Monoidal push-pull for local systems

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Contents

1 Introduction and conventions

1.1 The marked-scaled model structure

One common model for $(\infty, 2)$-categories is Lurie's theory of ∞-bicategories

In [2], a marked-biscaled model structure modelling cartesian fibrations of ∞-bicategories is defined. When taken over a point, the two scalings collapse into one, giving the following.

Definition 1.1.1. A marked-scaled simplicial set is a triple (X, E_X, T_X), where $E_X \subseteq X_1$ is a collection of edges of X containing all degenerate edges, and $T_X \subseteq X_2$ is a collection of triangles of X containing all degenerate triangles.

Notation 1.1.2.

- We will denote the category of marked-scaled simplicial sets by $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}$.
- To save on notation, we will sometimes denote the marked-scaled simplicial set (X, E_X, T_X) by $X_{T_X}^{E_X}$, particularly in the case that E_X or T_X are $\sharp$ or $\flat$, the maximum (resp. minimum) markings and scalings. For example, the bimarked simplicial set $X_\flat^\sharp$ has all 1-simplices marked and only degenerate 2-simplices scaled, etc.

Definition 1.1.3. The set of ms-anodyne morphisms is a saturated set of morphisms between marked-scaled simplicial sets containing the following classes of morphisms:

(A1) Inner horn inclusions

$$(\Lambda_i^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}) \to (\Delta^n, \flat, \{\Delta^{\{i-1,i,i+1\}}\}),$$

for $n \geq 2$ and $0 < i < n$.

(A2) Outer horn inclusions

$$(\Lambda_n^n, \{\Delta^{\{n-1,n\}}\}, \{\Delta^{\{0,n-1,n\}}\}),$$

for $n \geq 1$.

Theorem 1.1.4. There is a model structure on the category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{ms}}$, whose trivial cofibrations are given by the ms-anodyne maps, and whose fibrant objects are ∞-bicategories with precisely the equivalences marked.

2 A new Barwick's theorem

It turns out Barwick's construction can be obviously modified. Here is a form which is more useful to us. Little to no effort has been made to make this as general as possible, as long as it works. It is relatively clear from the proof that there are many sets of conditions that suffice, and probably no unique weakest one.

Theorem 2.0.1. Let $p\colon (\mathcal{C}, \mathcal{C}_\dagger, \mathcal{C}^\dagger) \to (\mathcal{D}, \mathcal{D}_\dagger, \mathcal{D}^\dagger)$ be a functor between adequate triples such that $p\colon \mathcal{C} \to \mathcal{D}$ is an inner fibration which satisfies the following conditions.

1. A morphism in $\mathcal{C}$ is egressive if and only if it is p-cartesian (hence also $p^\dagger$-cartesian).
2. Every egressive morphism in $\mathcal{D}$ admits a lift in $\mathcal{C}$ (given a lift of the target) which is both egressive and p-cartesian.
3. Consider a square

   $$\sigma = \begin{tikzcd} y' \arrow[r, "f'"] \arrow[d, two heads, "g'"'] & x' \arrow[d, two heads, "g"] \\ y \arrow[r, tail, "f"] & x \end{tikzcd}$$

   in $\mathcal{C}$ with egressive and ingressive morphisms as marked, in which f is p-cocartesian and g and g' are p-cartesian, lying over an ambigressive pullback square

   $$p(\sigma) = \begin{tikzcd} py' \arrow[r, tail] \arrow[d, two heads] & px' \arrow[d, two heads] \\ py \arrow[r, tail] & px \end{tikzcd}$$

   in $\mathcal{D}$. Then σ is pullback, and f' is ingressive, p-cocartesian, and and $p_\dagger$-cocartesian.

Then spans of the form

$$\begin{tikzcd} & z \arrow[dl, two heads, "g"'] \arrow[dr, tail, "f"] & \\ x & & y \end{tikzcd}$$

are cocartesian, where g is $p^\dagger$-cartesian and f is p-cocartesian.

Proof. Even simpler than the proof in my thesis of the ordinary Barwick's theorem. The square that we have to show is homotopy pullback factors as before, but this time we don't have to take the final homotopy pullback to find path components corresponding to cartesian 2-simplices, since our 2-simplices are automatically cartesian. □

3 Horn filling via left Kan extensions

Kan extensions of ∞-categories are usually defined pointwise via the colimit formula [4] [1]. In this section, we will show that such pointwise ∞-Kan extensions enjoy a horn filling property in $\mathbb{C}\mathrm{at}_\infty$ which generalizes the universal property for Kan extensions of functors between 1-categories.

Definition 3.0.1. We will call a 2-simplex $\tau\colon \Delta^2_\flat \to \mathbb{C}\mathrm{at}_\infty$ ***left Kan*** if it is of the form

$$\begin{tikzcd} X \arrow[rr, "F"] \arrow[dr, "f"'] & {} \arrow[d, phantom, "\overset{\eta}{\Downarrow}"] & \mathcal{C} \\ & Y \arrow[ur, "f_!F"'] & \end{tikzcd},$$

where $\mathcal{C}$ is cocomplete, X is small, $f_!F$ is a left Kan extension of F along f, and $\eta\colon F \Rightarrow f^*f_!F$ is the unit map.

The goal of this section is to prove the following.

Theorem 3.0.2. Let $\tau\colon \Delta^2 \to \mathbb{C}\mathrm{at}_\infty$ be a left Kan 2-simplex. Then for each $n \geq 2$, every solid lifting problem of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd} \tag{1}$$

admits a dashed solution.

Let us examine this in the case $n = 2$. In this case we have the data of categories and functors

$$\begin{tikzcd} & Y \arrow[drr, "f_!F"] \arrow[ddr, "g"] & & \\ X \arrow[ur, "f"] \arrow[rrr, "F"] \arrow[drr, "h"'] & & & \mathcal{C} \\ & & Z \arrow[ur, "G"'] & \end{tikzcd}$$

together with natural transformations $\eta\colon F \Rightarrow f_!F \circ f$, $\beta\colon h \Rightarrow g \circ f$, and $\delta\colon F \Rightarrow G \circ h$. We need to produce a natural transformation $\alpha\colon f_!F \Rightarrow G \circ g$ and a filling of the full 3-simplex, which is the data of a homomotopy-commutative diagram

$$\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_! F \circ f \arrow[d, "\alpha f"] \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}$$

in $\mathrm{Fun}(X, \mathcal{C})$.[1]

We can rephrase this as follows. We are given the data

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[d, "\delta"'] & f_! F \circ f \\ G \circ h \arrow[r, "G\beta"] & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_! F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

where the left-hand diagram is a map $LC^2 \to \mathrm{Fun}(X, C)$, where LC^2 is the simplicial subset of the boundary of $\Delta^1 \times \Delta^1$ except the right-hand face, and the right-hand diagram is a map $\partial\Delta^1 \to \mathrm{Fun}(Y, C)$. We need to construct a filler $\partial\Delta^1 \hookrightarrow \Delta^1$ on the right, and from it a filler $LC^2 \hookrightarrow \Delta^1 \times \Delta^1$ on the left. We do this in the following sequence of steps.

1. We first can fill the lower-left half of the diagram on the left simply by taking the composition $G\beta \circ \delta$. Doing this, we are left with the filling problem

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_! F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_! F \\ G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})} .$$

2. We note that we can extend the diagram on the right to a diagram which is adjunct to the diagram on the left under the adjunction $f_! \dashv f^*$:

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_! F \circ f \\ & G \circ g \circ f \end{tikzcd}}^{\text{in Fun}(X,\mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_! F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_! F \\ & G \circ g \end{tikzcd}}^{\text{in Fun}(Y,\mathcal{C})}$$

[1] In particular, if we take $h = f$, $g = \mathrm{id}_Y$, and β to be the identity $f \Rightarrow f$, we recover the classical universal property satisfied by left Kan extension.

3. The diagram on the right has an obvious filler, which is adjunct to a filler on the left.

$$\overbrace{\begin{tikzcd} F \arrow[r, "\eta"] \arrow[dr] & f_! F \circ f \arrow[d, "\alpha f"] \\ & G \circ g \circ f \end{tikzcd}}^{\text{in } \mathrm{Fun}(X, \mathcal{C})} \qquad \overbrace{\begin{tikzcd} f_! F \arrow[r, "\mathrm{id}"] \arrow[dr] & f_! F \arrow[d, "\alpha"] \\ & G \circ g \end{tikzcd}}^{\text{in } \mathrm{Fun}(Y, \mathcal{C})}$$

The lifting problems which we have to solve in Equation 1 for $n > 2$ amount to replacing $\Delta^1 \times \Delta^1$ by $(\Delta^1)^n$, etc; the basic process remains unchanged, but the combinatorics involved in filling the necessary cubes becomes more involved. In Subsection 3.1 we explain the combinatorics of filling cubes relative to their boundaries. In Subsection 3.2, we give a formalization the concept of adjunct data, and provide a means of for filling partial data to total adjunct data. In Subsection 3.3, we show that we can always solve the lifting problems of Equation 1.

3.1 Filling cubes relative to their boundaries

In this section, we give the combinatorics of filling the n-cube $(\Delta^1)^n$ relative to its boundary. Our main result is Proposition 3.1.21, which writes the inclusion the boundary of the n-cube missing a certain face into the full cube as a composition of an inner anodyne map and a marked anodyne map.

Definition 3.1.1. The ***n-cube*** is the simplicial set $C^n := (\Delta^1)^n$.

Note 3.1.2. We will consider the the factors Δ^1 of C^n to be ordered, so that we can speak about the first factor, the second factor, etc.

Our first step is to understand the nondegenerate simplices of the n-cube.

Definition 3.1.3. Denote by $a^i \colon \Delta^n \to \Delta^1$ the map which sends $\Delta^{\{0,\dots,i-1\}}$ to $\{0\}$, and $\Delta^{\{i,\dots,n\}}$ to $\{1\}$.

Note 3.1.4. The superscript of a^i counts how many vertices of Δ^n are sent to $\Delta^{\{0\}} \subset \Delta^1$.

Every nondegenerate simplex $\Delta^n \to C^n$ can be specified by giving a walk along the edges of C^n starting at $(0, \dots, 0)$ and ending at $(1, \dots, 1)$, and any simplex specified in this way is nondegenerate. We now use this to define a bijection between S_n, the symmetric group on the set $\{1, \dots, n\}$, and the nondegenerate simplices $\Delta^n \to C^n$ as follows.

Definition 3.1.5. For any permutation $\tau \colon \{1, \dots, n\} \to \{1, \dots, n\}$, we define an n-simplex

$$\phi(\tau) \colon \Delta^n \to C^n; \qquad \phi(\tau)_i = a^{\tau(i)}.$$

That is,

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)}).$$

Note 3.1.6. It is easy to check that the above definition really results in a bijection between S_n and the nondegenerate simplices of C^n.

Notation 3.1.7. We will denote any permutatation $\tau\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ by the corresponding n-tuple $(\tau(1), \ldots, \tau(n))$. Thus, the identity permutation is $(1, \ldots, n)$, and the permutation $\gamma_{1,2}$ which swaps 1 and 2 is $(2, 1, 3, \ldots, n)$.

Given a permutation $\tau \in S_n$ corresponding to an n-simplex

$$\phi(\tau) = (a^{\tau(1)}, \ldots, a^{\tau(n)})\colon \Delta^n \to C^n,$$

the ith face of $\phi(\tau)$ is a nondegenerate $(n-1)$-simplex in C^n, i.e. a map $\Delta^{n-1} \to C^n$. We calculate this as follows: for any $a^i\colon \Delta^n \to \Delta^1$ and any face map $\partial_j\colon \Delta^{n-1} \to \Delta^n$, we note that

$$\partial_j^* a^i = \begin{cases} a^{i-1}, & j < i \\ a^i, & j \geq i \end{cases},$$

where by minor abuse of notation we denote the map $\Delta^{n-1} \to \Delta^1$ sending $\Delta^{\{0,\ldots i-1\}}$ to $\{0\}$ and the rest to $\{1\}$ also by a^i. We then have that

$$d_i\tau = d_i(a^{\tau(1)}, \ldots, a^{\tau(n)}) := (\partial_i^* a^{\tau(1)}, \ldots, \partial_i^* a^{\tau(n)}).$$

Example 3.1.8. Consider the 3-simplex (a^1, a^2, a^3) in C^3. This has spine

$$(0, 0, 0) \to (1, 0, 0) \to (1, 1, 0) \to (1, 1, 1),$$

as is easy to see quickly:

- The function $\Delta^3 \to \Delta^1$ corresponding to the first coordinate is a^1, so the first coordinate of the first vertex is 0, and the first coordinate of the rest of the vertices are 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the second coordinate is a^2, so the second coordinates of the first two points are equal to zero, and the second coordinate of the remaining vertices is 1.
- The function $\Delta^3 \to \Delta^1$ corresponding to the third coordinate is a^3, so the third coordinates of the first three points is equal to zero, and the second coordinate of the final vertex is equal to 1.

Using Equation 3.1, we then calculate that

$$d_2(a^1, a^2, a^3) = (a^1, a^2, a^2).$$

This corresponds to the 2-simplex in C^n with spine

$$(0,0,0) \longrightarrow (1,0,0) \longrightarrow (1,1,1).$$

Our next task is to understand the boundary of the n-cube. We can view the boundary of the cube C^n as the union of its faces.

Definition 3.1.9. The ***boundary of the*** n***-cube*** is the simplicial subset

$$\partial C^n := \bigcup_{i=1}^{n} \bigcup_{j=0}^{1} \overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{j\}} \times \cdots \times \overset{(n)}{\Delta^1} \subset C^n. \tag{2}$$

The following is easy to see.

Proposition 3.1.10. An $(n-1)$-simplex

$$\gamma = (\gamma_1, \ldots, \gamma_n) \colon \Delta^{n-1} \longrightarrow C^n$$

lies entirely within the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^n = \mathrm{const}_0$, and to the face

$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(i)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}$$

if and only if $\gamma_i = a^0 = \mathrm{const}_1$.

Definition 3.1.11. The ***left box of the*** n***-cube***, denoted LC^n, is the simplicial subset of C^n given by the union of all of the faces in Equation 2 except for

$$\Delta^1 \times \cdots \times \Delta^1 \times \Delta^{\{1\}}.$$

Note that drawing the box with the final coordinate going from left to right, the right face is open here; the terminology is chosen to match with 'left horn'.

Example 3.1.12. The simplicial subset $LC^2 \subset C^2$ can be drawn as follows.

$$\begin{tikzcd} (0,0) \arrow[r] \arrow[d] & (0,1) \\ (1,0) \arrow[r] & (1,1) \end{tikzcd}$$

Note that in LC^2, the morphisms $(0,0) \to (1,0) \to (1,1)$ form an inner horn, which we can fill by pushing out along an inner horn inclusion. Our main goal in this section is to show that this is generically true: given a left cube LC^n, we can fill much of C^n using pushouts along inner horn inclusions. To this end, it will be helpful to know how each nondegenerate simplex in C^n intersects LC^n.

Lemma 3.1.13. Let $\phi\colon \Delta^n \to C^n$ be a nondegenerate n-simplex corresponding to the permutaton $\tau \in S_n$. We have the following.

1. The zeroth face $d_0\phi$ belongs to LC^n if and only if $\tau(n) \neq 1$.
2. For $0 < i < n$, the face $d_i\phi$ never belongs to LC^n.
3. The nth face $d_n\phi$ always belongs to LC^n.

Proof.
1. We have
$$d_0\phi = (a^{\tau(0)-1}, \dots a^{\tau(n)-1})$$
since $\tau(i) > 0$ for all i. For $j = \tau^{-1}(1)$, we have that $\tau(j) = 1$, so the jth entry of $d_0\phi$ is a^0. Thus, Proposition 3.1.10 guarantees that $d_0\phi$ is contained in the face
$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{1\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$
This face belongs to LC^n except when $j = n$.
2. In this case, the superscript of each entry of $d_i\phi$ is between 1 and $n-1$, hence not equal to n or 0.
3. In this case,
$$d_n(a^{\tau(1)}, \dots, a^{\tau(n)}) = (a^{\tau(1)}, \dots, a^{\tau(n)})$$
since $n \geq \tau(i)$ for all $1 \leq i \leq n$. Thus, the $\tau^{-1}(n) = j$th entry is a^n, so $d_n\phi$ belongs to the face
$$\overset{(1)}{\Delta^1} \times \cdots \times \overset{(j)}{\{0\}} \times \cdots \times \overset{(n)}{\Delta^1}.$$

□

This is promising: it means that for many simplices in C^n which we want to fill relative to LC^n, we already have the data of the first and last faces. The following lemma will allow us to take advantage of this.

Notation 3.1.14. For any subset $T \subseteq [n]$, write

$$\Lambda^n_T := \bigcup_{t \in T} d_t \Delta^n \subset \Delta^n.$$

Lemma 3.1.15. For any proper subset $T \subset [n]$ containing 0 and n, the inclusion $\Lambda^n_T \hookrightarrow \Delta^n$ is inner anodyne.

Proof. Induction. For $n = 2$, T must be equal to $\{0, 2\}$, so $\Delta^2_T = \Lambda^2_1$.

Assume the result holds for $n - 1$, and let $T \subset [n]$ be a proper subset containing 0 and n. Using the inductive step, we can fill all but one of the faces $d_1\Delta^n, \ldots, d_{n-1}\Delta^n$. The result follows □

We can use Lemma 3.1.15 to show that we can fill much of C^n as a sequence of inner anodyne pushouts, but to do this we need to pick an order in which to fill our simplices. We do this as follows.

Definition 3.1.16. We define a total order on S_n as follows. For any two permutations τ, $\tau' \in S_n$, we say that $\tau < \tau'$ if there exists $k \in [n]$ such that the following conditions are satisfied.

- For all $i < k$, we have that $\tau^{-1}(i) = \tau'^{-1}(i)$.
- We have that $\tau^{-1}(k) < \tau'^{-1}(k)$.

We then define a total order on the nondegenerate simplices of C^n by saying that $\phi(\tau) < \phi(\tau')$ if and only if $\tau < \tau'$.

Note that this is *not* the lexicographic order on S_n; instead, we have that $\tau < \tau'$ if and only if τ^{-1} is less than τ'^{-1} under the lexicographic order.

Example 3.1.17. The elements of the permutation group S_3 have the order

$$(1, 2, 3) < (1, 3, 2) < (2, 1, 3) < (3, 1, 2) < (2, 3, 1) < (3, 2, 1).$$

We use this ordering to define our filtration.

Notation 3.1.18. For $\tau \in S_n$, we mean by

$$\bigcup_{\tau' \in S_n}^{\tau}$$

the union over all $\tau' \in S_n$ for which $\tau' < \tau$ with respect to the ordering defined above. Note the strict inequality.

We would like to show that each step of the filtration

$$\begin{aligned} LC^n &\hookrightarrow LC^n \cup \phi(1, \ldots, n) \\ &\hookrightarrow \cdots \\ &\hookrightarrow LC^n \cup \bigcup_{\tau \in S_n}^{(2,\ldots,n,1)} \phi(\tau) \end{aligned} \tag{3}$$

is inner anodyne. To do this, it suffices to show that for each $\tau < (2, \ldots, n, 1)$, the intersection

$$B_\tau = \left(LC^n \cup \bigcup_{\tau' \in S_n}^{\tau} \phi(\tau') \right) \cap \phi(\tau) \tag{4}$$

is of the form Λ^n_T for some proper $T \subseteq [n]$ containing 0 and n.

Proposition 3.1.19. Each inclusion in Equation 3 is inner anodyne.

Proof. The first inclusion

$$LC^n \hookrightarrow LC^n \cup \phi(0, \ldots, n)$$

is inner anodyne because by Lemma 3.1.13 the intersection $LC^n \cap \phi(0, \ldots, n)$ is of the form $d_0\Delta^n \cup d_n\Delta^n$.

Each subsequent intersection contains the faces $d_0\Delta^n$ and $d_n\Delta^n$ (again by Lemma 3.1.13), so by Lemma 3.1.15, it suffices to show that for each τ under consideration, there is at least one face not shared with any previous τ'.

To this end, fix $0 < i < n$, and consider $\tau \in S_n$ such that $\tau \neq (0, \ldots, n)$. We consider the face $d_i\phi(\tau)$. Let $j = \tau^{-1}(i)$ and $j' = \tau^{-1}(i+1)$. Then $d_i a^{\tau(j)} = d_i a^{\tau(j')}$, so $d_i\phi(\tau) = d_i\phi(\tau \circ \gamma_{jj'})$, where $\gamma_{jj'} \in S_n$ is the permutation swapping j and j'. Thus, the face $d_i\sigma(\tau)$ is equal to the face $d_i\sigma(\tau \circ \gamma_{jj'})$, which is already contained in the union under consideration if and only $\tau \circ \gamma_{jj'} < \tau$. This in turn is true if and only if $j < j'$.

Assume that every face of $\phi(\tau)$ is contained in the union. Then

$$\tau^{-1}(1) < \tau^{-1}(2) < \cdots \tau^{-1}(n),$$

which implies that $\tau = (1, \ldots, n)$. This is a contradiction. Thus, each boundary inclusion under consideration is of the form $\Lambda^n_T \hookrightarrow \Delta^n$, for $T \subset [n]$ a proper subset containing 0 and n, and is thus inner anodyne. □

Each nondegenerate simplex of C^n which we have yet to fill is of the form $\phi(\tau)$, where $\tau(n) = 1$. Thus, each of their spines begins

$$(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1) \longrightarrow \cdots ,$$

and then remains confined to the face $\Delta^1 \times \cdots \times \Delta^1 \times \{1\}$. This implies that the part of LC^n yet to be filled is of the form $\Delta^0 * C^{n-1}$, and the boundary of this with respect to which we must do the filling is of the form $\Delta^0 * \partial C^{n-1}$.

Our next task is to show that we can also perform this filling, under the assumption that $(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1)$ is marked. We do not give the details of this proof, at is quite similar to the proof of Proposition 3.1.19. One continues to fill simplices in the order prescribed in Definition 3.1.16, and shows that each of these inclusions is marked anodyne. The main tool is the following lemma, proved using the same inductive argument as Lemma 3.1.15.

Lemma 3.1.20. For any subset $T \subset [n]$ containing 1 and n, and *not* containing 0, the inclusion

$$(\Lambda^n_T, \mathcal{E}) \hookrightarrow (\Delta^n, \mathcal{F})$$

is (cocartesian-)marked anodyne, where by $\mathcal{F}$ we mean the set of degenerate 1-simplces together with $\Delta^{\{0,1\}}$, and by $\mathcal{E}$ we mean the restriction of this marking to Λ^n_T.

We collect our major results from this section in the following proposition.

Proposition 3.1.21. Denote by $J^n \subseteq C^n$ the simplicial subset spanned by those nondegenerate n-simplices whose spines do not begin $(0, \ldots, 0, 0) \longrightarrow (0, \ldots, 0, 1)$. We can write the filling $LC^n \hookrightarrow C^n$ as a composition

$$LC^n \xhookrightarrow{i} J^n \xhookrightarrow{j} C^n ,$$

where i is inner anodyne, and j fits into a pushout square

$$\begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, hook] \arrow[d, hook] & \Delta^0 * C^{n-1} \arrow[d, hook] \\ J^n \arrow[r, hook] & C^n \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} ,$$

where the top inclusion underlies a marked anodyne morphism

$$(\Delta^0 * \partial C^{n-1}, \mathcal{G}) \hookrightarrow (\Delta^0 * C^{n-1}, \mathcal{G}),$$

where the marking $\mathcal{G}$ contains all degenerate morphisms together with the morphism $(0, \ldots, 0) \longrightarrow (0, \ldots 1)$.

3.2 Adjunct data

In this section, we give a formalization of the the notion of adjunct data. Our main result is Proposition 3.2.6, which shows that under certain conditions, we can use data on one side of an adjunction to solve lifting problems on the other side.

Definition 3.2.1. Let $s\colon K \to \Delta^1$ be a map of simplicial sets, and let $p\colon \mathcal{M} \to \Delta^1$ be a bicartesian fibration associated to the adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

via equivalences $h_0\colon \mathcal{C} \to \mathcal{M}_0$ and $h_1\colon \mathcal{D} \to \mathcal{M}_1$. We say that a map $\alpha\colon K \to \mathcal{C}$ is ***adjunct*** to a map $\tilde{\alpha}\colon K \to \mathcal{D}$ relative to s, and equivalently that $\tilde{\alpha}$ is adjunct to α relative to s, if there exists a map $A\colon K \times \Delta^1 \to \mathcal{M}$ such that the diagram

$$\begin{tikzcd} K \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d, "p"] \\ & \Delta^1 \end{tikzcd}$$

commutes, and such that the following conditions are satisfied.

1. The restriction $A|_{K\times\{0\}} = h_0 \circ \alpha$.
2. The restriction $A|_{K\times\{1\}} = h_1 \circ \tilde{\alpha}$.
3. For each vertex $k \in K$ in the fiber $s^{-1}\{0\}$, the image of the edge $\{k\} \times \Delta^1 \to \mathcal{M}$ is p-cocartesian.
4. For each vertex $k' \in K$ in the fiber $s^{-1}\{1\}$, the image of the edge $\{k'\} \times \Delta^1 \to \mathcal{M}$ is p-cartesian.

We often leave the equivalences h_0 and h_1 implicit to lighen notation.

In the next examples, fix an adjunction

$$f : \mathcal{C} \longleftrightarrow \mathcal{D} : g$$

corresponding to a bicartesian fibration $p\colon \mathcal{M} \to \Delta^1$.

Example 3.2.2. Any morphism in $\mathcal{D}$ of the form $fC \to D$ is adjunct to some morphism in $\mathcal{C}$

of the form $C \to gD$, witnessed by any square

$$\begin{tikzcd} C \arrow[r, "a"] \arrow[d] & fC \arrow[d] \\ gD \arrow[r, "b"] & D \end{tikzcd}$$

in $\mathcal{M}$ where a is p-cocartesian and b is p-cartesian. This corresponds to the map $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

Example 3.2.3. Pick some object $D \in \mathcal{D}$, and consider the identity morphism $\mathrm{id}\colon gD \to gD$ in $\mathcal{C}$. This morphism is adjunct to the component of the unit map $\eta_D\colon D \to fgD$ relative to $s = \mathrm{id}\colon \Delta^1 \to \Delta^1$.

The next example will follow from Proposition 3.2.6.

Example 3.2.4. Any diagram $K * K' \to \mathcal{D}$ such that K is in the image of f is adjunct to some diagram $K * K' \to \mathcal{C}$ such that K' is in the image of g.

Lemma 3.2.5. The inclusion of marked simplicial sets

$$\left(\Delta^n \times \Delta^{\{0\}} \coprod_{\Delta^n \times \Delta^{\{0\}}} \partial\Delta^n \times \Delta^{\{0\}}, \mathcal{E}\right) \hookrightarrow (\Delta^n \times \Delta^1, \mathcal{F}) \tag{5}$$

where the marking $\mathcal{F}$ is the flat marking together with the edge $\Delta^{\{0\}} \times \Delta^1$, and $\mathcal{E}$ is the restriction of this marking, is marked (cocartesian) anodyne.

The next proposition shows the power of adjunctions in lifting problems: given data on either side of an adjunction, we can fill in adjunct data on the other side.

Proposition 3.2.6. Let $\mathcal{M} \to \Delta^1$, $\mathcal{C}$, $\mathcal{D}$, f, g, h_0 and h_1 be as in Definition 3.2.1, and let

$$\begin{tikzcd} K' \arrow[rr, hook, "i"] \arrow[dr, "s\circ i"'] & & K \arrow[dl, "s"] \\ & \Delta^1 & \end{tikzcd},$$

be a commuting triangle of simplicial sets, where i is a cofibration such that $i|_{\{1\}}$ is an isomorphism. Let $\tilde{\alpha}'\colon K' \to \mathcal{D}$ and $\alpha\colon K \to \mathcal{C}$ be maps, and denote $\alpha' = \alpha \circ i$. Suppose that $\tilde{\alpha}'$ is adjunct to α' relative to $s \circ i$. Then there exists a dashed extension

$$\begin{tikzcd} K' \arrow[r, "\alpha'"] \arrow[d, hook, "i"'] & \mathcal{C} \\ K \arrow[ur, "\alpha"'] & \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} K' \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook, "i"'] & \mathcal{D} \\ K \arrow[ur, dashed, "\tilde{\alpha}"'] & \end{tikzcd} \tag{6}$$

such that $\tilde{\alpha}$ is adjunct to α relative to s.

Proof. Pick some commutative diagram

$$\begin{tikzcd} K' \times \Delta^1 \arrow[r, "A"] \arrow[dr, "\mathrm{pr}_{\Delta^1}"'] & \mathcal{M} \arrow[d] \\ & \Delta^1 \end{tikzcd}$$

displaying $\tilde{\alpha}'$ and α' as adjunct. We further consider the map $h_0 \circ \alpha \colon K \to \mathcal{M}_0$. From these two pieces of data we can construct the solid commutative square of cocartesian-marked simplicial sets

$$\begin{tikzcd} \left(K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times \Delta^1\right)^{\heartsuit} \arrow[r] \arrow[d] & \mathcal{M}^{\natural} \arrow[d] \\ (K \times \Delta^1)^{\heartsuit} \arrow[r] \arrow[ur, dashed, "\ell"] & (\Delta^1)^{\sharp} \end{tikzcd},$$

where $(K \times \Delta^1)^{\heartsuit}$ is the marked simplicial set where the only nondenerate morphisms marked are those of the form $\{k\} \times \Delta^1$, for $k \in K|_{s^{-1}\{0\}}$, and the $\heartsuit$-marked pushout-product denotes the restriction of this marking to the pushout-product. Starting from K' and adjoining one nondegenerate simplex of K at a time, we can write the left-hand map in the above square as a transfinite composition of pushouts of morphisms of the form given in Equation 5 (by assumption each simplex σ we adjoin is not completely contained in K', so the initial vertex of σ must lie in the fiber of s over 0). Each of these is marked anodyne, so the left-hand map is as well. Thus, a dashed lift ℓ exists, giving us a map $h_1^{-1} \circ \ell|_{K \times \Delta^{\{1\}}} \colon K \to \mathcal{D}$. This, together with the composition

$$K' \times \Delta^1 \xrightarrow{\tilde{\alpha}' \times \mathrm{id}} \mathcal{D} \times \Delta^1 \xrightarrow{c} \mathcal{D}$$

where $c \colon h_1^{-1} \circ h_1 \to \mathrm{id}_{\mathcal{D}}$ is a natural isomorphism, gives a solid diagram

$$\begin{tikzcd} K \times \Delta^{\{0\}} \coprod_{K' \times \Delta^{\{0\}}} K' \times (\Delta^1)^{\sharp} \arrow[r] \arrow[d] & \mathcal{D}^{\natural} \\ K \times (\Delta^1)^{\sharp} \arrow[ur, dashed, "m"'] & \end{tikzcd},$$

where $\mathcal{D}^{\natural}$ is the marked simplicial set whose underlying simplicial set is $\mathcal{D}$ and where all equivalences are marked. This admits a dashed lift, and restricting $m|_{K\times\Delta^{\{1\}}}$ gives us a map $\tilde{\alpha}\colon K \longrightarrow \mathcal{D}$ as in Equation 6. However, we still need to display this as adjunct to α: the lift ℓ has $\ell|_{K\times\Delta^{\{0\}}} = h_0 \circ \alpha$, but only $\ell|_{K\times\Delta^{\{1\}}} \simeq h_1 \circ \tilde{\alpha}$, not equality as we require.

Our last task is to remedy this. Denote the spine of the 3-simplex by

$$I_3 := \Delta^{\{0,1\}} \amalg_{\Delta^{\{1\}}} \Delta^{\{1,2\}} \amalg_{\Delta^{\{2\}}} \Delta^{\{2,3\}} \hookrightarrow \Delta^3.$$

We build a map $I_3 \times K \xrightarrow{b} \mathcal{M}$ which:

- On $\Delta^{\{0,1\}} \times K$ is given by the lift ℓ.
- On $\Delta^{\{1,2\}} \times K$ is given by the composition

  $$K \times \Delta^1 \xrightarrow{\ell|_{K\times\Delta^{\{1\}}}\times\mathrm{id}} \mathcal{M}_1 \times \Delta^1 \xrightarrow{H} \mathcal{M}_1 \hookrightarrow \mathcal{M},$$

  where H is a natural isomorphism $\mathrm{id}_{\mathcal{M}_1} \longrightarrow h_1 \circ h_1^{-1}$.
- On $\Delta^{\{2,3\}} \times K$ is given by the natural isomorphism

  $$K \times \Delta^1 \xrightarrow{m} \mathcal{D} \xrightarrow{h_1} \mathcal{M}_1 \hookrightarrow \mathcal{M}.$$

The map $I_3 \hookrightarrow \Delta^3$ is inner anodyne, so there exists a map $\Delta^3 \times K \longrightarrow \mathcal{M}$ whose restriction to $I_3 \times K$ is equal to b. The restriction of this filling to $\Delta^{\{0,3\}} \times K$ satisfies the conditions of Definition 3.2.1 (since the composition of a (co)cartesian morphism with an equivalence is again (co)cartesian), exhibiting $\tilde{\alpha}$ as adjunct to α. □

Note 3.2.7. The filler provided by Proposition 3.2.6 is not guaranteed to be unique (even up to equivalence). However, we have some say in how our filler should look. We construct our adjoint filler $\tilde{\alpha}$ by starting with the solid data of Equation 6 and building the dashed extension $\tilde{\alpha}\colon K \longrightarrow \mathcal{D}$ by filling a simplex at a time relative to its boundary, such that each new simplex making up the map $\tilde{\alpha}$ is adjunct to the corresponding one in α. We are free to choose each new adjunct simplex however we like.

Example 3.2.8. Let $f : \mathcal{C} \leftrightarrow \mathcal{D} : g$ be an adjunction of 1-categories, giving an adjunction between quasicategories upon taking nerves. Consider the diagram

$$\begin{tikzcd} K' = \Delta^{\{0,1,3\}} \cup \Delta^{\{0,2,3\}} \arrow[rr, hook, "i"] \arrow[dr] & & \Delta^3 = K \arrow[dl] \\ & \Delta^1 & \end{tikzcd},$$

where both downwards-facing arrows take $0, 1 \mapsto 0$ and $2, 3 \mapsto 1$, and fix a map $\tilde{\alpha}\colon K \longrightarrow N(\mathcal{C})$

and a map $\alpha' \colon K' \to N(\mathcal{D})$ such that $\tilde{\alpha} \circ i$ is adjunct to α'. This corresponds to the diagrams

$$\overbrace{\begin{tikzcd} C \arrow[r] \arrow[d] & gD \arrow[d] \\ C' \arrow[r] \arrow[ur, dashed] & gD' \end{tikzcd}}^{\text{in } \mathcal{C}} \qquad \overbrace{\begin{tikzcd} fC \arrow[r] \arrow[d] & D \arrow[d] \\ fC' \arrow[r] & D' \end{tikzcd}}^{\text{in } \mathcal{D}}$$

where the solid diagrams in each category are adjunct to one another. Proposition 3.2.6 implies that the solution to the lifting problem on the left yields one on the right. Similarly, its dual implies the converse. Thus, Proposition 3.2.6 implies in particular that such lifting problems are equivalent.

3.3 Left Kan implies globally left Kan

In this subsection, we will prove Theorem 3.0.2. In order to do that, we must show that we can solve lifting problems of the form

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd},$$

for $n \geq 2$, where τ is left Kan. Forgetting for a moment that have singled out some 2-simplex τ, and using the adjunction $\mathfrak{C}^{\mathrm{sc}} \dashv \mathrm{N}_{\mathrm{sc}}$, we will do this by solving equivalent lifting problems of the form[2]

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}] \arrow[r] \arrow[d, hook] & \mathbf{QCat} \\ \mathfrak{C}[\Delta^{n+1}] \arrow[ur, dashed, "\mathcal{F}"'] & \end{tikzcd} . \tag{7}$$

We expect that the reader is familiar with the basics of (scaled) rigidification, so we give only a rough description of this lifting problem here. The objects of the simplicial category $\mathfrak{C}[\Delta^{n+1}]$ are given by the set $\{0, \ldots, n+1\}$, and the mapping spaces are defined by

$$\mathfrak{C}[\Delta^{n+1}](i,j) = \begin{cases} N(P_{ij}), & i \leq j \\ \emptyset, & i > j \end{cases} .$$

[2]Note that we are implicitly taking $\mathfrak{C}[\Lambda_0^{n+1}]$ and $\mathfrak{C}[\Delta^{n+1}]$ to carry the flat markings on their mapping spaces.

where P_{ij} is the poset of subsets of the linearly ordered set $\{i, \dots, j\}$ containing i and j, ordered by inclusion. The simplicial category $\mathfrak{C}[\Lambda_0^{n+1}]$ has the same objects as $\mathfrak{C}[\Delta^{n+1}]$, and each morphism space $\mathfrak{C}[\Lambda_0^{n+1}](i,j)$ is a simplicial subset of $\mathfrak{C}[\Delta^{n+1}](i,j)$, as we shall soon describe.

The map $\mathfrak{C}[\Lambda_0^{n+1}](i,j) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](i,j)$ is an isomorphism except for $(i,j) = (1, n+1)$ and $(i,j) = (0, n+1)$. The missing data corresponds in the case of $(1, n+1)$ to the missing face $d_0\Delta^{n+1}$ of Λ_0^{n+1}, and in the case of $(0, n+1)$ to the missing interior. Thus, to find a filling as in Equation 7, we need to solve the lifting problems

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](1,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[ur, dashed, "\ell'"'] & \end{tikzcd} \tag{8}$$

and

$$\begin{tikzcd} \mathfrak{C}[\Lambda_0^{n+1}](0,n+1) \arrow[r] \arrow[d, hook] & \mathrm{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[ur, dashed, "\ell"'] & \end{tikzcd} . \tag{9}$$

However, these problems are not independent; the filling ℓ of the full simplex needs to agree with the filling ℓ' we found for the missing face of the horn, corresponding to the condition that the square

$$\begin{tikzcd} \mathfrak{C}[\Delta^{n+1}](1,n+1) \arrow[r, "\ell'"] \arrow[d, "\{0,1\}^*"'] & \mathrm{Fun}(\mathcal{F}(1), \mathcal{F}(n+1)) \arrow[d, "\mathcal{F}(\{0,1\})^*"] \\ \mathfrak{C}[\Delta^{n+1}](0,n+1) \arrow[r, "\ell"] & \mathrm{Fun}(\mathcal{F}(0), \mathcal{F}(n+1)) \end{tikzcd} \tag{10}$$

commute.

Notation 3.3.1. Recall our desired filling $\mathcal{F} \colon \mathfrak{C}[\Delta^{n+1}] \to \mathbf{QCat}$ of Equation 7. We will denote $\mathcal{F}(i)$ by X_i, and for each subset $S = \{i_1, \dots, i_k\} \subseteq [n]$, we will denote $\mathcal{F}(S) \in \mathrm{Map}(X_{i_1}, X_{i_k})$ by $f_{i_k \cdots i_1}$. For any inclusion $S' \subseteq S \subseteq [n]$ preserving minimim and maximum elements, we will denote the corresponding morphism by $\alpha_S^{S'}$.

Theorem 3.3.2. For any $n \geq 2$ and any globally left Kan 2-simplex τ, the solid extension problem

$$\begin{tikzcd} \Delta_\flat^{\{0,1,n+1\}} \arrow[d, hook] \arrow[dr, "\tau"] & \\ (\Lambda_0^{n+1})_\flat \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta_\flat^{n+1} \arrow[ur, dashed] & \end{tikzcd}$$

admits a dashed filler.

Proof. We need to solve the lifting problems of Equation 8 and Equation 9, and check that our solutions satisfy the condition of Equation 10. We note the following useful pieces of information.

- The information that the simplex $\tau\colon \Delta_\flat^{\{0,1,n+1\}} \to \mathbb{C}\mathrm{at}_\infty$ is left Kan tells us that $f_{n,1}$ is the left Kan extension of $f_{n,0}$ along $f_{1,0}$, and that $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}}\colon f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$ is the unit map.
- We have an adjunction

$$(f_{1,0})_! : \mathrm{Fun}(X_0, X_{n+1}) \longleftrightarrow \mathrm{Fun}(X_1, X_{n+1}) : f_{1,0}^*.$$

It is easy to see that we have the following succinct descriptions of the inclusions of Equation 8 and Equation 9 along which we have to extend.

- There is an isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](0, n+1) \xrightarrow{\cong} C^n$$

  specified completely by sending a subset $S \subseteq [n+1]$ to the point

$$(z_1, \dots, z_n) \in C^n, \qquad z_i = \begin{cases} 1, & i \in S \\ 0, & \text{otherwise.} \end{cases}$$

  The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](0, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](0, n+1)$ corresponds to the simplicial subset $LC^n \hookrightarrow C^n$.

- There is a similar isomorphism of simplicial sets

$$\mathfrak{C}[\Delta^{n+1}](1, n+1) \xrightarrow{\cong} C^{n-1}$$

  specified completely by sending $S \subseteq \{1, \dots, n+1\}$ to the point

$$(z_1, \dots, z_{n-1}) \in C^{n-1}, \qquad z_i = \begin{cases} 1, & i+1 \in S \\ 0, & \text{otherwise.} \end{cases}$$

  The inclusion $\mathfrak{C}[\Lambda^{n+1}_0](1, n+1) \hookrightarrow \mathfrak{C}[\Delta^{n+1}](1, n+1)$ corresponds to the inclusion $\partial C^{n-1} \hookrightarrow C^{n-1}$.

Using these descriptions, we can write our lifting problems in the more inviting form

$$(*) = \begin{tikzcd} LC^n \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ C^n \arrow[ur, dashed] & \end{tikzcd}, \qquad (**) = \begin{tikzcd} \partial C^{n-1} \arrow[r, "\tilde{\alpha}'"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ C^{n-1} \arrow[ur, dashed] & \end{tikzcd},$$

and our condition becomes that the square

$$(\star) = \begin{tikzcd} C^{n-1} \arrow[r] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \arrow[d, "f^*_{1,0}"] \\ C^n \arrow[r] & \mathrm{Fun}(X_0, X_{n+1}) \end{tikzcd}$$

commutes, where the left-hand vertical morphism is the inclusion of the right face.

Using Proposition 3.1.21, we can partially solve the lifting problem $(*)$, reducing it to the lifting problem

$$(*') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\alpha"] \arrow[d, hook] & \mathrm{Fun}(X_0, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed] & \end{tikzcd}.$$

The image of the 1-simplex $(0, \ldots, 0) \to (0, \ldots, 1)$ of $\Delta^0 * \partial C^{n-1} \subseteq C^n$ under α is the unit map $\alpha^{\{0,1,n+1\}}_{\{0,n+1\}} : f_{n+1,0} \to f_{n+1,1} \circ f_{1,0}$. This is not in general an equivalence, so we cannot use Proposition 3.1.21 to solve the lifting problem $(*')$ directly. However, the unit map is adjunct to the identity map $\mathrm{id}_{f_{n+1,1}}$ in $\mathrm{Fun}(X_1, X_{n+1})$ relative to $s = \mathrm{id}_{\Delta^1}$. Furthermore, the restriction of α to ∂C^{n-1} is in the image of $f^*_{1,0}$, so by Proposition 3.2.6 we can augment the map $\tilde{\alpha}'$ to a map $\tilde{\alpha} \colon \Delta^0 * \partial C^{n-1} \to \mathrm{Fun}(X_1, X_{n-1})$ which is adjunct to α relative to the map

$$s \colon \Delta^0 * \partial C^{n-1} \to \Delta^1$$

sending Δ^0 to $\Delta^{\{0\}}$ and ∂C^{n-1} to $\Delta^{\{1\}}$; by Note 3.2.7, we can choose $\tilde{\alpha}$ such that the image of the morphism $(0, \ldots, 0) \to (0, \ldots 1)$ is an equivalence. This allows us to replace the lifting problem $(**)$ by the superficially more complicated lifting problem

$$(**') = \begin{tikzcd} \Delta^0 * \partial C^{n-1} \arrow[r, "\tilde{\alpha}"] \arrow[d, hook] & \mathrm{Fun}(X_1, X_{n+1}) \\ \Delta^0 * C^{n-1} \arrow[ur, dashed, "\tilde{\beta}"'] & \end{tikzcd}$$

However, since image of $(0, \ldots 0) \to (0, \ldots, 1)$ is an equivalence, Proposition 3.1.21 implies that we can solve the lifting problem $(**')$. Again using (the dual to) Proposition 3.2.6, we can transport this filling to a solution to the lifting problem $(*')$. The condition $(\star)$ amounts to demanding that the restriction $\beta|_{C^{n-1}}$ be the image of $\tilde{\beta}|_{C^{n-1}}$ under $f^*_{1,0}$, which is true by construction. $\square$

4 Local systems

In [3], the twisted arrow category of an ∞-bicategory is defined.

4.1 The twisted arrow category

For a 2-category C, the twisted arrow 1-category $\mathbf{Tw}(\mathrm{C})$ has the following description.

- The objects of $\mathbf{Tw}(\mathrm{C})$ are the morphisms of C:

$$f\colon c \to c'.$$

- For two objects $(f\colon c \to c')$ and $(g\colon d \to d')$, the morphisms $f \to g$ are given by diagrams

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & d \arrow[d, "g"] \\ c' & d' \arrow[l, "a'"] \end{tikzcd},$$

  where α is a 2-morphism $f \Rightarrow a' \circ g \circ a$.

- The composition of morphisms is given by concatenating the corresponding diagrams.

$$\begin{tikzcd} c \arrow[r, "a"] \arrow[d, "f"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & d \arrow[r, "b"] \arrow[d, dashed, "g"] \arrow[dr, phantom, "\overset{\beta}{\Longrightarrow}"] & e \arrow[d, "h"] \\ c' & d' \arrow[l, "a'"] & e' \arrow[l, "b'"] \end{tikzcd}$$

In [3], a homotopy-coherent version of the twisted arrow category of an ∞-bicategory is defined. For any ∞-bicategory $\mathbb{C}$, the twisted arrow ∞-category of $\mathbb{C}$, denoted $\mathbf{Tw}(\mathbb{C})$, is a quasi-category whose n-simplices are diagrams $(\Delta^n)^{\mathrm{op}} \star \Delta^n \to \mathbb{C}$, together with a scaling to ensure that the information encoded in an n-simplex is determined, up to contractible choice, by data

$$\overbrace{\begin{tikzcd} c_0 \arrow[r, "a_0"] \arrow[d, "f_0"'] \arrow[dr, phantom, "\overset{\alpha_0}{\Longrightarrow}"] & c_1 \arrow[r, dotted] \arrow[d, dashed, "f_1"] & \cdots \arrow[r, dotted] & c_{n-1} \arrow[r, "a_{n-1}"] \arrow[d, dashed, "f_{n-1}"'] \arrow[dr, phantom, "\overset{\alpha_{n-1}}{\Longrightarrow}"] & c_n \arrow[d, "f_n"] \\ c'_0 & c'_1 \arrow[l, "a'_0"] & \cdots \arrow[l, dotted] & c'_{n-1} \arrow[l, dotted] & c'_n \arrow[l, "a'_{n-1}"] \end{tikzcd}}^{n \text{ squares}}$$

in $\mathbb{C}$. Here, the top row of morphisms corresponds to the spine of the n-simplex $\Delta^n \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$, and the bottom row of morphisms to the spine of $(\Delta^n)^{\mathrm{op}} \subset \Delta^n \star (\Delta^n)^{\mathrm{op}}$. To make this correspondence more clear, we will introduce the following notation.

Notation 4.1.1. For $i \in [n]$, we will write $\bar{i} := 2n + 1 - i$.

Thus, the ith column in the above diagram corresponds to the image of the morphism $i \to \bar{i}$ in $\Delta^n \star (\Delta^n)^{\mathrm{op}}$.

We are now ready to define this scaling.

Definition 4.1.2. We define a cosimplicial object $\tilde{Q} \colon \Delta \to \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ by sending

$$\tilde{Q}([n]) = (\Delta^n \star (\Delta^n)^{\mathrm{op}}, \dagger),$$

where $\dagger$ is the scaling consisting of all degenerate 2-simplices, together with all 2-simplices of the following kinds:

1. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through Δ^n.
2. All simplices $\Delta^2 \to \Delta^n \star (\Delta^n)^{\mathrm{op}}$ factoring through $(\Delta^n)^{\mathrm{op}}$.
3. All simplices $\Delta^{\{i,j,\bar{k}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.
4. All simplices $\Delta^{\{k,\bar{j},\bar{i}\}} \subseteq \Delta^n \star (\Delta^n)^{\mathrm{op}}$, $i < j \leq k$.

Note that $\Delta^n \star (\Delta^n)^{\mathrm{op}} \cong \Delta^{2n+1}$. We will use this identification freely.

This extends to a nerve-realization adjunction

$$Q : \mathcal{S}\mathrm{et}_\Delta \longleftrightarrow \mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}} : \mathbf{Tw},$$

where the functor Q is the extension by colimits of the functor $\tilde{Q}$, and for any scaled simplicial set X, the simplicial set $\mathbf{Tw}(X)$ has n-simplices

$$\mathrm{Hom}_{\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}}(Q(\Delta^n), X).$$

Notation 4.1.3. For any simplicial set X, $Q(X)$ carries the scaling given simplex-wise by that described above. We will denote this scaling by $\dagger_X$, or simply by $\dagger$ if the simplicial set X is clear from context. It will sometimes pay to notate the scaling $\dagger$ on $Q(X)$ explicitly, writing $Q(X) = Q(X)_\dagger$. A We can thus write $Q(\Delta^n)$ explicitly and compactly as $\Delta^{2n+1}_\dagger$, which we will often do.

The inclusion $\Delta^n \amalg (\Delta^n)^{\mathrm{op}} \hookrightarrow \Delta^n \star (\Delta^n)^{\mathrm{op}}$ provides, for any scaled simplicial set K, a functor $\mathbf{Tw}(K) \to K \times K^{\mathrm{op}}$. In fact, this assignment is functorial.

Proposition 4.1.4. This extends to a functor $\mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty^{\Delta^1}$.

4.2 Two lemmas about marked-scaled anodyne morphisms

In the section following this one, we provide a fairly explicit construction of a family of marked-scaled anodyne inclusions. Constructing these inclusions directly from the classes of generating marked-scaled anodyne inclusions given in Definition 1.1.3 horn fillings would be possible but tedious. To lighten this load somewhat, we reproduce in this section a lemma from [3], and provide a generalization. The majority of this section is thus lifted from [3, Section 1.2]. Note however that our needs will be rather different than those of [3], and this is reflected in some differences of notation.

We first need a compact notation for specifying simplicial subsets of Δ^n. Denote by $P\colon \mathcal{S}\mathrm{et} \to \mathcal{S}\mathrm{et}$ the power set functor.

Definition 4.2.1. Let T be a finite linearly ordered set, so $T \cong [n]$ for some $n \in \mathrm{N}$. For any subset $\mathcal{A} \subseteq P(T)$, define a simplicial subset

$$\mathcal{S}_T^{\mathcal{A}} := \bigcup_{S \in \mathcal{A}} \Delta^{T \smallsetminus S} \subseteq \Delta^T.$$

Notation 4.2.2.

- If the linearly ordered set T is clear from context, we will drop it, writing $\mathcal{S}^{\mathcal{A}}$.
- For any marked-scaled n-simplex (Δ^n, E, T) we will by minor abuse of notation reuse the same letters E and T to denote the restriction of the markings and scalings to any simplicial subset $S \subseteq \Delta^n$.

The assignment $\mathcal{A} \mapsto \mathcal{S}^{\mathcal{A}}$ does not induce is not a one-to-one correspondence between subsets $\mathcal{A} \subseteq P([n])$ and simplicial subsets $S^{\mathcal{A}} \subseteq \Delta^n$, since for $S \subsetneq S'$, $\Delta^{[n] \smallsetminus S'} \subseteq \Delta^{[n] \smallsetminus S}$. It will be useful to consider two subsets $\mathcal{A}$ and $\mathcal{A}' \subseteq P(T)$ to be equivalent if they produce the same simplicial subset of Δ^T.

Notation 4.2.3. We will write $\mathcal{A} \sim \mathcal{A}'$ if $\mathcal{S}^{\mathcal{A}} = \mathcal{S}^{\mathcal{A}'}$.

The set $P([n])$ forms a poset, ordered by inclusion, and any $\mathcal{A} \subseteq P([n])$ a subposet. An element q of a poset Q is said to be *minimal* if there are no elements of Q which are strictly less than q. By the above remark, only the minimal elements of $\mathcal{A}$ contribute to the union defining $\mathcal{S}^{\mathcal{A}}$. We have just shown the following.

Lemma 4.2.4. For any $\mathcal{A} \subseteq P([n])$, we have $\mathcal{A} \sim \min(\mathcal{A})$, where $\min(\mathcal{A}) \subseteq \mathcal{A}$ is the set of minimal elements of $\mathcal{A}$.

Simplicial subsets of the form $\mathcal{S}_T^{\mathcal{A}}$ enjoy the following easily-proved calculation rules.

Lemma 4.2.5. Let $\mathcal{A} \subseteq P([n])$ and $T \subseteq [n]$. Then

$$\mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T = \mathcal{S}_{[n]}^{\mathcal{A}\cup\{[n]\setminus T\}}.$$

Lemma 4.2.6. Let $\mathcal{A} \subset P([n])$, and let $T \subset [n]$. Then the square

$$\begin{tikzcd}
\mathcal{S}_T^{\mathcal{A}|T} \arrow[r, hook] \arrow[d, hook] & \Delta^T \arrow[d, hook] \\
\mathcal{S}_{[n]}^{\mathcal{A}} \arrow[r, hook] & \mathcal{S}_{[n]}^{\mathcal{A}} \cup \Delta^T
\end{tikzcd}$$

is bicartesian, where $\mathcal{A}|T$ is the subset of $P(T)$ given by

$$\mathcal{A}|T = \{S \cap T \mid S \in \mathcal{A}\}.$$

Proof. We have an equality

$$\mathcal{S}_T^{\mathcal{A}|T} = \mathcal{S}_{[n]}^{\mathcal{A}} \cap \Delta^T.$$

as subsets of Δ^n. □

The goal of the remainder of this section is to provide two criteria for the inclusion $i_{\mathcal{A}} \colon (\mathcal{S}^{\mathcal{A}}, E, T) \subseteq (\Delta^n, E, T)$ to be marked-scaled anodyne:

Definition 4.2.7 ([3], Definition 1.3)**.** We will call a subset $\mathcal{A} \subseteq P([n])$ ***dull*** if it satisfies the following conditions.

- it does not include the empty set; $\emptyset \notin \mathcal{A}$.
- There exists $0 < i < n$ such that for all $S \in \mathcal{A}$, we have $i \notin S$.
- It contains a pair of singletons $\{u\}, \{v\} \in \mathcal{A}$ such that $u < i < v$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.

The element $i \in [n]$ is known as the *pivot point.*

Definition 4.2.8. We will call a subset $\mathcal{A} \subset P([n])$ ***right dull*** if it satisfies the following conditions.

- it does not include the empty set; $\emptyset \notin \mathcal{A}$.
- For all $S \in \mathcal{A}$, we have $n \notin S$.
- It contains a singleton $\{u\}$ such that $u < n$.
- For every $S, T \in \mathcal{A}$, it follows that $S \cap T = \emptyset$.

In this case, we will refer to n as the pivot point.

Definition 4.2.9. Let $A \subset P([n])$ be either dull or right dull. We call $X \in P([n])$ an $\mathcal{A}$-basal set if it contains precisely one element from each $S \in \mathcal{A}$. We denote the set of all $\mathcal{A}$-basal sets by $\mathrm{Bas}(\mathcal{A})$.

Lemma 4.2.10 ([3], Lemma 1.10)**.** Let $\mathcal{A} \subset P([n])$ be a dull subset with pivot point i, and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that for every $Z \in \mathrm{Bas}(\mathcal{A})$ the following condition holds:

- Denote by ℓ_{i-1}^Z and ℓ_i^Z the consecutive elements of Z with $\ell_{i-1}^Z < i < \ell_i^Z$. For all $r, s \in [n]$ with $\ell_{i-1}^Z \leq r < i < s \leq \ell_i^Z$, the triangle $\Delta^{\{r,i,s\}}$ is scaled.

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

Lemma 4.2.11. Let $\mathcal{A} \subset P([n])$ be a right dull subset (whose pivot point is by definition n), and let (Δ^n, E, T) be a marked-scaled simplex. Further suppose that for every $Z \in \mathrm{Bas}(\mathcal{A})$ the following condition holds:

- Denote by ℓ_-^Z and ℓ_+^Z the minimal and maximal element of Z respectively. For all r, $s \in [n]$ with $r \leq \ell_-^Z \leq \ell_+^Z \leq s < n$, the triangle $\Delta^{\{r,s,n\}}$ is scaled, and the simplex $\Delta^{\{s,n\}}$ is marked.

Then the inclusion $(\mathcal{S}^{\mathcal{A}}, E, T) \hookrightarrow (\Delta^n, E, T)$ is marked-scaled anodyne.

4.3 The fibration governing local systems

For any ∞-category $\mathcal{C}$, a $\mathcal{C}$-local system on some space X is simply a functor $X \to \mathcal{C}$. Thus, the ∞-category of $\mathcal{C}$-local systems on X should simply be $\mathbf{LS}(\mathcal{C}, X) := \mathrm{Fun}(X, \mathcal{C})$. We now allow X to vary, defining the category $\mathbf{LS}(\mathcal{C})$ of $\mathcal{C}$-local systems.

Definition 4.3.1. For any cocomplete ∞-category $\mathcal{C}$, we define the ∞-category of $\mathcal{C}$-local systems $\mathbf{LS}(\mathcal{C})$ together with a map of simplicial sets $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ by the following pullback diagram.

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r, hook] \arrow[d, "p"'] & \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty) \arrow[d, "p'"] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd}$$

It is shown in [3] that p', hence also p, is a cartesian fibration. The cartesian fibration p classifies the functor $\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ sending

$$f\colon X \to Y \qquad \longmapsto \qquad f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ is cocomplete, each pullback map f^* has a left adjoint $f_!$, given by left Kan extension. It follows on abstract grounds that p is also a cocartesian fibration, whose cocartesian edges represent left Kan extension. In this section, we show this explicitly.

A general morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ corresponds to a map $Q(\Delta^1) = \Delta^3_\dagger \to \mathbb{C}\mathrm{at}_\infty$ satisfying the following conditions.

1. The objects $\Delta^{\{0\}}$ and $\Delta^{\{1\}}$ are mapped to spaces.
2. The image of $\Delta^{\{2,3\}} \subset \Delta^3$ is $\mathrm{id}_{\mathcal{C}}$.

Definition 4.3.2. We will call a 3-simplex $\Delta^3_\dagger \to \mathbb{C}\mathrm{at}_\infty$ satisfying conditions 1. and 2. above admissible. More generally, a $(2n+1)$-simplex $\sigma\colon \Delta^{2n+1}_\dagger \to \mathbb{C}\mathrm{at}_\infty$ will be called ***admissible*** if the following conditions are satisfied.

- The objects $\sigma(\Delta^{\{0\}}), \dots, \sigma(\Delta^{\{n\}})$ are spaces.
- The restriction of σ to the n-simplex $\Delta^{\{n+1,\dots,2n+1\}}$ is the constant functor with value $\mathcal{C} \in \mathbb{C}\mathrm{at}_\infty$.

Note 4.3.3. The admissible $(2n+1)$-simplicies $\Delta^{2n+1}_\dagger \to \mathbb{C}\mathrm{at}_\infty$ are precisely those which are adjunct to a map $\Delta^n \to \mathbf{LS}(\mathcal{C})$.

Definition 4.3.4. A $(2n+1)$-simplex $\sigma\colon \Delta^{2n+1}_\dagger \to \mathbb{C}\mathrm{at}_\infty$ is said to be ***left Kan*** if it has the following characteristies.

1. It is admissible; that is, it is adjunct to a simplex $\Delta^n \to \mathbf{LS}(\mathcal{C})$.
2. The restriction of σ to the 2-simplex $\Delta^{\{0,1,\bar{0}\}}_\flat \subset \Delta^{2n+1}_\dagger$ is left Kan.

Example 4.3.5. We draw the 'front' and 'back' of a general admissible 3-simplex $\sigma\colon \Delta^3_\dagger \to \mathbb{C}\mathrm{at}_\infty$ corresponding to some morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[dr, "\mathcal{H}"] \arrow[dr, phantom, shift right=3ex, "\Rightarrow\zeta"'] & Y \arrow[d, "\mathcal{G}"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, shift right=3ex, "\Rightarrow\eta"'] & Y \arrow[d, "\mathcal{G}"] \arrow[dl, "\mathcal{G}'"] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd}$$

Here, the 2-simplices which are notated without natural transformations are constrained by the scaling $\dagger$ to be thin. Such an admissible simplex σ is left Kan if and only if $\mathcal{G}'$ is a left Kan extension of $\mathcal{F}$ along f, and η is a unit map.

Our main goal in this section will be to prove the following.

Theorem 4.3.6. An edge $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is p-cocartesian if and only if it is left Kan.

This will imply Corollary 4.3.7 below. We provide a full proof of Corollary 4.3.7 before proving Theorem 4.3.6 because the strategy we will use to prove Theorem 4.3.6 is similar to, but more complicated than, that employed in the proof of Corollary 4.3.7.

Corollary 4.3.7. The map $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cocartesian fibration.

Notation 4.3.8. We endow the quasicategory $\mathbf{LS}(C)$ with the marking $\clubsuit \subseteq \mathbf{LS}(\mathcal{C})_1$ consisting of all left Kan edges.

Proof of Corollary 4.3.7. We know that p is a cartesian fibration, hence certainly an inner fibration. We need to show that p has enough cocartesian lifts, i.e. that any lifting problem

$$\begin{tikzcd} (\Delta^{\{0\}})^\flat \arrow[r] \arrow[d, hook] & \mathbf{LS}(\mathcal{C})^\clubsuit \arrow[d] \\ (\Delta^1)^\sharp \arrow[r] \arrow[ur, dashed] & (\mathcal{S})^\sharp \end{tikzcd}$$

has a dashed solution. Any such lifting problem is adjunct to a lifting problem of the form

$$\begin{tikzcd} K_\dagger \arrow[r] \arrow[d, hook] & \mathbb{C}\mathrm{at}_\infty \\ \Delta^3_\dagger \arrow[ur, dashed] & \end{tikzcd}, \qquad (11)$$

where $K_\dagger \cong (\Delta^{\{0,1\}} \cup \Delta^{\{0,3\}} \cup \Delta^{\{2,3\}}, \dagger)$, and $K_\dagger \to \mathbb{C}\mathrm{at}_\infty$ is the restriction of an admissible simplex (this tells us that $\Delta^{\{0\}}$ and $\Delta^{\{1\}}$ are mapped to spaces, and that $\Delta^{\{2,3\}}$ is mapped to $\mathrm{id}_{\mathcal{C}}$). We need to construct a dashed lift which is left Kan.

It is relatively easy to do this directly, but we take a different approach, which we will generalize in our proof of Theorem 4.3.6. We endow $\mathbb{C}\mathrm{at}_\infty$ with a marking $\natural$ consisting of all equivalences. We endow $\Delta^3_\dagger$ with the marking $\heartsuit$ consisting of the simplex $\Delta^{\{2,3\}}$ together with all degenerate 1-simplices. We denote the restriction of this marking to K also by $\heartsuit$. We then note that the lifting problem of Equation 11 is equivalent to the marked version

$$\begin{tikzcd} (K, \heartsuit, \dagger) \arrow[r] \arrow[d, hook, "a"'] & \mathbb{C}\mathrm{at}_\infty^\natural \\ (\Delta^3, \heartsuit, \dagger) \arrow[ur, dashed] & \end{tikzcd};$$

the only condition imposed by the markings we have just added is that the simplex $\Delta^{\{2,3\}} \subset K$ be sent to an equivalence, which is always satisfied for the lifting problem above since $\mathrm{id}_{\mathcal{C}}$ is an equivalence.

We note that there is a pullback square

$$\begin{tikzcd} (\Delta^{\{0,1\}} \cup \Delta^{\{0,3\}}, \flat, \flat) \arrow[r, hook, "u_1"] \arrow[d, hook, "b"'] & (K, \heartsuit, \dagger) \arrow[d, hook, "a"] \\ (\Delta^{\{0,1,3\}}, \flat, \flat) \arrow[r, hook, "v_1"] & (\Delta^3, \heartsuit, \dagger) \end{tikzcd} .$$

The map u_1 is a pushout along the map $(\Delta^{\{1\}}, \flat, \flat) \hookrightarrow (\Delta^1, \sharp, \flat)$, which is marked-scaled anodyne of type (A2). It is thus a weak equivalence. The map v_1 is a weak equivalence by Lemma 4.3.9. Thus, by [4, Prop. A.2.3.1], in order to find a lift as in Equation 11 it will suffice to find a lift

$$\begin{tikzcd} (\Delta^{\{0,1\}} \cup \Delta^{\{0,3\}}, \flat, \flat) \arrow[r] \arrow[d, hook, "b"'] & \mathbb{C}\mathrm{at}_\infty \\ (\Delta^{\{0,1,3\}}, \flat, \flat) \arrow[ur, dashed] & \end{tikzcd}$$

which is left Kan. However, the existence of such lifts is precisely the condition that for any functor between spaces $f : X \to Y$, and any functor $\mathcal{F} : X \to \mathcal{C}$, the left Kan extension $f_! \mathcal{F}$ exists. This is true by our assumption that $\mathcal{C}$ be cocomplete. □

Lemma 4.3.9. The map v_1 can be written as the composition

$$v_1 = (\Delta^{\{0,1,3\}}, \flat, \flat) \xhookrightarrow{v_1''} (\Delta^{\{0,1,3\}} \cup \Delta^{\{2,3\}}, \heartsuit, \flat) \xhookrightarrow{v_1'} (\Delta^3, \heartsuit, \dagger),$$

each of which is marked-scaled anodyne.

Proof. The morphism v_1'' is a pushout along a morphism of type (A2). We construct the morphism v_1' in the following way.

1. Then we fill the simplex $\Delta^{\{1,2,3\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A2).
2. Then we fill the simplex $\Delta^{\{0,1,2\}}$ (together with its marking and scaling) as a pushout along a morphism of type (A1).
3. Finally, we fill the full simplex Δ^3 (together with its marking and scaling) as a pushout along a morphism of type (A1).

□

We have thus shown that, should the left Kan edges really turn out to be precisely the p-cocartesian edges, then p is a cocartesian fibration. However, it remains to be verified that left Kan edges really are precisely the same as cocartesian edges. We first show one implication: that if an edge is left Kan, it is cocartesian.

In order to show that all left Kan edges are cocartesian, we have to show that lifting problems

$$\begin{tikzcd} (\Lambda^n_0)^{\mathcal{L}} \arrow[r] \arrow[d, hook] & \mathbf{LS}(\mathcal{C})^{\clubsuit} \arrow[d] \\ (\Delta^n)^{\mathcal{L}} \arrow[r] \arrow[ur, dashed] & \mathcal{S}^{\sharp} \end{tikzcd}, \qquad n \geq 2,$$

have solutions.[3] Unravelling the definitions, we find that these are adjunct to the lifting problems

$$\begin{tikzcd} K^n_{\dagger} \arrow[r, "\tau'"] \arrow[d, hook] & \mathbb{C}\mathrm{at}_{\infty} \\ \Delta^{2n+1}_{\dagger} \arrow[ur, dashed, "\tau"'] & \end{tikzcd}, \tag{12}$$

where:

- K^n is the simplicial subset of Δ^n on those simplices $\sigma\colon \Delta^k \to K^n$ which satisfy at least one of the following conditions.
  1. The k-simplex σ factors through the 'top face' $\Delta^n \subset \Delta^{2n+1} \cong \Delta^n \star (\Delta^n)^{\mathrm{op}}$.
  2. The k-simplex σ factors through the 'bottom face' $(\Delta^n)^{\mathrm{op}} \subset \Delta^{2n+1} \cong \Delta^n \star (\Delta^n)^{\mathrm{op}}$.
  3. There exists $\ell \neq 0$ such that the k-simplex σ hits neither $\Delta^{\{\ell\}}$ nor $\Delta^{\{\bar{\ell}\}}$.
- The filler τ is left Kan. (This implies in particular that the map τ' fulfills all conditions to be left Kan which apply to the scaled simplicial subset $K^n_{\dagger}$.)

It is actually easier to solve these lifting problems by generalizing them somewhat. Recall that any left Kan simplex $\tau\colon \Delta^{2n+1}_{\dagger} \to \mathbb{C}\mathrm{at}_{\infty}$ is in particular admissible, and thus has the property that the restriction $\tau|\Delta^{\{n+1,\ldots,2n+1\}}$ is the constant functor with value $\mathcal{C}$. We will relax this requirement, demanding only that each morphism in $\Delta^{\{n+1,\ldots,2n+1\}}$ be mapped to an equivalence. We can guarantee this by introducing a marking, in addition to the scaling. We can do this using the theory of marked-scaled simplicial sets, as developed in [2].

Notation 4.3.10.

1. Denote by $\natural$ the marking on $\mathbb{C}\mathrm{at}_{\infty}$ containing all equivalences. Denote $\mathbb{C}\mathrm{at}_{\infty}$ together with this marking by $\mathbb{C}\mathrm{at}_{\infty}^{\natural}$. Thus, $\mathbb{C}\mathrm{at}_{\infty}^{\natural}$ is a marked-scaled simplicial set.
2. Denote by $\heartsuit$ the marking on Δ^{2n+1} containing all 1-simplices in the 'bottom face' $(\Delta^n)^{\mathrm{op}} = \Delta^{\{n+1,\ldots,2n+1\}} \subset \Delta^{2n+1}$. Denote the marked-scaled simplicial set with this marking and the $\dagger$ scaling by $(\Delta^{2n+1})^{\heartsuit}_{\dagger}$.
3. Recall that for any simplicial subset $S \subseteq \Delta^{2n+1}$, we will denote the restrictions of $\dagger$ and

[3] Note that the case $n = 1$ corresponds to the condition that cocartesian lifts exist, proved above.

♡ to S also by † and ♡ respectively.

Definition 4.3.11. We will call a map $\sigma\colon (\Delta^{2n+1})^{\heartsuit}_{\dagger} \to \mathbb{C}\mathrm{at}^{\natural}_{\infty}$ ***weakly left Kan*** if it satisfies the following conditions.

- The objects $\sigma(\Delta^0), \dots, \sigma(\Delta^n) \in \mathbb{C}\mathrm{at}_\infty$ are spaces.
- The image of the simplex $(\Delta^{\{0,1,\bar{0}\}})^{\flat}_{\flat} \subset (\Delta^{2n+1})^{\heartsuit}_{\dagger}$ is left Kan.

We note that there is a pullback diagram of marked-scaled simplicial subsets of Δ^{2n+1}

$$\begin{tikzcd} (\Lambda_0^{\{0,\dots,n,\bar{0}\}})^{\heartsuit}_{\dagger} \arrow[r, hook, "u_n"] \arrow[d, hook] & (K^n)^{\heartsuit}_{\dagger} \arrow[d, hook] \\ (\Delta^{\{0,\dots,n,\bar{0}\}})^{\heartsuit}_{\dagger} \arrow[r, hook, "v_n"] & (\Delta^{2n+1})^{\heartsuit}_{\dagger} \end{tikzcd}. \tag{13}$$

Lemma 4.3.12. For all $n \geq 2$, the map v_n is marked-scaled anodyne, hence a weak equivalence.

Proof. In this proof, all simplicial subsets of Δ^{2n+1} will be assumed to carry the marking ♡ and the scaling † unless stated otherwise.

We can write each v_n as a composition

$$\Delta^{\{0,\dots,n,\bar{0}\}} \xhookrightarrow{v_n''} \Delta^{\{0,\dots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\dots,\bar{0}\}} \xhookrightarrow{v_n'} \Delta^{2n+1}.$$

More simply put, v_n'' is the addition of the bottom face $(\Delta^n)^{\mathrm{op}}$. This is a pushout along the inclusion $(\Delta^{\{n\}})^{\flat}_{\flat} \hookrightarrow (\Delta^n)^{\sharp}_{\sharp}$, which is marked-scaled anodyne.

It remains to show that the for each $n \geq 2$, v_n' is marked-scaled anodyne. We actually prove that this result holds for all $n \geq 1$. First, we introduce some notation.

Let $M_0^n = \Delta^{\{0,\dots,n,\bar{0}\}} \cup \Delta^{\{\bar{n},\dots\bar{0}\}} \subset \Delta^{2n+1}$, and for $1 \leq i \leq n$, define

$$M_i^n := M_0^n \cup \left(\bigcup_{k=1}^{i} \Delta^{\{[2n+1] \setminus \{i,\bar{i}\}\}} \right).$$

There is an obvious filtration

$$M_0^n \xhookrightarrow{i_0^n} M_1^n \xhookrightarrow{i_1^n} \cdots \xhookrightarrow{i_{n-1}^n} M_n^n \xhookrightarrow{i_n^n} \Delta^{2n+1}. \tag{14}$$

Define

$$j_k^n := i_n^n \circ \cdots \circ i_k^n \colon M_k^n \hookrightarrow \Delta^{2n+1}, \qquad 0 \leq k \leq n.$$

In particular, note that $j_0^n = v_n'$.

We aim to show that each i^n_k is marked-scaled anodyne for each $n \geq 1$ and $0 \leq i \leq n$. We proceed by induction on n.

For our base case, we take $n = 1$, where we have the filtration

$$M^n_0 \overset{i^1_0}{\cong} M^n_1 \overset{i^1_1}{\hookrightarrow} \Delta^3.$$

All isomorphisms are marked-scaled anodyne, and the map i^1_1 was proven in Lemma 4.3.9 to be marked-scaled anodyne (it coincides with the map v'_1). Thus, the maps i^1_0 and i^1_1 are marked-scaled anodyne; the result holds for $n = 1$.

We now suppose that the result holds true for $n-1$; that is, that the maps i^{n-1}_k are marked-scaled anodyne for all $0 \leq k \leq n-1$. We need to show that each of the maps i^n_k are marked-scaled anodyne. For $0 \leq k < n$, we can write i^n_k as a pushout along j^{n-1}_k, which is marked-scaled anodyne by the inductive assumption. Therefore, it remains only to show that i^n_n is marked-scaled anodyne.

We can write $i^n_n \colon M^n_n \hookrightarrow \Delta^{2n+1}$ as a simplicial inclusion

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \subseteq \Delta^{2n+1}, \qquad \mathcal{A} = \left\{ \begin{matrix} \{\overline{n-1},\ldots,\overline{0}\} \\ \{0,\ldots n\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix} \right\}.$$

We will express this inclusion as the following composition of fillings:

1. We first add the simplices $\Delta^{\{n,\overline{n},\ldots,\overline{0}\}}, \Delta^{\{n-1,n,\overline{n},\ldots,\overline{0}\}}, \ldots, \Delta^{\{2,\ldots,n,\overline{n},\ldots,\overline{0}\}}$.
2. We next add $\Delta^{\{1,\ldots,n,\overline{n},\overline{0}\}}, \Delta^{\{1,\ldots,n,\overline{n},\overline{n-1},\overline{0}\}}, \ldots, \Delta^{\{1,\ldots n,\overline{n},\ldots,\overline{0}\}}$.
3. We then add $\Delta^{\{0,\ldots,n,\overline{n},\overline{0}\}}, \Delta^{\{0,\ldots,n,\overline{n},\overline{n-1},\overline{0}\}}, \ldots, \Delta^{\{0,\ldots n,\overline{n},\ldots,\overline{0}\}}$.

We proceed.

1. - Using Lemma 4.2.6, we see that the square

$$\begin{tikzcd} \mathcal{S}^{\mathcal{A}|\{n,\overline{n},\ldots,\overline{0}\}}_{\{n,\overline{n},\ldots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} \arrow[d, hook] \\ \mathcal{S}^{\mathcal{A}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\ldots,\overline{0}\}} \end{tikzcd}$$

   is bicartesian. In order to show that the bottom inclusion is marked-scaled anodyne, it thus suffices to show that the top inclusion is marked-scaled anodyne. We have

$$\mathcal{A}|\{n,\overline{n},\dots,\overline{0}\} = \left\{\begin{matrix}\{\overline{n-1},\dots,\overline{0}\}\\ \{n\}\\ \{\overline{1}\}\\ \vdots\\ \{\overline{n-1}\}\\ \{n,\overline{n}\}\end{matrix}\right\} \sim \left\{\begin{matrix}\{n\}\\ \{\overline{n-1}\}\\ \vdots\\ \{\overline{1}\}\end{matrix}\right\}.$$

This is a dull subset of $\{n,\overline{n},\dots,\overline{0}\}$ with pivot $\overline{n}$. The only basal set is $\{n,\overline{n-1},\dots,1\}$, and the simplex $\{n,\overline{n},\overline{n-1}\}$ is scaled, so the top inclusion is marked-scaled anodyne by Lemma 4.2.10. We can write

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \Delta^{\{n,\overline{n},\dots,\overline{0}\}} = \mathcal{S}^{\mathcal{A}\cup([2n+1]\setminus\{n,\overline{n},\dots,\overline{0}\})}_{[2n+1]},$$

Note that

$$\mathcal{A}\cup([2n+1]\setminus\{n,\overline{n},\dots,\overline{0}\}) \sim \left\{\begin{matrix}\{\overline{n-1},\dots,\overline{0}\}\\ \{0,\dots n-1\}\\ \{1,\overline{1}\}\\ \vdots\\ \{n,\overline{n}\}\end{matrix}\right\} =: \mathcal{A}_{n-1}.$$

- We proceed inductively. Suppose we have added the simplices $\Delta^{\{n,\overline{n},\dots,\overline{0}\}}$, $\Delta^{\{n-1,n,\overline{n},\dots,\overline{0}\}}$, $\dots$, $\Delta^{\{k+1,\dots,n,\overline{n},\dots,\overline{0}\}}$, for $2 \le k \le n-1$. Using Lemma 4.2.5 and Lemma 4.2.4 can write the result of these additions as

$$\mathcal{S}^{\mathcal{A}}_{[2n+1]} \cup \left(\bigcup_{i=k+1}^{n} \Delta^{\{i,\dots,n,\overline{n},\dots,\overline{0}\}}\right) = \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]},$$

where

$$\mathcal{A}_{k+1} = \left\{\begin{matrix}\{\overline{n-1},\dots,\overline{0}\}\\ \{0,\dots,k\}\\ \{1,\overline{1}\}\\ \vdots\\ \{n,\overline{n}\}\end{matrix}\right\}.$$

Using Lemma 4.2.6, we see that the square

$$\begin{tikzcd}
\mathcal{S}^{\mathcal{A}_{k+1}|\{k,\dots,n,\overline{n},\dots,\overline{0}\}}_{\{k,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[r, hook] \arrow[d, hook] & \Delta^{\{k,\dots,n,\overline{n},\dots,\overline{0}\}} \arrow[d, hook] \\
\mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \arrow[r, hook] & \mathcal{S}^{\mathcal{A}_{k+1}}_{[2n+1]} \cup \Delta^{\{k,\dots,n,\overline{n},\dots,\overline{0}\}}
\end{tikzcd}$$

is pushout, so in order to show that the bottom morphism is marked-scaled anodyne, it suffices to show that the top morphism is. We see that

$$\mathcal{A}_{k+1}|\{k,\dots,n,\overline{n},\dots,\overline{0}\} = \left\{\begin{matrix} \{\overline{n-1},\dots,\overline{0}\} \\ \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k,\overline{k}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\} \sim \left\{\begin{matrix} \{k\} \\ \{\overline{1}\} \\ \vdots \\ \{\overline{k-1}\} \\ \{k+1,\overline{k+1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\}$$

.

This is a dull subset of $P(\{k,\dots,n,\overline{n},\dots,\overline{0}\})$ with pivot $\overline{k}$. The basal sets are of the form

$$\{k, a_1,\dots,a_{n-k},\overline{k-1},\dots,\overline{1}\},$$

where each a_1, …, a_{n-k} is of the form ℓ or $\overline{\ell}$ for $k+1 \leq \ell \leq n$. In each case, the simplex $\{a_{n-k},\overline{k},\overline{k-1}\}$ is scaled, so by Lemma 4.2.10, the top morphism is marked-scaled anodyne.

We have now added the simplices promised in part 1., and are left with the simplicial subset $\mathcal{S}^{\mathcal{A}_2}_{[2n+1]}$, where

$$\mathcal{A}_2 = \left\{\begin{matrix} \{\overline{n-1},\dots,\overline{0}\} \\ \{0,1\} \\ \{1,\overline{1}\} \\ \vdots \\ \{n,\overline{n}\} \end{matrix}\right\}.$$

2. •

□

Lemma 4.3.13. For all $n \geq 2$, the map u_n is marked-scaled anodyne, hence a weak equivalence.

We have just shown that left Kan morphisms $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ are p-cocartesian, and that enough left Kan lifts exist. It remains only to show that any p-cocartesian lift is left Kan. We use the fact that any two p-cocartesian lifts of a morphism with a specified source are equivalent.

5 The non-monoidal construction

5.1 The bare functor

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{S} \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd} .$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is a 3-simplex of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] & Y \arrow[d] \arrow[dl, phantom, "\circlearrowleft" near start] \\ \mathcal{C} \arrow[ur, phantom, "\Searrow" near end] & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow" near start] & Y \arrow[d] \arrow[dl] \\ \mathcal{C} & \mathcal{C} \arrow[l, "\mathrm{id}_{\mathcal{C}}"] \arrow[ul, phantom, "\circlearrowleft" near start] \end{tikzcd} , \tag{15}$$

To save on notation, we will generally denote such a morphism as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] \arrow[ur, phantom, "\Longrightarrow"] & \mathcal{C} \end{tikzcd} . \tag{16}$$

There is a potential for confusion here. The natural transformation notated in Diagram 16 above does not correspond to a single natural transformation in Diagram 15, but rather to both natural transformations simultaneously. However, both of these natural transformations agree up to a homotopy provided by the interior of the 3-simplex.

A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cartesian if and only if all 2-simplices forming the faces of the corresponding map $Q(1) \to \mathcal{C}\mathrm{at}_\infty$ are thin; that is, if the 3-simplex making up the twisted square weakly commutes. We will use the symbol $\circlearrowleft$ to denote a weakly commuting twisted morphism, and call such morphisms *thin:*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] \arrow[ur, phantom, "\circlearrowleft"] & \mathcal{C} \end{tikzcd} .$$

The functor $r\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{S}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad X \mapsto \mathrm{Fun}(X, \mathcal{C}),$$

which sends a map $f\colon X \to Y$ to the pullback

$$f^*\colon \mathrm{Fun}(Y, \mathcal{C}) \to \mathrm{Fun}(X, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. More explicitly, we have the following.

Proposition 5.1.1. A morphism $\Delta^1 \to \mathbf{LS}(\mathcal{C})$ is r-cocartesian if and only the 3-simplex $\sigma\colon Q(1) \to \mathbb{C}\mathrm{at}_\infty$ to which it corresponds has the property that $d_3\sigma$ is left Kan. More explicitly, the morphism is left Kan if and only if σ the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr] \arrow[dr, phantom, "\circlearrowleft", shift left=2] \arrow[dr, phantom, "\Rightarrow", shift right=2] & Y \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, "\overset{\eta}{\Rightarrow}", shift right=3] & Y \arrow[d] \arrow[dl, "f_!\mathcal{F}"] \arrow[dl, phantom, "\circlearrowleft", shift left=3] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd},$$

where $f_!\mathcal{F}$ is a left Kan extensions of $\mathcal{F}$ along f, and η is the counit map. Note that the data of the second diagram determines both the data of the first and the filling 3-simplex up to contractible choice.

Proof. First we show that morphisms of the promised form really are r-cocartesian. Since by assumption $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is a cartesian fibration (hence an isofibration) whose source and target are quasicategories, it suffices to show that we can solve the lifting problem

$$\begin{tikzcd} \Delta^{\{0,1\}} \arrow[d, hook] \arrow[dr, "a"] & \\ \Lambda^2_0 \arrow[r] \arrow[d] & \mathbf{LS}(\mathcal{C}) \arrow[d] \\ \Delta^2 \arrow[r] \arrow[ur, dashed] & \mathcal{S} \end{tikzcd}$$

where a is of the promised form, and that the space of such fillings is contractible. □

For short, we will denote r-cocartesian morphisms with an exclamation mark:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\mathcal{F}"'] \arrow[r, phantom, "=!\Rightarrow", shift right=5] & X' \arrow[d, "\mathrm{Lan}_f\mathcal{F}"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}.$$

5.2 The triple structures

5.2.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^{\simeq}$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 5.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 5.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

5.2.2 The triple structure on local systems

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Such a span will correspond to a cocartesian morphism if it is of the form

$$\begin{tikzcd}
\mathcal{D}' \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^*\mathcal{F}"] \arrow[dr, phantom, "=!\Rightarrow"] & \mathcal{D}'' \arrow[d, "f_!g^*\mathcal{F}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

Proposition 5.2.3. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 5.2.4. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\sigma = \begin{tikzcd} A \arrow[r] \arrow[d, "a"'] & B \arrow[d, "b"] \\ B' \arrow[r] & C \end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd} a \arrow[rr] \arrow[dr] \arrow[dd] & & b \arrow[dr] \arrow[dd, dash] & \\ & b' \arrow[rr] \arrow[dd] & & c \arrow[dd] \\ \bar{a} \arrow[r, phantom, "\Downarrow\alpha"] & & \bar{b} \arrow[ll] \arrow[r, phantom, "\Downarrow\beta"] & \\ & \bar{b}' \arrow[ul] & & \bar{c} \arrow[ll] \arrow[ul] \end{tikzcd}$$

in $\mathbb{C}$.[4] Suppose the right face is thin, the top face is a pullback, and the bottom face is a pushout. Then the square σ is pullback if and only if the left face is thin.

Proof. This corresponds to a square σ in $\mathbf{Tw}(\mathbb{C})$ lying over a pullback square in $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$, such that b is p-cartesian. It is then classically known that σ is pullback if and only if a is p-cartesian. □

Proof of Proposition 5.2.3. We need to show that we can form pullbacks in $\mathbf{Tw}(\mathbb{C})$ of cospans one of whose legs is a cartesian morphism. Mapping the cospan to $\mathcal{C}\times\mathcal{C}^{\mathrm{op}}$ using r, We can form the pullback, then fill to a relative pullback by finding a cartesian lift a, then filling an inner and an outer horn. This diagram is pullback by Lemma 5.2.4

We also need to show that any such pullback is of this form. This also follows from Lemma 5.2.4. □

[4]Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute by definition.

5.3 Building categories of spans

Lemma 5.3.1. Let

$$\begin{tikzcd} X \times^h_Y Y' \arrow[r] \arrow[d] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

be a homotopy pullback diagram of Kan complexes, and let $y' \in Y'$. Then

$$(X \times^h_Y Y')_{/y'} \simeq X \times^h_Y (Y'_{/y'}) \tag{17}$$

Proof. We can model the homotopy pullback $X \times^h_Y Y'$ as the strict pullback

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y').$$

The left-hand side of Equation 17 is then given by the pullback

$$\left(Y^{\Delta^1} \times_{Y \times Y} (X \times Y')\right) \times_{Y'} Y'_{/y'}.$$

But this is isomorphic to

$$Y^{\Delta^1} \times_{Y \times Y} (X \times Y'_{/y'}),$$

which is a model for the right-hand side. □

Proposition 5.3.2. The map $p\colon (\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \to (\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ of triples whose underlying map is $p\colon \mathbf{LS}(\mathcal{C}) \to \mathcal{S}$ is adequate, i.e. satisfies the conditions of our modified version of Barwick's theorem.

Proof. We need to show that for any square

$$\sigma = \begin{tikzcd} x' \arrow[r, "f'"] \arrow[d, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ lying over a pullback square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in $\mathcal{S}$ (corresponding to a *homotopy* pullback of Kan complexes), such that g and g' are p-

cartesian and f is p-cocartesian, f' is p-cocartesian. The square σ corresponds to a twisted cube

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dr, "g'"'] \arrow[dd] & & Y' \arrow[dr, "g"] \arrow[dd] & \\
& X \arrow[rr, "f"] \arrow[dd] \arrow[r, phantom, "\Longrightarrow"] & \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[dd] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] \arrow[ur, phantom, "\circlearrowleft"] & & \mathcal{C} \arrow[dr, equal] \arrow[ul, phantom, "\Longrightarrow"] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd}$$

in which the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face corresponds to a left Kan extension. We need to show that the back face corresponds to a left Kan extension. For ease of notation, we flatten this out and add some more labels.

$$\begin{tikzcd}
X' \arrow[rr, "f'"] \arrow[dd, "g'"'] \arrow[dr, "g'^*\mathcal{F}"] & & Y' \arrow[dd, "g"] \arrow[dl, "\mathcal{G}"'] \\
& \mathcal{C} & \\
X \arrow[rr, "f"'] \arrow[ur, "\mathcal{F}"] & & Y \arrow[ul, "f_!\mathcal{F}"']
\end{tikzcd}$$

We need to show that $\mathcal{G}$ is a left Kan extension of $g'^*\mathcal{F}$ along f', i.e. that for all $y' \in Y'$,

$$\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X'_{/y'} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y'_{/y'}) \to X \to \mathcal{C}\right],
\end{aligned}$$

where here we mean by $X' \simeq X \times_Y Y'$ the homotopy pullback, and we have used Lemma 5.3.1. We know that $\mathcal{G}$ is the pullback along g of $f_!\mathcal{F}$, i.e. that

$$\begin{aligned}
\mathcal{G}(y') &\simeq \operatorname{colim}\left[X_{/g(y')} \to X \to \mathcal{C}\right] \\
&\simeq \operatorname{colim}\left[X \times_Y (Y_{/g(y')}) \to X \to \mathcal{C}\right].
\end{aligned}$$

The map $X \times_Y (Y'_{/y'}) \to X$ factors through $X \times_Y (Y_{/g(y')})$ thanks to the map $s\colon Y'_{/y'} \to Y_{/g(y')}$. The map s is a weak equivalence between contractible Kan complexes (because both $Y'_{/y'}$ and $Y_{/g(y')}$ are Kan complexes with terminal objects). Thus, the map $X \times_Y (Y'_{/y'}) \to X \times_Y (Y_{/g(y')})$ is also a weak homotopy equivalence between Kan complexes, thus cofinal. $\square$

6 The monoidal construction

6.1 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I)\colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd}
\Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\
\Delta^I \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i
\end{tikzcd}$$

commutes.

Example 6.1.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n\rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n\rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

Example 6.1.2. A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1\rangle \leftarrow \langle 2\rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ consists[5] of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}'_i$, $i = 1, 2$.

[5]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2\rangle \to \langle 1\rangle$ in $\mathcal{F}\mathrm{in}_*$.

- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1 \times G_2"] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd} .$$

A morphism of the above form is cartesian if and only if it is thin. A general morphism is cartesian if and only if each component square is thin.

Lemma 6.1.3. Let $\mathcal{C}$ be a small 1-category (and, by abuse of notation, its nerve), let $F, G\colon \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta$ be functors, and let $\alpha\colon F \Rightarrow G$. Suppose that α satisfes the following conditions.

1. For each object $c \in \mathcal{C}$, $\alpha_c\colon F(c) \to G(c)$ is a cartesian fibration.
2. For each morphism $f\colon c \to d$ in $\mathcal{C}$, the map $Ff\colon F(c) \to F(d)$ takes α_c-cartesian morphisms to α_d-cartesian morphisms.

Then taking the relative nerve gives a diagram

$$\begin{tikzcd} N_F(\mathcal{C}) \arrow[rr, "\rho"] \arrow[dr, "\Phi"'] & & N_G(\mathcal{C}) \arrow[dl, "\Gamma"] \\ & \mathcal{C}^{\mathrm{op}} & \end{tikzcd}$$

with the following properties.

1. The maps Φ, Γ, and ρ are all cartesian fibrations.
2. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ admit the following description: a morphism in $N(F)(\mathcal{C})$ lying over a morphim $f\colon d \leftarrow c$ in $\mathcal{C}^{\mathrm{op}}$ consists of a triple (x, y, ϕ), where $x \in F(d)$, $y \in F(c)$, and $\phi\colon x \to Ff(y)$. Such a morphism is ρ-cartesian if the morphism ϕ is α_d-cartesian.
3. The map ρ sends Φ-cartesian morphisms in $N_F(\mathcal{C})$ to Γ-cartesian morphisms in $N_G(\mathcal{C})$.

Proof. We first prove 2. We already know (HTT 3.2.5.11) that ρ is an inner fibration, so in order to prove 2., we need to show that we can solve lifting problems

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & N_F(\mathcal{C}) \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] \arrow[dr, "\gamma"'] & N_G(\mathcal{C}) \arrow[d] \\ & \mathcal{C}^{\mathrm{op}} \end{tikzcd} ,$$

where $\Delta^{\{n-1,n\}} \subset \Lambda^n_n$ is mapped to a cartesian morphism as described above, i.e. a triple (x, y, ϕ), where $x \in F(\gamma(n-1))$, $y \in F(\gamma(n))$, and $\phi\colon x \to F(\gamma_n)(y)$ is $\alpha_{\gamma(n-1)}$-cartesian. This is equivalent to solving the lifting problem

$$\begin{tikzcd} \Lambda^n_n \arrow[r] \arrow[d] & F(\gamma(0)) \arrow[d, "\alpha_{\gamma(0)}"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & G(\gamma(0)) \end{tikzcd},$$

where Λ^n_n is $F(\gamma_{n-1,0})(\phi)$. But by assumption $F(\gamma_{n-1,0})(\phi)$ is $\alpha_{\gamma(0)}$-cartesian, so this lifting problem has a solution.

Now we show Claim 1. The assumption that each α_c is a cartesian fibration guarantees that we have enough cartesian lifts, so ρ is indeed a cartesian fibration. The maps Φ and Γ are cartesian fibrations by definition.

Now we show Claim 3. The ρ-cartesian morphisms in $N_F(\mathcal{C})$ are pairs (x, y, ϕ) as above, where ϕ is an equivalence. Claim 3. then follows because the functors α_c functors map equivalences to equivalences. □

Note that for each $\langle n \rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ there is a cartesian fibration $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n \to (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n$, which taken together give a natural transformation α from the functor

$$F\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)^n$$

to the functor

$$G\colon \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty; \qquad \langle n \rangle \mapsto (\mathcal{C}\mathrm{at}_\infty^n)^{\mathrm{op}} \times \mathcal{C}\mathrm{at}_\infty^n.$$

Because the pointwise product of any number of thin 1-simplices in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$ is again a thin 1-simplex in $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)$, the conditions of Lemma 6.1.3 are satisfied. Unrolling, we find that $\mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes$ admits a forgetful functor to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^{\times})^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^{\times}$ is the *cocartesian* relative nerve of the same,

such that each map in the commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathbb{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^{\times})^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd} \tag{18}$$

is a cartesian fibration.

Because both $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ and $\mathcal{C}\mathrm{at}_\infty^\times \to \mathcal{F}\mathrm{in}_*$ classify the same functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$, they are related by a symmetric monoidal equivalence. Composing a monoidal ∞-category $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty^\times$ with this symmetric monoidal equivalence gives:

Proposition 6.1.4. We can express a symmetric monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\widetilde{\mathcal{S}}_\times \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is induced by the inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ on the first component, and given by the composition

$$\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\widetilde{\mathcal{S}}_\times \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \widetilde{\mathcal{S}}_\times \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

By definition, one sees that the map p is a cartesian fibration, and the p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$ are precisely those representing products. Furthermore, because the map r is a pullback of the horizontal map in Diagram 18, we have the following.

Proposition 6.1.5. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{X} \arrow[r, "\vec{f}"] \arrow[d, "\vec{\mathcal{F}}"'] \arrow[dr, phantom, "\Longrightarrow"] & \vec{Y} \arrow[d, "\vec{\mathcal{G}}"] \\
\vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if each 2-simplex making up the above diagram is thin.

- q-cartesian if $\vec{f}$ exhibits the Y_i as products of the X_i, and each 2-simplex making up the above diagram is thin.

One sees immediately that r sends q-cartesian morphisms in $\mathbf{LS}(\mathcal{C})_\otimes$ to p-cartesian morphisms in $\widetilde{\mathcal{S}}_\times$, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

6.2 The triple structures

6.2.1 The triple structure on finite pointed sets

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 6.2.1. This is an adequate triple structure.

Proof. Trivial. □

Proposition 6.2.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

6.2.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \widetilde{\mathcal{S}}_\times$
- $\mathcal{P}_\dagger = \widetilde{\mathcal{S}}_\times \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \widetilde{\mathcal{S}}_\times$

Proposition 6.2.3. The cartesian fibration $\widetilde{\mathcal{S}}_\times \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proof. The fibers clearly admit pullbacks, and the pullback maps commute with pullbacks because products commute with pullbacks. □

Proposition 6.2.4. This is an adequate triple structure.

Proof. It suffices to show that $\widetilde{\mathcal{S}}_\times$ admits pullbacks, and that for any pullback square in $\widetilde{\mathcal{S}}_\times$ lying over a square

$$\begin{tikzcd} \langle n\rangle & \langle m\rangle \arrow[l] \\ \langle n'\rangle \arrow[u, "\phi"] & \langle m'\rangle \arrow[l] \arrow[u, "\simeq"'] \end{tikzcd}$$

in $\widetilde{\mathcal{S}}_\times$, the morphism ϕ is an isomorphism in $\mathcal{F}\mathrm{in}_*$. But $\widetilde{\mathcal{S}}_\times$ admits relative pullbacks and $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits pullbacks, so p preserves pullbacks. (Just take the proof from my thesis!) □

6.2.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms, i.e. thin morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd} \vec{X} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l] \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d] \\ \mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\ \vec{X} \arrow[d, dotted, no head] & \vec{Z} \arrow[l] \arrow[r] \arrow[d, dotted, no head] & \vec{Y} \arrow[d, dotted, no head] \\ \langle n\rangle \arrow[r] & \langle m\rangle & \langle m\rangle \arrow[l, "\cong"'] \end{tikzcd}$$

According to the modified Barwick's theorem, such a span will correspond to a cocartesian morphism in $\mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ if it is of the form

$$\begin{tikzcd} \vec{X} \arrow[d, "\mathcal{F}"'] \arrow[dr, phantom, "\circlearrowleft"] & \vec{Z} \arrow[l, "g"'] \arrow[r, "f"] \arrow[d, dashed, "g^* t\circ\mathcal{F}"] \arrow[r, phantom, shift right=4, "\Longrightarrow"] & \vec{Y} \arrow[d, "f_!(g^* t\mathcal{F})"] \\ \mathcal{C}^n \arrow[r] & \mathcal{C}^m & \mathcal{C}^m \arrow[l, "\cong"'] \end{tikzcd}.$$

In particular, in the fiber over $\langle 1\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, this is simply pull-push.

Proposition 6.2.5. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

Proof. We need to show that for any solid diagram in $\mathbf{LS}(\mathcal{C})_\otimes$ of the form

$$\begin{tikzcd} y \arrow[r, dashed, "f'"] \arrow[d, dashed, "g'"'] & y' \arrow[d, "g"] \\ x \arrow[r, "f"] & y \end{tikzcd},$$

where g is cartesian and f lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, a dashed pullback exists, and for each such dashed pullback, g' is cartesian and f' lies over an isomorphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. The existence of such a pullback is easy; one takes a pullback in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, takes a q-cartesian lift g', then fills an inner and an outer horn. That this is pullback is immediate because g and g' are q-cartesian. That any other pullback square has the necessary properties follows from the existence of a comparison equivalence. □

6.3 Building categories of spans

Proposition 6.3.1. The map p satisfies the conditions of the original Barwick's Theorem.

Proof. See thesis. □

Proposition 6.3.2. The map r satisifes the conditions of the modified Barwick's Theorem.

Proof. The first condition is obvious since r is a cartesian fibration. In order to check the second condition, we fix a square σ in $\mathbf{Tw}(\mathcal{C})_\otimes$, lying over a square $r(\sigma)$ in $\widetilde{\mathcal{S}}_\times$ and a square $q(\sigma)$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Fixing notation, we will let $q(\sigma)$ be the square

$$\begin{tikzcd} \langle n \rangle & \langle n \rangle \arrow[l, "\gamma'"'] \\ \langle m \rangle \arrow[u, "\tau'"] & \langle m \rangle \arrow[l, "\gamma"] \arrow[u, "\tau"'] \end{tikzcd},$$

where γ and γ' are isomorphisms.

The square $r(\sigma)$ thus has the form

$$\begin{tikzcd} \vec{X}' \arrow[r, "f'"] \arrow[d, "g'"'] & \vec{Y}' \arrow[d, "g"] \\ \vec{X} \arrow[r, "f"] & \vec{Y} \end{tikzcd},$$

where f has components

$$f_i \colon X_i \longrightarrow Y_{\gamma^{-1}(i)}, \qquad i \in \langle m \rangle^\circ,$$

f' has components

$$f'_j \colon X'_j \to Y'_{\gamma'^{-1}(j)}, \qquad j \in \langle n \rangle^\circ,$$

g has components

$$g_j \colon Y'_j \to \prod_{\tau(i)=j} Y_i, \qquad j \in \langle n \rangle^\circ,$$

and g' has components

$$g'_j \colon X'_j \to \prod_{\tau'(i)=j} X_i, \qquad j \in \langle n \rangle^\circ.$$

Note that the condition that $r(\sigma)$ be pullback is the condition that

$$X'_j \simeq Y'_j \times_{(\prod_{\tau'(i)=j} Y_i)} \left(\prod_{\tau'(i)=j} X_i \right),$$

and that the maps f'_j and g'_j be the canonical projections.

The data of the square σ is thus given by a diagram of the form

$$\begin{tikzcd}
\vec{X}' \arrow[rr, "f'"] \arrow[dr, "g'"] \arrow[dd, "\mathcal{F}'"'] & & \vec{Y}' \arrow[dr, "g"] \arrow[dd, "\mathcal{G}'"] & \\
& \vec{X} \arrow[rr, "f"] \arrow[dd, "\mathcal{F}"] & & \vec{Y} \arrow[dd, "\mathcal{G}"] \\
\mathcal{C}^{(n)} & & \mathcal{C}^{(n)} \arrow[ll, "\gamma'_\otimes"] & \\
& \mathcal{C}^{(m)} \arrow[ul, "\tau'_\otimes"] & & \mathcal{C}^{(m)} \arrow[ll, "\gamma_\otimes"] \arrow[ul, "\tau_\otimes"']
\end{tikzcd},$$

where the left and right faces are thin, the top face is a pullback, the bottom face is a pushout, and the front face is a left Kan extension (i.e. each component is a left Kan extension). We need to show that the back face is a left Kan extension, i.e. that for each $j \in \langle n \rangle^\circ$, the square

$$\begin{tikzcd}
X'_j \arrow[r, "f'_j"] \arrow[d, "\mathcal{F}'_i"'] & Y'_{\gamma^{-1}(j)} \arrow[d, "\mathcal{G}'_{\gamma^{-1}(i)}"] \\
\mathcal{C} \arrow[r, equal] & \mathcal{C}
\end{tikzcd}$$

is left Kan.

Using HTT 4.3.1.9, we can choose cartesian edges, transporting the pullback square in $\mathbf{LS}(\mathcal{C})_\otimes$

above to a pullback square entirely in the fiber over $\langle n\rangle \in \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$. Since $(\mathbf{LS}(\mathcal{C})_\otimes)_{\langle n\rangle} \cong (\mathbf{LS}(\mathcal{C}))^n$, this diagram consists of n diagrams in $\mathbf{LS}(\mathcal{C})$, one for each $j \in \langle n\rangle^\circ$, of the form

$$\begin{tikzcd}
X'_j \arrow[rr, "f'_j"] \arrow[dr] \arrow[dd, "\mathcal{F}'_i"'] & & Y'_{\gamma'^{-1}(j)} \arrow[dr] \arrow[dd, "\mathcal{G}'_{\gamma^{-1}(i)}"] & \\
& \prod_{\tau'(i)=j} X_i \arrow[rr] \arrow[dd, "\mathcal{A}"] & & \prod_{\tau(i)=j} Y_i \arrow[dd, "\mathcal{B}"] \\
\mathcal{C} \arrow[rr, equal] \arrow[dr, equal] & & \mathcal{C} \arrow[dr, equal] & \\
& \mathcal{C} \arrow[rr, equal] & & \mathcal{C}
\end{tikzcd},$$

where the top face is pullback, the bottom face is trivially a pushout, and the left and right faces are thin. We want to show that the back face of each of these diagrams is a left Kan extension. If we can show that the front face is a left Kan extension, we will be done. The map $\mathcal{A}$ is the composition of the product of each of the maps $\mathcal{F}_i : X_i \to \mathcal{C}$ with the tensor product $\mathcal{C}^{|\{\tau'(i)=j\}|} \to \mathcal{C}$, and similarly for $\mathcal{B}$. Unravelling the conditions, one finds that the condition that $\mathcal{B}$ be a pointwise left Kan extension of $\mathcal{A}$ along the front-top-horizontal map above is precisely the condition that the tensor product commute with colimits.

□

7 Constructing the symmetric monoidal functor

The rest of the work follows my thesis pretty much exactly. We are now justified in building

$$\begin{tikzcd}
\mathrm{Span}(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger) \arrow[rr] \arrow[dr] & & \mathrm{Span}(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger) \arrow[dl] \\
& \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger) &
\end{tikzcd}.$$

We pull back along $\mathcal{F}\mathrm{in}_* \xrightarrow{\simeq} \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$, giving us isofibrations

$$\begin{tikzcd}
P \arrow[rr] \arrow[dr] & & Q \arrow[dl] \\
& \mathcal{F}\mathrm{in}_* &
\end{tikzcd},$$

and it is easy to check that each of these maps has enough cocartesian lifts. The functor we are interested is the component of $P \to Q$ lying over $\langle 1 \rangle$; this classifies the functor

$$\mathrm{Span}(\mathcal{S}) \to \mathcal{C}\mathrm{at}_\infty$$

sending a span of spaces to push-pull, up to an $^{\mathrm{op}}$. Composing with $^{\mathrm{op}}\colon \mathcal{C}\mathrm{at}_\infty \to \mathcal{C}\mathrm{at}_\infty$ gives precisely the map we want. General abstract nonsense implies that this functor is lax monoidal.

References

[1] Denis-Charles Cisinski. *Higher categories and homotopical algebra*. Vol. 180. Cambridge University Press, 2019.

[2] Fernando Abellán García and Walker H Stern. "2-Cartesian fibrations I: A model for infty-bicategories fibred in infty-bicategories". In: *arXiv preprint arXiv:2106.03606* (2021).

[3] Fernando Abellán García and Walker H Stern. "Enhanced twisted arrow categories". In: *arXiv preprint arXiv:2009.11969* (2020).

[4] Jacob Lurie. *Higher Topos Theory*. Princeton University Press, 2009. ISBN: 978-0-691-14049-0.