1 Segal spaces of spans

1.1 Basic definitions

We recall the basic definitions of Segal spaces of spans. Note that spans as we will define them form not only a Segal space, but a *complete* Segal space; this will not concern us. For more information, we direct the reader to [1]. This section is intended to be a summary of the results found there which we will need.

The objects of our study will be ∞-categories whose morphisms are spans

$$\begin{tikzcd} & y \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ x & & x' \end{tikzcd}$$

in some quasicategory $\mathcal{C}$ (this means that x, x', and y are objects in $\mathcal{C}$, and that ϕ and ψ are morphisms in $\mathcal{C}$). We will want to be able to place certain conditions on the legs ϕ and ψ of our spans. For example, we may want to restrict our attention to spans such that ϕ is an equivalence, or has contractible fibers. More precisely, we pick out two subcategories of $\mathcal{C}$ to which the respective legs of our spans must belong.

Definition 1.1.1. A ***triple*** of categories is a triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, where $\mathcal{C}$ is a quasicategory where $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$ are subcategories (in the sense of [2, Sec. 1.2.11]), each of which contain all equivalences.

We will use the following terminology and notation for the morphsism in our subcategories.

- We denote the morphisms in $\mathcal{C}_\dagger$ with tails (as in $x \rightarrowtail y$), and call them *ingressive.*
- We denote the morphisms in $\mathcal{C}^\dagger$ with two heads (as in $x \twoheadrightarrow y$), and call them *egressive.*

In order for a triple of categories to be able to support a category of spans, it will have to satisfy certain properties. We will call such categories *adequate.*

Definition 1.1.2. A triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is said to be ***adequate*** if it has the following properties:

1. For any morphism $y \rightarrowtail x \in \mathcal{C}_\dagger$ and any morphism $x' \twoheadrightarrow x \in \mathcal{C}^\dagger$, there exists a pullback square

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d, two heads] \\ y \arrow[r, tail] & x \end{tikzcd} .$$

2. For any pullback square

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d] & x' \arrow[d] \\ y \arrow[r, "f"] & x \end{tikzcd},$$

if the arrow f belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$), then the arrow f' belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$).

We will call a square of the form

$$\begin{tikzcd} y' \arrow[r, tail] \arrow[d, two heads] & x' \arrow[d, two heads] \\ y \arrow[r, tail] & x \end{tikzcd}$$

ambigressive. If such an ambigressive square is also a pullback square, we will call it *ambigressive pullback.*

It is now time to set about building our categories of spans. We give only a sketch. For each $n \geq 0$, we define a poset Σ_n as follows:

- The elements of Σ_n are pairs of integers (i, j), with $0 \leq i \leq j \leq n$.
- We define a partial order via

$$(i, j) \leq (i', j') \iff i \leq i' \leq j' \leq j.$$

Example 1.1.3. We can draw Σ_4 as follows.

$$\begin{tikzcd}[column sep=tiny] &&&& (0,4) \arrow[dl] \arrow[dr] &&&& \\ &&& (0,3) \arrow[dl] \arrow[dr] && (1,4) \arrow[dl] \arrow[dr] &&& \\ && (0,2) \arrow[dl] \arrow[dr] && (1,3) \arrow[dl] \arrow[dr] && (2,4) \arrow[dl] \arrow[dr] && \\ & (0,1) \arrow[dl] \arrow[dr] && (1,2) \arrow[dl] \arrow[dr] && (2,3) \arrow[dl] \arrow[dr] && (3,4) \arrow[dl] \arrow[dr] & \\ (0,0) && (1,1) && (2,2) && (3,3) && (4,4) \end{tikzcd}$$

These Σ_n assemble into a cosimplicial object

$$\Delta \to \mathcal{S}\mathrm{et}_\Delta; \qquad [n] \mapsto N(\Sigma_n),$$

Left Kan extending yields a functor $\mathrm{sd}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$. General abstract nonsense gives us right adjoint $\mathrm{Span}'\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ forming an adjunction

$$\mathrm{sd} : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta : \mathrm{Span}'.$$

For any simplicial set A, the simplicial set $\mathrm{Span}'(A)$ has n-simplices

$$\mathrm{Span}'(A)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\mathrm{sd}(\Delta^n), A).$$

Fix some triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$. We will call a functor $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is called *ambigressive Cartesian* if the image of each square in $\mathrm{sd}(\Delta^n) = N(\Sigma_n)$ of the form

$$\begin{tikzcd} & (i,j) \arrow[dl] \arrow[dr] & \\ (i, j-\ell) \arrow[dr] & & (i+k, j) \arrow[dl] \\ & (i+k, j-\ell) & \end{tikzcd}$$

(where we include the possibilities $k = 0$ and $\ell = 0$) is an ambigressive pullback. The quasicategory $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is then defined level-wise to be the subset

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \subseteq \mathrm{Span}'(\mathcal{C})$$

on functors $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ which are ambigressive Cartesian.

Analogously, for an adequate triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ and any simplicial set A, define

$$\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(A), \mathcal{C})^{\simeq} \subseteq \mathrm{Map}(\mathrm{sd}(A), \mathcal{C})^{\simeq}$$

to be the full simplicial subset on functors $\mathrm{sd}(A)$ such that for each standard simplex σ of A, the image of $\mathrm{sd}(\sigma)$ in $\mathcal{C}$ is ambigressive Cartesian. The (complete) Segal space of spans $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is defined level-wise by

$$\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)_n = \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq}.$$

For a proof that this really is a Segal space, the reader is once again referred to [1].

Example 1.1.4. For any quasicategory, we can define a triple $(\mathcal{C}, \mathcal{C}_\dagger = \mathcal{C}, \mathcal{C}^\dagger = \mathcal{C}^{\simeq})$. The corresponding spans are of the form

$$\begin{tikzcd} & y \arrow[dl, "\simeq"'] \arrow[dr] & \\ x & & x' \end{tikzcd},$$

where $Y \to X$ is an equivalence in $\mathcal{C}$.

The resulting quasicategory of spans is categorically equivalent to the category $\mathcal{C}$ from which we started. To see this, note that for each $n \geq 0$ there is a retraction of posets

$$[n] \hookrightarrow \Sigma_n \twoheadrightarrow [n],$$

where the first map takes $i \mapsto (i,n)$, and the second takes $(i,j) \mapsto i$. Under the nerve, this corresponds to a retraction of simplicial sets

$$\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n) \twoheadrightarrow \Delta^n.$$

Pulling back along these maps gives us, for each n, a retraction

$$\mathrm{Fun}(\Delta^n, \mathcal{C}) \xhookrightarrow{i_n} \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C}) \xhookrightarrow{r_n} \mathrm{Fun}(\Delta^n, \mathcal{C}) \ .$$

Note that the image of the inclusion i

It is not hard to convince oneself that a map $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is ambigressive cartesian if and only if it is a right Kan extension of its restriction along $\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n)$. In the case $n = 2$, the inclusion $[2] \hookrightarrow \Sigma_2$ is of the form

$$\begin{tikzcd}
0 \arrow[dr] & & & & & & (0,2) \arrow[dl] \arrow[dr] & & \\
& 1 \arrow[dr] & & \hookrightarrow & & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & \\
& & 2 & & (0,0) & & (1,1) & & (2,2)
\end{tikzcd} \ .$$

The right Kan extension of a diagram $\Delta^2 \to \mathcal{C}$

The inclusions $\Delta^n \to \mathrm{sd}(\Delta^n)$ does not yield a map of cosimplicial objects $\Delta^\bullet \to \mathrm{sd}(\Delta^\bullet)$, but the projections $\mathrm{sd}(\Delta^n) \to \Delta^n$ do, giving us a map of cosimplicial objects $\mathrm{sd}(\Delta^\bullet) \to \Delta^\bullet$.

There is an equivalence of Segal spaces

$$\Gamma(\mathcal{C}) \to \mathbf{Span}(\mathcal{C}, \mathcal{C}, \mathcal{C}^{\simeq}),$$

where $\Gamma(\mathcal{C})$ is the Segal space defined level-wise by

$$\Gamma(\mathcal{C})_n = \mathrm{Fun}(\Delta^n, \mathcal{C}).$$

This sends a morphism $f\colon x \to y$ in $\mathcal{C}$ to the span

$$\begin{tikzcd}
& x \arrow[dl, equal] \arrow[dr] & \\
x & & x'
\end{tikzcd} ,$$

A functor between adequate triples $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ is a functor

$$p\colon \mathcal{C} \to \mathcal{D}$$

such that $p(\mathcal{C}_\dagger) \subseteq \mathcal{D}_\dagger$, $p(\mathcal{C}^\dagger) \subseteq \mathcal{D}^\dagger$, and p preserves ambigressive pullbacks. We will denote the restriction of p to $\mathcal{C}_\dagger$ by

$$p_\dagger \colon \mathcal{C}_\dagger \to \mathcal{D}_\dagger,$$

and similarly for $p^\dagger$. Clearly, a functor p between adequate triples gives functors $\mathrm{Span}(p)$ and $\mathbf{Span}(p)$ respectively between quasicategories and Segal spaces of spans.

The goal of the remainder of this document is to provide a proof of the following theorem. This is [1, Thm. 12.2].

Theorem 1.1.5. Let $p \colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of adequate triples. Further suppose that p satisfies the following conditions:

1. For any ingressive morphism $g \colon s \rightarrowtail t$ of $\mathcal{D}$ and any object $x \in \mathcal{C}_s$, there exists an ingressive morphism $f \colon x \rightarrowtail y$ covering g which is both p-cocartesian and $p_\dagger$-cocartesian.

2. Suppose σ is a commutative square

$$\begin{tikzcd} y' \arrow[r, tail, "f'"] \arrow[d, two heads] & x' \arrow[d, "\psi"] \\ y \arrow[r, tail, "f"'] & x \end{tikzcd}$$

in $\mathcal{C}$ such that $p(\sigma)$ is ambigressive pullback in $\mathcal{D}$, with in- and egressive morphisms as marked, and f is p-cocartesian. Then f' is p-cocartesian if and only if σ is an ambigressive pullback square (and in particular ψ is egressive).

Then if a morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is of the form

$$\begin{tikzcd} & y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\ x & & x' \end{tikzcd},$$

where ϕ is egressive and $p^\dagger$-cartesian, and ψ is ingressive and p-cocartesian, it is π-cocartesian.[1]

This theorem originally appears in [1] (although, as noted above, the assumptions are slightly different). However, the proof there is combinatorial in nature. Our proof uses the material developed in Section **??**, leveraging the fact that ∞-categories of spans have natural incarnations as (complete) Segal spaces.

Before we get to the meat of the proof, however, we have to prove some preliminary results.

[1] This is not precisely the same condition that Barwick gives: he demands that ϕ be p-cartesian rather than $p^\dagger$-cartesian. As far as I can tell, this is simply an error; Barwick seems to use the condition as stated above in his proof.

1.2 Housekeeping

In this section we recall some basic facts we will need in the course of our proof of the main theorem.

Isofibrations

We will need some results about isofibrations, sometimes called categorical fibrations. These are standard, and proofs can be found on Kerodon.

Proposition 1.2.1. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map between quasicategories. The following are equivalent:

- The map f is an isofibration.
- The map f is an inner fibration, and each equivalence in $\mathcal{D}$ has a lift in $\mathcal{C}$ which is an equivalence.

We will denote by $\mathrm{Map}(-,-)$ the internal hom on $\mathcal{S}\mathrm{et}_\Delta$.

Proposition 1.2.2. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $i\colon A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Map}(B,\mathcal{C}) \to \mathrm{Map}(A,\mathcal{C}) \times_{\mathrm{Map}(A,\mathcal{D})} \mathrm{Map}(B,\mathcal{D})$$

is an isofibration.

We will denote by $(-)^{\simeq}$ the *core* functor, i.e. the functor which takes a simplicial set X to the largest Kan complex it contains.

Proposition 1.2.3. Let $f\colon X \to Y$ be an isofibration of simplicial sets. Then

$$f^{\simeq}\colon X^{\simeq} \to Y^{\simeq}$$

is a Kan fibration.

We will denote the core of the mapping space functor $\mathrm{Map}(-,-)$ by $\mathrm{Fun}(-,-)$. That is, for simplicial sets A and B, we have

$$\mathrm{Fun}(A,B) \cong \mathrm{Map}(A,B)^{\simeq}.$$

Corollary 1.2.4. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(A,\mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B,\mathcal{D})$$

is a Kan fibration.

Connected components

Our proof will rely heavily on the fact that the many properties that morphisms can have (ingressive, egressive, p-cocartesian, etc.) are well-behaved with respect to the homotopical structure of the complete Segal spaces in which they live; for example, if a morphism is p-cocartesian, then every morphism in its path component is p-cocartesian.

Let X' and X be simplicial sets, and let $f\colon X' \hookrightarrow X$ be an inclusion. We will say that f is an *inclusion of connected components* if for all simplices $\sigma \in X$, if σ has any vertex in common with X', then σ is wholly contained in X'.

Lemma 1.2.5. Let A be a simplicial set such that between any two vertices x and y of A there exists a finite zig-zag of 1-simplices of A connecting x to y. Let $A_0 \subseteq$ A be a nonempty simplicial subset of A. Let $f\colon X' \hookrightarrow X$ be a morphism between simplicial sets which is an inclusion of connected components. Then the following dashed lift always exists.

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, hook] & X' \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[ur, dashed] & X
\end{tikzcd}$$

Proof. We have a map $f\colon A \to X$; in order to construct our dashed lift, it suffices to show that under the above assumptions, f takes every simplex of A to a simplex which belongs to X'. To this end, let $\sigma \in A$ be a simplex. There exists a finite zig-zag of 1-simplices connecting some vertex of σ to a vertex α of A_0; under f, this is mapped to a zig-zag of 1-simplices connecting $f(\alpha)$ to $f(\sigma)$. The vertex $f(\alpha)$ belongs to X'; proceeding inductively, we find that the image of every 1-simplex belonging to the zig-zag also belongs to X', and that $f(\sigma)$ therefore also belongs to X'. $\square$

Corollary 1.2.6. Given any commuting square of simplicial sets

$$\begin{tikzcd}
X' \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, hook] & Y
\end{tikzcd}$$

with monomorphisms as marked, where i is an inclusion of connected components between Kan complexes and f is a Kan fibration, the map f' is a Kan fibration.

Furthermore, if f is a trivial fibration and f' is surjective on vertices, then f' is a trivial fibration.

Proof. First, we show that if f is a Kan fibration, then f' is a Kan fibration. We need to show that a dashed lift below exists for each $n \geq 1$ and each $0 \leq i \leq n$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X' \arrow[r, hook, "i"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y' \arrow[r, hook] & Y \end{tikzcd}$$

We can always solve our outer lifting problem, and by Lemma 1.2.5, our lift factors through X'.

Now suppose that f is a trivial Kan fibration. The logic above says that f' has the right lifting property with respect to $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 1$; the right lifting property with respect to $\partial\Delta^0 \hookrightarrow \Delta^0$ is equivalent to the surjectivity of f' on vertices. □

Contractibility

We will need at certain points to show that various spaces of lifts are contractible. These are mostly common-sense results, but it will be helpful to have them written down somewhere so we can refer to them later.

Lemma 1.2.7. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Then for any commuting square

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd},$$

the fiber F in the pullback square below is contractible.

$$\begin{tikzcd} F \arrow[rr] \arrow[d] & & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(\alpha, f\alpha, \beta)}"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) & \end{tikzcd}$$

Proof. The right-hand map is a trivial Kan fibration. □

One should interpret this as telling us that given a Λ^2_2-horn α in $\mathcal{C}$ lying over a 2-simplex β in $\mathcal{D}$, the space of fillings of α lying over β is contractible. We will need a pantheon of similar results.

Lemma 1.2.8. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Let e be any f-cartesian edge. Then for any square

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd}$$

such that $\alpha|_{\{1,2\}} = e$, the fiber F in the pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(\alpha, f\alpha, \beta)}"] & \mathrm{Fun}(\Lambda^2_2, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) \end{tikzcd} \tag{1}$$

is contractible.

Proof. Denote $f(e) = \bar{e}$. Define a marking $\mathcal{E}'$ on $\mathcal{D}$ containing all degenerate edges and $\bar{e}$. Define a marking $\mathcal{E}$ on $\mathcal{C}$ containing the f-cocartesian lifts of the edges in $\mathcal{E}$. By [2, Prop. 3.1.1.6], the map $\mathcal{C}^{\mathcal{E}} \to \mathcal{D}^{\mathcal{E}'}$ has the right-lifting property with respect to all cocartesian-marked anodyne morphisms. It follows that

$$\begin{tikzcd} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \arrow[d] \\ \mathrm{Map}^\sharp((\Lambda^2_2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \times_{\mathrm{Map}^\sharp((\Lambda^2_1)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'})} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'}) \end{tikzcd}$$

is a trivial Kan fibration; the map on the right-hand side of Diagram 1 is the core of this map, and is thus also a trivial Kan fibration. □

Lemma 1.2.9. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be an inner fibration between triples satisfying the conditions of Theorem 1.1.5. Denote by $\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges $\Delta^1 \to \mathcal{C}^\dagger$ which are both p-cocartesian and $p^\dagger$-cocartesian. Then the fibers of the map

$$\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \to \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger)$$

are contractible.

Proof. Denote by $\mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges which are $p^\dagger$-cartesian. Then we have a square

$$\begin{tikzcd} \mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \arrow[r, hook] \arrow[d] & \mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger) \arrow[d] \\ \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger) \arrow[r, equal] & \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger) \end{tikzcd}$$

in which the top map is an inclusion of connected components, the right map is a trivial Kan fibration, and the left map is surjective on vertices. Thus, the left map is a trivial Kan fibration by Corollary 1.2.6 □

1.3 Main theorem

Our goal is now to prove Theorem 1.1.5. As in the statement of the theorem, we fix a map of triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ giving us a functor of quasicategories

$$\pi\colon \mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathrm{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

We wish to show that morphsims of the form

$$\begin{tikzcd} & y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\ x & & x' \end{tikzcd}, \tag{2}$$

are π-cocartesian, where ϕ is egressive and $p_\dagger$-cartesian, and ψ is ingressive and p-cocartesian.

The way forward is clear: the functor π is the zeroth row of a functor of Segal spaces

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

It will suffice to show by Corollary **??** that morphisms of the above form are π'-cocartesian. Morally, this result should not be surprising. We should think of the homotopy pullback condition defining cocartesian morphisms (Definition **??**) as telling us that we can fill relative Λ^2_2-horns in $\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, and that such fillings are unique up to contractible choice. Finding a filling

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \arrow[d, "\pi'"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger) \end{tikzcd}$$

is equivalent finding a filling

$$\begin{tikzcd} \mathrm{sd}(\Lambda^2_2) \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \mathrm{sd}(\Delta^2) \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

(such that the filling is ambigressive and the necessary square is pullback); the conditions on p guarantee us that we can perform this filling in a series of steps pictured in Figure 1, each of which is unique up to contractible choice.

Reedy fibration

First, however, we have to check that π' is a Reedy fibration. This follows from the following useful fact, implied by the first assumption of Theorem 1.1.5.

Lemma 1.3.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples satisfying the conditions of Theorem 1.1.5. Then the map $p\colon \mathcal{C} \to \mathcal{D}$ is an isofibration.

Proof. By assumption, p is an inner fibration and $\mathcal{C}$ and $\mathcal{D}$ are quasicategories, so it suffices by Proposition 1.2.1 to show that p admits lifts of equivalences. Each equivalence in $\mathcal{D}$ is in particular ingressive, and hence admits a p-cocartesian lift by the assumptions of Theorem 1.1.5; these are automatically equivalences. □

Proposition 1.3.2. For any functor of adequate triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, the map

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$$

is a Reedy fibration.

Proof. Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{D})
\end{tikzcd}.$$

The right-hand map is a Kan fibration by Corollary 1.2.4 (because sd preserves monomorphisms), and the top map is an inclusion of connected components, so Corollary 1.2.6 implies that the left-hand map is a Kan fibration, which is what we needed to show. □

We are now ready to begin in earnest our proof that morphisms of the form Equation 2 are π'-cocartesian. We first introduce some notation. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples.

- We will denote p-cartesian morphisms in $\mathcal{C}$ with a circle:

$$x \longrightarrow\!\!\!\!\!\!\!\circ\;\; y$$

- We will denote p-cocartesian morphisms in $\mathcal{C}$ with a bullet:

$$x \longrightarrow\!\!\!\!\!\!\!\bullet\;\; y$$

- We will denote $p^\dagger$-cartesian morphisms in $\mathcal{C}$ with a triangle:

$$x \longrightarrow\!\!\!\!\!\!\!\triangleright\;\; y$$

- We will denote $p_\dagger$-cocartesian morphisms in $\mathcal{C}$ with a filled triangle:

$$x \longrightarrow\!\!\!\!\!\!\!\blacktriangleright\;\; y$$

Thus, an ingressive morphism which is both p-cocartesian and $p_\dagger$-cocartesian will be denoted by

$$x \rightarrowtail\!\!\blacktriangleright\!\!\bullet\!\!\rightarrow y$$

Our proof will rest on the factorization $\mathrm{sd}(\Lambda^2_0) \hookrightarrow \mathrm{sd}(\Delta^2)$ pictured in Figure 1. Denote the underlying factorization of simplicial sets by

$$A_1 \overset{i_1}{\hookrightarrow} A_2 \overset{i_2}{\hookrightarrow} A_3 \overset{i_3}{\hookrightarrow} A_4 \overset{i_4}{\hookrightarrow} A_5 \overset{i_5}{\hookrightarrow} A_6$$

For each of the A_i above, denote by

$$\mathrm{Fun}'(A_i, \mathcal{C}) \subseteq \mathrm{Fun}(A_i, \mathcal{C}) \tag{3}$$

the full simplicial subset on those functors $A_i \to \mathcal{C}$ which respect each labelling in Figure 1:

- ingressive
- egressive
- p-cartesian
- p-cocartesian
- $p^\dagger$-cartesian
- $p_\dagger$-cocartesian

Note that because equivalences in $\mathcal{C}$ belong to each of these classes of morphisms, the inclusion in Equation 3 is an inclusion of connected components.

Similarly, we will denote by $\mathrm{Fun}'(A_i, \mathcal{D})$ the full simplicial subset on functors which respect the ingressive and egressive labellings.

Lemma 1.3.3. The square

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_0, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_0, \mathcal{D})
\end{tikzcd}$$

is homotopy pullback.

Proof. The factorization of Figure 1 gives us a factorization of the above square into five squares

$$\begin{tikzcd}
\mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_k, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_{k+1}, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_k, \mathcal{D})
\end{tikzcd}, \qquad 1 \leq k < 6,$$

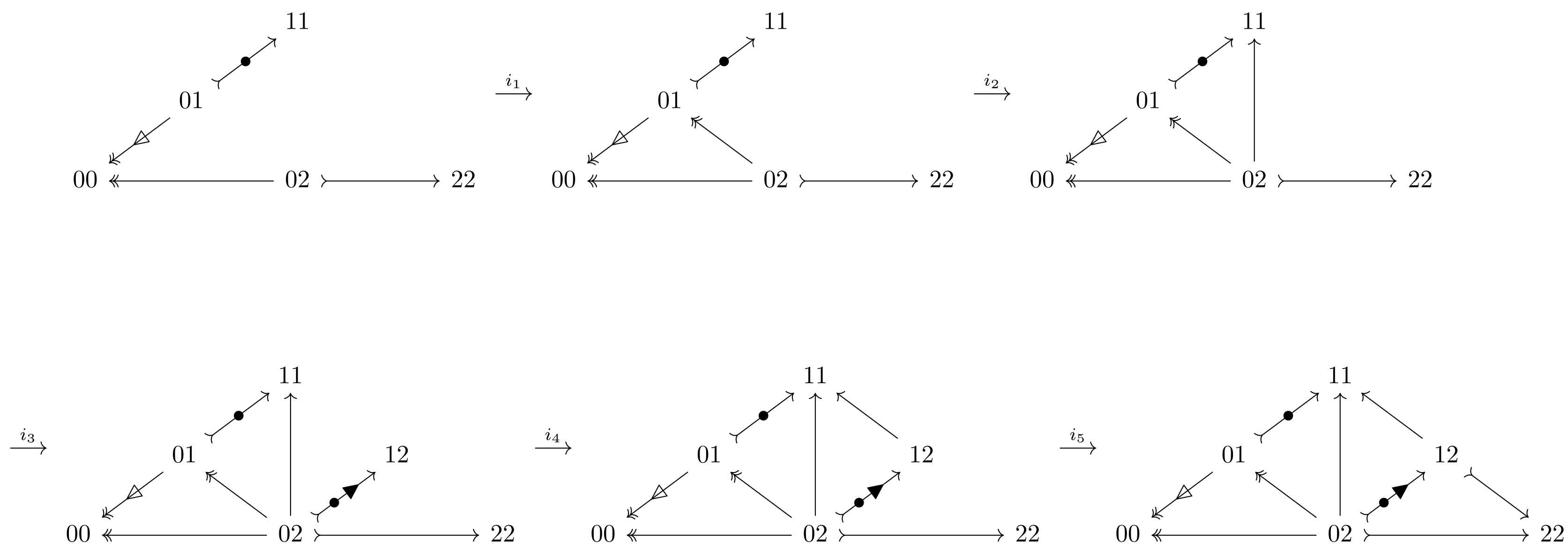

Figure 1: A factorization of the inclusion $\mathrm{sd}(\Lambda^2_0) \hookrightarrow \mathrm{sd}(\Delta^2)$, where certain morphisms have been labelled.

each corresponding to one of the inclusions i_k. We will be done if we can show that each of these squares is homotopy pullback. It suffices to show that for each k, the map

$$j_k\colon \mathrm{Fun}'(A_{k+1}, \mathcal{C}) \to \mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1},\mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D})$$

is a weak equivalence. First, we show that each of these maps is a Kan fibration. To see this, consider the square

$$\begin{tikzcd}
\mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r, hook] \arrow[d, "j_k"'] & \mathrm{Fun}(A_{k+1}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1},\mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(A_k, \mathcal{C}) \times_{\mathrm{Fun}(A_{k+1},\mathcal{D})} \mathrm{Fun}(A_k, \mathcal{D})
\end{tikzcd}.$$

The right-hand map is a Kan fibration because of Corollary 1.2.4, and we have already seen that the top map is an inclusion of connected componenents. Hence each j_k is a Kan fibration by Corollary 1.2.6. Thus, in order to show that each j_k is a weak equivalence, it suffices to show that the fibers are contractible.

First consider the case $k = 1$. For any map α below, consider the following pullback square.

$$\begin{tikzcd}
F_\alpha \arrow[r] \arrow[d] & \mathrm{Fun}'(A_1, \mathcal{C}) \arrow[d, "j_1"] \\
\Delta^0 \arrow[r, "\alpha"] & \mathrm{Fun}'(A_0, \mathcal{C}) \times_{\mathrm{Fun}'(A_1,\mathcal{D})} \mathrm{Fun}'(A_1, \mathcal{D})
\end{tikzcd}.$$

The fiber F_α over α is the space of ways of completing a diagram of shape A_0 in $\mathcal{C}$ to a diagram of shape A_1 in $\mathcal{C}$ given a diagram of shape A_1 in $\mathcal{D}$. This is the space of ways of filling $\Lambda^2_2 \hookrightarrow \Delta^2$ in $\mathcal{C}^\dagger$ lying over a 2-simplex in $\mathcal{D}^\dagger$. This is contractible by Lemma 1.2.8.

The case $k = 2$ is similar, using Lemma 1.2.7.

The case $k = 3$ is similar, using Lemma 1.2.9.

The cases $k = 4$ and $k = 5$ use the dual to Lemma 1.2.8. □

We are now ready to show that morphisms of the form Diagram 2 are π'-cocartesian.

Proposition 1.3.4. For any morphism e of the form given in Equation 2, the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\}
\end{tikzcd}$$

is homotopy pullback.

Proof. This square factors into the two squares

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_0, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_0, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}.$$

The first is homotopy pullback because the bottom map is an inclusion of connected components, and the second condition of Theorem 1.1.5 guarantees that the fiber over an ambigressive $\mathrm{sd}(\Delta^2) \to \mathcal{D}$ belonging to $\mathrm{Fun}'(A_6, \mathcal{D})$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is πe is precisely an ambigressive Cartesian functor $\mathrm{sd}(\Delta^2) \to \mathcal{C}$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is e.

That the second is homotopy pullback follows immediately from Lemma 1.3.3. □

This proves Theorem 1.1.5.