1 Categorical stuff

1.1 Kapranov-Voevodsky 2-vector spaces

In this section, we will denote by **Vect** the category $FinVect_{\mathbb{C}}$ of finite-dimensional complex vector spaces.

Definition 1 (Kapranov-Voevodsky 2-vector space). A \mathbb{C} -linear additive category is a category \mathcal{V} enriched in (Vect, \otimes) with finite biproducts (including zero objects). We call such biproducts *direct sums*.

An object $x \in \mathcal{V}$ is said to be *simple* if $\operatorname{Hom}(x, x) \cong \mathbb{C}$. An object is *semisimple* if it can be written as a finite direct sum simple objects. We say that \mathcal{V} is *semisimple* if all of its objects are semisimple.

A <u>Kapranov-Voevodsky 2-vector space</u> is a semisimple \mathbb{C} -linear additive category. A morphism of 2-vector spaces is a functor such that the maps $\operatorname{Hom}(x,y) \to \operatorname{Hom}(Fx,Fy)$ are linear.

Note that all 2-vector spaces are abelian, and all morphisms between 2-vector spaces are additive.

Example 2. Consider the category

1.2 Joyal model structure

Definition 3 (Joyal model structure). There is a model structure on $Set_{\mathbb{A}}$ with the following classes of morphisms.

- The cofibrations Cof are monomorphisms, i.e. level-wise injections.
- The weak equivalences W are *weak categorical equivalences*, i.e. morphisms $u \colon A \to B$ of simplicial sets such that for all quasi-categories X the induced map on internal homs

$$u^*: X^B \to X^A$$

induces an isomorphism...

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• The fibrations Fib are isofibrations, i.e. morphisms with the right lifting property with respect to inner horn inclusions and the morphism

$$N(\star \to (\bullet \to \star))$$
.

1.3 Kan extensions

1.4 Segal spaces

Definition 4 (Reedy model category). The category of simplicial objects in $Set_{\mathbb{A}}$ admits the following model structure, called the Reedy model structure.

- Weak equivalences are degree-wise weak equivalences.
- Fibrations

Definition 5 (category object). Let \mathcal{C} be an ∞ -category with finite limits. A <u>category</u> object in \mathcal{C} is a simplicial object

$$C_{\bullet} \colon \mathbb{\Delta}^{\mathrm{op}} \to \mathbb{C}$$

such that the maps

$$C_n \to C_1 \times_{C_0} \cdots \times_{C_0} C_1 \tag{1.1}$$

are equivalences¹ for all n.

The condition Equation 1.1 is known as the Segal condition.

Example 6. A category object in $N(\mathbf{Set})$ is the same thing as (the nerve of) a category; the segal condition can be interpreted as telling us that an n-simplex is completely determined by its spine.

The category of segal objects in **Set**, however, is not exactly the 2-category **Cat**: isomorphisms in the category of segal objects in **Set** are isomorphisms of categories, rather than equivalences. We need to pass to an ∞ -setting.

Definition 7 (segal space). A <u>Segal space</u> is a category object in the ∞ -category S of spaces. The category of Segal spaces, denoted Seg(S), is the full subcategory of simplicial objects in S satisfying the Segal condition.

Here we think of a category with a *space*, rather than a *set*, of operations.

Example 8. Any category, i.e. any category object in **Set**, can be thought of as a segal space by applying the functor **Set** \hookrightarrow S. In this case, the spaces, of morphisms are thought of as discrete sets of points.

 $^{^1}$ That is, descend to isomorphisms in the homotopy category h $^{\circ}$ C.

Since E^n is a category (hence a category object in **Set**), we can think of it as a Segal space using the functor **Set** $\hookrightarrow S$.

Definition 9 (classifying space of equivalences). Let E^n be the contractible groupoid with n objects and a unique morphism between each pair of objects, and let K be a Segal space. The classifying space of equivalences of K is the simplicial set

$$\iota K := \operatorname{colim} \iota_{\bullet} K, \qquad \iota_n K = \operatorname{Map}_{\operatorname{Seg}(\mathbb{S})}(E^n, K).$$

Example 10. Let K be an ordinary category. The zero-simplices of ιK are all maps

$$F: E^n \to K$$

for some $n \ge 0$. That is, they are clusters of n isomorphic objects in K. The 1-simplices are maps

$$E^n \times \mathbb{A}^1 \to K$$
,

i.e. pairs of clusters of isomorphic objects connected by a morphism.

Definition 11 (fully faithful, essentially surjective). We say that a morphism $f: X \to Y$ of Segal spaces is fully faithful and essentially surjective if

- 1. The map $\iota X \to \iota Y$ is an equivalence of spaces
- 2. The diagram

$$\begin{array}{ccc}
X_1 & \longrightarrow & Y_1 \\
\downarrow & & \downarrow \\
X_0 \times X_0 & \longrightarrow & Y_0 \times Y_0
\end{array}$$

is a pullback square.

Definition 12 (inert morphism). Let $\phi \colon [m] \to [n]$ be a morphism in Δ . We say that ϕ is intert if it is the inclusion of a subinterval, that is, if

$$\phi(i) = \phi(0) + i$$
 for all i .

We denote the subcategory of \triangle on inert morphisms by \triangle_{int} .

Let $Cell^1$ denote the full subcategory of Δ_{int} , i.e. the category

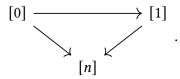
$$[0] \Rightarrow [1]$$
.

The objects are inert morphisms

$$[0] \rightarrow [n],$$
 and $[1] \rightarrow [n],$

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and the only non-identity morphisms are commuting triangles



We can draw this category as follows.

