Monoidal push-pull for local systems

Angus Rush

October 28, 2021

Contents

1 Introduction

Remaining gaps

- I still need a functorial twisted arrow category.
- Left horn fillings suffice to fill $Q(\Lambda^n_0)^R \hookrightarrow Q(\Delta^n)^R$.
- In showing that certain maps in $\mathbf{LS}(\mathcal{C})$ are left Kan extensions, I still need to show that the natural transformations involved are the unit maps.
- I'm ignoring size issues.

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. A morphism in $\mathcal{C}$ is egressive if and only if it is p-cartesian (hence also $p^\dagger$-cartesian).
2. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.
3. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ with egressive and ingressive morphisms as marked, in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, twoheadrightarrow, "g"'] \arrow[dr, rightarrowtail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Proof. Even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. □

3 Lifting properties of left Kan extensions

We provide a conjectural answer to the following question. Let $f\colon X \to Y'$ be a map of Kan complexes, and let $\mathcal{F}\colon X \to \mathcal{C}$ be a functor, where $\mathcal{C}$ is cocomplete. What universal property does the ∞-Kan extension $f_!\mathcal{F}\colon Y \to \mathcal{C}$ satisfy? More generally, we would like an answer to this question where X and Y are ∞-categories rather than just spaces.

A left Kan extension should be modelled by a 2-simplex in $\mathbb{C}\mathrm{at}_\infty$

$$\begin{tikzcd} X \arrow[rr, "\mathcal{F}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "f_!\mathcal{F}"'] & \end{tikzcd},$$

where η is the unit map. In analogy to the 1-categorical case, it is logical to assume that such 2-simplices would enjoy certain pleasant lifting properties. Here is the property we need to hold in order to make our construction work out.

This definition is terribly-phrased, but it'll get the job done.

Definition 3.0.1. A 2-simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ is ***left Kan*** if for each $n \geq 3$ and any

$$\alpha\colon (\Lambda^n_0)_\flat \to \mathbb{C}\mathrm{at}_\infty$$

such that

- $\alpha|_{\Delta^{\{0,1,n\}}} = \tau$,
- the image of each vertex $\Delta^{\{i\}}$ under α is a small Kan complex for $0 \leq i < n$, and
- the image of the vertex $\Delta^{\{n\}}$ under α is a cocomplete ∞-category,

a filling of α to a full n-simplex exists.

Definition 3.0.2. A 2-simplex $\sigma\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ is ***pointwise left Kan*** if it is of the form

$$\begin{tikzcd} X \arrow[rr, "\mathcal{F}"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "f_!\mathcal{F}"'] & \end{tikzcd},$$

where X and Y are small Kan complexes, $\mathcal{C}$ is a cocomplete ∞-category, $f_!F$ is a left Kan extension of F along f, and η is a unit map.

Proposition 3.0.3. Let $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ be pointwise left Kan. Then it is left Kan.

Proof. Let $(\Lambda^3_0)_\flat \to \mathbb{C}\mathrm{at}_\infty$ be a map satisfying the conditions of Definition 3.0.2, corresponding to a horn ◻

4 Horn filling via left Kan extensions

In this section, we show that a simplex $\Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ which is left Kan is also globally left Kan. This will amount to showing that a simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ of the form

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\eta\Downarrow"] & \mathcal{C} \\ & Y \arrow[ur, "f_!F"'] & \end{tikzcd},$$

where $\mathcal{C}$ is cocomplete, X is small, $f_!F$ is a left Kan extension of F along f, and $\eta\colon F \Rightarrow f^*f_!F$ is the unit map, has the property that for any $n \geq 2$, any solid filling problem

$$\begin{tikzcd} \Delta^{\{0,1,n+1\}}_\flat \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda^{n+1}_0)_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta^{n+1}_\flat \arrow[ur, dashed] & \end{tikzcd} \tag{1}$$

has a dashed solution.

Let us examine this in the case $n = 2$. In this case we have the data of categories and functors

$$\begin{tikzcd} & Y \arrow[drr, "f_!F"] \arrow[ddr, "g"] & & \\ X \arrow[ur, "f"] \arrow[rrr, "F"] \arrow[drr, "h"'] & & & \mathcal{C} \\ & & Z \arrow[ur, "G"'] & \end{tikzcd}$$

together with natural transformations $\eta\colon F \Rightarrow f_!F \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow G \circ h$. We need to produce a natural transformation $\alpha\colon f_!F \Rightarrow G \circ g$ and a filling of the full 3-simplex, which is the data of a homomotopy-commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \arrow[d, "\alpha f"] \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}$$

in $\mathrm{Fun}(X, \mathcal{C})$.

We can rephrase this as follows. We are given the data

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \begin{tikzcd} \overbrace{f_!F}^{\text{in Fun}(Y,\mathcal{C})} \\ G \circ g \end{tikzcd}$$

where the left-hand diagram is a map $LC^2 \to \mathrm{Fun}(X, C)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ except the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathrm{Fun}(Y, C)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

1. We first can fill the lower-left half of the diagram on the left simply by taking the composition $G\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \begin{tikzcd} \overbrace{f_!F}^{\text{in Fun}(Y,\mathcal{C})} \\ G \circ g \end{tikzcd} .$$

2. We note that we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left under the adjunction $f_! \dashv f^*$:

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

3. The diagram on the right has an obvious filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \arrow[d, "\alpha f"] \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \arrow[d, "\alpha"] \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

The lifting problems which we have to solve in Equation 1 for $n > 2$ amount to replacing $\Delta^1 \times \Delta^1$ by $(\Delta^1)^n$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved. In Subsection 4.1 we explain the combinatorics of filling cubes relative to their boundaries. In Subsection 4.2, we give a formalization the concept of adjunct data, and provide a means of for filling partial data to total adjunct data. In Subsection 4.3, we show that we can always solve the lifting problems of Equation 1.

4.1 Filling cubes relative to their boundary

In this section, we give the combinatorics of filling the n-cube $(\Delta^1)^n$ relative to its boundary. Our main result is Proposition 4.1.19, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map. Our first step is to understand the nondegenerate simplices of the n-cube.

Definition 4.1.1. The ***n*-cube** is the simplicial set $C^n := (\Delta^1)^n$.

Note 4.1.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Definition 4.1.3. Denote by $a^i : \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\ldots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\ldots,n\}}$ to $\{1\}$.

The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

We would now like to understand which n-simplices in C^n are nondegenerate. Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0, \ldots, 0)$ and ending at $(1, \ldots, 1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1, \ldots, n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition 4.1.4. For any permutation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$, we define an n-simplex

$$\phi(\tau)\colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)}).$$

Notation 4.1.5. We will denote any permutatation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ by the corresponding n-tuple $(\tau(1), \ldots, \tau(n))$. Thus, the identity permutation is $(1, \ldots, n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2, 1, 3, \ldots, n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)})\colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i\colon \Delta^n \to \Delta^1$ and any face map $\partial_j\colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\ldots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)}, \ldots, a^{\tau(n)}) := (\partial_i^* a^{\tau(1)}, \ldots, \partial_i^* a^{\tau(n)}).$$

Example 4.1.6. Consider the 3-simplex (a^1, a^2, a^3) in C^3. This has spine

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.

- The function $\Delta^3 \longrightarrow \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.
- The function $\Delta^3 \longrightarrow \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation 4.1, we then calculate that

$$d_2(a^1, a^2, a^3) = (a^1, a^2, a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0,0,0) \longrightarrow (1,0,0) \longrightarrow (1,1,1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition 4.1.7. The ***boundary of the n-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{2}$$

The following is easy to see.

Proposition 4.1.8. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \ldots, \gamma_n) \colon \Delta^{n-1} \longrightarrow C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition 4.1.9. The ***left box of the n-cube***, denoted LC^n, is the simplicial subset of C^n

given by the union of all of the faces in Equation 2 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example 4.1.10. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] & (0,1) \\
(1,0) \arrow[r] & (1,1)
\end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma 4.1.11. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

1. The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.
2. For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.
3. The nth face $d_n\phi$ always belongs to LC^n.

Proof. 1. We have

$$d_0\phi = (a^{\tau(0)-1}, \ldots a^{\tau(n)-1})$$

since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition 4.1.8 guarantees that $d_0\phi$ is contained in the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

This face belongs to LC^n except when $j = n$.

2. In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.
3. In this case,

$$d_n(a^{\tau(1)}, \ldots, a^{\tau(n)}) = (a^{\tau(1)}, \ldots, a^{\tau(n)})$$

since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation 4.1.12. For any subset $T \subseteq [n]$, write

$$\Lambda^n_T := \bigcup_{t \in T} d_t\Delta^n \subset \Delta^n.$$

Lemma 4.1.13. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n-1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \ldots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma 4.1.13 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition 4.1.14. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) = \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example 4.1.15. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation 4.1.16. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$\begin{aligned} LC^n &\hookrightarrow LC^n \cup \phi(0,\dots,n) \\ &\hookrightarrow \cdots \\ &\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\dots,n,1)} \phi(\tau) \end{aligned} \tag{3}$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2,\dots,n,1)$, the intersection

$$B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{4}$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition 4.1.17. Each inclusion in Equation 3 is inner anodyne.

Proof. The first inclusion

$$LC^n \hookrightarrow LC^n \cup \phi(0,\dots,n)$$

is inner anodyne because by Lemma 4.1.11 the intersection $LC^n \cap \phi(0,\dots,n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma 4.1.11), so by Lemma 4.1.13, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (0,\dots,n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),$$

which implies that $\tau = (1, \ldots, n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. $\square$

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$. Thus, each of their spines begins

$$(0, \ldots, 0, 0) \to (0, \ldots, 0, 1) \to \cdots,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that $(0, \ldots, 0, 0) \to (0, \ldots, 0, 1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition 4.1.17. One continues to fill simplices in the order prescribed in Definition 4.1.14, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma 4.1.13.

Lemma 4.1.18. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition 4.1.19. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0, \ldots, 0, 0) \to (0, \ldots, 0, 1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \overset{i}{\hookrightarrow} J^n \overset{j}{\hookrightarrow} C^n,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd}
\Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\
J^n \arrow[r, hook] & C^n \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd},$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0, \ldots, 0) \to (0, \ldots 1)$.

4.2 Adjunct data

In this section, we give a formalization of the the notion of adjunct data. Our main result is Proposition 4.2.6, which shows that under certain conditions, we can use data on one side of an adjunction to solve lifting problems on the other side.

Definition 4.2.1. Let $s\colon K \to \Delta^1$ be a map of simplicial sets, and let $p\colon \mathcal{M} \to \Delta^1$ be a bicartesian fibration associated to the adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via equivalences $h_0\colon \mathcal{C} \to \mathcal{M}_0$ and $h_1\colon \mathcal{D} \to \mathcal{M}_1$. We say that a map $\alpha\colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K\times\{0\}} = h_0 \circ \alpha$.
2. The restriction $A|_{K\times\{1\}} = h_1 \circ \tilde{\alpha}$.
3. For each vertex $k \in K$ in the fiber $s^{-1}\{0\}$, the image of the edge $\{k\} \times \Delta^1 \to \mathcal{M}$ is p-cocartesian.
4. For each vertex $k' \in K$ in the fiber $s^{-1}\{1\}$, the image of the edge $\{k'\} \times \Delta^1 \to \mathcal{M}$ is p-cartesian.

We often leave the equivalences h_0 and h_1 implicit to lighen notation.

In the next examples, fix an adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$.

Example 4.2.2. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$ of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is p-cocartesian and b is p-cartesian. This corresponds to the map $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

Example 4.2.3. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id}\colon gD \to gD$ in $\mathcal{C}$. This morphism is adjunct to the component of the unit map $\eta_D\colon D \to fgD$ relative to $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

The next example will follow from Proposition 4.2.6.

Example 4.2.4. Any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g.

Lemma 4.2.5. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^{\{0\}}, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{5}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition 4.2.6. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, h_0 and h_1 be as in Definition 4.2.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s\circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a cofibration such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$

is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{6}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. We further consider the map $h_0 \circ \alpha \colon K \to \mathcal{M}_0$. From these two pieces of data we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd} \left(K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1\right)^{\heartsuit} \arrow[r] \arrow[d] & \mathcal{M}^{\natural} \arrow[d] \\ (K \times \Delta^1)^{\heartsuit} \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^{\sharp} \end{tikzcd},$$

where $(K \times \Delta^1)^{\heartsuit}$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\} \times \Delta^1$, for $k \in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. Starting from K' and adjoining one nondegenerate simplex of K at a time, we can write the left-hand map in the above square as a transfinite composition of pushouts of morphisms of the form given in Equation 5 (by assumption each simplex σ we adjoin is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). Each of these is marked anodyne, so the left-hand map is as well. Thus, a dashed lift ℓ exists, giving us a map $h_1^{-1} \circ \ell|_{K \times \Delta^{\{1\}}} \colon K \to \mathcal{D}$. This, together with the composition

$$K' \times \Delta^1 \xrightarrow{\tilde{\alpha}' \times \mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{c} \mathcal{D}$$

where $c \colon h_1^{-1} \circ h_1 \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, gives a solid diagram

$$\begin{tikzcd} K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times (\Delta^1)^{\sharp} \arrow[r] \arrow[d] & \mathcal{D}^{\natural} \\ K \times (\Delta^1)^{\sharp} \arrow[ur, dashed, "m"'] & \end{tikzcd},$$

where $\mathcal{D}^\natural$ is the marked simplicial set whose underlying simplicial set is $\mathcal{D}$ and where all equivalences are marked. This admits a dashed lift, and restricting $m|_{K\times\Delta^{\{1\}}}$ gives us a map $\tilde{\alpha}\colon K \to \mathcal{D}$ as in Equation 6. However, we still need to display this as adjunct to α: the lift ℓ has $\ell|_{K\times\Delta^{\{0\}}} = h_0 \circ \alpha$, but only $\ell|_{K\times\Delta^{\{1\}}} \simeq h_1 \circ \tilde{\alpha}$, not equality as we require.

Our last task is to remedy this. Denote the spine of the 3-simplex by

$$I_3 := \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \hookrightarrow \Delta^3.$$

We build a map $I_3 \times K \xrightarrow{b} \mathcal{M}$ which:

- On $\Delta^{\{0,1\}} \times K$ is given by the lift ℓ.
- On $\Delta^{\{1,2\}} \times K$ is given by the composition

$$K \times \Delta^1 \xrightarrow{\ell|_{K\times\Delta^{\{1\}}}\times\mathrm{id}} \mathcal{M}_1 \times \Delta^1 \xrightarrow{H} \mathcal{M}_1 \hookrightarrow \mathcal{M},$$

  where H is a natural isomorphism $\mathrm{id}_{\mathcal{M}_1} \to h_1 \circ h_1^{-1}$.
- On $\Delta^{\{2,3\}} \times K$ is given by the natural isomorphism

$$K \times \Delta^1 \xrightarrow{m} \mathcal{D} \xrightarrow{h_1} \mathcal{M}_1 \hookrightarrow \mathcal{M}.$$

The map $I_3 \hookrightarrow \Delta^3$ is inner anodyne, so there exists a map $\Delta^3 \times K \to \mathcal{M}$ whose restriction to $I_3 \times K$ is equal to b. The restriction of this filling to $\Delta^{\{0,3\}} \times K$ satisfies the conditions of Definition 4.2.1 (since the composition of a (co)cartesian morphism with an equivalence is again (co)cartesian), exhibiting $\tilde{\alpha}$ as adjunct to α. $\square$

Note 4.2.7. The filler provided by Proposition 4.2.6 is not guaranteed to be unique (even up to equivalence). However, we have some say in how our filler should look. We construct our adjoint filler $\tilde{\alpha}$ by starting with the solid data of Equation 6 and building the dashed extension $\tilde{\alpha}\colon K \to \mathcal{D}$ by filling a simplex at a time relative to its boundary, such that each new simplex making up the map $\tilde{\alpha}$ is adjunct to the corresponding one in α. We are free to choose each new adjunct simplex however we like.

Example 4.2.8. Let $f : \mathcal{C} \leftrightarrow \mathcal{D} : g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K' = \Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \arrow[rr, hook, "i"] \arrow[dr] & & \Delta^3 = K \arrow[dl] \\ & \Delta^1 & \end{tikzcd},$$

where both downwards-facing arrows take $0, 1 \mapsto 0$ and $2, 3 \mapsto 1$, and fix a map $\tilde{\alpha}\colon K \to N(\mathcal{C})$ and a map $\alpha'\colon K' \to N(\mathcal{D})$ such that $\tilde{\alpha} \circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition 4.2.6 implies that the solution to the lifting problem on the left yields one on the right. Similarly, its dual implies the converse. Thus, Proposition 4.2.6 implies in particular that such lifting problems are equivalent.

4.3 Left Kan implies globally left Kan

In this subsection, we will show that we can solve lifting problems of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd},$$

for $n \geq 2$, where τ is left Kan. Forgetting for a moment that have singled out some 2-simplex τ, and using the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}_{\mathrm{sc}}$, we will do this by solving equivalent lifting problems of the form[1]

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}] \arrow[r] \arrow[d, hook] & \mathbf{QCat} \\ \mathfrak{C}[\Delta^{n+1}] \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}. \tag{7}$$

We expect that the reader is familiar with the basics of (scaled) rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category

[1]Note that we are implicitly taking $\mathfrak{C}[\Lambda_0^{n+1}]$ and $\mathfrak{C}[\Delta^{n+1}]$ to carry the flat markings on their mapping spaces.

$\mathfrak{C}[\Delta^{n+1}]$ are given by the set $\{0, \ldots, n+1\}$, and the mapping spaces are defined by

$$\mathfrak{C}[\Delta^{n+1}](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases}.$$

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \ldots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda^{n+1}_0]$ has the same objects as $\mathfrak{C}[\Delta^{n+1}]$, and each morphism space $\mathfrak{C}[\Lambda^{n+1}_0](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^{n+1}](i,j)$, as we shall now describe.

The map $\mathfrak{C}[\Lambda^{n+1}_0](i,j) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](i,j)$ is an isomorphism except for $(i,j) = (1, n+1)$ and $(i,j) = (0, n+1)$. The missing data corresponds in the case of $(1, n+1)$ to the missing face $d_0\Delta^{n+1}$ of Λ^{n+1}_0, and in the case of $(0, n+1)$ to the missing interior. Thus, to find a filling as in Equation 7, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda^{n+1}_0](1,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{8}$$

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda^{n+1}_0](0,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{9}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \mathrm{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[r, "\ell"] & \mathrm{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \end{tikzcd} \tag{10}$$

commute.

Notation 4.3.1. Recall our desired filling $\mathcal{F}\colon \mathfrak{C}[\Delta^{n+1}] \to \mathbf{QCat}$ of Equation 7. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \ldots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \mathrm{Map}(X_{i_1}, X_{i_k})$ by $f_{i_k \cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

Theorem 4.3.2. For any $n \geq 2$ and any globally left Kan 2-simplex τ, the solid extension problem

$$\begin{tikzcd}
\Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\
(\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\
\Delta_\flat^{n+1} \arrow[ur, dashed] &
\end{tikzcd}$$

admits a dashed filler.

Proof. We need to solve the lifting problems of Equation 8 and Equation 9, and check that our solutions satisfy the condition of Equation 10. We note the following useful pieces of information.

- The information that the simplex $\tau \colon \Delta_\flat^{\{0,1,n+1\}} \to \mathbb{C}\mathrm{at}_\infty$ is left Kan tells us that $f_{n,1}$ is the left Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha_{\{0,n+1\}}^{\{0,1,n+1\}} : f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$ is the unit map.
- We have an adjunction

$$(f_{1,0})_! : \mathrm{Fun}(X_0, X_{n+1}) \longleftrightarrow \mathrm{Fun}(X_1, X_{n+1}) : f_{1,0}^*.$$

It is easy to see that we have the following succinct descriptions of the inclusions of Equation 8 and Equation 9 along which we have to extend.

- There is an isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](0, n+1) \xrightarrow{\cong} C^n$$

  specified completely by sending a subset $S \subseteq [n+1]$ to the point

$$(z_1, \ldots, z_n) \in C^n, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

  The inclusion $\mathfrak{C}[\Lambda_0^{n+1}](0, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](0, n+1)$ corresponds to the simplicial subset $LC^n \hookrightarrow C^n$.

- There is a similar isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](1, n+1) \xrightarrow{\cong} C^{n-1}$$

specified completely by sending $S \subseteq \{1, \ldots, n+1\}$ to the point

$$(z_1, \ldots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$

The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](1, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](1, n+1)$ corresponds to the inclusion $\partial C^{n-1} \hookrightarrow C^{n-1}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^n \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ C^n \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-1} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$(\star) = \begin{tikzcd} C^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \arrow[d, "f^*_{1,0}"] \\ C^n \arrow[r] & \mathrm{Fun}(X_0, X_{n+1}) \end{tikzcd}$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition 4.1.19, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$(*') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed] & \end{tikzcd}.$$

The image of the 1-simplex $(0, \ldots, 0) \to (0, \ldots, 1)$ of $\Delta^0 * \partial C^{n-1} \subseteq C^n$ under α is the unit map $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}} \colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$. This is not an equivalence, so we cannot use Proposition 4.1.19 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to the identity map $\mathrm{id}_{f_{n+1,1}}$ in $\mathrm{Fun}(X_1, X_{n+1})$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-1} is in the image of $f^*_{1,0}$, so by Proposition 4.2.6 we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-1} \to \mathrm{Fun}(X_1, X_{n-1})$ which is adjunct to α relative to the map

$$s \colon \Delta^0 * \partial C^{n-1} \to \Delta^1$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-1} to $\Delta^{\{1\}}$; by Note 4.2.7, we can choose $\tilde{\alpha}$ such that the image of the morphism $(0, \ldots, 0) \to (0, \ldots 1)$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$(**') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed, "\tilde{\beta}"'] & \end{tikzcd}$$

However, since image of $(0, \ldots 0) \to (0, \ldots, 1)$ is an equivalence, Proposition 4.1.19 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition 4.2.6, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-1}}$ be the image of $\tilde{\beta}|_{C^{n-1}}$ under $f_{1,0}^*$, which is true by construction. □

5 The non-monoidal construction

5.1 The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} .$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \arrow[dl] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \arrow[ul, phantom, "\Rightarrow"] \end{tikzcd} , \tag{11}$$

To save on notation, we will generally denote such a morphism as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} . \tag{12}$$

There is a potential for confusion here. The natural transformation notated in Diagram 12 above does not correspond to a single natural transformation in Diagram 11, but rather to both natural transformations simultaneously. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cartesian if and only if all 2-simplices forming the faces of the corresponding map $Q(1) \to \mathbb{C}\mathrm{at}_\infty$ are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 5.1.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex $\sigma\colon Q(1) \to \mathbb{C}\mathrm{at}_\infty$ to which it corresponds has the property that $d_3\sigma$ is left Kan (Definition 3.0.1). More explicitly, the morphism is left Kan if and only if σ the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \arrow[dl, phantom, "\circlearrowleft", pos=0.3] \\ \mathcal{C} \arrow[r, equal] \arrow[ur, phantom, "\Rightarrow", pos=0.3] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] & Y \arrow[d] \arrow[dl, "f_!\mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] \arrow[ur, phantom, "\overset{\eta}{\Rightarrow}", pos=0.7] & \mathcal{C} \arrow[ul, phantom, "\circlearrowleft", pos=0.3] \end{tikzcd},$$

where $f_!\mathcal{F}$ is a left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines both the data of the first and the filling 3-simplex up to contractible choice.

Proof. First we show that morphisms of the promised form really are r-cocartesian. Since by assumption $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cartesian fibration (hence an isofibration) whose source and target are quasicategories, it suffices to show that we can solve the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "a"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{S} \end{tikzcd}$$

where a is of the promised form, and that the space of such fillings is contractible. ◻

For short, we will denote r-cocartesian morphisms with an exclamation mark:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "=!\Rightarrow"] & X' \arrow[d, "\mathrm{Lan}_f \mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} .$$

5.2 The triple structures

5.2.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 5.2.1. This is an adequate triple structure.

Proof. Trivial. ◻

Proposition 5.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

5.2.2 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_{\dagger} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^{\dagger}$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[l, phantom, "\circlearrowleft"] \arrow[r, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_! g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 5.2.3. The triple $(\mathcal{Q}, \mathcal{Q}_{\dagger}, \mathcal{Q}^{\dagger})$ is adequate.

To do this, we need to prove:

Lemma 5.2.4. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd}
A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\
B' \arrow[r] & C
\end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
a \arrow[rr] \arrow[dr] \arrow[dd] & & b \arrow[dr] \arrow[dd] & \\
& b' \arrow[rr] \arrow[dd] \arrow[r, phantom, "\Longrightarrow"] \arrow[dl, phantom, "\Downarrow\alpha"] & & c \arrow[dd] \arrow[dl, phantom, "\Downarrow\beta"] \\
\bar{a} & & \bar{b} \arrow[ll] & \\
& \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$.[2] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 5.2.3. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 5.2.4

We also need to show that any such pullback is of this form. This also follows from Lemma 5.2.4. □

5.3 Building categories of spans

Lemma 5.3.1. Let

$$\begin{tikzcd} X \times_Y^h Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times_Y^h Y')_{/y'} \simeq X \times_Y^h (Y'_{/y'}) \tag{13}$$

Proof. We can model the homotopy pullback $X \times_Y^h Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 13 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. □

[2]Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

Proposition 5.3.2. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] \arrow[ddr, phantom, "\circlearrowleft"] \arrow[ddrr, phantom, "\Rightarrow"] & & Y' \arrow[dr, "g"] \arrow[dd] \arrow[ddr, phantom, "\circlearrowleft"] & \\ & X \arrow[rr, "f"] \arrow[dd] \arrow[ddrr, phantom, "\Rightarrow"] & & Y \arrow[dd] \\ \mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\ & \mathcal{C} \arrow[rr, equal] & & \mathcal{C} \end{tikzcd}$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\ & \mathcal{C} & \\ X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"'] \end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 5.3.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\ &\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

6 The monoidal construction

6.1 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^{I} \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i} \end{tikzcd}$$

commutes.

Example 6.1.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 6.1.2. A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[3] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}'_i$, $i = 1, 2$.
- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' & \mathcal{D}'_1 \times \mathcal{D}'_2 \arrow[l, "\beta"] \end{tikzcd} \quad \Longrightarrow .$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 6.1.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

[3] Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d), y \in F(c)$, and $\phi\colon x \longrightarrow Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\
\Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\
 & \mathcal{C}^{\mathrm{op}}
\end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \longrightarrow F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd}
\Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0))
\end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. □

Note that for each $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \longrightarrow (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \longrightarrow \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n\rangle \mapsto (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n.$$

Because the pointwise product of any number of thin 1-simplices in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$, the conditions of Lemma 6.1.3 are satisfied. Unrolling, we find that $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{14}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 6.1.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\ \widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C})_{\otimes} \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_{\times} \arrow[dl, "p"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} .$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_{\times}$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 14, we have the following.

Proposition 6.1.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd} \vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \arrow[dl, phantom, "\Longrightarrow"] \\ \vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"] \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.
- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_{\otimes}$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_{\times}$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_{\otimes}$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

6.2 The triple structures

6.2.1 The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_{\dagger}, \mathcal{F}^{\dagger})$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_{\dagger} = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^{\dagger} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 6.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 6.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

6.2.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 6.2.3. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 6.2.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd}
\langle n \rangle & \langle m \rangle \arrow[l] \\
\langle n' \rangle \arrow[u, "\phi"] & \langle m' \rangle \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) □

6.2.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{X} \arrow[d] \arrow[r, phantom, "\circlearrowleft" description] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, "\Longrightarrow" description] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, "\circlearrowleft" description] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t \circ \mathcal{F}" description] \arrow[r, phantom, "\Longrightarrow" description] & \vec{Y} \arrow[d, "f_!(g^* t \mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd}.$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 6.2.5. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_\otimes$ of the form

$$\begin{tikzcd}
y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\
x \arrow[r, "f"] & y
\end{tikzcd},$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

6.3 Building categories of spans

Proposition 6.3.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 6.3.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_{\otimes}$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_{\times}$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd}
\langle n \rangle & \langle n \rangle \arrow[l, "\gamma'"'] \\
\langle m \rangle \arrow[u, "\tau'"] & \langle m \rangle \arrow[l, "\gamma"'] \arrow[u, "\tau"']
\end{tikzcd},$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd}
\vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\
\vec{X} \arrow[r, "f"] & \vec{Y}
\end{tikzcd},$$

where f has components

$$f_i \colon X_i \to Y_{\gamma^{-1}(i)}, \qquad i \in \langle m \rangle^{\circ},$$

f' has components

$$f'_j \colon X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n \rangle^{\circ},$$

g has components

$$g_j \colon Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n \rangle^{\circ},$$

and g' has components

$$g'_j \colon X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n \rangle^{\circ}.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{\left(\prod_{\tau'(i)=j} Y_i\right)} \left(\prod_{\tau'(i)=j} X_i \right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'" near start] & \\
& \vec{X} \arrow[rr, "f" near start] \arrow[dd, "\mathcal{F}" near start] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes" near start] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"']
\end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^\circ$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathbf{LS}(\mathcal{C})_\otimes$ above to a pullback square entirely in the fiber over $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathbf{LS}(\mathcal{C})_\otimes)_{\langle n \rangle} \cong (\mathbf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathbf{LS}(\mathcal{C})$, one for each $j \in \langle n \rangle^\circ$, of the form

$$\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}" near start] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr] \arrow[dd, "\mathcal{A}" near start] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd},$$

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i \colon X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

7 Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$\begin{tikzcd} \mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl] \\ & \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) & \end{tikzcd}.$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, giving us isofibrations

$$\begin{tikzcd} P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\ & \mathcal{F}\mathrm{in}_* & \end{tikzcd},$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an $^{\mathrm{op}}$. Composing with $^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.