Monoidal push-pull for local systems

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Contents

1 Introduction

This is not meant to be a finished document, as there are still some gaps (listed below). It follows very closely the structure of my thesis, and I use a lot of results from there. It will not yet make sense on its own, and you may need to refer to my thesis to find the definitions of certain terms.

The gaps include:

- I'm using some things that don't exist yet, or at least which I haven't been able to find anywhere:
  - An extrinsic description of left Kan extensions in $\mathbb{C}\mathrm{at}_\infty$. That is, if I know that a morphism in $\mathbb{C}\mathrm{at}_\infty$ is a left Kan extension, can I use this to solve lifting problems in $\mathbb{C}\mathrm{at}_\infty$? A more concrete description of the universal property that I need is in section 3.
  - The easy overcategory colimit formula for non-inclusions: In Ambidexterity, Lurie uses that the colimit formula for left Kan extensions can be given in terms of overcategories even along non-inclusions. I'm perfectly willing to believe this, but I haven't been able to find a reference.
  - A functorial twisted arrow category.
- In showing that certain maps in $\mathsf{LS}(\mathcal{C})$ are left Kan extensions, I still need to show that the natural transformations involved are the unit maps.
- I'm ignoring size issues.

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. A morphism in $\mathcal{C}$ is egressive if and only if it is p-cartesian (hence also $p^\dagger$-cartesian).
2. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.

3. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

   in $\mathcal{C}$ with egressive and ingressive morphisms as marked, in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

   in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_{\dagger}$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^{\dagger}$-cartesian and f is p-cocartesian.

Proof. Even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. □

3 Conjectural lifting properties of left Kan extensions

We provide a conjectural answer to the following question. Let $f\colon X \to Y'$ be a map of Kan complexes, and let $\mathcal{F}\colon X \to \mathcal{C}$ be a functor, where $\mathcal{C}$ is cocomplete. What universal property does the ∞-Kan extension $f_!\mathcal{F}\colon Y \to \mathcal{C}$ satisfy? More generally, we would like an answer to this question where X and Y are ∞-categories rather than just spaces.

A left Kan extension should be modelled by a 2-simplex in $\mathbb{C}\mathrm{at}_\infty$

$$\begin{tikzcd} X \arrow[rr, "\mathcal{F}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "f_!\mathcal{F}"'] & \end{tikzcd},$$

where η is the unit map. In analogy to the 1-categorical case, it is logical to assume that such

2-simplices would enjoy certain pleasant lifting properties. Here is the property we need to hold in order to make our construction work out.

This definition is terribly-phrased, but it'll get the job done.

Definition 3.0.1. A 2-simplex $\sigma\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ is ***left Kan*** if either of the following equivalent hold.

1. For all
$$\tau\colon (\Lambda^3_0)_\flat \to \mathbb{C}\mathrm{at}_\infty$$
such that
   - $d_2\tau = \sigma$, and
   - the image of each vertex $\Delta^{\{i\}}$ under τ is a small Kan complex[1] for $0 \leq i < 3$, and
   - the image of the vertex $\Delta^{\{3\}}$ under τ is a cocomplete ∞-category,

   The space
$$\left(\mathrm{Map}(\Delta^3_\flat, \mathbb{C}\mathrm{at}_\infty) \times_{\mathrm{Map}((\Lambda^3_0)_\flat, \mathbb{C}\mathrm{at}_\infty)} \{\tau\}\right)^{\simeq}$$
of fillings of τ to a full 3-simplex is contractible.

2. For each $n \geq 3$ and any
$$\tau\colon (\Lambda^n_0)_\flat \to \mathcal{C}\mathrm{at}_\infty$$
such that
   - $\tau|_{\Delta^{\{0,1,n\}}} = \sigma$,
   - the image of each vertex $\Delta^{\{i\}}$ under τ is a small Kan complex for $0 \leq i < 3$, and
   - the image of the vertex $\Delta^{\{3\}}$ under τ is a cocomplete ∞-category,

   a filling of τ to a full n-simplex exists.

Note 3.0.2. It's not yet clear to me which of these definitions will be easier to use. They really are equivalent because of the following fact: for any categorical fibration $p\colon \mathcal{C} \to \mathcal{D}$, a morphism f in $\mathcal{C}$ is p-cocartesian if either of the following equivalent holds:

[1] We don't need to assume Kan complex here, just ∞-category, but I don't wanna make my life any harder than it needs to be.

- For all $n \geq 2$, the space of solutions to the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^n_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

is nonempty.

- The space of solutions to the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d] \arrow[dr, "f"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

is contractible.

Definition 3.0.3. A 2-simplex $\sigma\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ is ***pointwise left Kan*** if it is of the form

$$\begin{tikzcd} X \arrow[rr, "\mathcal{F}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "f_!\mathcal{F}"'] & \end{tikzcd},$$

where X and Y are small Kan complexes, $\mathcal{C}$ is a cocomplete ∞-category, $f_!F$ is a left Kan extension of F along f, and η is a unit map.

Proposition 3.0.4. Let $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ be pointwise left Kan. Then it is left Kan.

Proof. Let $(\Lambda^3_0)_\flat \to \mathbb{C}\mathrm{at}_\infty$ be a map satisfying the conditions of Definition 3.0.3, corresponding to a horn $\square$

4 The non-monoidal construction

4.1 The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd}
\mathsf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d] \\
\mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}}
\end{tikzcd}.$$

A general morphism in $\mathsf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \\
\mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"]
\arrow[from=1-1, to=2-1, phantom, shift right=-6, "\Rightarrow"] \arrow[from=1-2, to=2-2, phantom, shift left=6, "\circlearrowleft"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl] \\
\mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"]
\arrow[from=1-1, to=2-1, phantom, shift right=-6, "\Rightarrow"] \arrow[from=1-2, to=2-2, phantom, shift left=6, "\circlearrowleft"]
\end{tikzcd}, \tag{1}$$

To save on notation, we will generally denote such a morphism as follows.

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & Y \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}. \tag{2}$$

There is a potential for confusion here. The natural transformation notated in Diagram 2 above does not correspond to a single natural transformation in Diagram 1, but rather to both natural transformations simultaneously. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

A morphism $\Delta^1 \to \mathsf{LS}(\mathcal{C})$ is r-cartesian if and only if all 2-simplices forming the faces of the corresponding map $Q(1) \to \mathbb{C}\mathrm{at}_\infty$ are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}.$$

The functor $r\colon \mathsf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 4.1.1. A morphism $\Delta^1 \to \mathsf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex $\sigma\colon Q(1) \to \mathbb{C}\mathrm{at}_\infty$ to which it corresponds has the property that $d_3\sigma$ is left Kan (Definition 3.0.1). More explicitly, the morphism is left Kan if and only if σ the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\Rightarrow"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, shift right=4, "\eta\Rightarrow"] & Y \arrow[d] \arrow[dl, "f_!\mathcal{F}"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd},$$

where $f_!\mathcal{F}$ is a left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines both the data of the first and the filling 3-simplex up to contractible choice.

Proof. First we show that morphisms of the promised form really are r-cocartesian. Since by assumption $p\colon \mathsf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cartesian fibration (hence an isofibration) whose source and target are quasicategories, it suffices to show that we can solve the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "a"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathsf{LS}(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{S} \end{tikzcd}$$

where a is of the promised form, and that the space of such fillings is contractible. □

For short, we will denote r-cocartesian morphisms with an exclamation mark:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "=!\Rightarrow"] & X' \arrow[d, "\mathrm{Lan}_f \mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} .$$

4.2 The triple structures

4.2.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 4.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 4.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

4.2.2 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathsf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathsf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathsf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd} \mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[l, phantom, "\circlearrowleft", shift right=4ex] \arrow[r, phantom, "=!\Rightarrow", shift right=4ex] & \mathcal{D}'' \arrow[d, "f_!g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 4.2.3. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 4.2.4. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}$$

in $\mathrm{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] & & b \arrow[dr] \arrow[dd] & \\
& b' \arrow[rr] \arrow[dd] \arrow[dl, phantom, "\Downarrow\alpha"] & & c \arrow[dd] \arrow[dl, phantom, "\Downarrow\beta"] \\
\bar{a} & & \bar{b} \arrow[ll] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$.[2] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathrm{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 4.2.3. We need to show that we can form pullbacks in $\mathrm{Tw}(\mathbb{C})$ of cospans

[2]Note that the top and bottom faces of any such diagram in $\mathrm{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{op}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 4.2.4

We also need to show that any such pullback is of this form. This also follows from Lemma 4.2.4. $\square$

4.3 Building categories of spans

Lemma 4.3.1. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{3}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 3 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. $\square$

Proposition 4.3.2. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathsf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathrm{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd}
X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\
& X \arrow[rr, "f"] \arrow[dd] & & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] & & \mathcal{C} \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],
\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 4.3.1.

We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

5 The monoidal construction

5.1 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathsf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor Tw comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^I \arrow[r] & \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i \end{tikzcd}$$

commutes.

Example 5.1.1. An object of $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 5.1.2. A morphism in $\mathrm{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[3] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.
- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' \arrow[ur, phantom, "\Longrightarrow"] & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd} .$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 5.1.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in

[3] Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^{\mathrm{op}} \end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0)) \end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. □

Note that for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \to (\mathcal{C}\mathrm{at}^n_\infty)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}^n_\infty$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto \mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto (\mathcal{C}\mathrm{at}^n_\infty)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}^n_\infty.$$

Because the pointwise product of any number of thin 1-simplices in $\mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty)$, the conditions of Lemma 5.1.3 are satisfied. Unrolling, we find that $\mathsf{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}^{\times}_\infty)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$

- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd}
\mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd} \tag{4}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 5.1.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathsf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathsf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathsf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 4, we have the following.

Proposition 5.1.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd} \vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \\ \vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"] \end{tikzcd}$$

in $\mathsf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.
- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathsf{LS}(\mathcal{C})_\otimes$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$, and has the sanity check that for any morphism f in $\mathsf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

5.2 The triple structures

5.2.1 The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 5.2.1. This is an adequate triple structure.

Proof. Trivial. $\square$

Proposition 5.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

5.2.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 5.2.3. The cartesian fibration $\widetilde{\mathcal{S}}_{\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. ◻

Proposition 5.2.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_{\times}$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_{\times}$ lying over a square

$$\begin{tikzcd}
\langle n \rangle & \langle m \rangle \arrow[l] \\
\langle n' \rangle \arrow[u, "\phi"] & \langle m' \rangle \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\widetilde{\mathcal{S}}_{\times}$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_{\times}$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) ◻

5.2.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathsf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathsf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathsf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t \circ \mathcal{F}"] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^* t \mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd} .$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 5.2.5. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathsf{LS}(\mathcal{C})_\otimes$ of the form

$$\begin{tikzcd}
y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd} ,$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

5.3 Building categories of spans

Proposition 5.3.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 5.3.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathsf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd}
\langle n \rangle & \langle n \rangle \arrow[l, "\gamma'"'] \\
\langle m \rangle \arrow[u, "\tau'"] & \langle m \rangle \arrow[l, "\gamma"'] \arrow[u, "\tau"']
\end{tikzcd} ,$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i \colon X_i \to Y_{\gamma^{-1}(i)}, \qquad i \in \langle m \rangle^\circ,$$

f' has components

$$f'_j \colon X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n \rangle^\circ,$$

g has components

$$g_j \colon Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n \rangle^\circ,$$

and g' has components

$$g'_j \colon X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n \rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{(\prod_{\tau'(i)=j} Y_i)} \left(\prod_{\tau'(i)=j} X_i \right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd} \vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'"] & \\ & \vec{X} \arrow[rr, "f"] \arrow[dd, "\mathcal{F}"] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\ \mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes"] & \\ & \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"'] \end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^\circ$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathsf{LS}(\mathcal{C})_\otimes$ above to a pullback square entirely in the fiber over $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathsf{LS}(\mathcal{C})_\otimes)_{\langle n \rangle} \cong (\mathsf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathsf{LS}(\mathcal{C})$, one for each $j \in \langle n \rangle^\circ$, of the form

$$\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}" near end] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr, crossing over] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] \arrow[from=uu, crossing over, "\mathcal{A}" near start] & & \mathcal{C}
\end{tikzcd}$$

,

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i \colon X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

6 Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl] \\
& \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) &
\end{tikzcd}.$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, giving us isofibrations

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\
& \mathcal{F}\mathrm{in}_* &
\end{tikzcd},$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an $^{\mathrm{op}}$. Composing with $^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.