Monoidal push-pull for local systems

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Contents

1 Introduction and conventions

1.1 General conventions

By $\mathcal{S}$, we mean the ∞-category of *small* spaces.

1.2 The marked-scaled model structure

One common model for $(\infty, 2)$-categories is Lurie's theory of ∞-bicategories

In [3], a marked-biscaled model structure modelling cartesian fibrations of ∞-bicategories is defined. When taken over a point, the two scalings collapse into one, giving the following.

Definition 1.2.1. A marked-scaled simplicial set is a triple (X, E_X, T_X), where $E_X \subseteq X_1$ is a collection of edges of X containing all degenerate edges, and $T_X \subseteq X_2$ is a collection of triangles of X containing all degenerate triangles.

Notation 1.2.2.

- We will denote the category of marked-scaled simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}$.
- To save on notation, we will sometimes denote the marked-scaled simplicial set (X, E_X, T_X) by $X_{T_X}^{E_X}$, particularly in the case that E_X or T_X are $\sharp$ or $\flat$, the maximum (resp. minimum) markings and scalings. For example, the bimarked simplicial set $X_\flat^\sharp$ has all 1-simplices marked and only degenerate 2-simplices scaled, etc.

We will use the following abuse of notation freely.

Notation 1.2.3. Let (Δ^n, E, T) be a marked-scaled n-simplex. For any simplicial subset $S \subseteq \Delta^n$, we will denote by (S, E, T) the simplicial subset S together with the inherited marking and scaling.

Notation 1.2.4. Given a cosimplicial object $Q\colon \Delta \to \mathcal{C}$, we will write $Q(n)$ instead of $Q([n])$.

Definition 1.2.5. The set of ms-anodyne morphisms is a saturated set of morphisms between marked-scaled simplicial sets containing the following classes of morphisms:

(A1) Inner horn inclusions

$$(\Lambda^n_i, \flat, \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}),$$

for $n \geq 2$ and $0 < i < n$.

(A2) Outer horn inclusions

$$(\Lambda^n_n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}),$$

for $n \geq 1$.

Theorem 1.2.6. There is a model structure on the category $\mathcal{S}\mathrm{et}_\Delta^{\mathsf{MS}}$, whose trivial cofibrations are given by the ms-anodyne maps, and whose fibrant objects are ∞-bicategories with precisely the equivalences marked.

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. A morphism in $\mathcal{C}$ is egressive if and only if it is p-cartesian (hence also $p^\dagger$-cartesian).
2. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.
3. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ with egressive and ingressive morphisms as marked, in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Proof. Even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. □

3 Horn filling via left Kan extensions

Kan extensions of ∞-categories are usually defined pointwise via the colimit formula [5] [2]. In this section, we will show that such pointwise ∞-Kan extensions enjoy a horn filling property in $\mathbb{C}\mathrm{at}_\infty$ which generalizes the universal property for Kan extensions of functors between 1-categories.

We will expand upon the following definition in Subsection 3.1.

Definition 3.0.1. We will say that a 2-simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$,

$$\tau = \begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{C} \\ & Y \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}, \tag{1}$$

is ***left Kan*** if it satisfies the following conditions.

- For all objects $y \in Y$, the category $\mathcal{C}$ admits all colimits of shape $X_{/y}$.
- The functor G is a left Kan extension of F along f.
- The natural transformation η is a unit map.

Example 3.0.2. Any commuting simplex $\Delta^2_\sharp \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{0,1\}}$ is mapped to an equivalence is left Kan.

The goal of this section is to prove the following.

Theorem 3.0.3. Let $\tau\colon \Delta^2 \to \mathbb{C}\mathrm{at}_\infty$ be a left Kan 2-simplex. Then for each $n \geq 2$, every solid lifting problem of the form

$$\begin{tikzcd}
\Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\
(\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\
\Delta_\flat^{n+1} \arrow[ur, dashed] &
\end{tikzcd} \tag{2}$$

admits a dashed solution.

Let us examine this in the case $n = 2$. In this case we have the data of categories and functors

$$\begin{tikzcd}
 & Y \arrow[ddr, "g"] \arrow[drr, "f_!F"] & & \\
X \arrow[ur, "f"] \arrow[rrr, "F"] \arrow[dr, "h"'] & & & \mathcal{C} \\
 & & Z \arrow[ur, "G"'] &
\end{tikzcd}$$

together with natural transformations $\eta\colon F \Rightarrow f_!F \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow G \circ h$. We need to produce a natural transformation $\alpha\colon f_!F \Rightarrow G \circ g$ and a filling of the full 3-simplex, which is the data of a homomotopy-commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \arrow[d, "\alpha f"] \\
G \circ h \arrow[r, "G\beta"] & G \circ g \circ f
\end{tikzcd}$$

in $\mathrm{Fun}(X, \mathcal{C})$.[1]

[1] In particular, if we take $h = f$, $g = \mathrm{id}_Y$, and β to be the identity $f \Rightarrow f$, we recover the classical universal property satisfied by left Kan extension.

We can rephrase this as follows. We are given the data

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

where the left-hand diagram is a map $LC^2 \to \mathrm{Fun}(X, C)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ except the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathrm{Fun}(Y, C)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

1. We first can fill the lower-left half of the diagram on the left simply by taking the composition $G\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})} .$$

2. We note that we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left under the adjunction $f_! \dashv f^*$:

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

3. The diagram on the right has an obvious filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \arrow[d, "\alpha f"] \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \arrow[d, "\alpha"] \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

The lifting problems which we have to solve in Equation 2 for $n > 2$ amount to replacing $\Delta^1 \times \Delta^1$

by $(\Delta^1)^n$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved. In Subsection 3.2 we explain the combinatorics of filling cubes relative to their boundaries. In Subsection 3.3, we give a formalization the concept of adjunct data, and provide a means of for filling partial data to total adjunct data. In Subsection 3.4, we show that we can always solve the lifting problems of Equation 2.

3.1 Basic facts about Kan extensions

In this section, we recall a few basic facts about Kan extensions. These can be found in Kerodon. Since Kan extensions are defined by the colimit formula, we will start by defining colimits.

Definition 3.1.1. Let X, Y, and $\mathcal{C}$ be ∞-categories, $f\colon X \to Y$, $F\colon X \to \mathcal{C}$ and $G\colon Y \to \mathcal{C}$ functors, and $\eta\colon F \to G \circ f$ a natural transformation.

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow \eta"] & \mathcal{C} \\ & Y \arrow[ur, "G"'] & \end{tikzcd}$$

We say that η ***exhibits*** G ***as the left Kan extension of*** F ***along*** f if for each object $x \in X$,

3.2 Filling cubes relative to their boundaries

In this section, we give the combinatorics of filling the n-cube $(\Delta^1)^n$ relative to its boundary. Our main result is Proposition 3.2.21, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map.

Definition 3.2.1. The n***-cube*** is the simplicial set $C^n := (\Delta^1)^n$.

Note 3.2.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Our first step is to understand the nondegenerate simplices of the n-cube.

Definition 3.2.3. Denote by $a^i\colon \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\ldots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\ldots,n\}}$ to $\{1\}$.

Note 3.2.4. The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0, \ldots, 0)$ and ending at $(1, \ldots, 1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1, \ldots, n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition 3.2.5. For any permutation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, we define an n-simplex

$$\phi(\tau)\colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)}).$$

Note 3.2.6. It is easy to check that the above definition really results in a bijection between S_n and the nondegenerate simplices of C^n.

Notation 3.2.7. We will denote any permutatation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ by the corresponding n-tuple $(\tau(1), \ldots, \tau(n))$. Thus, the identity permutation is $(1, \ldots, n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2, 1, 3, \ldots, n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)})\colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i\colon \Delta^n \to \Delta^1$ and any face map $\partial_j\colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\ldots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)}, \ldots, a^{\tau(n)}) := (\partial_i^* a^{\tau(1)}, \ldots, \partial_i^* a^{\tau(n)}).$$

Example 3.2.8. Consider the 3-simplex (a^1, a^2, a^3) in C^3. This has spine

$$(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.

- The function $\Delta^3 \to \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation 3.2, we then calculate that

$$d_2(a^1, a^2, a^3) = (a^1, a^2, a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0, 0, 0) \to (1, 0, 0) \to (1, 1, 1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition 3.2.9. The ***boundary of the*** n***-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{3}$$

The following is easy to see.

Proposition 3.2.10. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \ldots, \gamma_n) \colon \Delta^{n-1} \to C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition 3.2.11. The ***left box of the*** n***-cube***, denoted LC^n, is the simplicial subset of C^n given by the union of all of the faces in Equation 3 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example 3.2.12. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma 3.2.13. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

1. The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.
2. For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.
3. The nth face $d_n\phi$ always belongs to LC^n.

Proof.

1. We have
$$d_0\phi = (a^{\tau(0)-1}, \dots a^{\tau(n)-1})$$
since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition 3.2.10 guarantees that $d_0\phi$ is contained in the face
$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$
This face belongs to LC^n except when $j = n$.

2. In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.

3. In this case,
$$d_n(a^{\tau(1)}, \dots, a^{\tau(n)}) = (a^{\tau(1)}, \dots, a^{\tau(n)})$$
since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face
$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation 3.2.14. For any subset $T \subseteq [n]$, write

$$\Lambda^n_T := \bigcup_{t \in T} d_t \Delta^n \subset \Delta^n.$$

Lemma 3.2.15. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n - 1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \dots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma 3.2.15 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition 3.2.16. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) < \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example 3.2.17. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation 3.2.18. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$\begin{aligned} LC^n &\hookrightarrow LC^n \cup \phi(1,\dots,n) \\ &\hookrightarrow \cdots \\ &\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\dots,n,1)} \phi(\tau) \end{aligned} \tag{4}$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2,\dots,n,1)$, the intersection

$$B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{5}$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition 3.2.19. Each inclusion in Equation 4 is inner anodyne.

Proof. The first inclusion

$$LC^n \hookrightarrow LC^n \cup \phi(0,\dots,n)$$

is inner anodyne because by Lemma 3.2.13 the intersection $LC^n \cap \phi(0,\dots,n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma 3.2.13), so by Lemma 3.2.15, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (0,\dots,n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),$$

which implies that $\tau = (1,\dots,n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. $\square$

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$. Thus, each of their spines begins

$$(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1) \longrightarrow \cdots,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that $(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition 3.2.19. One continues to fill simplices in the order prescribed in Definition 3.2.16, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma 3.2.15.

Lemma 3.2.20. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition 3.2.21. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \xhookrightarrow{i} J^n \xhookrightarrow{j} C^n,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd}
\Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\
J^n \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & C^n
\end{tikzcd},$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0, \ldots, 0) \longrightarrow (0, \ldots 1)$.

3.3 Adjunct data

In this section, we give a formalization of the the notion of adjunct data. Our main result is Proposition 3.3.11, which shows that under certain conditions, we can use data on one side of an adjunction to solve lifting problems on the other side. First, we recall somewhat explicitly the definition of an adjunction given in [5].

Definition 3.3.1. An ***adjunction*** is a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$. We say that p is ***associated*** to functors $f\colon \mathcal{C} \to \mathcal{D}$ and $g\colon \mathcal{D} \to \mathcal{C}$ if there exist equivalences $h_0\colon \mathcal{C} \to \mathcal{M}_0$ and $h_1\colon \mathcal{D} \to \mathcal{M}_1$, and a commutative diagram

$$\begin{tikzcd} \mathcal{C} \times \Delta^1 \arrow[r, "u"] \arrow[dr, "\mathrm{pr}"'] & \mathcal{M} \arrow[d, "p"] & \mathcal{D} \times \Delta^1 \arrow[l, "v"'] \arrow[dl, "\mathrm{pr}"] \\ & \Delta^1 & \end{tikzcd}$$

such that the following conditions are satisfied.

- The map u is associated to the functor f: the restriction $u|\mathcal{C} \times \{0\} = h_0$, the restriction $u|\mathcal{C} \times \{1\} = f \circ h_1$, and for each $c \in \mathcal{C}$, the edge $u|\{c\} \times \Delta^1$ is p-cocartesian.
- The map v is associated to the functor g: the restriction $v|\mathcal{D} \times \{0\} = g \circ h_0$, the restriction $u|\mathcal{D} \times \{1\} = h_1$, and for each $d \in \mathcal{D}$, the edge $u|\{d\} \times \Delta^1$ is p-cartesian.

If f and g are functors to which an adjunction is associated as above, then we say that f is left adjoint to g, and equivalently that g is right adjoint to f.

Note 3.3.2. This terminology is slightly imprecise; it would be more correct to say that p is associated to f and g via the data (h_0, h_1, u, v).

Note 3.3.3. It is shown in [5] that if f and g are functors such that f is left adjoint to g, then we can choose an adjunction $p\colon \mathcal{M} \to \Delta^1$ and data (h_0, h_1, u, v) such that h_0 and h_1 are isomorphisms. We can use this to identify $\mathcal{M}_0$ with $\mathcal{C}$, and $\mathcal{M}_1$ with $\mathcal{D}$. In what follows we will always assume that we have chosen such data, and will thus leave the isomorphisms h_0 and h_1 implicit.

Definition 3.3.4. Let $s\colon K \to \Delta^1$ be a map of simplicial sets whose fibers we denote by K_0 and K_1, and let $p\colon \mathcal{M} \to \Delta^1$ be a bicartesian fibration associated to adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via data (u, v). We say that a map $\alpha\colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K\times\{0\}} = \alpha$.
2. The restriction $A|_{K\times\{1\}} = \tilde{\alpha}$.
3. The restriction $A|K_0 \times \Delta^1$ is equal to the composition

$$K_0 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\alpha\circ\times\mathrm{id}} \mathcal{C} \times \Delta^1 \xrightarrow{u} \mathcal{M}.$$

4. The restriction $A|K_1 \times \Delta^1$ is equal to the composition

$$K_1 \times \Delta^1 \hookrightarrow K \times \Delta^1 \xrightarrow{\tilde{\alpha}\times\mathrm{id}} \mathcal{C} \times \Delta^1 \xrightarrow{v} \mathcal{M}.$$

In the next examples, fix adjoint functors

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p \colon \mathcal{M} \to \Delta^1$ via data (u, v).

Example 3.3.5. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$ of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is $u|\{c\} \times \Delta^1$ and b is $v|\{d\} \times \Delta^1$. This corresponds to the map $s = \mathrm{id} \colon \Delta^1 \to \Delta^1$.

Example 3.3.6. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id} \colon gD \to gD$ in $\mathcal{C}$. This morphism is adjunct to the component of the unit map $\eta_D \colon D \to fgD$ relative to $s = \mathrm{id} \colon \Delta^1 \to \Delta^1$.

We will actually take this to be our working definition of the components of the unit map.

Definition 3.3.7. Let $f \dashv g$ be adjoint functors. A morphism in $\mathcal{C}$ is a ***component of a unit map*** if it is adjunct to an equivalence in $\mathcal{D}$ relative to $s = \mathrm{id} \colon \Delta^1 \to \Delta^1$.

Note 3.3.8. In the context of Kan extensions along a functor f, with unit natural transformation $\eta\colon \mathrm{id} \Rightarrow f^* f_!$, one often calls the component of the unit map at a functor F, namely $\eta_F\colon F \to f^* f_! F$, simply the *unit map,* dropping any mention of the word 'component.' We will adopt this terminology.

The next example will follow from Proposition 3.3.11.

Example 3.3.9. Any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g.

Lemma 3.3.10. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^{\{0\}}, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{6}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition 3.3.11. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, u and v be as in Definition 3.3.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s \circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a monomorphim such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{7}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s. Furthermore, any two such lifts are equivalent as functors $K \to \mathcal{D}$.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. From the map A and the map $\alpha\colon K \to \mathcal{C} \cong \mathcal{M}_0$, we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd} \left(K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1\right)^{\heartsuit} \arrow[r] \arrow[d, "w"'] & \mathcal{M}^{\natural} \arrow[d] \\ (K \times \Delta^1)^{\heartsuit} \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^{\sharp} \end{tikzcd},$$

where $(K \times \Delta^1)^{\heartsuit}$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\} \times \Delta^1$ for $k \in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. We can write the morphism of simplicial sets underlying w as a smash-product $(K' \hookrightarrow K) \wedge (\Delta^{\{0\}} \hookrightarrow \Delta^1)$. Building $K' \hookrightarrow K$ simplex by simplex, in increasing order of dimension, we can write w as a transfinite composition of pushouts along inclusions of the form given in Equation 6 (by assumption each simplex σ we adjoin to K is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). We can thus also build our lift ℓ by lifting against each of these in turn, and we can always choose each lift so that it is compatible with v. We can then take $\tilde{\alpha} = \ell|K \times \{1\}$.

It is manifest from this technique that the space of lifts is contractible; in particular, any two such lifts are equivalent in $\mathrm{Map}(K \times \Delta^1, \mathcal{M})$, so their restrictions to $K \times \{1\}$ are equivalent as functors $K \to \mathcal{D}$.

$\square$

Example 3.3.12. Let $f : \mathcal{C} \leftrightarrow \mathcal{D} : g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K' = \Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \arrow[r, hook, "i"] \arrow[dr] & \Delta^3 = K \arrow[d] \\ & \Delta^1 \end{tikzcd},$$

where both downwards-facing arrows take $0, 1 \mapsto 0$ and $2, 3 \mapsto 1$, and fix a map $\tilde{\alpha}\colon K \to N(\mathcal{C})$

and a map $\alpha' \colon K' \to N(\mathcal{D})$ such that $\tilde{\alpha} \circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition 3.3.11 implies that the solution to the lifting problem on the left yields one on the right. Similarly, its dual implies the converse. Thus, Proposition 3.3.11 implies in particular that such lifting problems are equivalent.

3.4 Left Kan implies globally left Kan

In this subsection, we will prove Theorem 3.0.3. In order to do that, we must show that we can solve lifting problems of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd},$$

for $n \geq 2$, where τ is left Kan. Forgetting for a moment that have singled out some 2-simplex τ, and using the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}_{\mathrm{sc}}$, we will do this by solving equivalent lifting problems of the form[2]

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}] \arrow[r] \arrow[d, hook] & \mathbf{QCat} \\ \mathfrak{C}[\Delta^{n+1}] \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}. \tag{8}$$

We expect that the reader is familiar with the basics of (scaled) rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category $\mathfrak{C}[\Delta^{n+1}]$ are given by the set $\{0, \ldots, n+1\}$, and the mapping spaces are defined by

$$\mathfrak{C}[\Delta^{n+1}](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases}.$$

[2]Note that we are implicitly taking $\mathfrak{C}[\Lambda_0^{n+1}]$ and $\mathfrak{C}[\Delta^{n+1}]$ to carry the flat markings on their mapping spaces.

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \ldots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda_0^{n+1}]$ has the same objects as $\mathfrak{C}[\Delta^{n+1}]$, and each morphism space $\mathfrak{C}[\Lambda_0^{n+1}](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^{n+1}](i,j)$, as we shall soon describe.

The map $\mathfrak{C}[\Lambda_0^{n+1}](i,j) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](i,j)$ is an isomorphism except for $(i,j) = (1, n+1)$ and $(i,j) = (0, n+1)$. The missing data corresponds in the case of $(1, n+1)$ to the missing face $d_0\Delta^{n+1}$ of Λ_0^{n+1}, and in the case of $(0, n+1)$ to the missing interior. Thus, to find a filling as in Equation 8, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](1,n+1) \arrow[r] \arrow[d, hook] & \operatorname{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{9}$$

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](0,n+1) \arrow[r] \arrow[d, hook] & \operatorname{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{10}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \operatorname{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[r, "\ell"] & \operatorname{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \end{tikzcd} \tag{11}$$

commute.

Notation 3.4.1. Recall our desired filling $\mathcal{F}\colon \mathfrak{C}[\Delta^{n+1}] \to \mathbf{QCat}$ of Equation 8. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \ldots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \operatorname{Map}(X_{i_1}, X_{i_k})$ by $f_{i_k \cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

Theorem 3.4.2. For any $n \geq 2$ and any globally left Kan 2-simplex τ, the solid extension problem

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

admits a dashed filler.

Proof. We need to solve the lifting problems of Equation 9 and Equation 10, and check that our solutions satisfy the condition of Equation 11. We note the following useful pieces of information.

- The information that the simplex $\tau\colon \Delta_\flat^{\{0,1,n+1\}} \to \mathbb{C}\mathrm{at}_\infty$ is left Kan tells us that $f_{n,1}$ is the left Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}}\colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$ is the unit map.
- We have an adjunction

$$(f_{1,0})_! : \mathrm{Fun}(X_0, X_{n+1}) \longleftrightarrow \mathrm{Fun}(X_1, X_{n+1}) : f_{1,0}^*.$$

It is easy to see that we have the following succinct descriptions of the inclusions of Equation 9 and Equation 10 along which we have to extend.

- There is an isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](0, n+1) \xrightarrow{\cong} C^n$$

  specified completely by sending a subset $S \subseteq [n+1]$ to the point

$$(z_1, \dots, z_n) \in C^n, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

  The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](0, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](0, n+1)$ corresponds to the simplicial subset $LC^n \hookrightarrow C^n$.
- There is a similar isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](1, n+1) \xrightarrow{\cong} C^{n-1}$$

  specified completely by sending $S \subseteq \{1, \dots, n+1\}$ to the point

$$(z_1, \dots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$

  The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](1, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](1, n+1)$ corresponds to the inclusion $\partial C^{n-1} \hookrightarrow C^{n-1}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^n \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ C^n \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-1} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$(\star) = \begin{tikzcd} C^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \arrow[d, "f_{1,0}^*"] \\ C^n \arrow[r] & \mathrm{Fun}(X_0, X_{n+1}) \end{tikzcd}$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition 3.2.21, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$(*') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed] & \end{tikzcd}.$$

The image of the 1-simplex $(0, \ldots, 0) \to (0, \ldots, 1)$ of $\Delta^0 * \partial C^{n-1} \subseteq C^n$ under α is the unit map $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}} \colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$. This is not in general an equivalence, so we cannot use Proposition 3.2.21 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to the identity map $\mathrm{id}_{f_{n+1,1}}$ in $\mathrm{Fun}(X_1, X_{n+1})$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-1} is in the image of $f_{1,0}^*$, so by Proposition 3.3.11 we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-1} \to \mathrm{Fun}(X_1, X_{n-1})$ which is adjunct to α relative to the map

$$s \colon \Delta^0 * \partial C^{n-1} \to \Delta^1$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-1} to $\Delta^{\{1\}}$; in particular, the image of the morphism $(0, \ldots, 0) \to (0, \ldots 1)$ under $\tilde{\alpha}'$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$(**') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed, "\tilde{\beta}"'] & \end{tikzcd}$$

However, since image of $(0, \ldots 0) \to (0, \ldots, 1)$ is an equivalence, Proposition 3.2.21 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition 3.3.11, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-1}}$ be the image of $\tilde{\beta}|_{C^{n-1}}$ under $f_{1,0}^*$, which is true by construction. □

3.5 A few other tricks with Kan extensions

We end this section with a few miscellaneous tricks which we can play with maps $\Delta^n \to \mathbb{C}\mathrm{at}_\infty$ involving Kan extensions.

Lemma 3.5.1. Denote by $\mathcal{E} \subseteq \mathrm{Hom}(\Delta^2, \Delta^3)$ the subset containing all degenerate 2-simplices together with the simplices $\Delta^{\{0,2,3\}}$ and $\Delta^{\{1,2,3\}}$. Let $\sigma \colon \Delta^3_{\mathcal{E}} \to \mathbb{C}\mathrm{at}_\infty$ such that $\Delta^{\{2,3\}}$ is mapped to an equivalence. Then $\sigma|\Delta^{\{0,1,2\}}$ is left Kan if and only if $\sigma|\Delta^{\{0,1,3\}}$ is left Kan.

Proof. Denote the image of the object $\Delta^{\{i\}}$ under σ by X_i, the image of $\Delta^{\{i,j\}}$ by f_{ji}, and the image of $\Delta^{\{i,j,k\}}$ by α_{jki}.

If either $\sigma|\Delta^{\{0,1,2\}}$ or $\sigma|\Delta^{\{0,1,3\}}$ is left Kan, then we have adjoint functors

$$(f_{10})_! : \mathrm{Fun}(X_0, X_3) \leftrightarrow \mathrm{Fun}(X_1, X_3) : (f_{10})^*.$$

We can extract from the data given to us by σ a commuting square in $\mathrm{Fun}(X_0, X_3)$

$$\begin{tikzcd}
f_{30} \arrow[r, "\alpha_{310}"] \arrow[d, "\alpha_{320}"', "\simeq"] & f_{31} f_{10} \arrow[d, "\alpha_{321} f_{10}"] \\
f_{32} f_{20} \arrow[r, "f_{32}\alpha_{210}"] & f_{32} f_{21} f_{10}
\end{tikzcd}$$

which is adjunct to a commuting square in $\mathrm{Fun}(X_1, X_3)$

$$\begin{tikzcd}
(f_{10})_! f_{30} \arrow[r, "\eta"] \arrow[d, "(f_{10})!\alpha_{320}"', "\simeq"] & f_{31} \arrow[d, "\alpha_{321}", "\simeq"'] \\
(f_{10})_! f_{32} f_{20} \arrow[r, "\eta'"] & f_{32} f_{21}
\end{tikzcd} .$$

Note that η is an equivalence if and only if η' is an equivalence; thus, that α_{310} is a unit map if and only if $f_{32}\alpha_{210}$ is a unit map. But f_{32} is an equivalence, so this in turn is true if and only if α_{210} is a unit map. □

4 Local systems

In [4], the twisted arrow category of an ∞-bicategory is defined.

4.1 The twisted arrow category

For a 2-category C, the twisted arrow 1-category $\mathbf{Tw}(\mathrm{C})$ has the following description.

- The objects of $\mathbf{Tw}(\mathrm{C})$ are the morphisms of C:
$$f\colon c \to c'.$$
- For two objects $(f\colon c \to c')$ and $(g\colon d \to d')$, the morphisms $f \to g$ are given by diagrams
$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & d \arrow[d, "g"] \\ c' & d' \arrow[l, "a'"] \end{tikzcd},$$
where α is a 2-morphism $f \Rightarrow a' \circ g \circ a$.
- The composition of morphisms is given by concatenating the corresponding diagrams.
$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & d \arrow[r, "b"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\overset{\beta}{\Longrightarrow}"] & e \arrow[d, "h"] \\ c' & d' \arrow[l, "a'"] & e' \arrow[l, "b'"] \end{tikzcd}$$

In [4], a homotopy-coherent version of the twisted arrow category of an ∞-bicategory is defined. For any ∞-bicategory $\mathbb{C}$, the twisted arrow ∞-category of $\mathbb{C}$, denoted $\mathbf{Tw}(\mathbb{C})$, is a quasi-category whose n-simplices are diagrams $(\Delta^n)^{\mathrm{op}} \star \Delta^n \to \mathbb{C}$, together with a scaling to ensure that the information encoded in an n-simplex is determined, up to contractible choice, by data

$$\overbrace{\begin{tikzcd} c_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\alpha_0}{\Longrightarrow}"] & c_1 \arrow[r, dotted] \arrow[d, dashed, "f_1"] & \cdots \arrow[r, dotted] & c_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, dashed, "f_{n-1}"] \arrow[dr, phantom, "\overset{\alpha_{n-1}}{\Longrightarrow}"] & c_n \arrow[d, "f_n"] \\ c_0' & c_1' \arrow[l, "a_0'"] & \cdots \arrow[l, dotted] & c_{n-1}' \arrow[l, dotted] & c_n' \arrow[l, "a_{n-1}'"] \end{tikzcd}}^{n \text{ squares}}$$

in $\mathbb{C}$. Here, the top row of morphisms corresponds to the spine of the n-simplex $\Delta^n \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$, and the bottom row of morphisms to the spine of $(\Delta^n)^{\mathrm{op}} \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$. To make this correspondence more clear, we will introduce the following notation.

Notation 4.1.1. For $i \in [n]$, we will write $\bar{i} := 2n + 1 - i$.

Thus, the ith column in the above diagram corresponds to the image of the morphism $i \to \bar{i}$ in $\Delta^n \star (\Delta^n)^{\mathrm{op}}$.

We are now ready to define this scaling.

Definition 4.1.2. We define a cosimplicial object $\tilde{Q}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ by sending

$$\tilde{Q}([n]) = (\Delta^n \star (\Delta^n)^{\mathrm{op}}, \dagger),$$

where $\dagger$ is the scaling consisting of all degenerate 2-simplices, together with all 2-simplices of the following kinds:

1. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through Δ^n.
2. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through $(\Delta^n)^{\mathrm{op}}$.
3. All simplices $\Delta^{\{i,j,\bar{k}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.
4. All simplices $\Delta^{\{k,\bar{j},\bar{i}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.

Note that $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$. We will use this identification freely.

This extends to a nerve-realization adjunction

$$Q : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} : \mathbf{Tw},$$

where the functor Q is the extension by colimits of the functor $\tilde{Q}$, and for any scaled simplicial set X, the simplicial set $\mathbf{Tw}(X)$ has n-simplices

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}}(Q(\Delta^n), X).$$

Notation 4.1.3. For any simplicial set X, $Q(X)$ carries the scaling given simplex-wise by that described above. We will denote this scaling by $\dagger_X$, or simply by $\dagger$ if the simplicial set X is clear from context. It will sometimes pay to notate the scaling $\dagger$ on $Q(X)$ explicitly, writing $Q(X) = Q(X)_\dagger$. We can thus write $Q(\Delta^n)$ explicitly and compactly as $\Delta^{2n+1}_\dagger$, which we will often do.

The inclusion $\Delta^n \amalg (\Delta^n)^{\mathrm{op}} \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ provides, for any ∞-bicategory $\mathbb{C}$ with underlying ∞-category $\mathcal{C}$, a morphism of simplicial sets

$$\mathbf{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

In [4], the following is shown.

Theorem 4.1.4. Let $\mathbb{C}$ be an ∞-bicategory (presented as a fibrant scaled simplicial set), and let $\mathcal{C}$ be the underlying ∞-category (presented as a quasicategory). Then the map

$$p_{\mathbb{C}} \colon \mathbf{Tw}(\mathbb{C}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}.$$

is a cartesian fibration between quasicategories, and a morphism $f \colon \Delta^1 \to \mathbf{Tw}(\mathbb{C})$ is $p_{\mathbb{C}}$-cartesian if and only if the morphism $\sigma \colon \Delta^3_{\dagger} \to \mathbb{C}$ to which it is adjunct is fully scaled, i.e. factors through the map $\Delta^3_{\dagger} \to \Delta^3_{\sharp}$.

In fact, this assignment is functorial.

Proposition 4.1.5. This extends to a functor $\mathbb{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty^{\Delta^1}$.

4.2 Two lemmas about marked-scaled anodyne morphisms

In the section following this one, we provide a fairly explicit construction of a family of marked-scaled anodyne inclusions. Constructing these inclusions directly from the classes of generating marked-scaled anodyne inclusions given in Definition 1.2.5 horn fillings would be possible but tedious. To lighten this load somewhat, we reproduce in this section two lemmas from [1]. The majority of this section is a retelling of sections 2.3 and 2.4 of [1]. Note however that our needs will be rather different than those of [1], and this is reflected in some subtle differences of notation.

We first need a compact notation for specifying simplicial subsets of Δ^n. Denote by $P \colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ the power set functor.

Definition 4.2.1. Let T be a finite linearly ordered set, so $T \cong [n]$ for some $n \in \mathrm{N}$. For any subset $\mathcal{A} \subseteq P(T)$, define a simplicial subset

$$\mathcal{S}_T^{\mathcal{A}} := \bigcup_{S \in \mathcal{A}} \Delta^{T \smallsetminus S} \subseteq \Delta^T.$$

Notation 4.2.2.

- If the linearly ordered set T is clear from context, we will drop it, writing $\mathcal{S}^{\mathcal{A}}$.
- For any marked-scaled n-simplex (Δ^n, E, T) we will by minor abuse of notation reuse the same letters E and T to denote the restriction of the markings and scalings to any simplicial subset $S \subseteq \Delta^n$.

Note 4.2.3. The assignment $\mathcal{A} \mapsto \mathcal{S}^{\mathcal{A}}$ does not induce is not a one-to-one correspondence between subsets $\mathcal{A} \subseteq P([n])$ and simplicial subsets $S^{\mathcal{A}} \subseteq \Delta^n$, since for $S \subsetneq S'$, we have that $\Delta^{[n] \smallsetminus S'} \subseteq \Delta^{[n] \smallsetminus S}$.

It will be useful to consider two subsets $\mathcal{A}$ and $\mathcal{A}' \subseteq P(T)$ to be equivalent if they produce the same simplicial subset of Δ^T.

Definition 4.2.4. We will write $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{S}^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}'}$.

The relation $\sim$ is an equivalence relation, but we will not use this.

The set $P([n])$ forms a poset, ordered by inclusion, and any $\mathcal{A} \subseteq P([n])$ a subposet. An element q of a poset Q is said to be *minimal* if there are no elements of Q which are strictly less than q. By Note 4.2.3 only the minimal elements of $\mathcal{A}$ contribute to the union defining $\mathcal{S}^{\mathcal{A}}$. We have just shown the following.

Lemma 4.2.5. For any $\mathcal{A} \subseteq P([n])$, we have $\mathcal{A} \sim \min(\mathcal{A})$, where $\min(\mathcal{A}) \subseteq \mathcal{A}$ is the set of minimal elements of $\mathcal{A}$.

Simplicial subsets of the form $\mathcal{S}_T^{\mathcal{A}}$ enjoy the following easily-proved calculation rules.

Lemma 4.2.6. Suppose $\mathcal{A} \subseteq P([n])$ is a subset such that for all $S, T \in \mathcal{A}$, it holds that $S \cap T = \emptyset$. Then $\mathcal{S}^{\mathcal{A}}$ contains each k-simplex of Δ^n for all $k < |\mathcal{A}|$.

Proof. Let $X \subseteq [n]$. The simplex $\Delta^X \to \Delta^n$ factors through $\mathcal{S}^{\mathcal{A}}$ if and only if it factors through $\Delta^{[n] \setminus T}$ for some (possibly not unique) $T \in \mathcal{A}$. This in turn is true if and only if X and T do not have any elements in common. For $|X| < |\mathcal{A}|$, there is always a set $T \in \mathcal{A}$ which does not have any elements in common with X. $\square$

Lemma 4.2.7. Let $\mathcal{A} \subseteq P([n])$ and $T \subseteq [n]$. Then

$$\mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T = \mathcal{S}_{[n]}^{\mathcal{A} \cup \{[n] \setminus T\}}.$$

The next lemma will be particularly useful in building inclusions $\mathcal{S}_{[n]}^{\mathcal{A}} \hookrightarrow \Delta^n$ simplex-by-simplex.

Lemma 4.2.8. Let $\mathcal{A} \subset P([n])$, and let $T \subset [n]$. Then the square

$$\begin{tikzcd}
\mathcal{S}_T^{\mathcal{A}|T} \arrow[r, hook] \arrow[d, hook] & \Delta^T \arrow[d, hook] \\
\mathcal{S}_{[n]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T
\end{tikzcd}$$

is bicartesian, where $\mathcal{A}|T$ is the subset of $P(T)$ given by

$$\mathcal{A}|T = \{S \cap T \mid S \in \mathcal{A}\}.$$

Proof. We have an equality
$$\mathcal{S}_T^{\mathcal{A}|T} = \mathcal{S}_{[n]}^{\mathcal{A}} \cap \Delta^T.$$
as subsets of Δ^n. □

In the remainder of this section we reproduce two lemmas from [1] which provide criteria for the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \subseteq (\Delta^n, E, T)$ to be marked-scaled anodyne.

Definition 4.2.9. Let $\mathcal{A} \subseteq P([n])$. We call $X \in P([n])$ an $\mathcal{A}$***-basal set*** if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of all $\mathcal{A}$-basal sets by $\mathrm{Bas}(\mathcal{A})$.

Definition 4.2.10 ([4], Definition 1.3)**.** We will call a subset $\mathcal{A} \subseteq P([n])$ ***inner dull*** if it satisfies the following conditions.

- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- There exists $0 < i < n$ such that for all $S \in \mathcal{A}$, we have $i \notin S$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.
- For each $\mathcal{A}$-basal set $X \in P([n])$, there exist $u, v \in X$ such that $u < i < v$.

The element $i \in [n]$ is known as the *pivot point.*

Note 4.2.11. The last condition of Definition 4.2.10 is always satisfied if $\mathcal{A}$ contains two singletons $\{u\}$ and $\{v\}$ such that $u < i < v$.

Definition 4.2.12. We will call a subset $\mathcal{A} \subseteq P([n])$ ***right dull*** if it satisfies the following conditions.

- It does not include the empty set; $\emptyset \notin \mathcal{A}$.
- For all $S \in \mathcal{A}$, we have $n \notin S$.
- There exists some $\ell \in [n]$ such that $\{s\} \in \mathcal{A}$ for some $s \leq \ell$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.

In this case, we will refer to n as the pivot point.

Definition 4.2.13. Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i. We will denote the adjacent elements of X surrounding i by $\ell^X < i < u^X$.

Lemma 4.2.14 (The Pivot Trick: [1], Lemma 2.3.5)**.** Let $\mathcal{A} \subseteq P([n])$ be an inner dull subset with pivot point i, and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that the following conditions hold:

1. Every marked edge $e \in E$ which does not contain i factors through $\mathcal{S}^{\mathcal{A}}$.

2. For all scaled simplices $\sigma = \{a < b < c\}$ of Δ^n which do not factor through $\mathcal{S}^{\mathcal{A}}$, and which do not contain the pivot point i, we have $a < i < c$, and $\sigma \cup \{i\}$ is fully scaled.
3. For all $X \in \mathrm{Bas}(\mathcal{A})$ and all $r, s \in [n]$ such that $\ell^X \le r < i < s \le u^X$, the triangle $\{r, i, s\}$ is scaled

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

Lemma 4.2.15. Let $\mathcal{A} \subset P([n])$ be a right dull subset (whose pivot point is by definition n), and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that for every $Z \in \mathrm{Bas}(\mathcal{A})$ the following condition holds:

- Denote by ℓ_-^Z and ℓ_+^Z the minimal and maximal element of Z respectively. For all r, $s \in [n]$ with $r \le \ell_-^Z \le \ell_+^Z \le s < n$, the triangle $\Delta^{\{r,s,n\}}$ is scaled, and the simplex $\Delta^{\{s,n\}}$ is marked.

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

4.3 Cosimplicial objects surrounding the infinity-category of local systems

4.3.1 The infinity-category of local systems

For any ∞-category $\mathcal{C}$, a $\mathcal{C}$-local system on some space X is simply a functor $X \to \mathcal{C}$. Thus, the ∞-category of $\mathcal{C}$-local systems on X should simply be $\mathbf{LS}(\mathcal{C}, X) := \mathrm{Fun}(X, \mathcal{C})$. We now allow X to vary, defining the category $\mathbf{LS}(\mathcal{C})$ of $\mathcal{C}$-local systems.

Definition 4.3.1. For any cocomplete ∞-category $\mathcal{C}$, we define the ∞-category of $\mathcal{C}$-local systems $\mathbf{LS}(\mathcal{C})$ together with a map of simplicial sets $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ by the following pullback diagram.

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} \tag{12}$$

It is shown in [4] that p', hence also p, is a cartesian fibration. The cartesian fibration p classifies the functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending

$$f\colon X \to Y \qquad \longmapsto \qquad f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ is cocomplete, each pullback map f^* has a left adjoint $f_!$, given by left Kan extension. It follows on abstract grounds that p is also a cocartesian fibration, whose cocartesian edges represent left Kan extension. In Subsection 4.4, we show this explicitly. In doing so, it will be useful to factor the pullback square in Equation 12 into the two pullback squares

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hook] \arrow[d, "p''"] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\
\mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{S} \times (\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}})^{\simeq} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}
\end{tikzcd} .$$

In order to show that p is a cocartesian fibration, it will help us to understand the map p''. To do this, we will have to define several cosimplicial objects.

4.3.2 Cosimplicial objects

The n-simplices of $\mathcal{R}$ are given by maps

$$Q(\Delta^n) = (\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger \to \mathbb{C}\mathrm{at}_\infty$$

such that

- each object in Δ^n is sent to a space, and
- each morphism in $(\Delta^n)^{\mathrm{op}}$ is mapped to an equivalence in $\mathbb{C}$.

We can more usefully encode the second condition by endowing $\mathbb{C}\mathrm{at}_\infty$ and Q with a marking.

Definition 4.3.2. We denote by $\mathbb{C}\mathrm{at}_\infty^\natural$ the marked-scaled simplicial set whose underlying scaled simplicial set is $\mathbb{C}\mathrm{at}_\infty$, and where all equivalences have been marked.

Definition 4.3.3. We denote by $\heartsuit$ the marking on $\Delta^n \star (\Delta^n)^{\mathrm{op}}$ consisting of all morphisms belonging to $(\Delta^n)^{\mathrm{op}}$, and by $(\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger^\heartsuit$ the corresponding marked-scaled simplicial set. We define a cosimplicial object

$$\tilde{R}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}; \qquad [n] \mapsto (\Delta^n \star (\Delta^n)^{\mathrm{op}})_\dagger^\heartsuit.$$

We denote the extension of $\tilde{R}$ by colimits by

$$R\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} R(n).$$

The marked-scaled simplicial sets $R(\Delta^n)$ are rather complicated, and showing that p'' is a cartesian fibration by solving the necessary lifting problems explicitly would be impractical. We will instead replace $R(\Delta^n)$ by something simpler, considering the simplicial subsets coming from the inclusions

$$\Delta^n \star \Delta^{\{\bar{0}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}},$$

which we understand to inherit the marking and scaling.

Definition 4.3.4. We will denote by $\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}$ the cosimplicial object

$$\tilde{J}\colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}; \qquad [n] \mapsto (\Delta^n \star \Delta^{\{\bar{0}\}})^{\heartsuit}_{\dagger},$$

and by J the extension by colimits

$$J\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}; \qquad X \mapsto \operatorname*{colim}_{\Delta^n \to X} \tilde{J}(n).$$

We have defined, for each $n \geq 0$, inclusions of marked-scaled simplicial sets

$$v_n\colon \tilde{J}(n) \to \tilde{R}(n).$$

However, extreme care is warranted: *the v_n are not the components of a natural transformation $u\colon \tilde{J} \Rightarrow \tilde{R}$!* The necessary squares simply do not commute for morphisms $\phi\colon [m] \to [n]$ in Δ such that $\phi(0) \neq 0$.

We will simply define this problem away. Denote by $\mathring{\Delta}$ the subcategory of Δ on morphisms $[m] \to [n]$ which send $0 \mapsto 0$. Denote by I the inclusion $\mathring{\Delta} \hookrightarrow \Delta$. The morphisms v_n *do* form a natural transformation $v\colon \tilde{J} \circ I \Rightarrow \tilde{R} \circ I$:

$$\begin{tikzcd} \mathring{\Delta} \arrow[r, bend left, "J\circ I"] \arrow[r, bend right, "Q\circ I"'] \arrow[r, phantom, "\Downarrow v"] & \mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}} \end{tikzcd}.$$

Our goal in the remainder of this section is to show that each component v_n is an equivalence in $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}$.

Lemma 4.3.5. For all $n \geq 0$, the map v_n is marked-scaled anodyne, hence a weak equivalence.

Proof. In this proof, all simplicial subsets of Δ^{2n+1} will be assumed to carry the marking $\heartsuit$ and the scaling $\dagger$.

We can write each v_n as a composition

$$\Delta^{\{0,\dots,n,\bar{0}\}} \overset{v_n''}{\hookrightarrow} \Delta^{\{0,\dots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\dots,\bar{0}\}} \overset{v_n'}{\hookrightarrow} \Delta^{2n+1}.$$

Here, the map v_n'' is a pushout along the inclusion $(\Delta^{\{n\}})^{\sharp}_{\sharp} \hookrightarrow (\Delta^n)^{\sharp}_{\sharp}$, which is marked-scaled anodyne. It remains to show that each of the maps v_n' is marked-scaled anodyne.

To this end, we introduce some notation. Let $M_0^n = \Delta^{\{0,\dots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\dots\bar{0}\}} \subset \Delta^{2n+1}$, and for

$1 \leq k \leq n$, define

$$M_k^n := M_0^n \cup \left(\bigcup_{\ell=1}^{k} \Delta^{[2n+1] \smallsetminus \{\ell, \bar{\ell}\}} \right).$$

There is an obvious filtration

$$M_0^n \overset{i_0^n}{\hookrightarrow} M_1^n \overset{i_1^n}{\hookrightarrow} \cdots \overset{i_{n-1}^n}{\hookrightarrow} M_n^n \overset{i_n^n}{\hookrightarrow} \Delta^{2n+1}. \tag{13}$$

Define

$$j_k^n := i_n^n \circ \cdots \circ i_k^n \colon M_k^n \hookrightarrow \Delta^{2n+1}, \qquad 0 \leq k \leq n.$$

In particular, note that $j_0^n = v_n'$.

This allows us to replace our goal to something superficially more difficult: we would like to show that, for each $n \geq 0$ and each $0 \leq k \leq n$, the map i_k^n is marked-scaled anodyne. We proceed by induction. We take as our base case $n = 0$, where we have the trivial filtration

$$M_0^0 \overset{i_0^0}{=} \Delta^1.$$

This is an equality of subsets of Δ^1, hence certainly an equivalence.

We now suppose that the result holds true for $n-1$; that is, that the maps i_k^{n-1} are marked-scaled anodyne for all $0 \leq k \leq n-1$. We aim to show that each i_k^n is marked-scaled anodyne for each $0 \leq i \leq n$.

For $0 \leq k < n$, we can write i_k^n as the inclusion

$$\mathcal{S}_{[2n+1]}^{\mathcal{A}} \hookrightarrow \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{[2n+1] \smallsetminus \{i, \bar{i}\}}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\bar{n}, \ldots, \bar{1}\} \\ \{0, \ldots n\} \\ \{1, \bar{1}\} \\ \vdots \\ \{k-1, \overline{k-1}\} \end{matrix} \right\}.$$

By Lemma 4.2.8, we have a pushout square

$$\begin{tikzcd}
\mathcal{S}_{[2n+1] \smallsetminus \{k, \bar{k}\}}^{\mathcal{A}|([2n+1] \smallsetminus \{k, \bar{k}\})} \arrow[r, hook] \arrow[d, hook] & \Delta^{[2n+1] \smallsetminus \{k, \bar{k}\}} \arrow[d, hook] \\
\mathcal{S}_{[2n+1]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{[2n+1] \smallsetminus \{k, \bar{k}\}}.
\end{tikzcd}$$

Therefore, it suffices to show that the top morphism is marked-scaled anodyne. One checks that this map is of the form j_k^{n-1}, and is thus marked-scaled anodyne by the inductive hypothesis. Therefore, it remains only to show that i_n^n is marked-scaled anodyne.

We treat the case $n = 1$ separately. In this case, v_1' takes the form

$$\Delta^{\{0,1,\bar{0}\}} \cup \Delta^{\{\bar{1},\bar{0}\}} \xhookrightarrow{v_1'} \Delta^3,$$

which we construct in the following way.

1. Then we fill the simplex $\Delta^{\{1,\bar{1},\bar{0}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A2).
2. Then we fill the simplex $\Delta^{\{0,1,\bar{1}\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A1).
3. Finally, we fill the full simplex Δ^3 (together with its marking and scaling) as a pushout along a morphism of type (A1).

Now we may assume that $n \geq 2$. In this case, we can write $i_n^n \colon M_n^n \hookrightarrow \Delta^{2n+1}$ as an inclusion

$$\mathcal{S}_{[2n+1]}^{\mathcal{A}} \subseteq \Delta^{2n+1}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\bar{n},\ldots,\bar{1}\} \\ \{0,\ldots n\} \\ \{1,\bar{1}\} \\ \vdots \\ \{n,\bar{n}\} \end{matrix} \right\}.$$

We will express this inclusion as the following composition of fillings:

1. We first add the simplices $\Delta^{\{n,\bar{n},\ldots,\bar{0}\}}, \Delta^{\{n-1,n,\bar{n},\ldots,\bar{0}\}}, \ldots, \Delta^{\{2,\ldots,n,\bar{n},\ldots,\bar{0}\}}$.
2. We next add $\Delta^{\{1,\ldots,n,\bar{n},\bar{0}\}}, \Delta^{\{1,\ldots,n,\bar{n},\overline{n-1},\bar{0}\}}, \ldots, \Delta^{\{1,\ldots,n,\bar{n},\ldots,\bar{2},\bar{0}\}}$
3. We next add $\Delta^{\{1,\ldots n,\bar{n},\ldots,\bar{0}\}}$.
4. We then add $\Delta^{\{0,\ldots,n,\bar{n},\bar{0}\}}, \Delta^{\{0,\ldots,n,\bar{n},\overline{n-1},\bar{0}\}}, \ldots, \Delta^{\{0,\ldots n,\bar{n},\ldots,\bar{0}\}}$.

We proceed.

1. • Using Lemma 4.2.8, we see that the square

$$\begin{tikzcd} \mathcal{S}_{\{n,\bar{n},\ldots,\bar{0}\}}^{\mathcal{A}|\{n,\bar{n},\ldots,\bar{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{n,\bar{n},\ldots,\bar{0}\}} \arrow[d, hook] \\ \mathcal{S}_{[2n+1]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[2n+1]}^{\mathcal{A}} \cup \Delta^{\{n,\bar{n},\ldots,\bar{0}\}} \end{tikzcd}$$

is pushout. In order to show that the bottom inclusion is marked-scaled anodyne, it thus suffices to show that the top inclusion is marked-scaled anodyne. We have

$$\mathcal{A}|\{n,\overline{n},\dots,\overline{0}\} = \left\{\begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{n\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{n-1}\} \\ \{n,\overline{n}\} \end{matrix}\right\} \sim \left\{\begin{matrix} \{n\} \\ \{\overline{n-1}\} \\ \vdots \\ \{\overline{1}\} \end{matrix}\right\} =: \mathcal{A}' \ .$$

This is a dull subset of $\{n,\overline{n},\dots,\overline{0}\}$ with pivot $\overline{n}$. The only $\mathcal{A}'$-basal set is $\{n,\overline{n-1},\dots,\overline{1}\}$. One checks that $\mathcal{A}'$, together with the marking $\heartsuit$ and scaling $\dagger$, satisfies the conditions of the Pivot Trick:

- For $n=2$, the only the marked edge not containing the pivot $\overline{2}$ is $\overline{1} \to \overline{0}$, which belongs to $\mathcal{S}^{\mathcal{A}'}_{\{n,\overline{n},\dots,\overline{0}\}}$. The only scaled simplex which does not contain $\overline{2}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{2 < \overline{1} < \overline{0}\}$. The simplex $\sigma \cup \{\overline{2}\}$ is fully scaled.
- For $n=3$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'}$. The only scaled triangle which does not contain the pivot $\overline{3}$, and which does not factor through $\mathcal{S}^{\mathcal{A}'}$, is $\sigma = \{3 < \overline{2} < \overline{1}\}$. The simplex $\sigma \cup \{\overline{3}\}$ is fully scaled.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 4.2.6.

In each case, the simplex $\{n,\overline{n},\overline{n-1}\}$ is scaled, so the top inclusion is marked-scaled anodyne by the Pivot Trick. We can write

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\dots,\overline{0}\}} = \mathcal{S}^{\mathcal{A}\cup([2n+1]\setminus\{n,\overline{n},\dots,\overline{0}\})}_{[2n+1]},$$

We see that

$$\mathcal{A} \cup ([2n+1] \setminus \{n,\overline{n},\dots,\overline{0}\}) \sim \left\{\begin{matrix} \{\overline{n},\dots,\overline{1}\} \\ \{0,\dots n-1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\},$$

which we denote by $\mathcal{A}_{n-1}$.

- We proceed inductively. Suppose we have added the simplices $\Delta^{\{n,\overline{n},\dots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\dots,\overline{0}\}}$, $\dots$, $\Delta^{\{k+1,\dots,n,\overline{n},\dots,\overline{0}\}}$, for $2 \leq k \leq n-1$. Using Lemma 4.2.7 and Lemma 4.2.5 can write the result of these additions as

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \left(\bigcup_{i=k+1}^{n} \Delta^{\{i,\dots,n,\overline{n},\dots,\overline{0}\}}\right) = \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]},$$

where

$$\mathcal{A}_{k+1} = \left\{\begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,\ldots,k\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\}.$$

Using Lemma 4.2.8, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{A}_{k+1}|\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}}_{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \cup \Delta^{\{k,\ldots,n,\overline{n},\ldots,\overline{0}\}} \end{tikzcd}$$

is pushout, so in order to show that the bottom morphism is marked-scaled anodyne, it suffices to show that the top morphism is. We see that

$$\mathcal{A}_{k+1}|\{k,\ldots,n,\overline{n},\ldots,\overline{0}\} = \left\{\begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k,\overline{k}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\} \sim \left\{\begin{matrix} \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k+1,\overline{k+1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\} =: \mathcal{A}'_{k+1}.$$

This is a dull subset of $P(\{k,\ldots,n,\overline{n},\ldots,\overline{0}\})$ with pivot $\overline{k}$, and one checks that the conditions of the Pivot Trick are satisfied:

- For $n = 3$, where the only value of k is $k = 2$, each marked edge is contained in $\mathcal{S}^{\mathcal{A}'_{k+1}}$ by Lemma 4.2.6, and one can check that each scaled triangle factors through $\mathcal{S}^{\mathcal{A}'_{k+1}}$.
- For $n \geq 4$, all scaled and marked simplices belong to $\mathcal{S}^{\mathcal{A}'}$ by Lemma 4.2.6.

The basal sets are of the form

$$\{k, a_1, \ldots, a_{n-k}, \overline{k-1}, \ldots, \overline{1}\},$$

where each a_1, …, a_{n-k} is of the form ℓ or $\overline{\ell}$ for $k+1 \leq \ell \leq n$. In each case, the simplex $\{a_{n-k}, \overline{k}, \overline{k-1}\}$ is scaled. Thus, the conditions of the Pivot Trick, the top morphism is marked-scaled anodyne.

We have now added the simplices promised in part 1., and are left with the simplicial subset $\mathcal{S}^{\mathcal{A}_2}_{[2n+1]}$, where

$$\mathcal{A}_2 = \left\{ \begin{matrix} \{\overline{n},\ldots,\overline{1}\} \\ \{0,1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

2. Each step in this sequence is solved exactly like those above. The calculations are omitted. The end result is the simplicial subset

□

4.4 The map governing local systems is a cocartesian fibration

Our aim is to show that the map of quasicategories $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ defined by the diagram

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathcal{R} \arrow[r, hook] \arrow[d, "p''"] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{S} \times (\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}})^{\simeq} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd},$$

Definition 4.4.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is said to be ***left Kan*** if the simplex $\sigma\colon \Delta^3_\dagger \to \mathbb{C}\mathrm{at}_\infty$ to which it is adjoint has the property that the restriction $\sigma|\Delta^{\{0,1,\overline{0}\}}$ is left Kan in the sense of Definition 3.0.1.

We draw the 'front' and 'back' of a general 3-simplex $\sigma\colon \Delta^3_\dagger \to \mathbb{C}\mathrm{at}_\infty$ corresponding to some morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, "\mathcal{H}"] & Y \arrow[d, "\mathcal{G}"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \arrow[ul, phantom, "\overset{\zeta}{\Rightarrow}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, "\overset{\eta}{\Rightarrow}"', shift right=3] & Y \arrow[d, "\mathcal{G}"] \arrow[dl, "\mathcal{G}'"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \end{tikzcd}$$

Here, the 2-simplices which are notated without natural transformations are constrained by the scaling $\dagger$ to be thin. Such an admissible simplex σ is left Kan if and only if $\mathcal{G}'$ is a left Kan extension of $\mathcal{F}$ along f, and η is a unit map. In this case, we will also call the morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ to which σ is adjunct *left Kan*.

Notation 4.4.2. We endow the quasicategory $\mathbf{LS}(C)$ with the marking $\clubsuit \subseteq \mathbf{LS}(\mathcal{C})_1$ consisting of all left Kan edges.

Theorem 4.4.3. The map $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cocartesian fibration, and an edge $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is p-cocartesian if and only if it is left Kan.

Proof. By [5, Prop. 3.1.1.6], it suffices to show that the map $\mathbf{LS}(\mathcal{C})^{\clubsuit} \to \mathcal{S}^{\sharp}$ has the right lifting property with respect to each of the classes of generating marked anodyne morphisms. We check these one-by-one.

1. We need to check that all lifting problems

$$\begin{tikzcd} (\Lambda^n_i)^\flat \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ (\Delta^n)^\flat \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}, \qquad n \geq 2, \quad 0 < i < n$$

admit solutions. This follows from Theorem 4.1.4.

2. We need to show that each of the lifting problems

$$\begin{tikzcd} (\Lambda^n_0)^{\mathcal{L}} \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ (\Delta^n)^{\mathcal{L}} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}, \qquad n \geq 1.$$

has a solution. This is the content of REF

3. We need to show that every lifting problem of the form

$$\begin{tikzcd} (\Lambda^2_1)^{\sharp} \amalg_{(\Lambda^2_1)^\flat} (\Delta^2)^{\sharp} \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ (\Delta^2)^{\sharp} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}$$

has a solution. This is the content of REF

4. We need to show that for all Kan complexes K, the lifting problem

$$\begin{tikzcd} K^\flat \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ K^{\sharp} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}$$

has a solution. To see this, note that each morphism in K must be mapped to an equivalence in $\mathbf{LS}(\mathcal{C})$, and a morphism $a \colon \mathcal{F} \to \mathcal{G}$ in $\mathbf{LS}(\mathcal{C})$ is an equivalence and only if it is p-cartesian, and lies over an equivalence in $\mathcal{S}$.

   - By Theorem 4.1.4, the morphism a is p-cartesian if and only if the map $\sigma \colon \Delta^3_{\dagger} \to \mathbb{C}\mathrm{at}_\infty$ to which it is adjoint factors through the map $\Delta^3_{\dagger} \to \Delta^3_{\sharp}$. In particular, the restriction $\sigma|\Delta^{\{0,1,\bar{0}\}}$ is thin.

- The image of a in $\mathcal{S}$ is the restriction $\sigma|\Delta^{\{0,1\}}$, which is therefore an equivalence.

It follows from Example 3.0.2 that the morphism a is automatically left Kan. Thus, each morphism in K is mapped to a left Kan morphism in $\mathbf{LS}(\mathcal{C})$, and our lifting problems admit solutions.

□

Everything past this point is old, and probably won't survive without major revision.

In order to show that all left Kan edges are cocartesian, we have to show that lifting problems

$$\begin{tikzcd} (\Lambda^n_0)^{\mathcal{L}} \arrow[r] \arrow[d, hook] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ (\Delta^n)^{\mathcal{L}} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}, \qquad n \geq 2,$$

have solutions.[3] Unravelling the definitions, we find that these are adjunct to the lifting problems

$$\begin{tikzcd} K^n_{\dagger} \arrow[r, "\tau'"] \arrow[d, hook] & \mathbb{C}\mathrm{at}_{\infty} \\ \Delta^{2n+1}_{\dagger} \arrow[ur, dashed, "\tau"'] & \end{tikzcd}, \tag{14}$$

where:

- K^n is the simplicial subset of Δ^n on those simplices $\sigma\colon \Delta^k \to K^n$ which satisfy at least one of the following conditions.
  1. The k-simplex σ factors through the 'top face' $\Delta^n \subset \Delta^{2n+1} \cong \Delta^n \star (\Delta^n)^{\mathrm{op}}$.
  2. The k-simplex σ factors through the 'bottom face' $(\Delta^n)^{\mathrm{op}} \subset \Delta^{2n+1} \cong \Delta^n \star (\Delta^n)^{\mathrm{op}}$.
  3. There exists $\ell \neq 0$ such that the k-simplex σ contains neither $\Delta^{\{\ell\}}$ nor $\Delta^{\{\bar{\ell}\}}$.
- The filler τ is left Kan. (This implies in particular that the map τ' fulfills all conditions to be left Kan which apply to the scaled simplicial subset $K^n_{\dagger}$.)

It is actually easier to solve these lifting problems by generalizing them somewhat. Recall that any left Kan simplex $\tau\colon \Delta^{2n+1}_{\dagger} \to \mathbb{C}\mathrm{at}_{\infty}$ is in particular admissible, and thus has the property that the restriction $\tau|\Delta^{\{n+1,\dots,2n+1\}}$ is the constant functor with value $\mathcal{C}$. We will relax this requirement, demanding only that each morphism in $\Delta^{\{n+1,\dots,2n+1\}}$ be mapped to an equivalence. We can guarantee this by introducing a marking, in addition to the scaling. We can do this using the theory of marked-scaled simplicial sets, as developed in [3].

[3]Note that the case $n = 1$ corresponds to the condition that cocartesian lifts exist, proved above.

Notation 4.4.4.

1. Denote by $\natural$ the marking on $\mathbb{C}\mathrm{at}_\infty$ containing all equivalences. Denote $\mathbb{C}\mathrm{at}_\infty$ together with this marking by $\mathbb{C}\mathrm{at}_\infty^\natural$. Thus, $\mathbb{C}\mathrm{at}_\infty^\natural$ is a marked-scaled simplicial set, where a triangle is scaled if and only if it commutes up to a specified homotopy.
2. Denote by $\heartsuit$ the marking on Δ^{2n+1} containing all 1-simplices in the 'bottom face' $(\Delta^n)^{\mathrm{op}} = \Delta^{\{n+1,\dots,2n+1\}} \subset \Delta^{2n+1}$. Denote the marked-scaled simplicial set with this marking and the $\dagger$ scaling by $(\Delta^{2n+1})^\heartsuit_\dagger$.
3. For any simplicial subset $S \subseteq \Delta^{2n+1}$, we will denote the restrictions of $\dagger$ and $\heartsuit$ to S also by $\dagger$ and $\heartsuit$ respectively.

Definition 4.4.5. We will call a map $\sigma\colon (\Delta^{2n+1})^\heartsuit_\dagger \to \mathbb{C}\mathrm{at}_\infty^\natural$ ***weakly left Kan*** if it satisfies the following conditions.

- The objects $\sigma(\Delta^0), \dots, \sigma(\Delta^n) \in \mathbb{C}\mathrm{at}_\infty$ are spaces.
- The image of the simplex $(\Delta^{\{0,1,\bar{0}\}})^\flat_\flat \subset (\Delta^{2n+1})^\heartsuit_\dagger$ is left Kan.

We note that there is a pullback diagram of marked-scaled simplicial subsets of Δ^{2n+1}

$$\begin{tikzcd} (\Lambda_0^{\{0,\dots,n,\bar{0}\}})^\heartsuit_\dagger \arrow[r, hook, "u_n"] \arrow[d, hook] & (K^n)^\heartsuit_\dagger \arrow[d, hook] \\ (\Delta^{\{0,\dots,n,\bar{0}\}})^\heartsuit_\dagger \arrow[r, hook, "v_n"] & (\Delta^{2n+1})^\heartsuit_\dagger \end{tikzcd} . \tag{15}$$

Lemma 4.4.6. For all $n \geq 2$, the map u_n is marked-scaled anodyne, hence a weak equivalence.

Proof. We begin with a few definitions.

- For each $n \geq 2$, denote by P_n the poset of strict subsets of $[n]$ which contain 0. Note that

$$\Lambda^n_0 \cong \operatorname*{colim}_{S \in P_n} \Delta^S.$$

In the remainder of this proof, all simplicial subsets of Δ^{2n+1} will carry the $\heartsuit$ marking and the $\dagger$ scaling.

Denote by S^n the colimit of the upper composition above, and by T^n the colimit of the lower composition.

$$\begin{tikzcd} P^n \arrow[r] & \mathring{\Delta} \arrow[r, bend left, "J \circ I"] \arrow[r, bend right, "Q \circ I"'] & \mathrm{Set}_\Delta^{\mathsf{MS}} \end{tikzcd}$$

Since the cofibrations of the marked-scaled model structure are precisely those morphisms of marked-scaled simplicial sets whose underlying morphisms of simplicial sets are monomorphisms, the strict colimit of each of these diagrams is a model for the homotopy colimit. We showed in Lemma 4.3.5 that each of the components of the natural transformation $v\colon J \circ I \to Q \circ I$ is a weak equivalence in the model structure on $\mathcal{S}\mathrm{et}_\Delta^{\mathbf{MS}}$. Therefore, this natural transformation furnishes a map between the colimits

We can write the inclusion $u_n\colon \Lambda_0^{\{0,\dots,n,\bar{0}\}} \hookrightarrow K^n$ as the map between the colimits induced by the diagrams

$$\Delta^n \amalg_{\Lambda_0^n} L^n \amalg_{(\Lambda_0^n)^{\mathrm{op}}} (\Delta^n)^{\mathrm{op}}$$

□

We have just shown that left Kan morphisms $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ are p-cocartesian, and that enough left Kan lifts exist. It remains only to show that any p-cocartesian lift is left Kan. We use the fact that any two p-cocartesian lifts of a morphism with a specified source are equivalent.

5 The non-monoidal construction

5.1 The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd}.$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd}, \tag{16}$$

To save on notation, we will generally denote such a morphism as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}. \tag{17}$$

There is a potential for confusion here. The natural transformation notated in Diagram 17 above does not correspond to a single natural transformation in Diagram 16, but rather to both natural transformations simultaneously. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cartesian if and only if all 2-simplices forming the faces of the corresponding map $Q(1) \to \mathbb{C}\mathrm{at}_\infty$ are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 5.1.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex $\sigma\colon Q(1) \to \mathbb{C}\mathrm{at}_\infty$ to which it corresponds has the property that $d_3\sigma$ is left Kan. More explicitly, the morphism is left Kan if and only if σ the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] & Y \arrow[d] \arrow[dl, "f_!\mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd},$$

where $f_!\mathcal{F}$ is a left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines both the data of the first and the filling 3-simplex up to contractible choice.

Proof. First we show that morphisms of the promised form really are r-cocartesian. Since by assumption $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cartesian fibration (hence an isofibration) whose source and target are quasicategories, it suffices to show that we can solve the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "a"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{S} \end{tikzcd}$$

where a is of the promised form, and that the space of such fillings is contractible. □

For short, we will denote r-cocartesian morphisms with an exclamation mark:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "=!\Rightarrow"] & X' \arrow[d, "\mathrm{Lan}_f \mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} .$$

5.2 The triple structures

5.2.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 5.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 5.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

5.2.2 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.

- $\mathcal{Q}_{\dagger} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^{\dagger}$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[dr, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_!g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 5.2.3. The triple $(\mathcal{Q}, \mathcal{Q}_{\dagger}, \mathcal{Q}^{\dagger})$ is adequate.

To do this, we need to prove:

Lemma 5.2.4. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\alpha"] & & b \arrow[dr] \arrow[dd, dash] \arrow[ddr, phantom, "\Downarrow\beta"] & \\
& b' \arrow[rr] \arrow[dd] & & c \arrow[dd] \\
\bar{a} & & \bar{b} \arrow[ll] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$.[4] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout.

[4]Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute

Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. $\square$

Proof of Proposition 5.2.3. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 5.2.4

We also need to show that any such pullback is of this form. This also follows from Lemma 5.2.4. $\square$

5.3 Building categories of spans

Lemma 5.3.1. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{18}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y\times Y} (X \times Y').$$

The left-hand side of Equation 18 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y\times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y\times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. $\square$

Proposition 5.3.2. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

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by definition.

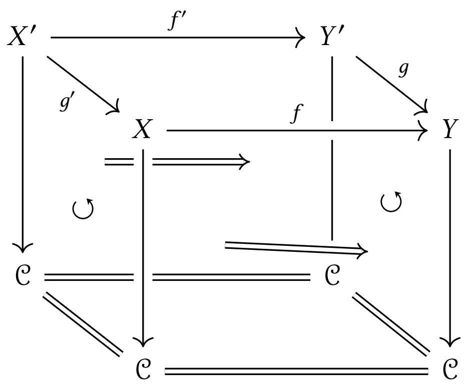
Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube



in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\ & \mathcal{C} & \\ X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"'] \end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 5.3.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

6 The monoidal construction

6.1 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^{I} \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i} \end{tikzcd}$$

commutes.

Example 6.1.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 6.1.2. A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[5] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.
- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd}.$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 6.1.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.

[5]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\
& \mathcal{C}^{\mathrm{op}}
\end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0))
\end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. $\square$

Note that for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \to (\mathcal{C}\mathrm{at}^n_\infty)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}^n_\infty$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n.$$

Because the pointwise product of any number of thin 1-simplices in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$, the conditions of Lemma 6.1.3 are satisfied. Unrolling, we find that $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd}
\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd} \tag{19}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 6.1.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 19, we have the following.

Proposition 6.1.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \\
\vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.
- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_\otimes$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

6.2 The triple structures

6.2.1 The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 6.2.1. This is an adequate triple structure.

Proof. Trivial. $\square$

Proposition 6.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

6.2.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 6.2.3. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 6.2.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd} \langle n\rangle & \langle m\rangle \arrow[l] \\ \langle n'\rangle \arrow[u, "\phi"] & \langle m'\rangle \arrow[l] \arrow[u, "\simeq"'] \end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) □

6.2.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_{\dagger}, \mathcal{P}^{\dagger})$ if it is of the form

$$\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t \circ \mathcal{F}"] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^* t \mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd} .$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 6.2.5. The triple $(\mathcal{Q}, \mathcal{Q}_{\dagger}, \mathcal{Q}^{\dagger})$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_{\otimes}$ of the form

$$\begin{tikzcd}
y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd} ,$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

6.3 Building categories of spans

Proposition 6.3.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 6.3.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd} \langle n\rangle & \langle n\rangle \arrow[l, "\gamma'"'] \\ \langle m\rangle \arrow[u, "\tau'"] & \langle m\rangle \arrow[l, "\gamma"] \arrow[u, "\tau"'] \end{tikzcd},$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i\colon X_i \to Y_{\gamma^{-1}(i)}, \qquad i \in \langle m\rangle^\circ,$$

f' has components

$$f'_j\colon X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n\rangle^\circ,$$

g has components

$$g_j\colon Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n\rangle^\circ,$$

and g' has components

$$g'_j\colon X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n\rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{(\prod_{\tau'(i)=j} Y_i)} \left(\prod_{\tau'(i)=j} X_i\right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'" near start] & \\
& \vec{X} \arrow[rr, "f" near start] \arrow[dd, "\mathcal{F}" near start] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_{\otimes}" near start] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_{\otimes}"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_{\otimes}"] \arrow[ul, "\tau_{\otimes}"']
\end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^{\circ}$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathbf{LS}(\mathcal{C})_{\otimes}$ above to a pullback square entirely in the fiber over $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathbf{LS}(\mathcal{C})_{\otimes})_{\langle n \rangle} \cong (\mathbf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathbf{LS}(\mathcal{C})$, one for each $j \in \langle n \rangle^{\circ}$, of the form

$$\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}" near start] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr] \arrow[dd, "\mathcal{A}" near start] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd},$$

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i \colon X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

7 Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$\begin{tikzcd} \mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl] \\ & \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) & \end{tikzcd} .$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, giving us isofibrations

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd} ,$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an $^{\mathrm{op}}$. Composing with $^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.

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