Monoidal push-pull for local systems

Angus Rush

July 16, 2021

Contents

1 Some problems

- I'm using some things that don't exist yet:

- An extrinsic description of left Kan extensions in $\mathbb{C}\mathrm{at}_\infty$
- The easy colimit formula for non-inclusions

2 Background

2.1 Scaled simplicial sets

Note: This is basically a summary of the parts of The Goodwillie Calculus that we're gonna need.

Definition 2.1.1. A ***scaled simplicial set*** is a pair $(X, A_X) = \overline{X}$, where X is a simplicial set, and $A \subseteq X_2$ is a set of 2-simplices of X which contains every degenerate simplex. A simplex belonging to A is known as *thin.* A morphism of scaled simplicial sets is a morphism of the underlying simplicial sets which takes thin simplices to thin simplices.

We will need the following facts about scaled simplicial sets.

1. For any category $\mathcal{C}$ enriched in marked simplicial sets, we can build a scaled simplicial set $\mathrm{N}_{\mathrm{sc}}(\mathcal{C}) = (\mathrm{N}(\mathcal{C}), T)$, the ***scaled nerve*** of $\mathcal{C}$, whose underlying simplicial set $\mathrm{N}(\mathcal{C})$ is the simplicial nerve of $\mathcal{C}$. The scaling T is defined in the following way. Let $\sigma\colon \Delta^2 \to \mathrm{N}(\mathcal{C})$ be any 2-simplex. This corresponds to a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

   in $\mathcal{C}$, together with a 1-simplex $\alpha_\sigma\colon \Delta^1 \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ corresponding to a map $gf \to h$. We will then define σ to be thin if and only if α_σ is marked in $\mathrm{Map}_{\mathcal{C}}(X, Z)$.

2. The functor N_{sc} admits a left adjoint $\mathfrak{C}^{\mathrm{sc}}\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathcal{C}\mathrm{at}_\Delta^+$, the ***scaled rigidification***, such that for any scaled simplicial set $\overline{X}$ with underlying simplicial set X, the underlying simplicially enriched category of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$ is simply $\mathfrak{C}(X)$, the rigidification of X. The scaling on $\mathfrak{C}^{\mathrm{sc}}(X)$ also admits a relatively simple description.

3. A map of scaled simplicial sets $\overline{X} \to \overline{Y}$ is said to be a ***bicategorical equivalence*** if the map $F\colon \mathfrak{C}^{\mathrm{sc}}(X) \to \mathfrak{C}^{\mathrm{sc}}(Y)$ is a weak equivalence of $\mathcal{S}\mathrm{et}_\Delta$-enriched categories; that is, if for all objects x, x' of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$, the map

$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{X})}(x, x') \to \mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{Y})}(F(x), F(x'))$$

is a weak equivalence of marked simplicial sets, and if for all $y \in \mathfrak{C}^{sc}(\overline{Y})$, there exists some $x \in \mathfrak{C}^{sc}(\overline{X})$ such that $F(x)$ and y are isomorphic in the homotopy category $h\mathfrak{C}^{sc}(\overline{Y})$.

4. There is a left proper, combinatorial model structure on $\mathcal{S}\mathrm{et}_\Delta^{sc}$, whose cofibrations are monomorphisms, and whose weak equivalences are bicategorical fibrations. This gives a Quillen equvialence
$$\mathfrak{C}^{sc} : \mathcal{S}\mathrm{et}_\Delta^{sc} \leftrightarrow \mathcal{C}\mathrm{at}_\Delta^+ : \mathrm{N}_{sc}.$$

5. An ∞-***bicategory*** is a fibrant object with respect to this model structure. In particular, for any fibrant object $\mathcal{C} \in \mathcal{C}\mathrm{at}_\Delta^+$ (i.e. any quasicategory-enriched category with equvialences marked), $\mathrm{N}_{sc}(\mathcal{C})$ is an ∞-bicategory.

6. The category of scaled simplicial sets has a natural simplicial enrichment as follows: For any two scaled simplicial sets $\overline{X}$ and $\overline{Y}$, we define
$$\mathrm{Map}^{sc}(\overline{X}, \overline{Y})_n = \mathrm{Hom}(\overline{X} \times \Delta^n_\sharp, \overline{Y}),$$
where $\Delta^n_\sharp$ is the scaled simplicial set (Δ^n, Δ^n_2). For $\overline{Y}$ an ∞-bicategory, $\mathrm{Map}^{sc}(\overline{X}, \overline{Y})$ is a quasicategory. Taking the core gives a Kan complex.

7. With this,

8. We will denote the simplicially enriched category whose objects are ∞-bicategories, with mapping spaces as above by $\mathbf{Bicat}_\infty$. We then define $\mathcal{C}\mathrm{at}_{(\infty,2)} = \mathrm{N}(\mathbf{Bicat}_\infty)$. This is a quasicategory.

9. We have also a simplicially enriched category whose objects are quasicategories, and whose morphisms are $\mathcal{S}\mathrm{et}_\Delta^+$-enriched mapping spaces. The coherent nerve of this is

We will need multiple notions of the quasicategory of $(\infty, 2)$-categories.

- The model category $\mathcal{S}\mathrm{et}_\Delta^{sc}$ of scaled simplicial sets, whose cofibrations are precisely the monomorphisms.
- The weak equivalences are bicategorical equivalences,

2.2 Enhanced twisted arrow categories

This is a summary of the parts of the theory that we'll need.

1. For each $n \in \mathrm{N}$, we consider the simplicial set $\Delta^n \star (\Delta^n)^{op}$. We will denote the vertices of Δ^n by i and the vertices of $(\Delta^n)^{op}$ by $\bar{i}$. This is really just a fancy notation for the

$(2n+1)$-simplex with vertices

$$1, \ldots, n, \bar{n}, \ldots, \bar{1}.$$

2. We define a cosimplicial object

$$Q\colon \Delta \longrightarrow \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}; \qquad [n] \mapsto (\Delta^n \star (\Delta^n)^{\mathrm{op}}, T_n),$$

where T_n is the set of 2-simplices $\sigma\colon \Delta^2 \longrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ which

- factor through Δ^n; or
- factor through $(\Delta^n)^{\mathrm{op}}$; or
- are of the form $\Delta^{\{i,j,\bar{k}\}}$, where $i < j \leq k$; or
- are of the form $\Delta^{\{k,\bar{j},\bar{i}\}}$, where $i < j \leq k$.

3. For any scaled simplicial set $\overline{X}$, we define a simplicial set

$$\mathbf{Tw}(\overline{X})_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}}(Q([n]), \overline{X}).$$

We further define a marking on $\mathbf{Tw}(\overline{X})$ by saying that a 1-simplex in $\mathbf{Tw}(\overline{X})$ is marked if and only if it corresponds to a map $\Delta^3_{\sharp} \longrightarrow \overline{X}$.

4. This construction clearly extends to an ordinary functor $\mathbf{Tw}\colon \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}} \longrightarrow \mathcal{S}\mathrm{et}_{\Delta}^{+}$. One checks that this functor preserves weak equivalences, and thus extends to a functor of quasi-categories or **Kan**-enriched categories or whatever your favorite model is

$$\mathbf{Tw}\colon \mathcal{C}\mathrm{at}_{(\infty,2)} \longrightarrow \mathcal{C}\mathrm{at}_{\infty}.$$

5. For any scaled simplicial set $\overline{X}$, a zero-simplex in $\mathbf{Tw}(\overline{X})$ is a 1-simplex in $\overline{X}$, and a 1-simplex in $\mathbf{Tw}(\overline{X})$ is a 3-simplex in $\overline{X}$ of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\Rrightarrow"] & x' \arrow[d] \\ y & y' \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d] & x' \arrow[d] \arrow[dl] \arrow[dl, phantom, shift left=3, "\Rightarrow"] \arrow[dl, phantom, shift right=3, "\circlearrowleft"] \\ y & y' \arrow[l] \end{tikzcd},$$

where the simplices marked with $\circlearrowleft$ are thin. The $\Rightarrow$-marked simplices are simply to remind us that given a $\mathcal{S}\mathrm{et}_{\Delta}^{+}$-enriched category $\mathcal{C}$, two-simplices in $\mathfrak{C}^{\mathrm{sc}}(\mathcal{C})$ correspond to diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & c' \arrow[dr, "g"] \arrow[d, phantom, "\Uparrow\eta"] & \\ c \arrow[ur, "f"] \arrow[rr, "h"'] & & c'' \end{tikzcd},$$

where η is a 2-morphism $h \Rightarrow g \circ f$.

Definition 2.2.1. We will call a 1-simplex $\Delta^1 \to \mathbf{Tw}(\overline{X})$ ***thin*** if it corresponds to a map $(\Delta^1 \star \Delta^{1\,\mathrm{op}})_\sharp \to \overline{X}$; that is, a 1-simplex in $\Delta^1 \to \mathbf{Tw}(\overline{X})$ is thin if and only if each of the faces of the adjunct map $\Delta^1 \star (\Delta^1)^{\mathrm{op}}$ is a thin 2-simplex.

3 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 3.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples which is an inner fibration, satisfying the following conditions.

1. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.
2. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$ is cocartesian and f is p-cartesian.

Proof. Even simpler than before. The square that we have to show is pullback factors as before, but this time we don't have to take the final pullback; the square is automatically pullback. □

4 The non-monoidal construction

4.1 Con extensions

We provide a conjectural answer to the following question. Let $f\colon \mathcal{D} \to \mathcal{D}'$ be a functor of quasicategories, and let $\mathcal{F}\colon \mathcal{D} \to \mathcal{C}$ be similar. What universal property does the ∞-Kan extension $f_!\mathcal{F}\colon \mathcal{D}' \to \mathcal{C}$ satisfy?

A Kan extension, conjecturally, should be modelled by a 2-simplex in $\mathbb{C}\mathrm{at}_\infty$

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "\mathcal{F}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta\Uparrow"] & \mathcal{C} \\ & \mathcal{D}' \arrow[ur, "f_!\mathcal{F}"'] & \end{tikzcd},$$

where η is the unit map.

Definition 4.1.1. A 2-simplex $\sigma\colon \Delta^2 \to \mathbb{C}\mathrm{at}_\infty$ is a ***left con extension*** if for all

$$\tau\colon \Lambda^3_0 \to \mathbb{C}\mathrm{at}_\infty$$

such that $d_2\tau = \sigma$, and such that $d_3\tau$ is thin, the space

$$\mathrm{Map}(\Delta^3, \mathbb{C}\mathrm{at}_\infty) \times_{\mathrm{Map}(\Lambda^3_0, \mathbb{C}\mathrm{at}_\infty)} \{\tau\}$$

of fillings to a full 3-simplex is contractible.

We conjecture that left con extensions are not really a con at all, but rather bona fide left Kan extensions; that is, a 2-simplex in $\mathcal{C}\mathrm{at}_\infty$ is a left con extension if and only if it is of the form pictured in Equation 4.1.

4.2 The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d] \\
\mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}
\end{tikzcd} .$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \arrow[dl, phantom, "\circlearrowleft" near start] \\
\mathcal{C} \arrow[r, equal] \arrow[ur, phantom, "\Rightarrow" near start] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow" near start] & Y \arrow[d] \arrow[dl] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[ul, phantom, "\circlearrowleft" near start]
\end{tikzcd} ,$$

We will generally denote such a morphism as follows.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & Y \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd} .$$

There is a potential to be confused here. The natural transformation notated above does not correspond to a single natural transformation, but rather to two natural transformations. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

Such a morphism is r-cartesian if and only if all 2-simplices are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd} .$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 4.2.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex it corresponds to in $\mathbb{C}\mathrm{at}_\infty$ has the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3ex, "\circlearrowleft"] \arrow[dr, phantom, shift right=3ex, "\Rightarrow"] & Y \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, shift right=3ex, "\eta\,\Rightarrow"] & Y \arrow[d, "f_!\mathcal{F}'"] \arrow[dl, "f_!\mathcal{F}"] \arrow[d, phantom, shift right=4ex, "\circlearrowleft"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd},$$

where $f_!\mathcal{F}$ and $f_!\mathcal{F}'$ are both left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines the data of the first, and the filling 3-simplex, up to contractible choice.

For short, we will denote cocartesian morphisms

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "=\,!\Rightarrow"] & X' \arrow[d, "\mathrm{Lan}_f\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}.$$

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$
\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[r, phantom, shift right=4, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_! g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}
$$

Proposition 4.2.2. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 4.2.3. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$
\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}
$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$
\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] & & b \arrow[dr] \arrow[dd] & \\
& b' \arrow[rr] \arrow[dd] \arrow[dl, phantom, "\Downarrow\alpha"] & & c \arrow[dd] \arrow[dl, phantom, "\Downarrow\beta"] \\
\bar{a} & & \bar{b} \arrow[ll] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}
$$

in $\mathbb{C}$.[1] Suppose b is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if a is thin.

Proof. Sorry for the writing, I'm really tired. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 4.2.2. This is literally the lemma right above us. □

[1] Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

Well, it's nice to see that this stuff works as intended. We know that the triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ of unconstrained spans of spaces works as intended, so let's just move on.

Lemma 4.2.4. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{1}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 1 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. □

Proposition 4.2.5. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\
& X \arrow[rr, "f"] \arrow[dd] \arrow[r, phantom, shift right=3ex, "\Longrightarrow"] & & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] & & \mathcal{C} \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] \arrow[l, phantom, shift left=3ex, "\Longrightarrow"] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}
$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$
\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}
$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$
\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],
\end{aligned}
$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 4.2.4. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$
\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].
\end{aligned}
$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. This is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects, and s sends the terminal object of one to the other.) Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between

Kan complexes, thus cofinal. $\square$

5 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd}
\Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\
\Delta^I \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i
\end{tikzcd}$$

Example 5.0.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[2] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.

[2]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

- A morphism
$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' & \mathcal{D}'_1 \times \mathcal{D}'_2 \arrow[l, "\beta"] \end{tikzcd}$$
where here we mean $\eta\colon F \Rightarrow \beta \circ (G_1 \times G_2) \circ \alpha$.

A morphism of the above form is cartesian if and only if η is an equivalence, in which case we call the morphism *thin.* A general morphism is cartesian if and only if each component of η is an equivalence.

Note that $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same.

Taking all this together gives a commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}.$$

Lemma 5.0.2. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where

$x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.

3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\
 & \mathcal{C}^{\mathrm{op}}
\end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0))
\end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration.

The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. □

Corollary 5.0.3. Everything in sight is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ and $\mathcal{C}\mathrm{at}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ gives:

Proposition 5.0.4. We can express a monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_{\times} \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_{\infty}^{\times})^{\mathrm{op}}$$

which is induced by the inclusion on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_{\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_{\infty}^{\times})^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C})_{\otimes} \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_{\infty})_{\otimes} \arrow[d] \\ \widetilde{\mathcal{S}}_{\times} \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_{\infty}^{\times})^{\mathrm{op}} \end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C})_{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_{\times} \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} .$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_{\times}$ are precisely those representing products.

Proposition 5.0.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd} \vec{\mathcal{D}} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \vec{\mathcal{D}}' \arrow[d, "G"] \\ \vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"] \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each component of η is an equivalence
- q-cartesian if α exhibits a product and η is an equivalence.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_{\otimes}$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_{\times}$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_{\otimes}$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

6 The triple structures

6.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 6.1.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 6.1.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

6.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 6.2.1. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 6.2.2. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd}
\langle n \rangle & \langle m \rangle \arrow[l] \\
\langle n' \rangle \arrow[u, "\phi"] & \langle m' \rangle \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) $\square$

6.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$
\begin{tikzcd}
\vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}
$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$
\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t \circ \mathcal{F}"] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^* t \mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd}
.
$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 6.3.1. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_\otimes$ of the form

$$
\begin{tikzcd}
y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd}
,
$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

7 Building categories of spans

Proposition 7.0.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 7.0.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd} \langle n\rangle & \langle n\rangle \arrow[l, "\gamma'"'] \\ \langle m\rangle \arrow[u, "\tau'"] & \langle m\rangle \arrow[l, "\gamma"'] \arrow[u, "\tau"'] \end{tikzcd},$$

where τ and τ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i\colon X_i \to Y_{\gamma^{-1}(i)}, \qquad i \in \langle m\rangle^\circ,$$

f' has components

$$f'_j\colon X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n\rangle^\circ,$$

g has components

$$g_j\colon Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n\rangle^\circ,$$

and g' has components

$$g'_j\colon X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n\rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{(\prod_{\tau'(i)=j} Y_i)} \left(\prod_{\tau'(i)=j} X_i\right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\tau'_\otimes \circ g'^* \mathcal{F}"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}" near start] & \\
& \vec{X} \arrow[rr, "f" near start] \arrow[dd, "\mathcal{F}" near start] & & \vec{Y} \arrow[dd, "f_! \mathcal{F}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes" near start] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"']
\end{tikzcd},$$

where the left and right faces (as well as the top and bottom faces) are thin, and the front face is a left Kan extension, i.e.

. Here $\mathcal{F}$ has components

$$\mathcal{F}_i\colon X_i \to \mathcal{C}_i\colon i \in \langle m\rangle^\otimes.$$

Kan extending $\mathcal{F}$ along f we find a map $f_!\mathcal{F}\colon \vec{Y} \to \mathcal{C}^{(m)}$, with components

$$(f_!\mathcal{F})_i \simeq (f_{\gamma(i)})_! \mathcal{F}_{\gamma(i)}\colon Y_i \to \mathcal{C}_i.$$

Thus, we find that

$$\mathcal{G} \simeq \tau_\otimes \circ g^* f_! \mathcal{F},$$

with components given by the composition

$$\mathcal{G}_j\colon Y'_j \to \prod_{\tau(i)=j} Y_j \to \prod_{\tau(i)=j} \mathcal{C}_i \to \mathcal{C}_j.$$

Alternatively, starting from $\mathcal{F}$ and pulling back along the left face gives a map

$$\tau'_{\otimes} \circ g'^{*}\mathcal{F}$$

with components

$$(\tau'_{\otimes} \circ g'^{*}\mathcal{F})_j : X'_j \to \prod_{\tau'(i)=j} X_j \to \prod_{\tau'(i)=j} \mathcal{C}_i \to \mathcal{C}_j.$$

We need to show that

$$\mathcal{G} \simeq f'_{!}(\tau'_{\otimes} \circ g'^{*}\mathcal{F})$$

or equivalently, componentwise, that

$$\mathcal{G}_j \simeq (f'_{\gamma'(j)})_{!}(\tau'_{\otimes} \circ g'^{*}\mathcal{F})_{\gamma'(j)}$$

□

8 Tilting at cosimplicial objects

Future Angus: This definitely isn't going to work. We're throwing away any semblance of coherence. This is just the twisted arrow category of **QCat** as a 2-category.

Let's consider $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$. The n-simplices are given by maps

$$\begin{aligned}\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)) &\simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_\Delta}(Q(n), \mathcal{C}\mathrm{at}_\infty)\\ &\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{at}^+_\Delta}(\mathfrak{C}^{\mathrm{sc}}Q(n), \mathbf{QCat}),\end{aligned}$$

so we can think of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$ as coming from the cosimplicial object $\mathfrak{C}^{\mathrm{sc}}Q \colon \Delta \to \mathcal{C}\mathrm{at}^+_\Delta$, with the following description:

- The objects of $\mathfrak{C}^{\mathrm{sc}}Q(n)$ are the objects of the linearly ordered set

  $$S_n = [n] \star [\bar{n}] = \{0 < 1 < \cdots < n < \bar{n} < \cdots < \bar{0}\}.$$

  We will denote a generic element of S_n by a, b, etc. We will denote an element of $[n] \subset S_n$ by i, and a generic element of $[\bar{n}] \subset S_n$ by $\bar{i}$. That is, for each $a \in S_n$ we always have either $a = i$ or $a = \bar{i}$ for some $0 \leq i \leq n$. We refer to i as the *digit* of a.
- For any $a, b \in S_n$, the underlying simplicial set of $\mathfrak{C}^{\mathrm{sc}}Q(n)(a, b)$ is the (nerve of the) poset of subsets of $\{a < \cdots < b\}$ containing a and b.
- A morphism $A \subseteq B \subseteq \{a < \cdots < b\}$ in such a mapping space is marked if and only if the highest digit between 0 and n appearing in both A and B is the same. For instance,

for mapping spaces between objects $i, j \in [n]$, the situation corresponding to A and B both belonging to $[n] \subseteq S_n$, everything is marked because by definition both A and B must contain i (and similarly if both belong to $[\bar{n}]$).

We then define a cosimplicial object $R\colon \Delta \to \mathcal{C}\mathrm{at}_\Delta^+$ as follows.

- The objects of $R(n)$ are the objects of the linearly ordered set
$$S_n = \{0 < 1 < \cdots < n < \bar{n} < \cdots < \bar{0}\}.$$
- The mapping space $R(n)(a, b)$ is given by the (nerve of the) linearly ordered set
$$\{\max(a, b) < \cdots < n\},$$
where by $\max(a, b)$, we mean the maximum digit appearing. In particular:
    - For $b < a$, the mapping simplicial sets are empty. To avoid having to repeatedly mention that this case is trivial, we implicitly consider only mapping spaces $\mathrm{Map}(a, b)$ with $a \leq b$.
    - For $a = i$ and $b = j$,
- Let $\phi\colon [m] \to [n]$. We define a map $R(m) \to R(n)$ as follows.
    - On objects, we send $i \mapsto \phi(i)$ and $\bar{i} \mapsto \overline{\phi(i)}$. We write both of these $a \mapsto \phi(a)$.
    - On morphisms, we define a map
    $$R(m)(a, b) \to R(n)(\phi(a), \phi(b))$$
    sending $i \in \{\max(a, b) < \cdots < m\}$ to $\phi(i) \in \{\max(\phi(a), \phi(b)) < \cdots < n\}$.

For each n, we get a map
$$\iota_n\colon R(n) \to \mathfrak{C}^{\mathrm{sc}}Q(n)$$
which is the identity on objects, and acts on morphisms in the following way:

- For $a = i$ and $b = j$, c is given by some digit k with $i \leq k \leq j$, and we send