A Appendix

A.1 A brief list of model structures

The category $\mathcal{S}\mathrm{et}_\Delta$ carries two model structures of which we will make frequent use:

- The *Kan* model structure, which has the following description.
  - The fibrations are the Kan fibrations.
  - The cofibrations are the monomorphisms.
  - The weak equivalences are weak homotopy equivalences.
- The *Joyal* model structure. We will not give a complete description, referring the reader to [2, Sec. 2.2.5]. We will make use of the following properties.
  - The cofibrations are monomorphisms.
  - The fibrant objects are quasicategories, and the fibrations between fibrant objects are isofibrations, i.e. inner fibrations with lifts of equivalences (cf. [2, Cor. 2.6.5]).

The category Δ^{op} has a Reedy structure, which gives us, together with the Kan model structure, a model structure on the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta) = \mathcal{S}\mathrm{et}_{\Delta^2}$ with the following properties.

- The cofibrations are monomorphisms.
- The weak equivalences are level-wise weak homotopy equivalences.
- The fibrations are *Reedy fibrations* (Definition **??**).

This model category has two important left Bousfeld localizations which we will need.

- The *Segal space model structure* on $\mathcal{S}\mathrm{et}_{\Delta^2}$ is the left Bousfeld localization of the Reedy model structure at the set of spine inclusions

  $$S = \left\{ \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \hookrightarrow \Delta^n \;\middle|\; n \geq 2 \right\},$$

  which has the following properties.
  - The cofibrations are the monomorphisms.
  - The fibrant objects are Segal spaces.
  - Every Reedy weak equivalence is a weak equivalence in the Segal space model structure.

- The *complete Segal space model structure* is the left Bousfeld localization of the Reedy model structure at the set S of spine inclusions together with the inclusion $\{0\} \hookrightarrow I$, where I is the (nerve of the) walking isomorphism. The complete Segal space model structure has the following properties.
  - The cofibrations are the monomorphisms.
  - The fibrant objects are complete Segal spaces.
  - Every weak equivalence in the Segal space model structure is a weak equivalence in the complete Segal space model structure.

Proposition A.1.1 (Joyal–Tierney)**.** There is a Quillen equivalence

$$p_1^* : \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{Joyal}} \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2}^{\mathrm{CSS}} : i_1^*$$

between the Joyal model structure on simplicial sets and the complete Segal space model structure on bisimplicial spaces. Here, the functors p_1^* and i_1^* are defined as follows.

- The functor p_1^* takes a simplicial set X to the bisimplicial space $(p_1^* X)_{mn} = X_m$. That is, the bisimplicial set $p_1^* X$ is constant in the the vertical direction, and each row is equal to the simplicial set X.
- The functor i_1^* takes a bisimplicial set K to the simplicial set $(i_1^* K)_n = K_{n0}$. That is, in the language of Subsection A.1, the simplicial set $i_1^* K$ is the 'zeroth row' of K.

Note A.1.2. In the notation of

For more information, the reader is directed to [1].

A.2 Divisibility of bifunctors

In this section, we recall some key results from [1]. We refer readers there for more information.

Let $\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of 1-categories. We will say that $\odot$ is *divisible on the left* if for each $A \in \mathcal{E}_1$, the functor $A \odot -$ admits a right adjoint $A\backslash -$. In this case, this construction turns out also to be functorial in A; that is, we get a functor

$$-\backslash - \colon \mathcal{E}_1^{\mathrm{op}} \times \mathcal{E}_3 \to \mathcal{E}_2.$$

Analogously, $\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the right* if for for each $B \in \mathcal{E}_2$, the functor $- \odot B$ admits a right adjoint $-/B$. In this case we get a of two variables

$$-/- \colon \mathcal{E}_3 \times \mathcal{E}_2^{\mathrm{op}} \to \mathcal{E}_1.$$

Example A.2.1. The reader may find it helpful to keep in mind the cartesian product

$$- \times -\colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta.$$

In this case, both $A\backslash X$ and X/A are the mapping space X^A.

If $\odot$ is divisible on both sides, then there is a bijection between maps of the following types:

$$A \odot B \to X, \qquad A \to X/B, \qquad B \to A\backslash X.$$

In particular, this implies that the functors $X/-$ and $-\backslash X$ are mutually right adjoint.

If both $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and $\mathcal{E}_3$ is finitely cocomplete, then from a map $u\colon A \to A'$ in $\mathcal{E}_1$, a map $v\colon B \to B'$ in $\mathcal{E}_2$, and a map $f\colon X \to Y$ in $\mathcal{E}_3$, we can build the following maps.

- From the square
$$\begin{tikzcd} A \odot B \arrow[r] \arrow[d] & A' \odot B \arrow[d] \\ A \odot B' \arrow[r] & A' \odot B' \end{tikzcd}$$
we get a map
$$u \odot' v\colon A \odot B' \amalg_{A \odot B} A' \odot B \to A' \odot B'.$$

- From the square
$$\begin{tikzcd} A'\backslash X \arrow[r] \arrow[d] & A\backslash X \arrow[d] \\ A'\backslash Y \arrow[r] & A\backslash Y \end{tikzcd}$$
we get a map
$$\langle u\backslash f\rangle\colon A'\backslash X \to A\backslash X \times_{A\backslash Y} A'\backslash Y$$

- From the square
$$\begin{tikzcd} X/B' \arrow[r] \arrow[d] & X/B \arrow[d] \\ Y/B' \arrow[r] & Y/B \end{tikzcd}$$
we get a map
$$\langle f/v\rangle\colon X/B' \to X/B \times_{Y/B} Y/B'.$$

Proposition A.2.2. With the above notation, the following are equivalent adjoint lifting problems:

$$
\begin{tikzcd}
A \odot B' \amalg_{A \odot B} A' \odot B \arrow[r] \arrow[d] & X \arrow[d] \\
A' \odot B' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r] \arrow[d] & X/B' \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X/B \times_{Y/B} Y/B'
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r] \arrow[d] & A' \backslash X \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & A \backslash X \times_{A \backslash Y} A' \backslash Y
\end{tikzcd}
$$