] \oplus \{x\} \oplus \{y\}) \times [1]$$
which never hit $(x, 0)$. The nondegenerate k-simplices, as usual, correspond to the *injective* maps of posets which never hit $(x, 0)$, i.e. strings of length k not containing $(x, 0)$.

- For $n \geq 1$, the nondegenerate k-simplices of $(\partial\Delta^n)^+$ are in bijective correspondence with strings
$$[k] \to ([n] \oplus \{x\} \oplus \{y\}) \times [1]$$
which never hit $(x, 0)$, and contain neither
$$(0,0) \to (1,0) \to \cdots \to (n,0) \qquad \text{nor} \qquad (0,1) \to \cdots \to (n,1) \tag{1.2}$$
as contiguous substrings.

  For $n = 0$, $(\partial\Delta^n)^+ \cong \emptyset^+ \cong \Lambda^2_2$.

Note that $(\partial\Delta^n)_k \cong (\Delta^n)_k$ for $k < n$, because no string of length less than n contain either of the strings in Equation 1.2 as substrings.

Define a simplicial set $\mathcal{C}$ by setting
$$\mathbf{Fun}(K, \mathcal{C}) = \{m \colon K^+ \to \mathcal{C} \mid m|_{(\{x\}\star\{y\})\times\{1\}} = f,\ m|_{\{y\}\times\Delta^1} = g\}.$$

The square of inclusions
$$\begin{tikzcd}
(\Delta^n \star \{y\}) \times \{1\} \arrow[r, hook] \arrow[d, hook] & (\Delta^n \star \{y\}) \times \Delta^1 \arrow[d, hook] \\
(\Delta^n \star \{x\} \star \{y\}) \times \{1\} \arrow[r, hook] & (\Delta^n)^+
\end{tikzcd}$$
gives a commuting square
$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & (\mathcal{O}_{\mathcal{S}})_{/g} \arrow[d] \\
\mathcal{S}_{/f} \arrow[r] & \mathcal{S}_{/Y}
\end{tikzcd}$$

This in turn gives us a map
$$q \colon \mathcal{C} \to (\mathcal{O}_{\mathcal{S}})_{/g} \times_{\mathcal{S}_{/Y}} \mathcal{S}_{/f}.$$

We claim that q is a trivial fibration. To check this, we show that for any $n \geq 0$

we can always find a lift ℓ as below.

$$\begin{tikzcd}
\partial\Delta^n \arrow[r, "a"] \arrow[d, hook] & \mathcal{C} \arrow[d, "q"] \\
\Delta^n \arrow[r, "b"'] \arrow[ur, dashed, "\ell"] & (\mathcal{O}_{\mathcal{S}})_{/g} \times_{\mathcal{S}_{/Y}} \mathcal{S}_{/f}
\end{tikzcd}$$

The case $n = 0$ is somewhat special since $\partial\Delta^n = \emptyset$, but as none of the simplices of $(\Delta^0)^+$ have dimension greater than three one can find a solution simply by drawing pictures. From now on, we fix $n \geq 1$.

We can read the solid diagram as equivalently specifying a map

$$\tilde{a}\colon (\partial\Delta^n)^+ \to \mathcal{S}$$

and a map

$$\tilde{b}\colon (\Delta^n \star \{x\} \star \{y\}) \times \{1\} \coprod_{(\Delta^n \star \{y\}) \times \{1\}} (\Delta^n \star \{y\}) \times \Delta^1 \equiv K_n \to \mathcal{S}$$

which are compatible on their overlap

$$(\partial\Delta^n)^+ \cap K_n \subset (\Delta^n)^+.$$

and which map $(\{x\} \star \{y\}) \times \{1\}$ to f and $\{y\} \times \Delta^1$ to g.

We would like to produce from this data a filler

$$\tilde{\ell}\colon (\Delta^n)^+ \to \mathcal{S}.$$

Let us first see which nondegenerate simplices we are missing.

- The simplicial set $(\Delta^n)^+$ does not contain any nondegenerate simplices of dimension $n + 4$ or higher.
- There are two nondegenerate $(n+3)$-simplices in $(\Delta^n)^+$ missing from $(\partial\Delta^n)^+$:

$$\sigma = \begin{tikzcd}
(0,0) \arrow[d] & & & & & \\
(0,1) \arrow[r] & (1,1) \arrow[r] & \cdots \arrow[r] & (n,1) \arrow[r] & (x,1) \arrow[r] & (y,1)
\end{tikzcd}$$

and

$$\tau = \begin{tikzcd}
(0,0) \arrow[r] & (1,0) \arrow[r] & \cdots \arrow[r] & (n,0) \arrow[d] & & \\
& & & (n,1) \arrow[r] & (x,1) \arrow[r] & (y,1)
\end{tikzcd}.$$

One finds that σ is contained in K_n, but τ is not.

- Apart from the faces of σ and τ, there is only one nondegenerate $(n+2)$-simplex in $(\Delta^n)^+$ not contained in $(\partial\Delta^n)^+$:

$$\chi = \begin{tikzcd} (0,0) \arrow[r] & (1,0) \arrow[r] & \cdots \arrow[r] & (n,0) \arrow[r, bend left] & (y,0) \arrow[d] \\ & & & & (y,1) \end{tikzcd}$$

  However, χ is contained in K_n.

- There are no nondegenerate simplices of $(\Delta^n)^+$ of dimension $n+1$ or lower missing from both $(\partial\Delta^n)^+$ and K_n.

Therefore, the only data we have to fill in is the simplex $\tau \to \mathcal{S}$.

- The faces $d_0\tau, d_1\tau, \ldots, d_n\tau$ are all contained in $(\partial\Delta^n)^+$.
- The face $d_{n+1}\tau$ is missing from both $(\partial\Delta^n)^+$ and K_n.
- The face $d_{n+2}\tau$ is contained in K_n.
- The face $d_{n+3}\tau$ is contained in K_n.

Thus, in order to find our lift ℓ we need only fill $\Lambda^{n+3}_{n+1} \hookrightarrow \Delta^{n+3}$, which we can do because $\mathcal{S}$ is an ∞-category.

2. This follows from the dual to [4, Cor. 2.4.7.12] with $f = \mathrm{id}_{\mathcal{S}}$.

□

1.3.1 Barwick is Kafka for simplicial sets

Lemma 20. Let $\mathcal{C}$ be an ∞-category, and consider a solid diagram

$$\begin{tikzcd} \Delta^n \star \Delta^1 \arrow[r] \arrow[d, hook, "\mathrm{id}\star\{1,2\}"'] & \mathcal{C} \\ \Delta^n \star \Delta^2 \arrow[ur, dashed] \end{tikzcd}$$

such that $\Delta^{\{1,2\}} \hookrightarrow \Delta^2$ is mapped to an equivalence. Then for any $n \geq 0$, we can find a dashed lift.

Proof. Induction. Call the images of the vertices of Δ^2 x, y, and z.

For $n = 0$, we need to fill the following diagram.

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First, fill $0 \to x \to y$. This is $\Lambda^2_2 \hookrightarrow \Delta^2$ with $\Delta^{\{1,2\}}$ an equivalence. Then fill $0 \to x \to$

$y \to z$, which is $\Lambda^3_2 \hookrightarrow \Delta^3$.

TBC $\square$

Lemma 21. Let $\mathcal{C}$ be an infinity category, and let σ be the below square in $\mathcal{C}$ with f an equivalence.

$$\begin{tikzcd} x \arrow[r, "f'"] \arrow[d] & y \arrow[d] \\ y' \arrow[r, "f"] & z \end{tikzcd}$$

Then σ is a pullback square if and only if f' is an equivalence.

Proof. First, suppose that f' is an equivalence. We need to show that $\mathcal{C}_{/\sigma} \hookrightarrow \mathcal{C}_{\sigma|\lambda^2_2}$ is a trivial fibration.

Denote by K the simplicial subset of $\Delta^1 \times \Delta^1$ given by the nerve of the poset

$$\begin{tikzcd} 0 \arrow[r] \arrow[dr] & 1 \arrow[d] \\ 1' \arrow[r] & 2 \end{tikzcd}.$$

We first show that

$$\mathcal{C}_{/\sigma|K} \to \mathcal{C}_{/\sigma|\Lambda^2_2}$$

is a trivial fibration. It is sufficient to be able to solve lifting problems

$$\partial$$

First, suppose that σ is pullback. Consider the following Λ^2_2 horn associated to σ.

$$\begin{tikzcd} & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & x' \end{tikzcd}$$

Let f^{-1} be any homotopy inverse to f, and pick a composite $f^{-1} \circ g$. By our previous work, the square

$$\tau = \begin{tikzcd} y \arrow[r, "\mathrm{id}"] \arrow[d, "f^{-1}\circ g"'] & y \arrow[d, "g"] \\ x' \arrow[r, "f"] & x \end{tikzcd}$$

is a pullback square in $\mathcal{C}$. Any other pullback square

$$\begin{tikzcd}
y \times_x x' \arrow[r, "f'"] \arrow[d] & y \arrow[d] \\
x' \arrow[r] & x
\end{tikzcd}$$

factors through τ via an equivalence g, giving us in particular a 2-simplex

$$\begin{tikzcd}
& x' \times_x y \arrow[dl, "f'"'] \arrow[dr, "g"] & \\
y \arrow[rr, "\mathrm{id}"] & & y
\end{tikzcd}.$$

But by 2/3, this means that f' is an equivalence.

□

Lemma 22. Denote $\mathbf{Fun}(\Delta^1, \mathcal{S})$ by $\mathcal{O}_\mathcal{S}$. The bicartesian fibration $p = \mathrm{ev}_1 \colon \mathcal{O}_\mathcal{S} \to \mathcal{S}$ satisfies the following conditions, which are conditions 12.2.1 and 12.2.2 of Barwick, with all morphisms ingressive and egressive.

1. For any morphism $g \colon X \to Y$ in $\mathcal{S}$ and any object $K \to X$ in $\mathcal{O}_\mathcal{S}$, there exists a morphism

$$\begin{tikzcd}
K \arrow[r] \arrow[d] & S \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

   in $\mathcal{O}_\mathcal{S}$ covering f which is p-coCartesian.

2. Suppose σ is a commutative square in $\mathcal{O}_\mathcal{S}$, corresponding to a commutative cube Σ in $\mathcal{S}$

$$\begin{tikzcd}
K' \arrow[rr, "F'"] \arrow[dr] \arrow[dd] & & S' \arrow[dr] \arrow[dd] & \\
& K \arrow[rr, equal, "F"] \arrow[dd] & & S \arrow[dd] \\
X' \arrow[rr, "f'"] \arrow[dr] & & Y' \arrow[dr] & \\
& X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

   such that the bottom face of Σ is a pullback and F is an equivalence. Then the back face of Σ is p-coCartesian if and only if σ is a pullback in $\mathcal{O}_\mathcal{S}$.

Proof. 1. The map p is a bicartesian fibration.

2. The square σ is a pullback square in $\mathcal{O}_\mathcal{S}$ if and only if both the top and bottom

faces of Σ are pullbacks. The bottom face of Σ is pullback by assumption, and by Lemma 21 the top face of Σ is a pullback if and only if F' is an equivalence. But F' is an equivalence if and only if the back face of Σ is p-coCartesian.

□

1.3.2 Lifting things we need to lift

Everything in this section is in [1].

Notation 23. We will distinguish integers expressed in binary notation with a subscript '2'. For example, $5 = 101_2$. We will often refer to the ith digit of a number N expressed in binary as d_i; in this notation,

$$N = (d_1 \dots d_n)_2.$$

We are trying to find lifts of the following form. The maps f, g, and ℓ should be Segal.

$$\begin{tikzcd}
\mathrm{asd}(\Lambda^n_k) \arrow[r, "f"] \arrow[d, hook] & \mathcal{O}_{\mathcal{S}} \arrow[d] \\
\mathrm{asd}(\Delta^n) \arrow[r, "g"] \arrow[ur, dashed, "\ell"] & \mathcal{S}
\end{tikzcd}$$

For the entirety of this section, fix $n \geq 2$.

As we have seen in Definition 3, we can think of $\mathrm{asd}(\Delta^n)$ as the nerve of the opposite of the poset of subintervals of $[n]$, which we have denoted $(I_n)^{\mathrm{op}}$. The poset $(I_n)^{\mathrm{op}}$ has objects ab, with $0 \leq a \leq b \leq n$, and $ab \leq a'b'$ if $a \leq a' \leq b' \leq b$.

Example 24. The subdivision of the 5-simplex, $\mathrm{asd}(\Delta^5)$, is the nerve of the poset $(I_5)^{\mathrm{op}}$, which can be drawn as follows.

$$\begin{tikzcd}[column sep=tiny]
 & & & & & 05 \arrow[dl] \arrow[dr] & & & & & \\
 & & & & 04 \arrow[dl] \arrow[dr] & & 15 \arrow[dl] \arrow[dr] & & & & \\
 & & & 03 \arrow[dl] \arrow[dr] & & 14 \arrow[dl] \arrow[dr] & & 25 \arrow[dl] \arrow[dr] & & & \\
 & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & 24 \arrow[dl] \arrow[dr] & & 35 \arrow[dl] \arrow[dr] & & \\
 & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & & 34 \arrow[dl] \arrow[dr] & & 45 \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22 & & 33 & & 44 & & 55
\end{tikzcd}$$

The nondegenerate n-simplices in $\mathrm{asd}(\Delta^n)$ are injective maps $[n] \to (I_n)^{\mathrm{op}}$, which we can think of as walks of length n starting at $0n$ and ending at ll for some $0 \leq l \leq n$.

These are in one-to-one correspondence with binary strings of length n, where the 0 or 1 in position i (read from left to right) corresponds to whether one goes left or right at the ith row.

Example 25. The nondegenerate 5-simplices of $\mathrm{asd}(\Delta^5)$ corresponding to $N = 00001_2$ and $N = 10110_2$ are below, in red and blue respectively.

$$\begin{tikzcd}[column sep=tiny]
 & & & & & 05 \arrow[dl] \arrow[dr] & & & & & \\
 & & & & 04 \arrow[dl] \arrow[dr] & & 15 \arrow[dl] \arrow[dr] & & & & \\
 & & & 03 \arrow[dl] \arrow[dr] & & 14 \arrow[dl] \arrow[dr] & & 25 \arrow[dl] \arrow[dr] & & & \\
 & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & 24 \arrow[dl] \arrow[dr] & & 35 \arrow[dl] \arrow[dr] & & \\
 & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & & 34 \arrow[dl] \arrow[dr] & & 45 \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22 & & 33 & & 44 & & 55
\end{tikzcd} \tag{1.3}$$

Binary strings of length n in turn correspond to numbers N with $0 \le N < 2^n$. We will make frequent use of the bijection between numbers $0 \le N < 2^n$ and nondegenerate simplices of $\mathrm{asd}(\Delta^n)$. It will be helpful to have a notation for this bijection.

Notation 26. For any $0 \le N < 2^n$, denote by $\sigma(N)$ the nodegenerate n-simplex of $\mathrm{asd}(\Delta^n)$ corresponding to N. We will denote the ith binary digit of N, read left-to-right, by d_i; that is, $N = (d_1 d_2 \ldots d_n)_2$.

Example 27. The red and blue simplices in Diagram 1.3 are $\sigma(00001_2)$ and $\sigma(10110_2)$ respectively.

To sum up, we may specify a nondegenerate n-simplex in $\mathrm{asd}(\Delta^n)$ by providing any one of the following equivalent data.

- A walk of length n on the poset $(I_n)^{\mathrm{op}}$ of the form

$$0n \to \cdots \to a_i b_i \to \cdots \to ll, \qquad 0 \le l \le n.$$

- A binary string $(d_1 \ldots d_n)_2$ of length n.
- An integer N with $0 \le N < 2^n$.

Armed with this information, our plan of attack is as follows: starting with $\mathrm{asd}(\Lambda^n_k)$, we will construct a filtration

$$\mathrm{asd}(\Lambda^n_k) \subset \mathrm{asd}(\Lambda^n_k) \cup \sigma(0) \subset \cdots \subset \mathrm{asd}(\Lambda^n_k) \cup \bigcup_{0 \le N < 2^n} \sigma(N) = \mathrm{asd}(\Delta^n). \tag{1.4}$$

It will be helpful to have a name for the Nth step of this filtration.

Definition 28. Let $n \geq 1$, and $0 \leq k < n$. Define

$$P_0(k) = \operatorname{asd}(\Lambda^n_k), \qquad P_N(k) = \sigma(N) \cup P_{N-1}(k), \quad 0 \leq N < 2^n.$$

That is,

$$P_N(k) = \operatorname{asd}(\Lambda^n_k) \cup \bigcup_{0 \leq N' < N} \sigma(N'). \tag{1.5}$$

We first need to figure out what each of these inclusions in Equation 1.4 looks like. More precisely, we must compute the intersections $\sigma(N) \cap P_N(k)$, $0 \leq N \leq 2^k - 1$.

From Equation 1.5 and the fact that $\Lambda^n_k = \bigcup_{j \neq k} d_j \Delta^n$, we have

$$\sigma(N) \cap P_N(k) = \overbrace{\left(\bigcup_{0 \leq K < N} \sigma(N) \cap \sigma(K) \right)}^{(a)} \cup \overbrace{\left(\bigcup_{j \neq k} \sigma(N) \cap \operatorname{asd}(d_j \Delta^n) \right)}^{(b)}. \tag{1.6}$$

We can understand (a) and (b) individually. First, let's tackle (a). We need some terminology.

Definition 29. Fix an integer N with $0 \leq N < 2^n$, with N expressed in binary notation as $N = (d_1 \dots d_n)_2$. A ***jut*** of $\sigma(N)$ is an integer $1 \leq j \leq n$ such that one of the following conditions is satisfied:

- For $1 \leq j < n$, j is a jut of $\sigma(N)$ if $d_j = 1$ and $d_{j+1} = 0$.
- For $j = n$, n is a jut of $\sigma(N)$ if $d_n = 1$.

The set of all juts of $\sigma(N)$ is denoted $Z(N)$. By definition, $Z(N) \subset \{1, \dots, n\}$.

Note 30. The juts of $\sigma(N)$ are the places where the walk corresponding to N is going right, and then either heads left or ends. In terms of the binary representation $N = (d_1 \dots d_n)_2$, an integer $j < n$ is a jut of $\sigma(N)$ every time the string '10' appears at the jth position, i.e. if $d_j d_{j+1} = 10$. There is a jut at n every time the last digit of N is a 1.

Example 31. The juts of the simplex $\sigma(01011_2) \subset \operatorname{asd}(\Delta^5)$ are 2 and 5, marked in red below.

$$\begin{tikzcd}
& & 05 \arrow[dl] \\
& 04 \arrow[dr] & \\
& & {\color{red}14} \arrow[dl] \\
& 13 \arrow[dl] & \\
12 \arrow[dr] & & \\
& {\color{red}22} &
\end{tikzcd}$$

Thus, $Z(01011_2) = \{2, 5\}$.

Knowledge of the juts of $\sigma(N)$ allows us to calculate (a).

Lemma 32. The expression (a) in Equation 1.6 can be written

$$\bigcup_{0 \le N' < N} \sigma(N) \cap \sigma(N') = \bigcup_{j \in Z(N)} d_j \sigma(N),$$

where $Z(N) \subset \{1, \ldots, n\}$ is the set of juts of $\sigma(N)$.

Proof. For any jut $j \in Z(N)$, define a number $N^j = (d'_1 \ldots d'_n)_2$ as follows.

- If $j < n$, define d'_i to be
$$d'_i = \begin{cases} d_i, & i \neq j, j+1 \\ 0, & i = j \\ 1, & i = j+1 \end{cases}.$$
- If $j = n$, define d'_i to be
$$d'_i = \begin{cases} d_i, & i \neq n \\ 0, & i = n \end{cases}.$$

The walk corresponding to $\sigma(N^j)$ agrees with the walk corresponding to $\sigma(N)$, except at the jth position, where it goes left instead of right.

For $N = 01011_2$, $\sigma(N)$ is the solid walk below, $\sigma(N^2)$ is the dashed walk, and $\sigma(N^5)$ is the dotted walk.

$$\begin{tikzcd}
 & & & 05 \arrow[dl] \\
 & & 04 \arrow[dl, dashed] \arrow[dr] & \\
 & 03 \arrow[dr, dashed] & & 14 \arrow[dl] \\
 & & 13 \arrow[dl] & \\
 & 12 \arrow[dl, dotted] \arrow[dr] & & \\
11 & & 22 &
\end{tikzcd}$$

Consider the poset of simplicial subsets of $\sigma(N)$ of the form

$$\sigma(N) \cap \sigma(K), \qquad 0 \le K < N.$$

The maximal elements of this poset are intersections $\sigma(N) \cap \sigma(N^j)$, where $j \in Z(N)$. Because $\sigma(N)$ and $\sigma(N^j)$ agree except at the jth vertex, the intersection $\sigma(N) \cap \sigma(N^j)$ is given by $d_j \sigma(N)$.

Thus, we find that

$$\bigcup_{0\leq N'<N} \sigma(N) \cap \sigma(N') = \bigcup_{j\in Z(N)} d_j\sigma(N).$$

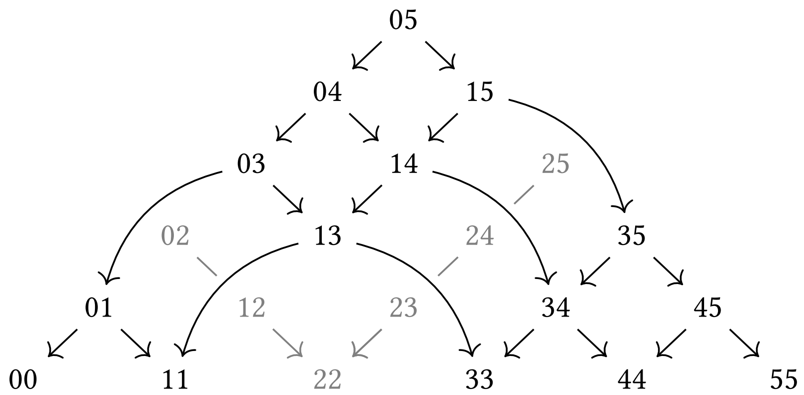
□

Now we turn our attention to (b). First, we should understand $\mathrm{asd}(\Lambda^n_k) = \bigcup_{j\neq k} \mathrm{asd}(d_j\Delta^n)$. To do this we must understand $\mathrm{asd}(d_j\Delta^n)$. This is the nerve of the full subcategory of $(I_n)^{\mathrm{op}}$ on objects ab where $a \neq j$ and $b \neq j$.

We will also pay attention to the full subcategory of $(I_n)^{\mathrm{op}}$ on objects containing js.

Definition 33. Denote the full subcategory of $(I_n)^{\mathrm{op}}$ on objects ab, where either $a = j$ or $b = j$, by V^n_j.

Example 34. The subdivision $\mathrm{asd}(d_2\Delta^5)$ is the nerve of the black diagram below. The category V^5_2 is pictured gray.



The ith face of any simplex $\sigma(N)$ corresponds to the walk on $(I_n)^{\mathrm{op}}$ which skips the ith position of the walk corresponding to $\sigma(N)$. That is, if $\sigma(N)$ corresponds to the walk

$$a_0b_0 \longrightarrow \cdots \longrightarrow a_nb_n,$$

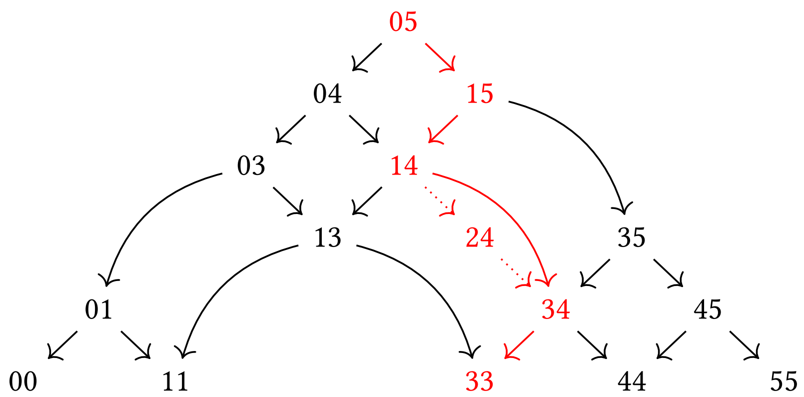
then $d_i\sigma(N)$ corresponds to the walk

$$a_0b_0 \longrightarrow \cdots \longrightarrow a_{i-1}b_{i-1} \longrightarrow a_{i+1}b_{i+1} \longrightarrow \cdots \longrightarrow a_nb_n.$$

Thus, the ith face of $\sigma(N)$ is contained in $\mathrm{asd}(d_j\Delta^n)$ if and only if

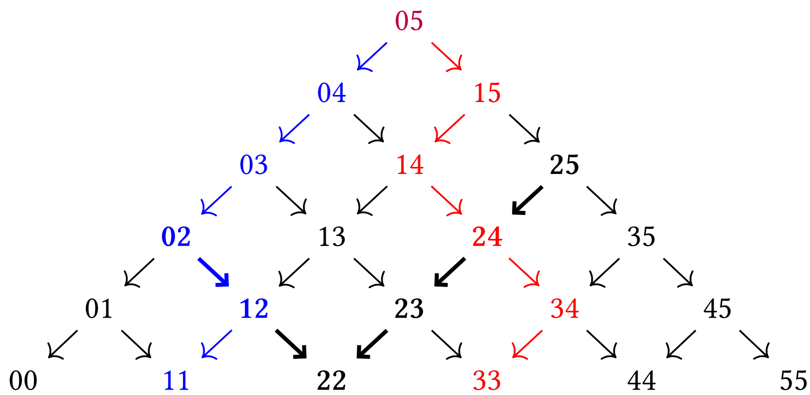
- $a_i = j$ or $b_i = j$ (or both); and
- $a_{i'} \neq j$ and $b_{i'} \neq j$ for $i' \neq i$.

Example 35. The simplex $d_3\sigma(10110_2)$ is contained in $\mathrm{asd}(d_2\Delta^n)$.



The faces of $\sigma(N)$ contained in $\mathrm{asd}(d_j\Delta^n)$ correspond to *crossings of* $\sigma(N)$ *at* j*:* places at which $\sigma(N)$ intersects V_j^n exactly once.

Example 36. The simplex $\sigma(10110_2)$, in red, has a crossing at 2. The simplex $\sigma(00010_2)$, in blue, does not. The subcategory $V_2^5 \subset (I_5)^{\mathrm{op}}$ is in bold.



Definition 37. Let $0 \leq j < n$, and fix some $0 \leq N < 2^n$. Let the binary representation of N be $(d_1 \dots d_n)_2$, and the walk corresponding to $\sigma(N)$ be

$$a_0b_0 \longrightarrow a_1b_1 \longrightarrow \cdots \longrightarrow a_nb_n.$$

An integer $0 \leq i \leq n-1$ is said to be a ***crossing of*** $\sigma(N)$ ***at*** j if one of the following holds:

- For $j = 0$, i is a crossing at j if $d_1 = 1$.
- For $0 < j < n$, i is a crossing at j if $d_i = d_{i+1} = 0$ and $b_i = j$, or if $d_i = d_{i+1} = 1$ and $a_i = j$.

We have already essentially shown the following.

Lemma 38. Let $0 \leq j < n$, and let $0 \leq N < 2^n$. The simplex $\sigma(N)$ has a face in common with $\mathrm{asd}(d_j\Delta^n)$ if and only if $\sigma(N)$ has a crossing at j.

We can now understand the faces of $\sigma(N)$ contained in $\bigcup_{j\neq k}\sigma(N)\cap \mathrm{asd}(d_j\Delta^n)$ as corresponding to *crossings of* $\sigma(N)$ *away from* k*;* that is, crossings at some $j \neq k$. This amounts to the following.

Definition 39. Let $0 \leq k \leq n$, and let $0 \leq N < 2^n$, with binary representation $(d_1 \ldots d_n)_2$ and walk

$$a_0b_0 \to \cdots \to a_nb_n.$$

An integer $0 < i < n$ is said to be a ***crossing of*** $\sigma(N)$ ***away from*** k if any of the following conditions are satisfied.

- Either $d_i = d_{i+1} = 1$ and $a_i \neq k$; or
- $d_i = d_{i+1} = 0$ and $b_i \neq k$.

The integer $i = 0$ is said to be a ***crossing of*** $\sigma(N)$ ***away from*** k if either of the following conditions are satisfied.

- $d_1 = 0$; or
- $d_1 = 1$ and $k \neq 0$.

We will denote the set of crossings of $\sigma(N)$ away from k by $X(N,k)$. By definition, $X(N,k) \subset \{0,\ldots,n-1\}$.

Example 40. Let $N = 10110_2$, and consider the simplex $\sigma(N)$.

$$\begin{tikzcd}[column sep=tiny]
& & & & & 05 \arrow[dl] \arrow[dr] & & & & & \\
& & & & 04 \arrow[dl] \arrow[dr] & & 15 \arrow[dl] \arrow[dr] & & & & \\
& & & 03 \arrow[dl] \arrow[dr] & & 14 \arrow[dl] \arrow[dr] & & 25 \arrow[dl] \arrow[dr] & & & \\
& & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & 24 \arrow[dl] \arrow[dr] & & 35 \arrow[dl] \arrow[dr] & & \\
& 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & & 34 \arrow[dl] \arrow[dr] & & 45 \arrow[dl] \arrow[dr] & \\
00 & & 11 & & 22 & & 33 & & 44 & & 55
\end{tikzcd}$$

The 0th vertex of the simplex $\sigma(N)$ is a crossing at 0, and the 3rd vertex is a crossing at 2. None of the other vertices are crossings. Therefore:

- $X(10110_2, 0) = \{3\}$,
- $X(10110_2, 1) = \{0, 3\}$,
- $X(10110_2, 2) = \{0\}$,
- $X(10110_2, 3) = \{0, 3\}$, and
- $X(10110_2, 4) = \{0, 3\}$.

We are now ready to understand the intersection $\sigma(N) \cap P_N(k)$.

Proposition 41. Let $0 \le k < N$, and let $0 \le N < 2^n$. The intersection $\sigma(N) \cap P_N(k)$ is given by the union of faces

$$\bigcup_{i \in Z(N) \cup X(N,k)} d_i \sigma(N).$$

Proof. Denote the walk corresponding to N by

$$a_0 b_0 \to \cdots \to a_n b_n.$$

Recall from Equation 1.6 the decomposition $\sigma(N) \cap P_N(k) = (a) \cup (b)$. We have seen in Lemma 32 that (a) is given by $\bigcup_{i \in Z(n)} d_i \sigma(N)$, and in Lemma 38 that the faces of $\sigma(N)$ contained in (b) are of the form $d_i \sigma(N)$, where $i \in X(N,k)$. Therefore, we will be done if we can show that (b) contains only faces, apart from lower-dimensional simplices which are also contained in (a). More explicitly, we need to show that any subsimplex of $\sigma(N)$ of codimension 2 or greater which

- is contained in (b), and
- is not contained in a higher-dimesional subsimplex of $\sigma(N)$ which is contained in (b),

must be contained in (a).

For any subsimplex τ of $\sigma(N)$ meeting the conditions above, there must exist a $j \neq k$ such that $\sigma(N)$ intersects V_j^n more than once; the subsimplex τ then has as its vertices the vertices s of $\sigma(N)$ such that $a_s \neq j$ and $b_s \neq j$. If τ has dimension m, then $\sigma(N)$ and V_j^n must intersect exactly m times. If a walk on $(I_n)^{\mathrm{op}}$ leaves V_j^n it cannot come back, so for any simplex which intersects V_j^n in $m < 1$ places, the intersections are contiguous; that is, there exists some i such that for each i' with $i \le i' < i + m$, either $a_{i'} = j$ or $b_{i'} = j$. It is easy to convince oneself that in this situation either i or $i + m$ must be a jut of $\sigma(N)$.[1] Without loss of generality, suppose i is a jut of $\sigma(N)$. Then $\tau \subset \sigma(N^i) \cap \sigma(N)$, which is contained in (a). □

Notation 42. Let T be a finite totally ordered set, and let $S \subset T$ be a proper subset. Define

$$\Lambda_S^T = \bigcup_{i \notin S} d_i \Delta^n.$$

Take $T = [n]$. We will write Λ_S^n instead of $\Lambda_S^{[n]}$. Interpreting $S \subset T$ as a set of vertices of Δ^n, Λ_S^n is the union of all faces of Δ^n which contain every vertex in S. This is because $d_i \Delta^n$ contains S if and only if $S \subseteq [n] \setminus \{i\}$, which in turn is true if and only if $i \notin S$.

Example 43. For $T = [m]$ and $S = \{i\}$ a singleton, $\Lambda_S^T = \Lambda_i^m$, the ith horn of Δ^n.

[1] Apart from the trivial case of $\sigma(0)$'s intersection with V_0^n.

Definition 44. Fix some $0 \le k < n$, and let $0 \le N < 2^n$. The ***exceptional vertices*** of $\sigma(N)$ are the subset

$$E(N,k) = [m] \setminus (Z(N) \cup X(N,k)).$$

To put it another way, a vertex of $\sigma(N)$ is exceptional if it is neither a jut nor a crossing away from k. In our new notation, Proposition 41 tells us that $\sigma(N) \cap P_N(k) = \Lambda^n_{E(N,k)}$. We are computing the intersections $\sigma(N) \cap P_N(k)$ in the hope that we can show that $P_N(k) \hookrightarrow P_{N+1}(k)$ is inner anodyne. We now have some idea when this might be so. We have a pushout square

$$\begin{tikzcd} \Lambda^n_{E(N,k)} \arrow[r, hook] \arrow[d, hook] & \Delta^n \arrow[d, hook] \\ P_N(k) \arrow[r, hook] & P_{N+1}(k) \end{tikzcd} .$$

If $\Lambda^n_{E(N,k)} \hookrightarrow \Delta^n$ is inner anodyne, then $P_N(k) \hookrightarrow P_{N+1}(k)$ will be as well. Our next goal will be to establish a sufficient condition for $\Lambda^n_{E(N,k)} \hookrightarrow \Delta^n$ to be inner anodyne.

Lemma 45. Let T be a finite, totally ordered set, and $S \subset T$ a nonempty proper subset. A sufficient condition for $\Lambda^T_S \hookrightarrow \Delta^T$ to be inner anodyne as follows:

($*$) There exist elements $a < s < b$ in T with $s \in S$ and $a, b \notin S$.

It will be useful to denote the elements of S by $\star$, and the elements of $T \setminus S$ by $\circ$. We can then draw the inclusion $S \subset T$ as a word on the letters $\star$ and $\circ$. Then Lemma 45 tells us that $\Lambda^T_S \hookrightarrow \Delta^T$ is inner anodyne if our drawing of $S \subset T$ is of the form

$$\cdots \circ \cdots \star \cdots \circ \cdots .$$

Example 46. With $T = [4]$ and $S = \{0, 2\}$, we can draw $S \subset T$ as

$$\star \circ \star \circ \circ.$$

Lemma 45 guarantees us that the inclusion $\Lambda^4_{\{0,2\}} \hookrightarrow \Delta^4$ is inner anodyne.

Definition 47. We will call a simplex $\sigma(N)$ ***exceptional*** if $E(N,k) \subset [n]$ does not satisfy Condition ($*$).

In Barwick's words, exceptional simplices are indeed exceptional. Any exceptional simplex cannot have more than one jut, since between any two juts there is a vertex which is neither a jut nor a crossing away from k.

$$\begin{tikzcd}
 & \ddots \arrow[dr] & \\
 & & \circ \arrow[dl] \\
 & \iddots \arrow[dl] & \\
\star \arrow[dr] & & \\
 & \ddots \arrow[dr] & \\
 & & \circ \arrow[dl] \\
 & \iddots &
\end{tikzcd}$$

As follows easily from Note 30, the only N such that $\sigma(N)$ contains at most one jut are of the form

- $N_r = \overbrace{1\cdots1}^{r}\overbrace{0\cdots0}^{n-r}$, for $0 \le r \le n$; or
- $N'_r = \overbrace{0\cdots0}^{r}\overbrace{1\cdots1}^{n-r}$, for $0 < r < n$.

In fact, we can easily see that all of the simplices $\sigma(N'_r)$ are not exceptional, since for any k and r, 0 is a crossing away from k of $\sigma(N'_r)$, n is a jut, and r is an exceptional vertex. This situation is pictured in Example 48.

Example 48. For $n = 5$, $\sigma(N'_3)$ is pictured below. Exceptional vertices are marked with a '$\star$' and juts and crossings away from k are marked with a '$\circ$'. The vertices marked '?' could be either exceptional or crossings away from k depending on k. However, the information we have is already enough to tell us that $\sigma(N'_3)$ is not exceptional.

$$\begin{tikzcd}
 & & & \circ \arrow[dl] \\
 & & ? \arrow[dl] & \\
 & ? \arrow[dl] & & \\
\star \arrow[dr] & & & \\
 & ? \arrow[dr] & & \\
 & & \circ &
\end{tikzcd}$$

Lemma 49. For $k = 0$, each simplex $\sigma(N_r)$ is exceptional, and we have

- $E(0, N_0) = \{n\}$;
- $E(0, N_r) = \{0, n\}$, for $0 \leq r \leq n$; and
- $E(0, N_n) = \{0\}$.

For $0 < k < n$, the simplices N_{k-1} and N_k are exceptional, and we have

- $E(k, N_{k-1}) = \{n-1, n\}$; and
- $E(k, N_k) = \{n\}$.

We do not prove this statement explicitly, but the example below should make clear why it is true.

Example 50. Let $n = 5$ and $k = 0$. The simplices $\sigma(N_0)$, $\sigma(N_2)$, and $\sigma(N_5)$ are pictured below.

$$\sigma(N_0) = \begin{tikzcd}
\star \arrow[dr] & & & & & \\
& \circ \arrow[dr] & & & & \\
& & \circ \arrow[dr] & & & \\
& & & \circ \arrow[dr] & & \\
& & & & \circ \arrow[dr] & \\
& & & & & \circ
\end{tikzcd}
\qquad
\sigma(N_2) = \begin{tikzcd}
& \star \arrow[dr] & & \\
& & \circ \arrow[dr] & \\
& & & \circ \arrow[dl] \\
& & \circ \arrow[dl] & \\
& \circ \arrow[dl] & & \\
\star & & &
\end{tikzcd}$$

$$\sigma(N_n) = \begin{tikzcd}
& & & & & \circ \arrow[dl] \\
& & & & \circ \arrow[dl] & \\
& & & \circ \arrow[dl] & & \\
& & \circ \arrow[dl] & & & \\
& \circ \arrow[dl] & & & & \\
\star & & & & &
\end{tikzcd}$$

For $k = 3$, $\sigma(N_1)$, $\sigma(N_2)$, $\sigma(N_3)$, and $\sigma(N_4)$ are pictured below.

$$\sigma(N_1) = \begin{tikzcd}[row sep=small, column sep=small]
 & & & \circ \arrow[dr] & \\
 & & & & \circ \arrow[dl] \\
 & & & \circ \arrow[dl] & \\
 & & \star \arrow[dl] & & \\
 & \circ \arrow[dl] & & & \\
 \star & & & &
\end{tikzcd}
\qquad
\sigma(N_2) = \begin{tikzcd}[row sep=small, column sep=small]
 & \circ \arrow[dr] & & \\
 & & \circ \arrow[dr] & \\
 & & & \circ \arrow[dl] \\
 & & \circ \arrow[dl] & \\
 & \star \arrow[dl] & & \\
 \star & & &
\end{tikzcd}$$

$$\sigma(N_3) = \begin{tikzcd}[row sep=small, column sep=small]
 \circ \arrow[dr] & & & \\
 & \circ \arrow[dr] & & \\
 & & \circ \arrow[dr] & \\
 & & & \circ \arrow[dl] \\
 & & \circ \arrow[dl] & \\
 & \star & &
\end{tikzcd}
\qquad
\sigma(N_4) = \begin{tikzcd}[row sep=small, column sep=small]
 \circ \arrow[dr] & & & & \\
 & \circ \arrow[dr] & & & \\
 & & \circ \arrow[dr] & & \\
 & & & \star \arrow[dr] & \\
 & & & & \circ \arrow[dl] \\
 & & & \star &
\end{tikzcd}$$

Recall our goal: we are trying to find Segal lifts ℓ as below, where f and g are Segal.

$$\begin{tikzcd}
\mathrm{asd}(\Lambda^n_k) \arrow[r, "f"] \arrow[d, hook] & \mathcal{O}_{\mathcal{S}} \arrow[d] \\
\mathrm{asd}(\Delta^n) \arrow[r, "g"] \arrow[ur, dashed, "\ell"] & \mathcal{S}
\end{tikzcd} \tag{1.7}$$

It turns out that we needn't worry ourselves that a lift might not be Segal; in most situations, any lift we can construct will be Segal automatically. Morally, this is because we are taking a limit over a very simple category, and demanding that $\mathrm{asd}(\Delta^n) \to \mathcal{C}$ be Segal is equivalent to demanding that each square

Lemma 51. Let $n \geq 3$, $k \neq 0$, and let ℓ be any lift as in Diagram 1.7. Then ℓ is automatically Segal.

Proof. A lift $\ell \colon \mathrm{asd}(\Delta^n) \to \mathcal{O}_{\mathcal{S}}$ is Segal if and only if the image of each of the squares

$$\begin{tikzcd}[column sep=small]
 & ij \arrow[dl] \arrow[dr] & \\
 i(j-1) \arrow[dr] & & (i+1)j \arrow[dl] \\
 & (i+1)(j-1) &
\end{tikzcd}$$

is pullback, for $0 \leq i < n$, $0 < j \leq n$.

The image of each square of the above form except the top square ($i = 0$, $j = n$) is contained in at least one of $\mathrm{asd}(d_0\Delta^n)$ and $\mathrm{asd}(d_n\Delta^n)$, and is hence pullback by assumption. Thus, we need only verify that either the top square is already in some $\mathrm{asd}(d_j\Delta^n)$, and hence is pullback by assumption, or that any way of filling the top square is guaranteed to be pullback. We have fixed some k, so either $k \neq 1$ or $k \neq n-1$. If $k \neq n-1$, then the outer square of

$$\begin{tikzcd}[column sep=small]
 & & 0n \arrow[dl] \arrow[dr] & \\
 & 0(n-1) \arrow[dl] \arrow[dr] & & 1n \arrow[dl] \\
0(n-1) \arrow[dr] & & 1(n-1) \arrow[dl] & \\
 & 1(n-2) & &
\end{tikzcd}$$

is contained in $\mathrm{asd}(d_{n-1}\Delta^n) \subset \mathrm{asd}(\Lambda^n_k)$, and is hence pullback, and the lower-left square is contained in $\mathrm{asd}(d_n(\Delta^n)) \subset \mathrm{asd}(\Lambda^n_k)$, and hence is also pullback. Thus, the pasting lemma implies that any way of filling in the top square will be pullback. Similar reasoning works in the case $k \neq 1$. □

1.3.3 Lifting things we need to lift the right way

Warm up:

Some classical knowledge: There is a Quillen equivalence

$$p_1^* : \mathbf{Set}_\Delta^{\mathrm{Joyal}} \longleftrightarrow \mathbf{Set}_{\Delta^2}^{\mathrm{CSS}} : i_1^*.$$

This means that the counit

$$p_1^* i_1^* \mathcal{C} \to \mathcal{C}$$

is a weak equivalence when evaluated on a fibrant object.

Definition 52. We say that a map $\mathcal{C} \to \mathcal{D}$ of complete Segal spaces is a ***inner fibration*** if the following square is homotopy pullback in the Kan model structure.

$$\begin{tikzcd}
\mathcal{C}_2 \arrow[r] \arrow[d] & \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \arrow[d] \\
\mathcal{D}_2 \arrow[r] & \mathcal{D}_1 \times_{\mathcal{D}_0} \mathcal{D}_1
\end{tikzcd}$$

Proposition 53. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories with pullbacks, and let $\pi\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration admitting relative pullbacks. Then $\mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is an inner fibration of complete Segal spaces.

Proof. The square □

Definition 54. Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a map of complete Segal spaces. We say that a morphism $f\colon x \to y$ in $\mathcal{C}$ is ***p*-coCartesian** if the following square is homotopy pullback.

$$\begin{tikzcd}
\mathcal{C}_2 \times_{\mathcal{C}_{\{0,1\}}} \{f\} \arrow[r] \arrow[d] & \mathcal{C}_1 \times_{\mathcal{C}_{\{0\}}} \{x\} \arrow[d] \\
\mathcal{D}_2 \times_{\mathcal{D}_{\{0,1\}}} \{\pi f\} \arrow[r] & \mathcal{D}_1 \times_{\mathcal{D}_{\{0\}}} \{\pi x\}
\end{tikzcd}$$

Proposition 55. Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration admitting relative pullbacks with the following property:

For any square

$$\sigma = \begin{tikzcd}
x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\
x' \arrow[r, "f'"] & y'
\end{tikzcd}$$

in $\mathcal{C}$, such that f' is p-Cartesian and $\pi(\sigma)$ is pullback in $\mathcal{D}$, the following are equivalent.

- The morphism f is π-Cartesian.
- The square σ is pullback.

Definition 56. Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a Reedy fibration of complete segal spaces.

Proposition 57. Let $\mathcal{C} \to \mathcal{D}$ be an inner fibration admitting relative pullbacks. Then $\mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is an inner fibration.