Segal spaces of spans

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Contents

1 Marked bisimplicial sets

We need a basic theory of marked bisimplicial sets. This is a very ugly formulation of something that's probably equivalent to something pretty. However, in the case we're interested, it's good enough.

Definition 1. A ***marked bisimplicial set*** is a bisimplicial set X together with a marking on $i_1^* X$. We interpret this as a subset of X_{10}.

Definition 2. For a marked simplicial set A and a marked bisimplicial set B, define a marking on the bisimplicial set $A \square B$ as follows: a simplex $(a, b) \in A_1 \times B_0$ is marked if and only if a is marked.

Definition 3. Let A denote a marked simplicial set, B an unmarked simplicial set, and X a marked bisimplicial set.

- Define an unmarked simplicial set $(A\backslash X)'$ by the universal property

$$\mathrm{Hom}_{\mathbf{Set}_\triangle}(B, (A\backslash X)) = \mathrm{Hom}_{\mathbf{Set}^+_{\triangle^2}}(A\square B, X).$$

- Define a marking on the simplicial set X/B as follows: a 1-simplex $\Delta^1 \to X/B$ is marked if and only if the corresponding map $\Delta^1 \square B \to X$ of unmarked bisimplicial sets comes from a map of marked bisimplicial sets $(\Delta^1)^\sharp \square B \to X$.

I still have to check with a fine-toothed comb that we have adjunctions

$$A\square- : \mathbf{Set}_\triangle \longleftrightarrow \mathbf{Set}_{\triangle^2} : A\backslash -$$

and

$$-\square B : \mathbf{Set}^+_\triangle \longleftrightarrow \mathbf{Set}^+_{\triangle^2} : -/B,$$

but I've sketched a proof.

Let $\mathcal{C}$ be an ∞-category, and let $\Gamma(\mathcal{C})$ be the complete Segal space

$$\Gamma(\mathcal{C})_n \cong \mathrm{Map}(\Delta^n, \mathcal{C})^\simeq.$$

Clearly, a marking on $\mathcal{C}$ gives a marking on $\Gamma(\mathcal{C})$.

Lemma 4. Let $\mathcal{C}$ be a quasicategory with a marking, and let A be a marked simplicial set. There is an isomorphism of simplicial sets

$$A\backslash\Gamma(\mathcal{C}) \cong \mathrm{Map}^+(A, \mathcal{C})^\simeq,$$

where Map^+ denotes the full subcategory of maps of marked simplicial sets. This isomorphism is natural in A. Furthermore, if the marking on $\mathcal{C}$ contains all equivalences and is closed under composition, then the inclusion

$$A\backslash\Gamma(\mathcal{C}) \cong \mathrm{Map}^+(A, \mathcal{C})^\simeq \subseteq \mathrm{Map}(A, \mathcal{C})^\simeq \cong A^\flat\backslash\Gamma(\mathcal{C})$$

is a full subcategory inclusion of connected components.

Corollary 5. Let $\mathcal{C} \to \mathcal{D}$ be a morphism of marked ∞-categories each of whose markings is closed under composition. Suppose that the underlying unmarked morphism $f\colon \Gamma(\mathcal{C}) \to \Gamma(\mathcal{D})$ is a Reedy fibration. Let $u\colon A \to B$ be a morphism of marked simplicial sets whose underlying morphism is a monomorphism. Then the map $\langle u\backslash f\rangle$ is a Kan fibration.

Definition 6. For $n \geq 1$, define the following simplicial subsets of Δ^n.

- Denote by I_n the **spine** of Δ^n, i.e. the space

$$\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}.$$

- Denote by Σ_n the space

$$\Sigma_n = \Delta^{\{0,1\}} \amalg_{\Delta^{\{0\}}} I_{\{0,\hat{1},2,\ldots,n\}}.$$

That is, Σ_n is the union of $\Delta^{\{0,1\}}$ with the spine of $d_1\Delta^n$.

We will call Σ_n the **left spine** of Δ^n.

Note that $\Sigma_2 \simeq \Lambda_0^2$.

Definition 7. Let $f \colon X \to Y$ be a Reedy fibration between complete Segal spaces. A morphism $e \in X_{10}$ is f-**cocartesian** if the square

$$
\begin{array}{ccc}
\Delta^2\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} & \longrightarrow & \Lambda_0^2\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \\
\downarrow & & \downarrow \\
\Delta^2\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} & \longrightarrow & \Lambda_0^2\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\}
\end{array}
$$

is homotopy pullback.

Lemma 8. Let $f \colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be a morphism with the property that the square

$$
\begin{array}{ccc}
\Delta^2\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} & \longrightarrow & \Lambda_0^2\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \\
\downarrow & & \downarrow \\
\Delta^2\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} & \longrightarrow & \Lambda_0^2\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\}
\end{array}
$$

is homotopy pullback. Then the square

$$
\begin{array}{ccc}
\Delta^n\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} & \longrightarrow & \Sigma_n\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \\
\downarrow & & \downarrow \\
\Delta^n\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} & \longrightarrow & \Sigma_n\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\}
\end{array}
$$

is homotopy pullback for all $n \geq 2$.

For any simplicial subset $A \subseteq \Delta^n$, denote the marking where the only marked nondegenerate edge is $\Delta^{\{0,1\}}$ by $A^{\mathcal{L}}$.

We will say a collection of morphisms $\mathcal{A} \subset \mathrm{Mor}(\mathbf{Set}_{\triangle}^+)$ has the *right cancellation property* if for all $u, v \in \mathrm{Mor}(\mathbf{Set}_{\triangle}^+)$,

$$u \in \mathcal{A}, \quad vu \in \mathcal{A} \quad \Longrightarrow \quad v \in A.$$

3

Lemma 9. Let $\mathcal{A}$ be a saturated set of morphisms of $\mathbf{Set}_\Delta^+$ all of whose underlying morphisms are monomorphisms, and which has the right cancellation property. Further suppose that $\mathcal{A}$ contains the following classes of morphisms.

1. Spine inclusions $(I_n)^\flat \hookrightarrow (\Delta^n)^\flat$, $n \geq 2$.

2. Left spine inclusions $(\Sigma_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$.

Then $\mathcal{A}$ contains the following classes of morphisms.

1'. Inner horn inclusions $(\Lambda_i^n)^\flat \hookrightarrow (\Delta^n)^\flat$, $n \geq 2$, $0 < i < n$.

2'. Left horn inclusions $(\Lambda_0^n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$.

Proof. It is shown in [1] that 1. $\implies$ 1'. It suffices $\qquad\qquad\square$

Theorem 10. Let $f \colon X \to Y$ be a Reedy fibration between complete Segal spaces such that over each morphism $\bar{e} \in Y_{10}$ there exists a cocartesian lift $e \in X_{10}$, i.e. a cocartesian morphism $e \in X_{10}$ such that $fe = \bar{e}$. Then the map $i_1^* X \to i_1^* Y$ is a cocartesian fibration of quasicategories.

Proof. It suffices to show that if a morphism in X_{10} is f-cocartesian, then it is $i_1^* f$-cocartesian. To see this, note that $\qquad\qquad\square$

2 Basics of fibrations complete Segal spaces

- The inclusion of Kan complexes into quasicategories has right adjoint $(-)^{\simeq}$, which sends a simplicial set to the largest kan complex it contains. sets.

- There is a Quillen equivalence

$$ p_1^* \colon \mathbf{Set}_\Delta^{\text{Joyal}} \longleftrightarrow \mathbf{Set}_{\Delta^2}^{\text{CSS}} \colon i_1^*, $$

where i_1^* sends a complete Segal space to its zeroth row, and p_1^* sends a quasicategory to the corresponding vertically constant bisimplicial set.

2.1 Inner fibrations of complete segal spaces

Definition 11. We say that a map $\mathcal{A} \to \mathcal{B}$ of complete Segal spaces is a ***inner fibration*** if the following square is homotopy pullback in the Kan model structure.

$$
\begin{array}{ccc}
\mathcal{A}_2 & \longrightarrow & \mathcal{A}_1 \times_{\mathcal{A}_0} \mathcal{A}_1 \\
\downarrow & & \downarrow \\
\mathcal{B}_2 & \longrightarrow & \mathcal{B}_1 \times_{\mathcal{B}_0} \mathcal{B}_1
\end{array} \quad .
$$

2.2 Cocartesian fibrations of complete segal spaces

Definition 12. Let $\pi\colon \mathcal{A} \to \mathcal{B}$ be a map of complete Segal spaces. We say that a morphism $f\colon x \to y$ in $\mathcal{A}$ is π-**cocartesian** if the following square is homotopy pullback in the Kan model structure.

$$
\begin{array}{ccc}
\mathcal{A}_2 \times_{\mathcal{A}_{\{0,1\}}} \{f\} & \longrightarrow & \mathcal{A}_{\{0,2\}} \times_{\mathcal{A}_{\{0\}}} \{x\} \\
\downarrow & & \downarrow \\
\mathcal{B}_2 \times_{\mathcal{B}_{\{0,1\}}} \{\pi f\} & \longrightarrow & \mathcal{B}_{\{0,2\}} \times_{\mathcal{B}_{\{0\}}} \{\pi x\}
\end{array}
$$

Definition 13. A map $\mathcal{A} \to \mathcal{B}$ between complete Segal spaces is a **cocartesian fibration** if it is an inner fibration and each morphism in $\mathcal{B}$ has a cocartesian lift.

3 Segal spaces of spans

3.1 Definitions

3.2 Inner fibrations

Theorem 14. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories with pullbacks, and let $\pi\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration preserving pullbacks. Then $\mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is an inner fibration of complete Segal spaces.

Proof. We need to show that

$$
\begin{array}{ccc}
\mathbf{Span}(\mathcal{C})_2 & \longrightarrow & \mathbf{Span}(\mathcal{C})_1 \times_{\mathbf{Span}(\mathcal{C})_0} \mathbf{Span}(\mathcal{C})_1 \\
\downarrow & & \downarrow \\
\mathbf{Span}(\mathcal{D})_2 & \longrightarrow & \mathbf{Span}(\mathcal{D})_1 \times_{\mathbf{Span}(\mathcal{D})_0} \mathbf{Span}(\mathcal{D})_1
\end{array}
$$

is a homotopy pullback square. This says that we need to show that

$$
\begin{array}{ccc}
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{C})^{\simeq} & \longrightarrow & \mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Lambda_1^2), \mathcal{C})^{\simeq} \\
\downarrow & & \downarrow \\
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{D})^{\simeq} & \longrightarrow & \mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Lambda_1^2), \mathcal{D})^{\simeq}
\end{array}
$$

is a homotopy pullback. Since all the objects involved are Kan complexes and the lower map is a Kan fibration, it suffices to show that the induced map from $\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{C})^{\simeq}$

into the fibered product is a weak equivalence. By

$$\mathbf{Fun}(\mathrm{asd}(\Delta^2), \mathcal{C}) \to \mathbf{Fun}(\mathrm{asd}(\Lambda_1^2), \mathcal{C}) \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^2), \mathcal{D})} \mathbf{Fun}(\mathrm{asd}(\Lambda_1^2), \mathcal{D})$$

Since Cartesian simplices connected by an equivalence are preserved, the map is □

3.3 Cocartesian fibrations

We start with some notation.

The main goal of this section is to prove the following theorem.

Theorem 15. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories admitting all pullbacks, and let $p\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration which preserves pullbacks. Further suppose that p has the following property:

For any square

$$\sigma = \begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow \\ x' & \xrightarrow{f'} & y' \end{array}$$

in $\mathcal{C}$ such that the morphism f' is p-cocartesian and $p(\sigma)$ is pullback in $\mathcal{D}$, the following are equivalent.

- The morphism f is p-cocartesian.
- The square σ is pullback.

Then the functor $\pi\colon \mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is a cocartesian fibration of Segal spaces, whose cocartesian morphisms are spans of the form

Note 16. Since the definition of $\mathbf{Span}(\mathcal{C})$ is self-dual, the functor $\mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is also a cartesian fibration with cartesian morphisms of the form

We prove Theorem 15 in several steps. Since we will be working with bicartesian fibrations, it will be helpful to adapt some of the tools of marked simplicial sets to our

purposes.

3.4 Bimarked simplicial sets

Our proof of Theorem 15 will involve working with with bicartesian fibrations. For this reason, it will be helpful to have results about simplicial sets with two markings, one of which controls the cocartesian structure and one of which controls the cartesian structure. This section essentially verifies that some key results about marked simplicial sets which can be found in [2, Sec. 3.1] hold in the bimarked case. The only results we will make use of are Proposition 26, Example 21, and Example 22.

Definition 17. A ***bimarked simplicial set*** is a triple $(X, \mathcal{E}, \mathcal{E}')$, where X is a simplicial set and $\mathcal{E}$ and $\mathcal{E}'$ are markings. We will often shorten this to $X^{(\mathcal{E}, \mathcal{E}')}$. A morphism of bimarked simplicial sets is a morphism of the underlying simplicial sets which preserves each class of markings separately. We will denote the category of bimarked simplicial sets by $\mathbf{Set}_{\triangle}^{++}$.

Example 18. For any simplicial set X we will denote

- The bimarked simplicial set where $\mathcal{E}$ and $\mathcal{E}'$ contain only the degenerate edges by $X^{(\flat, \flat)}$,

- The bimarked simplicial set where $\mathcal{E}$ contains only the degenerate edges and $\mathcal{E}'$ contains every edge by $X^{(\flat, \sharp)}$,

- The bimarked simplicial set where $\mathcal{E}$ contains every edge and $\mathcal{E}'$ contains only the degenerate edges by $X^{(\sharp, \flat)}$, and

- The bimarked simplicial set where $\mathcal{E}$ and $\mathcal{E}'$ contain every edge by $X^{(\sharp, \sharp)}$.

Example 19. Let $p \colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration. Denote by $\mathcal{C}^{\natural}$ the bimarked simplicial set where

- The set $\mathcal{E}$ is the set of all p-cocartesian morphisms, and

- The set $\mathcal{E}'$ is the set of all p-cartesian morphisms.

In this way every bicartesian fibration gives a morphism of bimarked simplicial sets.

The category $\mathbf{Set}_{\triangle}^{++}$ is closely connected to the category $\mathbf{Set}_{\triangle}^{+}$ and the category $\mathbf{Set}_{\triangle}$. We have the following obvious results.

- There is a forgetful functor $u \colon \mathbf{Set}_{\triangle}^{++} \to \mathbf{Set}_{\triangle}$ which forgets both markings. This has left adjoint $\iota \colon X \mapsto (X, \flat, \flat)$. The functor ι is a full subcategory inclusion.

- There is a forgetful functor $u_1 \colon \mathbf{Set}_{\triangle}^{++} \to \mathbf{Set}_{\triangle}^{+}$, which forgets the second marking, sending $(X, \mathcal{E}, \mathcal{E}') \mapsto (X, \mathcal{E})$. This has a left adjoint ι_1 given by the functor which sends $(X, \mathcal{E}) \mapsto (X, \mathcal{E}, \flat)$. The functor ι_1 is a full subcategory inclusion; there is

a bijection between maps between marked simplicial sets $(X, \mathcal{E}) \to (Y, \mathcal{E}')$ and maps $(X, \mathcal{E}, \flat) \to (Y, \mathcal{E}', \flat)$. The same is true of the functor u_2 which forgets the first marking and its left adjoint ι_2.

Just as in the marked case, we will a set of bimarked anodyne morphisms, the morphisms with the left lifting property with respect to bicartesian fibrations.

Definition 20. The class of **bimarked anodyne morphisms** is the saturated hull of the union of the following classes of morphisms.

(1) For each $0 < i < n$, the inner horn inclusions

$$(\Lambda_i^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}.$$

(2) For every $n > 0$, the inclusion

$$(\Lambda_0^n)^{(\mathcal{L},\flat)} \hookrightarrow (\Delta^n)^{(\mathcal{L},\flat)},$$

where $\mathcal{L}$ denotes the set of all degenerate edges of Δ^n together with the edge $\Delta^{\{0,1\}}$.

(2') For every $n > 0$, the inclusion

$$(\Lambda_n^n)^{(\flat,\mathcal{R})} \hookrightarrow (\Delta^n)^{(\flat,\mathcal{R})},$$

where $\mathcal{R}$ denotes the set of all degenerate edges of Δ^n together with the edge $\Delta^{\{n-1,n\}}$.

(3) The inclusion

$$(\Lambda_1^2)^{(\sharp,\flat)} \coprod_{(\Lambda_1^2)^{(\flat,\flat)}} (\Delta^2)^{(\flat,\flat)} \to (\Delta^2)^{(\sharp,\flat)}.$$

(3') The inclusion

$$(\Lambda_1^2)^{(\flat,\sharp)} \coprod_{(\Lambda_1^2)^{(\flat,\flat)}} (\Delta^2)^{(\flat,\flat)} \to (\Delta^2)^{(\flat,\sharp)}.$$

(4) For every Kan complex K, the map

$$K^{(\flat,\flat)} \to K^{(\sharp,\flat)}.$$

(4') For every Kan complex K, the map

$$K^{(\flat,\flat)} \to K^{(\flat,\sharp)}.$$

Example 21. Define a bimarked structure $(\mathrm{asd}(\Delta^1), \mathcal{E}, \mathcal{E}') = \mathrm{asd}(\Delta^1)^{\heartsuit}$ on $\mathrm{asd}(\Delta^1)$, where the morphism $01 \to 11$ is $\mathcal{E}$-marked, and the morphism $01 \to 00$ is $\mathcal{E}'$-marked.
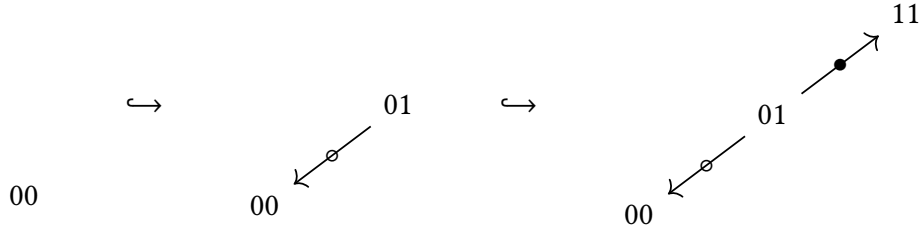
Denoting $\mathcal{E}$-marked morphisms with a $\bullet$ and $\mathcal{E}'$-marked morphisms with a $\circ$, we can draw this as follows.
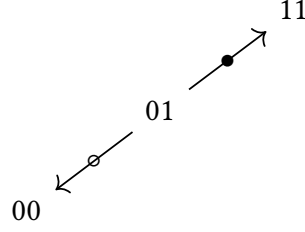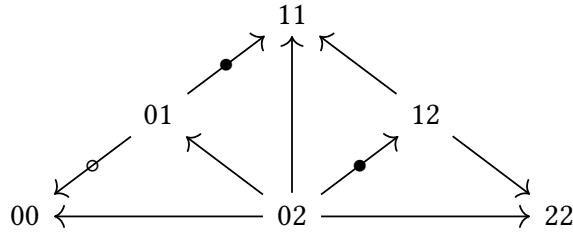

The inclusion $\{00\} = \mathrm{asd}(\Delta^{\{0\}}) \hookrightarrow \mathrm{asd}(\Delta^1)^\heartsuit$ is bimarked anodyne: we can factor it

$$\{00\} \hookrightarrow (\Delta^1)^{(\flat,\sharp)} \hookrightarrow \mathrm{asd}(\Delta^1)^\heartsuit,$$

where the first inclusion is of the form (2') and the second is a pushout of a morphism of the form (2). We can draw this process as follows.

Example 22. Define a bimarked structure $\mathrm{asd}(\Delta^2)^\heartsuit$ on $\mathrm{asd}(\Delta^2)$, following the notation of Example 21, as follows.

Note that the bimarking of Example 21 is the restriction of $\mathrm{asd}(\Delta^2)^\heartsuit$ to $\mathrm{asd}(\Delta^{\{0,1\}})$. Denote the restriction of the bimarking $\mathrm{asd}(\Delta^2)^\heartsuit$ to $\mathrm{asd}(\Delta^{\{0,2\}})$ by $\mathrm{asd}(\Delta^{\{0,2\}})^\heartsuit$; this agrees with the $(\flat,\flat)$-marking.

The inclusion $\mathrm{asd}(\Delta^{\{0,2\}})^\heartsuit \hookrightarrow \mathrm{asd}(\Delta^2)^\heartsuit$ is bimarked anodyne. To see this, note that... factorization...

Proposition 23. A map $p\colon (X, \mathcal{E}_X, \mathcal{E}'_X) \to (S, \mathcal{E}_S, \mathcal{E}'_S)$ of bimarked simplicial sets has the right lifting property with respect to bimarked anodyne morphisms if and only if the following conditions are satisifed.

(A) The map p is an inner fibration of simplicial sets.

(B) An edge e of X is $\mathcal{E}_X$-marked if and only if $p(e)$ is $\mathcal{E}_S$-marked and e is p-cocartesian.

(B') An edge e of X is $\mathcal{E}'_X$-marked if and only if $p(e)$ is $\mathcal{E}'_S$-marked and e is p-cartesian.

(C) For every object y of X and every $\mathcal{E}_S$-marked edge $\bar{e}\colon \bar{x} \to p(y)$ in S, there exists a $\mathcal{E}_X$ marked edge $e\colon x \to y$ of X with $p(e) = \bar{e}$.

(C') For every object y of X and every $\mathcal{E}'_S$-marked edge $\bar{e}\colon \bar{x} \to p(y)$ in S, there exists a $\mathcal{E}'_X$ marked edge $e\colon x \to y$ of X with $p(e) = \bar{e}$.

Proof. By [2, Prop. 3.1.1.6], (A), (B) and (C) are equivalent to (1), (2), and (3). By its dual, (A), (B') and (C') are equivalent to (1), (2'), and (3') $\qquad\square$

We would like to define cofibrations of bimarked simplicial sets to be maps of bimarked simplicial sets whose underlying map of simplicial sets is a monomorphism, and then show that the class of bimarked anodyne maps is stable under smash products with arbitrary cofibrations. Unfortunately, this turns out not to be quite true; the candidate class of cofibrations described above is generated by the following classes of maps.

(I) Boundary fillings $(\partial\Delta^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}$.

(II) Markings $(\Delta^1)^{(\flat,\flat)} \to (\Delta^1)^{(\sharp,\flat)}$.

(III) Markings $(\Delta^1)^{(\flat,\flat)} \to (\Delta^1)^{(\flat,\sharp)}$.

There is nothing that tells us, for example, that the smash product of a bimarked anodyne map of type (2) with a cofibration of type (III) should be bimarked anodyne. Denoting arrows with the first marking using a $\bullet$ and the second using a $\circ$, this amounts, in the case $n = 0$, to the statement that the map

should be bimarked anodyne, which it isn't. However, we have the following weaker statement.

Lemma 24. The class of bimarked anodyne maps in $\mathbf{Set}_{\triangle}^{++}$ is stable under smash products with flat monomorphisms, i.e. morphisms $A^{(\flat,\flat)} \to B^{(\flat,\flat)}$ such that the underlying morphism of simplicial sets $A \to B$ is a monomorphism. That is, if $f\colon X \to Y$ is bimarked anodyne and $A \to B$ is a monomorphism of simplicial sets, then

$$(X \times B^{(\flat,\flat)}) \coprod_{X \times A^{(\flat,\flat)}} (Y \times A^{(\flat,\flat)}) \to Y \times B^{(\flat,\flat)}$$

is bimarked anodyne.

Proof. It suffices to show that for any flat boundary inclusion $(\partial\Delta^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}$ and

any generating bimarked anodyne morphism $X \to Y$, the map

$$(X \times (\Delta^n)^{(\flat,\flat)}) \coprod_{X \times (\partial \Delta^n)^{(\flat,\flat)}} (Y \times (\partial \Delta^n)^{(\flat,\flat)}) \to Y \times (\Delta^n)^{(\flat,\flat)}$$

is bimarked anodyne. If $X \to Y$ belongs to one of the classes (1), (2'), (3'), or (4'), then this is true by the arguments of [2, Prop. 3.1.2.3]. If $X \to Y$ belongs to one of the classes (1), (2), (3), or (4), then it is true by the dual arguments. $\square$

Definition 25. For any bimarked simplicial sets X, Y, define a simplicial set $\mathrm{Map}^{(\flat,\flat)}(X, Y)$ by the following universal property: for any simplicial set Z, there is a bijection

$$\mathrm{Hom}_{\mathbf{Set}_\Delta}(Z, \mathrm{Map}^{(\flat,\flat)}(X, Y)) \cong \mathrm{Hom}_{\mathbf{Set}_\Delta^{++}}(Z^{(\flat,\flat)} \times X, Y).$$

Proposition 26. Let $p \colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration of quasicategories, and denote by $\mathcal{C}^\natural \to \mathcal{D}^{(\sharp,\sharp)}$ the associated map of bimarked simplicial sets as in Example 19. Let $X \to Y$ be any bimarked anodyne map of simplicial sets. Then the square

$$
\begin{array}{ccc}
\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^\natural)^\simeq & \longrightarrow & \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural)^\simeq \\
\downarrow & & \downarrow \\
\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{D}^{(\sharp,\sharp)})^\simeq & \longrightarrow & \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})^\simeq
\end{array}
$$

is a homotopy pullback in the Kan model structure.

Proof. First, we show that the right-hand map is a Kan fibration. In fact, the underlying map

$$\mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural) \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})$$

is a trivial Kan fibration, since by Lemma 24 together with Proposition 23 we can solve the necessary lifting problems. This, together with the fact that each of the objects is a Kan complex, implies that in order to show that the above square is homotopy pullback it suffices to check that the map

$$\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^\natural)^\simeq \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural)^\simeq \times_{\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{D}^{(\sharp,\sharp)})^\simeq} \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})^\simeq$$

is a trivial Kan fibration. Since the functor $(-)^\simeq$ is a right adjoint it preserves limits, so it again suffices to show that the underlying map

$$\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^\natural) \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural) \times_{\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{D}^{(\sharp,\sharp)})} \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})$$

is a trivial fibration, which follows from Lemma 24. $\square$

3.5 Cartesian morphisms

In this section, we show that we really have identified the cocartesian morphisms correctly. We will first show that morphisms of the form

$$
\begin{array}{ccc}
& y & \\
{}^{\varnothing}\swarrow & & \searrow^{\bullet} \\
x & & x'
\end{array}
\tag{1}
$$

are p-cocartesian.

Proposition 27. Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration of quasicategories which preserves pullbacks and satisfies the condition of Theorem 15, and let

$$
p\colon \mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})
$$

be the corresponding map between complete Segal spaces of spans. If a morphism in $\mathbf{Span}(\mathcal{C})$ is of the form

$$
\begin{array}{ccc}
& y & \\
{}^{\varnothing}\swarrow & & \searrow^{\bullet} \\
x & & x'
\end{array}
,
$$

where the morphism marked with a $\circ$ is π-cartesian and the morphism marked with a $\bullet$ is π-cocartesian, then it is p-cocartesian.

Proof. In order to show that a morphism $e\colon x \leftarrow y \to x'$ of the form given in Proposition 27 are cocartesian, we have to show that the square

$$
\begin{array}{ccc}
\mathbf{Span}(\mathcal{C})_2 \times_{\mathbf{Span}(\mathcal{C})_{\{0,1\}}} \{e\} & \longrightarrow & \mathbf{Span}(\mathcal{C})_{\{0,2\}} \times_{\mathbf{Span}(\mathcal{C})_{\{0\}}} \{x\} \\
\downarrow & & \downarrow \\
\mathbf{Span}(\mathcal{D})_2 \times_{\mathbf{Span}(\mathcal{D})_{\{0,1\}}} \{\pi e\} & \longrightarrow & \mathbf{Span}(\mathcal{D})_{\{0,2\}} \times_{\mathbf{Span}(\mathcal{D})_{\{0\}}} \{\pi x\}
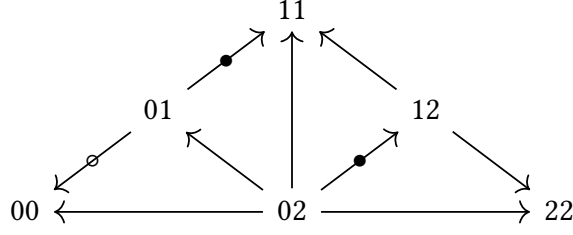\end{array}
$$

is homotopy pullback. Expanding, we have to show that the square

$$
\begin{array}{ccc}
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{C})^{\simeq}} \{e\} & \longrightarrow & \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}),\mathcal{C})^{\simeq}} \{x\} \\
\downarrow & & \downarrow \\
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{D})^{\simeq}} \{\pi e\} & \longrightarrow & \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}),\mathcal{D})^{\simeq}} \{\pi x\}
\end{array}
$$

is homotopy pullback.

Recall the bimarked structure $\mathrm{asd}(\Delta^2)^{\heartsuit} = (\mathrm{asd}(\Delta^2), \mathcal{E}, \mathcal{E}')$ on $\mathrm{asd}(\Delta^2)$ of Example 22, reproduced below, where the nondegenerate edges in $\mathcal{E}$ are distinguished with a $\bullet$, and

the nondegenerate edges in $\mathcal{E}'$ are distinguished with a $\circ$.

Denote the induced bimarked structures on $\mathrm{asd}(\Delta^{\{0,1\}})$, $\mathrm{asd}(\Delta^{\{0,2\}})$, and $\mathrm{asd}(\Delta^{\{0\}})$ also with a heart. Except in the case of $\mathrm{asd}(\Delta^{\{0,1\}})^\heartsuit$, these agree with the $(\flat, \flat)$-bimarked structure.

We now note that we can decompose the above square into two squares

$$
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{C})^{\simeq}} \{e\} \longrightarrow \mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^{\simeq} \times_{\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^\heartsuit,\mathcal{C}^\natural)^{\simeq}} \{e\}
$$

$$
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow
$$

$$
\mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{D})^{\simeq}} \{\pi e\} \to \mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\natural,\natural)})^{\simeq} \times_{\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^\heartsuit,\mathcal{D}^{(\natural,\natural)})^{\simeq}} \{\pi e\}
$$

and

$$
\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^{\simeq} \times_{\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^\heartsuit,\mathcal{C}^\natural)^{\simeq}} \{e\} \longrightarrow \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}),\mathcal{C})^{\simeq}} \{x\}
$$

$$
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow
$$

$$
\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\natural,\natural)})^{\simeq} \times_{\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^\heartsuit,\mathcal{D}^{(\natural,\natural)})^{\simeq}} \{\pi e\} \to \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}),\mathcal{D})^{\simeq}} \{\pi x\}
$$

The first square is a homotopy pullback because the bottom morphism is a full subcategory inclusion of connected components, and the fiber over a connected component corresponding to Cartesian functors $\mathrm{asd}(\Delta^2) \to \mathcal{D}$ is consists precisely of Cartesian functors $\mathrm{asd}(\Delta^2) \to \mathcal{C}$ by the condition of Theorem 15.

Therefore, we need to show that the second square is homotopy pullback. For this it suffices to show separately that the squares

$$
\begin{array}{ccc}
\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^{\simeq} & \longrightarrow & \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{C})^{\simeq} \\
\downarrow & & \downarrow \\
\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\natural,\natural)})^{\simeq} & \longrightarrow & \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,2\}}), \mathcal{D})^{\simeq}
\end{array} \qquad .
$$

and

$$\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^{\heartsuit}, \mathcal{C}^{\natural})^{\simeq} \longrightarrow \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}), \mathcal{C})^{\simeq}$$
$$\downarrow \qquad\qquad\qquad\qquad\qquad \downarrow \qquad\qquad .$$
$$\mathbf{Fun}^{(\flat,\flat)}(\mathrm{asd}(\Delta^{\{0,1\}})^{\heartsuit}, \mathcal{D}^{(\natural,\natural)})^{\simeq} \longrightarrow \mathbf{Fun}(\mathrm{asd}(\Delta^{\{0\}}), \mathcal{D})^{\simeq}$$

are homotopy pullback. But it is easy to see that these are of the form of the square in Proposition 26; the first has $X \to Y = \mathrm{asd}(\Delta^{\{0,2\}})^{\heartsuit} \to \mathrm{asd}(\Delta^2)^{\heartsuit}$, which we saw in Example 22 was bimarked anodyne, and the second has $X \to Y = \mathrm{asd}(\Delta^{\{0\}})^{\heartsuit} \to \mathrm{asd}(\Delta^{\{0,1\}})^{\heartsuit}$, which we saw in Example 21 was bimarked anodyne. Both are therefore homotopy pullback. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Thus we have shown that if a morphism has the form of Equation 1, it is p-cocartesian. It remains to show the reverse implication.

First note that since $\mathrm{asd}(\Delta^0)^{\heartsuit} \to \mathrm{asd}(\Delta^1)^{\heartsuit}$ is bimarked anodyne, every morphism has a lift of the form Equation 1, which we have just shown is p-cocartesian. Since any two p-cocartesian lifts are equivalent, any morphism which is a p-cocartesian lift is equivalent to such a morphism, and hence must be of that form.

References

[1] Andre Joyal and Myles Tierney. "Quasi-categories vs Segal spaces". In: (July 31, 2006). arXiv: math/0607820v2 [math.AT].

[2] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. ISBN: 978-0-691-14049-0.