Complete Segal spaces of spans

Angus Rush

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1. Cocartesian fibrations between complete Segal spaces

1.1. Marked bisimplicial sets

Note 1. Toby: At the moment, I'm assuming all the notation from Joyal/Tierney. That means in particular the functor $-\square-$ and all related constructions. If this is going to be turned into something other people read, I'll of course add this.

Definition 2. A ***marked bisimplicial set*** $(X, \mathcal{E})$ is a bisimplicial set X together with a distinguished subset $\mathcal{E} \subseteq (X_1)_0$ containing all degenerate edges, i.e. all edges in the image of $s_0\colon (X_0)_0 \to (X_1)_0$. Equivalently, a marked bisimplicial set is bisimplicial set X together with a marking $\mathcal{E}$ on the simplicial set X/Δ^0.

Definition 3. For a marked simplicial set A and an unmarked simplicial set B, define a marking on the bisimplicial set $A \square B$ as follows: a simplex $(a, b) \in A_1 \times B_0$ is marked if and only if a is marked.

This construction gives us a functor

$$- \square - : \mathbf{Set}_{\Delta}^{+} \times \mathbf{Set}_{\Delta} \to \mathbf{Set}_{\Delta^2}^{+}.$$

Our first order of business is to generalize the results of [1] (summarized in Section A.1.2) to the marked case. We will first show that the above functor is divisible on the left and on the right.

Notation 4. For any marked simplicial set A, denote the underlying unmarked simplicial set by $\mathring{A}$. Similarly, for any marked bisimplicial set X, denote the underlying unmarked bisimplicial set by $\mathring{X}$.

Definition 5. Let A denote a marked simplicial set, B an unmarked simplicial set, and X a marked bisimplicial set.

- Define an unmarked simplicial set $A \backslash X$ level-wise by

$$(A \backslash X)_n = \mathrm{Hom}_{\mathbf{Set}_{\Delta^2}^{+}}(A \square \Delta^n, X).$$

- Define a marked simplicial set X/B as follows. The underlying simplicial set is the same as $\mathring{X}/\mathring{B}$, and a 1-simplex $\Delta^1 \to X/B$ is marked if and only if the corresponding map $\Delta^1 \square B \to X$ of unmarked bisimplicial sets comes from a map of marked bisimplicial sets $(\Delta^1)^{\sharp} \square B \to X$.

Example 6. For any unmarked simplicial set A and marked bisimplicial set X, there is an isomorphism

$$A^{\flat} \backslash X \cong A \backslash \mathring{X}.$$

Similarly, for any unmarked bisimplicial set Y and marked simplicial set B, there is an isomorphism

$$B \backslash X^{\sharp} \cong \mathring{B} \backslash X.$$

Example 7. For any marked bisimplicial set X, the marked simplicial set X/Δ^0 is the first row of X, together with the obvious marking. In particular, for any quasicategory $\mathcal{C}$ with a marking $\mathcal{E}$, the bisimplicial set $\Gamma(\mathcal{C})$ defined level-wise by

$$\Gamma(\mathcal{C})_n = \mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq}$$

is a complete Segal space, and the marking on $\mathcal{C}$ gives a marking on $\Gamma(\mathcal{C})$ in the obvious way. Then the first row $\Gamma(\mathcal{C})/\Delta^0$ is isomorphic $\mathcal{C}$ as a marked simplicial set.

Proposition 8. We have the following adjunctions.

1. For each marked simplicial set $A \in \mathbf{Set}_{\Delta}^{+}$ there is an adjunction.

$$A\square -\colon \mathbf{Set}_{\Delta} \leftrightarrow \mathbf{Set}_{\Delta^2}^{+} : A\backslash -$$

2. For each unmarked simplicial set $B \in \mathbf{Set}_{\Delta}$ there is an adjunction.

$$-\square B\colon \mathbf{Set}_{\Delta}^{+} \leftrightarrow \mathbf{Set}_{\Delta^2} : -/B.$$

Proof. □

This shows that the marked version of □ is, in the language of [1], *divisible on the left and on the right.* This implies that $A\backslash X$ is a functor of both A and X, and that X/B is a functor of both X and B. This in turn implies that there is a bijection between maps

$$A\square B \to X, \qquad A \to X/B \qquad \text{and} \quad B \to A\backslash X.$$

Thus, the functors $-\backslash X$ and $X/-$ are mutually right adjoint. Since $\mathbf{Set}_{\Delta}$, $\mathbf{Set}_{\Delta}^{+}$, and $\mathbf{Set}_{\Delta^2}^{+}$ are all finitely complete and cocomplete, the results of [1, Sec. 7] imply that we get functors

Note 9. Toby: This would go on for a while. I'll probably reproduce the relevant parts of [1] in a separate section, and then reference stuff there.

Note 10. Toby: Is 'full subcategory inclusion' the right terminology for the below? I mean that an n-simplex of $A^\flat\backslash X$ belongs to in $A\backslash X$ if and only if each of its vertices belong to $A^\flat$.

Lemma 11. For any marked simplicial set A and marked bisimplicial set X, the simplicial set $A\backslash X$ is a Kan complex, and the inclusion $i\colon A\backslash X \hookrightarrow A^\flat\backslash X \cong \mathring{A}\backslash\mathring{X}$ is a full subcategory inclusion.

Proof. We first show that the map i is a full subcategory inclusion. The n-simplices of $A\backslash X$ are maps of marked simplicial sets $\tilde{\sigma}\colon A\square\Delta^n \to X$. A map of underlying bisimplicial sets gives a map of marked bisimplicial sets if and only if it respects the markings, i.e. if and only if for each $(a, i) \in A_1 \times (\Delta^n)_0$ with a marked, $\tilde{\sigma}(a, i)$ is marked in X. This is equivalent to demanding that $\sigma|_{\Delta^{\{i\}}}$ belong to $A\backslash X$.

To show that $A\backslash X$ is a Kan complex, we need to find dashed lifts below.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}.$$

For $n = 1$, the horn inclusion is of the form $\Delta^0 \hookrightarrow \Delta^1$, and we can take the lift to be degenerate. For $n \geq 2$, we can augment our diagram as follows.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \arrow[r] & A^\flat\backslash X \\
\Delta^n \arrow[urr, dashed]
\end{tikzcd}.$$

Since $A^\flat\backslash X$ is a Kan complex, we can always find such a dashed lift. The inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is full on vertices, so our lift factors through $A\backslash X$. ◻

Definition 12. Let X be a bisimplicial set, and let $\mathcal{E}$ be a marking on X. We will say that $\mathcal{E}$ ***respects path components*** if it has the following property: for any map $\Delta^1 \to X_1$ representing an edge $e \to e'$ between morphisms e and e', the morphism e is marked if and only if the morphism e' is marked.

Proposition 13. Let $f\colon X \to Y$ be a Reedy fibration between marked bisimplicial sets such that the marking on X respects path components, and let $u\colon A \to A'$ be a morphism of marked simplicial sets whose underlying morphism of unmarked simplicial sets is a monomorphism. Then the map $\langle u\backslash f\rangle$ is a Kan fibration.

Proof. We need to show that for each $n \geq 0$ and $0 \leq k \leq n$ we can solve the lifting problem

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & A\backslash X \times_{A'\backslash Y} A'\backslash Y
\end{tikzcd}.$$

First assume that $n \geq 2$. We can augment the above square as follows.

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[r] \arrow[d] & A^\flat\backslash X \arrow[d] \\
\Delta^n \arrow[r] & A\backslash X \times_{A'\backslash Y} A'\backslash Y \arrow[r] & A^\flat\backslash X \times_{(A')^\flat\backslash Y} (A')^\flat\backslash X
\end{tikzcd}.$$

Since the map on the right is a Kan fibration, we can solve the outer lifting problem. All the vertices of Δ^n belong to Λ^n_k, so a lift of the outside square factors through $A'\backslash X$.

Now take $n = 1$, $k = 0$, so our horn inclusion is $\Delta^{\{0\}} \hookrightarrow \Delta^1$. We need to be able to solve the lifting problem

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & X/\Delta^1 \arrow[d] \\
A' \arrow[r] \arrow[ur, dashed] & X/\Delta^0 \times_{Y/\Delta^0} Y/\Delta^1
\end{tikzcd}.$$

Note that a lift always exists on the level of simplicial sets, so we only need to check that any such lift respects the marking on X. To see this, consider the following triangle formed by some dashed lift.

$$\begin{tikzcd} & X/\Delta^1 \arrow[d] \\ A' \arrow[ur, dashed] \arrow[r] & X/\Delta^0 \end{tikzcd}$$

Let $a \in A_1$ be a marked 1-simplex, and consider the diagram

$$\begin{tikzcd} \Delta^1 \square \Delta^0 \arrow[r, "(a{,}\mathrm{id})"] \arrow[d] & A \square \Delta^0 \arrow[d] \arrow[dr] & \\ \Delta^1 \square \Delta^1 \arrow[r, "(a{,}\mathrm{id})"] & A \square \Delta^1 \arrow[r, dashed, "\ell"'] & X \end{tikzcd},$$

where the triangle on the right is the adjunct to the triangle above. In order to check that the dashed lift respects the marking on X, we have to show that for each $(a, b) \in (A\square\Delta^1)_{10} = A_1 \times \{0, 1\}$, the morphism $\ell(a, b)$ is marked in X. The commutativity of the triangle guarantees this for $b = 0$. The map $\Delta^1\square\Delta^1 \to X$ gives us a 1-simplex in X_1 connecting $\ell(a, 0)$ and $\ell(a, 1)$, which implies by that $\ell(a, 1)$ is also marked because each marking respects path components.

The case $n = 1$, $k = 1$ is essentially identical. □

1.2. Cocartesian morphisms

Definition 14. Let $f : X \to Y$ be a Reedy fibration between complete Segal spaces. A morphism $e \in X_{10}$ is ***f-cocartesian*** if the square

$$\begin{tikzcd} \Delta^2\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \arrow[r] \arrow[d] & \Lambda^2_0\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \arrow[d] \\ \Delta^2\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} \arrow[r] & \Lambda^2_0\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} \end{tikzcd}$$

is homotopy pullback.

Example 15. Identity morphisms are clearly f-cocartesian.

Definition 16. For $n \geq 1$, define the following simplicial subsets of Δ^n.

- For $n \geq 1$, denote by I_n the ***spine*** of Δ^n, i.e. the simplicial subset

$$\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n.$$

- For $n \geq 2$, denote by L_n the simplicial subset

$$L_n = \Delta^{\{0,1\}} \amalg_{\Delta^{\{0\}}} \Delta^{\{0,2\}} \amalg_{\Delta^{\{2\}}} \overbrace{\Delta^{\{2,3\}} \amalg_{\Delta^{\{3\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}}^{I_{\{2,\dots,n\}}} \subseteq \Delta^n.$$

  That is, L_n is the union of $\Delta^{\{0,1\}}$ with the spine of $d_1\Delta^n$. We will call L_n the ***left spine*** of Δ^n.

Note that $L_2 \cong \Lambda^2_0$.

Note 17. Toby: I don't know if the below proof is easy enough that it doesn't need to be expanded. I started typing out a more detailed proof, but the the notation became pretty terrible. This should probably be taken as another sign that better notation is needed.

Proposition 18. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then the square

$$\begin{tikzcd}
\Delta^n\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \arrow[r] \arrow[d] & L_n\backslash X \times_{\Delta^{\{0,1\}}\backslash X} \{e\} \arrow[d] \\
\Delta^n\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\} \arrow[r] & L_n\backslash Y \times_{\Delta^{\{0,1\}}\backslash Y} \{fe\}
\end{tikzcd}$$

is homotopy pullback for all $n \geq 2$.

Proof. We have the base case $n = 2$ because e is f-cocartesian. The higher cases follow from the Segal condition.

□

For any simplicial subset $A \subseteq \Delta^n$, denote the marking on A where the only marked nondegenerate edge is $\Delta^{\{0,1\}}$ by $A^{\mathcal{L}}$. If A does not contain the edge $\Delta^{\{0,1\}}$ then this agrees with the $\flat$-marking. For any simplicial set A, define a marked simplicial set $(\Delta^1 \star A, \mathcal{L}')$ where the only nondegenerate simplex belonging to $\mathcal{L}'$ is Δ^1.

Lemma 19. Let $A \hookrightarrow B$ be a monomorphism of simplicial sets, and suppose that B is n-skeletal (and therefore that A is n-skeletal). Then the map

$$(\Delta^{\{0\}} \star B)^\flat \amalg_{(\Delta^{\{0\}} \star A)^\flat} (\Delta^1 \star A)^{\mathcal{L}'} \hookrightarrow (\Delta^1 \star B)^{\mathcal{L}'}$$

is in the saturated hull of the morphisms

$$(\Lambda^k_0)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}, \qquad 2 \leq k \leq n+2.$$

Proof. It suffices to show this for $A \hookrightarrow B = \partial\Delta^m \hookrightarrow \Delta^m$ for $0 \le m \le n$. In this case the necessary map is of the form

$$(\Lambda_0^{m+2})^{\mathcal{L}} \hookrightarrow (\Delta^{m+2})^{\mathcal{L}}.$$

□

We will say a collection of morphisms $\mathcal{A} \subset \mathrm{Mor}(\mathbf{Set}_{\Delta}^{+})$ has the *right cancellation property* if for all $u, v \in \mathrm{Mor}(\mathbf{Set}_{\Delta}^{+})$,

$$u \in \mathcal{A}, \quad vu \in \mathcal{A} \quad \implies \quad v \in A.$$

Lemma 20. Let $\mathcal{A}$ be a saturated set of morphisms of $\mathbf{Set}_{\Delta}^{+}$ all of whose underlying morphisms are monomorphisms, and which has the right cancellation property. Further suppose that $\mathcal{A}$ contains the following classes of morphisms.

1. Maps $(A)^{\flat} \hookrightarrow (B)^{\flat}$, where $A \to B$ is inner anodyne.
2. Left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \ge 2$.

Then $\mathcal{A}$ contains left horn inclusions $(\Lambda_0^n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \ge 2$.

Proof. We have an isomorphism $(L_2)^{\mathcal{L}} \cong (\Lambda_1^2)^{\mathcal{L}}$, which belongs to $\mathcal{A}$ because saturated sets contain all isomorphisms.

We proceed by induction. Suppose we have shown the result for $n-1$ i.e. that all horn inclusions $(\Lambda_0^k)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}$ belong to $\mathcal{A}$ for $2 \le k \le n-1$. From now on on we will suppress the marking $(-)^{\mathcal{L}}$. All simplicial subsets of Δ^n below will have $\Delta^{\{0,1\}}$ marked if they contain it.

Consider the factorization

$$\begin{tikzcd} L_n \arrow[r, "u_n"] \arrow[rr, bend right, "v_n \circ u_n"'] & \Lambda_0^n \arrow[r, "v_n"] & \Delta^n \end{tikzcd} .$$

The morphism $v_n \circ u_n$ belongs to $\mathcal{A}$ by assumption, so in order to show that v_n belongs to $\mathcal{A}$, it suffices by right cancellation to show that u_n belongs to $\mathcal{A}$. Consider the factorization

$$L_n \xrightarrow{w_n'} L_n \cup d_1\Delta^n \xrightarrow{w_n} \Lambda_0^n.$$

The map w_n' is a pushout along a spine inclusion, and hence is inner anodyne. Hence, we need only show that w_n belongs to $\mathcal{A}$. Let

$$Q = d_2\Delta^n \cup \cdots \cup d_n\Delta^n,$$

and consider the following pushout diagram.

$$\begin{tikzcd} (L_n \cup d_1\Delta^n) \cap Q \arrow[r, hook] \arrow[d, hook] & Q \arrow[d, hook] \\ L_n \cup d_1\Delta^n \arrow[r, hook] & L_n \cup d_1\Delta^n \cup Q \end{tikzcd}.$$

Since $L_n \cup d_1\Delta^n \cup Q \cong \Lambda^n_0$, the bottom map is w_n, so it suffices to show that the top map belongs to $\mathcal{A}$. But this is isomorphic to

$$(\Delta^{\{0,1\}} \star \emptyset) \coprod_{(\Delta^{\{0\}} \star \emptyset)} (\Delta^{\{0\}} \star \partial\Delta^{\{2,3,\ldots,n\}}) \hookrightarrow \Delta^{\{0,1\}} \star \partial\Delta^{\{2,3,\ldots,n\}}.$$

The simplicial set $\partial\Delta^{\{2,\ldots,n\}}$ is $(n-3)$-skeletal, so this map belongs to $\mathcal{A}$ by Lemma 19. $\square$

Definition 21. Let $f\colon X \longrightarrow Y$ be a Reedy fibration between complete Segal spaces. We will say that f is a ***cocartesian fibration*** if each morphism in Y has an f-cocartesian lift in X.

Definition 22. Let $f\colon X \longrightarrow Y$ be a cocartesian fibration between complete Segal spaces. We define a marking on X such that a morphism is marked if and only if it is f-cocartesian. We will denote the corresponding marked simplicial by set $X^\natural$. We can view the map f as a map of marked bisimplicial sets $f\colon X^\natural \longrightarrow Y^\sharp$.

Note 23. Toby: In the rest of this subsection, we have to assume that the cocartesian marking respects path components. I'm convinced that this isn't necessary (i.e. that the cocartesian marking always respects path components), but this doesn't matter in our case as it is trivially true for spans.

Proposition 24. Let $f\colon X \longrightarrow Y$ be a cocartesian fibration of complete Segal spaces, and further assume that the cocartesian marking on X respects path components. Let $e \in X_{10}$ be an f-cocartesian morphism. Then e, viewed as a morphism in the quasicategory X/Δ^0, is f/Δ^0-cocartesian.

Proof. Let $u^n_0\colon (\Lambda^n_0 \hookrightarrow \Delta^n)^{\mathcal{L}}$ denote the $\mathcal{L}$-marked inclusion. First, let $n \geq 2$, and consider the set

$$S = \left\{ \begin{matrix} u\colon A \longrightarrow \mathrm{B} \text{ morphism of} \\ \text{marked simplicial sets} \\ \text{such that } \mathring{u} \text{ is mono} \end{matrix} \;\middle|\; \langle u \backslash f \rangle \text{ weak homotopy equivalence} \right\}.$$

It is clear that this set has the right cancellation property. By Proposition 13, a map u belonging to S is automatically a Kan fibration, hence is a trivial Kan fibration. Thus, we can equivalently say that $u \in S$ if and only if u has the left-lifting property with respect to all maps of the form $\langle X/v \rangle$, where v is a cofibration of simplicial sets. Thus, S is saturated.

The set S contains all flat-marked inner anodyne morphisms because f is a Reedy fibration. Proposition 18 implies that S contains all left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$. Thus, by Lemma 20, S contains all $\mathcal{L}$-marked left horn inclusions. $\square$

Corollary 25. Let $f\colon X \to Y$ be a cocartesian fibration of complete Segal spaces such that the cocartesian marking on X respects path components. Then the map

$$f/\Delta^0\colon X/\Delta^0 \to Y/\Delta^0$$

is a cocartesian fibration of quasicategories.

Proof. By [1], the map f/Δ^0 is an isofibration between quasicategories, and hence is certainly an inner fibration. By assumption, every morphism in Y has a f-cartesian lift. ByProposition 24, these lifts are also f/Δ^0-cocartesian. $\square$

2. Complete Segal spaces of spans

The main goal of this section is to prove the following theorem.

Theorem 26. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories admitting all pullbacks, and let $p\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration which preserves pullbacks. Further suppose that p has the following property:

> For any square
>
> $$\sigma = \begin{tikzcd} x \arrow[r, "f"] \arrow[d] & y \arrow[d] \\ x' \arrow[r, "f'"] & y' \end{tikzcd}$$
>
> in $\mathcal{C}$ such that the morphism f' is p-cocartesian and $p(\sigma)$ is pullback in $\mathcal{D}$, the following are equivalent.
>
> - The morphism f is p-cocartesian.
> - The square σ is pullback.

Then the functor $\pi\colon \mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is a cocartesian fibration of Segal spaces, and if a morphism has the form

$$\begin{tikzcd} & y \arrow[dl] \arrow[dr] & \\ x & & x' \end{tikzcd},$$

then it is p-cartesian.

Note 27. Since the definition of $\mathbf{Span}(\mathcal{C})$ is self-dual, the functor $\mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$ is also a cartesian fibration with cartesian morphisms of the form

$$\begin{tikzcd} & y \arrow[dl, "\bullet" marking] \arrow[dr, "\circ" marking] & \\ x & & x' \end{tikzcd} .$$

We prove Theorem 26 in several steps. Since we will be working with bicartesian fibrations, it will be helpful to adapt some of the tools of marked simplicial sets to our purposes.

2.1. Bimarked simplicial sets

Our proof of Theorem 26 will involve working with with bicartesian fibrations. For this reason, it will be helpful to have results about simplicial sets with two markings, one of which controls the cocartesian structure and one of which controls the cartesian structure. This section consists mainly of verifications that some key results about marked simplicial sets which can be found in [2, Sec. 3.1] hold in the bimarked case. The only results we will make use of are Proposition 37, Example 32, and Example 33.

Definition 28. A ***bimarked simplicial set*** is a triple $(X, \mathcal{E}, \mathcal{E}')$, where X is a simplicial set and $\mathcal{E}$ and $\mathcal{E}'$ are markings. We will often shorten this to $X^{(\mathcal{E},\mathcal{E}')}$. A morphism of bimarked simplicial sets is a morphism of the underlying simplicial sets which preserves each class of markings separately. We will denote the category of bimarked simplicial sets by $\mathbf{Set}_{\Delta}^{++}$.

Example 29. For any simplicial set X we will denote

- The bimarked simplicial set where $\mathcal{E}$ and $\mathcal{E}'$ contain only the degenerate edges by $X^{(\flat,\flat)}$,
- The bimarked simplicial set where $\mathcal{E}$ contains only the degenerate edges and $\mathcal{E}'$ contains every edge by $X^{(\flat,\sharp)}$,
- The bimarked simplicial set where $\mathcal{E}$ contains every edge and $\mathcal{E}'$ contains only the degenerate edges by $X^{(\sharp,\flat)}$, and
- The bimarked simplicial set where $\mathcal{E}$ and $\mathcal{E}'$ contain every edge by $X^{(\sharp,\sharp)}$.

Example 30. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration. Denote by $\mathcal{C}^{\natural}$ the bimarked simplicial set where

- The set $\mathcal{E}$ is the set of all p-cocartesian morphisms, and
- The set $\mathcal{E}'$ is the set of all p-cartesian morphisms.

In this way every bicartesian fibration gives a morphism of bimarked simplicial sets.

The category $\mathbf{Set}_\Delta^{++}$ is closely connected to the category $\mathbf{Set}_\Delta^{+}$ and the category $\mathbf{Set}_\Delta$. We have the following obvious results.

- There is a forgetful functor $u\colon \mathbf{Set}_\Delta^{++} \to \mathbf{Set}_\Delta$ which forgets both markings. This has left adjoint $\iota\colon X \mapsto (X, \flat, \flat)$. The functor ι is a full subcategory inclusion.
- There is a forgetful functor $u_1\colon \mathbf{Set}_\Delta^{++} \to \mathbf{Set}_\Delta^{+}$, which forgets the second marking, sending $(X, \mathcal{E}, \mathcal{E}') \mapsto (X, \mathcal{E})$. This has a left adjoint ι_1 given by the functor which sends $(X, \mathcal{E}) \mapsto (X, \mathcal{E}, \flat)$. The functor ι_1 is a full subcategory inclusion; there is a bijection between maps between marked simplicial sets $(X, \mathcal{E}) \to (Y, \mathcal{E}')$ and maps $(X, \mathcal{E}, \flat) \to (Y, \mathcal{E}', \flat)$. The same is true of the functor u_2 which forgets the first marking and its left adjoint ι_2.

Just as in the marked case, we will a set of bimarked anodyne morphisms, the morphisms with the left lifting property with respect to bicartesian fibrations.

Definition 31. The class of ***bimarked anodyne morphisms*** is the saturated hull of the union of the following classes of morphisms.

(1) For each $0 < i < n$, the inner horn inclusions

$$(\Lambda_i^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}.$$

(2) For every $n > 0$, the inclusion

$$(\Lambda_0^n)^{(\mathcal{L},\flat)} \hookrightarrow (\Delta^n)^{(\mathcal{L},\flat)},$$

where $\mathcal{L}$ denotes the set of all degenerate edges of Δ^n together with the edge $\Delta^{\{0,1\}}$.

(2') For every $n > 0$, the inclusion

$$(\Lambda_n^n)^{(\flat,\mathcal{R})} \hookrightarrow (\Delta^n)^{(\flat,\mathcal{R})},$$

where $\mathcal{R}$ denotes the set of all degenerate edges of Δ^n together with the edge $\Delta^{\{n-1,n\}}$.

(3) The inclusion

$$(\Lambda_1^2)^{(\sharp,\flat)} \coprod_{(\Lambda_1^2)^{(\flat,\flat)}} (\Delta^2)^{(\flat,\flat)} \to (\Delta^2)^{(\sharp,\flat)}.$$

(3') The inclusion

$$(\Lambda_1^2)^{(\flat,\sharp)} \coprod_{(\Lambda_1^2)^{(\flat,\flat)}} (\Delta^2)^{(\flat,\flat)} \to (\Delta^2)^{(\flat,\sharp)}.$$

(4) For every Kan complex K, the map

$$K^{(\flat,\flat)} \longrightarrow K^{(\sharp,\flat)}.$$

(4') For every Kan complex K, the map

$$K^{(\flat,\flat)} \longrightarrow K^{(\flat,\sharp)}.$$

Example 32. Define a bimarked structure $(\mathrm{asd}(\Delta^1), \mathcal{E}, \mathcal{E}') = \mathrm{asd}(\Delta^1)^{\heartsuit}$ on $\mathrm{asd}(\Delta^1)$, where the morphism $01 \longrightarrow 11$ is $\mathcal{E}$-marked, and the morphism $01 \longrightarrow 00$ is $\mathcal{E}'$-marked. Denoting $\mathcal{E}$-marked morphisms with a $\bullet$ and $\mathcal{E}'$-marked morphisms with a $\circ$, we can draw this as follows.

$$\begin{tikzcd} & & 11 \\ & 01 \arrow[ur, "\bullet" description] \arrow[dl, "\circ" description] & \\ 00 & & \end{tikzcd}$$

The inclusion $\{00\} = \mathrm{asd}(\Delta^{\{0\}}) \hookrightarrow \mathrm{asd}(\Delta^1)^{\heartsuit}$ is bimarked anodyne: we can factor it

$$\{00\} \hookrightarrow (\Delta^1)^{(\flat,\sharp)} \hookrightarrow \mathrm{asd}(\Delta^1)^{\heartsuit},$$

where the first inclusion is of the form (2') and the second is a pushout of a morphism of the form (2). We can draw this process as follows.

$$\begin{tikzcd} 00 \end{tikzcd} \hookrightarrow \begin{tikzcd} & 01 \arrow[dl, "\circ" description] \\ 00 & \end{tikzcd} \hookrightarrow \begin{tikzcd} & & 11 \\ & 01 \arrow[ur, "\bullet" description] \arrow[dl, "\circ" description] & \\ 00 & & \end{tikzcd}$$

Example 33. Define a bimarked structure $\mathrm{asd}(\Delta^2)^{\heartsuit}$ on $\mathrm{asd}(\Delta^2)$, following the notation of Example 32, as follows.

$$\begin{tikzcd} & & 11 & & \\ & 01 \arrow[ur, "\bullet" description] \arrow[dl, "\circ" description] & & 12 \arrow[ul] \arrow[dr] & \\ 00 & & 02 \arrow[ll] \arrow[rr] \arrow[ul] \arrow[uu] \arrow[ur, "\bullet" description] & & 22 \end{tikzcd}$$

Note that the bimarking of Example 32 is the restriction of $\mathrm{asd}(\Delta^2)^{\heartsuit}$ to $\mathrm{asd}(\Delta^{\{0,1\}})$. Denote the restriction of the bimarking $\operatorname{asd}(\Delta^2)^\heartsuit$ to $\operatorname{asd}(\Delta^{\{0,2\}})$ by $\operatorname{asd}(\Delta^{\{0,2\}})^\heartsuit$; this agrees with the $(\flat, \flat)$-marking.

The inclusion $\operatorname{asd}(\Delta^{\{0,2\}})^\heartsuit \hookrightarrow \operatorname{asd}(\Delta^2)^\heartsuit$ is bimarked anodyne. To see this, note that... factorization...

Proposition 34. A map $p\colon (X, \mathcal{E}_X, \mathcal{E}'_X) \to (S, \mathcal{E}_S, \mathcal{E}'_S)$ of bimarked simplicial sets has the right lifting property with respect to bimarked anodyne morphisms if and only if the following conditions are satisifed.

(A) The map p is an inner fibration of simplicial sets.

(B) An edge e of X is $\mathcal{E}_X$-marked if and only if $p(e)$ is $\mathcal{E}_S$-marked and e is p-cocartesian.

(B') An edge e of X is $\mathcal{E}'_X$-marked if and only if $p(e)$ is $\mathcal{E}'_S$-marked and e is p-cartesian.

(C) For every object y of X and every $\mathcal{E}_S$-marked edge $\bar{e}\colon \bar{x} \to p(y)$ in S, there exists a $\mathcal{E}_X$ marked edge $e\colon x \to y$ of X with $p(e) = \bar{e}$.

(C') For every object y of X and every $\mathcal{E}'_S$-marked edge $\bar{e}\colon \bar{x} \to p(y)$ in S, there exists a $\mathcal{E}'_X$ marked edge $e\colon x \to y$ of X with $p(e) = \bar{e}$.

Proof. By [2, Prop. 3.1.1.6], (A), (B) and (C) are equivalent to (1), (2), and (3). By its dual, (A), (B') and (C') are equivalent to (1), (2'), and (3') □

We would like to define cofibrations of bimarked simplicial sets to be maps of bimarked simplicial sets whose underlying map of simplicial sets is a monomorphism, and then show that the class of bimarked anodyne maps is stable under smash products with arbitrary cofibrations. Unfortunately, this turns out not to be quite true; the candidate class of cofibrations described above is generated by the following classes of maps.

(I) Boundary fillings $(\partial\Delta^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}$.

(II) Markings $(\Delta^1)^{(\flat,\flat)} \to (\Delta^1)^{(\sharp,\flat)}$.

(III) Markings $(\Delta^1)^{(\flat,\flat)} \to (\Delta^1)^{(\flat,\sharp)}$.

There is nothing that tells us, for example, that the smash product of a bimarked anodyne map of type (2) with a cofibration of type (III) should be bimarked anodyne. Denoting arrows with the first marking using a $\bullet$ and the second using a $\circ$, this amounts, in the case $n = 0$, to the statement that the map

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "\bullet" marking] \arrow[dr] & \cdot \arrow[d, "\bullet" marking] & & \cdot \arrow[r, "\circ" marking] \arrow[d, "\bullet" marking] \arrow[dr] & \cdot \arrow[d, "\bullet" marking] \\
\cdot \arrow[r, "\circ" marking] & \cdot \arrow[r, phantom, "\longrightarrow"] & \, & \cdot \arrow[r, "\circ" marking] & \cdot
\end{tikzcd}$$

should be bimarked anodyne, which it isn't. However, we have the following weaker statement.

Lemma 35. The class of bimarked anodyne maps in $\mathbf{Set}_\Delta^{++}$ is stable under smash products with flat monomorphisms, i.e. morphisms $A^{(\flat,\flat)} \to B^{(\flat,\flat)}$ such that the underlying morphism of simplicial sets $A \to B$ is a monomorphism. That is, if $f\colon X \to Y$ is bimarked anodyne and $A \to B$ is a monomorphism of simplicial sets, then

$$(X \times B^{(\flat,\flat)}) \coprod_{X \times A^{(\flat,\flat)}} (Y \times A^{(\flat,\flat)}) \to Y \times B^{(\flat,\flat)}$$

is bimarked anodyne.

Proof. It suffices to show that for any flat boundary inclusion $(\partial\Delta^n)^{(\flat,\flat)} \to (\Delta^n)^{(\flat,\flat)}$ and any generating bimarked anodyne morphism $X \to Y$, the map

$$(X \times (\Delta^n)^{(\flat,\flat)}) \coprod_{X \times (\partial\Delta^n)^{(\flat,\flat)}} (Y \times (\partial\Delta^n)^{(\flat,\flat)}) \to Y \times (\Delta^n)^{(\flat,\flat)}$$

is bimarked anodyne. If $X \to Y$ belongs to one of the classes (1), (2'), (3'), or (4'), then this is true by the arguments of [2, Prop. 3.1.2.3]. If $X \to Y$ belongs to one of the classes (1), (2), (3), or (4), then it is true by the dual arguments. $\square$

Definition 36. For any bimarked simplicial sets X, Y, define a simplicial set $\mathrm{Map}^{(\flat,\flat)}(X, Y)$ by the following universal property: for any simplicial set Z, there is a bijection

$$\mathrm{Hom}_{\mathbf{Set}_\Delta}(Z, \mathrm{Map}^{(\flat,\flat)}(X, Y)) \cong \mathrm{Hom}_{\mathbf{Set}_\Delta^{++}}(Z^{(\flat,\flat)} \times X, Y).$$

Proposition 37. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration of quasicategories, and denote by $\mathcal{C}^\natural \to \mathcal{D}^{(\sharp,\sharp)}$ the associated map of bimarked simplicial sets as in Example 30. Let $X \to Y$ be any bimarked anodyne map of simplicial sets. Then the square

$$\begin{tikzcd}
\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^\natural)^\simeq \arrow[r] \arrow[d] & \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural)^\simeq \arrow[d] \\
\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{D}^{(\sharp,\sharp)})^\simeq \arrow[r] & \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})^\simeq
\end{tikzcd}$$

is a homotopy pullback in the Kan model structure.

Proof. First, we show that the right-hand map is a Kan fibration. In fact, the underlying map

$$\mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^\natural) \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})$$

is a trivial Kan fibration, since by Lemma 35 together with Proposition 34 we can solve the necessary lifting problems. This, together with the fact that each of the objects is a Kan complex, implies that in order to show that the above square is homotopy pullback it suffices to check that the map

$$\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^{\natural})^{\simeq} \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^{\natural})^{\simeq} \times_{\mathbf{Fun}^{(\flat,\flat)}(Y,\mathcal{D}^{(\sharp,\sharp)})^{\simeq}} \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})^{\simeq}$$

is a trivial Kan fibration. Since the functor $(-)^{\simeq}$ is a right adjoint it preserves limits, so it again suffices to show that the underlying map

$$\mathbf{Fun}^{(\flat,\flat)}(Y, \mathcal{C}^{\natural}) \to \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{C}^{\natural}) \times_{\mathbf{Fun}^{(\flat,\flat)}(Y,\mathcal{D}^{(\sharp,\sharp)})} \mathbf{Fun}^{(\flat,\flat)}(X, \mathcal{D}^{(\sharp,\sharp)})$$

is a trivial fibration, which follows from Lemma 35. □

2.2. Cartesian morphisms

In this section, we show that we really have identified the cocartesian morphisms correctly. We will first show that morphisms of the form

$$\begin{tikzcd} & y \arrow[dl, "\circ" description] \arrow[dr, "\bullet" description] & \\ x & & x' \end{tikzcd} \quad . \tag{1}$$

are p-cocartesian.

Proposition 38. Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration of quasicategories which preserves pullbacks and satisfies the condition of Theorem 26, and let

$$p\colon \mathbf{Span}(\mathcal{C}) \to \mathbf{Span}(\mathcal{D})$$

be the corresponding map between complete Segal spaces of spans. If a morphism in $\mathbf{Span}(\mathcal{C})$ is of the form

$$\begin{tikzcd} & y \arrow[dl, "\circ" description] \arrow[dr, "\bullet" description] & \\ x & & x' \end{tikzcd} \quad ,$$

where the morphism marked with a $\circ$ is π-cartesian and the morphism marked with a $\bullet$ is π-cocartesian, then it is p-cocartesian.

Proof. In order to show that a morphism $e\colon x \leftarrow y \to x'$ of the form given in Proposition 38 are cocartesian, we have to show that the square

$$\begin{tikzcd} \mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{C})^{\simeq}} \{e\} \arrow[r] \arrow[d] & \mathbf{Fun}(\mathrm{asd}(\Lambda^2_0), \mathcal{C})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{C})^{\simeq}} \{e\} \arrow[d] \\ \mathbf{Fun}^{\mathrm{Cart}}(\mathrm{asd}(\Delta^2), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{D})^{\simeq}} \{\pi e\} \arrow[r] & \mathbf{Fun}(\mathrm{asd}(\Lambda^2_0), \mathcal{D})^{\simeq} \times_{\mathbf{Fun}(\mathrm{asd}(\Delta^{\{0,1\}}),\mathcal{D})^{\simeq}} \{\pi e\} \end{tikzcd}$$

is homotopy pullback.

Recall the bimarked structure $\operatorname{asd}(\Delta^2)^\heartsuit = (\operatorname{asd}(\Delta^2), \mathcal{E}, \mathcal{E}')$ on $\operatorname{asd}(\Delta^2)$ of Example 33, reproduced below, where the nondegenerate edges in $\mathcal{E}$ are distinguished with a $\bullet$, and the nondegenerate edges in $\mathcal{E}'$ are distinguished with a $\circ$.

$$\begin{tikzcd}
 & & 11 & & \\
 & 01 \arrow[ur, "\bullet"] \arrow[dl, "\circ"'] & & 12 \arrow[ul] \arrow[dr] & \\
00 & & 02 \arrow[ll] \arrow[ul] \arrow[uu] \arrow[ur, "\bullet"] \arrow[rr] & & 22
\end{tikzcd}$$

Denote the induced bimarked structure on $\operatorname{asd}(\Lambda^{\{2,1\}})$ also with a heart.

We now note that we can decompose the above square into two squares

$$\begin{tikzcd}
\mathbf{Fun}^{\mathrm{Cart}}(\operatorname{asd}(\Delta^2), \mathcal{C})^\simeq \times_{\mathbf{Fun}(\operatorname{asd}(\Delta^{\{0,1\}}), \mathcal{C})^\simeq} \{e\} \arrow[r] \arrow[d] & \mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^\simeq \times_{\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^{\{0,1\}})^\heartsuit, \mathcal{C}^\natural)^\simeq} \{e\} \arrow[d] \\
\mathbf{Fun}^{\mathrm{Cart}}(\operatorname{asd}(\Delta^2), \mathcal{D})^\simeq \times_{\mathbf{Fun}(\operatorname{asd}(\Delta^{\{0,1\}}), \mathcal{D})^\simeq} \{\pi e\} \arrow[r] & \mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\sharp,\sharp)})^\simeq \times_{\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^{\{0,1\}})^\heartsuit, \mathcal{D}^{(\sharp,\sharp)})^\simeq} \{\pi e\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^\simeq \times_{\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^{\{0,1\}})^\heartsuit, \mathcal{C}^\natural)^\simeq} \{e\} \arrow[r] \arrow[d] & \mathbf{Fun}(\operatorname{asd}(\Lambda^2_0), \mathcal{C})^\simeq \times_{\mathbf{Fun}(\operatorname{asd}(\Delta^{\{0,1\}}), \mathcal{C})^\simeq} \{e\} \arrow[d] \\
\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\sharp,\sharp)})^\simeq \times_{\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^{\{0,1\}})^\heartsuit, \mathcal{D}^{(\sharp,\sharp)})^\simeq} \{\pi e\} \arrow[r] & \mathbf{Fun}(\operatorname{asd}(\Lambda^2_0), \mathcal{D})^\simeq \times_{\mathbf{Fun}(\operatorname{asd}(\Delta^{\{0,1\}}), \mathcal{D})^\simeq} \{\pi e\}
\end{tikzcd}.$$

The first square is a homotopy pullback because the bottom morphism is a full subcategory inclusion of connected components, and the fiber over a connected component corresponding to Cartesian functors $\operatorname{asd}(\Delta^2) \to \mathcal{D}$ is consists precisely of Cartesian functors $\operatorname{asd}(\Delta^2) \to \mathcal{C}$ by the condition of Theorem 26.

Therefore, we need to show that the second square is homotopy pullback. For this it suffices to show that the square

$$\begin{tikzcd}
\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{C}^\natural)^\simeq \arrow[r] \arrow[d] & \mathbf{Fun}(\operatorname{asd}(\Lambda^2_0), \mathcal{C})^\simeq \arrow[d] \\
\mathbf{Fun}^{(\flat,\flat)}(\operatorname{asd}(\Delta^2)^\heartsuit, \mathcal{D}^{(\sharp,\sharp)})^\simeq \arrow[r] & \mathbf{Fun}(\operatorname{asd}(\Lambda^2_0), \mathcal{D})^\simeq
\end{tikzcd}.$$

is homotopy pullback. But it is easy to see that this is of the form of the square in Proposition 37, with $X \to Y = \operatorname{asd}(\Delta^{\{0,2\}})^\heartsuit \to \operatorname{asd}(\Delta^2)^\heartsuit$, which we saw in Example 33 was bimarked anodyne. $\square$

Theorem 39. Let $\mathcal{C}$ and $\mathcal{D}$ be quasicategories with pullbacks, and let $\pi\colon \mathcal{C} \to \mathcal{D}$ be a bicartesian fibration which sends pullbacks to pullbacks. Then the map

$$\mathbf{Span}(\mathcal{C})/\Delta^0 \to \mathbf{Span}(\mathcal{D})/\Delta^0$$

is a cocartesian fibration (hence a bicartesian fibration) between quasicategories.

Proof. Take $\mathbf{Span}(\mathcal{C})$ to have the cocartesian marking, which we have denoted by $\mathbf{Span}(C)^\natural$. In order to fulfill the requirements of Corollary 25, it suffices to show that this marking respects path components. But this is clear: a 1-simplex $g \to g'$ in $\mathbf{Span}(\mathcal{C})_1^\natural$ looks as follows.

$$\begin{tikzcd}
X \arrow[d, "\simeq"] & Y \arrow[l, "g_0"'] \arrow[r, "g_1"] \arrow[d, "\simeq"] & Z \arrow[d, "\simeq"] \\
X' & Y' \arrow[l, "g_0'"'] \arrow[r, "g_1'"] & Z'
\end{tikzcd}$$

Since equivalences are both cartesian and cocartesian, it is clear that f is cocartesian if and only if f' is cocartesian. ◻

A. Background results

A.1. Results from J+T

A.1.1. Bisimplicial sets

We will denote the simplex category by $\mathbb{\Delta}$, and the category of functors $\mathbb{\Delta}^{\mathrm{op}} \to \mathbf{Set}$ by $\mathbf{Set}_{\mathbb{\Delta}}$. This is the category of *simplicial sets.*

The category $\mathbf{Set}_{\mathbb{\Delta}}$ carries two model structures of which we will make frequent use:

- The *Kan* model structure, which has the following description.
  - The fibrations are the Kan fibrations.
  - The cofibrations are the monomorphisms.
  - The weak equivalences are weak homotopy equivalences.
- The *Joyal* model structure. We will not give a complete description, referring the reader to [2, Sec. 2.2.5]. We will make use of the following properties.
  - The cofibrations are monomorphisms.
  - The fibrant objects are quasicategories, and the fibrations between fibrant objects are isofibrations, i.e. inner fibrations with lifts of equivalences (cf [2, Cor. 2.6.5]).

The category $\mathbb{\Delta}^{\mathrm{op}}$ has a Reedy structure, which gives us, together with the Kan model structure, a model structure on the category $\mathbf{Fun}(\mathbb{\Delta}^{\mathrm{op}}, \mathbf{Set}_{\mathbb{\Delta}})$ with the following properties.

- The cofibrations are monomorphisms.
- The weak equivalences are level-wise weak homotopy equivalences.
- We will call the fibrations *Reedy fibrations,* and will come to them in a moment.

A.1.2. The box functor

In this section, we recall some key results from [1]. We refer readers there for more information.

Let $\odot\colon \mathcal{E}_2 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor. Following [1], we will say that $\odot$ is *divisible on the left* if for each $A \in \mathcal{E}_1$, the functor $A \odot -$ admits a right adjoint $A\backslash -$, and *divisible on the right* if for for each $B \in \mathcal{E}_2$, the functor $- \odot B$ admits a right adjoint $-/B$. In the second case, we get a functor of two variables

$$-/-\colon \mathcal{E}_3 \times \mathcal{E}_2^{\mathrm{op}} \to \mathcal{E}_1;$$

the first case is similar. Thus, there is a bijection between maps

$$A \square B \to X, \qquad A \to X/B, \qquad B \to A\backslash X,$$

implying that the functors $X/-$ and $-\backslash X$ are mutually right adjoint.

If both $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and $\mathcal{E}_3$ is finitely cocomplete, then from a map $u\colon A \to B$ in $\mathcal{E}_1$ and a map $f\colon X \to Y$ in $\mathcal{E}_3$, we get a map

$$X/B$$

Definition 40. Define a functor $-\square-\colon \mathbf{Set}_{\mathbb{\Delta}} \times \mathbf{Set}_{\mathbb{\Delta}} \to \mathbf{Set}_{\mathbb{\Delta}^2}$ by the formula

$$(X, Y) \mapsto X \square Y, \qquad (X \square Y)_{mn} = X_m \times Y_n.$$

We will call this functor the ***box functor.***

We provide a partial proof of the following result because we will need to refer to it later.

Proposition 41. The box functor is *divisible on the left and on the right.* This means that for each simplicial set A, there are adjunctions

$$A \square -\colon \mathbf{Set}_{\mathbb{\Delta}} \leftrightarrow \mathbf{Set}_{\mathbb{\Delta}^2} : A\backslash -,$$

where an n-simplex of the simplicial set $A\backslash X$ is a map $A \square \Delta^n \to X$.

There is a similar adjunction

$$-\square A \colon \mathbf{Set}_{\triangle} \leftrightarrow \mathbf{Set}_{\triangle^2} : -\backslash A.$$

Proof. We prove the first; the second is identical. We do this by explicitly exhibiting a bijection

$$\mathrm{Hom}_{\mathbf{Set}_{\triangle^2}}(A \square B, X) \cong \mathrm{Hom}_{\mathbf{Set}_{\triangle}}(B, A\backslash X).$$

Define a map

$$\Phi \colon \mathrm{Hom}_{\mathbf{Set}_{\triangle^2}}(A \square B, X) \to \mathrm{Hom}_{\mathbf{Set}_{\triangle}}(B, A\backslash X)$$

by sending a map $f \colon A \square B \to X$ to the map $\tilde{f} \colon B \to A\backslash X$ which sends an n-simplex $b \in B_n$ to the composition

$$A \square \Delta^n \xrightarrow{(\mathrm{id}, b)} A \square B \xrightarrow{f} X \ .$$

Now define a map $\mathrm{ev} \colon A \square (A\backslash X) \to X$ level-wise as follows: we take $(a, \sigma) \in A_m \times (A\backslash X)_n$ to

$$\sigma_{mn}(a, \mathrm{id}_{\Delta^n}) \in X_{mn}.$$

Then define a map

$$\Psi \colon \mathrm{Hom}_{\mathbf{Set}_{\triangle}}(B, A\backslash X) \to \mathrm{Hom}_{\mathbf{Set}_{\triangle^2}}(A \square B, X)$$

sending a map $g \colon B \to A\backslash X$ to the composition

$$A \square B \xrightarrow{(\mathrm{id}, g)} A \square (A\backslash X) \xrightarrow{\mathrm{ev}} X \ .$$

The maps Φ and Ψ are mutually inverse, and provide the necessary bijection. □

A.2. Results about equivalences

References

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