1 Introduction

1.1 Description

1.1.1 Categories of correspondences

- $\mathcal{C} \rightsquigarrow \mathbf{Corr}(\mathcal{C}) = \mathbf{Span}(\mathcal{C})$

$$\begin{tikzcd}
 & & y \times_{x'} y' \arrow[dl] \arrow[dr] & & \\
 & y \arrow[dl] \arrow[dr] & & y' \arrow[dl] \arrow[dr] & \\
x & & x' & & x''
\end{tikzcd}$$

- $\mathcal{C}^{\otimes} \rightsquigarrow \mathbf{Corr}(\mathcal{C})^{\otimes}$ (monoidal structure)
- $\mathcal{C} \rightsquigarrow \mathbf{Corr}_2(\mathcal{C}), \mathbf{Corr}_n(\mathcal{C})^{\otimes}$ (where perhaps $\otimes$ is a symmetric monoidal n-category)

$$\begin{tikzcd}
 & y \arrow[dl] \arrow[dr] & \\
x & z \arrow[l] \arrow[r] \arrow[u] \arrow[d] & x' \\
 & y' \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

TFT: $M \mapsto X^M$

$$\begin{tikzcd}
 & M & & & & X^M & \\
\partial^+ M \arrow[ur, hook] & & \partial^- M \arrow[ul, hook'] & \mapsto & X^{\partial^+ M} \arrow[ur, hook] & & X^{\partial^- M} \arrow[ul, hook']
\end{tikzcd}$$

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$$\mathbf{Bord}_n^{\otimes} \to \mathbf{Corr}_n(\mathcal{C})^{\otimes} \overset{?}{\dashrightarrow} \mathbf{Vect}^{\otimes}$$

1.1.2 Transfer theories

Let $F\colon \mathcal{C} \to \mathcal{D}$, and let $x \in \mathcal{C}$.

$$\begin{tikzcd} & y \arrow[dl, "p"'] \arrow[dr, "q"] & \\ x & & x' \end{tikzcd}$$

1.2 Motivation

Let X and Y be sets, say representing the physical states of some physical system. For example, imagine a photon sent out by some emitter, passing through a screen with holes, and then being detected. The set X might be

$$\{\text{left hole, middle hole, right hole}\},$$

and the set Y might be

$$\{\text{detected at } A, \text{detected at } B\}.$$

Consider a set T, equipped with maps $s\colon T \to X$ and $t\colon T \to Y$.

$$\begin{tikzcd} & T \arrow[dl, "s"'] \arrow[dr, "t"] & \\ X & & Y \end{tikzcd}$$

We can think of T as consisting of a set of ways of going from an element of X to an element of Y; that is, (using the universal property for products) the fiber $T_y^x := (s,t)^{-1}(x,y)$ can be thought of as the set of ways of going from $x \in X$ to $y \in Y$. In our example, the fiber T_y^x would correspond to the set of classical trajectories starting at hole x and ending at detector y.

Consider concatenating two such processes.

$$\begin{tikzcd} & T \arrow[dl, "s"'] \arrow[dr, "t"] & & T' \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\ X & & Y & & Z \end{tikzcd}$$

Taking the pullback in **Set** gives us a set of ways of going from X to Z.

$$
\begin{tikzcd}
 & & S \arrow[dl, "\sigma"'] \arrow[dr, "\tau"] & & \\
 & T \arrow[dl, "s"'] \arrow[dr, "t"] & & T' \arrow[dl, "s'"'] \arrow[dr, "t'"] & \\
X & & Y & & Z
\end{tikzcd}
$$

That is, the fibers S_z^x consist of pairs (t, t'), where t' is a process from x to y and t' is a process from y to z.

For a set X, denote by $L(X)$ the free $\mathbb{C}$-vector space on X. Consider a span $T\colon X \to Y$ as before. We can assign to T a linear map

$$L(T)\colon L(X) \to L(Y)$$