Cocartesian Fibrations, Categories of Spans, and Transfer Theories

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Master's Thesis

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1 Introduction

Denote by $\mathcal{S}$ the ∞-category of spaces. For any space $X \in \mathcal{S}$, denote by $\mathcal{P}(X)$ the ∞-category of presheaves (of spaces) on X. Recall that for any morphism $f\colon X \to Y$ of spaces, there are several associated functors between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. In particular:

- The functor $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ pulls back presheaves on Y to preasheaves on X, sending a presheaf $\mathcal{F}\colon Y \to \mathcal{S}$ to the presheaf $f^*\mathcal{F}$ given by the composition
$$X \xrightarrow{f} Y \xrightarrow{\mathcal{F}} \mathcal{S} .$$
- The functor $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ pushes forward presheaves via left Kan extension, sending a presheaf $\mathcal{G}\colon X \to \mathcal{S}$ to the left Kan extension $\mathrm{Lan}_f\mathcal{G}$.

$$\begin{tikzcd} X \arrow[rr, "\mathcal{G}"] \arrow[dr, "f"'] & & \mathcal{S} \\ & Y \arrow[ur, "\mathrm{Lan}_f\mathcal{G}"'] & \end{tikzcd}$$

This is a prototype of a common situation: one has a functor (say, of quasicategories) $F\colon \mathcal{C} \to \mathcal{D}$, which sends an object $c \in \mathcal{C}$ to an object $F(c) \in \mathcal{D}$, and a morphism $f\colon c \to c'$ in $\mathcal{C}$ to a morphism $f_!\colon F(c) \to F(c')$ in $\mathcal{D}$. One has additionally a 'wrong-way' map, which creates from a morphism $f\colon c \to c'$ a morphism $f^*\colon F(c') \to F(c)$. In this case, from the data of a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd} & c' \arrow[dl, "g"'] \arrow[dr, "f"] & \\ c & & c'' \end{tikzcd} , \tag{1}$$

one can produce a morphism $F(c) \to F(c'')$ in $\mathcal{D}$ via the composition

$$\begin{tikzcd} & F(c') \arrow[dr, "f_!"] & \\ F(c) \arrow[ur, "g^*"] \arrow[rr, "f_! \circ g^*"'] & & F(c'') \end{tikzcd} .$$

The data of Diagram 1 is known as a *span* in $\mathcal{C}$. It is natural to ask whether this construction can be extended to a functor $\mathrm{Span}(\mathcal{C}) \to \mathcal{D}$, where $\mathrm{Span}(\mathcal{C})$ is an ∞-category whose objects are the same as the objects of $\mathcal{C}$, and whose morphisms are spans in $\mathcal{C}$.

We will define two ∞-categorical models for our category of spans in $\mathcal{C}$: a Segal space $\mathbf{Span}(\mathcal{C})$, and a quasicategory $\mathrm{Span}(\mathcal{C})$. Let us for the moment concentrate on the quasicategorical model, although what we say would equally apply to the Segal space model. In defining the quasicategory $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$, it is necessary to specify a composition law for spans. There is a natural way of doing this: given two spans $X \leftarrow Y \to X'$ and $X' \leftarrow Y' \to X''$ in $\mathcal{C}$, we define their composition to be the pullback $X \leftarrow Y \times_{X'} Y' \to X''$ as below.

$$\begin{tikzcd}
 & & Y \times_{X'} Y' \arrow[dl, dashed, "q'"'] \arrow[dr, dashed, "f'"] & & \\
 & Y \arrow[dl, "g"'] \arrow[dr, "f"] & & Y' \arrow[dl, "q"'] \arrow[dr, "p"] & \\
X & & X' & & X''
\end{tikzcd}$$

In order for a functor $\mathrm{Span}(\mathcal{C}) \to \mathcal{D}$ to be well-defined, it will have to respect this composition law on $\mathrm{Span}(\mathcal{C})$. That is, we must have for all such pullback diagrams an equivalence

$$(p \circ f')_! \circ (g \circ q')^* \simeq p_! \circ q^* \circ f_! \circ g^*.$$

This is equivalent to the simpler condition that for any pullback square as above we must have an equivalence

$$f'_! \circ (q')^* \simeq q^* \circ f_!.$$

This is know as the *base change condtition,* or the *Beck–Chevalley condition.*

The aim of this work is twofold. First, we to provide a new proof of a theorem of Barwick [2, Thm. 12.2], which provides sufficient conditions for a functor of quasicategories $p \colon \mathcal{C} \to \mathcal{D}$ to yield a cocartesian fibration between ∞-categories of spans $\mathrm{Span}(p) \colon \mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$.

Second, we use this result to construct a functor $\hat{r} \colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$, which sends a space X to the category $\mathcal{P}(X)$ of presheaves on X, and a morphism in $\mathrm{Span}(\mathcal{S})$ represented by a span

$$\begin{tikzcd}
 & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\
X & & X'
\end{tikzcd}$$

to the functor $f_! \circ g^* \colon \mathcal{P}(X) \to \mathcal{P}(X')$. We further show that our functor is lax monoidal with respect to a monoidal structure on $\mathrm{Span}(\mathcal{S})$ induced by the cartesian structure on $\mathcal{S}$, and the cartesian structure on $\mathcal{C}\mathrm{at}_\infty$.

We begin by providing a new proof of the theorem of Barwick mentioned above. Barwick's proof is explicit, constructing the necessary horn fillings by enumerating the necessary simplices, and arguing one-by-one why each filling is possible. This is an impressive feat of simplicial combinatorics, but provides little intuition for why the result might be true.

Our proof is more homotopy-theoretic in character, relying on the fact that for any quasicategory $\mathcal{C}$ with pullbacks, the ∞-category of spans in $\mathcal{C}$ has a a natural incarnation as a complete Segal space $\mathbf{Span}(\mathcal{C})$; the quasicategory $\mathrm{Span}(\mathcal{C})$ is then the 'first row' of this complete Segal space. We define a notion of cocartesian fibration between complete Segal spaces, and show that any such cocartesian fibration gives a cocartesian fibration between first rows. The readily available homotopical data in complete Segal spaces allow us to define cocartesian morphisms purely via a condition on 2-simplices, where the combinatorics of horn filling in categories of spans is more manageable.

Our definition of a cocartesian fibration between Segal spaces is not new, although the form in which it is presented is original. The definition was first written down by De Brito in [3], and expanded by Rasekh in [8]. There, Rasekh defines a model structure whose fibrant objects model cocartesian fibrations between Segal spaces. In defining this model structure, it is necessary to distinguish certain morphisms, and Rasekh solves this problem by adding an extra simplicial dimension; the model structure for cocartesian fibrations defined there is thus a model structure on slice categories of trisimplicial sets. In this work we restrict our attention to cocartesian fibrations between Segal spaces. This allows us to approach the problem of controlling cocartesian morphisms differently, by introducing a marking. This method is much closer to that used to work with cocartesian fibrations in [6], and has the advantage that many of the results proved there can be leveraged in explicit calculations. In particular, we provide a definition of a cocartesian fibration between complete Segal spaces to which Rasekh's definition reduces in the case that both the domain and codomain are complete Segal spaces.

1.1 Outline

This work consists of three sections. In Section 2, we explore cocartesian fibrations between Segal spaces. After a review in Section 2.1 of some material in [1], most notably the box product $-\square-$ and its adjoints, we define marked bisimplicial sets in Section 2.2. We then define a marked version of the box functor, and prove some results analogous to the unmarked case. In Section 2.3, we prove some technical lemmas about simplicial sets with certain restrictions on individual morphisms. The main result is a lemma which allows us to translate a condition on marked bisimplicial sets for a 'pointwise condition' involving unmarked bisimplicial sets.

In Section 2.4, we define the notion of a cocartesian morphism via a condition on left horn filling of 2-simplices, and define a cocartesian fibration between complete Segal spaces to be a Reedy fibration admitting cocartesian lifts. We show that this implies all higher left horn filling conditions, and use this to show that any cocartesian fibration between complete segal spaces yields a cocartesian fibration (in the sense of quasicategories) between first rows.

In Section 3, we use these results to prove the theorem of Barwick mentioned above, originally published in [2, Thm. 12.2]. The proof leverages the fact, proven in Section 2 that, when checking that a morphism in a Segal space is cocartesian relative to some Reedy fibration, it suffices to check that the lowest lifting problem $\Lambda^2_1 \hookrightarrow \Delta^2$ has a contractible space of solutions, which is combinatorially tractible.

In Section 4, we construct the functor $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$, together with a lax monoidal structure on it.

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2 Cocartesian fibrations between Segal spaces

2.1 A review of bisimplicial sets

In this section we review the basic theory of bisimplicial sets as laid out in [1]. This is mainly to fix notation.

Bisimplicial sets can be defined in two equivalent ways:

- As functors $\Delta^{\mathrm{op}} \to \mathcal{S}\mathrm{et}_\Delta$
- As functors $(\Delta^{\mathrm{op}})^2 \to \mathcal{S}\mathrm{et}$.

In the former case, we think of a bisimplicial set X as an $\mathbb{N}$-indexed collection of simplicial sets X_n; in the latter, we think of a bisimplicial set as an $\mathbb{N} \times \mathbb{N}$-indexed collection of sets X_{mn}. Both points of view are useful, and we will rely on both of them. For this reason, we fix the following convention: with the second point of view in mind, we imagine a bisimplicial set X as a collection of sets, each located at an integer lattice point of the first quadrant, where the first coordinate increases in the x-direction and the second coordinate increases in the y-direction.

$$\begin{array}{cccc}
X_{03} & X_{13} & X_{23} & X_{33} \\
X_{02} & X_{12} & X_{22} & X_{32} \\
X_{01} & X_{11} & X_{21} & X_{31} \\
X_{00} & X_{10} & X_{20} & X_{30}
\end{array}$$

Thus, the nth row of X is the simplicial set $X_{\bullet n}$, and the mth column of X is the simplicial set $X_{m\bullet}$. When we think of bisimplicial sets as $\mathbb{N}$-indexed collections of simplicial sets X_m, we mean by X_m the mth *column* of X; that is, $X_m = X_{m,\bullet}$.

If X is Reedy fibrant (i.e. fibrant with respect to the Reedy model structure on $\mathcal{S}\mathrm{et}_{\Delta^2}$, see Subsection 2.1), then each simplicial set X_n is a Kan complex. Later, when we are interested in Segal spaces, we will interpret X_n as the space of n-simplices of X.

Definition 2.1.1. Define a functor $-\Box-\colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_{\Delta^2}$ by the formula

$$(X, Y) \mapsto X\Box Y, \qquad (X\Box Y)_{mn} = X_m \times Y_n.$$

We will call this functor the ***box product.***

Example 2.1.2. By the Yoneda lemma, providing a map $\Delta^m \Box \Delta^n \to X$ is equivalent to providing as an element of X_{mn}.

Definition 2.1.3. Let A denote a simplicial set, and X a bisimplicial set.

- Define a simplicial set $A\backslash X$ level-wise by

$$(A\backslash X)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(A \Box \Delta^n, X).$$

- Define a simplicial set X/A level-wise by

$$(X/A)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n \Box A, X).$$

Note that by Example 2.1.2, the simplicial set $\Delta^m\backslash X$ is the mth column of X, which we have agreed to call X_m. Similarly, the simplicial set X/Δ^n is the nth row of X. In particular X/Δ^0 is the zeroth row of X. (This, confusingly, is usually called the *first row* of X; this terminological inconsistency is somewhat justified by the fact that, unlike the columns, one tends to be interested mainly in the zeroth row.)

We provide here a partial proof of the following result because we will need to refer to it later, when we prove Proposition 2.2.8.

Proposition 2.1.4. The box product is *divisible on the left.* This means that for each simplicial set A, there is an adjunction

$$A \Box - \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2} : A\backslash - \,.$$

Similarly, the box product is *divisible on the right.* This means that for each simplicial set B there is an adjunction

$$-\Box B \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2} : -/B.$$

Proof. We prove divisibility on the left; because the Cartesian product is symmetric, divisibility on the right is identical. We do this by explicitly exhibiting a natural bijection

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A \Box B, X) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X).$$

Define a map

$$\Phi \colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A \Box B, X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X)$$

by sending a map $f \colon A \Box B \to X$ to the map $\tilde{f} \colon B \to A\backslash X$ which sends an n-simplex $b \in B_n$ to the composition

$$A \Box \Delta^n \xrightarrow{(\mathrm{id}, b)} A \Box B \xrightarrow{f} X \,.$$

Before we define our map in the other direction, we need an intermediate result. Define a map $\mathrm{ev}\colon A\Box(A\backslash X) \to X$ level-wise by taking $(a, \sigma) \in A_m \times (A\backslash X)_n$ to

$$\sigma_{mn}(a, \mathrm{id}_{\Delta^n}) \in X_{mn}.$$

Then define a map

$$\Psi\colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\Box B, X)$$

sending a map $g\colon B \to A\backslash X$ to the composition

$$A\Box B \xrightarrow{(\mathrm{id},g)} A\Box(A\backslash X) \xrightarrow{\mathrm{ev}} X \ .$$

The maps Φ and Ψ are mutually inverse, and provide the necessary natural bijection.

The other bijection is defined analogously, so we only fix notation which we will need later. We will call the mutually inverse maps

$$\Phi'\colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\Box B, X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(A, X/B)$$

and

$$\Psi'\colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(A, X/B) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\Box B, X),$$

where in defining Ψ' we use a map $\mathrm{ev}'\colon (X/B)\Box B \to X$ determined by sending

$$(\phi\colon \Delta^m\Box B \to X, b \in B_n) \mapsto \phi_{mn}(\mathrm{id}_{\Delta^m}, b). \qquad \Box$$

Proposition 2.1.4, together with the fact that $\mathcal{S}\mathrm{et}_\Delta$ and $\mathcal{S}\mathrm{et}_{\Delta^2}$ are finitely complete and cocomplete, implies all of the results of Appendix A.2 apply to the box product. In the notation found there, we can give a compact formulation of the definition of a Reedy fibration which we will use repeatedly.

Definition 2.1.5. Let $f\colon X \to Y$ be a map between bisimplicial sets. The map f is a ***Reedy fibration*** if either of the following equivalent conditions hold.

- For each monomorphism $u\colon A \to A'$, the map $\langle u\backslash f\rangle$ is a Kan fibration.
- For each anodyne map $v\colon B \to B'$, the map $\langle f/v\rangle$ is a trivial Kan fibration.

(For the notations $\langle u\backslash f\rangle$ and $\langle f/v\rangle$ see Appendix A.2.) That Definition 2.1.5 is equivalent to the usual definition is shown in [1, Prop. 3.4]. We will also make use of the following fact ([1, Prop. 3.10]).

Theorem 2.1.6. If $f\colon X \to Y$ is a Reedy fibration between Segal spaces, then for any monomorphism of simplicial sets v, the map $\langle f/v\rangle$ is an inner fibration.

Corollary 2.1.7. If $f\colon X \to Y$ is a Reedy fibration between Segal spaces, then $f/\Delta^0\colon X/\Delta^0 \to Y/\Delta^0$ is an inner fibration between quasicategories.

2.2 Marked bisimplicial sets

In this section, we define a basic theory of marked bisimplicial sets, in analogy to the theory of marked simplicial sets laid out in [6]. In the following, one should keep in mind that the case in which we are mostly interested is when our bisimplicial spaces are Segal spaces, and thus that only the first (horizontal) simplicial direction should be thought of as categorical. For this reason, only the first simplicial direction will carry a marking.

Definition 2.2.1. A ***marked bisimplicial set*** $(X, \mathcal{E})$ is a bisimplicial set X together with a distinguished subset $\mathcal{E} \subseteq X_{10}$ containing all degenerate edges, i.e. all edges in the image of $s_0 \colon X_{00} \to X_{10}$. Equivalently, a marked bisimplicial set is bisimplicial set X together with a marking $\mathcal{E}$ on the simplicial set X/Δ^0.

Definition 2.2.2. For a marked simplicial set A and an unmarked simplicial set B, define a marking on the bisimplicial set $A\Box B$ as follows: a simplex $(a, b) \in A_1 \times B_0$ is marked if and only if a is marked in A.

This construction gives us a functor

$$-\Box- \colon \mathcal{S}\mathrm{et}_\Delta^+ \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_{\Delta^2}^+.$$

This is potentially ambiguous: we are using the same notation for the marked and unmarked box constructions. However, there is no real chance of confusion: when we write $A\Box B$, we mean the marked construction if A is a marked simplicial set and the unmarked construction if A is an unmarked simplicial set.

Our first order of business is to generalize the results of [1] summarized in Section A.2 to the marked case. We will first show that the above functor is divisible on the left and on the right.

Notation 2.2.3. For any marked simplicial set A, denote the underlying unmarked simplicial set by $\mathring{A}$. Similarly, for any marked bisimplicial set X, denote the underlying unmarked bisimplicial set by $\mathring{X}$.

Definition 2.2.4. Let A denote a marked simplicial set, B an unmarked simplicial set, and X a marked bisimplicial set.

- Define an unmarked simplicial set $A\backslash X$ level-wise by

$$(A\backslash X)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}^+}(A\Box\Delta^n, X).$$

- Define a marked simplicial set X/B as follows. The underlying simplicial set is the same as $\mathring{X}/B$, and a 1-simplex $\Delta^1 \to X/B$ is marked if and only if the corresponding map $\Delta^1\Box B \to \mathring{X}$ of unmarked bisimplicial sets descends to a map of marked bisimplicial sets $(\Delta^1)^\sharp\Box B \to X$.

Again, we are overloading notation, so there is the potential for confusion. However, there is no real ambiguity; the symbol $A\backslash X$ means the marked construction if A and X are marked, and the unmarked construction if A and X are unmarked. We have tried to be clear in stating whether (bi)simplicial sets do or do not carry markings.

Example 2.2.5. Recall that we can think of a marked bisimplicial set X as an unmarked bisimplicial set $\mathring{X}$ together with a marking $\mathcal{E}$ on the simplicial set $\mathring{X}/\Delta^0$. The marking $\mathcal{E}$ agrees with the marking on X/Δ^0.

We will need the following analogs of the $\flat$- and $\sharp$-markings for marked simplicial sets.

Example 2.2.6. For any unmarked bisimplicial set X, we have the following canonical markings.

- The *sharp marking* $X^\sharp$, in which each element of X_{10} is marked.
- The *flat marking* $X^\flat$, in which only the edges in the image of $s_0 \colon X_{00} \to X_{10}$ are marked.

Example 2.2.7. For each unmarked simplicial set A and marked bisimplicial set X, there is an isomorphism

$$A^\flat \backslash X \cong A \backslash \mathring{X}.$$

Similarly, for any unmarked bisimplicial set Y and marked simplicial set B, there is an isomorphism

$$B \backslash Y^\sharp \cong \mathring{B} \backslash Y.$$

The marked constructions above have similar properties to the unmarked constructions from Section 2.1. In particular, we have the following.

Proposition 2.2.8. We have the following adjunctions.

1. For each marked simplicial set $A \in \mathcal{S}\mathrm{et}_\Delta^+$ there is an adjunction.

$$A \square - \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}^+_{\Delta^2} : A \backslash -$$

2. For each unmarked simplicial set $B \in \mathcal{S}\mathrm{et}_\Delta$ there is an adjunction.

$$- \square B \colon \mathcal{S}\mathrm{et}_\Delta^+ \longleftrightarrow \mathcal{S}\mathrm{et}^+_{\Delta^2} : -/B.$$

Proof. We start with the first, fixing a marked simplicial set A, an unmarked simplicial set B, and a marked bisimplicial set X. We have inclusions

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_{\Delta^2}}(A \square B, X) \overset{i_0}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(\mathring{A} \square B, \mathring{X})$$

and

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A \backslash X) \overset{i_1}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, \mathring{A} \backslash \mathring{X}).$$

We have a natural bijection between the right-hand sides of the above inclusions given by the maps Φ and Ψ of Proposition 2.1.4. To show that there is a natural bijection between the subsets, it suffices to show that Φ and Ψ restrict to maps between the subsets.

To this end, suppose we have a map of bimarked simplicial sets $f\colon A\Box B \to X$. The inclusion i_0 forgets the markings, sending this to the map

$$\mathring{f}\colon \mathring{A}\Box B \to \mathring{X}.$$

Under Φ, this is taken to a map $\Phi(\mathring{f})\colon B \to \mathring{A}\backslash\mathring{X}$. We would like to show that $\Phi(\mathring{f})$ factors through $A\backslash X$, giving a map $\tilde{f}\colon B \to A\backslash X$. The map $\Phi(\mathring{f})$ takes an n-simplex $b \in B_n$ to the composition

$$\mathring{A}\Box\Delta^n \xrightarrow{(\mathrm{id},b)} \mathring{A}\Box B \xrightarrow{\mathring{f}} \mathring{X}\ .$$

We need to check that this is an n-simplex in $A\backslash X$, and not just $\mathring{A}\backslash\mathring{X}$, i.e. that it respects the markings on $A\Box\Delta^n$ and X. That (id, b) respects the markings on $A\Box\Delta^n$ and $A\Box B$ is clear, and $\mathring{f}$ respects the markings on $A\Box B$ and X because f is a map of marked simplicial sets by assumption. Thus $\Phi(\mathring{f})$ restricts to a map $\tilde{f}\colon B \to A\backslash X$.

Now we show the other direction. Suppose we have a map $g\colon B \to A\backslash X$. The inclusion i_1 takes this to the composition

$$B \xrightarrow{g} A\backslash X \hookrightarrow A^\flat\backslash X \cong \mathring{A}\backslash\mathring{X},$$

which we denote by $\mathring{g}$ by mild abuse of notation. Under Ψ, this is mapped to the composition

$$\Psi(\mathring{g})\colon\ \mathring{A}\Box B \xrightarrow{\mathrm{id}\times\mathring{g}} \mathring{A}\Box(\mathring{A}\backslash\mathring{X}) \xrightarrow{\mathrm{ev}} \mathring{X}.$$

We need to check that this respects the markings on $A\Box B$ and X, i.e. that for each marked simplex $a \in A_1$ and each $b \in B_0$, the element $\Psi(\mathring{g})_{10}(a,b)$ is marked in X_{10}. But $\Psi(\mathring{g})_{10}(a,b) = g(b)_{10}(a, \mathrm{id}_{\Delta^0})$, which is marked because g lands in $A\backslash X$ by assumption. Thus, $\Psi(\mathring{g})$ descends to a map $\tilde{g}\colon A\Box B \to X$.

Now we show the other bijection. Unlike the unmarked case, because of the asymmetry of the marked box product, this is not precisely the same as what we have just shown. Again we have inclusions

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_{\Delta^2}}(A\Box B, X) \overset{j_0}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(\mathring{A}\Box B, \mathring{X})$$

and

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_{\Delta}}(A, B\backslash X) \overset{j_1}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(\mathring{A}, B\backslash\mathring{X}),$$

and a bijection between the right-hand sides given by the maps Φ' and Ψ' from

Proposition 2.1.4. As before, suppose that

$$f\colon A\Box B \to X$$

is a map of marked bisimplicial sets. Under j_0, this is sent to $\mathring{f}\colon \mathring{A}\Box B \to \mathring{X}$. Then $\Phi'(\mathring{f})\colon A \to X/B$ is defined by sending $\sigma \in A_n$ to the composition

$$\Delta^n\Box B \xrightarrow{(\sigma,\mathrm{id})} \mathring{A}\Box B \xrightarrow{\mathring{f}} \mathring{X}\ .$$

We need to show that for each marked $a \in A_1$, the corresponding map

$$\Phi'(\mathring{f})(a)\colon\ \Delta^1\Box B \xrightarrow{(a,\mathrm{id})} \mathring{A}\Box B \xrightarrow{\mathring{f}} \mathring{X}$$

descends to a map of marked bisimplicial sets $\tilde{f}\colon (\Delta^1)^\sharp\Box B \to X$, and thus corresponds a marked 1-simplex in to X/B. But that the first map has this property is clear because a is marked by assumption, and the map $\mathring{f}$ has this property because f is a map of marked simplicial sets by assumption.

Now, let $g\colon A \to X/B$ be a map of marked simplicial sets. We need to check that the composition

$$\Psi(\mathring{g})\colon\ \mathring{A}\Box B \xrightarrow{(\mathring{g},\mathrm{id})} (\mathring{X}/B)\Box B \xrightarrow{\mathrm{ev}'} \mathring{X}$$

takes marked edges to marked edges. Let $(a,b) \in A_1 \times B_0$, with a marked. This maps to

$$(a,b) \mapsto (g(a),b) \mapsto g(a)_{10}(\mathrm{id}_{\Delta^1},b) \in X_{10}.$$

By definition, $g(a)$ is a map of marked simplicial sets

$$(\Delta^1)^\sharp\Box B \to X$$

which therefore sends $(\mathrm{id}_{\Delta^1},b)$ to a marked edge in X by assumption. □

This shows that the marked version of the box product $\Box$ is, in the language of [1], *divisible on the left and on the right.* Thus, the results summarized in Section A.2 apply.

Definition 2.2.9. We will call an inclusion of unmarked simplicial sets $B \hookrightarrow B'$ *full* if it has the following property: an n-simplex $\sigma\colon \Delta^n \to B'$ factors through B if and only if each vertex of σ factors through B. That is, any n-simplex in B' whose vertices belong to B belongs to B.

Lemma 2.2.10. For any marked simplicial set A and Reedy-fibrant marked bisimplicial set X, the simplicial set $A\backslash X$ is a Kan complex, and the inclusion $i\colon A\backslash X \hookrightarrow A^\flat\backslash X \cong \mathring{A}\backslash\mathring{X}$ is full.

Proof. We first show that the map i is a full inclusion. The n-simplices of $A\backslash X$ are maps of marked simplicial sets $\tilde{\sigma}\colon A\Box\Delta^n \to X$. A map of underlying bisimplicial sets gives a map of marked bisimplicial sets if and only if it respects the markings, i.e. if and only if for each $(a, i) \in A_1 \times (\Delta^n)_0$ with a marked, $\tilde{\sigma}(a, i)$ is marked in X. This is equivalent to demanding that $\sigma|_{\Delta^{\{i\}}}$ belong to $A\backslash X$.

To show that $A\backslash X$ is a Kan complex, we need to find dashed lifts

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}, \qquad n \geq 1, \quad 0 \leq k \leq n.$$

For $n = 1$, the horn inclusion is of the form $\Delta^0 \hookrightarrow \Delta^1$, and we can take the lift to be degenerate. For $n \geq 2$, we can augment our diagram as follows.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \arrow[r] & A^\flat\backslash X \\ \Delta^n \arrow[urr, dashed] & & \end{tikzcd}.$$

Since $A^\flat\backslash X$ is a Kan complex, we can always find such a dashed lift. The inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is surjective on vertices, so our lift factors through $A\backslash X$. $\square$

Recall that when thinking of a bisimplicial set X as a cosimplicial object in $\mathcal{S}\mathrm{et}_\Delta$, we think of the simplicial set X_1 as the space of 1-simplices in X. In particular, if X is a Segal space, then X_1 should be thought of as the space of morphisms in X. We should think of morphisms which are in the same path component of X_1 as equivalent. Therefore, we would like to pay special attention to markings which respect this homotopical structure.

Definition 2.2.11. Let $(X, \mathcal{E})$ be a marked bisimplicial set. We will say that the marking $\mathcal{E}$ ***respects path components*** if it has the following property: for any map $\Delta^1 \to X_1$ representing an edge $e \to e'$ between morphisms e and e', the morphism e is marked if and only if the morphism e' is marked.

Proposition 2.2.12. Let $f\colon X \to Y$ be a Reedy fibration between marked bisimplicial sets such that the marking on X respects path components, and let $u\colon A \to A'$ be a morphism of marked simplicial sets whose underlying morphism of unmarked simplicial sets is a monomorphism. Then the map $\langle u\backslash f\rangle$ is a Kan fibration.

Proof. We need to show that for each $n \geq 0$ and $0 \leq k \leq n$ we can solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[d, "\langle u\backslash f\rangle"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & A\backslash X \times_{A'\backslash Y} A'\backslash Y \end{tikzcd}.$$

First assume that $n \geq 2$. We can augment the above square as follows.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[r] \arrow[d] & A^\flat\backslash X \arrow[d] \\
\Delta^n \arrow[r] & A\backslash X \times_{A'\backslash Y} A'\backslash Y \arrow[r] & A^\flat\backslash X \times_{(A')^\flat\backslash Y} (A')^\flat\backslash X
\end{tikzcd}.$$

Since the map on the right is a Kan fibration, we can solve the outer lifting problem. All the vertices of Δ^n belong to Λ^n_k, so a lift of the outside square factors through $A'\backslash X$.

Now take $n = 1$, $k = 0$, so our horn inclusion is $\Delta^{\{0\}} \hookrightarrow \Delta^1$. By Proposition A.2.2, the lifting problem we need to solve is equivalent to

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X/\Delta^1 \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X/\Delta^0 \times_{Y/\Delta^0} Y/\Delta^1
\end{tikzcd}.$$

Because f is a Reedy fibration, the underlying diagram

$$\begin{tikzcd}
\mathring{A} \arrow[r] \arrow[d] & \mathring{X}/\Delta^1 \arrow[d] \\
\mathring{A}' \arrow[r] \arrow[ur, dashed] & \mathring{X}/\Delta^0 \times_{\mathring{Y}/\Delta^0} \mathring{Y}/\Delta^1
\end{tikzcd}.$$

of unmarked simplicial sets always admits a lift. It therefore suffices to check that any such lift respects the marking on X. To see this, consider the following triangle formed by some dashed lift.

$$\begin{tikzcd}
& \mathring{X}/\Delta^1 \arrow[dd] \\
\mathring{A}' \arrow[ur, dashed] \arrow[dr] & \\
& \mathring{X}/\Delta^0
\end{tikzcd}$$

Let $a \in A'_1$ be a marked 1-simplex, and consider the diagram

$$\begin{tikzcd}
\Delta^1\square\Delta^0 \arrow[r, "{(a,\mathrm{id})}"] \arrow[dd] & \mathring{A}'\square\Delta^0 \arrow[dd] \arrow[dr, "\gamma"] & \\
& & \mathring{X} \\
\Delta^1\square\Delta^1 \arrow[r, "{(a,\mathrm{id})}"] & \mathring{A}'\square\Delta^1 \arrow[ur, dashed, "\ell"'] &
\end{tikzcd},$$

where the triangle on the right is the adjunct to the triangle above. In order to check that the dashed lift respects the marking on X, we have to show that for each $(a,b) \in (A'\square\Delta^1)_{10} = A'_1 \times \{0,1\}$ with a marked, the element $\ell(a,b) \in X_{10}$ is marked. Because the map γ comes from a map of marked simplicial sets, the commutativity of the triangle guarantees this for $b = 0$. The map $\Delta^1\square\Delta^1 \to \mathring{X}$ gives us a 1-simplex $\Delta^1 \to X_1$ representing a 1-simplex $\ell(a,0) \to \ell(a,1)$, which implies by that $\ell(a,1)$ is also marked because each marking respects path components.

The case $n = 1$, $k = 1$ is analogous. $\square$

2.3 Simplicial technology

In the next section, we will need to work in several different cases with simplicial subsets $A \subseteq \Delta^n$ with certain conditions placed on the edge $\Delta^{\{0,1\}}$. In this section we prove some technical results in this direction. The main result in this section is Lemma 2.3.5.

For the remainder of this section, fix $n \geq 2$.

Definition 2.3.1. Let X be an unmarked bisimplicial set, and let $e \in X_{10}$. For any simplicial subset $A \subseteq \Delta^n$ such that $\Delta^{\{0,1\}} \subseteq A$, we will use the notation

$$(A\backslash X)^e = A\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\}.$$

The simplicial set $(A\backslash X)^e$ should be thought of as the space of A-shaped diagrams in X with the edge $\Delta^{\{0,1\}}$ fixed. The m-simplices of the simplicial set $(A\backslash X)^e$ are maps $A\square\Delta^m \to X$ such that the pullback

$$\Delta^{\{0,1\}}\square\Delta^m \longrightarrow A\square\Delta^m \longrightarrow X$$

factors through the map $\Delta^{\{0,1\}}\square\Delta^0 \to X$ corresponding to the element $e \in X_{10}$ under the Yoneda embedding.

Comparing simplices level-wise, it is easy to see the following.

Lemma 2.3.2. The square

$$\begin{tikzcd}
(\Delta^n\backslash X)^e \arrow[r] \arrow[d] & \Delta^n\backslash X \arrow[d] \\
(\Delta^n\backslash Y)^{f(e)} \times_{(A\backslash Y)^{f(e)}} (A\backslash X)^e \arrow[r] & \Delta^n\backslash Y \times_{A\backslash Y} A\backslash X
\end{tikzcd}$$

is a (strict) pullback.

Definition 2.3.3. For any simplicial subset $A \subseteq \Delta^n$ containing $\Delta^{\{0,1\}}$, denote the marking on A where the only marked nondegenerate edge is $\Delta^{\{0,1\}}$ by $\mathcal{L}$, and the corresponding marked simplicial set by $A^{\mathcal{L}}$.

Again, comparing simplices level-wise shows the following.

Lemma 2.3.4. Let $f\colon X \to Y$ be a map of marked bisimplicial sets, and let $A \subseteq \Delta^n$ be a simplicial subset with $\Delta^{\{0,1\}} \subseteq A$. Then for any marked edge $e \in X_{10}$, the square

$$\begin{tikzcd}
(\Delta^n \backslash \mathring{X})^e \arrow[r] \arrow[d] & (\Delta^n)^{\mathcal{L}} \backslash X \arrow[d] \\
(\Delta^n \backslash \mathring{Y})^{f(e)} \times_{(A\backslash \mathring{Y})^{f(e)}} (A\backslash \mathring{X})^e \arrow[r] & (\Delta^n)^{\mathcal{L}} \backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X
\end{tikzcd}$$

is a (strict) pullback.

Lemma 2.3.5. Let $f\colon X \to Y$ be a Reedy fibration between marked bisimplicial sets, and let $i\colon A \subseteq \Delta^n$ be a simplicial subset containing $\Delta^{\{0,1\}}$. The following are equivalent:

1. The map
$$\langle i^{\mathcal{L}} \backslash f \rangle \colon (\Delta^n)^{\mathcal{L}} \backslash X \to (\Delta^n)^{\mathcal{L}} \backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X$$
is a trivial fibration.

2. For each marked $e \in X_{10}$, the map
$$p_e\colon (\Delta^n \backslash \mathring{X})^e \to (\Delta^n \backslash \mathring{Y})^{f(e)} \times_{(A\backslash \mathring{Y})^{f(e)}} (A\backslash \mathring{X})^e$$
is a trivial fibration.

Proof. Suppose the first holds. Then Lemma 2.3.4 implies the second.

Next, suppose that the second holds. By Proposition 2.2.12, the map $\langle i^{\mathcal{L}} \backslash f \rangle$ is a Kan fibration, so it is a trivial Kan fibration if and only if its fibers are contractible. Consider any map

$$\gamma\colon \Delta^0 \to (\Delta^n)^{\mathcal{L}} \backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X.$$

This gives us in particular a map $\Delta^0 \to A^{\mathcal{L}}\backslash X$, which is adjunct to a map $A^{\mathcal{L}} \square \Delta^0 \to X$. The pullback

$$(\Delta^{\{0,1\}})^{\sharp} \square \Delta^0 \longrightarrow A^{\mathcal{L}} \square \Delta^0 \longrightarrow X$$

gives us a marked morphism $e \in X_{10}$. The bottom composition in the below diagram is thus a factorization of γ, in which the left-hand square is a pullback.

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & (\Delta^n \backslash \mathring{X})^e \arrow[r] \arrow[d, "p_e"] & (\Delta^n)^{\mathcal{L}} \backslash X \arrow[d] \\
\Delta^0 \arrow[r] & (\Delta^n \backslash \mathring{Y})^{f(e)} \times_{(A\backslash \mathring{Y})^{f(e)}} (A\backslash \mathring{X})^e \arrow[r] & (\Delta^n)^{\mathcal{L}} \backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X
\end{tikzcd}$$

The right-hand square is a pullback by Lemma 2.3.4. Since by assumption p_e is a trivial fibration, F is contractible. But by the pasting lemma, F is the fiber of

$\langle i^{\mathcal{L}} \backslash f \rangle$ over γ. Thus, the fibers of $\langle i^{\mathcal{L}} \backslash f \rangle$ are contractible, so $\langle i^{\mathcal{L}} \backslash f \rangle$ is a trivial Kan fibration. $\square$

2.4 Cocartesian fibrations

Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. There are several equivalent ways of defining when a morphism in $\mathcal{C}$ is π-cocartesian. For our purposes, the following will be the most useful: a morphism $e \in \mathcal{C}_1$ is π-cocartesian if and only if, for all $n \geq 2$, a dashed lift in the below diagram exists.

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "e"] & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

We would like to find an analogous definition for a p-cocartesian morphism where $p\colon C \to D$ is a Reedy fibration between Segal spaces. Our definition should have the property that if a morphism in C_{10} is p-cocartesian in the sense of Segal spaces, then it is p/Δ^0-cocartesian in the sense of quasicategories.

To this end, replace π in the above diagram by p/Δ^0. Passing to the adjoint lifting problem, we see that the existence of the above lift is equivalent to demanding that the map

$$p_e\colon (\Delta^n \backslash C)^e \to (\Lambda^n_0 \backslash C)^e \times_{(\Lambda^n_0 \backslash D)^{p(e)}} (\Delta^n \backslash D)^{p(e)} \tag{2}$$

be surjective on vertices. One natural avenue of generalization of the concept of a cocartesian morphism to Segal spaces would be to upgrade the condition of surjectivity on vertices to an analogous, homotopy-invariant condition which implies it. One such condition is that p_e be a trivial fibration. Indeed, this is the defintion we will use. However, this turns out to be equivalent to demand something superficially weaker.

Definition 2.4.1. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces. A morphism $e \in X_{10}$ is ***f-cocartesian*** if the square

$$\begin{tikzcd} (\Delta^2 \backslash X)^e \arrow[r] \arrow[d] & (\Lambda^2_0 \backslash X)^e \arrow[d] \\ (\Delta^2 \backslash Y)^{f(e)} \arrow[r] & (\Lambda^2_0 \backslash Y)^{f(e)} \end{tikzcd}$$

is homotopy pullback.

Example 2.4.2. Identity morphisms are f-cocartesian. This is because for $e = \mathrm{id}$, the horizontal morphisms in Definition 2.4.1 are equivalences. More generally, by [9, Lemma 11.6], homotopy equivalences are f-cocartesian.

Cocartesian morphisms automatically respect path components in the following sense.

Proposition 2.4.3. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $\alpha\colon \Delta^1 \to X_1$ be a map representing a path $e \to e'$ between morphisms e and e' in X_1. Then e is f-cocartesian if and only if e' is f-cocartesian.

Proof. Since X and Y are Reedy fibrant and f is a Reedy fibration, it suffices to show that the map

$$(\Delta^2\backslash X)^e \to (\Lambda^2_0\backslash X)^e \times_{(\Lambda^2_0\backslash Y)^{f(e)}} (\Delta^2\backslash Y)^{f(e)} \tag{3}$$

is a weak equivalence if and only if the map

$$(\Delta^2\backslash X)^{e'} \to (\Lambda^2_0\backslash X)^{e'} \times_{(\Lambda^2_0\backslash Y)^{f(e')}} (\Delta^2\backslash Y)^{f(e')} \tag{4}$$

is a weak equivalence.

Consider the diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d] & \Delta^2\backslash X \arrow[d] \\
Q \arrow[r] \arrow[d] & \Lambda^2_0\backslash X \times_{\Lambda^2_0\backslash Y} \Delta^2\backslash Y \arrow[d] \\
\Delta^1 \arrow[r, "\alpha"] & \Delta^{\{0,1\}}\backslash X
\end{tikzcd},$$

where both squares are strict pullback. The maps on the right-hand side are Kan fibrations by Reedy fibrancy and the fact that f is a Reedy fibration, so we get in particular a diagram

$$\begin{tikzcd}
P \arrow[rr, "\phi"] \arrow[dr] & & Q \arrow[dl] \\
& \Delta^1 &
\end{tikzcd},$$

where both downward-facing maps are Kan fibrations. Note that the component of the map ϕ over Δ^0 is the map from Equation 3, and component over Δ^1 is the map from Equation 4. Kan fibrations are in particular left fibrations, so under the Grothendieck construction this corresponds to a diagram given by the following homotopy-commutative square in $\mathcal{S}\mathrm{et}_\Delta$ (with the Kan model structre), in which the rightward-pointing maps are weak equivalences because our maps

$P \to \Delta^1$ and $Q \to \Delta^1$ were Kan fibrations.

$$\begin{tikzcd}
(\Delta^2\backslash X)^e \arrow[r, "\simeq"] \arrow[d] & (\Delta^2\backslash X)^{e'} \arrow[d] \\
(\Lambda^2_0\backslash X)^e \times_{(\Lambda^2_0\backslash Y)^{f(e)}} (\Delta^2\backslash Y)^{f(e)} \arrow[r, "\simeq"] & (\Lambda^2_0\backslash X)^{e'} \times_{(\Lambda^2_0\backslash Y)^{f(e')}} (\Delta^2\backslash Y)^{f(e')}
\end{tikzcd}$$

By the 2/3 property for weak equivalences, the map on the left is a weak equivalence if and only if the map on the right is a weak equivalence, which is what we wanted to show. □

Definition 2.4.4. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be a cocartesian morphism. Denote by $\mathcal{E}$ the smallest marking which contains e and all degenerate morphisms in X, and which respects path components. More explicitly, the marking $\mathcal{E}$ contains:

- The morphism e;
- Each identity morphism; and
- Any morphism connected to e or any identity morphism by a path.

Our next order of business is to show that our definition of cocartesian morphisms in terms of lifting with respect to the morphism $\Lambda^2_0 \hookrightarrow \Delta^2$ implies lifting with respect to $\Lambda^n_0 \to \Delta^n$ for all $n \geq 2$.

Definition 2.4.5. Define the following simplicial subsets of Δ^n.

- For $n \geq 1$, denote by I_n the ***spine*** of Δ^n, i.e. the simplicial subset

$$\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n.$$

- For $n \geq 2$, denote by L_n the simplicial subset

$$L_n = \Delta^{\{0,1\}} \amalg_{\Delta^{\{0\}}} \overbrace{\Delta^{\{0,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \amalg_{\Delta^{\{3\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}}^{I_{\{0,\hat{1},2,\ldots,n\}}} \subseteq \Delta^n.$$

  That is, L_n is the union of $\Delta^{\{0,1\}}$ with the spine of $d_1\Delta^n$. We will call L_n the ***left spine*** of Δ^n.

Note that $L_2 \cong \Lambda^2_0$.

Proposition 2.4.6. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then the square

$$\begin{tikzcd}
(\Delta^n\backslash X)^e \arrow[r] \arrow[d] & (L_n\backslash X)^e \arrow[d] \\
(\Delta^n\backslash Y)^{f(e)} \arrow[r] & (L_n\backslash Y)^{f(e)}
\end{tikzcd}$$

is homotopy pullback for all $n \geq 2$.

Proof. We have the case $n = 2$ because e is f-cocartesian. Assume the result is true up to $n-1$. Then the square

$$\begin{tikzcd}
(\Delta^{n-1}\backslash X)^e \times_{\Delta^{\{n-1\}}\backslash X} \Delta^{\{n-1,n\}}\backslash X \arrow[r] \arrow[d] & (L_{n-1}\backslash X)^e \times_{\Delta^{\{n-1\}}\backslash X} \Delta^{\{n-1,n\}}\backslash X \arrow[d] \\
(\Delta^{n-1}\backslash Y)^{f(e)} \times_{\Delta^{\{n-1\}}\backslash Y} \Delta^{\{n-1,n\}}\backslash Y \arrow[r] & (L_{n-1}\backslash Y)^{f(e)} \times_{\Delta^{\{n-1\}}\backslash Y} \Delta^{\{n-1,n\}}\backslash Y
\end{tikzcd}$$

is homotopy pullback since each component is homotopy pullback. But this square is equivalent to

$$\begin{tikzcd}
(\Delta^n\backslash X)^e \arrow[r] \arrow[d] & (L_n\backslash X)^e \arrow[d] \\
(\Delta^n\backslash Y)^{f(e)} \arrow[r] & (L_n\backslash Y)^{f(e)}
\end{tikzcd} :$$

The left-hand equivalences come from the Segal condition, and the right-hand equivalences come from the definition of L_n. $\square$

Corollary 2.4.7. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be a f-cocartesian morphism. Then for all $n \geq 2$, the map

$$(\Delta^n)^{\mathcal{L}}\backslash X^{\mathcal{E}} \to L_n^{\mathcal{L}}\backslash X^{\mathcal{E}} \times_{(\Delta^n)^{\mathcal{L}}\backslash Y^\sharp} L_n^{\mathcal{L}}\backslash Y^\sharp$$

is a trivial Kan fibration.

Proof. Each edge $e' \in \mathcal{E}$ is f-cocartesian: the morphism e is f-cocartesian by assumption, each degenerate edge is f-cocartesian by Example 2.4.2, and any morphism in the path component of an f-cocartesian morphism is f-cocartesian by Proposition 2.4.3.

Therefore, for any $e' \in \mathcal{E}$, the map

$$(\Delta^n\backslash X)^{e'} \to (L_n\backslash X)^{e'} \times_{(\Delta^n\backslash Y)^{f(e')}} (L_n\backslash Y)^{f(e')}$$

is a weak equivalence by Proposition 2.4.6, and it is a Kan fibration by Lemma 2.3.2. The result follows from Lemma 2.3.5. $\square$

For any simplicial set A, define a marked simplicial set $(\Delta^1 \star A, \mathcal{L}')$ where the only nondegenerate simplex belonging to $\mathcal{L}'$ is Δ^1. This is a slight generalization of the $\mathcal{L}$-marking.

Lemma 2.4.8. Let $A \hookrightarrow B$ be a monomorphism of simplicial sets, and suppose that B is n-skeletal (and therefore that A is n-skeletal). Then the map

$$(\Delta^{\{0\}} \star B)^\flat \coprod_{(\Delta^{\{0\}}\star A)^\flat} (\Delta^1 \star A)^{\mathcal{L}'} \hookrightarrow (\Delta^1 \star B)^{\mathcal{L}'}$$

is in the saturated hull of the morphisms

$$(\Lambda^k_0)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}, \qquad 2 \le k \le n+2.$$

Proof. It suffices to show this for $A \hookrightarrow B = \partial\Delta^m \hookrightarrow \Delta^m$ for $0 \le m \le n$. In this case the necessary map is of the form

$$(\Lambda^{m+2}_0)^{\mathcal{L}} \hookrightarrow (\Delta^{m+2})^{\mathcal{L}}.$$

$\square$

Definition 2.4.9. We will say a collection of morphisms $\mathcal{A} \subset \mathrm{Mor}(\mathcal{S}\mathrm{et}^+_\Delta)$ has the ***right cancellation property*** if for all u, $v \in \mathrm{Mor}(\mathcal{S}\mathrm{et}^+_\Delta)$,

$$u \in \mathcal{A}, \quad vu \in \mathcal{A} \qquad \Longrightarrow \qquad v \in A.$$

Lemma 2.4.10. Let $\mathcal{A}$ be a saturated set of morphisms of $\mathcal{S}\mathrm{et}^+_\Delta$ all of whose underlying morphisms are monomorphisms, and which has the right cancellation property. Further suppose that $\mathcal{A}$ contains the following classes of morphisms.

1. Maps $(A)^\flat \hookrightarrow (B)^\flat$, where $A \to B$ is inner anodyne.
2. Left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \ge 2$.

Then $\mathcal{A}$ contains left horn inclusions $(\Lambda^n_0)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \ge 2$.

Proof. For $n = 2$, there is nothing to check: we have an isomorphism $(L_2)^{\mathcal{L}} \cong (\Lambda^2_0)^{\mathcal{L}}$.

We proceed by induction. Suppose we have shown that all horn inclusions $(\Lambda^k_0)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}$ belong to $\mathcal{A}$ for $2 \le k < n$. From now on on we will suppress the marking $(-)^{\mathcal{L}}$. All simplicial subsets of Δ^n below will have $\Delta^{\{0,1\}}$ marked if they contain it.

Consider the factorization

$$\begin{tikzcd} L_n \arrow[r, "u_n"] \arrow[rr, bend right, "v_n \circ u_n"'] & \Lambda^n_0 \arrow[r, "v_n"] & \Delta^n \end{tikzcd} .$$

The morphism $v_n \circ u_n$ belongs to $\mathcal{A}$ by assumption, so in order to show that v_n belongs to $\mathcal{A}$, it suffices by right cancellation to show that u_n belongs to $\mathcal{A}$. Consider the factorization

$$\begin{tikzcd} L_n \arrow[r, "w'_n"] \arrow[rr, bend right, "u_n"'] & L_n \cup d_1\Delta^n \arrow[r, "w_n"] & \Lambda^n_0 \end{tikzcd} .$$

The map w'_n is a pushout along the spine inclusion $I_{\{0,\hat{1},2,\dots,n\}} \hookrightarrow d_1\Delta^n$, and hence is inner anodyne. Hence, we need only show that w_n belongs to $\mathcal{A}$. Let

$$Q = d_2\Delta^n \cup \cdots \cup d_n\Delta^n,$$

and consider the following pushout diagram.

$$\begin{tikzcd}
(L_n \cup d_1\Delta^n) \cap Q \arrow[r, hook] \arrow[d, hook] & Q \arrow[d, hook] \\
L_n \cup d_1\Delta^n \arrow[r, hook] & L_n \cup d_1\Delta^n \cup Q
\end{tikzcd}$$

Since $L_n \cup d_1\Delta^n \cup Q \cong \Lambda^n_0$, the bottom map is w_n, so it suffices to show that the top map belongs to $\mathcal{A}$. But this is isomorphic to

$$(\Delta^{\{0,1\}} \star \emptyset) \coprod_{(\Delta^{\{0\}} \star \emptyset)} (\Delta^{\{0\}} \star \partial\Delta^{\{2,3,\dots,n\}}) \hookrightarrow \Delta^{\{0,1\}} \star \partial\Delta^{\{2,3,\dots,n\}}.$$

The simplicial set $\partial\Delta^{\{2,\dots,n\}}$ is $(n-3)$-skeletal, so this map belongs to $\mathcal{A}$ by Lemma 2.4.8. ☐

For each $n \geq 2$, denote by h^n the $\mathcal{L}$-marked inclusion

$$h^n \colon (\Lambda^n_0)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}.$$

Proposition 2.4.11. Let $f \colon X \to Y$ be a Reedy fibration of Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then for all $n \geq 2$, the map

$$\langle h^n \backslash f^{\mathcal{E}} \rangle \colon (\Delta^n)^{\mathcal{L}} \backslash X^{\mathcal{E}} \to (\Lambda^n_0)^{\mathcal{L}} \backslash X^{\mathcal{E}} \times_{(\Lambda^n_0)^{\mathcal{L}} \backslash Y^\sharp} (\Delta^n)^{\mathcal{L}} \backslash Y^\sharp$$

is a trivial fibration of simplicial sets.

Proof. Consider the set

$$S = \left\{ \begin{matrix} u \colon A \to B \text{ morphism of} \\ \text{marked simplicial sets} \\ \text{such that } \mathring{u} \text{ is mono} \end{matrix} \;\middle|\; \langle u \backslash f^{\mathcal{E}} \rangle \text{ weak homotopy equivalence} \right\}.$$

We claim that this set has the right cancellation property (Definition 2.4.9). The set of morphisms of marked simplicial sets whose underlying morphisms are monic clearly has the right-cancellation property. To show that S does, let $u \colon A \to B$ and $v \colon B \to C$ be such morphisms and consider the following diagram.

$$\begin{tikzcd}
C\backslash X \arrow[rr, "\langle vu \backslash f^{\mathcal{E}} \rangle"] \arrow[d, "\langle v \backslash f^{\mathcal{E}} \rangle"'] & & A\backslash X \times_{A\backslash Y} C\backslash Y \\
B\backslash X \times_{B\backslash Y} C\backslash Y \arrow[rr, "\langle u \backslash f^{\mathcal{E}} \rangle \times_{\mathrm{id}} \mathrm{id}"] & & \big(A\backslash X \times_{A\backslash Y} B\backslash Y\big) \times_{B\backslash Y} C\backslash Y \arrow[u, "\simeq"']
\end{tikzcd}$$

If $\langle u\backslash f^{\mathcal{E}}\rangle$ is a weak equivalence, then the bottom morphism is a weak equivalence. The right-hand morphism is a weak equivalence because it is an isomorphism, so if $\langle vu\backslash f^{\mathcal{E}}\rangle$ is a weak equivalence, then the $\langle v\backslash f^{\mathcal{E}}\rangle$ is a weak equivalence by 2/3.

By Proposition 2.2.12, a map u belonging to S automatically has the property that $\langle u\backslash f^{\mathcal{E}}\rangle$ a Kan fibration, hence is a trivial Kan fibration. Thus, we can equivalently say that $u \in S$ if and only if u has the left-lifting property with respect to all maps of the form $\langle X/v\rangle$, where v is a cofibration of simplicial sets. Since the set of all monomorphisms is saturated, S is saturated.

The set S contains all flat-marked inner anodyne morphisms because f is a Reedy fibration. Corollary 2.4.7 tells us that S contains all left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$. Thus, by Lemma 2.4.10, S contains all $\mathcal{L}$-marked left horn inclusions. $\square$

Corollary 2.4.12. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian edge. Then the map

$$(\Delta^n\backslash X)^e \to (\Lambda^n_0\backslash X)^e \times_{(\Lambda^n_0\backslash Y)^{f(e)}} (\Delta^n\backslash Y)^{f(e)}$$

is a trivial fibration.

Proof. Lemma 2.3.5. $\square$

Corollary 2.4.13. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then e is f/Δ^0-cocartesian.

Proof. By Corollary 2.4.12, for all $n \geq 2$ the map

$$(\Delta^n\backslash X)^e \to (\Lambda^n_0\backslash X)^e \times_{(\Lambda^n_0\backslash Y)^{f(e)}} (\Delta^n\backslash Y)^{f(e)}$$

is a trivial fibration. Thus, it certainly has the right-lifting property with respect to $\emptyset \hookrightarrow \Delta^0$. Passing to the adjoint lifting problem, we find that this is equivalent to the existence of a dashed lift in the diagram

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "e"] & \\
\Lambda^n_0 \arrow[r] \arrow[d, hook] & X/\Delta^0 \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y/\Delta^0
\end{tikzcd},$$

which tells us that e is f/Δ^0-cocartesian. $\square$

Definition 2.4.14. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces. We will say that f is a ***cocartesian fibration*** if each morphism in Y has an f-cocartesian lift in X. More explicitly, we demand that, for each edge $e\colon y \to y'$

in Y and each vertex $x \in X$ such that $f(x) = y$, there exists an f-cocartesian morphism $\tilde{e}\colon x \to x'$ such that $f(\tilde{e}) = e$.

Corollary 2.4.15. Let $f\colon X \to Y$ be a cocartesian fibration of Segal spaces. Then the map

$$f/\Delta^0\colon X/\Delta^0 \to Y/\Delta^0$$

is a cocartesian fibration of quasicategories, and if a morphism in X_1 is f-cocartesian, then it is f/Δ^0-cocartesian.

Proof. By Theorem 2.1.6, the map f/Δ^0 is an inner fibration between quasicategories. By assumption, every morphism in Y has a f-cocartesian lift, and these lifts are f/Δ^0-cocartesian by Corollary 2.4.13. $\square$

3 Segal spaces of spans

3.1 Basic definitions

We recall the basic definitions of Segal spaces of spans. Note that spans as we will define them form not only a Segal space, but a *complete* Segal space; for the most part, this will not concern us. For more information, we direct the reader to [2]. This section is intended to be a summary of the relevant results of loc. cit.

The objects of our study will be ∞-categories whose morphisms are spans in some quasicategory $\mathcal{C}$, i.e. diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & y \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ x & & x' \end{tikzcd} .$$

We will want to be able to place certain conditions on the legs ϕ and ψ of our spans. For example, we may want to restrict our attention to spans such that ϕ is an equivalence. More precisely, we pick out two subcategories of $\mathcal{C}$ to which the respective legs of our spans must belong.

Definition 3.1.1. A ***triple*** of categories is a triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, where $\mathcal{C}$ is a quasicategory where $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$ are subcategories, each of which contain all equivalences.

We will use the following terminology and notation for the morphsism in our subcategories.

- We denote the morphisms in $\mathcal{C}_\dagger$ with tails (as in $x \rightarrowtail y$), and call them *ingressive*.
- We denote the morphisms in $\mathcal{C}^\dagger$ with two heads (as in $x \twoheadrightarrow y$), and call them *egressive*.

The egressive morphisms will correspond to the backwards (i.e. leftwards) facing legs of our spans, and the ingressive morphisms will correspond to the forwards (i.e. rightwards) facing legs. We provide the following diagram as a summary of this terminology.

$$\begin{tikzcd} & Y \arrow[dl, two heads, "\substack{\text{egressive} \\ \mathcal{C}^\dagger}"'] \arrow[dr, tail, "\substack{\text{ingressive} \\ \mathcal{C}_\dagger}"] & \\ X & & X' \end{tikzcd}$$

In order for a triple of categories to be able to support a category of spans, it will have to satisfy certain properties. Following Barwick, we will call such triples *adequate*.

Definition 3.1.2. A triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is said to be ***adequate*** if it has the following properties:

1. For any ingressive morphism $f \in \mathcal{C}_\dagger$ and any egressive morphism $g \in \mathcal{C}^\dagger$, there exists a pullback square

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd} .$$

2. For any pullback square

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d] & x' \arrow[d] \\ y \arrow[r, "f"] & x \end{tikzcd} ,$$

   if the arrow f belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$), then the arrow f' belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$).

We will call a square of the form

$$\begin{tikzcd} y' \arrow[r, tail] \arrow[d, two heads] & x' \arrow[d, two heads] \\ y \arrow[r, tail] & x \end{tikzcd}$$

ambigressive. If such an ambigressive square is also a pullback square, we will call it *ambigressive pullback.*

It is now time to set about building our ∞-categories of spans. Fix some triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$. We will define two models for such a category: a Segal space $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, and the quasicategory $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$.

Given two 1-simplices in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ represented by spans $X \twoheadleftarrow Y \rightarrowtail X'$ and $X' \twoheadleftarrow Y' \rightarrowtail X''$ in $\mathcal{C}$, we need to specify what it means to compose them. We define the composition to be the span $X \twoheadleftarrow Y \times_{X'} Y' \rightarrowtail X''$ given by the diagram below, where the top square is pullback.

$$\begin{tikzcd} & & Y \times_{X'} Y' \arrow[dl, dashed, two heads, "q'"'] \arrow[dr, dashed, tail, "f'"] & & \\ & Y \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & & Y' \arrow[dl, two heads, "q"'] \arrow[dr, tail, "p"] & \\ X & & X' & & X'' \end{tikzcd}$$

Note that the conditions in Definition 3.1.2 tell us precisely that such a composition always exists.

Equivalently, we this means that a 2-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ should be a diagram in $\mathcal{C}$ of the above form. More generally, we would like an n-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ to correspond to n-fold composition of spans. For example, for $n = 4$, such an n-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ should consist of a diagram in $\mathcal{C}$ of the form

$$\begin{tikzcd}
& & & & X_{04} \arrow[dl, two heads] \arrow[dr, tail] & & & & \\
& & & X_{03} \arrow[dl, two heads] \arrow[dr, tail] & & X_{14} \arrow[dl, two heads] \arrow[dr, tail] & & & \\
& & X_{02} \arrow[dl, two heads] \arrow[dr, tail] & & X_{13} \arrow[dl, two heads] \arrow[dr, tail] & & X_{24} \arrow[dl, two heads] \arrow[dr, tail] & & \\
& X_{01} \arrow[dl, two heads] \arrow[dr, tail] & & X_{12} \arrow[dl, two heads] \arrow[dr, tail] & & X_{23} \arrow[dl, two heads] \arrow[dr, tail] & & X_{34} \arrow[dl, two heads] \arrow[dr, tail] & \\
X_{00} & & X_{11} & & X_{22} & & X_{33} & & X_{44}
\end{tikzcd},$$

where the morphisms are ingressive and egressive as shown, and each square is pullback.

All that remains is to make this definition rigorous. To this end, we make the following definition.

Definition 3.1.3. for each $n \geq 0$, we define a poset Σ_n as follows:

- The elements of Σ_n are pairs of integers (i, j), with $0 \leq i \leq j \leq n$.
- For (i, j), $(i', j') \in \Sigma_n$, we define

$$(i, j) \leq (i', j') \iff i \leq i' \leq j' \leq j.$$

Example 3.1.4. We can draw Σ_4 as follows.

$$\begin{tikzcd}
& & & & (0,4) \arrow[dl] \arrow[dr] & & & & \\
& & & (0,3) \arrow[dl] \arrow[dr] & & (1,4) \arrow[dl] \arrow[dr] & & & \\
& & (0,2) \arrow[dl] \arrow[dr] & & (1,3) \arrow[dl] \arrow[dr] & & (2,4) \arrow[dl] \arrow[dr] & & \\
& (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & & (3,4) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2) & & (3,3) & & (4,4)
\end{tikzcd}$$

These posets Σ_n assemble into a cosimplicial object

$$\Delta \to \mathcal{S}\mathrm{et}_\Delta; \qquad [n] \mapsto N(\Sigma_n),$$

Left Kan extending along the Yoneda embedding $\Delta \hookrightarrow \mathcal{S}\mathrm{et}_\Delta$ yields a functor $\mathrm{sd}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$. General abstract nonsense gives us right adjoint $\mathrm{Span}'\colon \mathcal{S}\mathrm{et}_\Delta \to$

$\mathcal{S}\mathrm{et}_\Delta$ forming an adjunction

$$\mathrm{sd} : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta : \mathrm{Span}'.$$

For any simplicial set A, the simplicial set $\mathrm{Span}'(A)$ has n-simplices

$$\mathrm{Span}'(A)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\mathrm{sd}(\Delta^n), A).$$

Note that for a quasicategory $\mathcal{C}$, the simplicial set $\mathrm{Span}'(\mathcal{C})$ is very close to what we want; its n-simplices are maps $\Sigma_n \to \mathcal{C}$, but the backward- and forward-facing legs of the spans do not necessarily belong to the categories $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$, and the squares are not necessarily pullback. We will use the term to refer to a map $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ which has the desired form.

Definition 3.1.5. For any adequate triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, we will call a functor $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ *ambigressive Cartesian* if each square in $\mathrm{sd}(\Delta^n) = N(\Sigma_n)$ of the form

$$\begin{tikzcd}
 & (i,j) \arrow[dl] \arrow[dr] & \\
(i, j-\ell) \arrow[dr] & & (i+k, j) \arrow[dl] \\
 & (i+k, j-\ell) &
\end{tikzcd}$$

(where we include the possibilities $k = 0$ and $\ell = 0$) is mapped to an ambigressive pullback square,

$$\begin{tikzcd}
 & X_{ij} \arrow[dl] \arrow[dr] & \\
X_{i(j-\ell)} \arrow[dr] & & X_{(i+k)(j)} \arrow[dl] \\
 & X_{(i+k)(j-\ell)} &
\end{tikzcd}$$

where the backwards-facing morphsims are egressive and the forwards-facing morphisms are ingressive.

We are now ready to define our category $\mathrm{Span}(\mathcal{C})$.

Definition 3.1.6. We define a simplicial set $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ level-wise to be the subset

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)_n \subseteq \mathrm{Span}'(\mathcal{C})_n$$

on functors $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ which are ambigressive Cartesian.

While it is not hard to check that the face and degeneracy maps on $\mathrm{Span}'(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ restrict to $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ (and thus that the above construction really produces a simplicial set), solving the lifting problems necessary to prove directly that $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a quasicategory turns out to be combinatorially strenuous. It turns out to be easier to switch to a new model of quasicategories, (complete) Segal spaces, which give easier access to homotopical data.

To that end, for an adequate triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, we make the following defintion.

Definition 3.1.7. For any simplicial set A, we define a simplicial set $\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(A), \mathcal{C})^{\simeq}$ to be the full simplicial subset

$$\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(A), \mathcal{C})^{\simeq} \subseteq \mathrm{Map}(\mathrm{sd}(A), \mathcal{C})^{\simeq}$$

on functors $\mathrm{sd}(A) \to \mathcal{C}$ such that for each standard simplex σ of A, the image of $\mathrm{sd}(\sigma)$ in $\mathcal{C}$ is ambigressive Cartesian.

Definition 3.1.8. The (complete) Segal space of spans $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is defined level-wise by

$$\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)_n = \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq}.$$

For a proof that this really is a Segal space, the reader is once again referred to [2]. Note that the quasicategory of spans $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is the bottom row of the complete Segal space of span $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$; that is,

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) = \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)/\Delta^0.$$

Proposition A.1.1 thus immediately implies that $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a quasicategory.

3.2 Important example: spans and equivalences

Let $\mathcal{C}$ be a quasicategory. One can think about spans in $\mathcal{C}$ as 'fractions' of morphisms in $\mathcal{C}$. Consider a span in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd} .$$

If the morphism g were invertible, we could fill this to a full 2-simplex

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "f \circ g^{-1}"'] & & X' \end{tikzcd} .$$

Even if g is not invertible, we can think of our span as formally representing a 'fraction of morphisms' $f \circ g^{-1} \colon X \to X'$, which may or may not exist in $\mathcal{C}$.

With this point of view in mind, if we build from a quasicategory $\mathcal{C}$ a category of spans whose forwards-facing legs are arbitrary morphisms in $\mathcal{C}$, and whose backward-facing legs are equivalences, then we have not really done anything; each span, thought of as a formal morphism in $\mathcal{C}$, is represented by an actual morphism in $\mathcal{C}$. Thus, the category of spans in $\mathcal{C}$ whose backward-facing legs are equivalences should be equivalent to the category $\mathcal{C}$. In the following example, we construct this equivalence. This example, as well as the following one, will be important in several places to the construction in Section 4.

Example 3.2.1. For any quasicategory $\mathcal{C}$, we can define a triple $(\mathcal{C}, \mathcal{C}^\dagger = \mathcal{C}, \mathcal{C}_\dagger = \mathcal{C}^\simeq)$. These choices of ingressive and egressive morphisms correspond to spans of the form

$$\begin{tikzcd} & Y \arrow[dl, "\simeq"'] \arrow[dr] & \\ X & & X' \end{tikzcd},$$

i.e. spans such that the backwards-facing map $Y \to X$ is an equivalence in $\mathcal{C}$. With this triple, a functor $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is ambigressive cartesian if and only if each backward-facing leg is an equivalence. Note that the triple $(\mathcal{C}, \mathcal{C}, \mathcal{C}^\simeq)$ defined above is adequate even if $\mathcal{C}$ does not admit pullbacks: for any solid diagram below, where g is an equivalence, the dashed square completion is a pullback, where g^{-1} is any homotopy inverse for g.

$$\begin{tikzcd} X \arrow[r, dashed, "g^{-1}\circ f"] \arrow[d, dashed, "\mathrm{id}"'] & X' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y' \end{tikzcd}$$

We will denote the resulting Segal space of spans by $\mathbf{Span}^\simeq(\mathcal{C})$, and the quasicategory $\mathbf{Span}^\simeq(\mathcal{C})/\Delta^0$ by $\mathrm{Span}^\simeq(\mathcal{C})$.

The quasicategory $\mathrm{Span}^\simeq(\mathcal{C})$ is categorically equivalent to the category $\mathcal{C}$ from which we started. More specifically, there exists a weak categorical equivalence $\mathcal{C} \to \mathrm{Span}^\simeq(\mathcal{C})$. We will now construct this map.

For each $n \geq 0$ there is a retraction of posets

$$[n] \hookrightarrow \Sigma_n \twoheadrightarrow [n],$$

where the first map takes $i \mapsto (i, n)$, and the second takes $(i, j) \mapsto i$. In the case $n = 2$, the inclusion can be drawn

$$\begin{tikzcd}[column sep=small] 0 \arrow[dr, very thick] & & & & & & (0,2) \arrow[dl] \arrow[dr, very thick] & & \\ & 1 \arrow[dr, very thick] & \hookrightarrow & & & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr, very thick] & \\ & & 2 & & (0,0) & & (1,1) & & (2,2) \end{tikzcd},$$

and the map $\mathrm{sd}(\Delta^n) \to \Delta^n$ 'collapses' the backward-facing legs of the spans. The general case is analogous.

Applying the nerve, we find a retraction of simplicial sets

$$\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n) \twoheadrightarrow \Delta^n.$$

Pulling back along these maps gives us, for each n, a retraction

$$\mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \xhookrightarrow{i'_n} \mathrm{Map}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq} \xhookrightarrow{r'_n} \mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} .$$

Note that the image of the inclusion i'_n consists of spans whose backwards-facing legs are identities (hence certainly equivalences), and hence that we can restrict the total space of our retraction:

$$\mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \xhookrightarrow{i_n} \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq} \xrightarrow{r_n} \mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} .$$

It is not hard to convince oneself that a map $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is ambigressive cartesian if and only if it is a right Kan extension of its restriction along $\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n)$. In the case $n = 2$, for example, it is clear from the limit formula that any right Kan extension of a diagram $\Delta^2 \to \mathcal{C}$ along the inclusion $\Delta^2 \hookrightarrow \mathrm{sd}(\Delta^2)$ is of the form

$$\begin{tikzcd}[column sep=small, row sep=small]
X \arrow[dr] & & \\
& Y \arrow[dr] & \\
& & Z
\end{tikzcd}
\quad \hookrightarrow \quad
\begin{tikzcd}[column sep=small, row sep=small]
& & & X \arrow[dl, "\simeq"'] \arrow[dr] & & & \\
& & X' \arrow[dl, "\simeq"'] \arrow[dr] & & Y \arrow[dl, "\simeq"'] \arrow[dr] & & \\
& X'' & & Y' & & Z &
\end{tikzcd} ,$$

(and conversely, any diagram of the above form is a right Kan extension), and diagrams of the above form are precisely those which are ambigressive cartesian. Put differently, $\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C})$ on functors which are right Kan extensions of their restrictions to Δ^n. This, together with [6, Prop. 4.3.2.15] implies that r_n is a trivial fibration for all n. The identity $\mathrm{id}_{\mathrm{Fun}(\Delta^n, \mathcal{C})}$ is certainly a weak Kan equivalence, so by the 2/3 property for weak equivalences, i_n is also a weak equivalence for all n.

The projections $\mathrm{sd}(\Delta^n) \to \Delta^n$ assemble into a cosimplicial object $\mathrm{sd}(\Delta^\bullet) \to \Delta^\bullet$. From this it easily follows that the maps i_n assemble into a map of Segal spaces

$$i \colon \Gamma(\mathcal{C}) = \mathrm{Fun}(\Delta^\bullet, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^\bullet), \mathcal{C}) = \mathbf{Span}^{\simeq}(\mathcal{C})$$

which is a level-wise Kan weak equivalence, and hence a weak equivalence in the complete Segal spaces model structure.[1] Here $\Gamma(\mathcal{C})$ is the complete Segal space defined in [1] in the discussion preceding Thm. 4.11. Thus, because $\Gamma(\mathcal{C})$

[1]For a brief review of the model structures in question, see Appendix A.1. For more information, see [1], especially the discussion following Thm. 4.1.

and $\mathbf{Span}^{\simeq}(\mathcal{C})$ are complete Segal spaces (hence fibrant-cofibrant objects in the complete Segal space model structure), the restricion of i to first rows i/Δ^0 is a weak categorical equivalence, which is what we wanted to show.

As Example 3.2.1 shows, taking spans whose backward-facing legs are equivalences allows us to find an equivalence between $\mathcal{C}$ and a quasicategory of spans in $\mathcal{C}$. Similarly, if we look at spans whose forward-facing legs are equivalences, we find a category of spans equivalent to $\mathcal{C}^{\mathrm{op}}$.

Example 3.2.2. For any quasicategory $\mathcal{C}$, we can define an adequate triple $(\mathcal{C}, \mathcal{C}_\dagger = \mathcal{C}^{\simeq}, \mathcal{C}^\dagger = \mathcal{C})$. These selections of ingressive and egressive morphisms correspond to spans of the form

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "\simeq"] & \\ X & & X' \end{tikzcd},$$

i.e. spans such that the forward-facing map $Y \to X$ is an equivalence in $\mathcal{C}$. We will denote the corresponding category of spans by $\mathrm{Span}_{\simeq}(\mathcal{C})$. Exactly analogous reasoning to that in Example 3.2.1 gives us a categorical equivalence

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_{\simeq}(\mathcal{C}).$$

3.3 Maps between triples

We have now seen that given any quasicategory $\mathcal{C}$ with pullbacks, one can create a quasicategory of spans $\mathrm{Span}(\mathcal{C})$, and that by defining an adequate triple structure $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ on $\mathcal{C}$, one can specify certain classes of morphisms to which the legs of the spans are allowed to belong. It is easy to see that any functor $\mathcal{C} \to \mathcal{D}$ between quasicategories $\mathcal{C}$ and $\mathcal{D}$ with pullbacks induces a functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$. Similarly, we make the following definition.

Definition 3.3.1. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ and $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be adequate triples. A functor $p\colon \mathcal{C} \to \mathcal{D}$ is said to be a ***functor between adequate triples*** $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ if the following conditions hold.

1. The functor p preserves ingressive morphisms. That is, $p(\mathcal{C}_\dagger) \subseteq \mathcal{D}_\dagger$
2. The functor p preserves egressive morphsims. That is, $p(\mathcal{C}^\dagger) \subseteq \mathcal{D}^\dagger$.
3. The functor p preserves ambigressive pullbacks.

We will denote the restriction of p to $\mathcal{C}_\dagger$ by

$$p_\dagger\colon \mathcal{C}_\dagger \to \mathcal{D}_\dagger,$$

and similarly for $p^\dagger$. Clearly, a functor p between adequate triples gives functors $\mathrm{Span}(p)$ and $\mathbf{Span}(p)$ respectively between quasicategories and Segal spaces of spans.

It is natural to wonder about the relationship between p and $\mathrm{Span}(p)$. The goal of the remainder of this section is to provide a proof of the following theorem, originally appearing as [2, Thm. 12.2]. This theorem provides conditions on a functor p between adequate triples such that the induced map between category of spans is an inner fibration, and provides the form of a p-cocartesian morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$.

Theorem 3.3.2. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of adequate triples which is an inner fibration. Further suppose that p satisfies the following conditions:

1. For any ingressive morphism $g\colon s \rightarrowtail t$ of $\mathcal{D}$ and any object $x \in \mathcal{C}_s$, there exists an ingressive morphism $f\colon x \rightarrowtail y$ covering g which is both p-cocartesian and $p_\dagger$-cocartesian.
2. Suppose σ is a commutative square

$$\begin{tikzcd} y' \arrow[r, tail, "f'"] \arrow[d, two heads] & x' \arrow[d, "\psi"] \\ y \arrow[r, tail, "f"'] & x \end{tikzcd}$$

in $\mathcal{C}$ such that $p(\sigma)$ is ambigressive pullback in $\mathcal{D}$, with in- and egressive morphisms as marked, and f is p-cocartesian. Then f' is p-cocartesian if and only if σ is an ambigressive pullback square (and in particular ψ is egressive).

Then π is an inner fibration. Further, if a morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is of the form

$$\begin{tikzcd} & y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\ x & & x' \end{tikzcd},$$

where ϕ is egressive and $p^\dagger$-cartesian, and ψ is ingressive and p-cocartesian, it is π-cocartesian.

This theorem originally appears in [2]. However, the proof there is combinatorial in nature. Our proof uses the material developed in Section 2, leveraging the fact that ∞-categories of spans have natural incarnations as (complete) Segal spaces.

Before we get to the meat of the proof, however, we have to prove some preliminary results.

3.4 Housekeeping

In this section we recall some basic facts we will need in the course of our proof of the main theorem. It is recommended to skip this section on first reading, and refer back to the results as necessary.

3.4.1 Isofibrations

We will need some results about isofibrations, sometimes called categorical fibrations. These are standard, and proofs can be found in [7].

Proposition 3.4.1. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map between quasicategories. The following are equivalent:

- The map f is an isofibration.
- The map f is an inner fibration, and each equivalence in $\mathcal{D}$ has a lift in $\mathcal{C}$ which is an equivalence.

We will denote by $\mathrm{Map}(-,-)$ the internal hom on $\mathcal{S}\mathrm{et}_\Delta$.

Proposition 3.4.2. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $i\colon A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Map}(B, \mathcal{C}) \to \mathrm{Map}(A, \mathcal{C}) \times_{\mathrm{Map}(A,\mathcal{D})} \mathrm{Map}(B, \mathcal{D})$$

is an isofibration.

We will denote by $(-)^{\simeq}$ the *core* functor, i.e. the functor which takes a simplicial set X to the largest Kan complex it contains.

Proposition 3.4.3. Let $f\colon X \to Y$ be an isofibration of simplicial sets. Then

$$f^{\simeq}\colon X^{\simeq} \to Y^{\simeq}$$

is a Kan fibration.

We will denote the core of the mapping space functor $\mathrm{Map}(-,-)$ by $\mathrm{Fun}(-,-)$. That is, for simplicial sets A and B, we have

$$\mathrm{Fun}(A, B) \cong \mathrm{Map}(A, B)^{\simeq}.$$

Corollary 3.4.4. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Fun}(B, \mathcal{C}) \to \mathrm{Fun}(A, \mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B, \mathcal{D})$$

is a Kan fibration.

3.4.2 Connected components

Our proof will rely heavily on the fact that the many properties that morphisms can have (ingressive, egressive, p-cocartesian, etc.) are well-behaved with respect to the homotopical structure of the complete Segal spaces in which they live; for example, if a morphism is p-cocartesian, then every morphism in its path component is p-cocartesian.

Let X' and X be simplicial sets, and let $f\colon X' \hookrightarrow X$ be an inclusion. We will say that f is an *inclusion of connected components* if for all simplices $\sigma \in X$, if σ has any vertex in common with X', then σ is wholly contained in X'.

Lemma 3.4.5. Let A be a simplicial set such that between any two vertices x and y of A there exists a finite zig-zag of 1-simplices of A connecting x to y. Let $A_0 \subseteq$ A be a nonempty simplicial subset of A. Let $f\colon X' \hookrightarrow X$ be a morphism between simplicial sets which is an inclusion of connected components. Then the following dashed lift always exists.

$$\begin{tikzcd}
A_0 \arrow[r, "f_0"] \arrow[d, hook] & X' \arrow[d, hook] \\
A \arrow[r, "f"] \arrow[ur, dashed] & X
\end{tikzcd}$$

Proof. We have a map $f\colon A \to X$; in order to construct our dashed lift, it suffices to show that under the above assumptions, f takes every simplex of A to a simplex which belongs to X'. To this end, let $\sigma \in A$ be a simplex. There exists a finite zig-zag of 1-simplices connecting some vertex of σ to a vertex α of A_0; under f, this is mapped to a zig-zag of 1-simplices connecting $f(\alpha)$ to $f(\sigma)$. The vertex $f(\alpha)$ belongs to X'; proceeding inductively, we find that the image of every 1-simplex belonging to the zig-zag also belongs to X', and that $f(\sigma)$ therefore also belongs to X'. ☐

Corollary 3.4.6. Given any commuting square of simplicial sets

$$\begin{tikzcd}
X' \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\
Y' \arrow[r, hook] & Y
\end{tikzcd}$$

with monomorphisms as marked, where i is an inclusion of connected components between Kan complexes and f is a Kan fibration, the map f' is a Kan fibration.

Furthermore, if f is a trivial fibration and f' is surjective on vertices, then f' is a trivial fibration.

Proof. First, we show that if f is a Kan fibration, then f' is a Kan fibration. We need to show that a dashed lift below exists for each $n \geq 1$ and each $0 \leq i \leq n$.

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d] & X' \arrow[r, hook, "i"] \arrow[d, "f'"] & X \arrow[d, "f"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & Y' \arrow[r, hook] & Y
\end{tikzcd}$$

We can always solve our outer lifting problem, and by Lemma 3.4.5, our lift factors through X'.

Now suppose that f is a trivial Kan fibration. The logic above says that f' has the right lifting property with respect to $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 1$; the right lifting property with respect to $\partial\Delta^0 \hookrightarrow \Delta^0$ is equivalent to the surjectivity of f' on vertices. $\square$

3.4.3 Contractibility

We will need at certain points to show that various spaces of lifts are contractible. These are mostly common-sense results, but it will be helpful to have them written down somewhere so we can refer to them later.

Lemma 3.4.7. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Then for any commuting square

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd},$$

the fiber F in the pullback square below is contractible.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(\alpha, f\alpha, \beta)}"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) \end{tikzcd}$$

Proof. The right-hand map is a trivial Kan fibration. $\square$

One should interpret this as telling us that given a Λ^2_2-horn α in $\mathcal{C}$ lying over a 2-simplex β in $\mathcal{D}$, the space of fillings of α lying over β is contractible. We will need a pantheon of similar results.

Lemma 3.4.8. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Let e be any f-cartesian edge. Then for any square

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd}$$

such that $\alpha|_{\{1,2\}} = e$, the fiber F in the pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "{(\alpha, f\alpha, \beta)}"] & \mathrm{Fun}(\Lambda^2_2, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) \end{tikzcd} \tag{5}$$

is contractible.

Proof. Denote $f(e) = \bar{e}$. Define a marking $\mathcal{E}'$ on $\mathcal{D}$ containing all degenerate edges and $\bar{e}$. Define a marking $\mathcal{E}$ on $\mathcal{C}$ containing the f-cocartesian lifts of the edges in $\mathcal{E}$. By [6, Prop. 3.1.1.6], the map $\mathcal{C}^{\mathcal{E}} \to \mathcal{D}^{\mathcal{E}'}$ has the right-lifting property with respect to all cocartesian-marked anodyne morphisms. It follows that

$$\begin{tikzcd} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \arrow[d] \\ \mathrm{Map}^\sharp((\Lambda^2_2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \times_{\mathrm{Map}^\sharp((\Lambda^2_1)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'})} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'}) \end{tikzcd}$$

is a trivial Kan fibration; the map on the right-hand side of Diagram 5 is the core of this map, and is thus also a trivial Kan fibration. □

Lemma 3.4.9. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be an inner fibration between triples satisfying the conditions of Theorem 3.3.2. Denote by $\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges $\Delta^1 \to \mathcal{C}^\dagger$ which are both p-cocartesian and $p^\dagger$-cocartesian. Then the fibers of the map

$$\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \to \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger)$$

are contractible.

Proof. Denote by $\mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges which are $p^\dagger$-cartesian. Then we have a square

$$\begin{tikzcd} \mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \arrow[r, hook] \arrow[d] & \mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger) \arrow[d] \\ \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger) \arrow[r, equal] & \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger) \end{tikzcd}$$

in which the top map is an inclusion of connected components, the right map is a trivial Kan fibration, and the left map is surjective on vertices. Thus, the left map is a trivial Kan fibration by Corollary 3.4.6 □

3.5 Proof of main theorem

Our goal is now to prove Theorem 3.3.2. As in the statement of the theorem, we fix a functor of triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ giving us a functor of quasicategories

$$\pi\colon \mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathrm{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

We wish to show that the following hold.

1. The map π is an inner fibration.

2. Morphsims of the form

$$\begin{tikzcd} & y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\ x & & x' \end{tikzcd}, \tag{6}$$

   are π-cocartesian, where ϕ is egressive and $p_\dagger$-cartesian, and ψ is ingressive and p-cocartesian.

The way forward is clear: the functor π is the zeroth row of a functor of Segal spaces

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

The results of Section 2 therefore guarantee the following.

1. In order to show that π is an inner fibration, it suffices to show that π' is a Reedy fibration.

2. In order to show that morphisms of the required form are π-cocartesian, it will suffice to show by Corollary 2.4.15 that morphisms of this form are π'-cocartesian.

3.5.1 The map π is an inner fibration

We begin by checking that the map π is an inner fibration. As stated above, we do this by checking that π' is a Reedy fibration. This follows from the following useful fact, implied by the first assumption of Theorem 3.3.2.

Lemma 3.5.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples satisfying the conditions of Theorem 3.3.2. Then the map $p\colon \mathcal{C} \to \mathcal{D}$ is an isofibration.

Proof. By assumption, p is an inner fibration and $\mathcal{C}$ and $\mathcal{D}$ are quasicategories, so it suffices by Proposition 3.4.1 to show that p admits lifts of equivalences. Each equivalence in $\mathcal{D}$ is in particular ingressive, and hence admits a p-cocartesian lift by the assumptions of Theorem 3.3.2; these are automatically equivalences. □

Proposition 3.5.2. For any functor of adequate triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, the map

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$$

is a Reedy fibration.

Proof. Consider the following diagram.

$$\begin{tikzcd} \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[d] \\ \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{D}) \end{tikzcd}.$$

The right-hand map is a Kan fibration by Corollary 3.4.4 (because sd preserves monomorphisms), and the top map is an inclusion of connected components, so Corollary 3.4.6 implies that the left-hand map is a Kan fibration, which is what we needed to show. □

Corollary 3.5.3. The map π is an inner fibration

Proof. Corollary 2.1.7. □

3.5.2 Cocartesian morphisms have the promised form

We are now ready to begin in earnest our proof that morphisms of the form Equation 6 are π'-cocartesian. Morally, this result should not be surprising. We should think of the homotopy pullback condition defining cocartesian morphisms (Definition 2.4.1) as telling us that we can fill relative Λ^2_2-horns in $\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, and that such fillings are unique up to contractible choice. Finding a filling

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \arrow[d, "\pi'"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger) \end{tikzcd}$$

is equivalent finding a filling

$$\begin{tikzcd} \mathrm{sd}(\Lambda^2_2) \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \mathrm{sd}(\Delta^2) \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

(such that the filling is ambigressive and the necessary square is pullback); the conditions on p guarantee us that we can perform this filling in a series of steps pictured in Figure 1, each of which is unique up to contractible choice.

In fact, our proof mainly consists of making this idea rigorous. To this end, we first introduce some notation. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples.

- We will denote p-cartesian morphisms in $\mathcal{C}$ with a circle:

$$x \relbar\!\!\circ\!\!\!\longrightarrow y$$

- We will denote p-cocartesian morphisms in $\mathcal{C}$ with a bullet:

$$x \relbar\!\!\bullet\!\!\!\longrightarrow y$$

- We will denote $p^\dagger$-cartesian morphisms in $\mathcal{C}$ with a triangle:

$$x \relbar\!\!\triangleright\!\!\!\longrightarrow y$$

- We will denote $p_\dagger$-cocartesian morphisms in $\mathcal{C}$ with a filled triangle:

$$x \relbar\!\!\blacktriangleright\!\!\!\longrightarrow y$$

Thus, an ingressive morphism which is both p-cocartesian and $p_\dagger$-cocartesian will be denoted by

$$x \rightarrowtail\!\!\blacktriangleright\!\!\relbar\!\!\bullet\!\!\!\longrightarrow y$$

Our proof will rest on the factorization $\mathrm{sd}(\Lambda^2_0) \hookrightarrow \mathrm{sd}(\Delta^2)$ pictured in Figure 1. Denote the underlying factorization of simplicial sets by

$$A_1 \overset{i_1}{\hookrightarrow} A_2 \overset{i_2}{\hookrightarrow} A_3 \overset{i_3}{\hookrightarrow} A_4 \overset{i_4}{\hookrightarrow} A_5 \overset{i_5}{\hookrightarrow} A_6$$

For each of the A_i above, denote by

$$\mathrm{Fun}'(A_i, \mathcal{C}) \subseteq \mathrm{Fun}(A_i, \mathcal{C}) \tag{7}$$

the full simplicial subset on those functors $A_i \to \mathcal{C}$ which respect each labelling in Figure 1:

- ingressive
- egressive
- p-cartesian
- p-cocartesian
- $p^\dagger$-cartesian
- $p_\dagger$-cocartesian

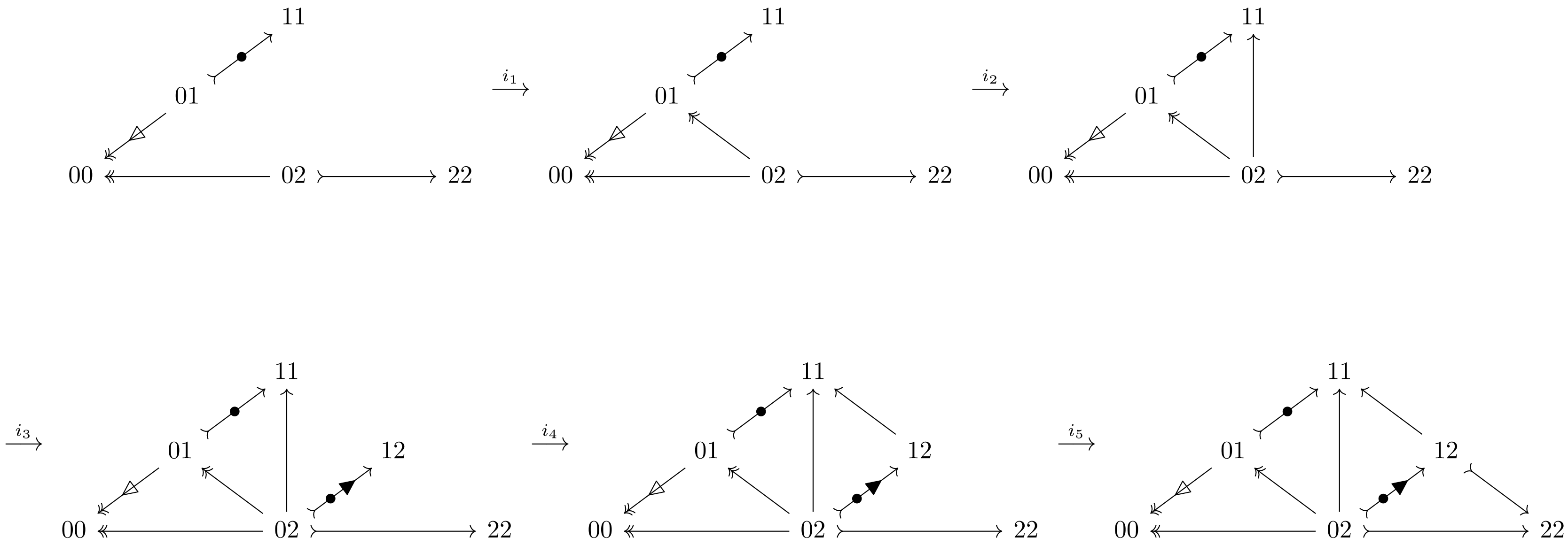

Figure 1: A factorization of the inclusion $\mathrm{sd}(\Lambda_0^2) \hookrightarrow \mathrm{sd}(\Delta^2)$, where certain morphisms have been labelled.

Note that because equivalences in $\mathcal{C}$ belong to each of these classes of morphisms, the inclusion in Equation 7 is an inclusion of connected components.

Similarly, we will denote by $\mathrm{Fun}'(A_i, \mathcal{D})$ the full simplicial subset on functors which respect the ingressive and egressive labellings.

Lemma 3.5.4. The square

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_1, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_1, \mathcal{D})
\end{tikzcd}$$

is homotopy pullback.

Proof. The factorization of Figure 1 gives us a factorization of the above square into five squares

$$\begin{tikzcd}
\mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_k, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_{k+1}, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_k, \mathcal{D})
\end{tikzcd}, \qquad 1 \leq k < 6,$$

each corresponding to one of the inclusions i_k. We will be done if we can show that each of these squares is homotopy pullback. It suffices to show that for each k, the map

$$j_k \colon \mathrm{Fun}'(A_{k+1}, \mathcal{C}) \to \mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1}, \mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D})$$

is a weak equivalence. First, we show that each of these maps is a Kan fibration. To see this, consider the square

$$\begin{tikzcd}
\mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r, hook] \arrow[d, "j_k"'] & \mathrm{Fun}(A_{k+1}, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1}, \mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(A_k, \mathcal{C}) \times_{\mathrm{Fun}(A_{k+1}, \mathcal{D})} \mathrm{Fun}(A_k, \mathcal{D})
\end{tikzcd}.$$

The right-hand map is a Kan fibration because of Corollary 3.4.4, and we have already seen that the top map is an inclusion of connected componenents. Hence each j_k is a Kan fibration by Corollary 3.4.6. Thus, in order to show that each j_k is a weak equivalence, it suffices to show that the fibers are contractible.

First consider the case $k = 1$. For any map α below, consider the following pullback square.

$$\begin{tikzcd}
F_\alpha \arrow[rr] \arrow[d] & & \mathrm{Fun}'(A_1, \mathcal{C}) \arrow[d, "j_1"] \\
\Delta^0 \arrow[r, "\alpha"] & \mathrm{Fun}'(A_0, \mathcal{C}) \times_{\mathrm{Fun}'(A_1, \mathcal{D})} \mathrm{Fun}'(A_1, \mathcal{D}) &
\end{tikzcd}.$$

The fiber F_α over α is the space of ways of completing a diagram of shape A_0 in $\mathcal{C}$ to a diagram of shape A_1 in $\mathcal{C}$ given a diagram of shape A_1 in $\mathcal{D}$. This is the space of ways of filling $\Lambda^2_2 \hookrightarrow \Delta^2$ in $\mathcal{C}^\dagger$ lying over a 2-simplex in $\mathcal{D}^\dagger$. This is contractible by Lemma 3.4.8.

The case $k = 2$ is similar, using Lemma 3.4.7.

The case $k = 3$ is similar, using Lemma 3.4.9.

The cases $k = 4$ and $k = 5$ use the dual to Lemma 3.4.8. □

We are now ready to show that morphisms of the form Diagram 6 are π'-cocartesian.

Proposition 3.5.5. For any morphism e of the form given in Equation 6, the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\}
\end{tikzcd}$$

is homotopy pullback.

Proof. This square factors horizontally into the two squares

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_0, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_0, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}.$$

The first is homotopy pullback because the bottom map is an inclusion of connected components, and the second condition of Theorem 3.3.2 guarantees that the fiber over an ambigressive $\mathrm{sd}(\Delta^2) \to \mathcal{D}$ belonging to $\mathrm{Fun}'(A_6, \mathcal{D})$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is πe is precisely an ambigressive Cartesian functor $\mathrm{sd}(\Delta^2) \to \mathcal{C}$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is e.

That the second is homotopy pullback follows immediately from Lemma 3.5.4. □

This proves Theorem 3.3.2.

4 Application to presheaves

4.1 Introduction

The goal of the remainder of this work is to construct a lax monoidal functor

$$\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty, \tag{8}$$

which takes a space X to the category $\mathcal{P}(X)$ of presheaves on X, and a span of spaces

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd}$$

to the map $\mathcal{P}(X) \to \mathcal{P}(X')$ given by the composition

$$\begin{tikzcd} & \mathcal{P}(Y) \arrow[dr, "f^*"] & \\ \mathcal{P}(X) \arrow[ur, "g_!"] \arrow[rr, "f^* \circ g_!"] & & \mathcal{P}(X'). \end{tikzcd}$$

In this section, we leverage Theorem 3.3.2 to construct $\hat{r}$. First, we give a sketch of the process, ignoring the lax monoidal structure.

4.1.1 Sketch of the construction of the functor underlying $\hat{r}$

Consider the category $\mathrm{Map}(\Delta^1, \mathcal{S})$, whose objects are morphisms $S \to X$ in $\mathcal{S}$, and whose morphisms $(S \to X) \to (S' \to X')$ are commuting squares in $\mathcal{S}$

$$\begin{tikzcd} S \arrow[r] \arrow[d] & S' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}.$$

There is a functor $\mathrm{ev}_1\colon \mathrm{Map}(\Delta^1, \mathcal{S}) \to \mathcal{S}$, which sends an n-simplex $\Delta^n \to \mathrm{Map}(\Delta^1, \mathcal{S})$ corresponding to a map $\Delta^n \times \Delta^1 \to \mathcal{S}$ to the n-simplex $\Delta^n \times \Delta^{\{1\}} \to \mathcal{S}$ corresponding to the 'bottom face'.

In [6] the following is shown.

Proposition 4.1.1. The functor

$$\mathrm{ev}_1\colon \mathrm{Map}(\Delta^1, \mathcal{S}) \to \mathcal{S} \tag{9}$$

is an inner fibration. Further, for any morphism f between objects $(S \to X)$

and $(S' \to X')$ in $\mathrm{Map}(\Delta^1, \mathcal{S})$ corresponding to a square

$$\sigma = \begin{tikzcd} S \arrow[r] \arrow[d] & S' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

in $\mathcal{S}$, the following hold.

- The morphism f is ev_1-cartesian if and only if σ is pullback.
- The morphism f is ev_1-cocartesian if and only if the morphism $S \to S'$ in σ is an equivalence.

Proof. [6, Lem. 6.1.1.1] and [6, Lemma 2.4.7.12]. □

With this in mind, the following follows immediately.

Corollary 4.1.2. The functor ev_1 of Equation 9 is a bicartesian fibration.

Proof. Fix a generic morphism $f\colon X \to X'$ in $\mathcal{S}$.

First, we check that this morphism has a cartesian lift given a lift of its target. Fix a lift $S' \to X'$ of X' in $\mathrm{Map}(\Delta^1, \mathcal{S})$. This gives us the following data.

$$\begin{tikzcd} & S' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

We can always find a cartesian lift because the pullback

$$\begin{tikzcd} S' \times_X X' \arrow[r] \arrow[d] & S' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

always exists.

Now we check that we can find a cocartesian lift given the data

$$\begin{tikzcd} S \arrow[d, "g"] & \\ X \arrow[r, "f"] & X' \end{tikzcd} .$$

Such a lift is given, for example, by the square

$$\begin{tikzcd} S \arrow[r, equal] \arrow[d, "g"'] & S' \arrow[d, "f \circ g"] \\ X \arrow[r, "f"] & X' \end{tikzcd} .$$

□

Thus, for each edge $f\colon \Delta^1 \to \mathcal{S}$, corresponding to a morphism $f\colon X \to Y$ of spaces, the fiber of ev_1 over f corresponds to a pair of adjoint functors

$$f_! : \mathcal{S}^{/X} \longleftrightarrow \mathcal{S}^{/Y} : f^*.$$

Interpreting $\mathcal{S}^{/X}$ as models for $\mathcal{P}(X)$, the ∞-categories of presheaves on X, the functor f^* corresponds to the functor which restricts along f, taking a presheaf $Y \to \mathcal{S}$ to the pullback $X \to Y \to \mathcal{S}$. The functor $f_!$ takes a preasheaf $X \to \mathcal{S}$ to its left Kan extension along $X \to Y$.

This gives us everything we need to build our functor $\hat{r}$. By taking spans in $\mathrm{Map}(\Delta^1, \mathcal{S})$ and $\mathcal{S}$ (and appropriately restricting the legs), Theorem 3.3.2 will provide us with a cocartesian fibration over $\mathrm{Span}(\mathcal{S})$, which will classify a functor $\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$. We thus define the following triple structures.

- In $\mathcal{S}$, we place no restriction on the frontward- or backward-facing legs of the spans, taking the triple $(\mathcal{S}, \mathcal{S}_\dagger = \mathcal{S}, \mathcal{S}^\dagger = \mathcal{S})$.
- In $\mathrm{Map}(\Delta^1, \mathcal{S}) = \mathcal{M}$, we allow arbitrary egressive morphisms, taking $\mathcal{M}^\dagger = \mathcal{M}$. For ingressive morphisms, we take only those morphisms $(S \to X) \to (S' \to X')$ corresponding to squares of the form

$$\begin{tikzcd} S \arrow[r, "\simeq"] \arrow[d] & S' \arrow[d] \\ X \arrow[r] & X' \end{tikzcd}$$

  in $\mathcal{C}$. That is, we take $\mathcal{M}_\dagger$ to be the subcategory of $\mathcal{M}$ on morphisms of this form.

One checks the conditions of Theorem 3.3.2 and finds that this yields a cocartesian fibration

$$\mathrm{Span}(\mathcal{M}, \mathcal{M}_\dagger, \mathcal{M}^\dagger) \to \mathrm{Span}(\mathcal{S}).$$

The functor $\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$ classified by this cocartesian fibration has the following description.

- Each object $X \in \mathrm{Span}(\mathcal{S})$ is sent, up to equivalence, to the fiber over X. This is equivalent to $(\mathcal{S}^{/X})^{\mathrm{op}} \simeq (\mathcal{P}(X))^{\mathrm{op}}$.
- A morphism in $\mathrm{Span}(\mathcal{S})$ corresponding to a span $X \leftarrow Y \to X'$ in $\mathcal{S}$ sends an object $S \to X$ to the object $S' \to X'$ defined as in the following diagram in which the left-hand square corresponds to an ev_1-cartesian morphism, and the right-hand square corresponds to an ev_1-cocartesian morphism.

$$\begin{tikzcd} S \arrow[d] & S \times_X Y \arrow[l] \arrow[r, "\simeq"] \arrow[d] & S' \arrow[d] \\ X & Y \arrow[l, "q"] \arrow[r, "p"] & X' \end{tikzcd}$$

Denote by $\mathcal{F}$ the presheaf $X \to \mathcal{S}$ classified by $S \to X$. Then the presheaf classified by $S \times_X Y \to Y$ classifies the presheaf $q^*\mathcal{F}$, and the presheaf classified by $S' \to X'$ classifies the presheaf $p_! q^* \mathcal{F}$. This is precisely what we wanted.

There is a small annoyance: the functor we have constructed sends a space X not to the category $\mathcal{P}(X)$ of presheaves on X, but to $\mathcal{P}(X)^{\mathrm{op}}$. This is an artifact of the way we constructed our category of spans, and is not an essential difficulty; composint with $\mathrm{op}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives us a functor sending $X \mapsto \mathcal{P}(X)$.

4.1.2 Sketch of the construction of the lax monoidal structure

Note that the category $\mathcal{S}$ has products. This induces a symmetric monoidal structure on $\mathcal{S}$, the cartesian monoidal structure (defined, for example, in [5, Sec. 2.4.1]), which (as we will show) induces a symmetric monoidal structure on $\mathrm{Span}(\mathcal{S})$. When we take $\mathcal{C}\mathrm{at}_\infty$ also to carry the cartesian monoidal structure, we will see that our functor $\hat{r}$ (Equation 8) even admits a lax monoidal structure.

In the theory of quasicategories, one packages up the coherence data involved in writing down a symmetric monoidal category using a cartesian fibration, and a

The idea behind our construction of the functor $\hat{r}$, in broad strokes, is as follows.

In Subsection 4.2, we define symmetric monoidal structures on $\mathrm{Map}(\Delta^1, \mathcal{S})$ and $\mathcal{S}$, and upgrade the functor ev_1 to a symmetric monoidal functor r as follows (the monoidal structures on $\mathrm{Map}(\Delta^1, \mathcal{S})^{\otimes}$ and $\mathcal{S}^{\otimes}$ are not exactly the standard cocartesian monoidal functors, as we will later discuss; they are represented by *cartesian,* rather than *cocartesian* fibrations). Schematically, this means we will define the following data.

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, \mathcal{S})^{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{S}^{\otimes} \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{10}$$

Note 4.1.3. Here, and in what follows, we notationally suppress the nerve in $N(\mathcal{F}\mathrm{in}_*)$, writing simply $\mathcal{F}\mathrm{in}_*$.

In Subsection 4.3 and Subsection 4.4, we take spans in each of the categories in Diagram 10, restricting the backwards- and forwards-facing legs judiciously so that we get a diagram

$$\begin{tikzcd} \mathrm{Span}(\mathrm{Map}(\Delta^1, \mathcal{S}))^{\otimes} \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & \mathrm{Span}(\mathcal{S})^{\otimes} \arrow[dl, "\pi"] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd}, \tag{11}$$

where the ϖ and π are symmetric monoidal structures and ρ is a cocartesian fibration. In the language of [5], ρ exhibits $\mathrm{Span}(\mathrm{Map}(\Delta^1, \mathcal{S}))$ as a $\mathrm{Span}(\mathcal{S})$-

monoidal category.

In Subsection 4.5, we will finally construct the functor $\hat{r}$. Restricting to the component of ρ over $\langle 1\rangle \in \mathcal{F}\mathrm{in}_*$, one finds precisely the cocartesian fibration described in Subsubsection 4.1.1. However, by [5], 2.4.2.4–2.4.2.6, the existence of ρ, π, and ϖ implies that the functor classified by this cartesian fibration is lax monoidal.

Much of the technology in this section is similar to work already done in [4]. However, the manner in which we present it allows for generalization in a different direction than that carried out there. This will be the subject of future work.

4.2 Defining the maps p, q, and r

In this section, we begin our construction of the functor $\hat{r}$, following the plan laid out in Subsubsection 4.1.2. Our first step is to define the maps p, q, and r as described in Diagram 10. The maps p and q are cartesian fibrations, exhibiting symmetric monoidal structures on $\mathrm{Map}(\Delta^1, \mathcal{S})$ and $\mathcal{S}$; the map r is a monoidal version of the map ev_1.

Recall that for any ∞-category $\mathcal{C}$ with finite products, one can construct a symmetric monoidal structure $\mathcal{C}^\times \to \mathcal{F}\mathrm{in}_*$ whose corresponding tensor product $\times\colon \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ is the cartesian product; for more details, see [5, Sec. 2.4.1]. Similarly, as described in [5, Sec. 2.4.2], for any ∞-category with coproducts there is a symmetric monoidal structure $\mathcal{C}^{\amalg} \to \mathcal{F}\mathrm{in}_*$, called the *cocartesian monoidal structure,* whose tensor product $\amalg\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is the coproduct. Because cocartesian fibrations give better control of the morphisms out of an object than morphisms into an object, the cocartesian monoidal structure is much more tractable than the cartesian monoidal structure.

Since taking coproducts in $\mathcal{C}^{\mathrm{op}}$ is the same as taking products in $\mathcal{C}$, we can express the cartesian monoidal structure on $\mathcal{C}$ as a cartesian fibration rather than a cocartesian fibration by taking the opposite of the cocartesian monoidal structure on $\mathcal{C}^{\mathrm{op}}$. In the case $\mathcal{C} = \mathcal{S}$, this gives us the monoidal structure we are interested in.

Definition 4.2.1. We define the map $p\colon \mathcal{S}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ to be the functor

$$((\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathcal{F}\mathrm{in}_*)^{\mathrm{op}},$$

where $(\mathcal{S}^{\mathrm{op}})^{\amalg} \to \mathcal{F}\mathrm{in}_*$ is the cocartesian monoidal structure, as described in [5, Sec. 2.4.2].

In low degrees, the category $\mathcal{S}_\times$ admits the following description.

- The objects are pairs $(\langle n\rangle, \vec{X})$, where $\langle n\rangle$ is an object of $\mathcal{F}\mathrm{in}_*$, and $\vec{X} = [X_1, \ldots, X_n]$ is an ordered n-tuple of objects of $\mathcal{S}$.

- The morphisms $(\langle m\rangle, \vec{X}) \to (\langle n\rangle, \vec{Y})$ are pairs (ϕ, Ψ), where $\phi\colon \langle n\rangle \to \langle m\rangle$ is a morphism in $\mathcal{F}\mathrm{in}_*$, and Ψ consists, for each $i \in \langle m\rangle^\circ$, of a collection of morphisms $X_i \to Y_j$ for each $j \in \langle n\rangle^\circ$ with $\phi(j) = i$.

The functor $\mathcal{S}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is the obvious forgetful functor.

Example 4.2.2. An object $(\langle 1\rangle, \vec{X})$ of $\mathcal{S}_\times$ in the fiber over $\langle 1\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is the same thing as an object $X \in \langle S\rangle$. An object $(\langle 2\rangle, \vec{Y})$ in the fiber over $\langle 2\rangle$ is the same as a pair of objects $[Y, Y']$ in $\mathcal{S}$.

A morphism $(\phi, \Psi)\colon (\langle 1\rangle, \vec{X}) \to (\langle 2\rangle, \vec{Y})$ where $\phi\colon \langle 2\rangle \to \langle 1\rangle$ is the morphism in $\mathcal{F}\mathrm{in}_*$ which sends $1 \mapsto 1$, $2 \mapsto 1$, consists of a pair of maps $X \to Y$, $X \to Y'$.

$$\begin{tikzcd} & 1 \arrow[dl, mapsto] \\ 1 & \\ & 2 \arrow[ul, mapsto] \end{tikzcd} \qquad \begin{tikzcd} & Y \\ X \arrow[ur] \arrow[dr] & \\ & Y' \end{tikzcd}$$

Lemma 4.2.3. The map p is a cartesian fibration, and a morphism $(\phi, \Psi)\colon (\langle m\rangle, \vec{X}) \to (\langle n\rangle, \vec{Y})$ in $\mathcal{S}_\times$ is p-cartesian if and only if it for each $i \in \langle m\rangle^\circ$, the maps $X_i \to Y_j$ exhibit X_i as the product of the Y_j.

Proof. This follows easily from [5, Rem. 2.4.3.4]. □

Example 4.2.4. The morphism described in Example 4.2.2 is cartesian if and only if $X \simeq Y \times Y'$, and the maps $X \to Y$ and $X \to Y'$ are the canonical projections.

We can now define the maps q and r rather easily.

Definition 4.2.5. We define maps q and r via the following diagram, where the left-hand square is pullback. In particular, the space $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ is defined to be the (strict) pullback below.

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, \mathcal{S})_\times \arrow[r, hook] \arrow[d, "q"'] \arrow[rr, bend left, "r"] & \mathrm{Map}(\Delta^1, \mathcal{S}_\times) \arrow[r, "\mathrm{ev}_1"] \arrow[d] & \mathcal{S}_\times \arrow[d, "p"] \\ \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \arrow[r, hook] \arrow[rr, bend right, "\mathrm{id}"'] & \mathrm{Map}(\Delta^1, \mathcal{F}\mathrm{in}_*^{\mathrm{op}}) \arrow[r, "\mathrm{ev}_1"] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \end{tikzcd}$$

Lemma 4.2.6. The map q is a cartesian fibration, and the cartesian morphisms are those which are level-wise cartesian morphisms in $\mathcal{S}_\times$.

Proof. The map q is given by the pullback

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, \mathcal{S})_\times \arrow[r, hook] \arrow[d, "q"'] & \mathrm{Map}(\Delta^1, \mathcal{S}_\times) \arrow[d, "q'"] \\ \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \arrow[r, hook] & \mathrm{Map}(\Delta^1, \mathcal{F}\mathrm{in}_*^{\mathrm{op}}) \end{tikzcd} .$$

By [6, Prop. 3.1.2.1], the map q' is a cartesian fibration whose cartesian morphisms are pointwise cartesian morphisms in $\mathcal{S}_\times$. The map q is also a cartesian fibration, where a morphism in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ is q-cartesian if and only if its image in $\mathrm{Map}(\Delta^1, \mathcal{S}_\times)$ is q'-cartesian. That every q cartesian morphism is q' cartesian follows from [6, Prop 2.4.1.3]; the converse follows from the existence of comparison equivalences between cartesian lifts of the same morphism. ☐

Lemma 4.2.7. The map r is a bicartesian fibration with r-cartesian morphisms pullback squares in $\mathcal{S}_\times$ and r-cocartesian morphisms squares of the form

$$\begin{tikzcd} X \arrow[r, "\simeq"] \arrow[d] & Y \arrow[d] \\ X' \arrow[r] & Y' \end{tikzcd} .$$

Proof. The map r factors into

$$\mathrm{Map}(\Delta^1, \mathcal{S})_\times \xhookrightarrow{j} \mathrm{Map}(\Delta^1, \mathcal{S}_\times) \xrightarrow{\mathrm{ev}_1} \mathcal{S}_\times$$

The map ev_1 is a bicartesian fibration with the cartesian and cocartesian morphisms described above; the map j is a full inclusion (in the sense of Definition 2.2.9), and therefore has the right-lifting property with respect to *all* horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, $n \geq 2$ (because Λ^n_i contains all vertices for $n \geq 2$). Every edge in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ is therefore both j-cartesian and j-cocartesian. [6, Prop 2.4.1.3.3] implies that an edge in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ is r-cartesian (resp. cocartesian) if and only if its image in $\mathrm{Map}(\Delta^1, \mathcal{S}_\times)$ is ev_1-cartesian (resp. cocartesian). ☐

Let us take stock. We have now defined the data of the commutative diagram below. Here, p and q are cartesian fibrations, and r is a bicartesian fibration.

$$\begin{tikzcd} \mathrm{Map}(\Delta^1, \mathcal{S})_\times \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{S}_\times \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{12}$$

Our next step is to take spans with judiciously-chosen legs in each of these quasicategories. To do this, we must define the subcategories to which the legs of our spans are allowed to belong. As in Section 3, we do this by defining triples of categories.

4.3 Defining the adequate triple structures

In this section, we continue our construction of the functor $\hat{r}$, following the plan laid out in Subsubsection 4.1.2. Our next step is to define triple structures on the quasicategories $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$, $\mathcal{S}_\times$, and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, which specify the categories of spans in Diagram 11.

4.3.1 Defining the triple structure on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Recall Subsection 4.1. We have cartesian fibrations p, q, and r as in Diagram 12; these play the role of the maps alluded to in Diagram 10. We would like a triple structure on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ such that, when passing to quasicategories of spans, we find cocartesian fibrations of the form Diagram 11.

This means, in particular, that we want a triple structure on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ which gives us something equivalent to $\mathcal{F}\mathrm{in}_*$ upon taking categories of spans. Thus, we should take spans whose forward-pointing legs are equivalences:

$$\langle m \rangle \longrightarrow \langle n \rangle \xleftarrow{\simeq} \langle n \rangle \ .$$

We have considered this situation before, in Example 3.2.2. There, we saw that by considering spans in some quasicategory $\mathcal{C}$ whose forward-facing legs were equivalences, we found a categorical equivalence $\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_{\simeq}(\mathcal{C})$.

Thus, we will take on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ the triple structure defined in Example 3.2.2.

Definition 4.3.1. We define a triple $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ as follows.

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

For posterity, we record the content of Example 3.2.2 in the following proposition.

Lemma 4.3.2. There is a weak Joyal equvialence

$$\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger),$$

which sends an n-simplex $\langle m \rangle \to \langle n \rangle \to \cdots \to \langle p \rangle$ to a diagram

$$\begin{tikzcd}
 & & & \langle p \rangle \arrow[dr, equal] & & & \\
 & & \cdots \arrow[ur] & & \cdots \arrow[dr, equal] & & \\
 & \langle n \rangle \arrow[ur] \arrow[dr, equal] & & \cdots \arrow[ur] & & \langle p \rangle \arrow[dr, equal] & \\
\langle m \rangle \arrow[ur] & & \langle n \rangle \arrow[ur] & & \cdots & & \langle p \rangle
\end{tikzcd} ,$$

where we have drawn the morphisms in the span as belonging to $\mathcal{F}\mathrm{in}_*$ rather than $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$.

4.3.2 Defining the triple structure on $\mathcal{S}_\times$

Next, we define a triple structure on $\mathcal{S}_\times$.

Definition 4.3.3. We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \mathcal{S}_\times$
- $\mathcal{P}_\dagger = \mathcal{S}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \mathcal{S}_\times$

Here, we do not restrict the legs of our spans at all, except to ensure that p restricts to a map $p_\dagger \colon \mathcal{P}_\dagger \to \mathcal{F}_\dagger$.

Next, we need to show that the triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ is adequate. For this we need a few lemmas.

Lemma 4.3.4. The cartesian fibration $p \colon \mathcal{S}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. By [6, Cor. 4.3.1.11], it suffices to show that each fiber $(\mathcal{S}_\times)_{\langle n \rangle}$ admits pullbacks, and that for each map $\phi \colon \langle m \rangle \leftarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the map $\phi^* \colon (\mathcal{S}_\times)_{\langle m \rangle} \to (\mathcal{S}_\times)_{\langle n \rangle}$ preserves pullbacks. By the defintion of the cocartesian monoidal structure on $\mathcal{S}^{\mathrm{op}}$, we have an isomorphism $(\mathcal{S}_\times)_{\langle n \rangle} \cong \mathcal{S}^n$, and $\mathcal{S}^n$ admits pullbacks. Hence, each fiber admits pullbacks.

Now we show that for any map $\phi \colon \langle m \rangle \leftarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the map ϕ^* preserves pullbacks. Unravelling the definitions, we find that ϕ^* sends an object $(X_1, \ldots, X_m) \in (\mathcal{S}_\times)_{\langle m \rangle}$ to an object $(Y_1, \ldots, Y_n) \in (\mathcal{S}_\times)_{\langle n \rangle}$ such that

$$X_i \simeq \prod_{\phi(j)=i} Y_j,$$

and the action on higher simplices is determined by the universal property for products. Since pullbacks in $\mathcal{S}^m$ are level-wise pullbacks in $\mathcal{S}$ and products commute with pullbacks, the result follows. $\square$

Lemma 4.3.5. Let $p \colon \mathcal{C} \to \mathcal{D}$ be a cartesian fibration between quasicategories, and let K be a simplicial set. Suppose that $\mathcal{D}$ admits all K-shaped limits, and that p admits all relative K-shaped limits. Then the following hold.

1. The quasicategory $\mathcal{C}$ admits all K-shaped limits.
2. The map p preserves all K-shaped limits.

Proof.

1. For any diagram $q\colon K \to \mathcal{C}$, the diagram $p \circ q\colon K \to \mathcal{D}$ admits a limit cone $\overline{p \circ q}\colon K^{\triangleleft} \to \mathcal{D}$, and this data admits a relative p-limit cone $\bar{q}$ as below.

$$\begin{tikzcd} K \arrow[r, "q"] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ K^{\triangleleft} \arrow[r, "\overline{p\circ q}"'] \arrow[ur, dashed, "\bar{q}"] & \mathcal{D} \end{tikzcd}$$

This is a limit cone for q by [6, Prop. 4.3.1.5].

2. Let $\bar{q}\colon K^{\triangleleft} \to \mathcal{C}$ be a limit cone, and write $q = \bar{q}|_K$. We need to show that $p \circ \bar{q}\colon K^{\triangleleft} \to \mathcal{D}$ is a limit cone in $\mathcal{D}$. The restriction $p \circ q\colon K \to \mathcal{D}$ admits a limit cone $\widehat{p \circ q}\colon K^{\triangleleft} \to \mathcal{D}$, which admits a relative lift to a limit cone $\hat{q}\colon K^{\triangleleft} \to \mathcal{C}$. But $\bar{q}$ and $\hat{q}$ are both limit cones of q, hence are equivalent as objects of $\mathrm{Map}(K^{\triangleleft}, \mathcal{C})$ (by [6, Prop. 4.3.1.5]). Therefore, $\widehat{p \circ q} = p \circ \hat{q}$ and $p \circ \bar{q}$ are equivalent as objects of $\mathrm{Map}(K^{\triangleleft}, \mathcal{D})$, so $p \circ \bar{q}$ is a limit cone since $\widehat{p \circ q}$ is.

□

Proposition 4.3.6. The triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ is adequate.

Proof. We check the conditions of Definition 3.1.2.

1. The category $\mathcal{S}_\times$ admits all pullbacks by Lemma 4.3.4, hence certainly admits ambigressive pullbacks.
2. We need to show that for any pullback square in $\mathcal{S}_\times$

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

such that f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, so does f'. But Lemma 4.3.5 implies that the above square lies over a pullback square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, and the pullback of an isomorphism is an isomorphism.

□

4.3.3 Defining the triple structure on $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$

Lastly, we define a triple structure on $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$.

Definition 4.3.7. We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows.

- $\mathcal{Q} = \mathrm{Map}(\Delta^1, \mathcal{S})_\times$.

- $\mathcal{Q}_\dagger$ is given by the pullback

$$\begin{tikzcd}
\mathcal{Q}_\dagger \arrow[rr] \arrow[d] & & \mathcal{Q} \arrow[d, "{(q,\mathrm{ev}_0)}"] \\
(\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq} \times \mathcal{S}_\times^{\simeq} \arrow[r] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \times \mathcal{S}_\times &
\end{tikzcd} .$$

- $\mathcal{Q}^\dagger = \mathcal{Q}$.

Lemma 4.3.8. The functor $q\colon \mathrm{Map}(\Delta^1, \mathcal{S})_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. We need to show that each fiber $(\mathrm{Map}(\Delta^1, \mathcal{S})_\times)_{\langle n\rangle}$ admits pullbacks. Using the isomorphisms $(\mathcal{S}_\times)_{\langle n\rangle} \simeq \mathcal{S}^n$, we find isomorphisms

$$(\mathrm{Map}(\Delta^1, \mathcal{S})_\times)_{\langle n\rangle} \cong \mathrm{Map}(\Delta^1, \mathcal{S})^n.$$

The proof of Lemma 4.3.4 then applies to our situation, with $\mathcal{S}$ replaced by $\mathrm{Map}(\Delta^1, \mathcal{S})$. $\square$

Lemma 4.3.9. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We check the conditions of Definition 3.1.2.

1. Since q admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, Lemma 4.3.5 implies that $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ admits all pullbacks, hence certainly ambigressive pullbacks.

2. Consider a pullback square in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ corresponding to a cube in $\mathcal{S}_\times$

$$\begin{tikzcd}
S' \arrow[rr, "F"] \arrow[dr] \arrow[dd] & & T' \arrow[dr] \arrow[dd] & \\
& S \arrow[rr, "f"] \arrow[dd] & & T \arrow[dd] \\
X' \arrow[rr] \arrow[dr] & & Y' \arrow[dr] & \\
& X \arrow[rr] & & Y
\end{tikzcd} , \tag{13}$$

and which is lying over a square

$$\begin{tikzcd}
\langle n\rangle & \langle n'\rangle \arrow[l, "\psi"'] \\
\langle m\rangle \arrow[u] & \langle m'\rangle \arrow[l, "\phi"'] \arrow[u]
\end{tikzcd} \tag{14}$$

in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. We need to show that (a) if f is an equivalence, then F is an equivalence, and (b) if ϕ is an equivalence, then ψ is an equivalence.

(a) By [6, Cor. 4.3.1.15] with $\mathcal{C} = \mathrm{Map}(\Delta^1, \mathcal{S}_\times)$, $\mathcal{D} = \mathcal{E} = \mathrm{Map}(\Delta^1, \mathcal{F}\mathrm{in}_*^{\mathrm{op}})$, and $\mathcal{E}_0 = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the cube in Diagram 13 corresponds to a pullback diagram in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ if and only if it corresponds to a pullback square in $\mathrm{Map}(\Delta^1, \mathcal{S}_\times)$, which in turn is true if and only if the top and bottom squares are pullback in $\mathcal{S}_\times$. The result follows because pullbacks preserve equivalences.

(b) Lemma 4.3.5 implies that the square in Diagram 14 is pullback (in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$), and the result again follows because pullbacks preserve equivalences.

$\square$

As we have defined them, it is clear that p, q, and r restrict to maps

$$\begin{tikzcd} \mathcal{M}_\dagger \arrow[r, "r_\dagger"] \arrow[rr, bend right, "q_\dagger"'] & \mathcal{P}_\dagger \arrow[r, "q_\dagger"] & \mathcal{F}_\dagger \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{M}^\dagger \arrow[r, "r^\dagger"] \arrow[rr, bend right, "q^\dagger"'] & \mathcal{P}^\dagger \arrow[r, "q^\dagger"] & \mathcal{F}^\dagger \end{tikzcd},$$

and preserve ambigressive pullback squares.

4.4 Building categories of spans

We have now constructued the following diagram of adequate triples and maps between them.

$$\begin{tikzcd} (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr, "r"] \arrow[dr, "q"'] & & (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl, "p"] \\ & (\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) & \end{tikzcd}$$

This gives us a commuting triangle

$$\begin{tikzcd} \mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr, "\rho'"] \arrow[dr, "\varpi'"'] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl, "\pi'"] \\ & \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) & \end{tikzcd} . \tag{15}$$

Our next step is to show that the maps π', ρ', and ϖ' are cocartesian fibrations. To do this, we need to show that they are inner fibrations, and that they admit enough cocartesian lifts; both of these will be assured by Theorem 3.3.2, assuming we can show that this theorem applies.

Lemma 4.4.1. The morphism p satisfies the conditions of Theorem 3.3.2.

Proof. We check the conditions.

1. We have to show that for any isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, there exists a p-cocartesian lift. But since every morphism in $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$ is an equivalence, it suffices to find a p-cartesian lift, which is possible because p is a cartesian fibration.

2. We need to show that for any commutative square

$$\sigma \quad = \quad \begin{tikzcd} S' \arrow[r, "f"] \arrow[d] & T' \arrow[d] \\ S \arrow[r, "\simeq"] & T \end{tikzcd}$$

   lying over a square

$$\begin{tikzcd} \langle n \rangle & \langle n \rangle \arrow[l, "\cong"'] \\ \langle m \rangle \arrow[u] & \langle m \rangle \arrow[l, "\cong"'] \arrow[u] \end{tikzcd}$$

   in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ (where we have drawn the arrows as belonging to $\mathcal{F}\mathrm{in}_*$), σ is a pullback square if and only if f is p-cocartesian. But f is lying over an equivalence, so it is p-cocartesian if and only if it is an equivalence, which in turn is true if and only if the square in question is pullback.

$\square$

Lemma 4.4.2. The morphism q satisfies the conditions of Theorem 3.3.2.

Proof. We have to check the conditions of the Theorem 3.3.2:

1. We have to show that for any isomorphism ϕ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, there exists a q-cocartesian lift. It follows from the 2/3 property for equivalences that such a lift will automatically be $q_\dagger$-cocartesian. But because ϕ is an equivalence, it suffices to find a q-cartesian lift, which is possible because q is a cartesian fibration.

2. We need to show that for any commutative square in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ corresponding to a cube in $\mathcal{S}_\times$ of the form

$$\begin{tikzcd} X' \arrow[rr, "F"] \arrow[dr] \arrow[dd] & & Y' \arrow[dr] \arrow[dd] & \\ & X \arrow[rr, crossing over, "\simeq"] & & Y \arrow[dd] \\ S' \arrow[rr, "f"] \arrow[dr] & & T' \arrow[dr] & \\ & S \arrow[rr, "\simeq"] \arrow[from=uu, crossing over] & & T \end{tikzcd}$$

and lying over a square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ of the form

$$\begin{tikzcd}
\langle n \rangle & \langle n \rangle \arrow[l, "\cong"'] \\
\langle m \rangle \arrow[u] & \langle m \rangle \arrow[l, "\cong"'] \arrow[u]
\end{tikzcd},$$

the morphism (F, f) is p-cocartesian if and only if the top and bottom squares are pullback and ψ is p-cocartesian. But this again follows because equivalences are p-cocartesian, and cocartesian morphisms are closed under composition.

$\square$

Lemma 4.4.3. The map r satisfies the conditions of Theorem 3.3.2.

Proof.

1. It suffices to show that for each morphism $T \to S'$ in $\mathcal{S}_\times$ lying over $\langle m \rangle \xleftarrow{\cong} \langle m \rangle$ and each $Y \to T$ in the fiber over $\langle m \rangle$, the square

   $$\begin{tikzcd}
   Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\
   T \arrow[r] & S'
   \end{tikzcd}$$

   is both p-cocartesian and $p_\dagger$-cocartesian, both of which follow from Lemma 4.2.7.

2. We need to show that for any commutative square in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ corresponding to a cube in $\mathcal{S}_\times$ of the form

   $$\begin{tikzcd}
   X' \arrow[rr, "F"] \arrow[dr] \arrow[dd] & & Y' \arrow[dr] \arrow[dd] & \\
   & X \arrow[rr, crossing over, "\simeq"] & & Y \arrow[dd] \\
   S' \arrow[rr, "f"] \arrow[dr] & & T' \arrow[dr] & \\
   & S \arrow[rr] \arrow[from=uu, crossing over] & & T
   \end{tikzcd}$$

   where the bottom square is pullback, the morphism (F, f) is r-cocartesian if and only if the top square is pullback and ψ is an equivalence. This is clear.

$\square$

Theorem 3.3.2 then tells us immediately that the maps in Diagram 15 are inner fibrations. Thus, in order to show that they are cocartesian fibrations, it suffices to exhibit sufficient cocartesian lifts.

Proposition 4.4.4. The maps π', ϖ', and ρ' are cocartesian fibrations.

Proof. We need to check that each of π', ϖ', and ρ' admits enough cocartesian lifts. Theorem 3.3.2 provides a sufficient condition that morphisms be cocartesian, so we can simply check that lifts of these forms exist. Fix be a morphism

$$\phi = \langle m\rangle \longrightarrow \langle n\rangle \xleftarrow{\simeq} \langle n\rangle$$

in $\mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, and a morphism

$$\Gamma = S \longleftarrow T \longrightarrow S'$$

in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$

- Given an object X in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ lying over $\langle m\rangle$, Theorem 3.3.2 tells us that a π'-cocartesian lift of ϕ is of the form

$$X \xleftarrow{f} Y \xrightarrow{\simeq} X' ,$$

  where f is p-cartesian. Such lifts always exist because p is a cartesian fibration.

- Given an object $S \to X$ in $\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ lying over $\langle m\rangle$, a ϖ'-cocartesian lift of ϕ is of the form

$$\begin{tikzcd} S \arrow[d] & T \arrow[l, "F"'] \arrow[r, "\simeq"] \arrow[d] & S' \arrow[d] \\ X & Y \arrow[l, "f"'] \arrow[r, "\simeq"] & X' \end{tikzcd},$$

  where F and f are p-cartesian (i.e. the left-hand square corresponds to a q-cartesian morphism in $\mathrm{Map}(\Delta^1, \mathcal{S})_\times$). Such lifts automatically exist because q is a cartesian fibration.

- Given an object $S \to X$ in $\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ lying over $\langle m\rangle$, a ρ'-cocartesian lift of ϕ is of the form

$$\begin{tikzcd} S \arrow[d] & S \times_X Y \arrow[l] \arrow[r, "\simeq"] \arrow[d] & S' \arrow[d] \\ X & Y \arrow[l] \arrow[r] & X' \end{tikzcd},$$

  where the left-hand square is pullback. Such lifts exist because $\mathcal{S}_\times$ admits pullbacks.

$\square$

Recall that there is an equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$. Pulling back Diagram 15 along this map gives a diagram of cocartesian fibrations

$$\begin{tikzcd} P \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & Q \arrow[dl, "\pi"] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd}.$$

Proposition 4.4.5. The maps π and ϖ exhibit P and Q as symmetric monoidal categories in the sense of [5, Def. 2.0.0.7].

Proof. We consider the case of π. We need to show that the pushforward functors corresponding to the inert maps $\rho_i\colon \langle n\rangle \to \langle 1\rangle$ induce an equivalence $P_{\langle n\rangle} \simeq P^n_{\langle 1\rangle}$. These maps send (up to equivalence) $[X_1, \ldots, X_n] \mapsto X_i$, giving an isomorphism $P_{\langle n\rangle} \cong P^n_{\langle 1\rangle}$.

The case of ϖ is similar. ☐

4.5 Building the functor $\hat{r}$

We are now ready to build the lax monoidal functor $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$. Restricting the functor ρ to the component lying over $\langle 1\rangle \in \mathcal{F}\mathrm{in}_*$ gives a cocartesian fibration $\rho_{\langle 1\rangle}\colon Q_{\langle 1\rangle} \to P_{\langle 1\rangle}$, which classifies a functor $P_{\langle 1\rangle} \to \mathcal{C}\mathrm{at}_\infty$. This will be almost the functor we need, as the following propositions show.

Proposition 4.5.1. There is an equivalence of simplicial sets $P_{\langle 1\rangle} \simeq \mathrm{Span}(\mathcal{S})$.

Proof. Unravelling the defintitions, the n-simplices of $P_{\langle 1\rangle}$ are ambigressive cartesian maps $\sigma\colon \mathrm{sd}(\Delta^n) \to \mathcal{S}_\times$ which make the diagram

$$\begin{tikzcd} \mathrm{sd}(\Delta^n) \arrow[rr, "\sigma"] \arrow[dr, "\mathrm{const}_{\langle 1\rangle}"'] & & \mathcal{S}_\times \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

commute. The condition that the diagram commute is the condition that σ land in $(\mathcal{S}_\times)_{\langle 1\rangle} \simeq \mathcal{S}$, and the ambigressiveness condition is that the forward-facing legs in σ lie over equivalences in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, which is automatic. This provides an isomorphism of simplicial sets $P_{\langle 1\rangle} \cong \mathrm{Span}(\mathcal{S})$, which is stronger than what we are looking for. ☐

Proposition 4.5.2. For any $X \in P_{\langle 1\rangle}$, there is an equivalence

$$(Q_{\langle 1\rangle})_X \simeq (\mathcal{S}_{/X})^{\mathrm{op}}.$$

Proof. The n-simplices of $(Q_{\langle 1\rangle})_X$ are ambigressive cartesian maps $\sigma\colon \mathrm{sd}(\Delta^n) \to$

$\mathrm{Map}(\Delta^1, \mathcal{S})_\times$ such that the diagrams

$$\begin{tikzcd} \mathrm{sd}(\Delta^n) \arrow[rr, "\sigma"] \arrow[dr, "\mathrm{const}_{\langle 1\rangle}"'] & & \mathrm{Map}(\Delta^1, \mathcal{S})_\times \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{sd}(\Delta^n) \arrow[r, "\sigma"] \arrow[rr, bend left, "\mathrm{const}_X"] & \mathrm{Map}(\Delta^1, \mathcal{S})_\times \arrow[r, "\mathrm{ev}_1"] & \mathcal{S}_\times \end{tikzcd}$$

commute. The first diagram ensures that σ lands entirely inside $(\mathrm{Map}(\Delta^1, \mathcal{S})_\times)_{\langle 1\rangle} \cong \mathrm{Map}(\Delta^1, \mathcal{S})$, so any such σ is uniquely determined by a map $\mathrm{sd}(\Delta^n) \times \Delta^1 \to \mathcal{S}$ such that:

- The composition

$$\mathrm{sd}(\Delta^n) \times \Delta^{\{1\}} \hookrightarrow \mathrm{sd}(\Delta^n) \times \Delta^1 \to \mathcal{S}$$

  is equal to const_X.

- The forward-facing legs in $\mathrm{sd}(\Delta^n) \times \Delta^{\{0\}}$ are mapped to equivalences in $\mathcal{S}$.

This is precisely the data of a map $\mathrm{sd}(\Delta^n) \diamond \Delta^0 \to \mathcal{S}$ sending $\Delta^0 \mapsto X$, and such that and the forward-facing legs of $\mathrm{sd}(\Delta^n)$ are sent to equivalences in $\mathcal{S}$. But these are precisely the n-simplices of $\mathrm{Span}_\simeq(\mathcal{S}^{/X})$. This provides an isomorphism $Q_{\langle 1\rangle} \cong \mathrm{Span}_\simeq(\mathcal{S}^{/X})$. The equvialences $(\mathcal{S}^{/X})^{\mathrm{op}} \to \mathrm{Span}_\simeq(\mathcal{S}^{/X})$ and $\mathcal{S}_{/X} \simeq \mathcal{S}^{/X}$ now give us the equivalence that we want. □

We now define R to be the functor classified by the cocartesian fibration ρ. On objects, this functor sends a space X to the category $(\mathcal{S}_{/X})^{\mathrm{op}}$. A morphism in $\mathrm{Span}(\mathcal{S})$ represented by a span of spaces

$$X \xleftarrow{g} Y \xrightarrow{f} X'$$

is sent to the functor $(\mathcal{S}_{/X})^{\mathrm{op}} \to (\mathcal{S}_{/X'})^{\mathrm{op}}$ defined by sending an object $S \to X$ to the object $S' \to X'$ defined up to equivalence as follows.

$$\begin{tikzcd} S \arrow[d] & S \times_X Y \arrow[l] \arrow[r, "\simeq"] \arrow[d] & S' \arrow[d] \\ X & Y \arrow[l, "g"'] \arrow[r, "f"] & X' \end{tikzcd}$$

This is almost what we want, up to an op: we would like our functor $\hat{r}$ to send $X \mapsto \mathcal{S}_{/X}$ rather than $(\mathcal{S}_{/X})^{\mathrm{op}}$. Fortunately, this is only an inconvenience. Recall that there is an autoequivalence of $\mathcal{C}\mathrm{at}_\infty$ sending $\mathcal{C} \mapsto \mathcal{C}^{\mathrm{op}}$.

Definition 4.5.3. We define the map $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$ to be the composition

$$\mathrm{Span}(\mathcal{S}) \xrightarrow{R} \mathcal{C}\mathrm{at}_\infty \xrightarrow{\mathrm{op}} \mathcal{C}\mathrm{at}_\infty \ .$$

The map R admits a lax monoidal structure by [5], 2.4.2.4–2.4.2.6, and the map op is an equivalence, hence certainly monoidal with respect to the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$. Thus, $\hat{r}$ is lax monoidal.

4.6 Conclusion and outlook

Let us meditate on our construction of the functor $\hat{r}$. Our starting point was the fact that the assignment of a space X to the quasicateogry $\mathcal{P}(X)$ of presheaves on X admits both covariant and contravariant functoriality. More specifically, we saw that for any map of spaces $f\colon X \to Y$, we could construct the following maps of presheaves.

- The left Kan extension functor $f_!\colon \mathcal{P}(X) \to \mathcal{P}(Y)$,
- The pullback functor $f^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$.

The existence of these two maps in opposite directions was the key ingredient that allowed us to construct our map $\hat{r}\colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$.

However, as we alluded to in Section 1, having a way of producing from a morphism in some category $\mathcal{C}$ a morphism in some category $\mathcal{D}$ in both a covariant and a contravariant way is a rather common situation. The following are examples.

- Let $p\colon E \to B$ be a covering space with finite fibers, say of cardinality n. This induces a map $p_*\colon H_*E \to H_*B$ on singular homology. Additionally, for any singular n-simplex $\sigma\colon |\Delta^n| \to B$, the homotopy lifting property allows us to construct an element $t(\sigma)$ of $S_n(E)$ such that $p_* \circ t(\sigma) = n$. Roughly, $t(\sigma)$ is a sum of singular n-simplices in E each lying above σ, one in each fiber. This extends to a map of singular chain complexes, and descends to a map $t_!\colon H_*B \to H_*E$.
- More generally, let $E \to B$ be a Hurewicz fibration.

Note that in both of these examples, not all morphisms admit wrong-way maps, and that it is therefore vital to have control over the classes of morphisms to which each leg is allowed to belong.

It is the hope of the author that the technology we have used in constructing the functor $\hat{r}$ will be useful in understanding similar constructions in other transfer theories. There are relatively simple generalizations, such as replacing presheaves valued in $\mathcal{S}$ with presheaves valued in other categories.

A Appendix

A.1 A brief list of model structures

The category $\mathcal{S}\mathrm{et}_\Delta$ carries two model structures of which we will make frequent use:

- The *Kan* model structure, which has the following description.
    - The fibrations are the Kan fibrations.
    - The cofibrations are the monomorphisms.
    - The weak equivalences are weak homotopy equivalences.
- The *Joyal* model structure. We will not give a complete description, referring the reader to [6, Sec. 2.2.5]. We will make use of the following properties.
    - The cofibrations are monomorphisms.
    - The fibrant objects are quasicategories, and the fibrations between fibrant objects are isofibrations, i.e. inner fibrations with lifts of equivalences (cf. [6, Cor. 2.6.5]).

The category Δ^{op} has a Reedy structure, which gives us, together with the Kan model structure, a model structure on the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta) = \mathcal{S}\mathrm{et}_{\Delta^2}$ with the following properties.

- The cofibrations are monomorphisms.
- The weak equivalences are level-wise weak homotopy equivalences.
- The fibrations are *Reedy fibrations* (Definition 2.1.5).

This model category has two important left Bousfeld localizations which we will need.

- The *Segal space model structure* on $\mathcal{S}\mathrm{et}_{\Delta^2}$ is the left Bousfeld localization of the Reedy model structure at the set of spine inclusions

  $$S = \left\{ \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \hookrightarrow \Delta^n \;\middle|\; n \geq 2 \right\},$$

  which has the following properties.
    - The cofibrations are the monomorphisms.
    - The fibrant objects are Segal spaces.
    - Every Reedy weak equivalence is a weak equivalence in the Segal space model structure.

- The *complete Segal space model structure* is the left Bousfeld localization of the Reedy model structure at the set S of spine inclusions together with the inclusion $\{0\} \hookrightarrow I$, where I is the (nerve of the) walking isomorphism. The complete Segal space model structure has the following properties.
    - The cofibrations are the monomorphisms.
    - The fibrant objects are complete Segal spaces.
    - Every weak equivalence in the Segal space model structure is a weak equivalence in the complete Segal space model structure.

Proposition A.1.1 (Joyal–Tierney)**.** There is a Quillen equivalence

$$p_1^* : \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{Joyal}} \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2}^{\mathrm{CSS}} : i_1^*$$

between the Joyal model structure on simplicial sets and the complete Segal space model structure on bisimplicial spaces. Here, the functors p_1^* and i_1^* are defined as follows.

- The functor p_1^* takes a simplicial set X to the bisimplicial space $(p_1^* X)_{mn} = X_m$. That is, the bisimplicial set $p_1^* X$ is constant in the the vertical direction, and each row is equal to the simplicial set X.
- The functor i_1^* takes a bisimplicial set K to the simplicial set $(i_1^* K)_n = K_{n0}$. That is, in the language of Subsection A.1, the simplicial set $i_1^* K$ is the 'zeroth row' of K.

Note A.1.2. In the notation of

For more information, the reader is directed to [1].

A.2 Divisibility of bifunctors

In this section, we recall some key results from [1]. We refer readers there for more information.

Let $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of 1-categories. We will say that $\odot$ is *divisible on the left* if for each $A \in \mathcal{E}_1$, the functor $A \odot -$ admits a right adjoint $A \backslash -$. In this case, this construction turns out also to be functorial in A; that is, we get a functor

$$-\backslash-\colon \mathcal{E}_1^{\mathrm{op}} \times \mathcal{E}_3 \to \mathcal{E}_2.$$

Analogously, $\odot\colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the right* if for for each $B \in \mathcal{E}_2$, the functor $- \odot B$ admits a right adjoint $-/B$. In this case we get a of two variables

$$-/-\colon \mathcal{E}_3 \times \mathcal{E}_2^{\mathrm{op}} \to \mathcal{E}_1.$$

Example A.2.1. The reader may find it helpful to keep in mind the cartesian product
$$- \times - \colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta.$$
In this case, both $A\backslash X$ and X/A are the mapping space X^A.

If $\odot$ is divisible on both sides, then there is a bijection between maps of the following types:
$$A \odot B \to X, \qquad A \to X/B, \qquad B \to A\backslash X.$$
In particular, this implies that the functors $X/-$ and $-\backslash X$ are mutually right adjoint.

If both $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and $\mathcal{E}_3$ is finitely cocomplete, then from a map $u\colon A \to A'$ in $\mathcal{E}_1$, a map $v\colon B \to B'$ in $\mathcal{E}_2$, and a map $f\colon X \to Y$ in $\mathcal{E}_3$, we can build the following maps.

- From the square
$$\begin{tikzcd} A \odot B \arrow[r] \arrow[d] & A' \odot B \arrow[d] \\ A \odot B' \arrow[r] & A' \odot B' \end{tikzcd}$$
we get a map
$$u \odot' v\colon A \odot B' \amalg_{A \odot B} A' \odot B \to A' \odot B'.$$
- From the square
$$\begin{tikzcd} A'\backslash X \arrow[r] \arrow[d] & A\backslash X \arrow[d] \\ A'\backslash Y \arrow[r] & A\backslash Y \end{tikzcd}$$
we get a map
$$\langle u\backslash f\rangle\colon A'\backslash X \to A\backslash X \times_{A\backslash Y} A'\backslash Y$$
- From the square
$$\begin{tikzcd} X/B' \arrow[r] \arrow[d] & X/B \arrow[d] \\ Y/B' \arrow[r] & Y/B \end{tikzcd}$$
we get a map
$$\langle f/v\rangle\colon X/B' \to X/B \times_{Y/B} Y/B'.$$

Proposition A.2.2. With the above notation, the following are equivalent adjoint lifting problems:

$$\begin{tikzcd}
A \odot B' \amalg_{A \odot B} A' \odot B \arrow[r] \arrow[d] & X \arrow[d] \\
A' \odot B' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & X/B' \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X/B \times_{Y/B} Y/B'
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r] \arrow[d] & A'\backslash X \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & A\backslash X \times_{A\backslash Y} A'\backslash Y
\end{tikzcd}$$

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