1 Limits

1.1 Limits and colimits

Definition 1 (diagram)**.** Let J and C be categories. A diagram of type J in C is a functor

$$\mathcal{D}\colon \mathsf{J} \to \mathsf{C}.$$

Example 2. Consider the category J with objects and morphisms as follows.

$$\begin{tikzcd} & \bullet \arrow[d] \\ \star \arrow[r] & * \end{tikzcd}$$

A diagram of type J in a category C is a functor J → C. Such a functor picks out three objects and two morphisms in C as follows.

$$\begin{tikzcd} & y \arrow[d, "f"] \\ x \arrow[r, "g"'] & z \end{tikzcd}$$

As this example makes clear, one thinks of J as specifying the shape of the diagram in which one is interested.

Definition 3 (cone)**.** Let Δ be the functor $\mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ (the category of functors $\mathsf{J} \to \mathsf{C}$, see Definition **??**) which assigns to each object $X \in \mathrm{Obj}(\mathsf{C})$ the constant functor $\Delta_X : \mathsf{J} \to \mathsf{C}$, i.e. the functor which maps every object of J to X and every morphism to id_X. A cone over $\mathcal{D}$ is then an object in the comma category (Definition **??**) $(\Delta \downarrow \mathcal{D})$ given by the diagram

$$\mathsf{C} \xrightarrow{\Delta} [\mathsf{J}, \mathsf{C}] \xleftarrow{\mathcal{D}} 1 \ .$$

The objects of this category are pairs (X, f), where $X \in \mathrm{Obj}(\mathsf{C})$ and $f\colon \Delta(X) \to \mathcal{D}$; that is to say, f is a natural transformation $\Delta_X \Rightarrow \mathcal{D}$.

Definition 4 (category of cones over a diagram)**.** Let C be a category, J an index category, and $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ a diagram. The category of cones over $\mathcal{D}$ is the category $(\Delta \downarrow \mathcal{D})$.

Definition 4 not only gives the definition of a cone, but even prescribes what morphisms between cones look like. Let (X, Φ) be a cone over a diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$, i.e.

- an object in the category $(\Delta \downarrow \mathcal{D})$, i.e.
- a pair (X, Φ), where $X \in \mathrm{Obj}(\mathsf{C})$ and $\Phi\colon \Delta_X \Rightarrow \mathcal{D}$ is a natural transformation, i.e.

- for each $J \in \mathrm{Obj}(\mathsf{J})$ a morphism $\Phi_J : X \to \mathcal{D}(J)$ such that for any other object $J' \in \mathrm{Obj}(\mathsf{J})$ and any morphism $f : J \to J'$ the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "\Phi_J"'] \arrow[dr, "\Phi_{J'}"] & \\ \mathcal{D}(J) \arrow[rr, "\mathcal{D}(f)"'] & & \mathcal{D}(J') \end{tikzcd}$$

Let (Y, Γ) be another cone over $\mathcal{D}$. Then a morphism $\Xi : (X, \Phi) \to (Y, \Gamma)$ is

- a morphism $\Xi \in \mathrm{Hom}_{(\Delta \downarrow \mathcal{D})}((X, \Phi), (Y, \Gamma))$, i.e.
- a natural transformation $\Xi : \Delta_X \Rightarrow \Delta_Y$ (i.e. a morphism $\Delta_X \to \Delta_Y$ in the category $[\mathsf{J}, \mathsf{C}]$) such that the diagram

$$\begin{tikzcd} \Delta_X \arrow[rr, Rightarrow, "\Xi"] \arrow[dr, Rightarrow, "\Phi"'] & & \Delta_Y \arrow[dl, Rightarrow, "\Gamma"] \\ & \mathcal{D} & \end{tikzcd}$$

  commutes, i.e.
- a morphism $\xi : X \to Y$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the diagram

$$\begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\Phi_J"'] & & Y \arrow[dl, "\Gamma_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

  commutes.

Cocones are the dual notion to cones. We make the following definition.

Definition 5 (cocone)**.** A cocone over a diagram $\mathcal{D}$ is an object in the comma category $(\mathcal{D} \downarrow \Delta)$.

Definition 6 (category of cocones)**.** The category of cocones over a diagram $\mathcal{D}$ is the category $(\mathcal{D} \downarrow \Delta)$.

The categorical definitions of cones and cocones allow us to define limits and colimits succinctly.

Definition 7 (limits, colimits)**.** A limit of a diagram $\mathcal{D} : \mathsf{J} \to \mathsf{C}$ is a final object in the category $(\Delta \downarrow \mathcal{D})$. A colimit is an initial object in the category $(\mathcal{D} \downarrow \Delta)$.

Note 8. The above definition of a limit unwraps as follows. The limit of a diagram $\mathcal{D} : \mathsf{J} \to \mathsf{C}$ is a cone (X, Φ) over $\mathcal{D}$ such that for any other cone (Y, Γ) over $\mathcal{D}$, there is a unique map $\xi : Y \to X$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the following diagram commutes.

$$\begin{tikzcd} Y \arrow[rr, "\xi"] \arrow[dr, "\Gamma_J"'] & & X \arrow[dl, "\Phi_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

Notation 9. We often denote the limit over the diagram $\mathcal{D} : \mathsf{J} \to \mathsf{C}$ by

$$\lim_{\leftarrow} \mathcal{D}.$$

Similarly, we will denote the colimit over $\mathcal{D}$ by

$$\lim_{\rightarrow} \mathcal{D}.$$

If there is notational confusion over which functor we are taking a (co)limit, we will add a dummy index:

$$\lim_{\leftarrow i} \mathcal{D}_i \qquad \text{or} \qquad \lim_{\rightarrow i} \mathcal{D}_i.$$

Example 10. Here is a definition of the product $A \times B$ equivalent to that given in Example **??**: it is the limit of the following somewhat trivial diagram.

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] & & B \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}$$

Let us unwrap this definition. We are saying that the product $A \times B$ is a cone over A and B.

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

But not just any cone: a cone which is universal in the sense that any *other* cone factors through it uniquely.

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, "\exists! f"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

You will recognize this as precisely the diagram defining the product.

In fact, this allows us to generalize the product: the product of n objects $\prod_{i=1}^n A_i$ is the limit over diagram consisting of all the A_i with no morphisms between them.

Definition 11 (equalizer)**.** Let J be the category with objects and morphisms as follows. (The necessary identity arrows are omitted.)

$$\begin{tikzcd} J \arrow[r, shift left, "1"] \arrow[r, shift right, "2"'] & J' \end{tikzcd}$$

A diagram $\mathcal{D}$ of shape J in some category C looks like the following.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

The <u>equalizer</u> of f and g is the limit of the diagram $\mathcal{D}$; that is to say, it is an object $\mathrm{eq} \in \mathrm{Obj}(\mathsf{C})$ and a morphism $e \colon \mathrm{eq} \to A$

$$\begin{tikzcd} \mathrm{eq} \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

such that for any *other* object Z and morphism $i \colon Z \to A$ such that $f \circ i = g \circ i$, there is a unique morphism $e \colon Z \to \mathrm{eq}$ making the following diagram commute.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! u"'] \arrow[dr, "i"] & & \\ \mathrm{eq} \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

1.2 Pullbacks and kernels

In what follows, C will be a category and A, B, etc. objects in Obj(C).

Definition 12 (pullback)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

A pullback of f along g (also called a pullback of g over f, sometimes notated $A \times_C B$) is a commuting square

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there is a unique morphism $h\colon V \to U$ such that the diagram

$$\begin{tikzcd} V \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

commutes.

Note 13. Here is another definition: the pullback $A \times_C B$ is the limit of the diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

This might at first seem odd; after all, don't we also need an arrow $A \times_C B \to C$? But this arrow is completely determined by the commutativity conditions, so it is superfluous.

Example 14. In Set, U is given (up to unique isomorphism) by

$$U = \left\{ (a, b) \in A \times B \,\middle|\, f(a) = g(b) \right\} .$$

The morphisms p and q are given by the projections $p(a, b) = a$, $q(a, b) = b$.

To see that this really does satisfy the universal property, consider any other set V and functions $s: V \to A$ and $t: V \to B$ making the above diagram commute. Then for all $v \in V$, $f(s(v)) = t(g(v))$.

Now consider the map $V \to U$ sending v to $(s(v), t(v))$. This certainly makes the above diagram commute; furthermore, any other map from V to U would not make the diagram commute. Thus U and h together satisfy the universal property.

Example 15. Suppose $f: X \to Y$ is a monomorphism. Then the square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square.

Conversely, if the above square is a pullback square, then f is a monomorphism.

Lemma 16. Let $f: X \to Y$ be a monomorphism. Then any pullback of f is a monomorphism.

Proof. Suppose we have the following pullback square.

$$\begin{tikzcd} X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Our aim is to show that p_2 is a monomorphism.

Suppose we are given an object Z and two morphisms $\alpha, \beta: Z \to X'$.

$$\begin{tikzcd} Z \arrow[dr, shift left, "\alpha"] \arrow[dr, shift right, "\beta"'] & & \\ & X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\ & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

We can compose α and β with p_2. Suppose that these agree, i.e.

$$p_2 \circ \alpha = p_2 \circ \beta.$$

We will be done if we can show that this implies that $p_1 = p_2$.

We can compose with g to find that

$$g \circ p_2 \circ \alpha = g \circ p_2 \circ \beta.$$

but since the pullback square commutes, we can replace $g \circ p_2$ by $f \circ p_1$.

$$f \circ p_1 \circ \alpha = f \circ p_1 \circ \beta.$$

since f is a monomorphism, this implies that

$$p_1 \circ \alpha = p_1 \circ \beta.$$

Now forget α and β for a minute. We have constructed a commuting square as follows.

$$\begin{tikzcd}
Z \arrow[rrd, bend left, "p_1\circ\alpha=p_1\circ\beta"] \arrow[rdd, bend right, "p_2\circ\alpha=p_2\circ\beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

the universal property for pullbacks tells us that there is a unique morphism $Z \to X'$ making the diagram commute. But either α or β will do! So $\alpha = \beta$. $\square$

Definition 17 (kernel of a morphism)**.** Let C be a category with an initial object (Definition **??**) $\emptyset$ and pullbacks. The kernel $\ker(f)$ of a morphism $f \colon A \to B$ is the pullback along f of the unique morphism $\emptyset \to B$.

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\iota"'] & \emptyset \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

That is to say, the kernel of f is a pair $(\ker(f), \iota)$, where $\ker(f) \in \mathrm{Obj}(\mathrm{C})$ and $\iota \colon \ker(f) \to A$ which satisfies the above universal property.

Note 18. Although the kernel of a morphism f is a pair $(\ker(f), \iota)$ as described above, we will sometimes sloppily say that the object $\ker(f)$ is the kernel of f, especially when the the morphism ι is obvious or understood. Such abuses of terminology are common; one occasionally even sees the morphism ι being called the kernel of f.

Example 19. In Vect_k, the initial object is the zero vector space $\{0\}$. For any vector spaces V and W and any linear map $f \colon V \to W$, the kernel of f is the pair $(\ker(f), \iota)$ where $\ker(f)$ is the vector space

$$\ker(f) = \left\{ v \in V \,\middle|\, f(v) = 0 \right\}$$

and ι is the obvious injection $\ker(f) \to V$.

Lemma 20. Let $f \colon A \to B$, and let $(\iota, \ker(f))$ be the kernel of f. Then ι is a monomorphism (Definition **??**).

Proof. Suppose we have an object $Z \in \mathrm{Obj}(\mathrm{C})$ and two morphisms $g_1, g_2 \colon Z \to \ker(f)$. We have the following diagram.

$$\begin{tikzcd}
Z \arrow[rd, shift left, "g_1"] \arrow[rd, shift right, "g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Further suppose that $\iota \circ g_1 = \iota \circ g_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Now pretend that we don't know about g_1 and g_2.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

The universal property for kernels tells us that there is a unique map $Z \to \ker(f)$ making the above diagram commute. But since g_1 and g_2 both make the diagram commute, g_1 and g_2 must be the same map, i.e. $g_1 = g_2$. □

1.3 Pushouts and cokernels

Pushouts are the dual notion to pullbacks.

Definition 21 (pushouts)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \\
A &
\end{tikzcd}$$

The pushout of f along g (or g along f) is a commuting square

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \\
A \arrow[r, "p"'] & U
\end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "t"] \\
A \arrow[r, "s"'] & V
\end{tikzcd}$$

there exists a unique morphism $h\colon U \to V$ such that the diagram

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\
A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\
& & V
\end{tikzcd}$$

commutes.

Note 22. As with pullbacks, we can also define a pushout as the colimit of the following diagram.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \\
A &
\end{tikzcd}$$

Example 23. Let us construct the pushout in Set. If we ignore the object C and the morphisms f and g, we discover that U must satisfy the universal property of the coproduct of A and B.

$$\begin{tikzcd}
& B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\
A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\
& & V
\end{tikzcd}$$

Let us therefore make the ansatz that $U = A \amalg B = A \sqcup B$ and see what happens when we add C, f, and g back in.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\
A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\
& & V
\end{tikzcd}$$

In doing so, we find that the square A-C-B-U must also commute, i.e. we must have that $(q \circ g)(c) = (p \circ f)(c)$ for all $c \in C$. Since p and q are just inclusions, we see that

$$U = A \amalg B/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $f(c) \sim g(c)$ for all $c \in C$.

Definition 24 (cokernel of a morphism)**.** Let C be a category with terminal object 1. The cokernel of a morphism $f\colon A \to B$ is the pushout of f along the unique morphism $A \to 1$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\pi"] \\
1 \arrow[r] & \operatorname{coker}(f)
\end{tikzcd}$$

Example 25. In Vect_k, the terminal object is the vector space $\{0\}$. If V and W are k-vector spaces and f is a linear map $V \to W$, then $\operatorname{coker}(f)$ is

$$W/\sim,$$

where $\sim$ is the relation generated by $f(v) \sim 0$ for all $v \in V$. But this relation is exactly the one which mods out by $\operatorname{im}(f)$, so $\operatorname{coker}(f) = W/\operatorname{im}(f)$.

Lemma 26. For any morphism $f\colon A \to B$, the canonical projection $\pi\colon B \to \operatorname{coker}(f)$ is an epimorphism.

Proof. The proof is dual to the proof that the canonical injection ι is mono (Lemma 20). □

Definition 27 (normal monomorphism)**.** A monomorphism (Definition ??) $f\colon A \to B$ is <u>normal</u> if it the kernel of some morphism. To put it more plainly, f is normal if there exists an object C and a morphism $g\colon B \to C$ such that (A, f) is the kernel of g.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Example 28. In Vect_k, monomorphisms are injective linear maps (Example ??). If f is injective then sequence

$$\{0\} \longrightarrow V \xrightarrow{f} W \xrightarrow{\pi} W/\operatorname{im}(f), \longrightarrow \{0\}$$

is exact, and we always have that $\operatorname{im}(f) = \ker(\pi)$. Thus in Vect_k, every monomorphism is normal.

Definition 29 (conormal epimorphism)**.** An epimorphism $f\colon A \to B$ is <u>conormal</u> if it is the cokernel of some morphism. That is to say, if there exists an object C and a morphism $g\colon C \to A$ such that (B, f) is the cokernel of g.

Example 30. In Vect_k, epimorphisms are surjective linear maps. If $f\colon V \to W$ is a surjective linear map, then the sequence

$$\{0\} \longrightarrow \ker(f) \xrightarrow{\iota} V \xrightarrow{f} W \longrightarrow \{0\}$$

is exact. But then $\operatorname{im}(\iota) = \ker(f)$, so f is conormal. Thus in Vect_k, every epimorphism is conormal.

Note 31. To show that in our proofs that in Vect_k monomorphisms were normal and epimorphisms were conormal, we showed that monomorphisms were the kernels of their cokernels, and epimorphisms were the cokernels of their kernels. This will be a general feature of Abelian categories.

Definition 32 (binormal category)**.** A category is <u>binormal</u> if all monomorphisms are normal and all epimorphisms are conormal.

Example 33. As we have seen, Vect_k is binormal.

1.4 A necessary and sufficient condition for the existence of small limits

In this section we prove a simple criterion to check that a category has all small limits (i.e. limits over diagrams with a small set of objects and between each two objects a small set of morphisms). The idea is as follows.

In general, adding more objects to a diagram makes the limit over it larger, and adding more morphisms makes the limit smaller. In the extreme case in which are no morphisms (apart from the identity morphisms), the limit is simply the categorical product. As we add morphisms, roughly speaking, we have to get rid of the parts of the product which prevent the triangles they form from commuting. Thus, to make the universal cone over a diagram, we can start with the product, and cut out the bare minimum we need to make everything commute: the way to do that is with an equalizer.

The following proof was adapted from [**?**].

Theorem 34. Let C be a category. Then if C has all small products and binary equalizers, C has all small limits.

Proof. Let $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ be a small diagram. We want to prove that $\mathcal{D}$ has a limit; we will do this by constructing a universal cone over it, i.e.

- an object $L \in \mathrm{Obj}(\mathsf{C})$, and
- for each $j \in \mathrm{Obj}(\mathsf{J})$, a morphism $P_j\colon L \to \mathcal{D}(j)$

such that

1. for any $i, j \in \mathrm{Obj}(\mathsf{J})$ and any $\alpha\colon i \to j$ the following diagram commutes,

$$\begin{tikzcd} & L \arrow[dl, "P_i"'] \arrow[dr, "P_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   and

2. for any other object $L' \in \mathrm{Obj}(\mathsf{C})$ and family of morphisms $Q_j\colon L' \to \mathcal{D}(j)$ which make the diagrams

$$\begin{tikzcd} & L' \arrow[dl, "Q_i"'] \arrow[dr, "Q_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   commute for all i, j, and α, there is a unique morphism $f\colon L' \to L$ such that $Q_j = P_j \circ f$ for all $j \in \mathrm{Obj}(\mathsf{J})$.

Denote by $\mathrm{Mor}(\mathsf{J})$ the set of all morphisms in J. For any $\alpha \in \mathrm{Mor}(\mathsf{J})$, denote by $\mathrm{dom}(\alpha)$ the domain of α, and by $\mathrm{cod}(\alpha)$ the codomain.

Consider the following small products:

$$A = \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \qquad \text{and} \qquad B = \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha)).$$

From the universal property for products, we know that we can construct a morphism $f : A \to B$ by specifying a family of morphisms $f_\alpha : A \to \mathcal{D}(\mathrm{cod}(\alpha))$, one for each $\alpha \in \mathrm{Mor}(\mathsf{J})$. We will define two morphisms $R, S : A \to B$ in this way:

$$R_\alpha = \pi_{\mathcal{D}(\mathrm{cod}(\alpha))}; \qquad S_\alpha = \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(\mathrm{dom}(\alpha))}.$$

Now let $e : L \to A$ be the equalizer of R and S (we are guaranteed the existence of this equalizer by assumption).

$$L \xhookrightarrow{e} \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \underset{S}{\overset{R}{\rightrightarrows}} \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha))$$

Further, define $P_j : L \to \mathcal{D}(j)$ by

$$P_j = \pi_{\mathcal{D}(j)} \circ e$$

for all $j \in \mathrm{Obj}(\mathsf{J})$.

The claim is that L together with the P_j is the limit of $\mathcal{D}$. We need to verify conditions 1 and 2 on L and P_j listed above.

1. We need to show that for all $i, j \in \mathrm{Obj}(\mathsf{J})$ and all $\alpha : i \to j$, we have the equality $\mathcal{D}(\alpha) \circ P_i = P_j$. Now, for every $\alpha : i \to j$ we have

$$\begin{aligned} \mathcal{D}(\alpha) \circ P_i &= \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(i)} \circ e \\ &= S_\alpha \circ e \\ &= R_\alpha \circ e \\ &= \pi_{\mathcal{D}(j)} \circ e \\ &= P_j. \end{aligned}$$

2. We need to show that for any other $L' \in \mathrm{Obj}(\mathsf{C})$ and any other family of morphisms $Q_j : L' \to \mathcal{D}(j)$ such that for all $\alpha : i \to j$, $Q_j = \mathcal{D}(\alpha) \circ Q_i$, there is a unique morphism $h : L' \to L$ such that $Q_j = P_j \circ h$ for all $j \in \mathrm{Obj}(\mathsf{J})$. Suppose we are given such an L' and Q_j.

   The universal property for products allows us to construct from the family of morphisms Q_j a morphism $Q : L' \to A$ such that $Q_j = \pi_{\mathcal{D}(j)} \circ Q$. Now, for any $\alpha : i \to j$,

$$\begin{aligned} R_\alpha \circ Q &= \pi_{\mathcal{D}(j)} \circ Q \\ &= Q_j \\ &= \mathcal{D}(\alpha) \circ Q_i \\ &= \mathcal{D}(\alpha) \circ \pi_i \circ Q \\ &= S_\alpha \circ Q. \end{aligned}$$

   Thus, $Q : L' \to A$ equalizes R and S. But the universal property for equalizers guarantees us a unique morphism $h : L' \to L$ such that $Q = P \circ h$. We can compose both sides of this equation on the left with $\pi_{\mathcal{D}(j)}$ to find

$$\pi_{\mathcal{D}(j)} \circ Q = \pi_{\mathcal{D}(j)} \circ P \circ h,$$

   i.e.

$$Q_j = P_j \circ h$$

   as required.

∎

1.5 The hom functor preserves limits

For any functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$, recall that we defined the category of cones over $\mathcal{F}$ to be the category $(\Delta \downarrow \mathcal{F})$. If we pick an object $d \in \mathsf{D}$, the set of cones with tip d is then $(\Delta_d \downarrow \mathcal{F})$.

Lemma 35. Let

Theorem 36. Let C be a locally small category. The hom functor $\mathrm{Hom}_{\mathsf{C}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$ preserves limits in the second argument, i.e. for $\mathcal{D} \colon \mathsf{J} \to \mathsf{C}$ a diagram in C we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}),$$

where the limit on the RHS is over the hom-set diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \to \mathsf{Set}.$$

Proof. Let L be the limit over the diagram $\mathcal{D}$. Then for any map $f \colon Y \to L$, there is a cone from Y to $\mathcal{D}$ by composition, and for any cone with tip Y over $\mathcal{D}$ we get a map $f \colon Y \to L$ from the universal property of limits. Thus, there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq (Y \downarrow \mathcal{D}),$$

which is natural in Y since the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow} \mathcal{D}) \arrow[r, "(-)\circ f"] \arrow[d] & \mathrm{Hom}_{\mathsf{C}}(Z, \lim_{\leftarrow} \mathcal{D}) \arrow[d] \\ (Y \downarrow \mathcal{D}) \arrow[r, "(-)\circ f"] & (Z \downarrow \mathcal{D}) \end{tikzcd}$$

trivially commutes. We will be done if we can show that there is also a natural isomorphism

$$(Y \downarrow \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}).$$

Let us understand the elements of the set $\lim_{\leftarrow} \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D})$. The diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \to \mathsf{Set}$$

maps $J \in \mathrm{Obj}(\mathsf{J})$ to $\mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J)) \in \mathrm{Obj}(\mathsf{Set})$. A universal cone over this diagram is a set S together with, for each $J_i \in \mathrm{Obj}(\mathsf{J})$, a function

$$f_i \colon S \to \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i))$$

such that for each $\alpha \in \mathrm{Hom}_{\mathsf{J}}(J_i, J_j)$, the diagram

$$\begin{tikzcd} & S \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i)) \arrow[rr, "\mathcal{D}(\alpha)\circ(-)"'] & & \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_j)) \end{tikzcd}$$

commutes.

Now pick any element $s \in S$. Each f_i maps this to a function $Y \to \mathcal{D}(J_i)$ which makes the diagram

$$\begin{tikzcd} & Y \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\ \mathcal{D}(J_i) \arrow[rr, "\mathcal{D}(\alpha)"'] & & \mathcal{D}(J_j) \end{tikzcd}$$

commute. But this is exactly an element of $(Y \downarrow \mathcal{D})$. Conversely, every element of $(Y \downarrow \mathcal{D})$ gives us an element of S, so we have a bijection

$$(Y \downarrow \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_C(Y, \mathcal{D}),$$

which is natural as required. ◻

Corollary 37. The first slot of the hom functor turns colimits into limits. That is,

$$\mathrm{Hom}_C(\varinjlim \mathcal{D}, Y) \simeq \varprojlim \mathrm{Hom}_C(\mathcal{D}, Y).$$

Proof. Dual to that of Theorem 36. ◻

1.6 Filtered colimits and ind-objects

Definition 38 (preorder)**.** Let S be a set. A preoreder on S is a binary relation $\leq$ which is

1. reflexive ($a \leq a$)
2. transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)

A preorder is said to be directed if for any two objects a and b, there exists an 'upper bound,' i.e. an object r such that $a \leq r$ and $b \leq r$.

Filtered categories are a generalization of filtered preorders.

Definition 39 (filtered category)**.** A category J is filtered if

- for each pair of objects $J, J' \in \mathrm{Obj}(\mathsf{J})$, there exists an object K and morphisms $J \to K$ and $J' \to K$.

$$\begin{tikzcd} J \arrow[dr] & \\ & K \\ J' \arrow[ur] & \end{tikzcd}$$

  That is, every diagram with two objects and no morphisms is the base of a cocone.

- For every pair of morphisms $i, j \colon J \to J'$, there exists an object K and a morphism $f \colon J' \to K$ such that $f \circ i = f \circ j$, i.e. the following diagram commutes.

$$\begin{tikzcd} J \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & J' \arrow[r, "f"] & K \end{tikzcd}$$

  That is, every diagram of the form $\bullet \rightrightarrows \bullet$ is the base of a cocone.

Example 40. Any poset is a filtered category.

Definition 41 (filtered colimit)**.** A filtered colimit is a colimit over a diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$, where J is a filtered category.

We now define the so-called *category of inductive objects* (or simply *ind-objects*).

Definition 42 (category of ind-objects)**.** Let C be a category. We define the category of ind-objects of C, denote $\mathsf{Ind}(\mathsf{C})$ as follows.

- The objects $F \in \mathrm{Obj}(\mathsf{Ind}(\mathsf{C}))$ are defined to be filtered colimits of objects of diagrams $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$.
- For two objects $F = \lim_{\to d} \mathcal{F}_d$ and $G = \lim_{\to e} \mathcal{G}_e$, the morphisms $\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G)$ are defined to be the set

$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G) = \lim_{\leftarrow d} \lim_{\to e} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}_d, \mathcal{G}_e).$$

Note 43. There is a fully faithful embedding $\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$ which exhibits any object A as the colimit over the trivial diagram $\bullet \circlearrowleft$.

The importance of the category of ind-objects can be seen in the following example.

Example 44. Let V be an infinite-dimensional vector space over some field k. Then V can be realized as an object in the category $\mathsf{Ind}(\mathsf{FinVect})$.

To see this, let V be any vector space. Consider the poset of finite-dimensional vector subspaces of V. We can express V as the colimit of this poset. Thus, we have an embedding $\mathsf{Vect}_k \hookrightarrow \mathsf{Ind}(\mathsf{FinVect}_k)$.

In fact, there is an equivalence of categories $\mathsf{Vect}_k \simeq \mathsf{Ind}(\mathsf{FinVect}_k)$. Similarly, there is an equivalence of categories $\mathsf{SVect}_k \simeq \mathsf{Ind}(\mathsf{FinSVect}_k)$

1.7 Miscellaneous results

Theorem 45. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor.

- If C has an initial object i, then $\lim_{\leftarrow \mathcal{F}} = F(i)$.
- If C has a terminal object i, then $\lim_{\to \mathcal{F}} = F(i)$.

Proof. For each $c \in \mathrm{Obj}(\mathsf{C})$, we get a unique morphism $i \to c$. Sticking these into $\mathcal{F}$ gives us a cone $\alpha_c\colon F(i) \to F(c)$ under $\mathcal{F}(i)$.

Now let (d, η) be any other cone over $\mathcal{F}$. In particular, η has a component

$$\eta_i\colon d \to \mathcal{F}(i).$$

In order to show that α is a limit cone, we need only show that this is the only possible morphism $d \longrightarrow F(i)$ such that the triangles

$$\begin{tikzcd} & d \arrow[d, "\eta_i"] \arrow[ddl, bend right, "\eta_j"'] \\ & F(i) \arrow[dl, "\alpha_c"] \\ F(c) & \end{tikzcd}$$

commute for all c. Let $f\colon d \longrightarrow F(i)$ be any other such morphism. The triangle above must in particular commute when $c = i$, so $\alpha_i = F(\mathrm{id}_i) = \mathrm{id}_{F(i)}$.

$$\begin{tikzcd} & d \arrow[d, "f"] \arrow[ddl, bend right, "\eta_i"'] \\ & F(i) \arrow[dl, "\mathrm{id}_{F(i)}"] \\ F(i) & \end{tikzcd}$$

Thus, $f = \eta_i$. □

Theorem 46. Let C and D be categories, with D small and C locally small.

- If D is complete, then so is [C, D].
- If D is cocomplete, then so is [C, D].

Proof. The idea is as follows. A cone in a functor category is built out of natural transformations. Evaluating on a component gives a cone in the target category. A cone in a functor category is a limit cone if and only if each component is a limit cone. □

Let $\mathcal{F}\colon J \longrightarrow \mathcal{C}$ be any functor. The functor $\mathcal{F}$ induces a functor $\mathcal{G}\colon \mathsf{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$ defined on objects via

$$\mathcal{G}\colon c \mapsto \lim \mathrm{Hom}_{\mathsf{C}}(c, F(-)).$$

To define $\mathcal{G}$ on morphisms, note that we can write the functor over which we are taking the limit as $h^c \circ F$, where h^c is the covariant Yoneda embedding (**??**). The functoriality of the covariant Yoneda embedding gives us, for any map $f\colon c \longrightarrow d$ a natural transformation $h^d \circ \mathcal{F} \longrightarrow h^c \circ \mathcal{F}$. This in turn induces a morphism of limits $\lim h^d \circ \mathcal{F} \longrightarrow \lim h^c \circ \mathcal{F}$, which we define to be $\mathcal{G}(f)$.

Theorem 47. Let $\mathcal{F}\colon J \longrightarrow \mathsf{C}$ be a functor as above. Then the limit $\lim \mathcal{F}$ exists if and only if the functor $\mathcal{G}$ is representable.

Proof. First, suppose that the limit $\lim \mathcal{F}$ exists. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(-, \lim \mathcal{F}) &\simeq \lim \mathrm{hom}_{\mathsf{C}}(\,\cdot\,, F(-)) \\ &\simeq \mathcal{G}, \end{aligned}$$

so $\mathcal{G}$ is represented by $\lim \mathcal{F}$.

Conversely, suppose that $\mathcal{G}$ is representable, i.e. there exists a natural isomorphism

$$\mathcal{G} \simeq \mathrm{Hom}_{\mathrm{C}}(-, X)$$

for some $X \in \mathrm{Obj}(\mathrm{C})$. By definition of $\mathcal{G}$, this is a natural isomorphism

$$\lim_{*} \mathrm{Hom}_{\mathrm{C}}(-, F(*)) \simeq \mathrm{Hom}_{\mathrm{C}}(-, X).$$

But by the above $\square$