# Monoidal push-pull for local systems

# Angus Rush

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# 1 Background

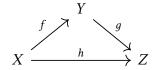
#### 1.1 Scaled simplicial sets

Note: This is basically a summary of the parts of The Goodwillie Calculus that we're gonna need.

**Definition 1.1.1.** A *scaled simplicial set* is a pair  $(X, A_X) = \overline{X}$ , where X is a simplicial set, and  $A \subseteq X_2$  is a set of 2-simplices of X which contains every degenerate simplex. A simplex belonging to A is known as *thin*. A morphism of scaled simplicial sets is a morphism of the underlying simplicial sets which takes thin simplices to thin simplices.

We will need the following facts about scaled simplicial sets.

1. For any category  $\mathcal{C}$  enriched in marked simplicial sets, we can build a scaled simplicial set  $N_{sc}(\mathcal{C}) = (N(\mathcal{C}), T)$ , the **scaled nerve** of  $\mathcal{C}$ , whose underlying simplicial set  $N(\mathcal{C})$  is the simplicial nerve of  $\mathcal{C}$ . The scaling T is defined in the following way. Let  $\sigma \colon \Delta^2 \to N(\mathcal{C})$  be any 2-simplex. This corresponds to a diagram



in  $\mathcal{C}$ , together with a 1-simplex  $\alpha_{\sigma} \colon \Delta^{1} \to \operatorname{Map}_{\mathcal{C}}(X, Z)$  corresponding to a map  $gf \to h$ . We will then define  $\sigma$  to be thin if and only if  $\alpha_{\sigma}$  is marked in  $\operatorname{Map}_{\mathcal{C}}(X, Z)$ .

- 2. The functor  $N_{sc}$  admits a left adjoint  $\mathfrak{C}^{sc} \colon \operatorname{Set}_{\Delta}^{sc} \to \operatorname{Cat}_{\Delta}^+$ , the *scaled rigidification*, such that for any scaled simplicial set  $\overline{X}$  with underlying simplicial set X, the underlying simplicially enriched category of  $\mathfrak{C}^{sc}(\overline{X})$  is simply  $\mathfrak{C}(X)$ , the rigidification of X. The scaling on  $\mathfrak{C}^{sc}(X)$  also admits a relatively simple description.
- 3. A map of scaled simplicial sets  $\overline{X} \to \overline{Y}$  is said to be a *bicategorical equivalence* if the map  $F \colon \mathfrak{C}^{\mathrm{sc}}(X) \to \mathfrak{C}^{\mathrm{sc}}(Y)$  is a weak equivalence of  $\operatorname{Set}_{\Delta}$ -enriched categories; that is, if for all objects x, x' of  $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$ , the map

$$\operatorname{Hom}_{\mathfrak{C}^{\operatorname{sc}}(\overline{X})}(x,x') \to \operatorname{Hom}_{\mathfrak{C}^{\operatorname{sc}}(\overline{Y})}(F(x),F(x'))$$

is a weak equivalence of marked simplicial sets, and if for all  $y \in \mathfrak{C}^{\mathrm{sc}}(\overline{Y})$ , there exists some  $x \in \mathfrak{C}^{\mathrm{sc}}(\overline{X})$  such that F(x) and y are isomorphic in the homotopy category  $h\mathfrak{C}^{\mathrm{sc}}(\overline{Y})$ .

4. There is a left proper, combinatorial model structure on  $\mathsf{Set}^\mathsf{sc}_\Delta$ , whose cofibrations are monomorphisms, and whose weak equivalences are bicategorical fibrations. This gives a Quillen equivalence

$$\mathfrak{C}^{sc}: \mathbb{S}et^{sc}_{\Lambda} \leftrightarrow \mathbb{C}at^{+}_{\Lambda}: N_{sc}.$$

- 5. An  $\infty$ -bicategory is a fibrant object with respect to this model structure. In particular, for any fibrant object  $\mathcal{C} \in \operatorname{Cat}_{\Delta}^+$  (i.e. any quasicategory-enriched category with equvialences marked),  $N_{sc}(\mathcal{C})$  is an  $\infty$ -bicategory.
- 6. The category of scaled simplicial sets has a natural simplicial enrichment as follows: For any two scaled simplicial sets  $\overline{X}$  and  $\overline{Y}$ , we define

$$\operatorname{Map^{sc}}(\overline{X}, \overline{Y})_n = \operatorname{Hom}(\overline{X} \times \Delta_{\sharp}^n, \overline{Y}),$$

where  $\Delta^n_{\sharp}$  is the scaled simplicial set  $(\Delta^n, \Delta^n_2)$ . For  $\overline{Y}$  an  $\infty$ -bicategory, Map<sup>sc</sup> $(\overline{X}, \overline{Y})$  is a quasicategory. Taking the core gives a Kan complex.

- 7. With this,
- 8. We will denote the simplicially enriched category whose objects are  $\infty$ -bicategories, with mapping spaces as above by  $\mathbf{Bicat}_{\infty}$ . We then define  $\mathbb{C}\mathrm{at}_{(\infty,2)} = N(\mathbf{Bicat}_{\infty})$ . This is a quasicategory.

We will need multiple notions of the quasicategory of  $(\infty, 2)$ -categories.

- The model category  $\text{Set}^{sc}_\Delta$  of scaled simplicial sets, whose cofibrations are precisely the monomorphisms.
- The weak equivalences are bicategorical equivalences,

#### 2 The non-monoidal construction

Let  $\mathcal C$  be a category admitting small colimits. We consider the pullback

$$\begin{array}{ccc} LS(\mathcal{C}) & \longrightarrow & Tw(\mathcal{C}at_{\infty}) \\ \downarrow & & \downarrow & \\ \mathcal{C}at_{\infty} \times \{\mathcal{C}\} & \longrightarrow & \mathcal{C}at_{\infty} \times \mathcal{C}at_{\infty}^{op} \end{array}.$$

A general morphism in  $LS(\mathcal{C})$  is of the form

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\alpha} & \mathcal{D}' \\
\downarrow F \downarrow & = \eta \Rightarrow & \downarrow F' \\
\mathcal{C} & = & \mathcal{C}
\end{array}$$

This is r-cartesian if and only if  $\eta$  is a natural isomorphism  $F \Rightarrow F' \circ \alpha$ ; that is, if the above square (weakly) commutes. We will denote a weakly commuting square via

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\alpha} & \mathcal{D}' \\
\downarrow^{F'} & & \downarrow^{F'} \\
\mathcal{C} & & & \mathcal{C}
\end{array}$$

The functor  $r: LS(\mathcal{C}) \to \mathcal{C}at_{\infty}$  thus classifies the functor

$$\mathfrak{C}at^{op}_{\infty} \to \mathfrak{C}at_{\infty}; \qquad \mathfrak{D} \mapsto \operatorname{Fun}(\mathfrak{D}, \mathfrak{C}),$$

which sends a functor  $f \colon \mathcal{D} \to \mathcal{D}'$  to the pullback

$$f^* : \operatorname{Fun}(\mathcal{D}', \mathcal{C}) \to \operatorname{Fun}(\mathcal{D}, \mathcal{C}).$$

Under the assumption that  $\mathcal{C}$  admits colimits, each of these functors admit a left adjoint  $f_!$  given by left Kan extension. The map r is thus a bicartesian fibration with cocartesian morphisms given by morphisms

$$\begin{array}{ccc}
\mathcal{D} & \xrightarrow{\alpha} & \mathcal{D}' \\
F \downarrow & = \eta \Rightarrow & \downarrow \operatorname{Lan}_{\alpha} F \\
\mathcal{C} & = & \mathcal{C}
\end{array}$$

We define a triple structure  $(\mathfrak{Q},\mathfrak{Q}_{\dagger},\mathfrak{Q}^{\dagger})$  on  $LS(\mathfrak{C})$  as follows.

- Q = LS(C).
- $Q_{\dagger} = LS(\mathcal{C}).$
- $Q^{\dagger}$  consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{array}{cccc}
\mathcal{D}' & \longleftarrow & \mathcal{D} & \longrightarrow & \mathcal{D}'' \\
\downarrow & \circlearrowleft & \downarrow & \Longrightarrow & \downarrow \\
\mathcal{C} & \longleftarrow & \mathcal{C} & \longleftarrow & \mathcal{C}
\end{array}$$

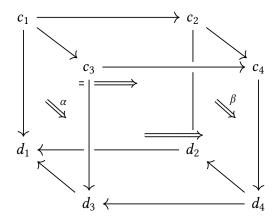
**Proposition 2.0.1.** The triple  $(Q, Q_{\dagger}, Q^{\dagger})$  is adequate.

To do this, we need to prove:

**Lemma 2.0.2.** Let  $\mathbb C$  be an  $\infty$ -bicategory such that the underlying quasicategory  $\mathcal C$  has pullbacks and pushouts. Consider a square

$$\begin{array}{ccc}
C_1 & \longrightarrow & C_2 \\
\downarrow a & & \downarrow b \\
C_3 & \longrightarrow & C_4
\end{array}$$

in  $Tw(\mathbb{C})$  corresponding to a twisted cube



in  $\mathbb{C}$ , where the top face is pullback and the bottom face is pushout. Note that the top and bottom faces of any such diagram in  $Tw(\mathbb{C})$  belong to  $\mathbb{C}$ , and hence must weakly commute, by definition of  $Tw(\mathbb{C})$ .

Suppose  $\alpha$  is an equivalence. Then

## 3 The base fibration

The starting point of our construction is the functor

$$\mathfrak{Fin}_* \xrightarrow{\mathfrak{C}at_\infty^{\times}} \mathfrak{C}at_{(\infty,2)} \xrightarrow{Tw} \mathfrak{C}at_\infty$$

Note that this gives a monoidal category since  $\mathbf{Tw}(\operatorname{Cat}_{\infty}^n) \cong \mathbf{Tw}(\operatorname{Cat}_{\infty})^n$ .

We're not gonna worry too much about where the functor  $T\mathbf{w}$  comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration  $T\mathbf{w}(\operatorname{Cat}_{\infty})_{\otimes} \to \operatorname{Fin}^{\operatorname{op}}_*$  with the following description: an n-simplex  $\sigma$  corresponding to a diagram

$$\Delta^n \xrightarrow{\sigma} \mathbf{LS}(\operatorname{Cat}_{\infty})_{\otimes}$$

$$fin^{\operatorname{op}}_*$$

corresponds to the data of, for each subset  $I \subseteq [n]$  having minimal element i, a map

$$\tau(I) \colon \Delta^I \to \mathbf{Tw}(\mathfrak{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets  $I' \subseteq I \subseteq [n]$ , the diagram

$$\Delta^{I'} \longrightarrow \mathbf{Tw}(\mathbf{Cat}_{\infty})^{i'} 
\downarrow \qquad \qquad \downarrow 
\Delta^{I} \longrightarrow \mathbf{Tw}(\mathbf{Cat}_{\infty})^{i}$$

**Example 3.0.1.** An object of  $\mathbf{Tw}(\mathbb{C}at_{\infty})_{\otimes}$  lying over  $\langle n \rangle$  corresponds to a collection of functors  $\mathbb{C}_i \to \mathbb{D}_i$ ,  $i \in \langle n \rangle^{\circ}$ . We will denote this  $\vec{\mathbb{C}} \to \vec{\mathbb{D}}$ .

A morphism in  $T\mathbf{w}(\operatorname{\mathfrak{C}at}_\infty)_\otimes$  lying over the active map  $\langle 1 \rangle \leftarrow \langle 2 \rangle$  in  $\operatorname{\mathfrak{F}in}^{\operatorname{op}_1}_*$  consists of

- A 'source' object  $F \colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects  $G_i : \mathcal{D}_i \to \mathcal{D}'_i$ , i = 1, 2.
- · A morphism

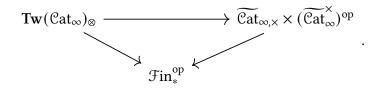
$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\alpha} & \mathcal{D}_1 \times D_2 \\
\downarrow F \downarrow & = \eta \Rightarrow & \downarrow G_1 \times G_2 \\
\mathbb{C}' & \longleftarrow_{\beta} & \mathcal{D}'_1 \times \mathcal{D}'_2
\end{array}$$

where here we mean  $\eta: F \Rightarrow \beta \circ G_1 \times G_2 \circ \alpha$ .

Note that  $Tw(\mathfrak{C}at_{\infty})_{\otimes}$  admits a map to  $\widetilde{\mathfrak{C}at}_{\infty,\times}\times (\widetilde{\mathfrak{C}at}_{\infty}^{\times})^{op}$ , where

- $\widetilde{\text{Cat}}_{\infty,\times}$  is the relative nerve (as a cartesian fibration) of the functor  $\mathfrak{F}\text{in}_* \to \mathbb{C}\text{at}_\infty$  giving the cartesian monoidal structure on  $\mathbb{C}\text{at}_\infty$
- $\widetilde{\operatorname{Cat}}_{\infty}^{\times}$  is the *cocartesian* relative nerve of the same.

Taking all this together gives a commuting triangle



where

**Proposition 3.0.2.** Everything in sight is a cartesian fibration.

**Proposition 3.0.3.** There is an equivalence  $\mathfrak{C}at_{\infty,\times} \simeq \widetilde{\mathfrak{C}at}_{\infty,\times}$  which is homotopic to the identity on  $\mathfrak{C}at_{\infty}$ .

<sup>&</sup>lt;sup>1</sup>Here we mean the morphism in  $\mathfrak{Fin}^{op}_*$  corresponding to the active map  $\langle 2 \rangle \rightarrow \langle 1 \rangle$  in  $\mathfrak{Fin}_*$ .

**Proposition 3.0.4.** We can express a monoidal  $\infty$ -category as a functor  $\mathfrak{F}in_* \to \widetilde{\mathfrak{C}at}_\infty^\times$ .

Fix some monoidal  $\infty$ -category  $\mathcal{C}$  which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\operatorname{Cat}_{\infty,\times} \to \widetilde{\operatorname{Cat}}_{\infty,\times} \times (\widetilde{\operatorname{Cat}}_{\infty}^{\times})^{\operatorname{op}}$$

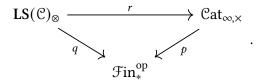
which is the equivalence on the first component, and given by the composition

$$\operatorname{\mathfrak{C}at}_{\infty,\times} \to \operatorname{\mathfrak{F}in}^{\operatorname{op}}_* \xrightarrow{\operatorname{\mathfrak{C}}} (\widetilde{\operatorname{\mathfrak{C}at}}_\infty^\times)^{\operatorname{op}}.$$

Forming the pullback

$$\begin{array}{ccc} LS(\mathcal{C})_{\otimes} & \longrightarrow & Tw(\mathcal{C}at_{\infty})_{\otimes} \\ \downarrow & & \downarrow \\ \mathcal{C}at_{\infty,\times} & \longrightarrow & \widetilde{\mathcal{C}at}_{\infty,\times} \times (\widetilde{\mathcal{C}at}_{\infty}^{\times})^{op} \end{array}$$

gives us a commutative triangle



**Proposition 3.0.5.** The map p is a cartesian fibration, and the p-cartesian morphisms in  $Cat_{\infty,\times}$  are precisely those representing products.

**Proposition 3.0.6.** The maps q and r are cartesian fibrations, and a generic morphism

$$\vec{\mathcal{D}} \xrightarrow{\alpha} \vec{\mathcal{D}}'$$

$$F \downarrow = \eta \Rightarrow \qquad \downarrow G_1 \times G_2$$

$$\vec{\mathcal{C}} \xleftarrow{\otimes} \vec{\mathcal{C}}$$

in LS(C) is

- r-cartesian if  $\eta$  is an equivalence
- *q*-cartesian if  $\alpha$  exhibits a product and  $\eta$  is an equivalence.

One sees immediately that r sends q-cartesian morphisms to p-cartesian morphisms, and has the sanity check that for any morphism f in  $LS(\mathcal{C})_{\otimes}$  whose image in  $\mathfrak{C}at_{\infty,\times}$  is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

# 4 The triple structures

#### 4.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between  $\mathcal{F}_{in_*}$  and  $N\mathcal{F}_{in_*}$ . Who has time for that these days?

We define a triple structure  $(\mathfrak{F}, \mathfrak{F}_{\dagger}, \mathfrak{F}^{\dagger})$  on  $\mathfrak{Fin}^{op}_*,$  where

- $\mathcal{F} = \mathcal{F}in_*^{op}$
- $\mathcal{F}_{\dagger} = (\mathcal{F}in_*^{op})^{\simeq}$
- $\mathcal{F}^{\dagger} = \mathcal{F}in_*^{op}$

**Proposition 4.1.1.** This is an adequate triple structure.

**Proposition 4.1.2.** There is a Joyal equivalence  $\mathcal{F}in_* \to Span(\mathcal{F}, \mathcal{F}_{\dagger}, \mathcal{F}^{\dagger})$ .

## 4.2 The triple structure on infinity-categories

We define a triple  $(\mathcal{P}, \mathcal{P}_{\dagger}, \mathcal{P}^{\dagger})$  as follows.

- $\mathcal{P} = \mathcal{C}at_{\infty} \times$
- $\mathcal{P}_{\dagger} = \operatorname{Cat}_{\infty, \times} \times_{\operatorname{Fin}^{\operatorname{op}}_*} (\operatorname{Fin}^{\operatorname{op}}_*)^{\simeq}$
- $\mathcal{P}^{\dagger} = Cat_{\infty,\times}$

**Proposition 4.2.1.** The cartesian fibration  $\mathfrak{C}at_{\infty,\times} \to \mathfrak{F}in_*^{op}$  admits relative pullbacks.

**Proposition 4.2.2.** This is an adequate triple structure.

## 4.3 The triple structure on the twisted arrow category

We define a triple  $(\mathfrak{Q},\mathfrak{Q}_{\dagger},\mathfrak{Q}^{\dagger})$  as follows:

- $Q = LS(\mathcal{C})_{\otimes}$
- $Q_{\dagger} = LS(\mathcal{C})_{\otimes} \times_{\mathfrak{F}in^{op}_{*}} (\mathfrak{F}in^{op}_{*})^{\simeq}$
- $Q^{\dagger}$  consists of the subcategory of LS( $\mathfrak{C}$ ) $_{\otimes}$  of r-cartesian morphisms.

These triple constructions correspond to spans of the following form.

$$\vec{D} \longleftarrow \vec{D}' \longrightarrow \vec{D}''$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$C^{n} \longrightarrow C^{m} \longleftarrow \stackrel{\simeq}{\longleftarrow} C^{m}$$

$$\langle n \rangle \longleftarrow \langle m \rangle \stackrel{\simeq}{\longrightarrow} \langle m \rangle$$