1 Abelian categories

This section draws heavily from [?].

1.1 Additive categories

Recall that Ab is the category of abelian groups.

Definition 1 (Ab-enriched category)**.** A category C is Ab-enriched if

1. for all objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition

$$\circ\colon \mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{C}(A, C)$$

   is additive in each slot: for any $f_1, f_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, we must have

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$

   and similarly in the second slot.

That is to say, Ab-enriched categories are categories enriched over $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, 0)$.

Note 2. In any Ab-enriched category, every hom-set has at least one element—the identity element of the hom-set taken as an abelian group.

Lemma 3. Let C be an Ab-enriched category with finite products. Then the product $X_1 \times \cdots \times X_n$ satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

Proof. First, we show that initial and terminal objects coincide. Let $\emptyset$ be initial and $*$ terminal in C.

Since $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on $\emptyset$ *and* the identity element of $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, $\mathrm{id}_* = 0$ in $\mathrm{Hom}_{\mathsf{C}}(*, *)$.

Since $\emptyset$ is initial and $*$ is final, $\mathrm{Hom}_{\mathsf{C}}(\emptyset, *)$ has exactly one element f. By Note 2, we are guaranteed that $\mathrm{Hom}_{\mathsf{C}}(*, \emptyset)$ has at least one element, g. Then $g \circ f = 0 = \mathrm{id}$ and $f \circ g = 0 = \mathrm{id}$, so $f\colon \emptyset \to *$ is an isomorphism. Thus, initial and terminal objects coincide, so C has a zero object 0.

We prove the lemma for binary products. The general case is preciesly the same.

For each X, $Y \in \mathcal{C}$, we can define $i_X \colon X \to X \times Y$ using the universal property for products as follows.

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}"'] \arrow[d, dashed, "i_X"] \arrow[dr, "0"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Similarly, get $i_Y \colon Y \to X \times Y$.

The claim is that $X \times Y$, together with i_X and i_Y, satisfies the universal property for coproducts; that is, that we can find a unique map ϕ making the below diagram commute.

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, dashed, "\phi"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

In fact, $\phi = f \circ \pi_X + g \circ \pi_Y$ satisfies the universal property. It is clear that this makes the diagram commute since $\pi_X \circ i_X = \mathrm{id}$, $\pi_X \circ i_Y = 0$, and similarly for Y.

Note that, by the universal property for products, there is a unique downward map. But both $\mathrm{id}_{X \times Y}$ and $i_X \circ \pi_X + i_Y \circ \pi_Y$ make it commute, so they must be equal.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_Y"'] \arrow[d] \arrow[dr, "\pi_X"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Now we are ready to show that ϕ is unique. Suppose we have another candidate ψ which also makes the above diagram commute. Then

$$\psi = \psi \circ \mathrm{id}_{X \times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

□

Definition 4 (additive category)**.** A category C is additive if it has products and is Ab-enriched.

Example 5. The category Ab of Abelian groups is an additive category. We have already seen that it has the direct sum $\oplus$ as biproduct. Given any two abelian groups A and B and morphisms $f, g \colon A \to B$, we can define the sum $f + g$ via

$$(f + g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$

Then for another abelian group C and a morphism $h \colon B \to C$, we have

$$[h \circ (f + g)]\,(a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g]\,(a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g,$$

and similarly in the other slot.

Example 6. The category Vect_k is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

Definition 7 (additive functor)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor between additive categories. We say that $\mathcal{F}$ is <u>additive</u> if for each $X, Y \in \mathrm{Obj}(\mathsf{C})$ the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of abelian groups.

Lemma 8. For any additive functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$, there exists a natural isomorphism

$$\Phi\colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

Proof. The commutativity of the following diagram is immediate.

$$\begin{tikzcd}
\mathcal{F}(X \oplus Y) \arrow[r, "\mathcal{F}(f \oplus g)"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{F}(X' \oplus Y') \arrow[d, "\Phi_{X',Y'}"] \\
\mathcal{F}(X) \oplus \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f) \oplus \mathcal{F}(g)"] & \mathcal{F}(X') \oplus \mathcal{F}(Y')
\end{tikzcd}$$

The (X, Y)-component $\Phi_{X,Y}$ is an isomorphism because □

1.2 Pre-abelian categories

Definition 9 (pre-abelian category)**.** A category C is <u>pre-abelian</u> if it is additive and every morphism has a kernel (Definition **??**) and a cokernel (Definition **??**).

Lemma 10. Pre-abelian categories have equalizers (Definition **??**).

Proof. We show that in an pre-abelian category, the equalizer of f and g coincides with the kernel of $f - g$. It suffices to show that the kernel of $f - g$ satisfies the universal property for the equalizer of f and g.

Here is the diagram for the universal property of the kernel of $f - g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists ! \bar{\imath}"] \arrow[drr, bend left] \arrow[ddr, bend right, "i"'] & & \\
& \ker(f - g) \arrow[r] \arrow[d, "\iota_{f-g}"'] & 0 \arrow[d] \\
& A \arrow[r, "f-g"] & B
\end{tikzcd}$$

The universal property tells us that for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $i\colon Z \to A$ with $i \circ (f - g) = 0$ (i.e. $i \circ f = i \circ g$), there exists a unique morphism $\bar{\imath}\colon Z \to \ker(f - g)$ such that $i = \bar{\imath} \circ \iota_{f-g}$. □

Corollary 11. Every pre-abelian category has all finite limits and colimits.

Proof. By Theorem **??**, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 10. The colimit case is dual. □

Recall from Section **??** the following definition.

Definition 12 (zero morphism)**.** Let C be a category with zero object 0. For any two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B \end{tikzcd}$$

Notation 13. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}.

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism: $f \circ 0 = 0$ and $0 \circ g = 0$.

Lemma 14. Every morphism $f\colon A \to B$ in a pre-abelian catgory has a canonical decomposition

$$A \xrightarrow{p} \mathrm{coker}(\ker(f)) \xrightarrow{\bar{f}} \ker(\mathrm{coker}(f)) \xrightarrow{i} B \ ,$$

where p is an epimorphism (Definition **??**) and i is a monomorphism (Definition **??**).

Proof. We start with a map

$$A \xrightarrow{f} B \ .$$

Since we are in a pre-abelian category, we are guaranteed that f has a kernel $(\ker(f), \iota)$ and a cokernel $(\mathrm{coker}(f), \pi)$. From the universality squares it is immediate that $f \circ \iota = 0$ and $\pi \circ f = 0$ This tells us that the composition $\pi \circ f \circ \iota = 0$, so the following commutes.

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & & & \mathrm{coker}(f) \\ & 0 \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

We know that π has a kernel $(\ker(\pi), i)$ and ι has a cokernel $(\mathrm{coker}(\iota), p)$, so we can add their commutativity squares as well.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

If we squint hard enough, we can see the following diagram.

$$\begin{tikzcd}
A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
 & \ker(\pi) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
 & B \arrow[r, "\pi"] & \operatorname{coker}(f)
\end{tikzcd}$$

The outer square commutes because $\pi \circ f = 0$, so the universal property for $\ker(\pi)$ gives us a unique morphism $A \to \ker(\pi)$. Let's add this to our diagram, along with a morphism $0 \to \ker(\pi)$ which trivially keeps everything commutative.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

Again, buried in the bowels of our new diagram, we find the following.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[d] & A \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & \operatorname{coker}(\iota) & \\
 & & \ker(\pi)
\end{tikzcd}$$

And again, the universal property of cokernels gives us a unique morphism $\bar{f}\colon \operatorname{coker}(\iota) \to$

$\ker(\pi)$.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

The fruit of our laborious construction is the following commuting square.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "i"] \\
\operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi)
\end{tikzcd}$$

We have seen (Lemma ??) that i is mono, and (Lemma ??) that p is epi.

Now we abuse terminology by calling $\iota = \ker(f)$ and $\pi = \operatorname{coker}(f)$. Then we have the required decomposition. □

Note 15. The abuse of notation above is ubiquitous in the literature.

1.3 Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

Definition 16 (abelian category)**.** A pre-abelian category C is abelian if every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

Note 17. The above piecemeal definition is equivalent to the following.

A category C is *abelian* if

1. it is Ab-enriched Definition 4, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
2. it admits finite coproducts, hence (by Lemma 3) biproducts and zero objects;
3. every morphism has a kernel and a cokernel;
4. every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Example 18.

- Let R be a ring. Then the categories R-Mod and Mod-R of left- and right R-modules respectively, are abelian.
- If A is abelian, then Ch(A), the category of chain complexes in A, is abelian.
- If I is a small category, then Fun(I, A) is abelian.

- ∘ With $I = BG$ for some group G, this gives the category of A-representations of G
- ∘ With X a topological space and Open(X) its category of opens, taking I = Open$^{\mathrm{op}}$) tells us that the category of presheaves on X, denoted Psh(X), is abelian.

Example 19. Here are two more tricky cases.

- Consider $R = \mathbb{C}[x_1, x_2, \dots]$ the ring of polynomials in countably many variables. Consider the full subcategory

$$(R\text{-Mod})^{\text{fin gen}} \hookrightarrow R\text{-Mod}.$$

This is additive (as a full subcategory), but does not have kernels in general. For example, consider the quotient

$$f : R \twoheadrightarrow R/(x_1, \dots) \overset{\text{as a vector space}}{\simeq} \mathbb{C}.$$

Both R and $R/(x_1, \dots)$ are finitely generated R-modules. It may at first seem that there cannot be a kernel since the kernel of f is $(x_1, \dots)$, which is not finitely generated. But this is not the kernel we are looking for—it is taken in the category of R-Mod, which satisfies a much stronger universal property than the hypothetical kernel in (R-Mod).

However, it turns out that there is no saving this construction. Suppose there was a kernel. It would have to be finitely generated, i.e. we could write

$$\ker f = R[g_1, \dots, g_n]/(\cdots).$$

Each of the g_i are polynomials in (finitely many) variables x_i; in particular, there is a highest i such that x_i appears in one of the g_i. But then the map

For the remainder of the section, let C be an abelian category.

Lemma 20. Let $f : A \to B$ be a morphism in an abelian category.

- If f is mono, then $\ker f = 0$.
- If f is epi, then $\operatorname{coker} f = 0$.

Proof. Suppose f is mono. The kernel of f has the property that for any morphism $g : C \to A$ such that $f \circ g = 0$, there exists a unique morphism $h : C \to \ker f$ such that $\iota \circ h = g$. However, if f is mono, then $f \circ g = 0 \implies g = 0$, so the zero morphism also has this property. Thus, 0 is a kernel of f.

$$\begin{tikzcd}
 & C \arrow[dl, "\exists!0"'] \arrow[d, "g=0"'] \arrow[dr, "0"] & \\
0 \arrow[r] & X \arrow[r, "f"'] & Y
\end{tikzcd}$$

The cokernel case is dual. □

Lemma 21. Let $\mathcal{A}$ be an abelian category, and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Then f is an isomorphism if and only if f is monic and epic.

Proof. The direction iso $\implies$ monic and epic is obvious. To see the other direction, suppose that f is monic and epic. By monicness, Lemma 20 implies that $\ker f = 0$. Because f is epic, f is the cokernel of the map $0 \to X$. ◻

Lemma 22. In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.

Proof. ◻

Note 23. The above decomposition is unique up to unique isomorphism.

Definition 24 (image of a morphism)**.** Let $f\colon A \to B$ be a morphism. The object $\ker(\operatorname{coker}(f))$ is called the image of f, and is denoted $\operatorname{im}(f)$.

Lemma 25.

1. A morphism $f\colon A \to B$ is mono iff for all $Z \in \mathrm{Obj(C)}$ and for all $g\colon Z \to A$, $f \circ g = 0$ implies $g = 0$.
2. A morphism $f\colon A \to B$ is epi iff for all $Z \in \mathrm{Obj(C)}$ and for all $g\colon B \to Z$, $g \circ f = 0$ implies $g = 0$.

Proof.

1. First, suppose f is mono. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

   If the above diagram commutes, i.e. if $f \circ g = 0$, then $g = 0$, so $1 \implies 2$.

   Now suppose that for all $Z \in \mathrm{Obj(C)}$ and all $g\colon Z \to A$, $f \circ g = 0$ implies $g = 0$.

   Let $g, g'\colon Z \to A$, and suppose that $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0$. But that means that $g - g' = 0$, i.e. $g = g'$. Thus, $2 \implies 1$.

2. Dual to the proof above.

◻

Lemma 26. Let $f\colon A \to B$. We have the following.

1. The morphism f is mono iff $\ker(f) = 0$
2. The morphism f is epi iff $\operatorname{coker}(f) = 0$.

Proof.

1. We first show that if $\ker(f) = 0$, then f is mono. Suppose $\ker(f) = 0$. By the universal property of kernels, we know that for any $Z \in \mathrm{Obj}(\mathsf{C})$ and any $g\colon Z \longrightarrow A$ with $f \circ g = 0$ there exists a unique map $\bar{g}\colon Z \longrightarrow \ker(f)$ such that $g = \iota \circ \bar{g}$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists! g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(f) = 0 \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

   But then g factors through the zero object, so we must have $g = 0$. This shows that $f \circ g = 0 \implies g = 0$, and by Lemma 25 f must be mono.

   Next, we show that if f is mono, then $\ker(f) = 0$. To do this, it suffices to show that $\ker(f)$ is final, i.e. that there exists a unique morphism from every object to $\ker(f)$.

   Since $\mathrm{Hom}_{\mathsf{C}}(Z, \ker(f))$ has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms h_1 and h_2.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

   Our aim is to show that $h_1 = h_2$. To this end, compose each with ι.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] \arrow[ddr, bend right, "\iota\circ h_1"] \arrow[ddr, bend right=40, "\iota\circ h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

   Since $f \circ \iota = 0$, we have $f \circ (\iota \circ h_1) = 0$ and $f \circ (\iota \circ h_2) = 0$. But since f is mono, by Lemma 25, we must have $\iota \circ h_1 = 0 = \iota \circ h_2$. But by Lemma **??**, ι is mono, so again we have

$$h_1 = h_2 = 0,$$

   and we are done.

2. Dual to the proof above.

$\square$

Lemma 27. We have the following.

1. The kernel of the zero morphism $0 : A \longrightarrow B$ is the pair (A, id_A).
2. The cokernel of the zero morphism $0 : A \longrightarrow B$ is the pair (B, id_B).

Proof.

1. We need only verify that the universal property is satisfied. That is, for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $h\colon Z \longrightarrow A$ such that $0 \circ g = 0$, there exists a unique morphism $\bar{g}\colon Z \longrightarrow \ker(0)$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\bar{g}=g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(0) = A \arrow[r, "0"] \arrow[d, "\iota=\mathrm{id}_A"'] & 0 \arrow[d] \\
& A \arrow[r, "0"] & B
\end{tikzcd}$$

   But this is pretty trivial: $\bar{g} = g$.

2. Dual to above.

□

Theorem 28. All abelian categories are binormal (Definition **??**). That is to say:

1. all monomorphisms are kernels
2. all epimorphisms are cokernels.

Proof.

1. Consider the following diagram taken Lemma 14, which shows the canonical factorization of any morphism f.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \mathrm{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \mathrm{coker}(f) \\
& 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

   By definition of a pre-abelian category, we know that $\bar{f}$ is an isomorphism.

□

Note 29. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial. See e.g. [**?**] for details.

Definition 30 (subobject, quotient object, subquotient object)**.** Let $Y \in \mathrm{Obj}(\mathsf{C})$.

1. A subobject of Y is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a monomorphism $i\colon X \hookrightarrow Y$. If X is a subobject of Y we will write $X \subseteq Y$.

2. A quotient object of Y is an object Z together with an epimorphism $p\colon Y \twoheadrightarrow Z$.
3. A subquotient object of Y is a quotient object of a subobject of Y.

Definition 31 (quotient)**.** Let $X \subseteq Y$, i.e. let there exist a monomorphism $f\colon X \hookrightarrow Y$. The quotient Y/X is the cokernel $(\mathrm{coker}(f), \pi_f)$.

Example 32. Let V be a vector space, $W \subseteq V$ a subspace. Then we have the canonical inclusion map $\iota\colon W \hookrightarrow V$, so W is a subobject of V in the sense of Definition 30.

According to Definition 31, the quotient V/W is the cokernel $(\mathrm{coker}(\iota), \pi_\iota)$ of ι. We saw in Example **??** that the cokernel of ι was $V/\mathrm{im}(\iota)$. However, $\mathrm{im}(\iota)$ is exactly W! So the categorical notion of the quotient V/W agrees with the linear algebra notion.

Definition 33 (k-linear category)**.** An abelian category Definition 16 C is k-linear if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of a k-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is k-linear.

Example 34. The category Vect_k is k-linear.

Definition 35 (k-linear functor)**.** let C and D be two k-linear categories, and $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ a functor. Suppose that for all objects $C, D \in \mathrm{Obj}(\mathsf{C})$ all morphisms $f, g\colon C \to D$, and all $\alpha, \beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{G}(g).$$

Then we say that $\mathcal{F}$ is k-linear.

1.4 Exact sequences

Definition 36 (exact sequence)**.** A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree i if the image (Definition 24) of f_{i-1} is equal to the kernel (Definition **??**) of f_i. A sequence is exact if it is exact in every degree.

Lemma 37. If a sequence

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree i, then $f_i \circ f_{i-1} = 0$.

Definition 38 (short exact sequence)**.** A short exact sequence is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Definition 39 (exact functor)**.** Let C, D be abelian categories, $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ an additive functor (Definition 7). We say that $\mathcal{F}$ is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

1.5 Length of objects

Definition 40 (simple object)**.** A nonzero object $X \in \mathrm{Obj}(\mathsf{C})$ is called simple if 0 and X are its only subobjects.

Example 41. In Vect_k, the only simple object (up to isomorphism) is k, taken as a one-dimensional vector space over itself.

Definition 42 (semisimple object)**.** An object $Y \in \mathrm{Obj}(\mathsf{C})$ is semisimple if it is isomorphic to a dirct sum of simple objects.

Example 43. In Vect_k, all finite-dimensional vector spaces are semisimple.

Definition 44 (semisimple category)**.** An abelian category C is semisimple if every object of C is semisimple.

Example 45. The category $\mathsf{FinVect}_k$ is semisimple.

Definition 46 (Jordan-Hölder series)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of X such that X_i/X_{i-1} is simple for all i is called a Jordan Hölder series for X. The integer n is called the length of the series X_i.

The importance of Jordan-Hölder series is the following.

Theorem 47 (Jordan-Hölder)**.** Let X_i and Y_i be two Jordan-Hölder series for some object $X \in \mathrm{Obj}(\mathsf{C})$. Then the length of X_i is equal to the length of Y_i, and the objects Y_i/Y_{i-1} are a reordering of X_i/X_{i-1}.

Proof. See [?], pg. 5, Theorem 1.5.4. □

Definition 48 (length)**.** The length of an object X is defined to be the length of any of its Jordan-Hölder series. This is well-defined by Theorem 47.