Monoidal push-pull for local systems

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Contents

1 Introduction and conventions

1.1 General conventions

By $\mathcal{S}$, we mean the ∞-category of *small* spaces.

1.2 The marked-scaled model structure

One common model for $(\infty, 2)$-categories is Lurie's theory of ∞-bicategories

In [4], a marked-biscaled model structure modelling cartesian fibrations of ∞-bicategories is defined. When taken over a point, the two scalings collapse into one, giving the following.

Definition 1.2.1. A marked-scaled simplicial set is a triple (X, E_X, T_X), where $E_X \subseteq X_1$ is a collection of edges of X containing all degenerate edges, and $T_X \subseteq X_2$ is a collection of triangles of X containing all degenerate triangles.

Notation 1.2.2.

- We will denote the category of marked-scaled simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}$.
- To save on notation, we will sometimes denote the marked-scaled simplicial set (X, E_X, T_X) by $X_{T_X}^{E_X}$, particularly in the case that E_X or T_X are $\sharp$ or $\flat$, the maximum (resp. minimum) markings and scalings. For example, the bimarked simplicial set $X_\flat^\sharp$ has all 1-simplices marked and only degenerate 2-simplices scaled, etc.

We will use the following abuse of notation freely.

Notation 1.2.3. Let (Δ^n, E, T) be a marked-scaled n-simplex. For any simplicial subset $S \subseteq \Delta^n$, we will denote by (S, E, T) the simplicial subset S together with the inherited marking and scaling.

Notation 1.2.4. Given a cosimplicial object $Q\colon \Delta \to \mathcal{C}$, we will write $Q(n)$ instead of $Q([n])$.

Definition 1.2.5. The set of ms-anodyne morphisms is a saturated set of morphisms between marked-scaled simplicial sets containing the following classes of morphisms:

(A1) Inner horn inclusions

$$(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}),$$

for $n \geq 2$ and $0 < i < n$.

(A2) Outer horn inclusions

$$(\Lambda^n_n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}),$$

for $n \geq 1$.

Proposition 1.2.6. For any right anodyne morphism $A \hookrightarrow B$, the morphism $A^\sharp_\sharp \hookrightarrow B^\sharp_\sharp$ is marked-scaled anodyne.

Theorem 1.2.7. There is a model structure on the category $\mathcal{S}\mathrm{et}^{\mathsf{MS}}_\Delta$, whose trivial cofibrations are given by the ms-anodyne maps, and whose fibrant objects are ∞-bicategories with precisely the equivalences marked.

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1 (Old Barwick). Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. Each morphism $g \in \mathcal{D}_\dagger$ admits a lift to a morphism in $\mathcal{C}_\dagger$ (given a lift of the source) which is both p-cocartesian and $p_\dagger$-cocartesian.

2. Consider a commutative square

$$\sigma = \begin{tikzcd} y' \arrow[r, tail, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ where g' belongs to $\mathcal{C}^\dagger$, and f and f' belong to $\mathcal{C}_\dagger$. Suppose that f is p-cocartesian. Then f' is p'-cocartesian if and only if σ is an ambigressive pullback square (and in particular $g \in \mathcal{C}^\dagger$).

Theorem 2.0.2 (New Barwick)**.** Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. The subcategory $\mathcal{C}^\dagger \subseteq \mathcal{C}$ consists of all p-cartesian morhisms in $\mathcal{C}$; that is, an n-simplex in $\mathcal{C}$ belongs to $\mathcal{C}^\dagger$ if and only if each 1-simplex it contains is p-cartesian.
2. The map $p^\dagger\colon \mathcal{C}^\dagger \to \mathcal{D}^\dagger$ is a cartesian fibration.
3. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ where g and g' belong to $\mathcal{C}^\dagger$, and f belongs to $\mathcal{C}_\dagger$. Further suppose that f is p-cocartesian. Then f' belongs to $\mathcal{C}_\dagger$, and is both p-cocartesian and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Proof. We first note that σ is automatically pullback; since the sides are p-cartesian, it is automatically a relative pullback, and it lies over a pullback square.

The rest is even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. $\square$

3 Horn filling via left Kan extensions

Kan extensions of ∞-categories are usually defined pointwise via the colimit formula [8] [3]. In this section, we will show that such pointwise ∞-Kan extensions enjoy a horn filling property in $\mathbb{C}\mathrm{at}_\infty$ which generalizes the universal property for Kan extensions of functors between 1-categories.

We will expand upon the following definition in Subsection 3.1.

Definition 3.0.1. We will say that a 2-simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$,

$$\tau = \begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "G"'] & \end{tikzcd}, \tag{1}$$

is ***left Kan*** if the natural transformation η exhibits G as the left Kan extension of F along f.

The goal of this section is to prove the following.

Theorem 3.0.2. Let $\tau\colon \Delta^2 \to \mathbb{C}\mathrm{at}_\infty$ be a left Kan 2-simplex. Then for each $n \geq 2$, every solid lifting problem of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda^{n+1}_0)_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta^{n+1}_\flat \arrow[ur, dashed] & \end{tikzcd} \tag{2}$$

admits a dashed solution.

Let us examine this in the case $n = 2$. In this case we have the data of categories and functors

$$\begin{tikzcd} & Y \arrow[drr, "f_!F"] \arrow[ddr, "g"] & & \\ X \arrow[ur, "f"] \arrow[rrr, "F"] \arrow[drr, "h"'] & & & \mathcal{C} \\ & & Z \arrow[ur, "G"'] & \end{tikzcd}$$

together with natural transformations $\eta\colon F \Rightarrow f_!F \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow G \circ h$. We need to produce a natural transformation $\alpha\colon f_!F \Rightarrow G \circ g$ and a filling of the full 3-simplex, which is the data of a homomotopy-commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \arrow[d, "\alpha f"] \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}$$

in $\mathrm{Fun}(X, \mathcal{C})$.[1]

We can rephrase this as follows. We are given the data

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

where the left-hand diagram is a map $LC^2 \to \mathrm{Fun}(X, C)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ except the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathrm{Fun}(Y, C)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

1. We first can fill the lower-left half of the diagram on the left simply by taking the composition $G\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}.$$

2. We note that we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left (in the sense to be described in Subsection 3.3):

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

[1] In particular, if we take $h = f$, $g = \mathrm{id}_Y$, and β to be the identity $f \Rightarrow f$, we recover the classical universal property satisfied by left Kan extension.

3. The diagram on the right has an obvious filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_! F \circ f \arrow[d, "\alpha f"] \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_! F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_! F \arrow[d, "\alpha"] \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

The lifting problems which we have to solve in Equation 2 for $n > 2$ amount to replacing $\Delta^1 \times \Delta^1$ by $(\Delta^1)^n$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved. In Subsection 3.2 we explain the combinatorics of filling cubes relative to their boundaries. In Subsection 3.3, we give a formalization the concept of adjunct data, and provide a means of for filling partial data to total adjunct data. In Subsection 3.4, we show that we can always solve the lifting problems of Equation 2.

3.1 Basic facts about Kan extensions

In this section, we recall a few basic facts about Kan extensions. These can be found in Kerodon. Since Kan extensions are defined by the colimit formula, we will start by defining colimits. Clasically, colimits in ∞-categories are defined using colimit cones.

Definition 3.1.1. Let $F: K \to \mathcal{C}$ be a diagram, where $\mathcal{C}$ is an ∞-category. A cocone $\tilde{F}: K^{\triangleright} \to \mathcal{C}$ is a ***colimit cone*** if it is an initial object in the ∞-category $\mathcal{C}_{F/}$.

It is sometimes more convenient to define colimits via natural transformation.

Definition 3.1.2. Let $f: K \to \mathcal{C}$ be a map between simplicial sets, where $\mathcal{C}$ is an ∞-category. Let $c \in \mathcal{C}$ be an object, and denote by $\underline{c}$ the constant functor $K \to \mathcal{C}$ with value c. A natural transformation $f \Rightarrow \underline{c}$ ***exhibits*** c ***as the colimit of*** f if it is an initial object in the ∞-category $\mathcal{C}^{F/}$.

Fortunately, these definitions are compatible: an object $K^{\triangleright} \to \mathcal{C}$ in $\mathcal{C}_{/F}$ is a colimit cone with cone tip c if and only if the composite map

$$K \times \Delta^1 \to K \times \Delta^1 \amalg_{K \times \{1\}} \Delta^0 \to K \star \Delta^0 \to \mathcal{C} \tag{3}$$

exhibits c as a colimit of F. This follows from the fact that for any ∞-category $\mathcal{C}$ and any functor $F: K \to \mathcal{C}$, the comparison map $\mathcal{C}_{/F} \to \mathcal{C}^{/F}$ constructed analogously to that above is a categorical equivalence.

Note 3.1.3. If a natural transformation η exhibits some object as a colimit of some diagram, than any natural transformation η' which is homotopic to η will do just as well.

Notation 3.1.4. Let $f: X \to Y$ be a map of simplicial sets, and $y \in Y$ an object. We will use the following notation.

- Denote by $X_{/y}$ the fiber product $X \times_Y Y_{/y}$.
- Denote by $\pi: X_{/y} \to X$ the projection map.
- Denote by $\alpha: f \circ \pi \Rightarrow \underline{y}$ the natural transformation $X_{/y} \times \Delta^1 \to X$ coming from the comparison map $X_{/y} \to X^{/y}$.

Kerodon 7.3.1.5

Definition 3.1.5. Let X, Y, and $\mathcal{C}$ be ∞-categories, $f: X \to Y$, $F: X \to \mathcal{C}$ and $G: Y \to \mathcal{C}$ functors, and $\eta: F \Rightarrow G \circ f$ a natural transformation.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{C} \\ & Y \arrow[ur, "G"'] & \end{tikzcd}$$

We say that η ***exhibits*** G ***as the left Kan extension of*** F ***along*** f if for each object $y \in Y$, the natural transformation $F \circ \pi \Rightarrow \underline{G(y)}$ furnished by the pasting diagram

$$\begin{tikzcd} X_{/y} \arrow[r, "\pi"] \arrow[d] \arrow[dr, phantom, "\Downarrow\alpha"] & X \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\eta\Downarrow"] & \mathcal{C} \\ \{y\} \arrow[r, hook] & Y \arrow[ur, "G"'] & \end{tikzcd}$$

exhibits $G(y)$ as the colimit of the functor $F \circ \pi$.

The following is Kerodon 7.3.5.1.

Theorem 3.1.6 (Existence of local left Kan extensions)**.** Suppose that $f: X \to Y$ is a map of simplicial sets, and suppose $F: X \to \mathcal{C}$ is a map of simplicial sets such that $\mathcal{C}$ is a quasicategory. The functor F admits a left Kan extension along f if and only if for all objects $x \in X$, the colimit of the functor

$$X_{/y} \xrightarrow{\pi} X \xrightarrow{F} \mathcal{C}$$

exists in $\mathcal{C}$.

The following is a combination of special cases of 7.3.1.15 and 7.3.1.16.

Example 3.1.7. For any homotopy-commuting diagram of simplicial sets

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta \Downarrow \simeq"] & \mathcal{C} \\ & Y \arrow[ur, "G"'] & \end{tikzcd}$$

where X, Y, and $\mathcal{C}$ are ∞-categories and $f \colon X \to Y$ is a categorical equivalence, η exhibits G as a left Kan extension of F along f.

The following is Kerodon 7.3.6.3

Theorem 3.1.8 (Existence of global left Kan extensions)**.** Let $f \colon X \to Y$ be a map of simplicial sets, and suppose that $\mathcal{C}$ is a quasicategory which admits $X_{/y}$-shaped colimits for all $y \in Y$. Then the restriction functor $f^* \colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C})$ admits a left adjoint $f_!$, sending a functor $F \colon X \to \mathcal{C}$ to $f_! F \colon Y \to \mathcal{C}$, its left Kan extension along f.

The following is an easy consequence of 7.3.1.11.

Proposition 3.1.9. Let $f \colon X \to Y$ be a map of simplicial sets, and let $\mathcal{C}$ be a category such that all functors $X \to \mathcal{C}$ admit left Kan extensions along f, so that we have an adjunction $f_! \vdash f^*$. Let $G \colon Y \to \mathcal{C}$ and $\eta \colon F \Rightarrow G \circ f$. The natural transformation η exhibits G as a left Kan extension of F along f if and only if it is adjunct (in the sense of Definition 3.3.4) to an equivalence in $\mathrm{Fun}(Y, \mathcal{C})$ relative to $s = \mathrm{id}_{\Delta^1}$.

The following is Kerodon 7.3.7.18

Proposition 3.1.10. Consider ∞-categories and functors

$$\begin{tikzcd} X \arrow[rrr, "F"] \arrow[dr, "f"'] & & & \mathcal{C} \\ & Y \arrow[urr, "G"] \arrow[dr, "g"'] & & \\ & & Z \arrow[uur, "H"'] & \end{tikzcd},$$

and natural transformations $\alpha \colon F \Rightarrow G \circ f$ and $\beta \colon H \circ g \Rightarrow G$, and suppose that α exhibits G as the left Kan extension of F along f. Then β exhibits H as the left Kan extension of G along g if and only if $\beta f \circ \alpha$ exhibits H as the left Kan extension of F along $g \circ f$.

The following is Kerodon 7.3.1.12.

Proposition 3.1.11. Suppose $H\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, $f\colon X \to Y$ is a map of simplicial sets, $G\colon Y \to \mathcal{C}$ is a functor, and $\eta\colon F \Rightarrow G \circ f$ is a natural transformation. Then η exhibits G as a left Kan extension of F along f if and only if $H\eta\colon H \circ F \Rightarrow H \circ G \circ f$ exhibits $H \circ G$ as a left Kan extension of $H \circ F$ along f.

This is Kerodon 7.3.1.9.

Proposition 3.1.12. The property that η exhibits G as a left Kan extension of F along f is homotopy-invariant; any homotopic η will do just as well.

3.2 Filling cubes relative to their boundaries

In this section, we give the combinatorics of filling the n-cube $(\Delta^1)^n$ relative to its boundary. Our main result is Proposition 3.2.21, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map.

Definition 3.2.1. The ***n*-cube** is the simplicial set $C^n := (\Delta^1)^n$.

Note 3.2.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Our first step is to understand the nondegenerate simplices of the n-cube.

Definition 3.2.3. Denote by $a^i\colon \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\ldots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\ldots,n\}}$ to $\{1\}$.

Note 3.2.4. The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0,\ldots,0)$ and ending at $(1,\ldots,1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1,\ldots,n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition 3.2.5. For any permutation $\tau\colon \{1,\ldots,n\} \to \{1,\ldots,n\}$, we define an n-simplex

$$\phi(\tau)\colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)}).$$

Note 3.2.6. It is easy to check that the above definition really results in a bijection between S_n and the nondegenerate simplices of C^n.

Notation 3.2.7. We will denote any permutatation $\tau\colon \{1,\dots,n\} \to \{1,\dots,n\}$ by the corresponding n-tuple $(\tau(1),\dots,\tau(n))$. Thus, the identity permutation is $(1,\dots,n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2,1,3,\dots,n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)},\dots,a^{\tau(n)})\colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i\colon \Delta^n \to \Delta^1$ and any face map $\partial_j\colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\dots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)},\dots,a^{\tau(n)}) := (\partial_i^* a^{\tau(1)},\dots,\partial_i^* a^{\tau(n)}).$$

Example 3.2.8. Consider the 3-simplex (a^1,a^2,a^3) in C^3. This has spine

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation 3.2, we then calculate that

$$d_2(a^1,a^2,a^3) = (a^1,a^2,a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0,0,0) \longrightarrow (1,0,0) \longrightarrow (1,1,1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition 3.2.9. The ***boundary of the*** n***-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{4}$$

The following is easy to see.

Proposition 3.2.10. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \dots, \gamma_n)\colon \Delta^{n-1} \longrightarrow C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition 3.2.11. The ***left box of the*** n***-cube***, denoted LC^n, is the simplicial subset of C^n given by the union of all of the faces in Equation 4 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example 3.2.12. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma 3.2.13. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

1. The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.
2. For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.
3. The nth face $d_n\phi$ always belongs to LC^n.

Proof. 1. We have

$$d_0\phi = (a^{\tau(0)-1}, \dots a^{\tau(n)-1})$$

since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition 3.2.10 guarantees that $d_0\phi$ is contained in the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

This face belongs to LC^n except when $j = n$.

2. In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.
3. In this case,

$$d_n(a^{\tau(1)}, \dots, a^{\tau(n)}) = (a^{\tau(1)}, \dots, a^{\tau(n)})$$

since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation 3.2.14. For any subset $T \subseteq [n]$, write

$$\Lambda^n_T := \bigcup_{t \in T} d_t\Delta^n \subset \Delta^n.$$

Lemma 3.2.15. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n - 1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \ldots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma 3.2.15 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition 3.2.16. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) < \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example 3.2.17. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation 3.2.18. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$
\begin{aligned}
LC^n &\hookrightarrow LC^n \cup \phi(1,\ldots,n) \\
&\hookrightarrow \cdots \\
&\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\ldots,n,1)} \phi(\tau)
\end{aligned}
\tag{5}
$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2,\ldots,n,1)$, the intersection

$$
B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{6}
$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition 3.2.19. Each inclusion in Equation 5 is inner anodyne.

Proof. The first inclusion

$$
LC^n \hookrightarrow LC^n \cup \phi(0,\ldots,n)
$$

is inner anodyne because by Lemma 3.2.13 the intersection $LC^n \cap \phi(0,\ldots,n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma 3.2.13), so by Lemma 3.2.15, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (0,\ldots,n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$
\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),
$$

which implies that $\tau = (1,\ldots,n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. □

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$.

Thus, each of their spines begins

$$(0,\dots,0,0) \to (0,\dots,0,1) \to \cdots,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that $(0,\dots,0,0) \to (0,\dots,0,1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition 3.2.19. One continues to fill simplices in the order prescribed in Definition 3.2.16, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma 3.2.15.

Lemma 3.2.20. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition 3.2.21. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0,\dots,0,0) \to (0,\dots,0,1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \xhookrightarrow{i} J^n \xhookrightarrow{j} C^n,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd}
\Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\
J^n \arrow[r, hook] & C^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0,\dots,0) \to (0,\dots 1)$.

3.3 Adjunct data

In this section, we give a formalization of the the notion of adjunct data. Our main result is Proposition 3.3.9, which shows that under certain conditions, we can use data on one side of an adjunction to solve lifting problems on the other side. First, we recall somewhat explicitly the definition of an adjunction given in [8].

Definition 3.3.1. An ***adjunction*** is a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$. We say that p is ***associated*** to functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ if there exist equivalences $h_0\colon \mathcal{C} \to \mathcal{M}_0$ and $h_1\colon \mathcal{D} \to \mathcal{M}_1$, and a commutative diagram

$$\begin{tikzcd} \mathcal{C}\times\Delta^1 \arrow[r, "u"] \arrow[dr, "\mathrm{pr}"'] & \mathcal{M} \arrow[d, "p"] & \mathcal{D}\times\Delta^1 \arrow[l, "v"'] \arrow[dl, "\mathrm{pr}"] \\ & \Delta^1 & \end{tikzcd}$$

such that the following conditions are satisfied.

- The map u is associated to the functor f: the restriction $u|\mathcal{C}\times\{0\} = h_0$, the restriction $u|\mathcal{C}\times\{1\} = f\circ h_1$, and for each $c\in\mathcal{C}$, the edge $u|\{c\}\times\Delta^1$ is p-cocartesian.
- The map v is associated to the functor g: the restriction $v|\mathcal{D}\times\{0\} = g\circ h_0$, the restriction $u|\mathcal{D}\times\{1\} = h_1$, and for each $d\in\mathcal{D}$, the edge $u|\{d\}\times\Delta^1$ is p-cartesian.

If f and g are functors to which an adjunction is associated as above, then we say that f is left adjoint to g, and equivalently that g is right adjoint to f.

Note 3.3.2. This terminology is slightly imprecise; it would be more correct to say that p is associated to f and g via the data (h_0, h_1, u, v).

Note 3.3.3. It is shown in [8] that if f and g are functors such that f is left adjoint to g, then we can choose an adjunction $p\colon \mathcal{M}\to\Delta^1$ and data (h_0, h_1, u, v) such that h_0 and h_1 are isomorphisms. We can use this to identify $\mathcal{M}_0$ with $\mathcal{C}$, and $\mathcal{M}_1$ with $\mathcal{D}$. In what follows we will always assume that we have chosen such data, and will thus leave the isomorphisms h_0 and h_1 implicit.

Definition 3.3.4. Let $s\colon K\to\Delta^1$ be a map of simplicial sets whose fibers we denote by K_0 and K_1, and let $p\colon\mathcal{M}\to\Delta^1$ be a bicartesian fibration associated to adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via data (u, v). We say that a map $\alpha\colon K\to\mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K\to\mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K\times\Delta^1\to\mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K\times\{0\}} = \alpha$.
2. The restriction $A|_{K\times\{1\}} = \tilde{\alpha}$.
3. The restriction $A|K_0 \times \Delta^1$ is equal to the composition

$$K_0 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\alpha\circ\times\mathrm{id}} \mathcal{C} \times \Delta^1 \xrightarrow{u} \mathcal{M}.$$

4. The restriction $A|K_1 \times \Delta^1$ is equal to the composition

$$K_1 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\tilde{\alpha}\times\mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{v} \mathcal{M}.$$

In the next examples, fix adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$ via data (u, v).

Example 3.3.5. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$ of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is $u|\{c\} \times \Delta^1$ and b is $v|\{d\} \times \Delta^1$. This corresponds to the map $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

Example 3.3.6. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id}\colon gD \to gD$ in $\mathcal{C}$. This morphism is adjunct to the component of the unit map $\eta_D\colon D \to fgD$ relative to $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

The next example will follow from Proposition 3.3.9.

Example 3.3.7. Any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g.

Lemma 3.3.8. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\partial\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^{\{0\}}, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{7}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition 3.3.9. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, u and v be as in Definition 3.3.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s\circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a monomorphim such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}' \colon K' \to \mathcal{D}$ and $\alpha \colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{8}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s. Furthermore, any two such lifts are equivalent as functors $K \to \mathcal{D}$.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. From the map A and the map $\alpha \colon K \to \mathcal{C} \cong \mathcal{M}_0$, we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd} (K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1)^\heartsuit \arrow[r] \arrow[d, "w"'] & \mathcal{M}^\natural \arrow[d] \\ (K \times \Delta^1)^\heartsuit \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^\sharp \end{tikzcd},$$

where $(K\times\Delta^1)^\heartsuit$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\}\times\Delta^1$ for $k\in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. We can write the morphism of simplicial sets underlying w as a smash-product $(K'\hookrightarrow K)\wedge(\Delta^{\{0\}}\hookrightarrow\Delta^1)$. Building $K'\hookrightarrow K$ simplex by simplex, in increasing order of dimension, we can write w as a transfinite composition of pushouts along inclusions of the form given in Equation 7 (by assumption each simplex σ we adjoin to K is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). We can thus also build our lift ℓ by lifting against each of these in turn, and we can always choose each lift so that it is compatible with v. We can then take $\tilde{\alpha}=\ell|K\times\{1\}$.

It is manifest from this technique that the space of lifts is contractible; in particular, any two such lifts are equivalent in $\mathrm{Map}(K\times\Delta^1,\mathcal{M})$, so their restrictions to $K\times\{1\}$ are equivalent as functors $K\to\mathcal{D}$.

□

Example 3.3.10. Let $f:\mathcal{C}\leftrightarrow\mathcal{D}:g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K'=\Delta^{\{0,1,3\}}\cup\Delta^{\{0,2,3\}} \arrow[rr, hook, "i"] \arrow[dr] & & \Delta^3=K \arrow[dl] \\ & \Delta^1 & \end{tikzcd},$$

where both downwards-facing arrows take $0,1\mapsto 0$ and $2,3\mapsto 1$, and fix a map $\tilde{\alpha}\colon K\to N(\mathcal{C})$ and a map $\alpha'\colon K'\to N(\mathcal{D})$ such that $\tilde{\alpha}\circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition 3.3.9 implies that the solution to the lifting problem on the left yields one on the right. Similarly, its dual implies the converse. Thus, Proposition 3.3.9 implies in particular that such lifting problems are equivalent.

We note that in the proof of Proposition 3.3.9, we did not need to use the full power of the statement that f was left adjoint to g (i.e. that $p\colon\mathcal{M}\to\Delta^1$ was a bicartesian fibration), only that f was *locally* left adjoint to g, (i.e. that p admitted cocartesian lifts at certain objects). This allows us to generalize Proposition 3.3.9 somewhat.

Definition 3.3.11. Let $g\colon \mathcal{D} \to \mathcal{C}$ be a functor between quasicategories, and pick some cartesian fibration $p\colon \mathcal{M} \to \Delta^1$ which classifies it. We assume without loss of generality that $h_0\colon \mathcal{M}_0 \cong \mathcal{C}$ and $h_1\colon \mathcal{M}_1 \cong \mathcal{D}$ are isomorphisms, and we will notationally suppress them.

We say that g ***admits a left adjoint at*** $c \in \mathcal{C}$ if p admits a cocartesian lift at c.

Definition 3.3.12. Let $s\colon K \to \Delta^1$ be a map of simplicial sets whose fibers we denote by K_0 and K_1, and let $p\colon \mathcal{M} \to \Delta^1$ be a cartesian fibration associated to $f\colon \mathcal{D} \to \mathcal{C}$ via a map $v\colon \mathcal{D} \times \Delta^1 \to \mathcal{M}$. We say that a map $\alpha\colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K\times\{0\}} = \alpha$.
2. The restriction $A|_{K\times\{1\}} = \tilde{\alpha}$.
3. For each vertex $k \in K_0$, the image of $\{k\} \times \Delta^1$ under A in $\mathcal{M}$ is p-cartesian.
4. The restriction $A|K_1 \times \Delta^1$ is equal to the composition

$$K_1 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\tilde{\alpha}\times\mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{v} \mathcal{M}.$$

Example 3.3.13. Suppose $f\colon \mathcal{C} \to \mathcal{C}'$ and $F\colon \mathcal{C} \to \mathcal{D}$ are functors. The statement that the left Kan extension $f_!F\colon \mathcal{C}' \to \mathcal{D}$ exists is equivalent to the statement that f^* admits a left adjoint at F.

Proposition 3.3.14. Let $p\colon \mathcal{M} \to \Delta^1$ classify $g\colon \mathcal{D} \to \mathcal{C}$ via $v\colon \mathcal{D} \times \Delta^1 \to \mathcal{M}$, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s\circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a monomorphim such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Further suppose that g admits a left adjoint at all vertices belonging to the image $\alpha(K_0) \subseteq \mathcal{C}$.

Then there exists a dashed extension

$$
\begin{tikzcd}
K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\
K \arrow[ur, "\alpha"'] &
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\
K \arrow[ur, dashed, "\tilde{\alpha}"'] &
\end{tikzcd}
\tag{9}
$$

such that $\tilde{\alpha}$ is adjunct to α relative to s. Furthermore, any two such lifts are equivalent as functors $K \to \mathcal{D}$.

3.4 Left Kan implies globally left Kan

In this subsection, we will prove Theorem 3.0.2. In order to do that, we must show that we can solve lifting problems of the form

$$
\begin{tikzcd}
\Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\
(\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\
\Delta_\flat^{n+1} \arrow[ur, dashed] &
\end{tikzcd},
$$

for $n \geq 2$, where τ is left Kan. Forgetting for a moment that have singled out some 2-simplex τ, and using the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}_{\mathrm{sc}}$, we will do this by solving equivalent lifting problems of the form[2]

$$
\begin{tikzcd}
\mathfrak{C}[\Lambda_0^{n+1}] \arrow[r] \arrow[d, hook] & \mathbf{QCat} \\
\mathfrak{C}[\Delta^{n+1}] \arrow[ur, dashed, "\mathcal{F}"'] &
\end{tikzcd}.
\tag{10}
$$

We expect that the reader is familiar with the basics of (scaled) rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category $\mathfrak{C}[\Delta^{n+1}]$ are given by the set $\{0, \dots, n+1\}$, and the mapping spaces are defined by

$$
\mathfrak{C}[\Delta^{n+1}](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases}.
$$

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \dots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda_0^{n+1}]$ has the same objects as $\mathfrak{C}[\Delta^{n+1}]$, and each morphism space $\mathfrak{C}[\Lambda_0^{n+1}](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^{n+1}](i,j)$, as we shall soon describe.

[2] Note that we are implicitly taking $\mathfrak{C}[\Lambda_0^{n+1}]$ and $\mathfrak{C}[\Delta^{n+1}]$ to carry the flat markings on their mapping spaces.

The map $\mathfrak{C}[\Lambda_0^{n+1}](i,j) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](i,j)$ is an isomorphism except for $(i,j) = (1,n+1)$ and $(i,j) = (0,n+1)$. The missing data corresponds in the case of $(1,n+1)$ to the missing face $d_0\Delta^{n+1}$ of Λ_0^{n+1}, and in the case of $(0,n+1)$ to the missing interior. Thus, to find a filling as in Equation 10, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](1,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(1),\mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{11}$$

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](0,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(0),\mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{12}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \mathrm{Fun}(\mathcal{F}(1),\mathcal{F}(n+1)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[r, "\ell"] & \mathrm{Fun}(\mathcal{F}(0),\mathcal{F}(n+1)) \end{tikzcd} \tag{13}$$

commute.

Notation 3.4.1. Recall our desired filling $\mathcal{F}\colon \mathfrak{C}[\Delta^{n+1}] \to \mathbf{QCat}$ of Equation 10. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \dots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \mathrm{Map}(X_{i_1}, X_{i_k})$ by $f_{i_k \cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

Theorem 3.4.2. For any $n \geq 2$ and any globally left Kan 2-simplex τ, the solid extension problem

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

admits a dashed filler.

The author would like to offer the friendly recommendation that in reading the proof below, it is helpful to follow along with the explanation given at the beginning of Section 3.

Proof. We need to solve the lifting problems of Equation 11 and Equation 12, and check that our solutions satisfy the condition of Equation 13. We note that the information that the simplex $\tau\colon \Delta_\flat^{\{0,1,n+1\}} \to \mathbb{C}\mathrm{at}_\infty$ is left Kan tells us the following.

- Concretely, it tells us that the map $f_{n,1}$ is the left Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}}\colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$ is the unit map.
- In particular, Example 3.3.13 then tells us that the map $f_{1,0}^*$ admits a left adjoint at $f_{n+1,0}$.

It is easy to see that we have the following succinct descriptions of the inclusions of Equation 11 and Equation 12 along which we have to extend.

- There is an isomorphism of simplicial sets
$$\mathfrak{C}[\Delta^{n+1}](0, n+1) \xrightarrow{\cong} C^n$$
specified completely by sending a subset $S \subseteq [n+1]$ to the point
$$(z_1, \dots, z_n) \in C^n, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$
The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](0, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](0, n+1)$ corresponds to the simplicial subset $LC^n \hookrightarrow C^n$.
- There is a similar isomorphism of simplicial sets
$$\mathfrak{C}[\Delta^{n+1}](1, n+1) \xrightarrow{\cong} C^{n-1}$$
specified completely by sending $S \subseteq \{1, \dots, n+1\}$ to the point
$$(z_1, \dots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$
The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](1, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](1, n+1)$ corresponds to the inclusion $\partial C^{n-1} \hookrightarrow C^{n-1}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^n \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ C^n \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-1} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$
(\star) = \begin{tikzcd}
C^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \arrow[d, "f_{1,0}^*"] \\
C^n \arrow[r] & \mathrm{Fun}(X_0, X_{n+1})
\end{tikzcd}
$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition 3.2.21, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$
(*') = \begin{tikzcd}
\Delta^0 * \partial C^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\
\Delta^0 * C^{n-1} \arrow[ur, dashed] &
\end{tikzcd} .
$$

The image of the 1-simplex $(0, \ldots, 0) \to (0, \ldots, 1)$ of $\Delta^0 * \partial C^{n-1} \subseteq C^n$ under α is the unit map $\alpha_{\{0,n+1\}}^{\{0,1,n+1\}} \colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$. This is not in general an equivalence, so we cannot use Proposition 3.2.21 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to an equivalence in $\mathrm{Fun}(X_1, X_{n+1})$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-1} is in the image of $f_{1,0}^*$, so by Proposition 3.3.14 and our assumption that f_{10}^* admits a left adjoint at $f_{n+1,0}$, we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-1} \to \mathrm{Fun}(X_1, X_{n-1})$ which is adjunct to α relative to the map

$$
s \colon \Delta^0 * \partial C^{n-1} \to \Delta^1
$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-1} to $\Delta^{\{1\}}$; in particular, the image of the morphism $(0, \ldots, 0) \to (0, \ldots 1)$ under $\tilde{\alpha}'$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$
(**') = \begin{tikzcd}
\Delta^0 * \partial C^{n-1} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\
\Delta^0 * C^{n-1} \arrow[ur, dashed, "\tilde{\beta}"'] &
\end{tikzcd}
$$

However, since image of $(0, \ldots 0) \to (0, \ldots, 1)$ is an equivalence, Proposition 3.2.21 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition 3.3.9, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-1}}$ be the image of $\tilde{\beta}|_{C^{n-1}}$ under $f_{1,0}^*$, which is true by construction. □

3.5 A few other tricks with Kan extensions

We end this section with a few miscellaneous tricks which we can play with maps $\Delta^n \to \mathbb{C}\mathrm{at}_\infty$ involving left Kan simplices.

Lemma 3.5.1. Denote by $\mathcal{E} \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices together with the simplices $\Delta^{\{0,2,3\}}$ and $\Delta^{\{1,2,3\}}$. Let $\sigma\colon \Delta^3_{\mathcal{E}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{2,3\}}$ is mapped to an equivalence. Then $\sigma|\Delta^{\{0,1,2\}}$ is left Kan if and only if $\sigma|\Delta^{\{0,1,3\}}$ is left Kan.

Proof. Replacing thin simplices by strict compositions using Proposition 3.1.12, this statement reduces to that of Proposition 3.1.11. □

The next lemma is similar but easier.

Lemma 3.5.2. Denote by $\mathcal{E} \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices together with the simplices $\Delta^{\{0,2,3\}}$ and $\Delta^{\{1,2,3\}}$. Let $\sigma\colon \Delta^3_{\mathcal{E}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{2,3\}}$ is mapped to an equivalence. Then $\sigma|\Delta^{\{0,1,2\}}$ is thin if and only if $\sigma|\Delta^{\{0,1,3\}}$ is thin.

Lemma 3.5.3. Denote by $\mathcal{E}' \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices, together with the simplex $\Delta^{\{0,1,2\}}$. Let $\sigma\colon \Delta^3_{\mathcal{F}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\sigma|\Delta^{\{0,1,3\}}$ is left Kan. Then $\sigma|\Delta^{\{0,2,3\}}$ is left Kan if and only if $\sigma|\Delta^{\{1,2,3\}}$ is left Kan.

Proof. Replacing thin simplices by strict compositions using Proposition 3.1.12, this reduces to Propsition 3.1.10. □

4 Local systems

For any ∞-category $\mathcal{C}$, a $\mathcal{C}$-local system on some space X is simply a functor $X \to \mathcal{C}$. Thus, the ∞-category of $\mathcal{C}$-local systems on X should simply be $\mathbf{LS}(\mathcal{C})_X := \mathrm{Fun}(X, \mathcal{C})$. Our goal is to consider local systems on all spaces X simultaneously, defining an ∞-category of $\mathcal{C}$-local systems.

We follow the general strategy laid out in [6]. We will consider a cartesian fibration

$$p'\colon \int \mathrm{Fun}(-,-) \to \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}$$

which classifies the functor

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty; \qquad (\mathcal{C}, \mathcal{D}) \mapsto \mathrm{Fun}(\mathcal{C}, \mathcal{D}).$$

The strict pullback p of p' in the diagram

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "p"'] & \int \mathrm{Fun}(-,-) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd}$$

will then classify the functor

$$\mathcal{S} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathbf{LS}(\mathcal{C})_X.$$

The total space $\mathbf{LS}(\mathcal{C})$ of p will thus be our candidate for our ∞-category of local systems.

In [5], a suitable model for $\int \mathrm{Fun}(-,-)$ is given: the so-called *enhanced twisted arrow category* $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$. In Subsection 4.1, we reproduce the salient points of [5], defining the enhanced twisted arrow category. In Subsection 4.3, we use the enhanced twisted arrow category to define the category $\mathbf{LS}(\mathcal{C})$ of local systems, together with a cartesian fibration $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ classifying the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \to \mathrm{Fun}(X, \mathcal{C}).$$

We note that this functor sends a map of spaces $f\colon X \to Y$ to the pullback functor

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

If $\mathcal{C}$ is cocomplete, each functor f^* has a left adjoint $f_!$ given by left Kan extension. By abstract nonsense, the cartesian fibration p is also a cocartesian fibration, whose cocartesian edges correspond to left Kan extension. In Subsection 4.4 we will show this explicitly.

4.1 The enhanced twisted arrow category

For a 2-category C, the twisted arrow 1-category $\mathbf{Tw}(\mathrm{C})$ has the following description.

- The objects of $\mathbf{Tw}(\mathrm{C})$ are the morphisms of C:

$$f\colon c \to c'.$$

- For two objects $(f\colon c \to c')$ and $(g\colon d \to d')$, the morphisms $f \to g$ are given by diagrams

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] & d \arrow[d, "g"] \arrow[dl, phantom, "\overset{\alpha}{\Longrightarrow}"] \\ c' & d' \arrow[l, "a'"] \end{tikzcd},$$

where α is a 2-morphism $f \Rightarrow a' \circ g \circ a$.

- The composition of morphisms is given by concatenating the corresponding diagrams.

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] & d \arrow[r, "b"] \arrow[d, dashed, "g"] \arrow[l, phantom, "\overset{\alpha}{\Longrightarrow}"] & e \arrow[d, "h"] \arrow[l, phantom, "\overset{\beta}{\Longrightarrow}"] \\ c' & d' \arrow[l, "a'"] & e' \arrow[l, "b'"] \end{tikzcd}$$

In [5], a homotopy-coherent version of the twisted arrow category of an ∞-bicategory is defined. For any ∞-bicategory $\mathbb{C}$, the twisted arrow ∞-category of $\mathbb{C}$, denoted $\mathbf{Tw}(\mathbb{C})$, is a quasi-category whose n-simplices are diagrams $\Delta^n \star (\Delta^n)^{\mathrm{op}} \to \mathbb{C}$, together with a scaling to ensure that the information encoded in an n-simplex is determined, up to contractible choice, by data

$$\overbrace{\begin{tikzcd} c_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] & c_1 \arrow[r, dotted] \arrow[d, dashed, "f_1"] \arrow[l, phantom, "\overset{\alpha_0}{\Longrightarrow}"] & \cdots \arrow[r, dotted] & c_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, dashed, "f_{n-1}"] & c_n \arrow[d, "f_n"] \arrow[l, phantom, "\overset{\alpha_{n-1}}{\Longrightarrow}"] \\ c_0' & c_1' \arrow[l, "a_0'"] & \cdots \arrow[l, dotted] & c_{n-1}' \arrow[l, dotted] & c_n' \arrow[l, "a_{n-1}'"] \end{tikzcd}}^{n \text{ squares}}$$

in $\mathbb{C}$. Here, the top row of morphisms corresponds to the spine of the n-simplex $\Delta^n \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$, and the bottom row of morphisms to the spine of $(\Delta^n)^{\mathrm{op}} \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$. To make this correspondence more clear, we will introduce the following notation.

Notation 4.1.1. For $i \in [n] \subset [2n+1]$, we will write $\bar{i} := 2n+1-i$. Thus, $\bar{0} = 2n+1$, $\bar{1} = 2n$, etc.

Thus, the ith column in the above diagram corresponds to the image of the morphism $i \to \bar{i}$ in $\Delta^n \star (\Delta^n)^{\mathrm{op}}$.

We are now ready to define this scaling.

Definition 4.1.2. We define a cosimplicial object $\tilde{Q}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ by sending

$$\tilde{Q}([n]) = (\Delta^n \star (\Delta^n)^{\mathrm{op}}, \dagger),$$

where $\dagger$ is the scaling consisting of all degenerate 2-simplices, together with all 2-simplices of the following kinds:

1. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through Δ^n.
2. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through $(\Delta^n)^{\mathrm{op}}$.
3. All simplices $\Delta^{\{i,j,\bar{k}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.
4. All simplices $\Delta^{\{k,\bar{j},\bar{i}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.

Note that $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$. We will use this identification freely.

This extends to a nerve-realization adjunction

$$Q : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} : \mathbf{Tw},$$

where the functor Q is the extension by colimits of the functor $\tilde{Q}$, and for any scaled simplicial set X, the simplicial set $\mathbf{Tw}(X)$ has n-simplices

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}}(Q(\Delta^n), X).$$

Notation 4.1.3. For any simplicial set X, $Q(X)$ carries the scaling given simplex-wise by that described above. We will denote this also by $\dagger$. Using the identification $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$, we can write $Q(\Delta^n)$ explicitly and compactly as $\Delta^{2n+1}_\dagger$, which we will often do.

The inclusion $\Delta^n \amalg (\Delta^n)^{\mathrm{op}} \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ provides, for any ∞-bicategory $\mathbb{C}$ with underlying ∞-category $\mathcal{C}$, a morphism of simplicial sets

$$\mathbf{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

In [5], the following is shown.

Theorem 4.1.4. Let $\mathbb{C}$ be an ∞-bicategory (presented as a fibrant scaled simplicial set), and let $\mathcal{C}$ be the underlying ∞-category (presented as a quasicategory). Then the map

$$p_{\mathbb{C}} \colon \mathbf{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

is a cartesian fibration between quasicategories, and a morphism $f \colon \Delta^1 \to \mathbf{Tw}(\mathbb{C})$ is $p_{\mathbb{C}}$-cartesian if and only if the morphism $\sigma \colon \Delta^3_\dagger \to \mathbb{C}$ to which it is adjunct is fully scaled, i.e. factors through the map $\Delta^3_\dagger \to \Delta^3_\sharp$.

4.2 Two lemmas about marked-scaled anodyne morphisms

In the section following this one, we provide a fairly explicit construction of a family of marked-scaled anodyne inclusions. Constructing these inclusions directly from the classes of generating marked-scaled anodyne inclusions given in Definition 1.2.5 horn fillings would be possible but tedious. To lighten this load somewhat, we reproduce in this section two lemmas from [1]. The majority of this section is a retelling of sections 2.3 and 2.4 of [1]. Note however that our needs will be rather different than those of [1], and this is reflected in some subtle differences of notation.

We first need a compact notation for specifying simplicial subsets of Δ^n. Denote by $P \colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ the power set functor.

Definition 4.2.1. Let T be a finite linearly ordered set, so $T \cong [n]$ for some $n \in \mathbb{N}$. For any subset $\mathcal{A} \subseteq P(T)$, define a simplicial subset

$$\mathcal{S}_T^{\mathcal{A}} := \bigcup_{S \in \mathcal{A}} \Delta^{T \smallsetminus S} \subseteq \Delta^T.$$

Notation 4.2.2. If the linearly ordered set T is clear from context, we will drop it, writing $\mathcal{S}^{\mathcal{A}}$.

Note 4.2.3. The assignment $\mathcal{A} \mapsto \mathcal{S}^{\mathcal{A}}$ does not induce is not a one-to-one correspondence between subsets $\mathcal{A} \subseteq P([n])$ and simplicial subsets $S^{\mathcal{A}} \subseteq \Delta^n$, since for any two subsets $S \subsetneq S' \subset [n]$, we have that $\Delta^{[n] \smallsetminus S'} \subsetneq \Delta^{[n] \smallsetminus S}$.

It will be useful to consider two subsets $\mathcal{A}$ and $\mathcal{A}' \subseteq P(T)$ to be equivalent if they produce the same simplicial subset of Δ^T.

Definition 4.2.4. We will write $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{S}^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}'}$.

The relation $\sim$ is an equivalence relation, but we will not use this.

The set $P([n])$ forms a poset, ordered by inclusion, and any $\mathcal{A} \subseteq P([n])$ a subposet. An element q of a poset Q is said to be *minimal* if there are no elements of Q which are strictly less than q. By Note 4.2.3 only the minimal elements of $\mathcal{A}$ contribute to the union defining $\mathcal{S}^{\mathcal{A}}$. We have just shown the following.

Lemma 4.2.5. For any $\mathcal{A} \subseteq P([n])$, we have $\mathcal{A} \sim \min(\mathcal{A})$, where $\min(\mathcal{A}) \subseteq \mathcal{A}$ is the set of minimal elements of $\mathcal{A}$.

Simplicial subsets of the form $\mathcal{S}_T^{\mathcal{A}}$ enjoy the following easily-proved calculation rules.

Lemma 4.2.6. Suppose $\mathcal{A} \subseteq P([n])$ is a subset such that for all $S, T \in \mathcal{A}$, it holds that $S \cap T = \emptyset$. Then $\mathcal{S}^{\mathcal{A}}$ contains each k-simplex of Δ^n for all $k < |\mathcal{A}| - 1$.

Proof. Let $X \subseteq [n]$. The simplex $\Delta^X \to \Delta^n$ factors through $\mathcal{S}^{\mathcal{A}}$ if and only if it factors through $\Delta^{[n] \smallsetminus T}$ for some (possibly not unique) $T \in \mathcal{A}$. This in turn is true if and only if X and T do not have any elements in common. For $|X| < |\mathcal{A}|$, there is always a set $T \in \mathcal{A}$ which does not have any elements in common with X. □

Lemma 4.2.7. Let $\mathcal{A} \subseteq P([n])$ and $T \subseteq [n]$. Then

$$\mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T = \mathcal{S}_{[n]}^{\mathcal{A} \cup \{[n] \smallsetminus T\}}.$$

The next lemma will be particularly useful in building inclusions $\mathcal{S}_{[n]}^{\mathcal{A}} \hookrightarrow \Delta^n$ simplex-by-simplex.

Lemma 4.2.8. Let $\mathcal{A} \subset P([n])$, and let $T \subset [n]$. Then the square

$$\begin{tikzcd} \mathcal{S}_T^{\mathcal{A}|T} \arrow[r, hook] \arrow[d, hook] & \Delta^T \arrow[d, hook] \\ \mathcal{S}_{[n]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T \end{tikzcd}$$

is bicartesian, where $\mathcal{A}|T$ is the subset of $P(T)$ given by

$$\mathcal{A}|T = \{S \cap T \mid S \in \mathcal{A}\}.$$

Proof. We have an equality

$$\mathcal{S}_T^{\mathcal{A}|T} = \mathcal{S}_{[n]}^{\mathcal{A}} \cap \Delta^T.$$

as subsets of Δ^n. □

Notation 4.2.9. For any marked-scaled n-simplex (Δ^n, E, T) we will by minor abuse of notation reuse the same letters E and T to denote the restriction of the markings and scalings to any simplicial subset $S \subseteq \Delta^n$.

In the remainder of this section we reproduce two lemmas from [1] which provide criteria for the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \subseteq (\Delta^n, E, T)$ to be marked-scaled anodyne.

Definition 4.2.10. Let $\mathcal{A} \subseteq P([n])$. We call $X \in P([n])$ an $\mathcal{A}$***-basal set*** if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of all $\mathcal{A}$-basal sets by $\mathrm{Bas}(\mathcal{A})$.

Definition 4.2.11 ([5], Definition 1.3)**.** We will call a subset $\mathcal{A} \subseteq P([n])$ ***inner dull*** if it satisfies the following conditions.

- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- There exists $0 < i < n$ such that for all $S \in \mathcal{A}$, we have $i \notin S$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.
- For each $\mathcal{A}$-basal set $X \in P([n])$, there exist $u, v \in X$ such that $u < i < v$.

The element $i \in [n]$ is known as the *pivot point.*

Note 4.2.12. The last condition of Definition 4.2.11 is always satisfied if $\mathcal{A}$ contains two singletons $\{u\}$ and $\{v\}$ such that $u < i < v$.

Definition 4.2.13. Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i, and let $X \in \mathcal{A}$. We will denote the adjacent elements of X surrounding i by $\ell^X < i < u^X$.

Lemma 4.2.14 (The Pivot Trick: [1], Lemma 2.3.5)**.** Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i, and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that the following conditions hold:

1. Every marked edge $e \in E$ which does not contain i factors through $\mathcal{S}^{\mathcal{A}}$.
2. For all scaled simplices $\sigma = \{a < b < c\}$ of Δ^n which do not factor through $\mathcal{S}^{\mathcal{A}}$, and which do not contain the pivot point i, we have $a < i < c$, and $\sigma \cup \{i\}$ is fully scaled.
3. For all $X \in \mathrm{Bas}(\mathcal{A})$ and all $r, s \in [n]$ such that $\ell^X \leq r < i < s \leq u^X$, the triangle $\{r, i, s\}$ is scaled

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

Definition 4.2.15. We will call a subset $\mathcal{A} \subseteq P([n])$ ***right dull*** if it satisfies the following conditions.

- It is nonempty; $\mathcal{A} \neq \emptyset$.
- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- For all $S \in \mathcal{A}$, we have $n \notin S$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.

In this case, we will refer to n as the pivot point.

Lemma 4.2.16 (Right-anodyne pivot trick)**.** Let $\mathcal{A} \subset P([n])$ be a right dull subset (whose pivot point is by definition n), and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that the following conditions hold:

- Every marked edge $e \in E$ which does not contain n factors through $\mathcal{S}^{\mathcal{A}}$.
- Let $\sigma = \{a < b < c\}$ be a scaled simplex not containing n. Then either σ factors through $\mathcal{S}^{\mathcal{A}}$, or $\sigma \cup \{n\}$ is fully scaled, and $c \to n$ is marked.
- For all Z in $\mathrm{Bas}(\mathcal{A})$, For all $r, s \in [n]$ with $r \leq \min(Z) \leq \max(Z) \leq s < n$, the triangle $\Delta^{\{r,s,n\}}$ is scaled, and the simplex $\Delta^{\{s,n\}}$ is marked.

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

4.3 The infinity-category of local systems

4.3.1 The basic idea

Definition 4.3.1. For any cocomplete ∞-category $\mathcal{C}$, we define the ∞-category of $\mathcal{C}$-local systems $\mathbf{LS}(\mathcal{C})$ together with a map of simplicial sets $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ by the following pullback diagram.

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} \tag{14}$$

It is shown in [5] that p', hence also p, is a cartesian fibration. The cartesian fibration p classifies the functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending

$$f\colon X \to Y \qquad \longmapsto \qquad f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ is cocomplete, each pullback map f^* has a left adjoint $f_!$, given by left Kan extension. It follows on abstract grounds that p is also a cocartesian fibration, whose cocartesian edges represent left Kan extension. In Subsection 4.4, we show this explicitly.

4.3.2 Cosimplicial objects

Our main goal is to show that the map $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cocartesian fibration. In doing this, it will be useful to factor the pullback square in Equation 14 into the two pullback squares

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hook] \arrow[d, "p''"] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}] \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd},$$

where $[\mathcal{C}]$ denotes the path component of $\mathcal{C}$ in $(\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}})^{\simeq}$. In order to show that p is a cocartesian fibration, it will help us to understand the map p''. The n-simplices of $\mathcal{R}$ are given by maps

$$Q(\Delta^n) = (\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger \to \mathcal{C}\mathrm{at}_\infty$$

such that

- each object in $(\Delta^n)^{\mathrm{op}} \subseteq Q(\Delta^n)$ is sent to a quasicategory which is equivalent to $\mathcal{C}$ (and in particular cocomplete), and
- each morphism in $(\Delta^n)^{\mathrm{op}} \subseteq Q(\Delta^n)$ is mapped to an equivalence in $\mathcal{C}\mathrm{at}_\infty$.

We can more usefully encode the second condition by endowing $\mathcal{C}\mathrm{at}_\infty$ and Q with a marking.

Definition 4.3.2. We define the following markings.

- We denote by $\mathcal{C}\mathrm{at}_\infty^{\natural}$ the marked-scaled simplicial set whose underlying scaled simplicial set is $\mathcal{C}\mathrm{at}_\infty$, and where all equivalences have been marked.
- We denote by $\heartsuit$ the marking on $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ consisting of all morphisms belonging to

$(\Delta^n)^{\mathrm{op}}$, and by $(\Delta^n \star (\Delta^n)^{\mathrm{op}})^{\heartsuit}_{\dagger}$ the corresponding marked-scaled simplicial set.

Definition 4.3.3. We define a cosimplicial object

$$\tilde{R}\colon \Delta \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathbf{MS}}; \qquad [n] \mapsto (\Delta^n \star (\Delta^n)^{\mathrm{op}})^{\heartsuit}_{\dagger}.$$

We denote the extension of $\tilde{R}$ by colimits by

$$R\colon \mathcal{S}\mathrm{et}_{\Delta} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathbf{MS}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} R(n).$$

The marked-scaled simplicial sets $R(\Delta^n)$ are rather complicated, and showing that p'' is a cartesian fibration by solving the necessary lifting problems explicitly would be impractical. We will instead replace $R(\Delta^n)$ by something simpler, considering the simplicial subsets coming from the inclusions

$$\Delta^n \star \Delta^{\{\bar{0}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}, \tag{15}$$

which we understand to inherit the marking and scaling.

Definition 4.3.4. We will denote by $\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathbf{MS}}$ the cosimplicial object

$$\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathbf{MS}}; \qquad [n] \mapsto (\Delta^n \star \Delta^{\{\bar{0}\}})^{\heartsuit}_{\dagger},$$

and by J the extension by colimits

$$J\colon \mathcal{S}\mathrm{et}_{\Delta} \to \mathcal{S}\mathrm{et}_{\Delta}^{\mathbf{MS}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} \tilde{J}(n).$$

The inclusions of Equation 15 induce for each $n \geq 0$ an inclusion of marked-scaled simplicial sets

$$v_n\colon \tilde{J}(n) \to \tilde{R}(n).$$

Lemma 4.3.5. For all $n \geq 0$, the map v_n is marked-scaled anodyne, hence a weak equivalence.

We postpone the proof of Lemma 4.3.5 until the end of this section.

Note 4.3.6. Some care is warranted: the v_n are not the components of a natural transformation $\tilde{J} \Rightarrow \tilde{R}$! The necessary squares simply do not commute for morphisms $\phi\colon [m] \to [n]$ in Δ such that $\phi(0) \neq 0$. This means that we do not, for a general simplicial set X, get a canonical weak equivalence of marked-scaled simplicial sets $J(X) \to R(X)$.

Despite the warning given in Note 4.3.6, it is still possible to get maps $J(X) \to R(X)$ in some cases. In the remainder of this section, we check that we can produce a marked-scaled weak equivalence $J(\Lambda^n_0) \to R(\Lambda^n_0)$.

Notation 4.3.7. Denote by $\mathring{\Delta}$ the subcategory of Δ on morphisms $[m] \to [n]$ which send $0 \mapsto 0$. Denote by I the inclusion $\mathring{\Delta} \hookrightarrow \Delta$.

It is easy to check the following.

Lemma 4.3.8. The morphisms v_n form a natural transformation $v\colon \tilde{J} \circ I \Rightarrow \tilde{R} \circ I$, each of whose components is an equivalence in the model structure on marked-scaled simplicial sets.

Lemma 4.3.9. For all $n \geq 1$, there is a marked-scaled equivalence $J(\Lambda^n_0) \to R(\Lambda^n_0)$.

Proof. We can write $J(\Lambda^n_0)$ as a colimit of the composition $\tilde{J} \circ b$ coming from the bottom of the diagram

$$\begin{tikzcd} P_n \arrow[r, "a"] \arrow[d, hook] & \mathring{\Delta} \arrow[d, "I"] \arrow[dr, "\tilde{J}\circ I"] & \\ (\Delta \downarrow \Lambda^n_0)^{\mathrm{nd}} \arrow[r, "b"] & \Delta \arrow[r, "\tilde{J}"] & \mathcal{S}\mathrm{et}^{\mathrm{MS}}_{\Delta} \end{tikzcd},$$

where $(\Delta \downarrow \Lambda^n_0)^{\mathrm{nd}}$ is the category of nondegenerate simplices of Λ^n_0. Denote by P_n the poset of proper subsets of $[n]$ such that $0 \in S$. There is an obvious inclusion $P_n \hookrightarrow (\Delta \downarrow \Lambda^n_0)^{\mathrm{nd}}$, and one readily checks using Quillen's Theorem A that this inclusion is cofinal. Further note that the functor $P_n \to (\Delta \downarrow \Lambda^n_0) \to \Delta$ factors through $\mathring{\Delta}$ via a map $a\colon P_n \to \mathring{\Delta}$. Thus, we can equally express $J(\Lambda^n_0)$ as the colimit of the functor $\tilde{J} \circ I \circ a$. Precisely the same reasoning tells us that we can express $R(\Lambda^n_0)$ as the colimit of $\tilde{R} \circ I \circ a$. The result now follows from Lemma 4.3.8, and the fact that each strict colimit is the model for the homotopy colimit. $\square$

Proof of Lemma 4.3.5. In this proof, all simplicial subsets of Δ^{2n+1} will be assumed to carry the marking $\heartsuit$ and the scaling $\dagger$.

We can write each v_n as a composition

$$\Delta^{\{0,\ldots,n,\bar{0}\}} \overset{v''_n}{\hookrightarrow} \Delta^{\{0,\ldots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\ldots,\bar{0}\}} \overset{v'_n}{\hookrightarrow} \Delta^{2n+1}.$$

Here, the map v''_n is a pushout along the inclusion $(\Delta^{\{n\}})^{\sharp}_{\sharp} \hookrightarrow (\Delta^n)^{\sharp}_{\sharp}$, which is marked-scaled anodyne by Proposition 1.2.6. It remains to show that each of the maps v'_n is marked-scaled anodyne.

To this end, we introduce some notation. Let $M^n_0 = \Delta^{\{0,\ldots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\ldots\bar{0}\}} \subset \Delta^{2n+1}$, and for $1 \leq k \leq n$, define

$$M^n_k := M^n_0 \cup \left(\bigcup_{\ell=1}^{k} \Delta^{[2n+1]\setminus\{\ell,\bar{\ell}\}} \right).$$

There is an obvious filtration

$$M_0^n \xhookrightarrow{i_0^n} M_1^n \xhookrightarrow{i_1^n} \cdots \xhookrightarrow{i_{n-1}^n} M_n^n \xhookrightarrow{i_n^n} \Delta^{2n+1}. \tag{16}$$

Define

$$j_k^n := i_n^n \circ \cdots \circ i_k^n \colon M_k^n \hookrightarrow \Delta^{2n+1}, \qquad 0 \leq k \leq n.$$

In particular, note that $j_0^n = v_n'$.

This allows us to replace our goal to something superficially more difficult: we would like to show that, for each $n \geq 0$ and each $0 \leq k \leq n$, the map i_k^n is marked-scaled anodyne. We proceed by induction. We take as our base case $n = 0$, where we have the trivial filtration

$$M_0^0 \overset{i_0^0}{=} \Delta^1.$$

This is an equality of subsets of Δ^1, hence certainly an equivalence.

We now suppose that the result holds true for $n-1$; that is, that the maps i_k^{n-1} are marked-scaled anodyne for all $0 \leq k \leq n-1$. We aim to show that each i_k^n is marked-scaled anodyne for each $0 \leq k \leq n$.

For $0 \leq k < n$, we can write i_k^n as the inclusion

$$\mathcal{S}_{[2n+1]}^{\mathcal{A}} \hookrightarrow \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{[2n+1] \smallsetminus \{k,\bar{k}\}}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\bar{n},\ldots,\bar{1}\} \\ \{0,\ldots n\} \\ \{1,\bar{1}\} \\ \vdots \\ \{k-1,\overline{k-1}\} \end{matrix} \right\}.$$

By Lemma 4.2.8, we have a pushout square

$$\begin{tikzcd}
\mathcal{S}_{[2n+1] \smallsetminus \{k,\bar{k}\}}^{\mathcal{A}|([2n+1] \smallsetminus \{k,\bar{k}\})} \arrow[r, hook] \arrow[d, hook] & \Delta^{[2n+1] \smallsetminus \{k,\bar{k}\}} \arrow[d, hook] \\
\mathcal{S}_{[2n+1]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{[2n+1] \smallsetminus \{k,\bar{k}\}}.
\end{tikzcd}$$

Therefore, it suffices to show that the top morphism is marked-scaled anodyne. One checks that this map is of the form j_k^{n-1}, and is thus marked-scaled anodyne by the inductive hypothesis. Therefore, it remains only to show that i_n^n is marked-scaled anodyne.

The case $n = 0$ is an isomorphism, so there is nothing to show. We treat the case $n = 1$ separately. In this case, i_1^1 takes the form

$$\Delta^{\{0,1,\bar{0}\}} \cup \Delta^{\{\bar{1},\bar{0}\}} \xhookrightarrow{v_1'} \Delta^3,$$

which we construct in the following way.

1. First we fill the simplex $\Delta^{\{1,\overline{1},\overline{0}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A2).
2. Then we fill the simplex $\Delta^{\{0,1,\overline{1}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A1).
3. Finally, we fill the full simplex Δ^3 (together with its marking and scaling) as a pushout along a morphism of type (A1).

Now we may assume that $n \geq 2$. In this case, we can write $i_n^n \colon M_n^n \hookrightarrow \Delta^{2n+1}$ as an inclusion

$$\mathcal{S}_{[2n+1]}^{\mathcal{A}} \subseteq \Delta^{2n+1}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,\ldots n\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

We will express this inclusion as the following composition of fillings:

1. We first add the simplices $\Delta^{\{n,\overline{n},\ldots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\ldots,\overline{0}\}}$, $\ldots$, $\Delta^{\{2,\ldots,n,\overline{n},\ldots,\overline{0}\}}$.
2. We next add $\Delta^{\{0,\ldots,n,\overline{n},\overline{0}\}}$, $\Delta^{\{0,\ldots,n,\overline{n},\overline{n-1},\overline{0}\}}$, $\ldots$, $\Delta^{\{0,\ldots,n,\overline{n},\ldots,\overline{2},\overline{0}\}}$
3. We next add $\Delta^{\{1,\ldots n,\overline{n},\ldots,\overline{0}\}}$.
4. We finally add $\Delta^{\{0,\ldots n,\overline{n},\ldots,\overline{0}\}}$.

We proceed.

1. • Using Lemma 4.2.8, we see that the square

$$\begin{tikzcd} \mathcal{S}_{\{n,\overline{n},\ldots,\overline{0}\}}^{\mathcal{A}|\{n,\overline{n},\ldots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}_{[2n+1]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} \end{tikzcd}$$

is pushout. In order to show that the bottom inclusion is marked-scaled anodyne, it thus suffices to show that the top inclusion is marked-scaled anodyne. We have

$$\mathcal{A}|\{n,\overline{n},\ldots,\overline{0}\} = \left\{ \begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{n\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{n-1}\} \\ \{n,\overline{n}\} \end{matrix} \right\} \sim \left\{ \begin{matrix} \{n\} \\ \{\overline{n-1}\} \\ \vdots \\ \{\overline{1}\} \end{matrix} \right\} =: \mathcal{A}' \ .$$

This is a dull subset of $\{n,\overline{n},\ldots,\overline{0}\}$ with pivot $\overline{n}$. The only $\mathcal{A}'$-basal set is $\{n,\overline{n-1},\ldots,\overline{1}\}$.

One checks that $\mathcal{A}'$, together with the marking $\heartsuit$ and scaling $\dagger$, satisfies the conditions of the Pivot Trick:

- For $n = 2$, the only the marked edge not containing the pivot $\overline{2}$ is $\overline{1} \to \overline{0}$, which belongs to $\mathcal{S}^{\mathcal{A}'}_{\{n,\overline{n},\ldots,\overline{0}\}}$. The only scaled simplex which does not contain $\overline{2}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{2 < \overline{1} < \overline{0}\}$. The simplex $\sigma \cup \{\overline{2}\}$ is fully scaled.
- For $n = 3$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'}$. The only scaled triangle which does not contain the pivot $\overline{3}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{3 < \overline{2} < \overline{1}\}$. The simplex $\sigma \cup \{\overline{3}\}$ is fully scaled.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 4.2.6.

In each case, the simplex $\{n, \overline{n}, \overline{n-1}\}$ is scaled, so the top inclusion is marked-scaled anodyne by the Pivot Trick. We can write

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} = \mathcal{S}^{\mathcal{A}\cup([2n+1]\setminus\{n,\overline{n},\ldots,\overline{0}\})}_{[2n+1]},$$

We see that

$$\mathcal{A} \cup ([2n+1] \setminus \{n, \overline{n}, \ldots, \overline{0}\}) \sim \left\{ \begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,\ldots n-1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\},$$

which we denote by $\mathcal{A}_{n-1}$.

- We proceed inductively. Suppose we have added the simplices $\Delta^{\{n,\overline{n},\ldots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\ldots,\overline{0}\}}$, $\ldots$, $\Delta^{\{k+1,\ldots,n,\overline{n},\ldots,\overline{0}\}}$, for $2 \leq k \leq n-1$. Using Lemma 4.2.7 and Lemma 4.2.5 can write the result of these additions as

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \left(\bigcup_{i=k+1}^{n} \Delta^{\{i,\ldots,n,\overline{n},\ldots,\overline{0}\}} \right) = \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]},$$

where

$$\mathcal{A}_{k+1} = \left\{ \begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,\ldots,k\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

Using Lemma 4.2.8, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{A}_{k+1}|\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}}_{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \cup \Delta^{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \end{tikzcd}$$

is pushout, so in order to show that the bottom morphism is marked-scaled anodyne, it suffices to show that the top morphism is. We see that

$$\mathcal{A}_{k+1}|\{k,\dots,n,\overline{n},\dots,\overline{0}\} = \left\{ \begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k,\overline{k}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\} \sim \left\{ \begin{matrix} \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k+1,\overline{k+1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\} =: \mathcal{A}'_{k+1}.$$

This is a dull subset of $P(\{k,\dots,n,\overline{n},\dots,\overline{0}\})$ with pivot $\overline{k}$, and one checks that the conditions of the Pivot Trick are satisfied:

- For $n = 3$, where the only value of k is $k = 2$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'_{k+1}}$ by Lemma 4.2.6, and one can check that each scaled triangle factors through $\mathcal{S}^{\mathcal{A}'_{k+1}}$.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 4.2.6.

The basal sets are of the form

$$\{k, a_1, \dots, a_{n-k}, \overline{k-1}, \dots, \overline{1}\},$$

where each $a_1, \dots, a_{n-k}$ is of the form ℓ or $\overline{\ell}$ for $k+1 \leq \ell \leq n$. In each case, the simplex $\{a_{n-k}, \overline{k}, \overline{k-1}\}$ is scaled. Thus, the conditions of the Pivot Trick, the top morphism is marked-scaled anodyne.

We have now added the simplices promised in part 1., and are left with the simplicial subset $\mathcal{S}^{\mathcal{A}_2}_{[2n+1]}$, where

$$\mathcal{A}_2 = \left\{ \begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

2. Each step in this sequence is solved exactly like those above. The calculations are omitted. The end result is the simplicial subset

$$\mathcal{B}_2 = \left\{ \begin{matrix} \{\overline{1}\} \\ \{0,1\} \\ \{2,\overline{2}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

3. Using Lemma 4.2.8, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{B}_2|\{1,\dots,n,\overline{n},\dots,\overline{0}\}}_{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{B}_2}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{B}_2}_{[2n+1]} \cup \Delta^{\{1,\dots,n,\overline{n},\dots,\overline{0}\}} \end{tikzcd}$$

is pushout. We thus have to show that the top morphism is marked-scaled anodyne. We have

$$\mathcal{B}_2|\{1,\dots,n,\overline{n},\dots,\overline{0}\} \sim \left\{\begin{matrix} \{\overline{1}\} \\ \{2,\overline{2}\} \\ \vdots \\ \{n,\overline{n}\} \\ \{\overline{1}\} \end{matrix}\right\}.$$

One readily sees that this is a right-dull subset, and that the conditions of the Right-Anodyne Pivot Trick are satisfied.

4. One solves this as before, checking that the conditions of the Pivot Trick are satisfied, with pivot point 2.

□

4.4 The map governing local systems is a cocartesian fibration

Our aim is to show that the map of quasicategories $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ defined by the diagram

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hook] \arrow[d, "p''"] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}] \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd},$$

where both squares are pullback, is a cocartesian fibration. We now define the class of morphisms which we claim are p-cocartesian.

Definition 4.4.1. A morphism $\tilde{\sigma}\colon \Delta^1 \to \mathbf{LS}(\mathcal{C})$ is said to be ***left Kan*** if the simplex $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ to which it is adjoint has the property that the restriction $\sigma|\Delta^{\{0,1,\overline{0}\}}$ is left Kan in the sense of Definition 3.0.1.

We draw the 'front' and 'back' of a general 3-simplex $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ corresponding to some morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, "\mathcal{H}"] & Y \arrow[d, "\mathcal{G}"] \\
\mathcal{C} \arrow[r, phantom, "\Rrightarrow \zeta"] & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, "\eta \Rrightarrow"', shift right=3] & Y \arrow[d, "\mathcal{G}"] \arrow[dl, "\mathcal{G}'"] \\
\mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"]
\end{tikzcd}
$$

Here, the 2-simplices which are notated without natural transformations are constrained by the scaling $\dagger$ to be thin. The morphism $\tilde{\sigma}$ is left Kan if and only if $\mathcal{G}'$ is a left Kan extension of $\mathcal{F}$ along f, and η is a unit map.

Notation 4.4.2. We endow the quasicategory $\mathbf{LS}(\mathcal{C})$ with the marking $\clubsuit \subseteq \mathbf{LS}(\mathcal{C})_1$ consisting of all left Kan edges.

Theorem 4.4.3. The map $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cocartesian fibration, and an edge $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is p-cocartesian if and only if it is left Kan.

Proof. By [8, Prop. 3.1.1.6], it suffices to show that the map $\mathbf{LS}(\mathcal{C})^\clubsuit \to \mathcal{S}^\sharp$ has the right lifting property with respect to each of the classes of generating marked anodyne morphisms. We check these one-by-one.

1. We need to check that all lifting problems

$$
\begin{tikzcd}
(\Lambda^n_i)^\flat \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^\clubsuit \arrow[d] \\
(\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp
\end{tikzcd}
, \qquad n \geq 2, \quad 0 < i < n
$$

   admit solutions. This follows from Theorem 4.1.4.

2. We need to show that each of the lifting problems

$$
\begin{tikzcd}
(\Lambda^n_0)^\mathcal{L} \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^\clubsuit \arrow[d] \\
(\Delta^n)^\mathcal{L} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp
\end{tikzcd}
, \qquad n \geq 1.
$$

   has a solution. Given any such lifting problem, it suffices to find a solution to the outer lifting problem

$$
\begin{tikzcd}
(\Lambda^n_0)^\mathcal{L} \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d] & \mathcal{R} \arrow[d] \\
(\Delta^n)^\mathcal{L} \arrow[r] \arrow[urr, dashed] & \mathcal{S} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times [\mathcal{C}]
\end{tikzcd}
.
$$

Passing to the adjoint lifting problem, we need to find a solution to the lifting problem

$$\begin{tikzcd} R(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp \arrow[r, "\sigma"] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ R(\Delta^n) \arrow[ur, dashed] & \end{tikzcd} \tag{17}$$

such that $\sigma|\{0, 1, \bar{0}\}$ is left Kan. Here R is the functor defined in Definition 4.3.3.

We note the existence of a commutative square

$$\begin{tikzcd} J(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\bar{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\bar{0}\}})^\sharp_\sharp \arrow[r, hook, "b"] \arrow[d, hook] & R(\Lambda^n_0) \coprod_{(\Lambda^n_0)^\flat_\sharp \amalg ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp \arrow[d, hook] \\ J(\Delta^n) \arrow[r, hook, "v_n"] & R(\Delta^n) \end{tikzcd},$$

where J is the functor defined in Definition 4.3.4. Here, v_n is the morphism of Lemma 4.3.5. The morphism b is defined component-wise via the following morphisms.

- The morphism $J(\Lambda^n_0) \to R(\Lambda^n_0)$ comes from Lemma 4.3.9, where it is proven to be a weak equivalence in the marked-scaled model structure.
- The map $(\Delta^{\{\bar{0}\}})^\sharp_\sharp \to ((\Lambda^n_0)^{\mathrm{op}})^\sharp_\sharp$ is marked-scaled anodyne by Proposition 1.2.6.
- The maps $(\Delta^{\{\bar{0}\}})^\sharp_\sharp \to ((\Delta^n)^{\mathrm{op}})^\sharp_\sharp$ is marked-scaled anodyne by Proposition 1.2.6.
- The rest of the morphisms connecting the components are isomorphisms.

We showed in Lemma 4.3.5 that the lower morphism in this square is a marked-scaled equivalence. We would now like to show that the upper morphism is a marked-scaled equivalence. Recall that, since the cofibrations in the model structure on marked-scaled simplicial sets are simply those morphisms whose underlying morphism of simplicial sets is a monomorphism, the colimits defining b are models for the homotopy colimits, so the result follows from the observation that each component is a weak equivalence in the marked-scaled model structure.

Using [8, Prop. A.2.3.1], we see that in order to solve the lifting problem of Equation 17, it suffices to show that the lifting problem

$$\begin{tikzcd} J(\Lambda^n_0) \coprod\limits_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp \arrow[r, "b\circ\sigma"] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ J(\Delta^n) \arrow[ur, dashed, "\ell"'] & \end{tikzcd}$$

has a solution whose restriction $\ell|\{0,1,\overline{0}\}$ is left Kan. However, we note that there exists a pushout square

$$\begin{tikzcd} (\Lambda_0^{\{0,\ldots,n,\overline{0}\}})^\flat_\flat \arrow[r] \arrow[d] & J(\Lambda^n_0) \coprod\limits_{(\Lambda^n_0)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp} (\Delta^n)^\flat_\sharp \amalg (\Delta^{\{\overline{0}\}})^\sharp_\sharp \arrow[d] \\ (\Delta^{\{0,\ldots,n,\overline{0}\}})^\flat_\flat \arrow[r] & J(\Delta^n) \end{tikzcd}$$

in which the rightward-facing morphisms are isomorphisms on underlying simplicial sets, and where the only new decorations being added are $(\Delta^{\{0,\ldots,n\}})^\flat_\flat \hookrightarrow (\Delta^{\{0,\ldots,n\}})^\flat_\sharp$. Therefore, it suffices to solve the lifting problems

$$\begin{tikzcd} (\Lambda_0^{\{0,\ldots,n,\overline{0}\}})^\flat_\flat \arrow[r] \arrow[d] & \mathbb{C}\mathrm{at}_\infty \\ (\Delta^{\{0,\ldots,n,\overline{0}\}})^\flat_\flat \arrow[ur, dashed, "\ell'"'] & \end{tikzcd}, \qquad n \geq 1$$

such that $\ell'|\{0,1,\overline{0}\}$ is left Kan. That these lifting problems admit solutions for $n \geq 2$ is the content of Theorem 3.0.2. The case $n = 1$ is the statement that left Kan extensions of functors into cocomplete categories exist along functors between small categories.

3. We need to show that every lifting problem of the form

$$\begin{tikzcd} (\Lambda^2_1)^\sharp \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^\sharp \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ (\Delta^2)^\sharp \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp \end{tikzcd}$$

has a solution. Considering the adjoint lifting problem, we find that it suffices to show that for any $\sigma\colon \Delta^5_\dagger \to \mathbb{C}\mathrm{at}_\infty$ such that the restrictions $\sigma|\{0,1,\overline{0}\}$ and $\sigma|\{1,2,\overline{1}\}$ are left Kan and the morphisms belonging to $\sigma|\{\overline{2},\overline{1},\overline{0}\}$ are equivalences, the restriction $\sigma|\{0,2,\overline{0}\}$

is left Kan. Applying Lemma 3.5.1 to $\sigma|\{1,2,\overline{1},\overline{0}\}$, we see that $\sigma|\{1,2,\overline{1}\}$ is left Kan if and only if $\sigma|\{1,2,\overline{0}\}$ is left Kan. Applying Lemma 3.5.3 to $\sigma|\{0,1,2,\overline{0}\}$ guarantees that $\sigma|\{0,2,\overline{0}\}$ is left Kan as required.

4. We need to show that for all Kan complexes K, the lifting problem

$$\begin{tikzcd} K^\flat \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ K^\sharp \arrow[r] \arrow[ur, dashed] & \mathcal{S}^\sharp \end{tikzcd}$$

has a solution. To see this, note that each morphism in K must be mapped to an equivalence in $\mathbf{LS}(\mathcal{C})$, and a morphism $a\colon \mathcal{F} \to \mathcal{G}$ in $\mathbf{LS}(\mathcal{C})$ is an equivalence and only if it is p-cartesian, and lies over an equivalence in $\mathcal{S}$.

- By Theorem 4.1.4, the morphism a is p-cartesian if and only if the map $\sigma\colon \Delta^3_\dagger \to \mathcal{C}\mathrm{at}_\infty$ to which it is adjoint factors through the map $\Delta^3_\dagger \to \Delta^3_\sharp$. In particular, the restriction $\sigma|\Delta^{\{0,1,\overline{0}\}}$ is thin.
- The image of a in $\mathcal{S}$ is the restriction $\sigma|\Delta^{\{0,1\}}$, which is therefore an equivalence.

It follows from Example 3.1.7 that the morphism a is automatically left Kan. Thus, each morphism in K is mapped to a left Kan morphism in $\mathbf{LS}(\mathcal{C})$, and we are justified in marking them; our lifting problems admit solutions.

□

5 Push-pull of local systesms

For any cocomplete ∞-category $\mathcal{C}$, we have constructed an ∞-category $\mathbf{LS}(\mathcal{C})$ of local systems on $\mathcal{C}$, together with a map $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$. We have shown that this map is a bicartesian fibration.

- As a cartesian fibration, it classifies the pullback functor $(-)^*\colon \mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$; for any morphism $X \to Y$ in $\mathcal{S}$, this functoriality gives us a map

$$f^*\colon \mathrm{Fun}(Y,\mathcal{C}) \to \mathrm{Fun}(X,\mathcal{C}).$$

- As a cocartesian fibration, it classifies the left Kan extension functor $(-)_!\colon \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$; for any morphism $X \to Y$, in $\mathcal{S}$, this functoriality gives us a map

$$f_!\colon \mathrm{Fun}(X,\mathcal{C}) \to \mathrm{Fun}(Y,\mathcal{C}).$$

We might wonder if we can combine these functorialities. Given a *span* of morphisms in $\mathcal{S}$, i.e. a diagram in $\mathcal{S}$ of the form

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd}, \tag{18}$$

we can pull back along g and push forward along f, giving a map $f_! \circ g^* \colon \mathrm{Fun}(X, \mathcal{C}) \to \mathrm{Fun}(X', \mathcal{C})$ via the composition

$$\begin{tikzcd} & \mathrm{Fun}(Y, \mathcal{C}) \arrow[dr, "f_!"] & \\ \mathrm{Fun}(X, \mathcal{C}) \arrow[ur, "g^*"] \arrow[rr, dashed, "f_! \circ g^*"'] & & \mathrm{Fun}(X', \mathcal{C}) \end{tikzcd}.$$

More concretely, we would like to upgrade this construction to a functor $\hat{r} \colon \mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$, where $\mathrm{Span}(\mathcal{S})$ is an ∞-category with the following rough description.

- The objects of $\mathrm{Span}(\mathcal{S})$ are the same as the objects of $\mathcal{S}$, i.e. spaces X, Y, etc.
- The morphisms from X to X' are given by spans in $\mathcal{S}$, i.e. diagrams of the form given in Equation 18.
- The 2-simplices witnessing the composition of two morphisms

  $$X \xleftarrow{h} Y \xrightarrow{f} X' \quad \text{and} \quad X' \xleftarrow{g} Y \xrightarrow{j} X''$$

  are diagrams of the form

  $$\begin{tikzcd} & & Z \arrow[dl, "g'"'] \arrow[dr, "f'"] & & \\ & Y \arrow[dl, "h"'] \arrow[dr, "f"] & & Y' \arrow[dl, "g"'] \arrow[dr, "j"] & \\ X & & X' & & X'' \end{tikzcd}, \tag{19}$$

  where the square formed is pullback. The corresponding composition is then given by the span

  $$\begin{tikzcd} & Z \arrow[dl, "h \circ g'"'] \arrow[dr, "j \circ f'"] & \\ X & & X'' \end{tikzcd}.$$

The non-trivial part of constructing such a functor $\hat{r}$ will be showing that it respects composition; we need that for any diagram of the form given in Equation 19, both ways of composing morphisms from left to right in the diagram

$$\begin{tikzcd}[column sep=small]
 & & & \mathrm{Fun}(Z,\mathcal{C}) \arrow[dr, "(f')_!"] & & & \\
 & \mathrm{Fun}(Y,\mathcal{C}) \arrow[urr, "(g')^*"] \arrow[dr, "f_!"] & & & \mathrm{Fun}(Y',\mathcal{C}) \arrow[drr, "j_!"] & & \\
\mathrm{Fun}(X,\mathcal{C}) \arrow[ur, "h^*"] & & \mathrm{Fun}(X',\mathcal{C}) \arrow[urr, "g^*"] & & & & \mathrm{Fun}(X'',\mathcal{C})
\end{tikzcd},$$

agree up to a specified natural equivalence. This condition can be distilled down to the so-called *Beck-Chevalley condition:*

- For any pullback square in $\mathcal{S}$

  $$\begin{tikzcd} Z \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ Y \arrow[r, "f"] & X \end{tikzcd},$$

  the comparison map

  $$f_! \circ g^* \overset{\eta}{\Rightarrow} f_! \circ g^* \circ (f')^* \circ (f')_! \overset{\simeq}{\Rightarrow} f_! \circ f^* \circ (g')^* \circ (f')_! \overset{\epsilon}{\Rightarrow} (g')^* \circ (f')_!$$

  is an equivalence.

One can phrase the Beck-Chevalley condition at the level of fibrations rather than functors into $\mathcal{C}\mathrm{at}_\infty$. This is a classical definition, here more or less lifted from [6].

Definition 5.0.1. A bicartesian fibration of quasicategories $p\colon \mathcal{X} \to \mathcal{T}$ such that $\mathcal{T}$ admits pullbacks is called a ***Beck-Chevalley fibration*** if it has the following property.

(BC) For any commuting square in $\mathcal{X}$

$$\begin{tikzcd} z \arrow[r, "f'"] \arrow[d, "g'"'] & y \arrow[d, "g"] \\ y' \arrow[r, "f"] & x \end{tikzcd}$$

lying over a pullback square in $\mathcal{T}$, if the morphism f is p-cocartesian and the morphisms g and g' are p-cartesian, then the morphism f' is p-cocartesian.

In fact, it turns out that this condition is sufficient to guarantee that the the push-pull procedure is functorial.

Proposition 5.0.2. Let $p\colon \mathcal{X} \to \mathcal{T}$ be a Beck-Chevalley fibration. Then there is a functor

$\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_\infty$ sending an object $t \in \mathrm{Span}(\mathcal{T})$ to the fiber $\mathcal{X}_t$, and a span $t \xleftarrow{b} s \xrightarrow{a} t'$ to the composition $a_! \circ b^* : \mathcal{X}_t \to \mathcal{X}_{t'}$.

Proof. It follows immediately from Theorem 2.0.2 that there is a cocartesian fibration

$$\tilde{p} : \mathrm{Span}^{\mathrm{cart}}(\mathcal{X}) \to \mathrm{Span}(\mathcal{T}),$$

where $\mathrm{Span}^{\mathrm{cart}}$ denotes the category of spans whose backwards-facing legs are constrained to be p-cartesian; that is, $\mathcal{X}^\dagger = \mathcal{X}^{\mathrm{cart}}$. Straightening gives the result that we want. □

The proof of the following proposition will come at the end of this section.

Proposition 5.0.3. The functor $p : \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a Beck-Chevalley fibration.

We next prove a few lemmata which will help us in our proof.

The following is simply a reformulation of the definition of a left Kan extension (Definition 3.1.5) in the special case that we consider left Kan extensions along maps of Kan complexes.

Lemma 5.0.4. Let $f : X \to Y$ be a map between Kan complexes, let $F : X \to \mathcal{C}$ and $G : Y \to \mathcal{C}$ be functors, and let $\eta : F \to G \circ y$ be a natural transformation. Then η exhibits G as a left Kan extension of F along f if and only if, for all $y \in Y$, the natural transformatin $F \circ \pi \Rightarrow \underline{G(y)}$ defined by the pasting diagram

$$\begin{tikzcd} X_{/y} \arrow[r, "\pi"] \arrow[d] & X \arrow[r, "F"] \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow\eta"] & \mathcal{C} \\ \{y\} \arrow[r, hook] & Y \arrow[ur, "G"'] & \end{tikzcd},$$

exhibits $G(y)$ as the colimit of $F \circ \pi$, where the left-hand square is homotopy pullback.

Proof. We note that we can factor the above square into three squares

$$\begin{tikzcd} X_{/y} \arrow[r, "\simeq"] \arrow[d] & X^{/y} \arrow[r] \arrow[d] & Y^{\Delta^1} \times_Y X \arrow[r, "\simeq"] \arrow[d] & X \arrow[d, "f"] \\ \{y\} \arrow[r, equal] & \{y\} \arrow[r] & Y \arrow[r, equal] & Y \end{tikzcd},$$

where the middle square is a strict pullback. Since the map $Y^{\Delta^1} \times_Y X \to Y$ is a Kan fibration, the middle square is also a homotopy pullback. We note that the left and right squares are also homotopy pullbacks, since the horizontal morphisms are weak equivalences. Thus, the outer square is a homotopy pullback; one easily checks that the natural equivalence $f \circ \pi \Rightarrow \underline{y}$ defined by pasting the three squares agrees with the natural transformation α defined in Notation 3.1.4. □

Proof of Theorem 5.0.3. We have already shown that p is a bicartesian fibration, and we know that $\mathcal{S}$ admits pullbacks. Therefore, it suffices to show that p has Property (BC).

We consider a square $\sigma\colon \Delta^1 \times \Delta^1 \cong \Delta^{\{0,1,2\}} \amalg_{\Delta^{\{0,2\}}} \Delta^{\{0,1',2\}} \to \mathbf{LS}(\mathcal{C})$ with the following properties.

1. The restriction $\sigma|\Delta^{\{0,1\}}$ is p-cartesian.
2. The restriction $\sigma|\Delta^{\{1',2\}}$ is p-cartesian.
3. The restriction $\sigma|\Delta^{\{1,2\}}$ is p-cocartesian.
4. The square σ lies over a pullback square

$$p(\sigma) = \begin{tikzcd} X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[d, "g"] \\ X_1 \arrow[r, "f"] & X_2 \end{tikzcd}$$

   in $\mathcal{S}$.

We need to show that $\sigma|\Delta^{\{0,1'\}}$ is p-cocartesian.

The map σ adjunct to a map

$$\tau\colon \Delta_{\dagger}^{\{0,1,2,\bar{2},\bar{1},\bar{0}\}} \amalg_{\Delta_{\dagger}^{\{0,2,\bar{2},\bar{0}\}}} \Delta_{\dagger}^{\{0,1',2,\bar{2},\bar{1}',\bar{0}\}} \to \mathbb{C}\mathrm{at}_\infty$$

such that $\tau|\Delta^{\{\bar{2},\bar{1},\bar{0}\}} \amalg_{\Delta^{\{\bar{2},\bar{0}\}}} \Delta^{\{\bar{2},\bar{1}',\bar{0}\}}$ is the constant functor with value $\mathcal{C}$. We can read off the following further properties of τ, corresponding to the properties of σ above.

1. The simplices $\tau|\Delta^{\{0,1,\bar{0}\}}$ and $\tau|\Delta^{\{0,\bar{1},\bar{0}\}}$ are thin.
2. The simplices $\tau|\Delta^{\{1',2,\bar{1}'\}}$ and $\tau|\Delta^{\{1',\bar{2},\bar{1}'\}}$ are thin.
3. The simplex $\tau|\Delta^{\{1,2,\bar{1}\}}$ is left Kan.
4. The restriction $\tau|\Delta^{\{0,1,2\}} \amalg_{\Delta^{\{0,2\}}} \Delta^{\{0,1',2\}}$ is equal to $p(\sigma)$.

The condition that $\sigma|\Delta^{\{0,1'\}}$ is p-cocartesian corresponds to the condition that the simplex $\tau|\Delta^{\{0,1,\bar{0}\}}$ is left Kan.

The diagram τ contains a lot of redundant data, which we would now like to consolidate. We first shuffle some data around our diagram. Applying Lemma 3.5.1 to the simplex $\tau|\Delta^{\{1,2,\bar{1},\bar{0}\}}$, we see that $\tau|\Delta^{\{1,2,\bar{0}\}}$ is left Kan. Applying Lemma 3.5.2 to the simplex $\tau|\Delta^{\{1',2,\bar{1}',\bar{0}\}}$, we see that

$\tau|\Delta^{\{1',2,\bar{0}\}}$ is thin.

We can now study a subdiagram which contains all the information we need. We consider the restriction

$$\tau|\Delta^{\{0,1,2,\bar{0}\}} \amalg_{\Delta^{\{0,2,\bar{0}\}}} \Delta^{\{0,1',2,\bar{0}'\}}.$$

This is determined, up to specific choices of compositions, by the diagram

$$\begin{tikzcd} X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[r, "F"] \arrow[d, "g"'] & \mathcal{C} \arrow[dl, phantom, "\overset{\eta}{\Swarrow}"] \\ X_1 \arrow[r, "f"] & X_2 \arrow[ur, "G"'] & \end{tikzcd}.$$

Here, the square pictured is σ, and the triangle is $\tau|\Delta^{\{1',2,\bar{0}\}}$.

Since the simplices $\tau|\Delta^{\{0,1,\bar{0}\}}$ and $\tau|\Delta^{\{1,2,\bar{0}\}}$ are thin, the pasting of the square and the triangle above is homotopic to the restriction $\tau|\Delta^{\{0,1,\bar{0}\}}$. This is the triangle which we wish to show is left Kan. According to Lemma 5.0.4 it suffices to show that for all $x \in X_1$, the pasting diagram

$$\begin{tikzcd} (X_0)_{/x} \arrow[r, "\pi"] \arrow[d] & X_0 \arrow[r, "f'"] \arrow[d, "g'"'] & X_{1'} \arrow[r, "F"] \arrow[d, "g"'] & \mathcal{C} \arrow[dl, phantom, "\overset{\eta}{\Swarrow}"] \\ \{x\} \arrow[r, hook] & X_1 \arrow[r, "f"] & X_2 \arrow[ur, "G"'] & \end{tikzcd}$$

exhibits $G(f(x))$ as the colimit of $F \circ f' \circ \pi$. But applying the pasting law for homotopy pullbacks, this follows directly from the assumption that η exhibits G as a left Kan extension of F along f. $\square$

6 Monoidal push-pull of local systems

Suppose $p\colon \mathcal{C} \to \mathcal{D}$ is a cartesian fibration of 1-categories which classifies a pseudofunctor $\hat{p}\colon \mathcal{D}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}$. Further suppose the following:

- The category $\mathcal{C}$ carries a symmetric monoidal structure $\otimes$.
- The category $\mathcal{D}$ carries a symmetric monoidal structure $\boxtimes$.
- The functor p is strong monoidal: for $x, y \in \mathcal{C}$, $p(x \otimes y) = p(x) \boxtimes p(y)$.[3]
- The monoidal product $\otimes$ preserves p-cartesian morphisms: if f and g are p-cartesian

[3]The reason for the strict equality in this condition is that we have chosen specific monoidal functors $\otimes$ and $\times$. Later, we will replace these functors by fibrations, and the condition of strict equality will be replaced by the more familiar condition of homotopy coherence.

morphisms, then $f \otimes g$ is also p-cartesian.

Under these conditions, the pseudofunctor $\hat{p}$ carries a lax monoidal structure $(\mathcal{D}^{\mathrm{op}}, \boxtimes) \to (\mathcal{C}\mathrm{at}, \times)$ [9], described as follows. Restricting the tensor product

$$\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$$

to the fibers $\mathcal{C}_d$ and $\mathcal{C}_{d'}$ of p over d and d' yields the structure maps

$$\hat{p}(d) \times \hat{p}(d') \to \hat{p}(d \boxtimes d').$$

That these structure maps form a pseudonatural transformation, i.e. that the necessary squares commute up to specified homotopy, follows from the assumption that the monoidal product preserves p-cartesian morphisms in each slot.

If we want to study covariant rather than contravariant pseudofunctors, we should replace cartesian fibrations by cocartesian fibrations; of course, now we must assume that $\otimes$ preserves cocartesian edges rather than cartesian. The rest of the theory remains unchanged. These results also remain true in the ∞-categorical case, as we will show.

As we saw in the last chapter, if we have a bicartesian fibration subject which satisfies the Beck-Chevalley condition (a Beck-Chevalley fibration), we can combine both cartesian and cocartesian functorialities into pull-push functoriality. Our aim is to show that the theory of Beck-Chevalley fibrations admits a monoidal generalization. Consider a Beck-Chevalley fibration $r \colon \mathcal{X} \to \mathcal{T}$ such that $\mathcal{X}$ carries a symmetric monoidal structure $\otimes$, and $\mathcal{T}$ carries a symmetric monoidal structure $\boxtimes$.[4] Our first purpose in this chapter is to show that under the assumption that r is strong monoidal, and that $\otimes$ preserves both cartesian and cocartesian edges, one finds that the functor

$$\mathrm{Span}(\mathcal{T}) \to \mathcal{C}\mathrm{at}_\infty$$

constructed in the previous chapter carries a lax monoidal structure. Note that this statement is not entirely original. A generalized version in a somewhat different context is proved in [2]. We provide an elementary account of the portions of the theory which we will need.

We will then apply our results to local systems. We will show that if $\mathcal{C}$ is a symmetric monoidal ∞-category, then the category of $\mathcal{C}$-local systems carries a symmetric monoidal structure, defined on objects by

$$(\mathcal{F} \colon X \to \mathcal{C}) \otimes (\mathcal{G} \colon Y \to \mathcal{C}) \quad = \quad X \times Y \xrightarrow{\mathcal{F}\times\mathcal{G}} \mathcal{C} \times \mathcal{C} \xrightarrow{\otimes} \mathcal{C}$$

and that the functor $\mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a monoidal Beck-Chevalley fibration, thus classifying a lax

[4] It turns out that the only monoidal structure that is compatible with the creation of the category of spans is the cartesian monoidal structure, so we may assume without loss of generality that $\boxtimes$ is the cartesian product.

monoidal functor

$$(\mathrm{Span}(\mathcal{S}), \tilde{\times}) \to (\mathcal{C}\mathrm{at}_\infty, \times).$$

6.1 The lax monoidal Grothendieck construction

Our theory of symmetric monoidal ∞-categories exactly mirrors Lurie's [7], except that we will use cartesian rather than cocartesian fibrations.

Notation 6.1.1. Denote by $\rho_i \colon \langle n \rangle \to \langle 1 \rangle$ the map in $\mathcal{F}\mathrm{in}_*$ sending $i \mapsto 1$ and everything else to $*$.

Definition 6.1.2. A ***CSMC*** (contravariantly-presented symmetric monoidal ∞-category) is a cocartesian fibration $p \colon \mathcal{O}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ such that the pullback maps $\rho_i^* \colon (\mathcal{O}_\otimes)_{\langle n \rangle} \to (\mathcal{O}_\otimes)_{\langle 1 \rangle}$ are the canonical projections exhibiting $(\mathcal{O}_\otimes)_{\langle n \rangle}$ as an n-fold product.

Note 6.1.3. This means, among other things, that for any simplicial set K and diagrams $a_i \colon K \to (\mathcal{C}_\otimes)_{\langle 1 \rangle}$ for $1 \leq i \leq n$, we can find a diagram $a \colon K \to (\mathcal{C}_\otimes)_{\langle n \rangle}$ such that $\rho_i^*(a) = a_i$ for all i, and furthermore, that the space of all such a is contractible (and in particular any two are equivalent).

Definition 6.1.4. A ***map*** between CSMCs $\mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}^{\mathrm{op}}$ and $\mathcal{D}_\boxtimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is a functor r making the diagram

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

commute. Given a map of CSMCs as above, we will further make use of the following terminology.

- The map r is a ***monoidal functor*** if it sends q-cartesian morphisms to p-cartesian morphisms.

  This implies, for example, that $r(x \otimes y) \simeq r(x) \boxtimes r(y)$.

- An edge f in $(\mathcal{C}_\otimes)_{\langle n \rangle}$ is ***componentwise cartesian (resp. componentwise cocartesian)*** if for all inert $\rho_i \colon \langle 1 \rangle \leftarrow \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, $\rho_i^*(f)$ is $r|\langle 1 \rangle$-cartesian (resp. cocartesian).

  We can think of a morphism f in $(\mathcal{C}_\otimes)_{\langle n \rangle}$ as an n-tuple of morphisms f_i in $\mathcal{C}$. We say that f is componentwise (co)cartesian if each component f_i is (co)cartesian as an edge of the underlying fibration $\mathcal{C} \to \mathcal{D}$.

- The tensor product $\otimes$ ***preserves cartesian (resp. cocartesian) edges*** if for all $\phi \colon \langle n \rangle \leftarrow \langle m \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the associated functor $\phi^* \colon (\mathcal{C}_\otimes)_{\langle m \rangle} \to (\mathcal{C}_\otimes)_{\langle n \rangle}$ sends componentwise cartesian (resp. componentwise cocartesian) morphisms contained in the fiber $(\mathcal{C})_{\langle m \rangle}$ to componentwise cartesian (resp. componentwise cocartesian) morphisms contained in the fiber $(\mathcal{C}_{\otimes})_{\langle n \rangle}$.

We will show below that this is equivalent to demanding that if morphisms f and g in $\mathcal{C}$ are $r|\langle 1 \rangle$-(co)cartesian, then $f \otimes g$ is as well.

- The CSMC $q \colon \mathcal{C}_{\otimes} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is ***cartesian-compatible (resp. cocartesian-compatible)*** an edge f in the fiber $(\mathcal{C}_{\otimes})_{\langle n \rangle}$ is r-cartesian (resp. cocartesian) if and only if it is componentwise cartesian.

  This is a technical condition which is true for all CSMCs which we will consider; it says that the way we encode our monoidal category into a cartesian fibration is well-behaved with respect to the fibration r.

Lemma 6.1.5. Let

$$\begin{tikzcd} \mathcal{C}_{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_{\boxtimes} \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

be a map between CSMCs. The tensor product $\otimes$ preserves cartesian (resp. cocartesian) morphisms if and only if for all $r|\langle 1 \rangle$-cartesian (resp. $r|\langle 1 \rangle$-cocartesian) morphisms f and g in $\mathcal{C}$, the morphism $f \otimes g$ is also $r|\langle 1 \rangle$-cartesian (resp. $r|\langle 1 \rangle$-cocartesian).

Proof. We consider the cartesian case. The cocartesian case is identical.

Suppose that $\otimes$ preserves r-cartesian morphisms, and let f and g be $r|\langle 1 \rangle$-cartesian morphisms in $\mathcal{C}$. Then there is a morphism $[f, g]$ in $(\mathcal{C}_{\otimes})_{\langle 2 \rangle}$ with components f and g. Taking ϕ to be the active map $\langle 2 \rangle \to \langle 1 \rangle$ shows that $f \otimes g$ is $r|\langle 1 \rangle$-cartesian.

Now suppose that for any $r|\langle 1 \rangle$-cartesian morphisms f and g, the morphism $f \otimes g = \phi^*([f, g])$ in $\mathcal{C}$ is $r|\langle 1 \rangle$-cartesian. By associativity of the tensor product and induction, we have that the n-ary product of $r|\langle 1 \rangle$-cartesian morphisms is again $r|\langle 1 \rangle$-cartesian for $n \geq 2$. The image of the map α^*, for $\alpha \colon \langle 0 \rangle \to \langle 1 \rangle$, i.e. the 0-ary tensor product, is equivalent to id_I, where I is the unit object of $\mathcal{C}$, and is thus an equivalence, hence also $r|\langle 1 \rangle$-cartesian.

Now let $\psi \colon \langle m \rangle \to \langle n \rangle$ be a general map in $\mathcal{F}\mathrm{in}_*$, and let $f = [f_1, \ldots, f_m]$ be a componentwise cartesian map. The ith component of $\psi^*(f)$ is given by $\bigotimes_{\psi(j)=i} f_j$, which is $r|\langle 1 \rangle$-cartesian no matter the cardinality of $\psi^{-1}(i)$. Thus, $\psi^*(f)$ is r-cartesian. □

Lemma 6.1.6. Let $q \colon \mathcal{C}_{\otimes} \to \mathcal{F}\mathrm{in}^{\mathrm{op}}$ and $p \colon \mathcal{D}_{\otimes} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ be CSMCs as below, and let r be an inner fibration $\mathcal{C}_\otimes \to \mathcal{D}_\boxtimes$ such that the diagram

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

commutes. Then the following are equivalent.

1. The map r is a cartesian fibration.
2. The map r has the following properties.
   a) The restriction $r|\langle 1\rangle$ is a cartesian fibration.
   b) The map r is a monoidal functor.
   c) The tensor product $\otimes$ preserves cartesian edges.
   d) The CSMC q is cartesian-compatible.

Proof. Suppose that 1. holds, i.e. that r is a cartesian fibration. Then a) holds: $r|\langle 1\rangle$ is a cartesian fibration because the pullback of a cartesian fibration is again a cartesian fibration.

We now show that b) holds, i.e. that the map r sends q-cartesian morphisms to p-cartesian morphisms. To this end, let $f\colon c \to c' \in \mathcal{C}_\otimes$ be a q-cartesian morphism, and consider the image $p(f)\colon r(c) \to r(c')$ in $\mathcal{D}_\otimes$. Let $g\colon \tilde{d} \to r(c')$ be a p-cartesian lift of $q(f) \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, and $\hat{g}\colon \hat{d} \to c'$ a r-cartesian lift of g. By [8, Prop. 2.4.1.3], $\hat{g}$ is a q-cartesian lift of $q(f)$, so f and $\hat{g}$ are equivalent as morphisms in $\mathcal{C}_\otimes$. Thus $r(f)$ and g are equivalent as morphisms in $\mathcal{D}_\otimes$, so $r(f)$ is p-cartesian since g is. This proves b).

We now show that d) holds. We do this in two steps. First, we show that if some morphism f in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is r-cartesian, then for all $\phi\colon \langle n\rangle \to \langle m\rangle$, the morphism $\phi^*(f)$ is r-cartesian. This follows from the fact that for a square in $\mathcal{C}_\otimes$

$$\begin{tikzcd} c_0' \arrow[r, "\phi^*(f)"] \arrow[d, "u"'] & c_1' \arrow[d, "v"] \\ c_0 \arrow[r, "f"] & c_1 \end{tikzcd}$$

in which u and v are q-cartesian (and hence r-cartesian) and f is r-cartesian, $\phi^*(f)$ is also r-cartesian. Applying this result to the inert maps $\langle n\rangle \to \langle 1\rangle$ implies that if a morphism f in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is r-cartesian, then the components f_i are r-cartesian, hence also $r|\langle 1\rangle$-cartesian. This is one direction of d).

We now prove d) by showing the converse of the above statement: if a morphism in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is componentwise cartesian, then it is r-cartesian. To see this, let f_i be $r|\langle 1\rangle$-cartesian morphisms,

$1 \le i \le n$. We can find a morphism $f \in (\mathcal{C}_\otimes)_{\langle n\rangle}$ with components f_i. We would like to show that f is r-cartesian. To this end, consider the morphisms $r(f_i) = \overline{f}_i$. We can find a morphism $\overline{f}$ in $(\mathcal{D}_\boxtimes)_{\langle n\rangle}$ with components $\overline{f}_i$, and can take a r-cartesian lift $\hat{f}$ of $\overline{f}$. The components of $\hat{f}$ are equivalent to the f_i, and are hence r-cartesian. But then f is equivalent to $\hat{f}$, so f is r-cartesian as promised. This proves d).

We have now shown that under the assumption that 1. holds, a the pushforward maps preserve r-cartesian morphisms. This, together with d), proves c).

Now, suppose that 2. holds. We immediately note that c) implies that for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, the restriction $r|\langle n\rangle$ is a cartesian fibration, and an edge is $r|\langle n\rangle$-cartesian if and only if it is r-cartesian.

We now show that r admits cartesian lifts. Let $f\colon d \to d'$ be an edge in $\mathcal{D}_\boxtimes$ lying over an edge $\phi\colon \langle n\rangle \leftarrow \langle m\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, and let c' be a lift of d' to $\mathcal{C}_\otimes$. We can take a q-cartesian lift $g\colon c'' \to c$ of f, whose image in $\mathcal{D}_\boxtimes$ is by b) a p-cartesian map $h\colon d'' \to d'$. This gives us the solid data

$$\begin{tikzcd} & c'' \arrow[dr, "g"] & \\ c \arrow[ur, dashed, "k"] \arrow[rr, dashed, "\ell"'] & & c' \end{tikzcd}$$

$$\begin{tikzcd} & d'' \arrow[dr, "h"] & \\ d \arrow[ur, dashed, "j"] \arrow[rr, "f"'] & & d' \end{tikzcd} .$$

$$\begin{tikzcd} & \langle n\rangle & \\ \langle n\rangle \arrow[ur, equal] & & \langle m\rangle \arrow[ul, "\phi"'] \arrow[ll, "\phi"] \end{tikzcd}$$

Using the fact that h is p-cartesian, we can fill the 2-simplex in $\mathcal{D}_\boxtimes$ as above, giving us in particular a map $j\colon d \to d''$. We can lift j to an $r|\langle n\rangle$-cartesian morphism $k\colon c \to c''$, which is therefore also r-cartesian. Using that r is an inner fibration, we can compose g and k relative to the simplex in $\mathcal{D}_\boxtimes$, giving us a lift ℓ of f. But ℓ is the composition of two r-cartesian morphisms, and hence itself r-cartesian. ◻

Note 6.1.7. Lemma 6.1.6 gives us an ∞-categorical version of the monoidal Grothendieck construction of [9] discussed at the begininning of this section. There, a monoidal structure is modelled as tuple $(\mathcal{C}, \otimes, \dots)$. Because of the specific choice of a bifunctor $\otimes$ and the surrounding corresponding coherence data, strong conditions must be placed on the functor r: it must be monoidal 'on the nose.' Here, we model our monoidal structure as a cartesian fibration, leaving these choices unmade. The strong notion of monoidality is thus replaced by the usual one.

The connection between the above result and the monoidal Grothendieck construction, somewhat explicitly, is as follows. Lemma 6.1.6 tells us that the data of a functor $(\mathcal{C}, \otimes) \to (\mathcal{D}, \boxtimes)$ satisfying conditions analogous to those given in [9] is equivalent to a cartesian fibration between the corresponding CSMCs. A cartesian fibration r between CSMCs p and q as above straightens to a $\mathcal{D}_{\boxtimes}^{\mathrm{op}}$-monoid in $\mathcal{C}\mathrm{at}_\infty$. But a $\mathcal{D}_{\boxtimes}^{\mathrm{op}}$-monoid in $\mathcal{C}\mathrm{at}_\infty$ can be essentially uniquely extended to an $\mathcal{D}_{\boxtimes}^{\mathrm{op}}$-algebra object in $\mathcal{C}\mathrm{at}_{\infty,\times}$ [7, Prop 2.4.2.4–2.4.2.6], which is to say, a lax monoidal functor $(\mathcal{D}^{\mathrm{op}}, \boxtimes) \to (\mathcal{C}\mathrm{at}_\infty, \times)$.

In the remainder of this section, we study a special type of CSMC, those whose tensor product comes from the cartesian product.

Definition 6.1.8. A ***CSMC*** $p\colon \mathcal{D}_{\boxtimes} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is ***cartesian*** if for all objects X and Y in $\mathcal{D}$, the canonical maps $X \leftarrow X \boxtimes Y \to Y$ exhibit $X \boxtimes Y$ as the product of X and Y.

Example 6.1.9. Let $\mathcal{T}$ be a category with finite products. Then $\mathcal{T}^{\mathrm{op}}$ admits finite coproducts, and we can consider the cocartesian monoidal structure

$$(\mathcal{T}^{\mathrm{op}})^{\amalg} \to \mathcal{F}\mathrm{in}_*$$

of [7, Construction 2.4.3.1]. Taking the opposite of this functor gives us a cartesian fibration

$$\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}.$$

This is a cartesian CSMC; the map $(\mathcal{T}_\times)_{\langle n \rangle} \to (\mathcal{T}_\times)_{\langle 1 \rangle}^n$ coming from the pullbacks ρ_i^* is even an isomorphism.

In any category $\mathcal{T}$ with pullbacks, one can form a category $\mathrm{Span}(\mathcal{T})$ of spans in $\mathcal{T}$. If $\mathcal{T}$ also has a terminal object, hence all finite limits (including, of course, finite products), then the category of spans inherits a monoidal structure via

$$\left(\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array} \right) \otimes \left(\begin{array}{ccc} & Z' & \\ \swarrow & & \searrow \\ X' & & Y' \end{array} \right) = \begin{array}{ccc} & Z \times Z' & \\ \swarrow & & \searrow \\ X \times X' & & Y \times Y' \end{array}$$

We now construct the monoidal structure on this category of spans explicitly, starting from any cartesian CSMC $\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$.

Proposition 6.1.10. Suppose $\mathcal{T}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ is a cartesian CSMC whose underlying category $\mathcal{T}$ admits pullbacks. Then there exists a (cocartesian-presented) symmetric monoidal category $\mathrm{Span}(\mathcal{T})^\times \to \mathcal{F}\mathrm{in}_*$.

Proof. We upgrade this functor to a functor of triples. We will consider the following triples.

- We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where
  - $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
  - $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
  - $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

  This is obviously adequate.
- We define a triple $(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$ as follows.
  - $\mathcal{T} = \mathcal{T}_\times$
  - $\mathcal{T}_\dagger = \mathcal{T}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
  - $\mathcal{T}^\dagger = \mathcal{T}_\times$

  To see that this is adequate, we first note that p admits relative pullbacks. To see this, note that each fiber admits pullbacks by virtue our assumption that $\mathcal{T}$ admits pullbacks, and the identifications $(\mathcal{T}_\times)_{\langle n\rangle} \simeq \mathcal{T}^n$. The functoriality coming from p implements products, and therefore commute with pullbacks. Thus, $\mathcal{T}_\times$ admits pullbacks, and p preserves pullbacks. This immediately implies that $(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$ is adequate:

  1. Pullbacks of this form are simply squares with horizontal morphisms given by equivalences.
  2. Any pullback square lies over a pullback square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, so this reduces to the lemma about cartesian morphisms.

  We thus consider the map of triples

  $$\pi\colon (\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \to (\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger).$$

  One readily checks that π satisfies the conditions of Theorem 2.0.1:

  - The first condition holds because cocartesian morphisms lying over equivalences are themselves equivalences, hence also cartesian, so we can solve the relevant lifting problems using cartesian lifts.
  - The second condition holds because a square whose bottom-horizontal morphism is an equivalence is pullback if and only if its top-horizontal morphism is an equivalence.

This gives us a functor $\mathrm{Span}(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \to \mathrm{Span}(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger)$. Pulling back along the equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger)$ gives us the cartesian fibration functor $\mathrm{Span}(\mathcal{T})_\times \to \mathcal{F}\mathrm{in}_*$. □

6.2 Monoidal Beck-Chevalley fibrations

Note 6.2.1. For this to work out, I think we probably need to modify the second condition to be an adequate triple so that we only need to consider pullbacks of the type given in the first condition, i.e:

For any pullback square

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, "g'"'] & x' \arrow[d, "g"] \\ y \arrow[r, "f"] & x \end{tikzcd}$$

where $f \in \mathcal{C}_\dagger$ and $g \in \mathcal{C}^\dagger$, we have that $f' \in \mathcal{C}_\dagger$ and $g' \in \mathcal{C}^\dagger$.

This means we also need to add the condition:

For any square

$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, "g'"'] & x' \arrow[d, "g"] \\ y \arrow[r, "f"] & x \end{tikzcd}$$

where g and g' are equivalences, if $f \in \mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$), then $f' \in \mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$). Wait, no, we don't. The subcategories $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$ contain all equivalences by assumption. Barwick, why would you make your conditions like this?

Lemma 6.2.2. Let p and q be CSMCs as below. Then for monoidal functor r, the CSMC q is cocartesian-compatible.

$$\begin{tikzcd} \mathcal{C}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{D}_\boxtimes \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

Proof. We need to show that f in $(\mathcal{C}_\otimes)_{\langle n \rangle}$ is r-cocartesian if and only if it is componentwise cocartesian.

Consider the following diagram of pullback squares.

$$\begin{tikzcd} (\mathcal{C}_\otimes)_{\langle n \rangle} \arrow[r] \arrow[d, "r|\langle n \rangle"'] & \mathcal{C}_\otimes \arrow[d, "r"] \\ (\mathcal{D}_\otimes)_{\langle n \rangle} \arrow[r] \arrow[d, "p|\langle n \rangle"'] & \mathcal{D}_\otimes \arrow[d, "p"] \\ \{\langle n \rangle\} \arrow[r] & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \end{tikzcd}$$

It follows from the dual to [8, Cor. 4.3.1.15][5] that a morphism in $(\mathcal{C}_\otimes)_{\langle n\rangle}$ is $r|\langle n\rangle$-cocartesian if and only if its image in $\mathcal{C}_\otimes$ is r-cocartesian. But a morphism is $r|\langle n\rangle$-cocartesian if and only if each component is $r|\langle 1\rangle$-cocartesian. □

Definition 6.2.3. A ***monoidal Beck-Chevalley fibration*** is a functor r of CSMCs

$$\begin{tikzcd} \mathcal{X}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{T}_\times \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd},$$

where $\mathcal{T}_\times$ is a cartesian CSMC and $\mathcal{X}_\otimes$ is cartesian-compatible, with the following characteristics.

(M1) The map $r|\langle 1\rangle$ is a Beck-Chevalley fibration.

(M2) The map r is monoidal.

(M3) The tensor product $\otimes$ preserves r-cartesian morphisms.

(M4) The tensor product $\otimes$ preserves r-cocartesian morphisms.

(M5) The CSMC q is cartesian-compatible.[6]

The above conditions are more or less to do with the properties of the functor $r|\langle 1\rangle$, together with properties of the tensor product $\otimes$, and how it is encoded as a cartesian fibration. We can also express these properties in terms directly in terms of the fibrations p, q, and r.

Lemma 6.2.4. Consider a functor r of CSMCs, where $\mathcal{T}_\times$ is a cartesian CSMC.

$$\begin{tikzcd} \mathcal{X}_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{T}_\times \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

The following are equivalent.

1. The map r is a monoidal Beck-Chevalley fibration.
2. The map r has the following properties.
   a) The category $\mathcal{T}$ underlying $\mathcal{T}_\times$ admits pullbacks.
   b) The map r is a cartesian fibration.
   c) The functor $r|\langle 1\rangle$ is a cocartesian fibration.

[5]Note an unfortunate notational clash: our maps p, q, and r do not agree with Lurie's.
[6]By Lemma 6.2.2, q is automatically cocartesian-compatible, so we mustn't add this as a separate condition.

d) The functor r obeys the following interchange law: for any diagram

$$\begin{tikzcd}
\vec{z} \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{y} \arrow[d, "g"] \\
\vec{y}' \arrow[r, "f"] & \vec{x}
\end{tikzcd} \tag{20}$$

in $\mathcal{X}_\otimes$ whose image in $\mathcal{T}_\boxtimes$ is pullback, and which lies over a square

$$\begin{tikzcd}
\langle n \rangle & \langle n \rangle \arrow[l, "\alpha"', "\simeq"] \\
\langle m \rangle \arrow[u, "\phi"] & \langle m \rangle \arrow[l, "\beta"', "\simeq"] \arrow[u, "\psi"']
\end{tikzcd} , \tag{21}$$

in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ such that α and β are equivalences, if g and g' are r-cartesian and f is r-cocartesian, then f' is r-cocartesian.

Proof. That 1. implies a) and c) is clear, and b) follows from Lemma 6.1.6.

We now show 1. implies d). Consider the diagram

$$\begin{tikzcd}
\langle n \rangle \arrow[rr, equal] \arrow[dd, equal] \arrow[dr, equal] & & \langle n \rangle \arrow[dd, equal] & \\
& \langle n \rangle & & \langle n \rangle \arrow[ll, "\alpha"'] \arrow[ul, "\alpha"'] \\
\langle n \rangle \arrow[rr, equal] & & \langle n \rangle & \\
& \langle m \rangle \arrow[uu, "\phi"] \arrow[ul, "\phi"] & & \langle m \rangle \arrow[ll, "\beta"] \arrow[uu, "\psi"'] \arrow[ul, "\phi\circ\beta"']
\end{tikzcd} \quad \text{in } \mathcal{F}\mathrm{in}_*^{\mathrm{op}},$$

which we should think of as a natural transformation from Diagram 21 to a constant diagram.

Beginning with the solid diagram below coming from Diagram 20, we can find q-cartesian lifts of the diagonal arrows. Filling we find a dashed cube

$$\begin{tikzcd}
\vec{u} \arrow[rr, dashed, "h'"] \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & & \vec{v} \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & \\
& \vec{x} \arrow[rr, "f'"] \arrow[dd] & & \vec{y} \arrow[dd] \\
\vec{v}' \arrow[rr, dashed, "h"] \arrow[dr, dashed] & & \vec{w} \arrow[dr, dashed] & \\
& \vec{y}' \arrow[rr, "f"] & & \vec{z}
\end{tikzcd} \quad \text{in } \mathcal{X}_{\otimes}$$

lying over the cube in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, whose diagonal arrows are q-cartesian, and with equivalences as marked. Mapping this cube down to $\mathcal{T}_{\times}$, we find a cube

$$\begin{tikzcd}
r(\vec{u}) \arrow[rr, dashed] \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & & r(\vec{v}) \arrow[dr, dashed, "\simeq"] \arrow[dd, dashed] & \\
& r(\vec{x}) \arrow[rr] \arrow[dd] & & r(\vec{y}) \arrow[dd] \\
r(\vec{v}') \arrow[rr, dashed] \arrow[dr, dashed] & & r(\vec{w}) \arrow[dr, dashed] & \\
& r(\vec{y}') \arrow[rr] & & r(\vec{z})
\end{tikzcd} \quad \text{in } \mathcal{T}_{\times}$$

lying over the cube in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, whose diagonal morphisms are p-cartesian, and whose front face is pullback by assumption. The bottom face is pullback since it lies over a pullback square in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ and the diagonal morphsims are p-cartesian, and the top face is pullback because the diagonal morphisms are equivalences. The cube lemma thus implies that the back square is pullback. Note that the back square is entirely contained in the fiber over $\langle n \rangle$, and thus can be thought of as consisting of n component squares in $\mathcal{T}$; by [8, Prop. 4.3.1.10] these component squares are themselves pullback.

We now return our attention to the diagram in $\mathcal{X}_{\otimes}$. The morphism f is r-cocartesian by assumption. That the tensor product $\otimes$ preserves r-cocartesian morphisms implies that h is r-cocartesian. Applying the Beck-Chevalley condition componentwise to the back face yields that h' is componentwise cocartesian, hence r-cocartesian. Since f' is equivalent to h', f' is also r-cocartesian. Thus, d) holds.

We now show that 2. implies 1. Restricting each of the conditions of 2. to the fibers over $\langle 1 \rangle$ immediately implies that $r|\langle 1 \rangle$ is a Beck-Chevalley fibration, and b) implies that r is monoidal, and that q is cartesian-compatible. It remains to show that the tensor product $\otimes$ preserves

r-cartesian and r-cocartesian morphisms. To this end, consider a square

$$\sigma = \begin{tikzcd} \vec{x} \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{y} \arrow[d, "g"] \\ \vec{y}' \arrow[r, "f"] & \vec{z} \end{tikzcd} \qquad \text{in } \mathcal{X}_{\otimes}$$

such that g and g' are q-cartesian, lying over a square

$$\begin{tikzcd} \langle n \rangle \arrow[r, equal] & \langle n \rangle \\ \langle m \rangle \arrow[r, equal] \arrow[u, "\phi"] & \langle m \rangle \arrow[u, "\phi"'] \end{tikzcd} \qquad \text{in } \mathcal{F}\mathrm{in}_*^{\mathrm{op}}. \tag{22}$$

Note that since r is monoidal, the image of σ in $\mathcal{T}_{\times}$ is automatically pullback. We now note that if f is r-cartesian, then f' is r-cartesian by [8, Prop. 2.4.1.7], and if f is r-cocartesian, then f' is r-cocartesian (hence $r|\langle 1 \rangle$-cocartesian) by the interchange law. □

Lemma 6.2.5. The category $\mathcal{F}\mathrm{in}_*$ admits finite limits and colimits.

Proof. Denote by $\mathcal{F}\mathrm{in}$ the category of finite sets.

- A finite diagram $j\colon K^{\triangleleft} \to \mathcal{F}\mathrm{in}_*$ is a limit diagram if and only if the restriction $j|K$ is a limit diagram.
- A diagram $K^{\triangleright} \to \mathcal{F}\mathrm{in}_*$ is a colimit diagram if and only if the corresponding diagram $\{*\} \star (K^{\triangleright}) \to \mathcal{F}\mathrm{in}$ is a colimit diagram.

The result then follows from the fact that the category $\mathcal{F}\mathrm{in}$ admits all finite limits and colimits. □

Proposition 6.2.6. For any monoidal Beck-Chevalley fibration, there is a diagram

$$\begin{tikzcd} \mathrm{Span}'(\mathcal{X})^{\otimes} \arrow[rr, "\rho"] \arrow[dr, "\varpi"'] & & \mathrm{Span}(\mathcal{T})^{\otimes} \arrow[dl, "\pi"] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd}$$

where π and ϖ are symmetric monoidal categories and ρ exhibits $\mathrm{Span}'(\mathcal{X})^{\otimes}$ as a $\mathrm{Span}(\mathcal{T})^{\otimes}$-monoidal category. Straightening, one finds a lax monoidal functor

$$\hat{r}\colon (\mathrm{Span}(\mathcal{T}), \widetilde{\times}) \to (\mathcal{C}\mathrm{at}_{\infty}, \times)$$

with the following description up to equivalence.

- On objects, the functor $\hat{r}$ sends $t \in \mathrm{Span}(\mathcal{T})$ to the fiber $\mathcal{X}_t \in \mathcal{C}\mathrm{at}_\infty$
- On morphisms, $\hat{r}$ sends a span $t \xleftarrow{g} s \xrightarrow{f} t'$ to the composition $g^* \circ f_! \colon \mathcal{X}_t \to \mathcal{X}_{t'}$.

Proof. We define a triple structure $(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger)$ as follows.

- $\mathcal{X} = \mathcal{X}_\otimes$.
- $\mathcal{X}_\dagger = (\mathcal{X}_\otimes)_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}}(\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$.
- $\mathcal{X}^\dagger$ consists only of r-cartesian morphisms.

One sees that this is adequate, since pullbacks of the necessary form exist by the usual procedure:

- Map the diagram down to $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, take the pullback there.
- Take a relative pullback in $\mathcal{T}_\times$.
- Take an r-cartesian lift to produce an r-relative pullback in $\mathcal{X}_\times$. This lies over a pullback in $\mathcal{T}_\times$, hence is a pullback.

Thus, we have a maps of adequate triples

$$\rho \colon (\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \to (\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger).$$

giving us a map

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{X}, \mathcal{X}_\dagger, \mathcal{X}^\dagger) \arrow[rr, "\rho"] \arrow[dr, "\pi\circ\rho"'] & & \mathrm{Span}(\mathcal{T}, \mathcal{T}_\dagger, \mathcal{T}^\dagger) \arrow[dl, "\pi"] \\
& \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) &
\end{tikzcd}.$$

One readily checks that π satisfies the conditions of Theorem 2.0.1:

- The first condition holds because cocartesian morphisms lying over equivalences are themselves equivalences.
- The second condition holds because a square whose bottom-horizontal morphism is an equivalence is pullback if and only if its top-horizontal morphism is an equivalence.

Morphisms of the form

We thus need to check that the map ρ satisfies the conditions of Theorem 2.0.2. Again, this follows from Lemma 6.2.4.

□

6.3 The monoidal twisted arrow category

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I) \colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^I \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i \end{tikzcd}$$

commutes.

Example 6.3.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 6.3.2. A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[7] of

- A 'source' object $F \colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i \colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.

[7]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1 \times G_2"] \arrow[l, phantom, "\Longrightarrow"] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd}.$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 6.3.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^{\mathrm{op}} \end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0)) \end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. ◻

Note that for each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n \to (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n.$$

Because the pointwise product of any number of thin 1-simplices in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$, the conditions of Lemma 6.3.3 are satisfied. Unrolling, we find that $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}{}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}{}_\infty^\times$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}{}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{23}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 6.3.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 23, we have the following.

Proposition 6.3.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \\
\vec{\mathcal{C}} \arrow[ur, phantom, "\Longrightarrow"] & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.

- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_\otimes$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

6.4 The triple structures

6.4.1 The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 6.4.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 6.4.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

6.4.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 6.4.3. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 6.4.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd}
\langle n \rangle & \langle m \rangle \arrow[l] \\
\langle n' \rangle \arrow[u, "\phi"] & \langle m' \rangle \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) □

6.4.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t \circ \mathcal{F}"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^* t\mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd}.$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 6.4.5. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_\otimes$ of the form

$$\begin{tikzcd} y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd},$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

6.5 Building categories of spans

Proposition 6.5.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 6.5.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd} \langle n \rangle & \langle n \rangle \arrow[l, "\gamma'"'] \\ \langle m \rangle \arrow[u, "\tau'"] & \langle m \rangle \arrow[l, "\gamma"] \arrow[u, "\tau"'] \end{tikzcd},$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i \colon X_i \to Y_{\gamma^{-1}(i)}, \qquad i \in \langle m \rangle^\circ,$$

f' has components

$$f'_j : X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n \rangle^\circ,$$

g has components

$$g_j : Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n \rangle^\circ,$$

and g' has components

$$g'_j : X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n \rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{\left(\prod_{\tau'(i)=j} Y_i\right)} \left(\prod_{\tau'(i)=j} X_i \right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'"] & \\
& \vec{X} \arrow[rr, "f"] \arrow[dd, "\mathcal{F}"] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes"] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"']
\end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^\circ$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathbf{LS}(\mathcal{C})_\otimes$

above to a pullback square entirely in the fiber over $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathbf{LS}(\mathcal{C})_\otimes)_{\langle n\rangle} \cong (\mathbf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathbf{LS}(\mathcal{C})$, one for each $j \in \langle n\rangle^\circ$, of the form

$$\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}" near end] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr] \arrow[dd, "\mathcal{A}" near start] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd},$$

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i\colon X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

7 Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl] \\
& \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) &
\end{tikzcd}.$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, giving us isofibrations

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\
& \mathcal{F}\mathrm{in}_* &
\end{tikzcd},$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an ${}^{\mathrm{op}}$. Composing with ${}^{\mathrm{op}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.

8 Miscellaneous stuff to be deleted or absorbed

9 What does Barwick mean?

A few bits of terminology we'll refer to again and again:

- An ∞-category is ***disjunctive*** if it admits all limits and finite coproducts, and if finite coproducts are:
  - **Disjoint:** Pushout diagrams

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ B \arrow[r] & A \amalg B \end{tikzcd}$$

    where $\emptyset$ is an initial object are also pullback.
  - **Universal:** We have natural equivalences

$$(X \amalg Y) \times_B C \cong (X \times_B C) \amalg (Y \times_B C)$$

  This looks like topos stuff. It is certainly satisfied for $\mathcal{S}$.

In section 1, Barwick defines some monadic generalization of CSMCs. In the first part of section 2, Barwick shows in some generality that for any ∞-category $\mathcal{T}$ with some cartesian-like product, a monoidal structure is induced on $\mathrm{Span}(\mathcal{T})$.

9.1 The triple structures

9.1.1 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.

- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[dr, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_! g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 9.1.1. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 9.1.2. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\alpha"] & & b \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\beta"] & \\
& b' \arrow[rr] \arrow[dd] \arrow[r, phantom, "\Longrightarrow"] & & c \arrow[dd] \\
\bar{a} & & \bar{b} \arrow[ll] \arrow[l, phantom, "\Longrightarrow"] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$.[8] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout.

[8]Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute

Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 9.1.1. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 9.1.2

We also need to show that any such pullback is of this form. This also follows from Lemma 9.1.2. □

9.2 Building categories of spans

Lemma 9.2.1. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{24}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y\times Y} (X \times Y').$$

The left-hand side of Equation 24 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y\times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y\times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. □

Proposition 9.2.2. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

---

by definition.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\
& X \arrow[rr, "f", near start] \arrow[dd] \arrow[d, phantom, "\circlearrowleft"'] \arrow[r, phantom, "\Rightarrow"] & & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & \arrow[r, phantom, "\Rightarrow"] & \mathcal{C} \arrow[dr, equal] & \arrow[u, phantom, "\circlearrowleft"] \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 9.2.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

10 Monoidal categories as cartesian fibrations

Definition 10.0.1 (contravariantly presented monoidal category)**.** A ***contravariantly presented monoidal category*** is a cartesian fibration

$$\mathcal{C}_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$$

such that the pullback maps lying over the inert maps $\langle 1 \rangle \leftarrow \langle n \rangle$ exhibit $(\mathcal{C}_\otimes)_{\langle n \rangle}$ as the n-fold product of $(\mathcal{C}_\otimes)_{\langle 1 \rangle}$.

Example 10.0.2. The ∞-category $\mathcal{C}\mathrm{at}_\infty$ has a cartesian product, giving us a monoid

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