Monoidal push-pull for local systems

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2.1. A review of bisimplicial sets

In this section we review the basic theory of bisimplicial sets as laid out in [1]. This is mainly to fix notation.

Bisimplicial sets can be defined in two equivalent ways:

- As functors $\Delta^{op} \longrightarrow \mathcal{S}\mathrm{et}_\Delta$,
- As functors $(\Delta^{op})^2 \longrightarrow \mathcal{S}\mathrm{et}$.

In the former case, we think of a bisimplicial set X as an $\mathbb{N}$-indexed collection of simplicial sets X_n; in the latter, we think of a bisimplicial set as an $\mathbb{N} \times \mathbb{N}$-indexed collection of sets X_{mn}. Both points of view are useful, and we will rely on both of them. For this reason, we fix the following convention: with the second point of view in mind, we imagine a bisimplicial set X as a collection of sets, each located at an integer lattice point of the first quadrant, where the first coordinate increases in the x-direction and the second coordinate increases in the y-direction.

$$\begin{array}{cccc}
X_{03} & X_{13} & X_{23} & X_{33} \\
X_{02} & X_{12} & X_{22} & X_{32} \\
X_{01} & X_{11} & X_{21} & X_{31} \\
X_{00} & X_{10} & X_{20} & X_{30}
\end{array}$$

Thus, the nth row of X is the simplicial set $X_{\bullet n}$, and the mth column of X is the simplicial set $X_{m\bullet}$. When we think of bisimplicial sets as $\mathbb{N}$-indexed collections of simplicial sets X_m, we mean by X_m the mth *column* of X; that is, $X_m = X_{m,\bullet}$.

If X is Reedy fibrant (i.e. fibrant with respect to the Reedy model structure on $\mathcal{S}\mathrm{et}_{\Delta^2}$, see Subsection 2.1), then each simplicial set X_n is a Kan complex. Later, when we are interested in Segal spaces, we will interpret X_n as the space of n-simplices of X.

Definition 2.1.1. Define a functor $-\square-\colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_{\Delta^2}$ by the formula

$$(X, Y) \mapsto X\square Y, \qquad (X\square Y)_{mn} = X_m \times Y_n.$$

We will call this functor the ***box product.***

Example 2.1.2. By the Yoneda lemma, providing a map $\Delta^m\square\Delta^n \to X$ is equivalent to providing as an element of X_{mn}.

Definition 2.1.3. Let A denote a simplicial set, and X a bisimplicial set.

- Define a simplicial set $A\backslash X$ level-wise by

$$(A\backslash X)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(A\square\Delta^n, X).$$

- Define a simplicial set X/A level-wise by

$$(X/A)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n \square A, X).$$

Note that by Example 2.1.2, the simplicial set $\Delta^m\backslash X$ is the mth column of X, which we have agreed to call X_m. Similarly, the simplicial set X/Δ^n is the nth row of X. In particular X/Δ^0 is the zeroth row of X. (This, confusingly, is usually called the *first row* of X; this terminological inconsistency is somewhat justified by the fact that, unlike the columns, one tends to be interested mainly in the zeroth row.)

We provide here a partial proof of the following result because we will need to refer to it later, when we prove Proposition 2.2.8.

Proposition 2.1.4. The box product is *divisible on the left.* This means that for each simplicial set A, there is an adjunction

$$A\square - \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2} : A\backslash - .$$

Similarly, the box product is *divisible on the right.* This means that for each simplicial set B there is an adjunction

$$-\square B \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2} : -/B.$$

Proof. We prove divisibility on the left; because the Cartesian product is symmetric, divisibility on the right is identical. We do this by explicitly exhibiting a natural bijection

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\square B, X) \cong \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X).$$

Define a map

$$\Phi \colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\square B, X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X)$$

by sending a map $f \colon A\square B \to X$ to the map $\tilde{f} \colon B \to A\backslash X$ which sends an n-simplex $b \in B_n$ to the composition

$$A\square\Delta^n \xrightarrow{(\mathrm{id},b)} A\square B \xrightarrow{f} X .$$

Before we define our map in the other direction, we need an intermediate result. Define a map $\mathrm{ev} \colon A\square(A\backslash X) \to X$ level-wise by taking $(a,\sigma) \in A_m \times (A\backslash X)_n$ to

$$\sigma_{mn}(a, \mathrm{id}_{\Delta^n}) \in X_{mn}.$$

Then define a map

$$\Psi \colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(B, A\backslash X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\square B, X)$$

sending a map $g\colon B \to A\backslash X$ to the composition

$$A\square B \xrightarrow{(\mathrm{id},g)} A\square(A\backslash X) \xrightarrow{\mathrm{ev}} X \ .$$

The maps Φ and Ψ are mutually inverse, and provide the necessary natural bijection.

The other bijection is defined analogously, so we only fix notation which we will need later. We will call the mutually inverse maps

$$\Phi'\colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\square B, X) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(A, X/B)$$

and

$$\Psi'\colon \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(A, X/B) \to \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(A\square B, X),$$

where in defining Ψ' we use a map $\mathrm{ev}'\colon (X/B)\square B \to X$ determined by sending

$$(\phi\colon \Delta^m\square B \to X, b \in B_n) \mapsto \phi_{mn}(\mathrm{id}_{\Delta^m}, b). \qquad \square$$

Proposition 2.1.4, together with the fact that $\mathcal{S}\mathrm{et}_\Delta$ and $\mathcal{S}\mathrm{et}_{\Delta^2}$ are finitely complete and cocomplete, implies all of the results of Appendix A.2 apply to the box product. In the notation found there, we can give a compact formulation of the definition of a Reedy fibration which we will use repeatedly.

Definition 2.1.5. Let $f\colon X \to Y$ be a map between bisimplicial sets. The map f is a ***Reedy fibration*** if either of the following equivalent conditions hold.

- For each monomorphism $u\colon A \to A'$, the map $\langle u\backslash f\rangle$ is a Kan fibration.
- For each anodyne map $v\colon B \to B'$, the map $\langle f/v\rangle$ is a trivial Kan fibration.

(For the notations $\langle u\backslash f\rangle$ and $\langle f/v\rangle$ see Appendix A.2.) That Definition 2.1.5 is equivalent to the usual definition is shown in [1, Prop. 3.4]. We will also make use of the following fact ([1, Prop. 3.10]).

Theorem 2.1.6. If $f\colon X \to Y$ is a Reedy fibration between Segal spaces, then for any monomorphism of simplicial sets v, the map $\langle f/v\rangle$ is an inner fibration.

Corollary 2.1.7. If $f\colon X \to Y$ is a Reedy fibration between Segal spaces, then $f/\Delta^0\colon X/\Delta^0 \to Y/\Delta^0$ is an inner fibration between quasicategories.

2.2. Marked bisimplicial sets

In this section, we define a basic theory of marked bisimplicial sets, in analogy to the theory of marked simplicial sets laid out in [5]. In the following, one should keep in mind that the case in which we are mostly interested is when our bisimplicial spaces are Segal spaces, and thus that only the first (horizontal) simplicial direction should be thought of as categorical. For this reason, only the first simplicial direction will carry a marking.

Definition 2.2.1. A ***marked bisimplicial set*** $(X, \mathcal{E})$ is a bisimplicial set X together with a distinguished subset $\mathcal{E} \subseteq X_{10}$ containing all degenerate edges, i.e. all edges in the image of $s_0 \colon X_{00} \to X_{10}$. Equivalently, a marked bisimplicial set is bisimplicial set X together with a marking $\mathcal{E}$ on the simplicial set X/Δ^0.

Definition 2.2.2. For a marked simplicial set A and an unmarked simplicial set B, define a marking on the bisimplicial set $A\Box B$ as follows: a simplex $(a, b) \in A_1 \times B_0$ is marked if and only if a is marked in A.

This construction gives us a functor

$$-\Box- \colon \mathcal{S}\mathrm{et}_\Delta^+ \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_{\Delta^2}^+.$$

This is potentially ambiguous: we are using the same notation for the marked and unmarked box constructions. However, there is no real chance of confusion: when we write $A\Box B$, we mean the marked construction if A is a marked simplicial set and the unmarked construction if A is an unmarked simplicial set.

Our first order of business is to generalize the results of [1] summarized in Section A.2 to the marked case. We will first show that the above functor is divisible on the left and on the right.

Notation 2.2.3. For any marked simplicial set A, denote the underlying unmarked simplicial set by $\mathring{A}$. Similarly, for any marked bisimplicial set X, denote the underlying unmarked bisimplicial set by $\mathring{X}$.

Definition 2.2.4. Let A denote a marked simplicial set, B an unmarked simplicial set, and X a marked bisimplicial set.

- Define an unmarked simplicial set $A\backslash X$ level-wise by

$$(A\backslash X)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}^+}(A\Box\Delta^n, X).$$

- Define a marked simplicial set X/B as follows. The underlying simplicial set is the same as $\mathring{X}/B$, and a 1-simplex $\Delta^1 \to X/B$ is marked if and only if the corresponding map

$\Delta^1 \square B \to \mathring{X}$ of unmarked bisimplicial sets descends to a map of marked bisimplicial sets $(\Delta^1)^\sharp \square B \to X$.

Again, we are overloading notation, so there is the potential for confusion. However, there is no real ambiguity; the symbol $A \backslash X$ means the marked construction if A and X are marked, and the unmarked construction if A and X are unmarked. We have tried to be clear in stating whether (bi)simplicial sets do or do not carry markings.

Example 2.2.5. Recall that we can think of a marked bisimplicial set X as an unmarked bisimplicial set $\mathring{X}$ together with a marking $\mathcal{E}$ on the simplicial set $\mathring{X}/\Delta^0$. The marking $\mathcal{E}$ agrees with the marking on X/Δ^0.

We will need the following analogs of the $\flat$- and $\sharp$-markings for marked simplicial sets.

Example 2.2.6. For any unmarked bisimplicial set X, we have the following canonical markings.

- The *sharp marking* $X^\sharp$, in which each element of X_{10} is marked.
- The *flat marking* $X^\flat$, in which only the edges in the image of $s_0 \colon X_{00} \to X_{10}$ are marked.

Example 2.2.7. For each unmarked simplicial set A and marked bisimplicial set X, there is an isomorphism

$$A^\flat \backslash X \cong A \backslash \mathring{X}.$$

Similarly, for any unmarked bisimplicial set Y and marked simplicial set B, there is an isomorphism

$$B \backslash Y^\sharp \cong \mathring{B} \backslash Y.$$

The marked constructions above have similar properties to the unmarked constructions from Section 2.1. In particular, we have the following.

Proposition 2.2.8. We have the following adjunctions.

1. For each marked simplicial set $A \in \mathcal{S}\mathrm{et}_\Delta^+$ there is an adjunction.

$$A \square - \colon \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2}^+ : A \backslash -$$

2. For each unmarked simplicial set $B \in \mathcal{S}\mathrm{et}_\Delta$ there is an adjunction.

$$- \square B \colon \mathcal{S}\mathrm{et}_\Delta^+ \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2}^+ : -/B.$$

Proof. We start with the first, fixing a marked simplicial set A, an unmarked simplicial set B, and a marked bisimplicial set X. We have inclusions

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_{\Delta^2}}(A\square B, X) \overset{i_0}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(\mathring{A}\square B, \mathring{X})$$

and

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(B, A\backslash X) \overset{i_1}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta}}(B, \mathring{A}\backslash \mathring{X}).$$

We have a natural bijection between the right-hand sides of the above inclusions given by the maps Φ and Ψ of Proposition 2.1.4. To show that there is a natural bijection between the subsets, it suffices to show that Φ and Ψ restrict to maps between the subsets.

To this end, suppose we have a map of bimarked simplicial sets $f\colon A\square B \to X$. The inclusion i_0 forgets the markings, sending this to the map

$$\mathring{f}\colon \mathring{A}\square B \to \mathring{X}.$$

Under Φ, this is taken to a map $\Phi(\mathring{f})\colon B \to \mathring{A}\backslash\mathring{X}$. We would like to show that $\Phi(\mathring{f})$ factors through $A\backslash X$, giving a map $\tilde{f}\colon B \to A\backslash X$. The map $\Phi(\mathring{f})$ takes an n-simplex $b \in B_n$ to the composition

$$\mathring{A}\square\Delta^n \xrightarrow{(\mathrm{id},b)} \mathring{A}\square B \xrightarrow{\mathring{f}} \mathring{X}\ .$$

We need to check that this is an n-simplex in $A\backslash X$, and not just $\mathring{A}\backslash\mathring{X}$, i.e. that it respects the markings on $A\square\Delta^n$ and X. That (id, b) respects the markings on $A\square\Delta^n$ and $A\square B$ is clear, and $\mathring{f}$ respects the markings on $A\square B$ and X because f is a map of marked simplicial sets by assumption. Thus $\Phi(\mathring{f})$ restricts to a map $\tilde{f}\colon B \to A\backslash X$.

Now we show the other direction. Suppose we have a map $g\colon B \to A\backslash X$. The inclusion i_1 takes this to the composition

$$B \xrightarrow{g} A\backslash X \hookrightarrow A^\flat\backslash X \cong \mathring{A}\backslash\mathring{X},$$

which we denote by $\mathring{g}$ by mild abuse of notation. Under Ψ, this is mapped to the composition

$$\Psi(\mathring{g})\colon\ \ \mathring{A}\square B \xrightarrow{\mathrm{id}\times\mathring{g}} \mathring{A}\square(\mathring{A}\backslash\mathring{X}) \xrightarrow{\mathrm{ev}} \mathring{X}.$$

We need to check that this respects the markings on $A\square B$ and X, i.e. that for each marked simplex $a \in A_1$ and each $b \in B_0$, the element $\Psi(\mathring{g})_{10}(a, b)$ is marked in X_{10}. But $\Psi(\mathring{g})_{10}(a, b) = g(b)_{10}(a, \mathrm{id}_{\Delta^0})$, which is marked because g lands in $A\backslash X$ by assumption. Thus, $\Psi(\mathring{g})$ descends to a map $\tilde{g}\colon A\square B \to X$.

Now we show the other bijection. Unlike the unmarked case, because of the asymmetry of the marked box product, this is not precisely the same as what we have just shown. Again we have inclusions

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}^+_{\Delta^2}}(A\square B, X) \overset{j_0}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_{\Delta^2}}(\mathring{A}\square B, \mathring{X})$$

and

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^+}(A, B\backslash X) \overset{j_1}{\subseteq} \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\mathring{A}, B\backslash \mathring{X}),$$

and a bijection between the right-hand sides given by the maps Φ' and Ψ' from Proposition 2.1.4. As before, suppose that

$$f\colon A\square B \to X$$

is a map of marked bisimplicial sets. Under j_0, this is sent to $\mathring{f}\colon \mathring{A}\square B \to \mathring{X}$. Then $\Phi'(\mathring{f})\colon A \to X/B$ is defined by sending $\sigma \in A_n$ to the composition

$$\Delta^n\square B \xrightarrow{(\sigma,\mathrm{id})} \mathring{A}\square B \xrightarrow{\mathring{f}} \mathring{X}\ .$$

We need to show that for each marked $a \in A_1$, the corresponding map

$$\Phi'(\mathring{f})(a)\colon\ \ \Delta^1\square B \xrightarrow{(a,\mathrm{id})} \mathring{A}\square B \xrightarrow{\mathring{f}} \mathring{X}$$

descends to a map of marked bisimplicial sets $\tilde{f}\colon (\Delta^1)^\sharp\square B \to X$, and thus corresponds a marked 1-simplex in to X/B. But that the first map has this property is clear because a is marked by assumption, and the map $\mathring{f}$ has this property because f is a map of marked simplicial sets by assumption.

Now, let $g\colon A \to X/B$ be a map of marked simplicial sets. We need to check that the composition

$$\Psi(\mathring{g})\colon\ \ \mathring{A}\square B \xrightarrow{(\mathring{g},\mathrm{id})} (\mathring{X}/B)\square B \xrightarrow{\mathrm{ev}'} \mathring{X}$$

takes marked edges to marked edges. Let $(a, b) \in A_1 \times B_0$, with a marked. This maps to

$$(a, b) \mapsto (g(a), b) \mapsto g(a)_{10}(\mathrm{id}_{\Delta^1}, b) \in X_{10}.$$

By definition, $g(a)$ is a map of marked simplicial sets

$$(\Delta^1)^\sharp\square B \to X$$

which therefore sends $(\mathrm{id}_{\Delta^1}, b)$ to a marked edge in X by assumption. □

This shows that the marked version of the box product □ is, in the language of [1], *divisible on the left and on the right.* Thus, the results summarized in Section A.2 apply.

Definition 2.2.9. We will call an inclusion of unmarked simplicial sets $B \hookrightarrow B'$ *full* if it has the following property: an n-simplex $\sigma\colon \Delta^n \to B'$ factors through B if and only if each vertex of σ factors through B. That is, any n-simplex in B' whose vertices belong to B belongs to B.

Lemma 2.2.10. For any marked simplicial set A and Reedy-fibrant marked bisimplicial set X, the simplicial set $A\backslash X$ is a Kan complex, and the inclusion $i\colon A\backslash X \hookrightarrow A^\flat\backslash X \cong \mathring{A}\backslash\mathring{X}$ is full.

Proof. We first show that the map i is a full inclusion. The n-simplices of $A\backslash X$ are maps of marked simplicial sets $\tilde{\sigma}\colon A\Box\Delta^n \to X$. A map of underlying bisimplicial sets gives a map of marked bisimplicial sets if and only if it respects the markings, i.e. if and only if for each $(a,i) \in A_1 \times (\Delta^n)_0$ with a marked, $\tilde{\sigma}(a,i)$ is marked in X. This is equivalent to demanding that $\sigma|_{\Delta^{\{i\}}}$ belong to $A\backslash X$.

To show that $A\backslash X$ is a Kan complex, we need to find dashed lifts

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \\ \Delta^n \arrow[ur, dashed] \end{tikzcd}, \qquad n \geq 1, \quad 0 \leq k \leq n.$$

For $n = 1$, the horn inclusion is of the form $\Delta^0 \hookrightarrow \Delta^1$, and we can take the lift to be degenerate. For $n \geq 2$, we can augment our diagram as follows.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & A\backslash X \arrow[r, hook] & A^\flat\backslash X \\ \Delta^n \arrow[urr, dashed] \end{tikzcd}.$$

Since $A^\flat\backslash X$ is a Kan complex, we can always find such a dashed lift. The inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ is surjective on vertices, so our lift factors through $A\backslash X$. $\square$

Recall that when thinking of a bisimplicial set X as a cosimplicial object in $\mathcal{S}\mathrm{et}_\Delta$, we think of the simplicial set X_1 as the space of 1-simplices in X. In particular, if X is a Segal space, then X_1 should be thought of as the space of morphisms in X. We should think of morphisms which are in the same path component of X_1 as equivalent. Therefore, we would like to pay special attention to markings which respect this homotopical structure.

Definition 2.2.11. Let $(X, \mathcal{E})$ be a marked bisimplicial set. We will say that the marking $\mathcal{E}$ ***respects path components*** if it has the following property: for any map $\Delta^1 \to X_1$ representing an edge $e \to e'$ between morphisms e and e', the morphism e is marked if and only if the morphism e' is marked.

Proposition 2.2.12. Let $f\colon X \to Y$ be a Reedy fibration between marked bisimplicial sets such that the marking on X respects path components, and let $u\colon A \to A'$ be a morphism of marked simplicial sets whose underlying morphism of unmarked simplicial sets is a monomorphism. Then the map $\langle u\backslash f\rangle$ is a Kan fibration.

Proof. We need to show that for each $n \geq 0$ and $0 \leq k \leq n$ we can solve the lifting problem

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[d, "\langle u\backslash f\rangle"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & A\backslash X \times_{A'\backslash Y} A'\backslash Y \end{tikzcd} .$$

First assume that $n \geq 2$. We can augment the above square as follows.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & A'\backslash X \arrow[r] \arrow[d] & A^\flat\backslash X \arrow[d] \\ \Delta^n \arrow[r] & A\backslash X \times_{A'\backslash Y} A'\backslash Y \arrow[r] & A^\flat\backslash X \times_{(A')^\flat\backslash Y} (A')^\flat\backslash X \end{tikzcd} .$$

Since the map on the right is a Kan fibration, we can solve the outer lifting problem. All the vertices of Δ^n belong to Λ^n_k, so a lift of the outside square factors through $A'\backslash X$.

Now take $n = 1$, $k = 0$, so our horn inclusion is $\Delta^{\{0\}} \hookrightarrow \Delta^1$. By Proposition A.2.2, the lifting problem we need to solve is equivalent to

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X/\Delta^1 \arrow[d] \\ A' \arrow[r] \arrow[ur, dashed] & X/\Delta^0 \times_{Y/\Delta^0} Y/\Delta^1 \end{tikzcd} .$$

Because f is a Reedy fibration, the underlying diagram

$$\begin{tikzcd} \mathring{A} \arrow[r] \arrow[d] & \mathring{X}/\Delta^1 \arrow[d] \\ \mathring{A}' \arrow[r] \arrow[ur, dashed] & \mathring{X}/\Delta^0 \times_{\mathring{Y}/\Delta^0} \mathring{Y}/\Delta^1 \end{tikzcd} .$$

of unmarked simplicial sets always admits a lift. It therefore suffices to check that any such lift respects the marking on X. To see this, consider the following triangle formed by some dashed lift.

$$\begin{tikzcd} & \mathring{X}/\Delta^1 \arrow[d] \\ \mathring{A}' \arrow[ur, dashed] \arrow[r] & \mathring{X}/\Delta^0 \end{tikzcd}$$

Let $a \in A'_1$ be a marked 1-simplex, and consider the diagram

$$\begin{tikzcd} \Delta^1 \square \Delta^0 \arrow[r, "(a,\mathrm{id})"] \arrow[d] & \mathring{A}' \square \Delta^0 \arrow[d] \arrow[dr, "\gamma"] & \\ & & \mathring{X} \\ \Delta^1 \square \Delta^1 \arrow[r, "(a,\mathrm{id})"] & \mathring{A}' \square \Delta^1 \arrow[ur, dashed, "\ell"'] & \end{tikzcd},$$

where the triangle on the right is the adjunct to the triangle above. In order to check that the dashed lift respects the marking on X, we have to show that for each $(a, b) \in (A' \square \Delta^1)_{10} = A'_1 \times \{0, 1\}$ with a marked, the element $\ell(a, b) \in X_{10}$ is marked. Because the map γ comes from a map of marked simplicial sets, the commutativity of the triangle guarantees this for $b = 0$. The map $\Delta^1 \square \Delta^1 \to \mathring{X}$ gives us a 1-simplex $\Delta^1 \to X_1$ representing a 1-simplex $\ell(a, 0) \to \ell(a, 1)$, which implies by that $\ell(a, 1)$ is also marked because each marking respects path components.

The case $n = 1$, $k = 1$ is analogous. □

2.3. Simplicial technology

In the next section, we will need to work in several different cases with simplicial subsets $A \subseteq \Delta^n$ with certain conditions placed on the edge $\Delta^{\{0,1\}}$. In this section we prove some technical results in this direction. The main result in this section is Lemma 2.3.5.

For the remainder of this section, fix $n \geq 2$.

Definition 2.3.1. Let X be an unmarked bisimplicial set, and let $e \in X_{10}$. For any simplicial subset $A \subseteq \Delta^n$ such that $\Delta^{\{0,1\}} \subseteq A$, we will use the notation

$$(A \backslash X)^e = A \backslash X \times_{\Delta^{\{0,1\}} \backslash X} \{e\}.$$

The simplicial set $(A \backslash X)^e$ should be thought of as the space of A-shaped diagrams in X with the edge $\Delta^{\{0,1\}}$ fixed. The m-simplices of the simplicial set $(A \backslash X)^e$ are maps $A \square \Delta^m \to X$ such that the pullback

$$\Delta^{\{0,1\}} \square \Delta^m \longrightarrow A \square \Delta^m \longrightarrow X$$

factors through the map $\Delta^{\{0,1\}} \square \Delta^0 \to X$ corresponding to the element $e \in X_{10}$ under the Yoneda embedding.

Comparing simplices level-wise, it is easy to see the following.

Lemma 2.3.2. The square

$$\begin{tikzcd}
(\Delta^n\backslash X)^e \arrow[r] \arrow[d] & \Delta^n\backslash X \arrow[d] \\
(\Delta^n\backslash Y)^{f(e)} \times_{(A\backslash Y)^{f(e)}} (A\backslash X)^e \arrow[r] & \Delta^n\backslash Y \times_{A\backslash Y} A\backslash X
\end{tikzcd}$$

is a (strict) pullback.

Definition 2.3.3. For any simplicial subset $A \subseteq \Delta^n$ containing $\Delta^{\{0,1\}}$, denote the marking on A where the only marked nondegenerate edge is $\Delta^{\{0,1\}}$ by $\mathcal{L}$, and the corresponding marked simplicial set by $A^{\mathcal{L}}$.

Again, comparing simplices level-wise shows the following.

Lemma 2.3.4. Let $f\colon X \to Y$ be a map of marked bisimplicial sets, and let $A \subseteq \Delta^n$ be a simplicial subset with $\Delta^{\{0,1\}} \subseteq A$. Then for any marked edge $e \in X_{10}$, the square

$$\begin{tikzcd}
(\Delta^n\backslash \mathring{X})^e \arrow[r] \arrow[d] & (\Delta^n)^{\mathcal{L}}\backslash X \arrow[d] \\
(\Delta^n\backslash \mathring{Y})^{f(e)} \times_{(A\backslash \mathring{Y})^{f(e)}} (A\backslash \mathring{X})^e \arrow[r] & (\Delta^n)^{\mathcal{L}}\backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X
\end{tikzcd}$$

is a (strict) pullback.

Lemma 2.3.5. Let $f\colon X \to Y$ be a Reedy fibration between marked bisimplicial sets, and let $i\colon A \subseteq \Delta^n$ be a simplicial subset containing $\Delta^{\{0,1\}}$. The following are equivalent:

1. The map
$$\langle i^{\mathcal{L}}\backslash f\rangle\colon (\Delta^n)^{\mathcal{L}}\backslash X \to (\Delta^n)^{\mathcal{L}}\backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X$$
is a trivial fibration.
2. For each marked $e \in X_{10}$, the map
$$p_e\colon (\Delta^n\backslash \mathring{X})^e \to (\Delta^n\backslash \mathring{Y})^{f(e)} \times_{(A\backslash \mathring{Y})^{f(e)}} (A\backslash \mathring{X})^e$$
is a trivial fibration.

Proof. Suppose the first holds. Then Lemma 2.3.4 implies the second.

Next, suppose that the second holds. By Proposition 2.2.12, the map $\langle i^{\mathcal{L}}\backslash f\rangle$ is a Kan fibration, so it is a trivial Kan fibration if and only if its fibers are contractible. Consider any map

$$\gamma\colon \Delta^0 \to (\Delta^n)^{\mathcal{L}}\backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X.$$

This gives us in particular a map $\Delta^0 \to A^{\mathcal{L}}\backslash X$, which is adjunct to a map $A^{\mathcal{L}}\square\Delta^0 \to X$. The pullback

$$(\Delta^{\{0,1\}})^\sharp\square\Delta^0 \longrightarrow A^{\mathcal{L}}\square\Delta^0 \longrightarrow X$$

gives us a marked morphism $e \in X_{10}$. The bottom composition in the below diagram is thus a factorization of γ, in which the left-hand square is a pullback.

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & (\Delta^n\backslash\mathring{X})^e \arrow[r] \arrow[d, "p_e"] & (\Delta^n)^{\mathcal{L}}\backslash X \arrow[d] \\
\Delta^0 \arrow[r] & (\Delta^n\backslash\mathring{Y})^{f(e)} \times_{(A\backslash\mathring{Y})^{f(e)}} (A\backslash\mathring{X})^e \arrow[r] & (\Delta^n)^{\mathcal{L}}\backslash Y \times_{A^{\mathcal{L}}\backslash Y} A^{\mathcal{L}}\backslash X
\end{tikzcd}$$

The right-hand square is a pullback by Lemma 2.3.4. Since by assumption p_e is a trivial fibration, F is contractible. But by the pasting lemma, F is the fiber of $\langle i^{\mathcal{L}}\backslash f\rangle$ over γ. Thus, the fibers of $\langle i^{\mathcal{L}}\backslash f\rangle$ are contractible, so $\langle i^{\mathcal{L}}\backslash f\rangle$ is a trivial Kan fibration. $\square$

2.4. Cocartesian fibrations

Let $\pi\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. There are several equivalent ways of defining when a morphism in $\mathcal{C}$ is π-cocartesian. For our purposes, the following will be the most useful: a morphism $e \in \mathcal{C}_1$ is π-cocartesian if and only if, for all $n \geq 2$, a dashed lift in the below diagram exists.

$$\begin{tikzcd}
\Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "e"] & \\
\Lambda^n_0 \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "\pi"] \\
\Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{D}
\end{tikzcd}$$

We would like to find an analogous definition for a p-cocartesian morphism where $p\colon C \to D$ is a Reedy fibration between Segal spaces. Our definition should have the property that if a morphism in C_{10} is p-cocartesian in the sense of Segal spaces, then it is p/Δ^0-cocartesian in the sense of quasicategories.

To this end, replace π in the above diagram by p/Δ^0. Passing to the adjoint lifting problem, we see that the existence of the above lift is equivalent to demanding that the map

$$p_e\colon (\Delta^n\backslash C)^e \to (\Lambda^n_0\backslash C)^e \times_{(\Lambda^n_0\backslash D)^{p(e)}} (\Delta^n\backslash D)^{p(e)} \tag{1}$$

be surjective on vertices. One natural avenue of generalization of the concept of a cocartesian morphism to Segal spaces would be to upgrade the condition of surjectivity on vertices to an analogous, homotopy-invariant condition which implies it. One such condition is that p_e be a trivial fibration. Indeed, this is the defintion we will use. However, this turns out to be equivalent to demand something superficially weaker.

Definition 2.4.1. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces. A morphism $e \in X_{10}$ is f***-cocartesian*** if the square

$$\begin{tikzcd} (\Delta^2\backslash X)^e \arrow[r] \arrow[d] & (\Lambda^2_0\backslash X)^e \arrow[d] \\ (\Delta^2\backslash Y)^{f(e)} \arrow[r] & (\Lambda^2_0\backslash Y)^{f(e)} \end{tikzcd}$$

is homotopy pullback.

Example 2.4.2. Identity morphisms are f-cocartesian. This is because for $e = \mathrm{id}$, the horizontal morphisms in Definition 2.4.1 are equivalences. More generally, by [7, Lemma 11.6], homotopy equivalences are f-cocartesian.

Cocartesian morphisms automatically respect path components in the following sense.

Proposition 2.4.3. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $\alpha\colon \Delta^1 \to X_1$ be a map representing a path $e \to e'$ between morphisms e and e' in X_1. Then e is f-cocartesian if and only if e' is f-cocartesian.

Proof. Since X and Y are Reedy fibrant and f is a Reedy fibration, it suffices to show that the map

$$(\Delta^2\backslash X)^e \to (\Lambda^2_0\backslash X)^e \times_{(\Lambda^2_0\backslash Y)^{f(e)}} (\Delta^2\backslash Y)^{f(e)} \tag{2}$$

is a weak equivalence if and only if the map

$$(\Delta^2\backslash X)^{e'} \to (\Lambda^2_0\backslash X)^{e'} \times_{(\Lambda^2_0\backslash Y)^{f(e')}} (\Delta^2\backslash Y)^{f(e')} \tag{3}$$

is a weak equivalence.

Consider the diagram

$$\begin{tikzcd} P \arrow[r] \arrow[d] & \Delta^2\backslash X \arrow[d] \\ Q \arrow[r] \arrow[d] & \Lambda^2_0\backslash X \times_{\Lambda^2_0\backslash Y} \Delta^2\backslash Y \arrow[d] \\ \Delta^1 \arrow[r, "\alpha"] & \Delta^{\{0,1\}}\backslash X \end{tikzcd},$$

where both squares are strict pullback. The maps on the right-hand side are Kan fibrations by

Reedy fibrancy and the fact that f is a Reedy fibration, so we get in particular a diagram

$$\begin{tikzcd} P \arrow[rr, "\phi"] \arrow[dr] & & Q \arrow[dl] \\ & \Delta^1 & \end{tikzcd},$$

where both downward-facing maps are Kan fibrations. Note that the component of the map ϕ over Δ^0 is the map from Equation 2, and component over Δ^1 is the map from Equation 3. Kan fibrations are in particular left fibrations, so under the Grothendieck construction this corresponds to a diagram given by the following homotopy-commutative square in $\mathcal{S}\mathrm{et}_\Delta$ (with the Kan model structre), in which the rightward-pointing maps are weak equivalences because our maps $P \to \Delta^1$ and $Q \to \Delta^1$ were Kan fibrations.

$$\begin{tikzcd} (\Delta^2\backslash X)^e \arrow[r, "\simeq"] \arrow[d] & (\Delta^2\backslash X)^{e'} \arrow[d] \\ (\Lambda^2_0\backslash X)^e \times_{(\Lambda^2_0\backslash Y)^{f(e)}} (\Delta^2\backslash Y)^{f(e)} \arrow[r, "\simeq"] & (\Lambda^2_0\backslash X)^{e'} \times_{(\Lambda^2_0\backslash Y)^{f(e')}} (\Delta^2\backslash Y)^{f(e')} \end{tikzcd}$$

By the 2/3 property for weak equivalences, the map on the left is a weak equivalence if and only if the map on the right is a weak equivalence, which is what we wanted to show. □

Definition 2.4.4. Let $f: X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be a cocartesian morphism. Denote by $\mathcal{E}$ the smallest marking which contains e and all degenerate morphisms in X, and which respects path components. More explicitly, the marking $\mathcal{E}$ contains:

- The morphism e;
- Each identity morphism; and
- Any morphism connected to e or any identity morphism by a path.

Our next order of business is to show that our definition of cocartesian morphisms in terms of lifting with respect to the morphism $\Lambda^2_0 \hookrightarrow \Delta^2$ implies lifting with respect to $\Lambda^n_0 \to \Delta^n$ for all $n \geq 2$.

Definition 2.4.5. Define the following simplicial subsets of Δ^n.

- For $n \geq 1$, denote by I_n the ***spine*** of Δ^n, i.e. the simplicial subset

$$\Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \subseteq \Delta^n.$$

- For $n \geq 2$, denote by L_n the simplicial subset

$$L_n = \Delta^{\{0,1\}} \amalg_{\Delta^{\{0\}}} \overbrace{\Delta^{\{0,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \amalg_{\Delta^{\{3\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}}}^{I_{\{0,\hat{1},2,\ldots,n\}}} \subseteq \Delta^n.$$

That is, L_n is the union of $\Delta^{\{0,1\}}$ with the spine of $d_1\Delta^n$. We will call L_n the ***left spine*** of Δ^n.

Note that $L_2 \cong \Lambda^2_0$.

Proposition 2.4.6. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then the square

$$\begin{tikzcd} (\Delta^n\backslash X)^e \arrow[r] \arrow[d] & (L_n\backslash X)^e \arrow[d] \\ (\Delta^n\backslash Y)^{f(e)} \arrow[r] & (L_n\backslash Y)^{f(e)} \end{tikzcd}$$

is homotopy pullback for all $n \geq 2$.

Proof. We have the case $n = 2$ because e is f-cocartesian. Assume the result is true up to $n-1$. Then the square

$$\begin{tikzcd} (\Delta^{n-1}\backslash X)^e \times_{\Delta^{\{n-1\}}\backslash X} \Delta^{\{n-1,n\}}\backslash X \arrow[r] \arrow[d] & (L_{n-1}\backslash X)^e \times_{\Delta^{\{n-1\}}\backslash X} \Delta^{\{n-1,n\}}\backslash X \arrow[d] \\ (\Delta^{n-1}\backslash Y)^{f(e)} \times_{\Delta^{\{n-1\}}\backslash Y} \Delta^{\{n-1,n\}}\backslash Y \arrow[r] & (L_{n-1}\backslash Y)^{f(e)} \times_{\Delta^{\{n-1\}}\backslash Y} \Delta^{\{n-1,n\}}\backslash Y \end{tikzcd}$$

is homotopy pullback since each component is homotopy pullback. But this square is equivalent to

$$\begin{tikzcd} (\Delta^n\backslash X)^e \arrow[r] \arrow[d] & (L_n\backslash X)^e \arrow[d] \\ (\Delta^n\backslash Y)^{f(e)} \arrow[r] & (L_n\backslash Y)^{f(e)} \end{tikzcd} :$$

The left-hand equivalences come from the Segal condition, and the right-hand equivalences come from the definition of L_n. ◻

Corollary 2.4.7. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be a f-cocartesian morphism. Then for all $n \geq 2$, the map

$$(\Delta^n)^{\mathcal{L}}\backslash X^{\mathcal{E}} \to L_n^{\mathcal{L}}\backslash X^{\mathcal{E}} \times_{(\Delta^n)^{\mathcal{L}}\backslash Y^\sharp} L_n^{\mathcal{L}}\backslash Y^\sharp$$

is a trivial Kan fibration.

Proof. Each edge $e' \in \mathcal{E}$ is f-cocartesian: the morphism e is f-cocartesian by assumption, each degenerate edge is f-cocartesian by Example 2.4.2, and any morphism in the path component of an f-cocartesian morphism is f-cocartesian by Proposition 2.4.3.

Therefore, for any $e' \in \mathcal{E}$, the map

$$(\Delta^n \backslash X)^{e'} \to (L_n \backslash X)^{e'} \times_{(\Delta^n \backslash Y)^{f(e')}} (L_n \backslash Y)^{f(e')}$$

is a weak equivalence by Proposition 2.4.6, and it is a Kan fibration by Lemma 2.3.2. The result follows from Lemma 2.3.5. ◻

For any simplicial set A, define a marked simplicial set $(\Delta^1 \star A, \mathcal{L}')$ where the only nondegenerate simplex belonging to $\mathcal{L}'$ is Δ^1. This is a slight generalization of the $\mathcal{L}$-marking.

Lemma 2.4.8. Let $A \hookrightarrow B$ be a monomorphism of simplicial sets, and suppose that B is n-skeletal (and therefore that A is n-skeletal). Then the map

$$(\Delta^{\{0\}} \star B)^\flat \coprod_{(\Delta^{\{0\}} \star A)^\flat} (\Delta^1 \star A)^{\mathcal{L}'} \hookrightarrow (\Delta^1 \star B)^{\mathcal{L}'}$$

is in the saturated hull of the morphisms

$$(\Lambda^k_0)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}, \qquad 2 \le k \le n+2.$$

Proof. It suffices to show this for $A \hookrightarrow B = \partial\Delta^m \hookrightarrow \Delta^m$ for $0 \le m \le n$. In this case the necessary map is of the form

$$(\Lambda^{m+2}_0)^{\mathcal{L}} \hookrightarrow (\Delta^{m+2})^{\mathcal{L}}.$$

◻

Definition 2.4.9. We will say a collection of morphisms $\mathcal{A} \subset \mathrm{Mor}(\mathcal{S}\mathrm{et}^+_\Delta)$ has the ***right cancellation property*** if for all $u, v \in \mathrm{Mor}(\mathcal{S}\mathrm{et}^+_\Delta)$,

$$u \in \mathcal{A}, \quad vu \in \mathcal{A} \quad \Longrightarrow \quad v \in A.$$

Lemma 2.4.10. Let $\mathcal{A}$ be a saturated set of morphisms of $\mathcal{S}\mathrm{et}^+_\Delta$ all of whose underlying morphisms are monomorphisms, and which has the right cancellation property. Further suppose that $\mathcal{A}$ contains the following classes of morphisms.

1. Maps $(A)^\flat \hookrightarrow (B)^\flat$, where $A \to B$ is inner anodyne.
2. Left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \ge 2$.

Then $\mathcal{A}$ contains left horn inclusions $(\Lambda_0^n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$.

Proof. For $n = 2$, there is nothing to check: we have an isomorphism $(L_2)^{\mathcal{L}} \cong (\Lambda_0^2)^{\mathcal{L}}$.

We proceed by induction. Suppose we have shown that all horn inclusions $(\Lambda_0^k)^{\mathcal{L}} \hookrightarrow (\Delta^k)^{\mathcal{L}}$ belong to $\mathcal{A}$ for $2 \leq k < n$. From now on on we will suppress the marking $(-)^{\mathcal{L}}$. All simplicial subsets of Δ^n below will have $\Delta^{\{0,1\}}$ marked if they contain it.

Consider the factorization

$$\begin{tikzcd} L_n \arrow[r, "u_n"] \arrow[rr, bend right, "v_n \circ u_n"'] & \Lambda_0^n \arrow[r, "v_n"] & \Delta^n \end{tikzcd} .$$

The morphism $v_n \circ u_n$ belongs to $\mathcal{A}$ by assumption, so in order to show that v_n belongs to $\mathcal{A}$, it suffices by right cancellation to show that u_n belongs to $\mathcal{A}$. Consider the factorization

$$\begin{tikzcd} L_n \arrow[r, "w_n'"] \arrow[rr, bend right, "u_n"'] & L_n \cup d_1\Delta^n \arrow[r, "w_n"] & \Lambda_0^n \end{tikzcd} .$$

The map w_n' is a pushout along the spine inclusion $I_{\{0,\hat{1},2,\ldots,n\}} \hookrightarrow d_1\Delta^n$, and hence is inner anodyne. Hence, we need only show that w_n belongs to $\mathcal{A}$. Let

$$Q = d_2\Delta^n \cup \cdots \cup d_n\Delta^n,$$

and consider the following pushout diagram.

$$\begin{tikzcd} (L_n \cup d_1\Delta^n) \cap Q \arrow[r, hook] \arrow[d, hook] & Q \arrow[d, hook] \\ L_n \cup d_1\Delta^n \arrow[r, hook] & L_n \cup d_1\Delta^n \cup Q \end{tikzcd}$$

Since $L_n \cup d_1\Delta^n \cup Q \cong \Lambda_0^n$, the bottom map is w_n, so it suffices to show that the top map belongs to $\mathcal{A}$. But this is isomorphic to

$$(\Delta^{\{0,1\}} \star \emptyset) \coprod_{(\Delta^{\{0\}} \star \emptyset)} (\Delta^{\{0\}} \star \partial\Delta^{\{2,3,\ldots,n\}}) \hookrightarrow \Delta^{\{0,1\}} \star \partial\Delta^{\{2,3,\ldots,n\}}.$$

The simplicial set $\partial\Delta^{\{2,\ldots,n\}}$ is $(n-3)$-skeletal, so this map belongs to $\mathcal{A}$ by Lemma 2.4.8. □

For each $n \geq 2$, denote by h^n the $\mathcal{L}$-marked inclusion

$$h^n \colon (\Lambda_0^n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}.$$

Proposition 2.4.11. Let $f \colon X \to Y$ be a Reedy fibration of Segal spaces, and let $e \in X_{10}$ be an

f-cocartesian morphism. Then for all $n \geq 2$, the map

$$\langle h^n \backslash f^{\mathcal{E}} \rangle \colon (\Delta^n)^{\mathcal{L}} \backslash X^{\mathcal{E}} \longrightarrow (\Lambda^n_0)^{\mathcal{L}} \backslash X^{\mathcal{E}} \times_{(\Lambda^n_0)^{\mathcal{L}} \backslash Y^\sharp} (\Delta^n)^{\mathcal{L}} \backslash Y^\sharp$$

is a trivial fibration of simplicial sets.

Proof. Consider the set

$$S = \left\{ \begin{matrix} u \colon A \to \text{B morphism of} \\ \text{marked simplicial sets} \\ \text{such that } \mathring{u} \text{ is mono} \end{matrix} \;\middle|\; \langle u \backslash f^{\mathcal{E}} \rangle \text{ weak homotopy equivalence} \right\}.$$

We claim that this set has the right cancellation property (Definition 2.4.9). The set of morphisms of marked simplicial sets whose underlying morphisms are monic clearly has the right-cancellation property. To show that S does, let $u \colon A \to B$ and $v \colon B \to C$ be such morphisms and consider the following diagram.

$$\begin{tikzcd}
C \backslash X \arrow[rr, "\langle vu \backslash f^{\mathcal{E}} \rangle"] \arrow[d, "\langle v \backslash f^{\mathcal{E}} \rangle"'] & & A \backslash X \times_{A \backslash Y} C \backslash Y \\
B \backslash X \times_{B \backslash Y} C \backslash Y \arrow[r, "\langle u \backslash f^{\mathcal{E}} \rangle \times_{\mathrm{id}} \mathrm{id}"] & \big(A \backslash X \times_{A \backslash Y} B \backslash Y\big) \times_{B \backslash Y} C \backslash Y \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

If $\langle u \backslash f^{\mathcal{E}} \rangle$ is a weak equivalence, then the bottom morphism is a weak equivalence. The right-hand morphism is a weak equivalence because it is an isomorphism, so if $\langle vu \backslash f^{\mathcal{E}} \rangle$ is a weak equivalence, then the $\langle v \backslash f^{\mathcal{E}} \rangle$ is a weak equivalence by 2/3.

By Proposition 2.2.12, a map u belonging to S automatically has the property that $\langle u \backslash f^{\mathcal{E}} \rangle$ a Kan fibration, hence is a trivial Kan fibration. Thus, we can equivalently say that $u \in S$ if and only if u has the left-lifting property with respect to all maps of the form $\langle X/v \rangle$, where v is a cofibration of simplicial sets. Since the set of all monomorphisms is saturated, S is saturated.

The set S contains all flat-marked inner anodyne morphisms because f is a Reedy fibration. Corollary 2.4.7 tells us that S contains all left spine inclusions $(L_n)^{\mathcal{L}} \hookrightarrow (\Delta^n)^{\mathcal{L}}$, $n \geq 2$. Thus, by Lemma 2.4.10, S contains all $\mathcal{L}$-marked left horn inclusions. □

Corollary 2.4.12. Let $f \colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian edge. Then the map

$$(\Delta^n \backslash X)^e \longrightarrow (\Lambda^n_0 \backslash X)^e \times_{(\Lambda^n_0 \backslash Y)^{f(e)}} (\Delta^n \backslash Y)^{f(e)}$$

is a trivial fibration.

Proof. Lemma 2.3.5. □

Corollary 2.4.13. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces, and let $e \in X_{10}$ be an f-cocartesian morphism. Then e is f/Δ^0-cocartesian.

Proof. By Corollary 2.4.12, for all $n \geq 2$ the map

$$(\Delta^n \backslash X)^e \to (\Lambda^n_0 \backslash X)^e \times_{(\Lambda^n_0 \backslash Y)^{f(e)}} (\Delta^n \backslash Y)^{f(e)}$$

is a trivial fibration. Thus, it certainly has the right-lifting property with respect to $\emptyset \hookrightarrow \Delta^0$. Passing to the adjoint lifting problem, we find that this is equivalent to the existence of a dashed lift in the diagram

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "e"] & \\ \Lambda^n_0 \arrow[r] \arrow[d, hook] & X/\Delta^0 \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y/\Delta^0 \end{tikzcd},$$

which tells us that e is f/Δ^0-cocartesian. □

Definition 2.4.14. Let $f\colon X \to Y$ be a Reedy fibration between Segal spaces. We will say that f is a ***cocartesian fibration*** if each morphism in Y has an f-cocartesian lift in X. More explicitly, we demand that, for each edge $e\colon y \to y'$ in Y and each vertex $x \in X$ such that $f(x) = y$, there exists an f-cocartesian morphism $\tilde{e}\colon x \to x'$ such that $f(\tilde{e}) = e$.

Corollary 2.4.15. Let $f\colon X \to Y$ be a cocartesian fibration of Segal spaces. Then the map

$$f/\Delta^0\colon X/\Delta^0 \to Y/\Delta^0$$

is a cocartesian fibration of quasicategories, and if a morphism in X_1 is f-cocartesian, then it is f/Δ^0-cocartesian.

Proof. By Theorem 2.1.6, the map f/Δ^0 is an inner fibration between quasicategories. By assumption, every morphism in Y has a f-cocartesian lift, and these lifts are f/Δ^0-cocartesian by Corollary 2.4.13. □

3. Segal spaces of spans

3.1. Basic definitions

We recall the basic definitions of Segal spaces of spans. Note that spans as we will define them form not only a Segal space, but a *complete* Segal space; for the most part, this will not concern us. For more information, we direct the reader to [2]. This section is intended to be a summary of the relevant results of loc. cit.

The objects of our study will be ∞-categories whose morphisms are spans in some quasicategory $\mathcal{C}$, i.e. diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & y \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ x & & x' \end{tikzcd}.$$

We will want to be able to place certain conditions on the legs ϕ and ψ of our spans. For example, we may want to restrict our attention to spans such that ϕ is an equivalence. More precisely, we pick out two subcategories of $\mathcal{C}$ to which the respective legs of our spans must belong.

Definition 3.1.1. A ***triple*** of categories is a triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, where $\mathcal{C}$ is a quasicategory where $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$ are subcategories, each of which contain all equivalences.

We will use the following terminology and notation for the morphsism in our subcategories.

- We denote the morphisms in $\mathcal{C}_\dagger$ with tails (as in $x \rightarrowtail y$), and call them *ingressive.*
- We denote the morphisms in $\mathcal{C}^\dagger$ with two heads (as in $x \twoheadrightarrow y$), and call them *egressive.*

The egressive morphisms will correspond to the backwards (i.e. leftwards) facing legs of our spans, and the ingressive morphisms will correspond to the forwards (i.e. rightwards) facing legs. We provide the following diagram as a summary of this terminology.

$$\begin{tikzcd} & Y \arrow[dl, two heads, "\substack{\text{egressive} \\ \mathcal{C}^\dagger}"'] \arrow[dr, tail, "\substack{\text{ingressive} \\ \mathcal{C}_\dagger}"] & \\ X & & X' \end{tikzcd}$$

In order for a triple of categories to be able to support a category of spans, it will have to satisfy certain properties. Following Barwick, we will call such triples *adequate.*

Definition 3.1.2. A triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is said to be ***adequate*** if it has the following properties:

1. For any ingressive morphism $f \in \mathcal{C}_\dagger$ and any egressive morphism $g \in \mathcal{C}^\dagger$, there exists a pullback square

$$\begin{tikzcd} y' \arrow[r] \arrow[d] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}.$$

2. For any pullback square
$$\begin{tikzcd} y' \arrow[r, "f'"] \arrow[d] & x' \arrow[d] \\ y \arrow[r, "f"] & x \end{tikzcd},$$
if the arrow f belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$), then the arrow f' belongs to $\mathcal{C}_\dagger$ (resp. $\mathcal{C}^\dagger$).

We will call a square of the form
$$\begin{tikzcd} y' \arrow[r, tail] \arrow[d, two heads] & x' \arrow[d, two heads] \\ y \arrow[r, tail] & x \end{tikzcd}$$
ambigressive. If such an ambigressive square is also a pullback square, we will call it *ambigressive pullback.*

It is now time to set about building our ∞-categories of spans. Fix some triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$. We will define two models for such a category: a Segal space $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, and the quasicategory $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$.

Given two 1-simplices in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ represented by spans $X \twoheadleftarrow Y \rightarrowtail X'$ and $X' \twoheadleftarrow Y' \rightarrowtail X''$ in $\mathcal{C}$, we need to specify what it means to compose them. We define the composition to be the span $X \twoheadleftarrow Y \times_{X'} Y' \rightarrowtail X''$ given by the diagram below, where the top square is pullback.

$$\begin{tikzcd} & & Y \times_{X'} Y' \arrow[dl, dashed, two heads, "q'"'] \arrow[dr, dashed, tail, "f'"] & & \\ & Y \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & & Y' \arrow[dl, two heads, "q"'] \arrow[dr, tail, "p"] & \\ X & & X' & & X'' \end{tikzcd}$$

Note that the conditions in Definition 3.1.2 tell us precisely that such a composition always exists.

Equivalently, we this means that a 2-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ should be a diagram in $\mathcal{C}$ of the above form. More generally, we would like an n-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ to correspond to n-fold composition of spans. For example, for $n = 4$, such an n-simplex in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$

should consist of a diagram in $\mathcal{C}$ of the form

$$
\begin{tikzcd}[column sep=small]
 & & & & X_{04} \arrow[dl, two heads] \arrow[dr, tail] & & & & \\
 & & & X_{03} \arrow[dl, two heads] \arrow[dr, tail] & & X_{14} \arrow[dl, two heads] \arrow[dr, tail] & & & \\
 & & X_{02} \arrow[dl, two heads] \arrow[dr, tail] & & X_{13} \arrow[dl, two heads] \arrow[dr, tail] & & X_{24} \arrow[dl, two heads] \arrow[dr, tail] & & \\
 & X_{01} \arrow[dl, two heads] \arrow[dr, tail] & & X_{12} \arrow[dl, two heads] \arrow[dr, tail] & & X_{23} \arrow[dl, two heads] \arrow[dr, tail] & & X_{34} \arrow[dl, two heads] \arrow[dr, tail] & \\
X_{00} & & X_{11} & & X_{22} & & X_{33} & & X_{44}
\end{tikzcd},
$$

where the morphisms are ingressive and egressive as shown, and each square is pullback.

All that remains is to make this definition rigorous. To this end, we make the following definition.

Definition 3.1.3. for each $n \geq 0$, we define a poset Σ_n as follows:

- The elements of Σ_n are pairs of integers (i, j), with $0 \leq i \leq j \leq n$.
- For $(i, j), (i', j') \in \Sigma_n$, we define

$$(i, j) \leq (i', j') \iff i \leq i' \leq j' \leq j.$$

Example 3.1.4. We can draw Σ_4 as follows.

$$
\begin{tikzcd}[column sep=small]
 & & & & (0,4) \arrow[dl] \arrow[dr] & & & & \\
 & & & (0,3) \arrow[dl] \arrow[dr] & & (1,4) \arrow[dl] \arrow[dr] & & & \\
 & & (0,2) \arrow[dl] \arrow[dr] & & (1,3) \arrow[dl] \arrow[dr] & & (2,4) \arrow[dl] \arrow[dr] & & \\
 & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr] & & (2,3) \arrow[dl] \arrow[dr] & & (3,4) \arrow[dl] \arrow[dr] & \\
(0,0) & & (1,1) & & (2,2) & & (3,3) & & (4,4)
\end{tikzcd}
$$

These posets Σ_n assemble into a cosimplicial object

$$\Delta \to \mathcal{S}\mathrm{et}_\Delta; \qquad [n] \mapsto N(\Sigma_n),$$

Left Kan extending along the Yoneda embedding $\Delta \hookrightarrow \mathcal{S}\mathrm{et}_\Delta$ yields a functor $\mathrm{sd}\colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$.

General abstract nonsense gives us right adjoint $\mathrm{Span}' \colon \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta$ forming an adjunction

$$\mathrm{sd} : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta : \mathrm{Span}'.$$

For any simplicial set A, the simplicial set $\mathrm{Span}'(A)$ has n-simplices

$$\mathrm{Span}'(A)_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\mathrm{sd}(\Delta^n), A).$$

Note that for a quasicategory $\mathcal{C}$, the simplicial set $\mathrm{Span}'(\mathcal{C})$ is very close to what we want; its n-simplices are maps $\Sigma_n \to \mathcal{C}$, but the backward- and forward-facing legs of the spans do not necessarily belong to the categories $\mathcal{C}_\dagger$ and $\mathcal{C}^\dagger$, and the squares are not necessarily pullback. We will use the term to refer to a map $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ which has the desired form.

Definition 3.1.5. For any adequate triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, we will call a functor $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ *ambigressive Cartesian* if each square in $\mathrm{sd}(\Delta^n) = N(\Sigma_n)$ of the form

$$\begin{tikzcd} & (i,j) \arrow[dl] \arrow[dr] & \\ (i, j-\ell) \arrow[dr] & & (i+k, j) \arrow[dl] \\ & (i+k, j-\ell) & \end{tikzcd}$$

(where we include the possibilities $k = 0$ and $\ell = 0$) is mapped to an ambigressive pullback square,

$$\begin{tikzcd} & X_{ij} \arrow[dl] \arrow[dr] & \\ X_{i(j-\ell)} \arrow[dr] & & X_{(i+k)(j)} \arrow[dl] \\ & X_{(i+k)(j-\ell)} & \end{tikzcd}$$

where the backwards-facing morphsims are egressive and the forwards-facing morphisms are ingressive.

We are now ready to define our category $\mathrm{Span}(\mathcal{C})$.

Definition 3.1.6. We define a simplicial set $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ level-wise to be the subset

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)_n \subseteq \mathrm{Span}'(\mathcal{C})_n$$

on functors $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ which are ambigressive Cartesian.

While it is not hard to check that the face and degeneracy maps on $\mathrm{Span}'(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ restrict to $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ (and thus that the above construction really produces a simplicial set), solving the lifting problems necessary to prove directly that $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a quasicategory turns out to be combinatorially strenuous. It turns out to be easier to switch to a new model of quasicategories, (complete) Segal spaces, which give easier access to homotopical data.

To that end, for an adequate triple $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$, we make the following defintion.

Definition 3.1.7. For any simplicial set A, we define a simplicial set $\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(A), \mathcal{C})^{\simeq}$ to be the full simplicial subset

$$\mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(A), \mathcal{C})^{\simeq} \subseteq \mathrm{Map}(\mathrm{sd}(A), \mathcal{C})^{\simeq}$$

on functors $\mathrm{sd}(A) \to \mathcal{C}$ such that for each standard simplex σ of A, the image of $\mathrm{sd}(\sigma)$ in $\mathcal{C}$ is ambigressive Cartesian.

Definition 3.1.8. The (complete) Segal space of spans $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is defined level-wise by

$$\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)_n = \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq}.$$

For a proof that this really is a Segal space, the reader is once again referred to [2]. Note that the quasicategory of spans $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is the bottom row of the complete Segal space of span $\mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$; that is,

$$\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) = \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)/\Delta^0.$$

Proposition A.1.1 thus immediately implies that $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is a quasicategory.

3.2. Important example: spans and equivalences

Let $\mathcal{C}$ be a quasicategory. One can think about spans in $\mathcal{C}$ as 'fractions' of morphisms in $\mathcal{C}$. Consider a span in $\mathcal{C}$

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X & & X' \end{tikzcd}.$$

If the morphism g were invertible, we could fill this to a full 2-simplex

$$\begin{tikzcd} & Y \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X \arrow[rr, dashed, "f \circ g^{-1}"'] & & X' \end{tikzcd}.$$

Even if g is not invertible, we can think of our span as formally representing a 'fraction of morphisms' $f \circ g^{-1} \colon X \to X'$, which may or may not exist in $\mathcal{C}$.

With this point of view in mind, if we build from a quasicategory $\mathcal{C}$ a category of spans whose forwards-facing legs are arbitrary morphisms in $\mathcal{C}$, and whose backward-facing legs are equivalences, then we have not really done anything; each span, thought of as a formal morphism in $\mathcal{C}$, is represented by an actual morphism in $\mathcal{C}$. Thus, the category of spans in $\mathcal{C}$ whose backward-facing legs are equivalences should be equivalent to the category $\mathcal{C}$. In the following example, we construct this equivalence.

Example 3.2.1. For any quasicategory $\mathcal{C}$, we can define a triple $(\mathcal{C}, \mathcal{C}^{\dagger} = \mathcal{C}, \mathcal{C}_{\dagger} = \mathcal{C}^{\simeq})$. These choices of ingressive and egressive morphisms correspond to spans of the form

$$\begin{tikzcd} & Y \arrow[dl, "\simeq"'] \arrow[dr] & \\ X & & X' \end{tikzcd},$$

i.e. spans such that the backwards-facing map $Y \to X$ is an equivalence in $\mathcal{C}$. With this triple, a functor $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is ambigressive cartesian if and only if each backward-facing leg is an equivalence. Note that the triple $(\mathcal{C}, \mathcal{C}, \mathcal{C}^{\simeq})$ defined above is adequate even if $\mathcal{C}$ does not admit pullbacks: for any solid diagram below, where g is an equivalence, the dashed square completion is a pullback, where g^{-1} is any homotopy inverse for g.

$$\begin{tikzcd} X \arrow[r, dashed, "g^{-1}\circ f"] \arrow[d, dashed, "\mathrm{id}"'] & X' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y' \end{tikzcd}$$

We will denote the resulting Segal space of spans by $\mathbf{Span}^{\simeq}(\mathcal{C})$, and the quasicategory $\mathbf{Span}^{\simeq}(\mathcal{C})/\Delta^0$ by $\mathrm{Span}^{\simeq}(\mathcal{C})$.

The quasicategory $\mathrm{Span}^{\simeq}(\mathcal{C})$ is categorically equivalent to the category $\mathcal{C}$ from which we started. More specifically, there exists a weak categorical equivalence $\mathcal{C} \to \mathrm{Span}^{\simeq}(\mathcal{C})$. We will now construct this map.

For each $n \geq 0$ there is a retraction of posets

$$[n] \hookrightarrow \Sigma_n \twoheadrightarrow [n],$$

where the first map takes $i \mapsto (i, n)$, and the second takes $(i, j) \mapsto i$. In the case $n = 2$, the inclusion can be drawn

$$
\begin{tikzcd}[column sep=small, row sep=small]
0 \arrow[dr, thick] & & & & & & & (0,2) \arrow[dl] \arrow[dr, thick] & & & \\
& 1 \arrow[dr, thick] & & \hookrightarrow & & & (0,1) \arrow[dl] \arrow[dr] & & (1,2) \arrow[dl] \arrow[dr, thick] & & , \\
& & 2 & & & (0,0) & & (1,1) & & (2,2) &
\end{tikzcd}
$$

and the map $\mathrm{sd}(\Delta^n) \to \Delta^n$ 'collapses' the backward-facing legs of the spans. The general case is analogous.

Applying the nerve, we find a retraction of simplicial sets

$$\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n) \twoheadrightarrow \Delta^n.$$

Pulling back along these maps gives us, for each n, a retraction

$$\mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \xhookrightarrow{i'_n} \mathrm{Map}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq} \xhookrightarrow{r'_n} \mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \ .$$

Note that the image of the inclusion i'_n consists of spans whose backwards-facing legs are identities (hence certainly equivalences), and hence that we can restrict the total space of our retraction:

$$\mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \xhookrightarrow{i_n} \mathrm{Map}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})^{\simeq} \xtwoheadrightarrow{r_n} \mathrm{Map}(\Delta^n, \mathcal{C})^{\simeq} \ .$$

It is not hard to convince oneself that a map $\mathrm{sd}(\Delta^n) \to \mathcal{C}$ is ambigressive cartesian if and only if it is a right Kan extension of its restriction along $\Delta^n \hookrightarrow \mathrm{sd}(\Delta^n)$. In the case $n = 2$, for example, it is clear from the limit formula that any right Kan extension of a diagram $\Delta^2 \to \mathcal{C}$ along the inclusion $\Delta^2 \hookrightarrow \mathrm{sd}(\Delta^2)$ is of the form

$$
\begin{tikzcd}[column sep=small, row sep=small]
X \arrow[dr] & & & & & & & X \arrow[dl, "\simeq"'] \arrow[dr] & & & \\
& Y \arrow[dr] & & \hookrightarrow & & & X' \arrow[dl, "\simeq"'] \arrow[dr] & & Y \arrow[dl, "\simeq"'] \arrow[dr] & & , \\
& & Z & & & X'' & & Y' & & Z &
\end{tikzcd}
$$

(and conversely, any diagram of the above form is a right Kan extension), and diagrams of the above form are precisely those which are ambigressive cartesian. Put differently, $\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C})$ is the full subcategory of $\mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C})$ on functors which are right Kan extensions of their restrictions to Δ^n. This, together with [5, Prop. 4.3.2.15] implies that r_n is a trivial fibration for all n. The identity $\mathrm{id}_{\mathrm{Fun}(\Delta^n, \mathcal{C})}$ is certainly a weak Kan equivalence, so by the 2/3 property for weak equivalences, i_n is also a weak equivalence for all n.

The projections $\mathrm{sd}(\Delta^n) \to \Delta^n$ assemble into a cosimplicial object $\mathrm{sd}(\Delta^\bullet) \to \Delta^\bullet$. From this it easily follows that the maps i_n assemble into a map of Segal spaces

$$i\colon \Gamma(\mathcal{C}) = \mathrm{Fun}(\Delta^\bullet, \mathcal{C}) \to \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^\bullet), \mathcal{C}) = \mathbf{Span}^{\simeq}(\mathcal{C})$$

which is a level-wise Kan weak equivalence, and hence a weak equivalence in the complete Segal spaces model structure.[1] Here $\Gamma(\mathcal{C})$ is the complete Segal space defined in [1] in the discussion preceding Thm. 4.11. Thus, because $\Gamma(\mathcal{C})$ and $\mathbf{Span}^{\simeq}(\mathcal{C})$ are complete Segal spaces (hence fibrant-cofibrant objects in the complete Segal space model structure), the restricion of i to first rows i/Δ^0 is a weak categorical equivalence, which is what we wanted to show.

As Example 3.2.1 shows, taking spans whose backward-facing legs are equivalences allows us to find an equivalence between $\mathcal{C}$ and a quasicategory of spans in $\mathcal{C}$. Similarly, if we look at spans whose forward-facing legs are equivalences, we find a category of spans equivalent to $\mathcal{C}^{\mathrm{op}}$.

Example 3.2.2. For any quasicategory $\mathcal{C}$, we can define an adequate triple $(\mathcal{C}, \mathcal{C}_\dagger = \mathcal{C}^{\simeq}, \mathcal{C}^\dagger = \mathcal{C})$. These selections of ingressive and egressive morphisms correspond to spans of the form

$$\begin{tikzcd} & Y \arrow[dl] \arrow[dr, "\simeq"] & \\ X & & X' \end{tikzcd},$$

i.e. spans such that the forward-facing map $Y \to X$ is an equivalence in $\mathcal{C}$. We will denote the corresponding category of spans by $\mathrm{Span}_{\simeq}(\mathcal{C})$. Exactly analogous reasoning to that in Example 3.2.1 gives us a categorical equivalence

$$\mathcal{C}^{\mathrm{op}} \to \mathrm{Span}_{\simeq}(\mathcal{C}).$$

3.3. Maps between triples

We have now seen that given any quasicategory $\mathcal{C}$ with pullbacks, one can create a quasicategory of spans $\mathrm{Span}(\mathcal{C})$, and that by defining an adequate triple structure $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ on $\mathcal{C}$, one can specify certain classes of morphisms to which the legs of the spans are allowed to belong. It is easy to see that any functor $\mathcal{C} \to \mathcal{D}$ between quasicategories $\mathcal{C}$ and $\mathcal{D}$ with pullbacks induces a functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$. Similarly, we make the following definition.

Definition 3.3.1. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ and $(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be adequate triples. A functor $p\colon \mathcal{C} \to \mathcal{D}$ is said to be a ***functor between adequate triples*** $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ if the following conditions hold.

[1] For a brief review of the model structures in question, see Appendix A.1. For more information, see [1], especially the discussion following Thm. 4.1.

1. The functor p preserves ingressive morphisms. That is, $p(\mathcal{C}_\dagger) \subseteq \mathcal{D}_\dagger$
2. The functor p preserves egressive morphsims. That is, $p(\mathcal{C}^\dagger) \subseteq \mathcal{D}^\dagger$.
3. The functor p preserves ambigressive pullbacks.

We will denote the restriction of p to $\mathcal{C}_\dagger$ by

$$p_\dagger \colon \mathcal{C}_\dagger \to \mathcal{D}_\dagger,$$

and similarly for $p^\dagger$. Clearly, a functor p between adequate triples gives functors $\mathrm{Span}(p)$ and $\mathbf{Span}(p)$ respectively between quasicategories and Segal spaces of spans.

It is natural to wonder about the relationship between p and $\mathrm{Span}(p)$. The goal of the remainder of this section is to provide a proof of the following theorem, originally appearing as [2, Thm. 12.2]. This theorem provides conditions on a functor p between adequate triples such that the induced map between category of spans is an inner fibration, and provides the form of a p-cocartesian morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$.

Theorem 3.3.2. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of adequate triples which is an inner fibration. Further suppose that p satisfies the following conditions:

1. For any ingressive morphism $g\colon s \rightarrowtail t$ of $\mathcal{D}$ and any object $x \in \mathcal{C}_s$, there exists an ingressive morphism $f\colon x \rightarrowtail y$ covering g which is both p-cocartesian and $p_\dagger$-cocartesian.
2. Suppose σ is a commutative square

$$\begin{tikzcd} y' \arrow[r, tail, "f'"] \arrow[d, two heads] & x' \arrow[d, "\psi"] \\ y \arrow[r, tail, "f"'] & x \end{tikzcd}$$

   in $\mathcal{C}$ such that $p(\sigma)$ is ambigressive pullback in $\mathcal{D}$, with in- and egressive morphisms as marked, and f is p-cocartesian. Then f' is p-cocartesian if and only if σ is an ambigressive pullback square (and in particular ψ is egressive).

Then π is an inner fibration. Further, if a morphism in $\mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger)$ is of the form

$$\begin{tikzcd} & y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\ x & & x' \end{tikzcd},$$

where ϕ is egressive and $p^\dagger$-cartesian, and ψ is ingressive and p-cocartesian, it is π-cocartesian.

This theorem originally appears in [2]. However, the proof there is combinatorial in nature.

Our proof uses the material developed in Section 2, leveraging the fact that ∞-categories of spans have natural incarnations as (complete) Segal spaces.

Before we get to the meat of the proof, however, we have to prove some preliminary results.

3.4. Housekeeping

In this section we recall some basic facts we will need in the course of our proof of the main theorem. It is recommended to skip this section on first reading, and refer back to the results as necessary.

3.4.1. Isofibrations

We will need some results about isofibrations, sometimes called categorical fibrations. These are standard, and proofs can be found in [6].

Proposition 3.4.1. Let $f\colon \mathcal{C} \to \mathcal{D}$ be a map between quasicategories. The following are equivalent:

- The map f is an isofibration.
- The map f is an inner fibration, and each equivalence in $\mathcal{D}$ has a lift in $\mathcal{C}$ which is an equivalence.

We will denote by $\mathrm{Map}(-,-)$ the internal hom on $\mathcal{S}\mathrm{et}_\Delta$.

Proposition 3.4.2. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $i\colon A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Map}(B, \mathcal{C}) \to \mathrm{Map}(A, \mathcal{C}) \times_{\mathrm{Map}(A,\mathcal{D})} \mathrm{Map}(B, \mathcal{D})$$

is an isofibration.

We will denote by $(-)^{\simeq}$ the *core* functor, i.e. the functor which takes a simplicial set X to the largest Kan complex it contains.

Proposition 3.4.3. Let $f\colon X \to Y$ be an isofibration of simplicial sets. Then

$$f^{\simeq}\colon X^{\simeq} \to Y^{\simeq}$$

is a Kan fibration.

We will denote the core of the mapping space functor $\mathrm{Map}(-,-)$ by $\mathrm{Fun}(-,-)$. That is, for simplicial sets A and B, we have

$$\mathrm{Fun}(A,B) \cong \mathrm{Map}(A,B)^{\simeq}.$$

Corollary 3.4.4. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an isofibration between quasicategories, and let $A \hookrightarrow B$ be an inclusion of simplicial sets. Then the map

$$\mathrm{Fun}(B,\mathcal{C}) \to \mathrm{Fun}(A,\mathcal{C}) \times_{\mathrm{Fun}(A,\mathcal{D})} \mathrm{Fun}(B,\mathcal{D})$$

is a Kan fibration.

3.4.2. Connected components

Our proof will rely heavily on the fact that the many properties that morphisms can have (ingressive, egressive, p-cocartesian, etc.) are well-behaved with respect to the homotopical structure of the complete Segal spaces in which they live; for example, if a morphism is p-cocartesian, then every morphism in its path component is p-cocartesian.

Let X' and X be simplicial sets, and let $f\colon X' \hookrightarrow X$ be an inclusion. We will say that f is an *inclusion of connected components* if for all simplices $\sigma \in X$, if σ has any vertex in common with X', then σ is wholly contained in X'.

Lemma 3.4.5. Let A be a simplicial set such that between any two vertices x and y of A there exists a finite zig-zag of 1-simplices of A connecting x to y. Let $A_0 \subseteq A$ be a nonempty simplicial subset of A. Let $f\colon X' \hookrightarrow X$ be a morphism between simplicial sets which is an inclusion of connected components. Then the following dashed lift always exists.

$$\begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, hook] & X' \arrow[d, hook] \\ A \arrow[r, "f"'] \arrow[ur, dashed] & X \end{tikzcd}$$

Proof. We have a map $f\colon A \to X$; in order to construct our dashed lift, it suffices to show that under the above assumptions, f takes every simplex of A to a simplex which belongs to X'. To this end, let $\sigma \in A$ be a simplex. There exists a finite zig-zag of 1-simplices connecting some vertex of σ to a vertex α of A_0; under f, this is mapped to a zig-zag of 1-simplices connecting $f(\alpha)$ to $f(\sigma)$. The vertex $f(\alpha)$ belongs to X'; proceeding inductively, we find that the image of every 1-simplex belonging to the zig-zag also belongs to X', and that $f(\sigma)$ therefore also belongs to X'. $\square$

Corollary 3.4.6. Given any commuting square of simplicial sets

$$\begin{tikzcd} X' \arrow[r, hook, "i"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, hook] & Y \end{tikzcd}$$

with monomorphisms as marked, where i is an inclusion of connected components between Kan complexes and f is a Kan fibration, the map f' is a Kan fibration.

Furthermore, if f is a trivial fibration and f' is surjective on vertices, then f' is a trivial fibration.

Proof. First, we show that if f is a Kan fibration, then f' is a Kan fibration. We need to show that a dashed lift below exists for each $n \geq 1$ and each $0 \leq i \leq n$.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & X' \arrow[r, hook, "i"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y' \arrow[r, hook] & Y \end{tikzcd}$$

We can always solve our outer lifting problem, and by Lemma 3.4.5, our lift factors through X'.

Now suppose that f is a trivial Kan fibration. The logic above says that f' has the right lifting property with respect to $\partial\Delta^n \hookrightarrow \Delta^n$ for $n \geq 1$; the right lifting property with respect to $\partial\Delta^0 \hookrightarrow \Delta^0$ is equivalent to the surjectivity of f' on vertices. $\square$

3.4.3. Contractibility

We will need at certain points to show that various spaces of lifts are contractible. These are mostly common-sense results, but it will be helpful to have them written down somewhere so we can refer to them later.

Lemma 3.4.7. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Then for any commuting square

$$\begin{tikzcd} \Lambda^2_1 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd},$$

the fiber F in the pullback square below is contractible.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(\alpha, f\alpha, \beta)"] & \mathrm{Fun}(\Lambda^2_1, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) \end{tikzcd}$$

Proof. The right-hand map is a trivial Kan fibration. □

One should interpret this as telling us that given a Λ^2_2-horn α in $\mathcal{C}$ lying over a 2-simplex β in $\mathcal{D}$, the space of fillings of α lying over β is contractible. We will need a pantheon of similar results.

Lemma 3.4.8. Let $f\colon \mathcal{C} \to \mathcal{D}$ be an inner fibration between quasicategories. Let e be any f-cartesian edge. Then for any square

$$\begin{tikzcd} \Lambda^2_2 \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{C} \arrow[d, "f"] \\ \Delta^2 \arrow[r, "\beta"] & \mathcal{D} \end{tikzcd}$$

such that $\alpha|_{\{1,2\}} = e$, the fiber F in the pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \mathrm{Fun}(\Delta^2, \mathcal{C}) \arrow[d] \\ \Delta^0 \arrow[r, "(\alpha, f\alpha, \beta)"] & \mathrm{Fun}(\Lambda^2_2, \mathcal{C}) \times_{\mathrm{Fun}(\Lambda^2_1, \mathcal{D})} \mathrm{Fun}(\Delta^2, \mathcal{D}) \end{tikzcd} \tag{4}$$

is contractible.

Proof. Denote $f(e) = \bar{e}$. Define a marking $\mathcal{E}'$ on $\mathcal{D}$ containing all degenerate edges and $\bar{e}$. Define a marking $\mathcal{E}$ on $\mathcal{C}$ containing the f-cocartesian lifts of the edges in $\mathcal{E}$. By [5, Prop. 3.1.1.6], the map $\mathcal{C}^{\mathcal{E}} \to \mathcal{D}^{\mathcal{E}'}$ has the right-lifting property with respect to all cocartesian-marked anodyne morphisms. It follows that

$$\begin{tikzcd} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \arrow[d] \\ \mathrm{Map}^\sharp((\Lambda^2_2)^{\mathcal{L}}, \mathcal{C}^{\mathcal{E}}) \times_{\mathrm{Map}^\sharp((\Lambda^2_1)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'})} \mathrm{Map}^\sharp((\Delta^2)^{\mathcal{L}}, \mathcal{D}^{\mathcal{E}'}) \end{tikzcd}$$

is a trivial Kan fibration; the map on the right-hand side of Diagram 4 is the core of this map, and is thus also a trivial Kan fibration. □

Lemma 3.4.9. Let $(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be an inner fibration between triples satisfying the conditions of Theorem 3.3.2. Denote by $\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges $\Delta^1 \to \mathcal{C}^\dagger$ which are both p-cocartesian and $p^\dagger$-cocartesian. Then the fibers of the map

$$\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \to \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger)$$

are contractible.

Proof. Denote by $\mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger)$ the full simplicial subset of $\mathrm{Fun}(\Delta^1, \mathcal{C}^\dagger)$ on edges which are $p^\dagger$-cartesian. Then we have a square

$$\begin{tikzcd}
\mathrm{Fun}'(\Delta^1, \mathcal{C}^\dagger) \arrow[r, hook] \arrow[d] & \mathrm{Fun}''(\Delta^1, \mathcal{C}^\dagger) \arrow[d] \\
\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger) \arrow[r, equal] & \mathrm{Fun}(\Delta^{\{0\}}, \mathcal{C}^\dagger) \times_{\mathrm{Fun}(\Delta^{\{0\}}, \mathcal{D}^\dagger)} \mathrm{Fun}(\Delta^1, \mathcal{D}^\dagger)
\end{tikzcd}$$

in which the top map is an inclusion of connected components, the right map is a trivial Kan fibration, and the left map is surjective on vertices. Thus, the left map is a trivial Kan fibration by Corollary 3.4.6 ☐

3.5. Proof of main theorem

Our goal is now to prove Theorem 3.3.2. As in the statement of the theorem, we fix a functor of triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ giving us a functor of quasicategories

$$\pi\colon \mathrm{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathrm{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

We wish to show that the following hold.

1. The map π is an inner fibration.
2. Morphsims of the form
$$\begin{tikzcd}
& y \arrow[dl, two heads, "\phi"'] \arrow[dr, tail, "\psi"] & \\
x & & x'
\end{tikzcd}, \tag{5}$$
are π-cocartesian, where ϕ is egressive and $p_\dagger$-cartesian, and ψ is ingressive and p-cocartesian.

The way forward is clear: the functor π is the zeroth row of a functor of Segal spaces

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger).$$

The results of Section 2 therefore guarantee the following.

1. By Corollary 2.1.7, in order to show that π is an inner fibration, it suffices to show that π' is a Reedy fibration.
2. In order to show that morphisms of the required form are π-cocartesian, it will suffice to show by Corollary 2.4.15 that morphisms of this form are π'-cocartesian.

3.5.1. The map π is an inner fibration

We begin by checking that the map π is an inner fibration. As stated above, we do this by checking that π' is a Reedy fibration. This follows from the following useful fact, implied by the first assumption of Theorem 3.3.2.

Lemma 3.5.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples satisfying the conditions of Theorem 3.3.2. Then the map $p\colon \mathcal{C} \to \mathcal{D}$ is an isofibration.

Proof. By assumption, p is an inner fibration and $\mathcal{C}$ and $\mathcal{D}$ are quasicategories, so it suffices by Proposition 3.4.1 to show that p admits lifts of equivalences. Each equivalence in $\mathcal{D}$ is in particular ingressive, and hence admits a p-cocartesian lift by the assumptions of Theorem 3.3.2; these are automatically equivalences. □

Proposition 3.5.2. For any functor of adequate triples $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, the map

$$\pi'\colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$$

is a Reedy fibration.

Proof. Consider the following diagram.

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[r, hook] \arrow[d] & \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{C}) \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^n), \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\partial\Delta^n), \mathcal{D})} \mathrm{Fun}(\mathrm{sd}(\Delta^n), \mathcal{D})
\end{tikzcd}.$$

The right-hand map is a Kan fibration by Corollary 3.4.4 (because sd preserves monomorphisms), and the top map is an inclusion of connected components, so Corollary 3.4.6 implies that the left-hand map is a Kan fibration, which is what we needed to show. □

Corollary 3.5.3. The map π is an inner fibration

Proof. Corollary 2.1.7. □

3.5.2. Cocartesian morphisms have the promised form

We are now ready to begin in earnest our proof that morphisms of the form Equation 5 are π'-cocartesian. Morally, this result should not be surprising. We should think of the homotopy pullback condition defining cocartesian morphisms (Definition 2.4.1) as telling us that we can fill relative Λ^2_2-horns in $\pi' \colon \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$, and that such fillings are unique up to contractible choice. Finding a filling

$$\begin{tikzcd} \Lambda^2_2 \arrow[r] \arrow[d, hook] & \mathbf{Span}(\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \arrow[d, "\pi'"] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathbf{Span}(\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger) \end{tikzcd}$$

is equivalent finding a filling

$$\begin{tikzcd} \mathrm{sd}(\Lambda^2_2) \arrow[r] \arrow[d, hook] & \mathcal{C} \arrow[d, "p"] \\ \mathrm{sd}(\Delta^2) \arrow[r] \arrow[ur, dashed] & \mathcal{D} \end{tikzcd}$$

(such that the filling is ambigressive and the necessary square is pullback); the conditions on p guarantee us that we can perform this filling in a series of steps pictured in Figure 1, each of which is unique up to contractible choice.

In fact, our proof mainly consists of making this idea rigorous. To this end, we first introduce some notation. Let $p \colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor of triples.

- We will denote p-cartesian morphisms in $\mathcal{C}$ with a circle:

$$x \longrightarrow\!\!\!\!\!\!\!\circ\;\; y$$

- We will denote p-cocartesian morphisms in $\mathcal{C}$ with a bullet:

$$x \longrightarrow\!\!\!\!\!\!\!\bullet\;\; y$$

- We will denote $p^\dagger$-cartesian morphisms in $\mathcal{C}$ with a triangle:

$$x \longrightarrow\!\!\!\!\!\!\!\triangleright\;\; y$$

- We will denote $p_\dagger$-cocartesian morphisms in $\mathcal{C}$ with a filled triangle:

$$x \longrightarrow\!\!\!\!\!\!\!\blacktriangleright\;\; y$$

Thus, an ingressive morphism which is both p-cocartesian and $p_\dagger$-cocartesian will be denoted by

$$x \rightarrowtail\!\!\blacktriangleright\!\!\bullet\!\!\rightarrow y$$

Our proof will rest on the factorization $\mathrm{sd}(\Lambda^2_0) \hookrightarrow \mathrm{sd}(\Delta^2)$ pictured in Figure 1. Denote the underlying factorization of simplicial sets by

$$A_1 \overset{i_1}{\hookrightarrow} A_2 \overset{i_2}{\hookrightarrow} A_3 \overset{i_3}{\hookrightarrow} A_4 \overset{i_4}{\hookrightarrow} A_5 \overset{i_5}{\hookrightarrow} A_6$$

For each of the A_i above, denote by

$$\mathrm{Fun}'(A_i, \mathcal{C}) \subseteq \mathrm{Fun}(A_i, \mathcal{C}) \tag{6}$$

the full simplicial subset on those functors $A_i \to \mathcal{C}$ which respect each labelling in Figure 1:

- ingressive
- egressive
- p-cartesian
- p-cocartesian
- $p^\dagger$-cartesian
- $p_\dagger$-cocartesian

Note that because equivalences in $\mathcal{C}$ belong to each of these classes of morphisms, the inclusion in Equation 6 is an inclusion of connected components.

Similarly, we will denote by $\mathrm{Fun}'(A_i, \mathcal{D})$ the full simplicial subset on functors which respect the ingressive and egressive labellings.

Lemma 3.5.4. The square

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_1, \mathcal{C}) \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_1, \mathcal{D})
\end{tikzcd}$$

is homotopy pullback.

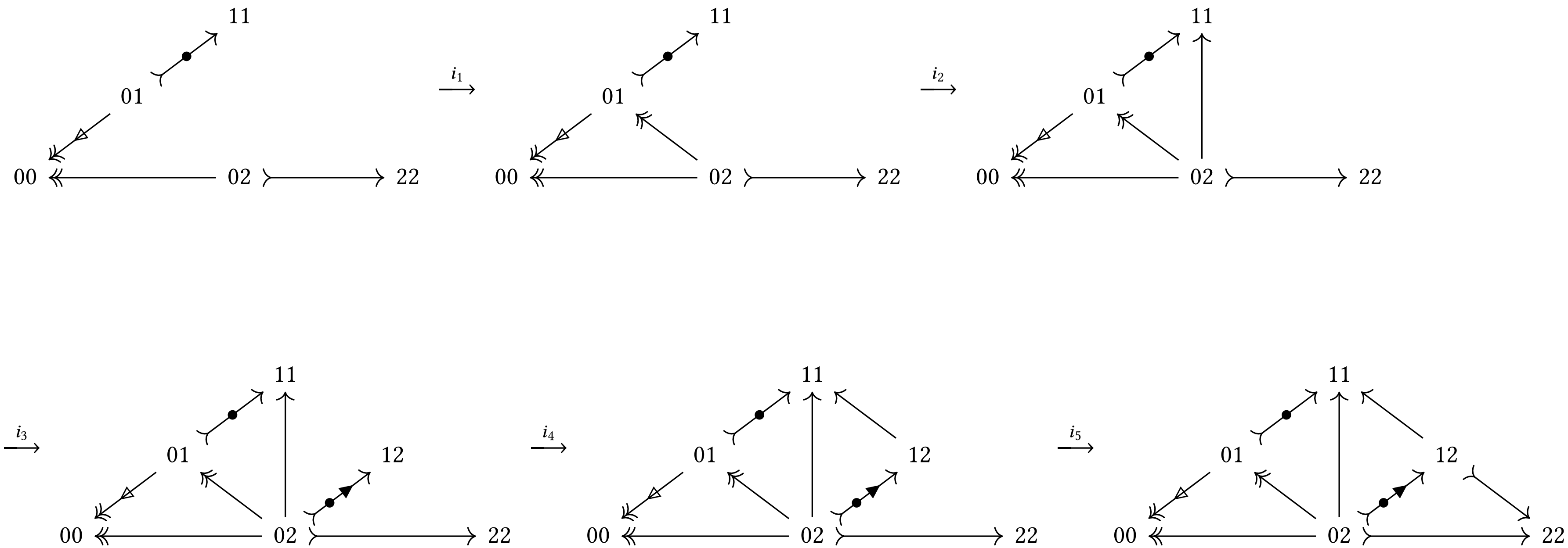

Figure 1: A factorization of the inclusion $\mathrm{sd}(\Lambda_0^2) \hookrightarrow \mathrm{sd}(\Delta^2)$, where certain morphisms have been labelled.

Proof. The factorization of Figure 1 gives us a factorization of the above square into five squares

$$\begin{tikzcd} \mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Fun}'(A_k, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}'(A_{k+1}, \mathcal{D}) \arrow[r] & \mathrm{Fun}'(A_k, \mathcal{D}) \end{tikzcd}, \qquad 1 \leq k < 6,$$

each corresponding to one of the inclusions i_k. We will be done if we can show that each of these squares is homotopy pullback. It suffices to show that for each k, the map

$$j_k \colon \mathrm{Fun}'(A_{k+1}, \mathcal{C}) \to \mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1}, \mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D})$$

is a weak equivalence. First, we show that each of these maps is a Kan fibration. To see this, consider the square

$$\begin{tikzcd} \mathrm{Fun}'(A_{k+1}, \mathcal{C}) \arrow[r, hook] \arrow[d, "j_k"'] & \mathrm{Fun}(A_{k+1}, \mathcal{C}) \arrow[d] \\ \mathrm{Fun}'(A_k, \mathcal{C}) \times_{\mathrm{Fun}'(A_{k+1}, \mathcal{D})} \mathrm{Fun}'(A_k, \mathcal{D}) \arrow[r, hook] & \mathrm{Fun}(A_k, \mathcal{C}) \times_{\mathrm{Fun}(A_{k+1}, \mathcal{D})} \mathrm{Fun}(A_k, \mathcal{D}) \end{tikzcd}.$$

The right-hand map is a Kan fibration because of Corollary 3.4.4, and we have already seen that the top map is an inclusion of connected componenents. Hence each j_k is a Kan fibration by Corollary 3.4.6. Thus, in order to show that each j_k is a weak equivalence, it suffices to show that the fibers are contractible.

First consider the case $k = 1$. For any map α below, consider the following pullback square.

$$\begin{tikzcd} F_\alpha \arrow[r] \arrow[d] & \mathrm{Fun}'(A_1, \mathcal{C}) \arrow[d, "j_1"] \\ \Delta^0 \arrow[r, "\alpha"] & \mathrm{Fun}'(A_0, \mathcal{C}) \times_{\mathrm{Fun}'(A_1, \mathcal{D})} \mathrm{Fun}'(A_1, \mathcal{D}) \end{tikzcd}.$$

The fiber F_α over α is the space of ways of completing a diagram of shape A_0 in $\mathcal{C}$ to a diagram of shape A_1 in $\mathcal{C}$ given a diagram of shape A_1 in $\mathcal{D}$. This is the space of ways of filling $\Lambda^2_2 \hookrightarrow \Delta^2$ in $\mathcal{C}^\dagger$ lying over a 2-simplex in $\mathcal{D}^\dagger$. This is contractible by Lemma 3.4.8.

The case $k = 2$ is similar, using Lemma 3.4.7.

The case $k = 3$ is similar, using Lemma 3.4.9.

The cases $k = 4$ and $k = 5$ use the dual to Lemma 3.4.8. □

We are now ready to show that morphisms of the form Diagram 5 are π'-cocartesian.

Proposition 3.5.5. For any morphism e of the form given in Equation 5, the square

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{C}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Lambda^2_0), \mathcal{D}) \times_{\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\}
\end{tikzcd}$$

is homotopy pullback.

Proof. This square factors horizontally into the two squares

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}^{\mathrm{aCart}}(\mathrm{sd}(\Delta^2), \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{D})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Fun}'(A_6, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[r] \arrow[d] & \mathrm{Fun}'(A_1, \mathcal{C}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{e\} \arrow[d] \\
\mathrm{Fun}'(A_6, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\} \arrow[r] & \mathrm{Fun}'(A_1, \mathcal{D}) \times_{\mathrm{Fun}(\mathrm{sd}(\Delta^{\{0,1\}}),\mathcal{C})} \{\pi e\}
\end{tikzcd}.$$

The first is homotopy pullback because the bottom map is an inclusion of connected components, and the second condition of Theorem 3.3.2 guarantees that the fiber over an ambigressive $\mathrm{sd}(\Delta^2) \to \mathcal{D}$ belonging to $\mathrm{Fun}'(A_6, \mathcal{D})$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is πe is precisely an ambigressive Cartesian functor $\mathrm{sd}(\Delta^2) \to \mathcal{C}$ whose restriction to $\mathrm{sd}(\Delta^{\{0,1\}})$ is e.

That the second is homotopy pullback follows immediately from Lemma 3.5.4. □

This proves Theorem 3.3.2.

4. A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 4.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. A morphism in $\mathcal{C}$ is egressive if and only if it is p-cartesian (hence also $p^\dagger$-cartesian).
2. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.
3. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

   in $\mathcal{C}$ with egressive and ingressive morphisms as marked, in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

   in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Proof. Even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. □

5. Horn filling via left Kan extensions

Kan extensions of ∞-categories are usually defined pointwise via the colimit formula [5] [3]. In this section, we will show that such pointwise ∞-Kan extensions enjoy a horn filling property in $\mathbb{C}\mathrm{at}_\infty$ which generalizes the universal property for Kan extensions of functors between 1-categories.

Definition 5.0.1. We will call a 2-simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ ***left Kan*** if it is of the form

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{C} \\ & Y \arrow[ur, "f_!F"'] & \end{tikzcd},$$

where $\mathcal{C}$ is cocomplete, X is small, $f_!F$ is a left Kan extension of F along f, and $\eta\colon F \Rightarrow f^*f_!F$ is the unit map.

The goal of this section is to prove the following.

Theorem 5.0.2. Let $\tau\colon \Delta^2 \to \mathbb{C}\mathrm{at}_\infty$ be left Kan. Then for each $n \geq 2$, every solid lifting problem of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd} \tag{7}$$

admits a dashed solution.

Let us examine this in the case $n = 2$. In this case we have the data of categories and functors

$$\begin{tikzcd} & Y \arrow[drr, "f_!F"] \arrow[ddr, "g"] & & \\ X \arrow[ur, "f"] \arrow[rrr, "F"] \arrow[drr, "h"'] & & & \mathcal{C} \\ & & Z \arrow[ur, "G"'] & \end{tikzcd}$$

together with natural transformations $\eta\colon F \Rightarrow f_!F \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow G \circ h$. We need to produce a natural transformation $\alpha\colon f_!F \Rightarrow G \circ g$ and a filling of the full 3-simplex, which is the data of a homomotopy-commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \arrow[d, "\alpha f"] \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}$$

in $\mathrm{Fun}(X, \mathcal{C})$.[2]

[2]In particular, if we take $g = \mathrm{id}$ and β to be a natural isomorphism, we recover the classical universal property

We can rephrase this as follows. We are given the data

$$\overbrace{\begin{array}{c}\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_!F \circ f \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}\end{array}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{array}{c} f_!F \\ \\ G \circ g \end{array}}^{\text{in Fun}(Y,\mathcal{C})}$$

where the left-hand diagram is a map $LC^2 \to \mathrm{Fun}(X, C)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ except the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathrm{Fun}(Y, C)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

1. We first can fill the lower-left half of the diagram on the left simply by taking the composition $G\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{array}{c}\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}\end{array}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{array}{c} f_!F \\ \\ G \circ g \end{array}}^{\text{in Fun}(Y,\mathcal{C})} .$$

2. We note that we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left under the adjunction $f_! \dashv f^*$:

$$\overbrace{\begin{array}{c}\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \\ & G \circ g \circ f \end{tikzcd}\end{array}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{array}{c}\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \\ & G \circ g \end{tikzcd}\end{array}}^{\text{in Fun}(Y,\mathcal{C})}$$

3. The diagram on the right has an obvious filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{array}{c}\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_!F \circ f \arrow[d, "\alpha f"] \\ & G \circ g \circ f \end{tikzcd}\end{array}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{array}{c}\begin{tikzcd} f_!F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_!F \arrow[d, "\alpha"] \\ & G \circ g \end{tikzcd}\end{array}}^{\text{in Fun}(Y,\mathcal{C})}$$

---

satisfied by left Kan extension.

The lifting problems which we have to solve in Equation 7 for $n > 2$ amount to replacing $\Delta^1 \times \Delta^1$ by $(\Delta^1)^n$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved. In Subsection 5.1 we explain the combinatorics of filling cubes relative to their boundaries. In Subsection 5.2, we give a formalization the concept of adjunct data, and provide a means of for filling partial data to total adjunct data. In Subsection 5.3, we show that we can always solve the lifting problems of Equation 7.

5.1. Filling cubes relative to their boundary

In this section, we give the combinatorics of filling the n-cube $(\Delta^1)^n$ relative to its boundary. Our main result is Proposition 5.1.21, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map.

Definition 5.1.1. The ***n-cube*** is the simplicial set $C^n := (\Delta^1)^n$.

Note 5.1.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Our first step is to understand the nondegenerate simplices of the n-cube.

Definition 5.1.3. Denote by $a^i : \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\dots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\dots,n\}}$ to $\{1\}$.

Note 5.1.4. The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0, \dots, 0)$ and ending at $(1, \dots, 1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1, \dots, n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition 5.1.5. For any permutation $\tau\colon \{1, \dots, n\} \to \{1, \dots, n\}$, we define an n-simplex

$$\phi(\tau)\colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)}, \dots, a^{\tau(n)}).$$

Note 5.1.6. It is easy to check that the above definition really results in a bijection between S_n and the nondegenerate simplices of C^n.

Notation 5.1.7. We will denote any permutatation $\tau\colon \{1,\dots,n\} \to \{1,\dots,n\}$ by the corresponding n-tuple $(\tau(1),\dots,\tau(n))$. Thus, the identity permutation is $(1,\dots,n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2,1,3,\dots,n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)},\dots,a^{\tau(n)})\colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i\colon \Delta^n \to \Delta^1$ and any face map $\partial_j\colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\dots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)},\dots,a^{\tau(n)}) := (\partial_i^* a^{\tau(1)},\dots,\partial_i^* a^{\tau(n)}).$$

Example 5.1.8. Consider the 3-simplex (a^1,a^2,a^3) in C^3. This has spine

$$(0,0,0) \to (1,0,0) \to (1,1,0) \to (1,1,1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation 5.1, we then calculate that

$$d_2(a^1,a^2,a^3) = (a^1,a^2,a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0,0,0) \to (1,0,0) \to (1,1,1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition 5.1.9. The ***boundary of the*** n***-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{8}$$

The following is easy to see.

Proposition 5.1.10. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \ldots, \gamma_n) \colon \Delta^{n-1} \to C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition 5.1.11. The ***left box of the*** n***-cube***, denoted LC^n, is the simplicial subset of C^n given by the union of all of the faces in Equation 8 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example 5.1.12. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma 5.1.13. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

1. The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.
2. For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.
3. The nth face $d_n\phi$ always belongs to LC^n.

Proof.

1. We have
$$d_0\phi = (a^{\tau(0)-1}, \dots a^{\tau(n)-1})$$
since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition 5.1.10 guarantees that $d_0\phi$ is contained in the face
$$\overset{(1)}{\Delta^1} \times \dots \times \overset{(j)}{\{1\}} \times \dots \times \overset{(n)}{\Delta^1}.$$
This face belongs to LC^n except when $j = n$.
2. In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.
3. In this case,
$$d_n(a^{\tau(1)}, \dots, a^{\tau(n)}) = (a^{\tau(1)}, \dots, a^{\tau(n)})$$
since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face
$$\overset{(1)}{\Delta^1} \times \dots \times \overset{(j)}{\{0\}} \times \dots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation 5.1.14. For any subset $T \subseteq [n]$, write
$$\Lambda^n_T := \bigcup_{t\in T} d_t\Delta^n \subset \Delta^n.$$

Lemma 5.1.15. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n-1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \dots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma 5.1.15 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition 5.1.16. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) < \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example 5.1.17. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation 5.1.18. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$\begin{aligned} LC^n &\hookrightarrow LC^n \cup \phi(1, \dots, n) \\ &\hookrightarrow \cdots \\ &\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\dots,n,1)} \phi(\tau) \end{aligned} \tag{9}$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2, \dots, n, 1)$, the intersection

$$B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{10}$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition 5.1.19. Each inclusion in Equation 9 is inner anodyne.

Proof. The first inclusion

$$LC^n \hookrightarrow LC^n \cup \phi(0, \dots, n)$$

is inner anodyne because by Lemma 5.1.13 the intersection $LC^n \cap \phi(0, \dots, n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma 5.1.13), so by Lemma 5.1.15, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (0, \dots, n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),$$

which implies that $\tau = (1, \dots, n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. □

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$. Thus, each of their spines begins

$$(0, \dots, 0, 0) \longrightarrow (0, \dots, 0, 1) \longrightarrow \cdots,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that $(0, \dots, 0, 0) \longrightarrow (0, \dots, 0, 1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition 5.1.19. One continues to fill simplices in the order prescribed in Definition 5.1.16, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma 5.1.15.

Lemma 5.1.20. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition 5.1.21. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0, \dots, 0, 0) \to (0, \dots, 0, 1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \xhookrightarrow{i} J^n \xhookrightarrow{j} C^n \ ,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\ J^n \arrow[r, hook] & C^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} ,$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0, \dots, 0) \to (0, \dots 1)$.

5.2. Adjunct data

In this section, we give a formalization of the the notion of adjunct data. Our main result is Proposition 5.2.6, which shows that under certain conditions, we can use data on one side of an adjunction to solve lifting problems on the other side.

Definition 5.2.1. Let $s\colon K \to \Delta^1$ be a map of simplicial sets, and let $p\colon \mathcal{M} \to \Delta^1$ be a bicartesian fibration associated to the adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via equivalences $h_0 \colon \mathcal{C} \to \mathcal{M}_0$ and $h_1 \colon \mathcal{D} \to \mathcal{M}_1$. We say that a map $\alpha \colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha} \colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A \colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K \times \{0\}} = h_0 \circ \alpha$.
2. The restriction $A|_{K \times \{1\}} = h_1 \circ \tilde{\alpha}$.
3. For each vertex $k \in K$ in the fiber $s^{-1}\{0\}$, the image of the edge $\{k\} \times \Delta^1 \to \mathcal{M}$ is p-cocartesian.
4. For each vertex $k' \in K$ in the fiber $s^{-1}\{1\}$, the image of the edge $\{k'\} \times \Delta^1 \to \mathcal{M}$ is p-cartesian.

We often leave the equivalences h_0 and h_1 implicit to lighen notation.

In the next examples, fix an adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p \colon \mathcal{M} \to \Delta^1$.

Example 5.2.2. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$ of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is p-cocartesian and b is p-cartesian. This corresponds to the map $s = \mathrm{id} \colon \Delta^1 \to \Delta^1$.

Example 5.2.3. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id} \colon gD \to gD$ in $\mathcal{C}$. This morphism is adjunct to the component of the unit map $\eta_D \colon D \to fgD$ relative to $s = \mathrm{id} \colon \Delta^1 \to \Delta^1$.

The next example will follow from Proposition 5.2.6.

Example 5.2.4. Any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g.

Lemma 5.2.5. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^{\{0\}}, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{11}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition 5.2.6. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, h_0 and h_1 be as in Definition 5.2.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s \circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a cofibration such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}' \colon K' \to \mathcal{D}$ and $\alpha \colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{12}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. We further consider the map $h_0 \circ \alpha \colon K \to \mathcal{M}_0$. From these two pieces of data we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd}
\left(K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1\right)^{\heartsuit} \arrow[r] \arrow[d] & \mathcal{M}^{\natural} \arrow[d] \\
(K \times \Delta^1)^{\heartsuit} \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^{\sharp}
\end{tikzcd},$$

where $(K \times \Delta^1)^{\heartsuit}$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\} \times \Delta^1$, for $k \in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. Starting from K' and adjoining one nondegenerate simplex of K at a time, we can write the left-hand map in the above square as a transfinite composition of pushouts of morphisms of the form given in Equation 11 (by assumption each simplex σ we adjoin is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). Each of these is marked anodyne, so the left-hand map is as well. Thus, a dashed lift ℓ exists, giving us a map $h_1^{-1} \circ \ell|_{K \times \Delta^{\{1\}}} \colon K \to \mathcal{D}$. This, together with the composition

$$K' \times \Delta^1 \xrightarrow{\tilde{\alpha}' \times \mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{c} \mathcal{D}$$

where $c \colon h_1^{-1} \circ h_1 \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, gives a solid diagram

$$\begin{tikzcd}
K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times (\Delta^1)^{\sharp} \arrow[r] \arrow[d] & \mathcal{D}^{\natural} \\
K \times (\Delta^1)^{\sharp} \arrow[ur, dashed, "m"'] &
\end{tikzcd},$$

where $\mathcal{D}^{\natural}$ is the marked simplicial set whose underlying simplicial set is $\mathcal{D}$ and where all equivalences are marked. This admits a dashed lift, and restricting $m|_{K \times \Delta^{\{1\}}}$ gives us a map $\tilde{\alpha} \colon K \to \mathcal{D}$ as in Equation 12. However, we still need to display this as adjunct to α: the lift ℓ has $\ell|_{K \times \Delta^{\{0\}}} = h_0 \circ \alpha$, but only $\ell|_{K \times \Delta^{\{1\}}} \simeq h_1 \circ \tilde{\alpha}$, not equality as we require.

Our last task is to remedy this. Denote the spine of the 3-simplex by

$$I_3 := \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \hookrightarrow \Delta^3.$$

We build a map $I_3 \times K \xrightarrow{b} \mathcal{M}$ which:

- On $\Delta^{\{0,1\}} \times K$ is given by the lift ℓ.
- On $\Delta^{\{1,2\}} \times K$ is given by the composition

$$K \times \Delta^1 \xrightarrow{\ell|_{K \times \Delta^{\{1\}}} \times \mathrm{id}} \mathcal{M}_1 \times \Delta^1 \xrightarrow{H} \mathcal{M}_1 \hookrightarrow \mathcal{M},$$

  where H is a natural isomorphism $\mathrm{id}_{\mathcal{M}_1} \to h_1 \circ h_1^{-1}$.

- On $\Delta^{\{2,3\}} \times K$ is given by the natural isomorphism

$$K \times \Delta^1 \xrightarrow{m} \mathcal{D} \xrightarrow{h_1} \mathcal{M}_1 \hookrightarrow \mathcal{M}.$$

The map $I_3 \hookrightarrow \Delta^3$ is inner anodyne, so there exists a map $\Delta^3 \times K \to \mathcal{M}$ whose restriction to $I_3 \times K$ is equal to b. The restriction of this filling to $\Delta^{\{0,3\}} \times K$ satisfies the conditions of Definition 5.2.1 (since the composition of a (co)cartesian morphism with an equivalence is again (co)cartesian), exhibiting $\tilde{\alpha}$ as adjunct to α. ◻

Note 5.2.7. The filler provided by Proposition 5.2.6 is not guaranteed to be unique (even up to equivalence). However, we have some say in how our filler should look. We construct our adjoint filler $\tilde{\alpha}$ by starting with the solid data of Equation 12 and building the dashed extension $\tilde{\alpha}\colon K \to \mathcal{D}$ by filling a simplex at a time relative to its boundary, such that each new simplex making up the map $\tilde{\alpha}$ is adjunct to the corresponding one in α. We are free to choose each new adjunct simplex however we like.

Example 5.2.8. Let $f : \mathcal{C} \leftrightarrow \mathcal{D} : g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K' = \Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \arrow[rr, hook, "i"] \arrow[dr] & & \Delta^3 = K \arrow[dl] \\ & \Delta^1 & \end{tikzcd},$$

where both downwards-facing arrows take $0, 1 \mapsto 0$ and $2, 3 \mapsto 1$, and fix a map $\tilde{\alpha}\colon K \to N(\mathcal{C})$ and a map $\alpha'\colon K' \to N(\mathcal{D})$ such that $\tilde{\alpha} \circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition 5.2.6 implies that the solution to the lifting problem on the left yields one on the right. Similarly, its dual implies the converse. Thus, Proposition 5.2.6 implies in particular that such lifting problems are equivalent.

5.3. Left Kan implies globally left Kan

In this subsection, we will prove Theorem 5.0.2. In order to do that, we must show that we can solve lifting problems of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd},$$

for $n \geq 2$, where τ is left Kan. Forgetting for a moment that have singled out some 2-simplex τ, and using the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}_{\mathrm{sc}}$, we will do this by solving equivalent lifting problems of the form[3]

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}] \arrow[r] \arrow[d, hook] & \mathbf{QCat} \\ \mathfrak{C}[\Delta^{n+1}] \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd}. \tag{13}$$

We expect that the reader is familiar with the basics of (scaled) rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category $\mathfrak{C}[\Delta^{n+1}]$ are given by the set $\{0, \dots, n+1\}$, and the mapping spaces are defined by

$$\mathfrak{C}[\Delta^{n+1}](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases}.$$

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \dots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda_0^{n+1}]$ has the same objects as $\mathfrak{C}[\Delta^{n+1}]$, and each morphism space $\mathfrak{C}[\Lambda_0^{n+1}](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^{n+1}](i,j)$, as we shall soon describe.

The map $\mathfrak{C}[\Lambda_0^{n+1}](i,j) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](i,j)$ is an isomorphism except for $(i,j) = (1, n+1)$ and $(i,j) = (0, n+1)$. The missing data corresponds in the case of $(1, n+1)$ to the missing face $d_0\Delta^{n+1}$ of Λ_0^{n+1}, and in the case of $(0, n+1)$ to the missing interior. Thus, to find a filling as in Equation 13, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](1, n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](1, n+1) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{14}$$

[3]Note that we are implicitly taking $\mathfrak{C}[\Lambda_0^{n+1}]$ and $\mathfrak{C}[\Delta^{n+1}]$ to carry the flat markings on their mapping spaces.

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](0,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(0),\mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{15}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \mathrm{Fun}(\mathcal{F}(1),\mathcal{F}(n+1)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[r, "\ell"] & \mathrm{Fun}(\mathcal{F}(0),\mathcal{F}(n+1)) \end{tikzcd} \tag{16}$$

commute.

Notation 5.3.1. Recall our desired filling $\mathcal{F}\colon \mathfrak{C}[\Delta^{n+1}] \to \mathbf{QCat}$ of Equation 13. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \ldots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \mathrm{Map}(X_{i_1}, X_{i_k})$ by $f_{i_k\cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

Theorem 5.3.2. For any $n \geq 2$ and any globally left Kan 2-simplex τ, the solid extension problem

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

admits a dashed filler.

Proof. We need to solve the lifting problems of Equation 14 and Equation 15, and check that our solutions satisfy the condition of Equation 16. We note the following useful pieces of information.

- The information that the simplex $\tau\colon \Delta_\flat^{\{0,1,n+1\}} \to \mathbb{C}\mathrm{at}_\infty$ is left Kan tells us that $f_{n,1}$ is the left Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha_{\{0,n+1\}}^{\{0,1,n+1\}} : f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$ is the unit map.
- We have an adjunction

$$(f_{1,0})_! : \mathrm{Fun}(X_0, X_{n+1}) \longleftrightarrow \mathrm{Fun}(X_1, X_{n+1}) : f_{1,0}^*.$$

It is easy to see that we have the following succinct descriptions of the inclusions of Equation 14 and Equation 15 along which we have to extend.

- There is an isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](0, n+1) \xrightarrow{\cong} C^n$$

specified completely by sending a subset $S \subseteq [n+1]$ to the point

$$(z_1, \ldots, z_n) \in C^n, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

The inclusion $\mathfrak{C}[\Lambda_0^{n+1}](0, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](0, n+1)$ corresponds to the simplicial subset $LC^n \hookrightarrow C^n$.

- There is a similar isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](1, n+1) \xrightarrow{\cong} C^{n-1}$$

specified completely by sending $S \subseteq \{1, \ldots, n+1\}$ to the point

$$(z_1, \ldots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$

The inclusion $\mathfrak{C}[\Lambda_0^{n+1}](1, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](1, n+1)$ corresponds to the inclusion $\partial C^{n-1} \hookrightarrow C^{n-1}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^n \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ C^n \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-1} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$(\star) = \begin{tikzcd} C^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \arrow[d, "f_{1,0}^*"] \\ C^n \arrow[r] & \mathrm{Fun}(X_0, X_{n+1}) \end{tikzcd}$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition 5.1.21, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$(*') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \operatorname{Fun}(X_0, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed] & \end{tikzcd} .$$

The image of the 1-simplex $(0, \dots, 0) \to (0, \dots, 1)$ of $\Delta^0 * \partial C^{n-1} \subseteq C^n$ under α is the unit map $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}} \colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$. This is not in general an equivalence, so we cannot use Proposition 5.1.21 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to the identity map $\mathrm{id}_{f_{n+1,1}}$ in $\operatorname{Fun}(X_1, X_{n+1})$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-1} is in the image of $f^*_{1,0}$, so by Proposition 5.2.6 we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-1} \to \operatorname{Fun}(X_1, X_{n-1})$ which is adjunct to α relative to the map

$$s \colon \Delta^0 * \partial C^{n-1} \to \Delta^1$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-1} to $\Delta^{\{1\}}$; by Note 5.2.7, we can choose $\tilde{\alpha}$ such that the image of the morphism $(0, \dots, 0) \to (0, \dots 1)$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$(**') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \operatorname{Fun}(X_1, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed, "\tilde{\beta}"'] & \end{tikzcd}$$

However, since image of $(0, \dots 0) \to (0, \dots, 1)$ is an equivalence, Proposition 5.1.21 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition 5.2.6, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-1}}$ be the image of $\tilde{\beta}|_{C^{n-1}}$ under $f^*_{1,0}$, which is true by construction. □

6. Local systems

In [4],

6.1. The fibration

6.2. Cocartesian edges

In this section, we will show that

7. The non-monoidal construction

7.1. The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd}.$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \\ \mathcal{C} \arrow[ur, phantom, "\Rightarrow"] & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Rightarrow"] & Y \arrow[d] \arrow[dl] \\ \mathcal{C} \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{C} \arrow[l, "\mathrm{id}_\mathcal{C}"] \end{tikzcd}, \tag{17}$$

To save on notation, we will generally denote such a morphism as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}. \tag{18}$$

There is a potential for confusion here. The natural transformation notated in Diagram 18 above does not correspond to a single natural transformation in Diagram 17, but rather to both natural transformations simultaneously. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cartesian if and only if all 2-simplices forming the faces of the corresponding map $Q(1) \to \mathcal{C}\mathrm{at}_\infty$ are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 7.1.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex $\sigma\colon Q(1) \to \mathbb{C}\mathrm{at}_\infty$ to which it corresponds has the property that $d_3\sigma$ is left Kan. More explicitly, the morphism is left Kan if and only if σ the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, shift left=3, "\circlearrowleft"] \arrow[dr, phantom, shift right=3, "\Rrightarrow"] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] & Y \arrow[d] \arrow[dl, "f_!\mathcal{F}"] \arrow[dl, phantom, shift right=3, "\overset{\eta}{\Rrightarrow}"] \arrow[dl, phantom, shift left=3, "\circlearrowleft"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd},$$

where $f_!\mathcal{F}$ is a left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines both the data of the first and the filling 3-simplex up to contractible choice.

Proof. First we show that morphisms of the promised form really are r-cocartesian. Since by assumption $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cartesian fibration (hence an isofibration) whose source and target are quasicategories, it suffices to show that we can solve the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "a"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{S} \end{tikzcd}$$

where a is of the promised form, and that the space of such fillings is contractible. $\square$

For short, we will denote r-cocartesian morphisms with an exclamation mark:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "=!\Rightarrow"] & X' \arrow[d, "\mathrm{Lan}_f\mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

7.2. The triple structures

7.2.1. The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 7.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 7.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

7.2.2. The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_! g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 7.2.3. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 7.2.4. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \quad \begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\ B' \arrow[r] & C \end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd} a \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\alpha"] & & b \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\beta"] & \\ & b' \arrow[rr] \arrow[dd] & & c \arrow[dd] \\ \bar{a} & & \bar{b} \arrow[ll] & \\ & \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul] \end{tikzcd}$$

in $\mathbb{C}$.[4] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 7.2.3. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 7.2.4

We also need to show that any such pullback is of this form. This also follows from Lemma 7.2.4. □

[4]Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

7.3. Building categories of spans

Lemma 7.3.1. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{19}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 19 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. □

Proposition 7.3.2. The map $p \colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p \colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-

cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] \arrow[ddr, phantom, "\circlearrowleft"] & & Y' \arrow[dr, "g"] \arrow[dd] \arrow[ddr, phantom, "\circlearrowleft"] & \\
& X \arrow[rr, "f"] \arrow[dd] \arrow[r, phantom, "\Rightarrow"] & & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'^*\mathcal{F}"] \arrow[dd, "g'"'] & & Y' \arrow[dl, "\mathcal{G}"'] \arrow[dd, "g"] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],
\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 7.3.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].
\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

8. The monoidal construction

8.1. The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I) \colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd}
\Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\
\Delta^I \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i
\end{tikzcd}$$

commutes.

Example 8.1.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 8.1.2. A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[5] of

- A 'source' object $F \colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i \colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.

[5]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1 \times G_2"] \arrow[l, phantom, ""] \\ \mathcal{C}' & \mathcal{D}'_1 \times \mathcal{D}'_2 \arrow[l, "\beta"] \end{tikzcd} \quad \Longrightarrow .$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 8.1.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^{\mathrm{op}} \end{tikzcd},$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0)) \end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. □

Note that for each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \to (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n.$$

Because the pointwise product of any number of thin 1-simplices in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$, the conditions of Lemma 8.1.3 are satisfied. Unrolling, we find that $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{20}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 8.1.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 20, we have the following.

Proposition 8.1.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \\
\vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.

- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_\otimes$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

8.2. The triple structures

8.2.1. The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 8.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 8.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

8.2.2. The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 8.2.3. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 8.2.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd}
\langle n\rangle & \langle m\rangle \arrow[l] \\
\langle n'\rangle \arrow[u, "\phi"] & \langle m'\rangle \arrow[l] \arrow[u, "\simeq"']
\end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) □

8.2.3. The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\
\langle n\rangle \arrow[r] & \langle m\rangle & \langle m\rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$\begin{tikzcd}
\vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*t\circ\mathcal{F}"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^*t\mathcal{F})"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd}.$$

In particular, in the fiber over $\langle 1\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 8.2.5. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_\otimes$ of the form

$$\begin{tikzcd} y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd},$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

8.3. Building categories of spans

Proposition 8.3.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 8.3.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd} \langle n \rangle & \langle n \rangle \arrow[l, "\gamma'"'] \\ \langle m \rangle \arrow[u, "\tau'"] & \langle m \rangle \arrow[l, "\gamma"'] \arrow[u, "\tau"'] \end{tikzcd},$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i \colon X_i \longrightarrow Y_{\gamma^{-1}(i)}, \qquad i \in \langle m \rangle^\circ,$$

f' has components

$$f'_j : X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n \rangle^\circ,$$

g has components

$$g_j : Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n \rangle^\circ,$$

and g' has components

$$g'_j : X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n \rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{(\prod_{\tau'(i)=j} Y_i)} \left(\prod_{\tau'(i)=j} X_i \right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'"] & \\
& \vec{X} \arrow[rr, "f"] \arrow[dd, "\mathcal{F}"] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes"] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"']
\end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^\circ$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathbf{LS}(\mathcal{C})_\otimes$

above to a pullback square entirely in the fiber over $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathbf{LS}(\mathcal{C})_{\otimes})_{\langle n\rangle} \cong (\mathbf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathbf{LS}(\mathcal{C})$, one for each $j \in \langle n\rangle^{\circ}$, of the form

$$
\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}" near end] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr] \arrow[dd, "\mathcal{A}" near start] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd},
$$

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i\colon X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

9. Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$
\begin{tikzcd}
\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_{\dagger}, \mathcal{Q}^{\dagger}) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_{\dagger}, \mathcal{P}^{\dagger}) \arrow[dl] \\
& \mathrm{Span}(\mathcal{F}, \mathcal{F}_{\dagger}, \mathcal{F}^{\dagger}) &
\end{tikzcd}.
$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_{\dagger}, \mathcal{F}^{\dagger})$, giving us isofibrations

$$
\begin{tikzcd}
P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\
& \mathcal{F}\mathrm{in}_* &
\end{tikzcd},
$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an $^{\mathrm{op}}$. Composing with $^{\mathrm{op}} \colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.

10. Brutal combinatorics

We are considering subsets of $\Delta^n \star (\Delta^n)^{\mathrm{op}}$, where the following simplices are scaled:

- Every 2-simplex factoring through Δ^n
- Every 2-simplex factoring through $(\Delta^n)^{\mathrm{op}}$
- The simplices $ij\bar{k}$ for $i < j \leq k$
- The simplices $k\bar{j}\bar{i}$ for $i < j \leq k$.

Any simplicial subset of this is completely determined by the vertices it contains. We consider any such subset to inherit the scaling and marking.

We further consider each 1-simplex in $(\Delta^n)^{\mathrm{op}}$ to be marked.

Lemma 10.0.1. For $n = 2$, the inclusion $01\bar{0} \cup \bar{1}\bar{0} \hookrightarrow 01\bar{1}\bar{0}$ is mk-anodyne.

A. Appendix

A.1. A brief list of model structures

The category $\mathcal{S}\mathrm{et}_\Delta$ carries two model structures of which we will make frequent use:

- The *Kan* model structure, which has the following description.
    - The fibrations are the Kan fibrations.
    - The cofibrations are the monomorphisms.
    - The weak equivalences are weak homotopy equivalences.
- The *Joyal* model structure. We will not give a complete description, referring the reader to [5, Sec. 2.2.5]. We will make use of the following properties.
    - The cofibrations are monomorphisms.

- The fibrant objects are quasicategories, and the fibrations between fibrant objects are isofibrations, i.e. inner fibrations with lifts of equivalences (cf. [5, Cor. 2.6.5]).

The category Δ^{op} has a Reedy structure, which gives us, together with the Kan model structure, a model structure on the category $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{S}\mathrm{et}_\Delta) = \mathcal{S}\mathrm{et}_{\Delta^2}$ with the following properties.

- The cofibrations are monomorphisms.
- The weak equivalences are level-wise weak homotopy equivalences.
- The fibrations are *Reedy fibrations* (Definition 2.1.5).

This model category has two important left Bousfeld localizations which we will need.

- The *Segal space model structure* on $\mathcal{S}\mathrm{et}_{\Delta^2}$ is the left Bousfeld localization of the Reedy model structure at the set of spine inclusions

$$S = \left\{ \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \cdots \amalg_{\Delta^{\{n-1\}}} \Delta^{\{n-1,n\}} \hookrightarrow \Delta^n \;\middle|\; n \geq 2 \right\},$$

  which has the following properties.

  - The cofibrations are the monomorphisms.
  - The fibrant objects are Segal spaces.
  - Every Reedy weak equivalence is a weak equivalence in the Segal space model structure.

- The *complete Segal space model structure* is the left Bousfeld localization of the Reedy model structure at the set S of spine inclusions together with the inclusion $\{0\} \hookrightarrow I$, where I is the (nerve of the) walking isomorphism. The complete Segal space model structure has the following properties.

  - The cofibrations are the monomorphisms.
  - The fibrant objects are complete Segal spaces.
  - Every weak equivalence in the Segal space model structure is a weak equivalence in the complete Segal space model structure.

Proposition A.1.1 (Joyal–Tierney)**.** There is a Quillen equivalence

$$p_1^* : \mathcal{S}\mathrm{et}_\Delta^{\mathrm{Joyal}} \longleftrightarrow \mathcal{S}\mathrm{et}_{\Delta^2}^{\mathrm{CSS}} : i_1^*$$

between the Joyal model structure on simplicial sets and the complete Segal space model structure on bisimplicial spaces. Here, the functors p_1^* and i_1^* are defined as follows.

- The functor p_1^* takes a simplicial set X to the bisimplicial space $(p_1^*X)_{mn} = X_m$. That is, the bisimplicial set p_1^*X is constant in the the vertical direction, and each row is equal to the simplicial set X.

- The functor i_1^* takes a bisimplicial set K to the simplicial set $(i_1^* K)_n = K_{n0}$. That is, in the language of Subsection A.1, the simplicial set $i_1^* K$ is the 'zeroth row' of K.

Note A.1.2. In the notation of

For more information, the reader is directed to [1].

A.2. Divisibility of bifunctors

In this section, we recall some key results from [1]. We refer readers there for more information.

Let $\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ be a functor of 1-categories. We will say that $\odot$ is *divisible on the left* if for each $A \in \mathcal{E}_1$, the functor $A \odot -$ admits a right adjoint $A \backslash -$. In this case, this construction turns out also to be functorial in A; that is, we get a functor

$$-\backslash - \colon \mathcal{E}_1^{\mathrm{op}} \times \mathcal{E}_3 \to \mathcal{E}_2.$$

Analogously, $\odot \colon \mathcal{E}_1 \times \mathcal{E}_2 \to \mathcal{E}_3$ is *divisible on the right* if for for each $B \in \mathcal{E}_2$, the functor $- \odot B$ admits a right adjoint $-/B$. In this case we get a of two variables

$$-/- \colon \mathcal{E}_3 \times \mathcal{E}_2^{\mathrm{op}} \to \mathcal{E}_1.$$

Example A.2.1. The reader may find it helpful to keep in mind the cartesian product

$$- \times - \colon \mathcal{S}\mathrm{et}_\Delta \times \mathcal{S}\mathrm{et}_\Delta \to \mathcal{S}\mathrm{et}_\Delta.$$

In this case, both $A \backslash X$ and X/A are the mapping space X^A.

If $\odot$ is divisible on both sides, then there is a bijection between maps of the following types:

$$A \odot B \to X, \qquad A \to X/B, \qquad B \to A \backslash X.$$

In particular, this implies that the functors $X/-$ and $-\backslash X$ are mutually right adjoint.

If both $\mathcal{E}_1$ and $\mathcal{E}_2$ are finitely complete and $\mathcal{E}_3$ is finitely cocomplete, then from a map $u \colon A \to A'$ in $\mathcal{E}_1$, a map $v \colon B \to B'$ in $\mathcal{E}_2$, and a map $f \colon X \to Y$ in $\mathcal{E}_3$, we can build the following maps.

- From the square

$$\begin{tikzcd} A \odot B \arrow[r] \arrow[d] & A' \odot B \arrow[d] \\ A \odot B' \arrow[r] & A' \odot B' \end{tikzcd}$$

we get a map

$$u \odot' v\colon A \odot B' \amalg_{A \odot B} A' \odot B \to A' \odot B'.$$

- From the square

$$\begin{tikzcd} A'\backslash X \arrow[r] \arrow[d] & A\backslash X \arrow[d] \\ A'\backslash Y \arrow[r] & A\backslash Y \end{tikzcd}$$

we get a map

$$\langle u\backslash f\rangle\colon A'\backslash X \to A\backslash X \times_{A\backslash Y} A'\backslash Y$$

- From the square

$$\begin{tikzcd} X/B' \arrow[r] \arrow[d] & X/B \arrow[d] \\ Y/B' \arrow[r] & Y/B \end{tikzcd}$$

we get a map

$$\langle f/v\rangle\colon X/B' \to X/B \times_{Y/B} Y/B'.$$

Proposition A.2.2. With the above notation, the following are equivalent adjoint lifting problems:

$$\begin{tikzcd} A \odot B' \amalg_{A \odot B} A' \odot B \arrow[r] \arrow[d] & X \arrow[d] \\ A' \odot B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d] & X/B' \arrow[d] \\ A' \arrow[r] \arrow[ur, dashed] & X/B \times_{Y/B} Y/B' \end{tikzcd}$$

$$\begin{tikzcd} B \arrow[r] \arrow[d] & A'\backslash X \arrow[d] \\ B' \arrow[r] \arrow[ur, dashed] & A\backslash X \times_{A\backslash Y} A'\backslash Y \end{tikzcd}$$