1 Categorical stuff

1.1 Kapranov-Voevodsky 2-vector spaces

In this section, we will denote by **Vect** the category $\mathbf{FinVect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces.

Definition 1 (Kapranov-Voevodsky 2-vector space)**.** A $\mathbb{C}$*-linear additive category* is a category $\mathcal{V}$ enriched in $(\mathbf{Vect}, \otimes)$ with finite biproducts (including zero objects). We call such biproducts *direct sums.*

An object $x \in \mathcal{V}$ is said to be *simple* if $\mathrm{Hom}(x, x) \cong \mathbb{C}$. An object is *semisimple* if it can be written as a finite direct sum simple objects. We say that $\mathcal{V}$ is *semisimple* if all of its objects are semisimple.

A Kapranov-Voevodsky 2-vector space is a semisimple $\mathbb{C}$-linear additive category. A morphism of 2-vector spaces is a functor such that the maps $\mathrm{Hom}(x, y) \to \mathrm{Hom}(Fx, Fy)$ are linear.

Note that all 2-vector spaces are abelian, and all morphisms between 2-vector spaces are additive.

Example 2. Consider the category

1.2 Joyal model structure

Definition 3 (Joyal model structure)**.** There is a model structure on $\mathbf{Set}_{\Delta}$ with the following classes of morphisms.

- The cofibrations Cof are monomorphisms, i.e. level-wise injections.
- The weak equivalences W are *weak categorical equivalences*, i.e. morphisms $u\colon A \to B$ of simplicial sets such that for all quasi-categories X the induced map on internal homs
$$u^*\colon X^B \to X^A$$
induces an isomorphism…

- The fibrations Fib are isofibrations, i.e. morphisms with the right lifting property with respect to inner horn inclusions and the morphism

$$N(\star \to (\bullet \to \star)).$$

1.3 Kan extensions

1.4 Segal spaces

Definition 4 (Reedy model category). The category of simplicial objects in $\mathbf{Set}_\Delta$ admits the following model structure, called the Reedy model structure.

- Weak equivalences are degree-wise weak equivalences.
- Fibrations

Definition 5 (category object). Let $\mathcal{C}$ be an ∞-category with finite limits. A category object in $\mathcal{C}$ is a simplicial object

$$C_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{C}$$

such that the maps

$$C_n \longrightarrow C_1 \times_{C_0} \cdots \times_{C_0} C_1 \tag{1.1}$$

are equivalences[1] for all n.

The condition Equation 1.1 is known as the *Segal condition.*

Example 6. A category object in $N(\mathbf{Set})$ is the same thing as (the nerve of) a category; the segal condition can be interpreted as telling us that an n-simplex is completely determined by its spine.

The category of segal objects in **Set**, however, is not exactly the 2-category **Cat**: isomorphisms in the category of segal objects in **Set** are isomorphisms of categories, rather than equivalences. We need to pass to an ∞-setting.

Definition 7 (segal space). A Segal space is a category object in the ∞-category $\mathcal{S}$ of spaces. The category of Segal spaces, denoted $\mathbf{Seg}(\mathcal{S})$, is the full subcategory of simplicial objects in $\mathcal{S}$ satsifying the Segal condition.

Here we think of a category with a *space,* rather than a *set,* of operations.

Example 8. Any category, i.e. any category object in **Set**, can be thought of as a segal space by applying the functor $\mathbf{Set} \hookrightarrow \mathcal{S}$. In this case, the spaces, of morphisms are thought of as discrete sets of points.

[1] That is, descend to isomorphisms in the homotopy category $h\mathcal{C}$.

Since E^n is a category (hence a category object in **Set**), we can think of it as a Segal space using the functor $\mathbf{Set} \hookrightarrow \mathcal{S}$.

Definition 9 (classifying space of equivalences)**.** Let E^n be the contractible groupoid with n objects and a unique morphism between each pair of objects, and let K be a Segal space. The classifying space of equivalences of K is the simplicial set

$$\iota K := \operatorname{colim} \iota_\bullet K, \qquad \iota_n K = \operatorname{Map}_{\mathbf{Seg}(\mathcal{S})}(E^n, K).$$

Example 10. Let K be an ordinary category. The zero-simplices of ιK are all maps

$$F \colon E^n \to K$$

for some $n \geq 0$. That is, they are clusters of n isomorphic objects in K. The 1-simplices are maps

$$E^n \times \mathbb{\Delta}^1 \to K,$$

i.e. pairs of clusters of isomorphic objects connected by a morphism.

Definition 11 (fully faithful, essentially surjective)**.** We say that a morphism $f : X \to Y$ of Segal spaces is fully faithful and essentially surjective if

1. The map $\iota X \to \iota Y$ is an equivalence of spaces
2. The diagram
$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & Y_1 \arrow[d] \\ X_0 \times X_0 \arrow[r] & Y_0 \times Y_0 \end{tikzcd}$$
is a pullback square.

Definition 12 (inert morphism)**.** Let $\phi \colon [m] \to [n]$ be a morphism in $\mathbb{\Delta}$. We say that ϕ is intert if it is the inclusion of a subinterval, that is, if

$$\phi(i) = \phi(0) + i \qquad \text{for all } i.$$

We denote the subcategory of $\mathbb{\Delta}$ on inert morphisms by $\mathbb{\Delta}_{\text{int}}$.

Let $\mathbf{Cell}^1$ denote the full subcategory of $\mathbb{\Delta}_{\text{int}}$, i.e. the category

$$[0] \rightrightarrows [1] \ .$$

The objects are inert morphisms

$$[0] \to [n], \qquad \text{and} \qquad [1] \to [n],$$

and the only non-identity morphisms are commuting triangles

$$\begin{tikzcd} {[0]} \arrow[rr] \arrow[dr] & & {[1]} \arrow[dl] \\ & {[n]} & \end{tikzcd} .$$

We can draw this category as follows.

$$\begin{tikzcd} \{0\} \arrow[dr] & & \{1\} \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & \{n\} \arrow[dl] \\ & \{0,1\} & & \{1,2\} & & \{n-1,n\} & \end{tikzcd}$$