1 Miscellaneous stuff to be deleted or absorbed

1.1 The triple structures

1.1.1 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}" description] \arrow[dr, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_! g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 1.1.1. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 1.1.2. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\alpha"] & & b \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\beta"] & \\
& b' \arrow[rr] \arrow[dd] & & c \arrow[dd] \\
\bar{a} & & \bar{b} \arrow[ll] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$.[1] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 1.1.1. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 1.1.2

We also need to show that any such pullback is of this form. This also follows from Lemma 1.1.2. □

1.2 Building categories of spans

Lemma 1.2.1. Let

$$\begin{tikzcd}
X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r] & Y
\end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{1}$$

[1] Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 1 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. $\square$

Proposition 1.2.2. The map $p \colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p \colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\
& X \arrow[rr, "f"] \arrow[dd] \arrow[r, phantom, "\Rightarrow"] & & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] & & \mathcal{C} \arrow[dr, equal] \arrow[u, phantom, "\Rightarrow"] \arrow[ur, phantom, "\circlearrowleft"] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}
$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}
$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$
\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],
\end{aligned}
$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 1.2.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$
\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].
\end{aligned}
$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □