

# 1 Miscellaneous stuff to be deleted or absorbed

## 1.1 The triple structures

### 1.1.1 The triple structure on local systems

We define a triple structure  $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$  on  $\mathbf{LS}(\mathcal{C})$  as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$ .
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$ .
- $\mathcal{Q}^\dagger$  consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{array}{ccccc} \mathcal{D}' & \longleftarrow & \mathcal{D} & \longrightarrow & \mathcal{D}'' \\ \downarrow & \circlearrowleft & \downarrow & \Longrightarrow & \downarrow \\ \mathcal{C} & \xlongequal{\quad} & \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \end{array}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{array}{ccccc} \mathcal{D}' & \xleftarrow{g} & \mathcal{D} & \xrightarrow{f} & \mathcal{D}'' \\ \mathcal{F} \downarrow & \circlearrowleft & g^* \mathcal{F} \downarrow & \xrightarrow{f!} & \downarrow f_* g^* \mathcal{F} \\ \mathcal{C} & \xlongequal{\quad} & \mathcal{C} & \xlongequal{\quad} & \mathcal{C} \end{array}$$

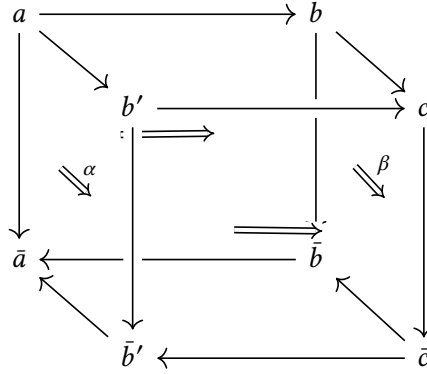
**Proposition 1.1.1.** The triple  $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$  is adequate.

To do this, we need to prove:

**Lemma 1.1.2.** Let  $\mathbb{C}$  be an  $\infty$ -bicategory such that the underlying quasicategory  $\mathcal{C}$  has pullbacks and pushouts. Consider a square

$$\sigma = \begin{array}{ccc} A & \longrightarrow & B \\ a \downarrow & & \downarrow b \\ B' & \longrightarrow & C \end{array}$$

in  $\mathbf{Tw}(\mathbb{C})$  corresponding to a twisted cube



in  $\mathbb{C}$ .<sup>1</sup> Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square  $\sigma$  is pullback if and only if the left face is thin.

*Proof.* This corresponds to a square  $\sigma$  in  $\mathbf{Tw}(\mathbb{C})$  lying over a pullback square in  $\mathbb{C} \times \mathbb{C}^{\text{op}}$ , such that  $b$  is  $p$ -cartesian. It is then classically known that  $\sigma$  is pullback if and only if  $a$  is  $p$ -cartesian.  $\square$

*Proof of Proposition 1.1.1.* We need to show that we can form pullbacks in  $\mathbf{Tw}(\mathbb{C})$  of cospans one of whose legs is a cartesian morphism. Mapping the cospan to  $\mathbb{C} \times \mathbb{C}^{\text{op}}$  using  $r$ , We can form the pullback, then fill to a relative pullback by finding a cartesian lift  $a$ , then filling an inner and an outer horn. This diagram is pullback by [Lemma 1.1.2](#)

We also need to show that any such pullback is of this form. This also follows from [Lemma 1.1.2](#).  $\square$

## 1.2 Building categories of spans

**Lemma 1.2.1.** Let

$$\begin{array}{ccc} X \times_Y^h Y' & \longrightarrow & Y' \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

be a homotopy pullback diagram of Kan complexes, and let  $y' \in Y'$ . Then

$$(X \times_Y^h Y')_{/y'} \simeq X \times_Y^h (Y'_{/y'}) \quad (1)$$

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<sup>1</sup>Note that the top and bottom faces of any such diagram in  $\mathbf{Tw}(\mathbb{C})$  belong to  $\mathbb{C}$ , and hence must weakly commute by definition.

*Proof.* We can model the homotopy pullback  $X \times_Y^h Y'$  as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 1 is then given by the pullback

$$\left( Y^{\Delta^1} \times_{Y \times Y} (X \times Y') \right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side.  $\square$

**Proposition 1.2.2.** The map  $p: (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \rightarrow (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$  of triples whose underlying map is  $p: \mathbf{LS}(\mathcal{C}) \rightarrow \mathcal{S}$  is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

*Proof.* We need to show that for any square

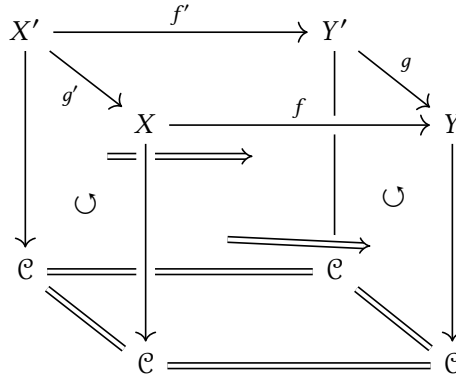
$$\sigma = \begin{array}{ccc} x' & \xrightarrow{f'} & y' \\ g' \downarrow & & \downarrow g \\ x & \xrightarrow{f} & y \end{array}$$

in  $\mathbf{LS}(\mathcal{C})$  lying over a pullback square

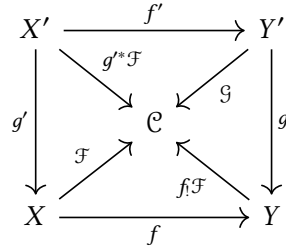
$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ g' \downarrow & & \downarrow g \\ X & \xrightarrow{f} & Y \end{array}$$

in  $\mathcal{S}$  (corresponding to a *homotopy* pullback of Kan complexes), such that  $g$  and  $g'$  are  $p$ -cartesian and  $f$  is  $p$ -cocartesian,  $f'$  is  $p$ -cocartesian. The square  $\sigma$  corresponds to a twisted

cube



in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.



We need to show that  $\mathcal{G}$  is a left Kan extension of  $g'^*\mathcal{F}$  along  $f'$ , i.e. that for all  $y' \in Y'$ ,

$$\begin{aligned} \mathcal{G}(y') &\simeq \operatorname{colim} [X'_{/y'} \rightarrow X \rightarrow \mathcal{C}] \\ &\simeq \operatorname{colim} [X \times_Y (Y'_{/y'}) \rightarrow X \rightarrow \mathcal{C}], \end{aligned}$$

where here we mean by  $X' \simeq X \times_Y Y'$  the homotopy pullback, and we have used [Lemma 1.2.1](#). We know that  $\mathcal{G}$  is the pullback along  $g$  of  $f!\mathcal{F}$ , i.e. that

$$\begin{aligned} \mathcal{G}(y') &\simeq \operatorname{colim} [X_{/g(y')} \rightarrow X \rightarrow \mathcal{C}] \\ &\simeq \operatorname{colim} [X \times_Y (Y_{/g(y')}) \rightarrow X \rightarrow \mathcal{C}]. \end{aligned}$$

The map  $X \times_Y (Y'_{/y'}) \rightarrow X$  factors through  $X \times_Y (Y_{/g(y')})$  thanks to the map  $s: Y'_{/y'} \rightarrow Y_{/g(y')}$ . The map  $s$  is a weak equivalence between contractible Kan complexes (because both  $Y'_{/y'}$  and  $Y_{/g(y')}$  are Kan complexes with terminal objects). Thus, the map  $X \times_Y (Y'_{/y'}) \rightarrow X \times_Y (Y_{/g(y')})$  is also a weak homotopy equivalence between Kan complexes, thus cofinal.  $\square$