Monoidal push-pull for local systems

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Contents

1 Background

1.1 Scaled simplicial sets

Note: This is basically a summary of the parts of The Goodwillie Calculus that we're gonna need.

Definition 1.1.1. A ***scaled simplicial set*** is a pair $(X, A_X) = \overline{X}$, where X is a simplicial set, and $A \subseteq X_2$ is a set of 2-simplices of X which contains every degenerate simplex. A simplex belonging to A is known as *thin.* A morphism of scaled simplicial sets is a morphism of the underlying simplicial sets which takes thin simplices to thin simplices.

We will need the following facts about scaled simplicial sets.

1. For any category $\mathcal{C}$ enriched in marked simplicial sets, we can build a scaled simplicial set $\mathrm{N}_{\mathrm{sc}}(\mathcal{C}) = (\mathrm{N}(\mathcal{C}), T)$, the ***scaled nerve*** of $\mathcal{C}$, whose underlying simplicial set $\mathrm{N}(\mathcal{C})$ is the simplicial nerve of $\mathcal{C}$. The scaling T is defined in the following way. Let $\sigma\colon \Delta^2 \to \mathrm{N}(\mathcal{C})$ be any 2-simplex. This corresponds to a diagram
$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$
in $\mathcal{C}$, together with a 1-simplex $\alpha_\sigma\colon \Delta^1 \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ corresponding to a map $gf \to h$. We will then define σ to be thin if and only if α_σ is marked in $\mathrm{Map}_{\mathcal{C}}(X, Z)$.

2. The functor N_{sc} admits a left adjoint $\mathfrak{C}^{\mathrm{sc}}\colon \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \to \mathcal{C}\mathrm{at}_\Delta^+$, the ***scaled rigidification***, such that for any scaled simplicial set $\overline{X}$ with underlying simplicial set X, the underlying simplicially enriched category of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$ is simply $\mathfrak{C}(X)$, the rigidification of X. The scaling on $\mathfrak{C}^{\mathrm{sc}}(X)$ also admits a relatively simple description.

3. A map of scaled simplicial sets $\overline{X} \to \overline{Y}$ is said to be a ***bicategorical equivalence*** if the map $F\colon \mathfrak{C}^{\mathrm{sc}}(X) \to \mathfrak{C}^{\mathrm{sc}}(Y)$ is a weak equivalence of $\mathcal{S}\mathrm{et}_\Delta$-enriched categories; that is, if for all objects x, x' of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$, the map
$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{X})}(x, x') \to \mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{Y})}(F(x), F(x'))$$
is a weak equivalence of marked simplicial sets, and if for all $y \in \mathfrak{C}^{\mathrm{sc}}(\overline{Y})$, there exists some $x \in \mathfrak{C}^{\mathrm{sc}}(\overline{X})$ such that $F(x)$ and y are isomorphic in the homotopy category $\mathrm{h}\mathfrak{C}^{\mathrm{sc}}(\overline{Y})$.

4. There is a left proper, combinatorial model structure on $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$, whose cofibrations are monomorphisms, and whose weak equivalences are bicategorical fibrations. This gives a Quillen equvialence
$$\mathfrak{C}^{\mathrm{sc}} : \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} \leftrightarrow \mathcal{C}\mathrm{at}_\Delta^+ : \mathrm{N}_{\mathrm{sc}}.$$

5. An ∞***-bicategory*** is a fibrant object with respect to this model structure. In particular, for any fibrant object $\mathcal{C} \in \mathcal{C}\mathrm{at}_\Delta^+$ (i.e. any quasicategory-enriched category with equvialences marked), $\mathrm{N}_{\mathrm{sc}}(\mathcal{C})$ is an ∞-bicategory.

6. The category of scaled simplicial sets has a natural simplicial enrichment as follows:

For any two scaled simplicial sets $\overline{X}$ and $\overline{Y}$, we define

$$\mathrm{Map}^{\mathrm{sc}}(\overline{X}, \overline{Y})_n = \mathrm{Hom}(\overline{X} \times \Delta^n_\sharp, \overline{Y}),$$

where $\Delta^n_\sharp$ is the scaled simplicial set (Δ^n, Δ^n_2). For $\overline{Y}$ an ∞-bicategory, $\mathrm{Map}^{\mathrm{sc}}(\overline{X}, \overline{Y})$ is a quasicategory. Taking the core gives a Kan complex.

7. With this,

8. We will denote the simplicially enriched category whose objects are ∞-bicategories, with mapping spaces as above by $\mathbf{Bicat}_\infty$. We then define $\mathcal{C}\mathrm{at}_{(\infty,2)} = \mathrm{N}(\mathbf{Bicat}_\infty)$. This is a quasicategory.

We will need multiple notions of the quasicategory of $(\infty, 2)$-categories.

- The model category $\mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta$ of scaled simplicial sets, whose cofibrations are precisely the monomorphisms.
- The weak equivalences are bicategorical equivalences,

1.2 Enhanced twisted arrow categories

This is a summary of the parts of the theory that we'll need.

1. For each $n \in \mathrm{N}$, we consider the simplicial set $\Delta^n \star (\Delta^n)^{\mathrm{op}}$. We will denote the vertices of Δ^n by i and the vertices of $(\Delta^n)^{\mathrm{op}}$ by $\bar{i}$. This is really just a fancy notation for the $(2n+1)$-simplex with vertices

$$1, \ldots, n, \bar{n}, \ldots, \bar{1}.$$

2. We define a cosimplicial object

$$Q\colon \Delta \to \mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta; \qquad [n] \mapsto (\Delta^n \star (\Delta^n)^{\mathrm{op}}, T_n),$$

where T_n is the set of 2-simplices $\sigma\colon \Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ which

   - factor through Δ^n; or
   - factor through $(\Delta^n)^{\mathrm{op}}$; or
   - are of the form $\Delta^{\{i,j,\bar{k}\}}$, where $i < j \leq k$; or
   - are of the form $\Delta^{\{k,\bar{j},\bar{i}\}}$, where $i < j \leq k$.

3. For any scaled simplicial set $\overline{X}$, we define a simplicial set

$$\mathbf{Tw}(\overline{X})_n = \mathrm{Hom}_{\mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta}(Q([n]), \overline{X}).$$

We further define a marking on $\mathbf{Tw}(\overline{X})$ by saying that a 1-simplex in $\mathbf{Tw}(\overline{X})$ is marked if and only if it corresponds to a map $\Delta^3_\sharp \to \overline{X}$.

4. This construction clearly extends to an ordinary functor $\mathbf{Tw}\colon \mathcal{S}\mathrm{et}^{\mathrm{sc}}_\Delta \to \mathcal{S}\mathrm{et}^+_\Delta$. One checks that this functor preserves weak equivalences, and thus extends to a functor of quasi-categories or **Kan**-enriched categories or whatever your favorite model is

$$\mathbf{Tw}\colon \mathcal{C}\mathrm{at}_{(\infty,2)} \to \mathcal{C}\mathrm{at}_\infty.$$

5. For any scaled simplicial set $\overline{X}$, a zero-simplex in $\mathbf{Tw}(\overline{X})$ is a 1-simplex in $\overline{X}$, and a 1-simplex in $\mathbf{Tw}(\overline{X})$ is a 3-simplex in $\overline{X}$ of the form

$$\begin{tikzcd} x \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\circlearrowleft", shift left=3ex] \arrow[dr, phantom, "\Rightarrow", shift right=3ex] & x' \arrow[d] \\ y & y' \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r] \arrow[d] \arrow[r, phantom, "\Rightarrow", shift right=4ex] & x' \arrow[d] \arrow[dl] \arrow[d, phantom, "\circlearrowleft", shift right=4ex] \\ y & y' \arrow[l] \end{tikzcd},$$

where the simplices marked with $\circlearrowleft$ are thin. The $\Rightarrow$-marked simplices are simply to remind us that given a $\mathcal{S}\mathrm{et}^+_\Delta$-enriched category $\mathcal{C}$, two-simplices in $\mathfrak{C}^{\mathrm{sc}}(\mathcal{C})$ correspond to diagrams in $\mathcal{C}$ of the form

$$\begin{tikzcd} & c' \arrow[dr, "g"] & \\ c \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, "\Uparrow\eta", shift left=3ex] & & c'' \end{tikzcd},$$

where η is a 2-morphism $h \Rightarrow g \circ f$.

Definition 1.2.1. We will call a 1-simplex in $\Delta^1 \to \mathbf{Tw}(\overline{X})$ ***svelte*** if it corresponds to a map $(\Delta^1 \star \Delta^{1\,\mathrm{op}})_\sharp \to \overline{X}$.

We would like to understand (relative) pullbacks in $p\colon \mathbf{Tw}(\overline{X}) \to \mathcal{C} \times \mathcal{C}^{\mathrm{op}}$. More specifically, we would like to have the following.

Proposition 1.2.2. Let $\sigma\colon \Lambda^2_2 \to \mathbf{Tw}(\mathbb{C})$ corresponding to a diagram

$$\begin{tikzcd} & C' \arrow[d, "F"] \\ C'' \arrow[r] & C \end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$, where F is svelte. Then a filling

$$\overline{\sigma} = \begin{tikzcd} \tilde{C} \arrow[r] \arrow[d, "F'"'] & C' \arrow[d, "F"] \\ C'' \arrow[r] & C \end{tikzcd}$$

to a square is a pullback if and only if the following conditions are satisfied:

- The image of $\overline{\sigma}$ in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$ is pullback (that is, the top square of $\overline{\sigma}$ is pullback, and the bottom square is pushout)
- The morphism F' is svelte.

Proof. We need to show that we can lift

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathbf{Tw}(\mathbb{C})_{/\overline{\sigma}} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathbf{Tw}(\mathbb{C})_{/\sigma} \times_{(\mathcal{C}\times\mathcal{C}^{\mathrm{op}})_{/\sigma}} (\mathcal{C} \times \mathcal{C}^{\mathrm{op}})_{/\overline{\sigma}} \end{tikzcd},$$

or equivalently

$$\begin{tikzcd} \partial\Delta^n \star (\Lambda^2_0)^{\triangleleft} \coprod_{\partial\Delta^n\star\Lambda^2_0} \Delta^n \star (\Lambda^2_0)^{\triangleleft} \arrow[r] \arrow[d] & \mathbf{Tw}(\mathbb{C}) \arrow[d] \\ \Delta^n \star (\Lambda^2_0)^{\triangleleft} \arrow[r] \arrow[ur, dashed] & \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \end{tikzcd}.$$

We can write $(\Lambda^2_0) \cong \Delta^{\{012\}} \amalg_{\Delta^{\{02\}}} \Delta^{\{01'2\}}$. □

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_{\dagger}, \mathcal{C}^{\dagger}) \to (\mathcal{D}, \mathcal{D}_{\dagger}, \mathcal{D}^{\dagger})$ be a functor between adequate triples which is an inner fibration, satisfying the following conditions.

1. Every egressive morphism in $\mathcal{D}$ admits a lift which is both egressive and p-cartesian.

2. Consider a square

$$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

in $\mathcal{C}$ in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

$$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$ is cocartesian and f is p-cartesian.

Proof. Even simpler than before. The square that we have to show is pullback factors as before, but this time we don't have to take the final pullback; the square is automatically pullback. □

3 The non-monoidal construction

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{C}\mathrm{at}_\infty \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{S} \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} .$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is of the form

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & Y \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

This is r-cartesian if and only if η is a natural isomorphism $F \Rightarrow F' \circ \alpha$; that is, if the above square (weakly) commutes. We will use the symbol $\circlearrowleft$ denote a weakly commuting square:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} .$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad \mathcal{D} \mapsto \mathrm{Fun}(\mathcal{D}, \mathcal{C}),$$

which sends a functor $f\colon \mathcal{D} \to \mathcal{D}'$ to the pullback

$$f^*\colon \mathrm{Fun}(\mathcal{D}', \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. The map r is thus a bicartesian fibration with cocartesian morphisms given by morphisms

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}' \arrow[d, "\mathrm{Lan}_\alpha F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd} \mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd} \mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d, "f_!g^*\mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Proposition 3.0.1. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 3.0.2. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\ B' \arrow[r] & C \end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd} a \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\alpha"] & & b \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow\beta"] & \\ & b' \arrow[rr] \arrow[dd] & & c \arrow[dd] \\ \bar{a} & & \bar{b} \arrow[ll] & \\ & \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul] \end{tikzcd}$$

in $\mathbb{C}$.[1] Suppose β is an equivalence, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if α is an equivalence.

Proof. Sorry for the writing, I'm really tired. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C} \times \mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 3.0.1. This is literally the lemma right above us. □

Well, it's nice to see that this stuff works as intended. We know that the triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ of unconstrained spans of spaces works as intended, so let's just move on.

Proposition 3.0.3. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

[1] Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\ & X \arrow[rr, "f"] \arrow[dd] \arrow[r, phantom, "\Rightarrow"] & & Y \arrow[dd] \\ \mathcal{C} \arrow[rr, equal] \arrow[dr, equal] \arrow[r, phantom, "\circlearrowleft"] & & \mathcal{C} \arrow[dr, equal] \arrow[r, phantom, "\circlearrowleft"] & \\ & \mathcal{C} \arrow[rr, equal] & & \mathcal{C} \end{tikzcd}$$

in which the left and right faces are thin (as are the top and bottom faces), and the front face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd} X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\ & \mathcal{C} & \\ X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"'] \end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}\mathcal{G}(y') &\cong \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\ &\cong \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],\end{aligned}$$

where here we mean by $X' \cong X \times_Y Y'$ the homotopy pullback. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}\mathcal{G}(y') &\cong \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\ &\cong \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. This is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects, and s sends the terminal object of one to the other.) Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. □

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4 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd} \Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^{I} \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i} \end{tikzcd}$$

Example 4.0.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[2] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}'_i$, $i = 1, 2$.
- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1\times G_2"] \\ \mathcal{C}' & \mathcal{D}'_1 \times \mathcal{D}'_2 \arrow[l, "\beta"] \end{tikzcd}$$

  where here we mean $\eta\colon F \Rightarrow \beta \circ (G_1 \times G_2) \circ \alpha$.

Note that $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ admits a map to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}^{\times}_\infty)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}^{\times}_\infty$ is the *cocartesian* relative nerve of the same.

Taking all this together gives a commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}^{\times}_\infty)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}.$$

where

Proposition 4.0.2. Everything in sight is a cartesian fibration.

Proof. The left-hand map is a cartesian fibration by construction, as is the right-hand map. The □

[2] Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

Proposition 4.0.3. There is an equivalence $\mathcal{C}\mathrm{at}_{\infty,\times} \simeq \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ which is homotopic to the identity on $\mathcal{C}\mathrm{at}_\infty$.

Proposition 4.0.4. We can express a monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\mathcal{C}\mathrm{at}_{\infty,\times} \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is the equivalence on the first component, and given by the composition

$$\mathcal{C}\mathrm{at}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\mathcal{C}\mathrm{at}_{\infty,\times} \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{C}\mathrm{at}_{\infty,\times} \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

Proposition 4.0.5. The map p is a cartesian fibration, and the p-cartesian morphisms in $\mathcal{C}\mathrm{at}_{\infty,\times}$ are precisely those representing products.

Proposition 4.0.6. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{\mathcal{D}} \arrow[r, "\alpha"] \arrow[d, "F"'] & \vec{\mathcal{D}}' \arrow[d, "G"] \\
\vec{\mathcal{C}} \arrow[ur, phantom, "=\eta\Rightarrow"] & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if η is an equivalence
- q-cartesian if α exhibits a product and η is an equivalence.

One sees immediately that r sends q-cartesian morphisms to p-cartesian morphisms, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

5 The triple structures

5.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 5.1.1. This is an adequate triple structure.

Proposition 5.1.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

5.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \mathcal{C}\mathrm{at}_{\infty,\times}$
- $\mathcal{P}_\dagger = \mathcal{C}\mathrm{at}_{\infty,\times} \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \mathcal{C}\mathrm{at}_{\infty,\times}$

Proposition 5.2.1. The cartesian fibration $\mathcal{C}\mathrm{at}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proposition 5.2.2. This is an adequate triple structure.

5.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$

- $\mathcal{Q}^{\dagger}$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_{\otimes}$ of r-cartesian morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{\mathcal{D}} \arrow[d] & \vec{\mathcal{D}}' \arrow[l] \arrow[r] \arrow[d] \arrow[l, phantom, "\circlearrowleft"] & \vec{\mathcal{D}}'' \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] \arrow[u, phantom, shift left=6ex, "\Longrightarrow"] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\vec{\mathcal{D}} \arrow[d, dotted, no head] & \vec{\mathcal{D}}' \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{\mathcal{D}}'' \arrow[d, dotted, no head] \\
\langle n \rangle \arrow[r] & \langle m \rangle & \langle m \rangle \arrow[l, "\cong"']
\end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_{\dagger}, \mathcal{P}^{\dagger})$ if it is of the form

$$\begin{tikzcd}
\vec{\mathcal{D}} \arrow[d, "\mathcal{F}"'] & \vec{\mathcal{D}}' \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}t"] \arrow[l, phantom, shift left=0ex, "\circlearrowleft"] & \vec{\mathcal{D}}'' \arrow[d, "f_!(t \circ g^*\mathcal{F}t)"] \\
\mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"']
\end{tikzcd}.$$

In particular, in the fiber over $\langle 1 \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

6 Tilting at cosimplicial objects

Future Angus: This definitely isn't going to work. We're throwing away any semblance of coherence. This is just the twisted arrow category of **QCat** as a 2-category.

Let's consider $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$. The n-simplices are given by maps

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(\Delta^n, \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)) &\simeq \mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^+}(Q(n), \mathcal{C}\mathrm{at}_\infty) \\
&\simeq \mathrm{Hom}_{\mathcal{C}\mathrm{at}_\Delta^+}(\mathfrak{C}^{\mathrm{sc}}Q(n), \mathbf{QCat}),
\end{aligned}$$

so we can think of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)$ as coming from the cosimplicial object $\mathfrak{C}^{\mathrm{sc}}Q \colon \Delta \to \mathcal{C}\mathrm{at}_\Delta^+$, with the following description:

- The objects of $\mathfrak{C}^{\mathrm{sc}}Q(n)$ are the objects of the linearly ordered set

$$S_n = [n] \star [\bar{n}] = \{0 < 1 < \cdots < n < \bar{n} < \cdots < \bar{0}\}.$$

  We will denote a generic element of S_n by a, b, etc. We will denote an element of

$[n] \subset S_n$ by i, and a generic element of $[\bar{n}] \subset S_n$ by $\bar{i}$. That is, for each $a \in S_n$ we always have either $a = i$ or $a = \bar{i}$ for some $0 \le i \le n$. We refer to i as the *digit* of a.

- For any $a, b \in S_n$, the underlying simplicial set of $\mathfrak{C}^{\mathrm{sc}}Q(n)(a,b)$ is the (nerve of the) poset of subsets of $\{a < \cdots < b\}$ containing a and b.
- A morphism $A \subseteq B \subseteq \{a < \cdots < b\}$ in such a mapping space is marked if and only if the highest digit between 0 and n appearing in both A and B is the same. For instance, for mapping spaces between objects $i, j \in [n]$, the situation corresponding to A and B both belonging to $[n] \subseteq S_n$, everything is marked because by definition both A and B must contain i (and similarly if both belong to $[\bar{n}]$).

We then define a cosimplicial object $R \colon \Delta \to \mathcal{C}\mathrm{at}^+_\Delta$ as follows.

- The objects of $R(n)$ are the objects of the linearly ordered set

$$S_n = \{0 < 1 < \cdots < n < \bar{n} < \cdots < \bar{0}\}.$$

- The mapping space $R(n)(a,b)$ is given by the (nerve of the) linearly ordered set

$$\{\max(a,b) < \cdots < n\},$$

  where by $\max(a,b)$, we mean the maximum digit appearing. In particular:
  - For $b < a$, the mapping simplicial sets are empty. To avoid having to repeatedly mention that this case is trivial, we implicitly consider only mapping spaces $\mathrm{Map}(a,b)$ with $a \le b$.
  - For $a = i$ and $b = j$,
- Let $\phi \colon [m] \to [n]$. We define a map $R(m) \to R(n)$ as follows.
  - On objects, we send $i \mapsto \phi(i)$ and $\bar{i} \mapsto \overline{\phi(i)}$. We write both of these $a \mapsto \phi(a)$.
  - On morphisms, we define a map

$$R(m)(a,b) \to R(n)(\phi(a), \phi(b))$$

    sending $i \in \{\max(a,b) < \cdots < m\}$ to $\phi(i) \in \{\max(\phi(a), \phi(b)) < \cdots < n\}$.

For each n, we get a map

$$\iota_n \colon R(n) \to \mathfrak{C}^{\mathrm{sc}}Q(n)$$

which is the identity on objects, and acts on morphisms in the following way:

- For $a = i$ and $b = j$, c is given by some digit k with $i \le k \le j$, and we send