Monoidal push-pull for local systems

Angus Rush

June 15, 2021

Contents

1 Background

1.1 Scaled simplicial sets

Note: This is basically a summary of the parts of The Goodwillie Calculus that we're gonna need.

Definition 1.1.1. A ***scaled simplicial set*** is a pair $(X, A_X) = \overline{X}$, where X is a simplicial set, and $A \subseteq X_2$ is a set of 2-simplices of X which contains every degenerate simplex. A simplex belonging to A is known as *thin.* A morphism of scaled simplicial sets is a morphism of the underlying simplicial sets which takes thin simplices to thin simplices.

We will need the following facts about scaled simplicial sets.

1. For any category $\mathcal{C}$ enriched in marked simplicial sets, we can build a scaled simplicial set $\mathrm{N}_{\mathrm{sc}}(\mathcal{C}) = (\mathrm{N}(\mathcal{C}), T)$, the ***scaled nerve*** of $\mathcal{C}$, whose underlying simplicial set $\mathrm{N}(\mathcal{C})$ is the simplicial nerve of $\mathcal{C}$. The scaling T is defined in the following way. Let $\sigma\colon \Delta^2 \to \mathrm{N}(\mathcal{C})$ be any 2-simplex. This corresponds to a diagram

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$, together with a 1-simplex $\alpha_\sigma\colon \Delta^1 \to \mathrm{Map}_{\mathcal{C}}(X, Z)$ corresponding to a map $gf \to h$. We will then define σ to be thin if and only if α_σ is marked in $\mathrm{Map}_{\mathcal{C}}(X, Z)$.

2. The functor N_{sc} admits a left adjoint $\mathfrak{C}^{\mathrm{sc}}\colon \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}} \to \mathcal{C}\mathrm{at}_{\Delta}^{+}$, the ***scaled rigidification***, such that for any scaled simplicial set $\overline{X}$ with underlying simplicial set X, the underlying simplicially enriched category of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$ is simply $\mathfrak{C}(X)$, the rigidification of X. The scaling on $\mathfrak{C}^{\mathrm{sc}}(X)$ also admits a relatively simple description.

3. A map of scaled simplicial sets $\overline{X} \to \overline{Y}$ is said to be a ***bicategorical equivalence*** if the map $F\colon \mathfrak{C}^{\mathrm{sc}}(X) \to \mathfrak{C}^{\mathrm{sc}}(Y)$ is a weak equivalence of $\mathcal{S}\mathrm{et}_{\Delta}$-enriched categories; that is, if for all objects x, x' of $\mathfrak{C}^{\mathrm{sc}}(\overline{X})$, the map

$$\mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{X})}(x, x') \to \mathrm{Hom}_{\mathfrak{C}^{\mathrm{sc}}(\overline{Y})}(F(x), F(x'))$$

is a weak equivalence of marked simplicial sets, and if for all $y \in \mathfrak{C}^{\mathrm{sc}}(\overline{Y})$, there exists some $x \in \mathfrak{C}^{\mathrm{sc}}(\overline{X})$ such that $F(x)$ and y are isomorphic in the homotopy category $\mathrm{h}\mathfrak{C}^{\mathrm{sc}}(\overline{Y})$.

4. There is a left proper, combinatorial model structure on $\mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}}$, whose cofibrations are monomorphisms, and whose weak equivalences are bicategorical fibrations. This gives a Quillen equvialence

$$\mathfrak{C}^{\mathrm{sc}} : \mathcal{S}\mathrm{et}_{\Delta}^{\mathrm{sc}} \leftrightarrow \mathcal{C}\mathrm{at}_{\Delta}^{+} : \mathrm{N}_{\mathrm{sc}}.$$

5. An ∞***-bicategory*** is a fibrant object with respect to this model structure. In particular, for any fibrant object $\mathcal{C} \in \mathcal{C}\mathrm{at}_{\Delta}^{+}$ (i.e. any quasicategory-enriched category with equvialences marked), $\mathrm{N}_{\mathrm{sc}}(\mathcal{C})$ is an ∞-bicategory.

6. The category of scaled simplicial sets has a natural simplicial enrichment as follows: For any two scaled simplicial sets $\overline{X}$ and $\overline{Y}$, we define

$$\mathrm{Map}^{\mathrm{sc}}(\overline{X}, \overline{Y})_n = \mathrm{Hom}(\overline{X} \times \Delta^n_{\sharp}, \overline{Y}),$$

where $\Delta^n_{\sharp}$ is the scaled simplicial set (Δ^n, Δ^n_2). For $\overline{Y}$ an ∞-bicategory, $\mathrm{Map}^{\mathrm{sc}}(\overline{X}, \overline{Y})$ is a quasicategory. Taking the core gives a Kan complex.

7. With this,
8. We will denote the simplicially enriched category whose objects are ∞-bicategories, with mapping spaces as above by $\mathbf{Bicat}_\infty$. We then define $\mathcal{C}\mathrm{at}_{(\infty,2)} = \mathrm{N}(\mathbf{Bicat}_\infty)$. This is a quasicategory.

We will need multiple notions of the quasicategory of $(\infty, 2)$-categories.

- The model category $\mathcal{S}\mathrm{et}_\Delta^{\mathrm{sc}}$ of scaled simplicial sets, whose cofibrations are precisely the monomorphisms.
- The weak equivalences are bicategorical equivalences,

2 The non-monoidal construction

Let $\mathcal{C}$ be a category admitting small colimits. We consider the pullback

$$\begin{tikzcd} \mathbf{LS}(\mathcal{C}) \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty) \arrow[d] \\ \mathcal{C}\mathrm{at}_\infty \times \{\mathcal{C}\} \arrow[r, hook] & \mathcal{C}\mathrm{at}_\infty \times \mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \end{tikzcd}.$$

A general morphism in $\mathbf{LS}(\mathcal{C})$ is of the form

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}' \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

This is r-cartesian if and only if η is a natural isomorphism $F \Rightarrow F' \circ \alpha$; that is, if the above square (weakly) commutes. We will denote a weakly commuting square via

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D}' \arrow[d, "F'"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

The functor $r \colon \mathbf{LS}(\mathcal{C}) \to \mathcal{C}\mathrm{at}_\infty$ thus classifies the functor

$$\mathcal{C}\mathrm{at}_\infty^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty; \qquad \mathcal{D} \mapsto \mathrm{Fun}(\mathcal{D}, \mathcal{C}),$$

which sends a functor $f\colon \mathcal{D} \to \mathcal{D}'$ to the pullback

$$f^*\colon \mathrm{Fun}(\mathcal{D}', \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C}).$$

Under the assumption that $\mathcal{C}$ admits colimits, each of these functors admit a left adjoint $f_!$ given by left Kan extension. The map r is thus a bicartesian fibration with cocartesian morphisms given by morphisms

$$\begin{tikzcd} \mathcal{D} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}' \arrow[d, "\mathrm{Lan}_\alpha F"] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

We define a triple structure $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ on $\mathbf{LS}(\mathcal{C})$ as follows.

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})$.
- $\mathcal{Q}^\dagger$ consists only of cartesian morphisms.

This corresponds to spans of the form

$$\begin{tikzcd} \mathcal{D}' \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{D} \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \mathcal{D}'' \arrow[d] \\ \mathcal{C} \arrow[r, equal] & \mathcal{C} \arrow[r, equal] & \mathcal{C} \end{tikzcd}$$

Proposition 2.0.1. The triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ is adequate.

To do this, we need to prove:

Lemma 2.0.2. Let $\mathbb{C}$ be an ∞-bicategory such that the underlying quasicategory $\mathcal{C}$ has pullbacks and pushouts. Consider a square

$$\begin{tikzcd} C_1 \arrow[r] \arrow[d, "a"'] & C_2 \arrow[d, "b"] \\ C_3 \arrow[r] & C_4 \end{tikzcd}$$

in $\mathbf{Tw}(\mathbb{C})$ corresponding to a twisted cube

$$\begin{tikzcd}
c_1 \arrow[rr] \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow \alpha"] & & c_2 \arrow[dr] \arrow[dd] \arrow[ddr, phantom, "\Downarrow \beta"] & \\
& c_3 \arrow[rr] \arrow[dd] & & c_4 \arrow[dd] \\
d_1 & & d_2 \arrow[ll] & \\
& d_3 \arrow[ul] & & d_4 \arrow[ll] \arrow[ul]
\end{tikzcd}$$

in $\mathbb{C}$, where the top face is pullback and the bottom face is pushout. Note that the top and bottom faces of any such diagram in $\mathbf{Tw}(\mathbb{C})$ belong to $\mathcal{C}$, and hence must weakly commute, by definition of $\mathbf{Tw}(\mathbb{C})$.

Suppose α is an equivalence. Then

3 The base fibration

The starting point of our construction is the functor

$$\mathcal{F}\mathrm{in}_* \xrightarrow{\mathcal{C}\mathrm{at}_\infty^\times} \mathcal{C}\mathrm{at}_{(\infty,2)} \xrightarrow{\mathbf{Tw}} \mathcal{C}\mathrm{at}_\infty$$

Note that this gives a monoidal category since $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty^n) \cong \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^n$.

We're not gonna worry too much about where the functor $\mathbf{Tw}$ comes from, just keep notes about what properties we're using. The relative nerve of this functor is a Cartesian fibration $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ with the following description: an n-simplex σ corresponding to a diagram

$$\begin{tikzcd}
\Delta^n \arrow[rr, "\sigma"] \arrow[dr, "\phi"'] & & \mathbf{LS}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[dl, "p'"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}$$

corresponds to the data of, for each subset $I \subseteq [n]$ having minimal element i, a map

$$\tau(I) \colon \Delta^I \to \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^i$$

such that For nonempty subsets $I' \subseteq I \subseteq [n]$, the diagram

$$\begin{tikzcd} \Delta^{I'} \arrow[r] \arrow[d, hook] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i'} \arrow[d] \\ \Delta^{I} \arrow[r] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)^{i} \end{tikzcd}$$

Example 3.0.1. An object of $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over $\langle n \rangle$ corresponds to a collection of functors $\mathcal{C}_i \to \mathcal{D}_i$, $i \in \langle n \rangle^\circ$. We will denote this $\vec{\mathcal{C}} \to \vec{\mathcal{D}}$.

A morphism in $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ lying over the active map $\langle 1 \rangle \leftarrow \langle 2 \rangle$ in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$[1] consists of

- A 'source' object $F\colon \mathcal{C} \to \mathcal{C}'$
- A pair of 'target' objects $G_i\colon \mathcal{D}_i \to \mathcal{D}_i'$, $i = 1, 2$.
- A morphism

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \mathcal{D}_1 \times D_2 \arrow[d, "G_1 \times G_2"] \\ \mathcal{C}' & \mathcal{D}_1' \times \mathcal{D}_2' \arrow[l, "\beta"] \end{tikzcd}$$

  where here we mean $\eta\colon F \Rightarrow \beta \circ G_1 \times G_2 \circ \alpha$.

Note that $\mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes$ admits a map to $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$, where

- $\widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ is the relative nerve (as a cartesian fibration) of the functor $\mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty$ giving the cartesian monoidal structure on $\mathcal{C}\mathrm{at}_\infty$
- $\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$ is the *cocartesian* relative nerve of the same.

Taking all this together gives a commuting triangle

$$\begin{tikzcd} \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[rr] \arrow[dr] & & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}} \arrow[dl] \\ & \mathcal{F}\mathrm{in}_*^{\mathrm{op}} & \end{tikzcd}.$$

where

Proposition 3.0.2. Everything in sight is a cartesian fibration.

Proposition 3.0.3. There is an equivalence $\mathcal{C}\mathrm{at}_{\infty,\times} \simeq \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times}$ which is homotopic to the identity on $\mathcal{C}\mathrm{at}_\infty$.

[1]Here we mean the morphism in $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ corresponding to the active map $\langle 2 \rangle \to \langle 1 \rangle$ in $\mathcal{F}\mathrm{in}_*$.

Proposition 3.0.4. We can express a monoidal ∞-category as a functor $\mathcal{F}\mathrm{in}_* \to \widetilde{\mathcal{C}\mathrm{at}}_\infty^\times$.

Fix some monoidal ∞-category $\mathcal{C}$ which admits colimits, and such that the monoidal structure preserves colimits in each slot. We can thus define a functor

$$\mathcal{C}\mathrm{at}_{\infty,\times} \to \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}$$

which is the equivalence on the first component, and given by the composition

$$\mathcal{C}\mathrm{at}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}} \xrightarrow{\mathcal{C}} (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}.$$

Forming the pullback

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[r] \arrow[d, "r"'] & \mathbf{Tw}(\mathcal{C}\mathrm{at}_\infty)_\otimes \arrow[d] \\
\mathcal{C}\mathrm{at}_{\infty,\times} \arrow[r] & \widetilde{\mathcal{C}\mathrm{at}}_{\infty,\times} \times (\widetilde{\mathcal{C}\mathrm{at}}_\infty^\times)^{\mathrm{op}}
\end{tikzcd}$$

gives us a commutative triangle

$$\begin{tikzcd}
\mathbf{LS}(\mathcal{C})_\otimes \arrow[rr, "r"] \arrow[dr, "q"'] & & \mathcal{C}\mathrm{at}_{\infty,\times} \arrow[dl, "p"] \\
& \mathcal{F}\mathrm{in}_*^{\mathrm{op}} &
\end{tikzcd}.$$

Proposition 3.0.5. The map p is a cartesian fibration, and the p-cartesian morphisms in $\mathcal{C}\mathrm{at}_{\infty,\times}$ are precisely those representing products.

Proposition 3.0.6. The maps q and r are cartesian fibrations, and a generic morphism

$$\begin{tikzcd}
\vec{\mathcal{D}} \arrow[r, "\alpha"] \arrow[d, "F"'] \arrow[dr, phantom, "=\eta\Rightarrow"] & \vec{\mathcal{D}}' \arrow[d, "G_1\times G_2"] \\
\vec{\mathcal{C}} & \vec{\mathcal{C}} \arrow[l, "\otimes"]
\end{tikzcd}$$

in $\mathbf{LS}(\mathcal{C})$ is

- r-cartesian if η is an equivalence
- q-cartesian if α exhibits a product and η is an equivalence.

One sees immediately that r sends q-cartesian morphisms to p-cartesian morphisms, and has the sanity check that for any morphism f in $\mathbf{LS}(\mathcal{C})_\otimes$ whose image in $\mathcal{C}\mathrm{at}_{\infty,\times}$ is p-cartesian, f is q-cartesian if and only if it is r-cartesian.

4 The triple structures

4.1 The triple structure on finite pointed sets

We're not gonna notationally distinguish between $\mathcal{F}\mathrm{in}_*$ and $N\mathcal{F}\mathrm{in}_*$. Who has time for that these days?

We define a triple structure $(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$ on $\mathcal{F}\mathrm{in}_*^{\mathrm{op}}$, where

- $\mathcal{F} = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$
- $\mathcal{F}_\dagger = (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{F}^\dagger = \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$

Proposition 4.1.1. This is an adequate triple structure.

Proposition 4.1.2. There is a Joyal equivalence $\mathcal{F}\mathrm{in}_* \to \mathrm{Span}(\mathcal{F}, \mathcal{F}_\dagger, \mathcal{F}^\dagger)$.

4.2 The triple structure on infinity-categories

We define a triple $(\mathcal{P}, \mathcal{P}_\dagger, \mathcal{P}^\dagger)$ as follows.

- $\mathcal{P} = \mathcal{C}\mathrm{at}_{\infty,\times}$
- $\mathcal{P}_\dagger = \mathcal{C}\mathrm{at}_{\infty,\times} \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{P}^\dagger = \mathcal{C}\mathrm{at}_{\infty,\times}$

Proposition 4.2.1. The cartesian fibration $\mathcal{C}\mathrm{at}_{\infty,\times} \to \mathcal{F}\mathrm{in}_*^{\mathrm{op}}$ admits relative pullbacks.

Proposition 4.2.2. This is an adequate triple structure.

4.3 The triple structure on the twisted arrow category

We define a triple $(\mathcal{Q}, \mathcal{Q}_\dagger, \mathcal{Q}^\dagger)$ as follows:

- $\mathcal{Q} = \mathbf{LS}(\mathcal{C})_\otimes$
- $\mathcal{Q}_\dagger = \mathbf{LS}(\mathcal{C})_\otimes \times_{\mathcal{F}\mathrm{in}_*^{\mathrm{op}}} (\mathcal{F}\mathrm{in}_*^{\mathrm{op}})^\simeq$
- $\mathcal{Q}^\dagger$ consists of the subcategory of $\mathbf{LS}(\mathcal{C})_\otimes$ of r-cartesian morphisms.

These triple constructions correspond to spans of the following form.

$$\begin{tikzcd}
\vec{\mathcal{D}} \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \vec{\mathcal{D}}' \arrow[l] \arrow[r] \arrow[d] \arrow[dr, phantom, "\Longrightarrow"] & \vec{\mathcal{D}}'' \arrow[d] \\
\mathcal{C}^n \arrow[r] \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[d, dotted, no head] & \mathcal{C}^m \arrow[l, "\simeq"'] \arrow[d, dotted, no head] \\
\langle n \rangle & \langle m \rangle \arrow[l] \arrow[r, "\simeq"] & \langle m \rangle
\end{tikzcd}$$