# 1 Abelian categories

This section draws heavily from [?].

## 1.1 Additive categories

Recall that Ab is the category of abelian groups.

**Definition 1** (Ab-enriched category). A category C is Ab-enriched if

- 1. for all objects  $A, B \in \text{Obj}(C)$ , the hom-set  $\text{Hom}_C(A, B)$  has the structure of an abelian group (i.e. one can add morphisms), such that
- 2. the composition

$$\circ: \operatorname{Hom}_{\mathbb{C}}(B, \mathbb{C}) \times \operatorname{Hom}_{\mathbb{C}}(A, \mathbb{B}) \to \operatorname{Hom}_{\mathbb{C}}(A, \mathbb{C})$$

is additive in each slot: for any  $f_1$ ,  $f_2 \in \text{Hom}_{\mathbb{C}}(B, C)$  and  $g \in \text{Hom}_{\mathbb{C}}(A, B)$ , we must have

$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$

and similarly in the second slot.

*Note* 2. In any Ab-enriched category, every hom-set has at least one element—the identity element of the hom-set taken as an abelian group.

**Lemma 3.** Let C be an Ab-enriched category with finite products. Then the product  $X_1 \times \cdots \times X_n$  satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

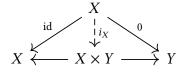
*Proof.* First, we show that initial and terminal objects coincide. Let  $\emptyset$  be initial and \* terminal in C.

Since  $\operatorname{Hom}_{\mathbb{C}}(\emptyset, \emptyset)$  has only one element, it must be both the identity morphism on  $\emptyset$  and the identity element of  $\operatorname{Hom}_{\mathbb{C}}(\emptyset, \emptyset)$  as an abelian group. Similarly,  $\operatorname{id}_* = 0$  in  $\operatorname{Hom}_{\mathbb{C}}(*, *)$ .

Since  $\emptyset$  is initial and \* is final,  $\operatorname{Hom}_{\mathbb{C}}(\emptyset,*)$  has exactly one element f. By Note 2, we are guaranteed that  $\operatorname{Hom}_{\mathbb{C}}(*,\emptyset)$  has at least one element, g. Then  $g \circ f = 0 = \operatorname{id}$  and  $f \circ g = 0 = \operatorname{id}$ , so  $f : \emptyset \to *$  is an isomorphism. Thus, initial and terminal objects coincide, so  $\mathbb{C}$  has a zero object 0.

We prove the lemma for binary products. The general case is preciesly the same.

For each  $X, Y \in \mathcal{C}$ , we can define  $i_X \colon X \to X \times Y$  using the universal property for products as follows.



Similarly, get  $i_Y : Y \to X \times Y$ .

The claim is that  $X \times Y$ , together with  $i_X$  and  $i_Y$ , satisfies the universal property for coproducts; that is, that we can find a unique map  $\phi$  making the below diagram commute.

In fact,  $\phi = f \circ \pi_X + g \circ \pi_Y$  satisfies the universal property. It is clear that this makes the diagram commute since  $\pi_X \circ i_X = \mathrm{id}$ ,  $\pi_X \circ i_Y = 0$ , and similarly for Y.

Note that, by the universal property for products, there is a unique downward map. But both  $id_{X\times Y}$  and  $i_X \circ \pi_X + i_Y \circ \pi_Y$  make it commute, so they must be equal.

$$X \times Y$$

$$X \stackrel{\pi_Y}{\longleftarrow} X \times Y \stackrel{\pi_X}{\longrightarrow} Y$$

Now we are ready to show that  $\phi$  is unique. Suppose we have another candidate  $\psi$  which also makes the above diagram commute. Then

$$\psi = \psi \circ \mathrm{id}_{X \times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

**Definition 4** (additive category). A category C is additive if it has products and is Ab-enriched.

**Example 5.** The category Ab of Abelian groups is an additive category. We have already seen that it has the direct sum  $\oplus$  as biproduct. Given any two abelian groups A and B and morphisms  $f, g: A \to B$ , we can define the sum f + g via

$$(f+q)(a) = f(a) + q(a)$$
 for all  $a \in A$ .

Then for another abelian group C and a morphism  $h: B \to C$ , we have

$$[h \circ (f+q)](a) = h(f(a)+g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g](a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g$$

and similarly in the other slot.

**Example 6.** The category  $Vect_k$  is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

**Definition** 7 (additive functor). Let  $\mathcal{F}: C \to D$  be a functor between additive categories. We say that  $\mathcal{F}$  is additive if for each  $X, Y \in \text{Obj}(C)$  the map

$$\operatorname{Hom}_{\mathsf{C}}(X,Y) \to \operatorname{Hom}_{\mathsf{D}}(\mathfrak{F}(X),\mathfrak{F}(Y))$$

is a homomorphism of abelian groups.

**Lemma 8.** For any additive functor  $\mathcal{F} \colon C \to D$ , there exists a natural isomorphism

$$\Phi \colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

*Proof.* The commutativity of the following diagram is immediate.

The (X, Y)-component  $\Phi_{X,Y}$  is an isomorphism because

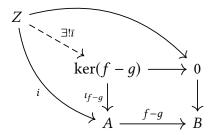
# 1.2 Pre-abelian categories

**Definition 9** (pre-abelian category). A category C is <u>pre-abelian</u> if it is additive and every morphism has a kernel (Definition ??) and a cokernel (Definition ??).

**Lemma 10.** Pre-abelian categories have equalizers (Definition ??).

*Proof.* We show that in an pre-abelian category, the equalizer of f and g coincides with the kernel of f - g. It suffices to show that the kernel of f - g satisfies the universal property for the equalizer of f and g.

Here is the diagram for the universal property of the kernel of f - g.



The universal property tells us that for any object  $Z \in \text{Obj}(\mathbb{C})$  and any morphism  $i: Z \to A$  with  $i \circ (f - g) = 0$  (i.e.  $i \circ f = i \circ g$ ), there exists a unique morphism  $\bar{\imath}: Z \to \ker(f - g)$  such that  $i = \bar{\imath} \circ \iota_{f-g}$ .

**Corollary 11.** Every pre-abelian category has all finite limits and colimits.

*Proof.* By Theorem ??, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 10. The colimit case is dual.

Recall from Section ?? the following definition.

**Definition 12** (zero morphism). Let C be a category with zero object 0. For any two objects  $A, B \in \text{Obj}(C)$ , the <u>zero morphism</u>  $0_{A,B}$  is the unique morphism  $A \to B$  which factors through 0.

$$A \xrightarrow{0_{A,B}} B$$

*Notation* 13. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of  $0_{AB}$ .

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism:  $f \circ 0 = 0$  and  $0 \circ q = 0$ .

**Lemma 14.** Every morphism  $f: A \to B$  in a pre-abelian catgory has a canonical decomposition

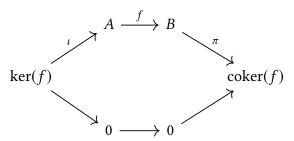
$$A \xrightarrow{p} \operatorname{coker}(\ker(f)) \xrightarrow{\bar{f}} \ker(\operatorname{coker}(f)) \xrightarrow{i} B$$
,

where p is an epimorphism (Definition ??) and i is a monomorphism (Definition ??).

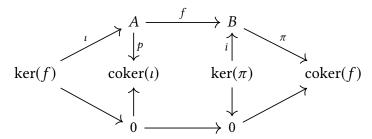
*Proof.* We start with a map

$$A \xrightarrow{f} B$$
.

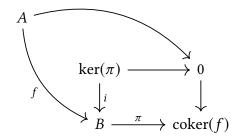
Since we are in a pre-abelian category, we are guaranteed that f has a kernel  $(\ker(f), \iota)$  and a cokernel  $(\operatorname{coker}(f), \pi)$ . From the universality squares it is immediate that  $f \circ \iota = 0$  and  $\pi \circ f = 0$  This tells us that the composition  $\pi \circ f \circ \iota = 0$ , so the following commutes.



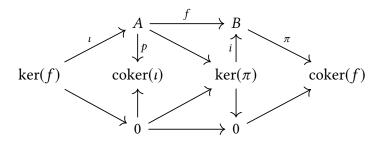
We know that  $\pi$  has a kernel (ker( $\pi$ ), i) and  $\iota$  has a cokernel (coker( $\iota$ ), p), so we can add their commutativity squares as well.



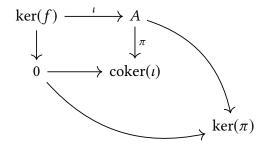
If we squint hard enough, we can see the following diagram.



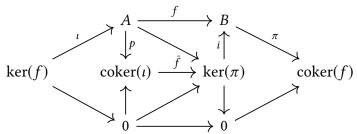
The outer square commutes because  $\pi \circ f = 0$ , so the universal property for  $\ker(\pi)$  gives us a unique morphism  $A \to \ker(\pi)$ . Let's add this to our diagram, along with a morphism  $0 \to \ker(\pi)$  which trivially keeps everything commutative.



Again, buried in the bowels of our new diagram, we find the following.



And again, the universal property of cokernels gives us a unique morphism  $\bar{f}$ :  $\operatorname{coker}(\iota) \to \ker(\pi)$ .



The fruit of our laborious construction is the following commuting square.

$$\begin{array}{ccc}
A & \xrightarrow{f} & B \\
\downarrow p & \downarrow i \\
\operatorname{coker}(\iota) & \xrightarrow{\bar{f}} & \ker(\pi)
\end{array}$$

We have seen (Lemma  $\ref{lem:initial}$ ) that i is mono, and (Lemma  $\ref{lem:initial}$ ) that p is epi.

Now we abuse terminology by calling  $\iota = \ker(f)$  and  $\pi = \operatorname{coker}(f)$ . Then we have the required decomposition.

*Note* 15. The abuse of notation above is ubiquitous in the literature.

# 1.3 Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

**Definition 16** (abelian category). A pre-abelian category C is <u>abelian</u> if every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

*Note* 17. The above piecemeal definition is equivalent to the following.

A category C is abelian if

- 1. it is Ab-enriched Definition 4, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
- 2. it admits finite coproducts, hence (by Lemma 3) biproducts and zero objects;
- 3. every morphism has a kernel and a cokernel;
- 4. every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

### Example 18.

- Let *R* be a ring. Then the categories *R*-Mod and Mod-*R* of left- and right *R*-modules respectively, are abelian.
- If A is abelian, then Ch(A), the category of chain complexes in A, is abelian.
- If I is a small category, then Fun(I, A) is abelian.
  - With I = BG for some group G, this gives the category of A-representations of G
  - With X a topological space and Open(X) its category of opens, taking  $I = Open^{op}$ ) tells us that the category of presheaves on X, denoted Psh(X), is abelian.

### **Example 19.** Here are two more tricky cases.

• Consider  $R = \mathbb{C}[x_1, x_2, ...]$  the ring of polynomials in countably many variables. Consider the full subcategory

$$(R\text{-Mod})^{\text{fin gen}} \hookrightarrow R\text{-Mod}.$$

This is additive (as a full subcategory), but does not have kernels in general. For example, consider the quotient

$$f: R woheadrightarrow R/(x_1, \ldots) \overset{\text{as a vector space}}{\simeq} \mathbb{C}.$$

Both R and  $R/(x_1, \ldots)$  are finitely generated R-modules. It may at first seem that there cannot be a kernel since the kernel of f is  $(x_1, \ldots)$ , which is not finitely generated. But this is not the kernel we are looking for—it is taken in the category of R-Mod, which satisfies a much stronger universal property than the hypothetical kernel in (R-Mod).

However, it turns out that there is no saving this construction. Suppose there was a kernel. It would have to be finitely generated, i.e. we could write

$$\ker f = R[g_1, \ldots, g_n]/(\cdots).$$

Each of the  $g_i$  are polynomials in (finitely many) variables  $x_i$ ; in particular, there is a highest i such that  $x_i$  appears in one of the  $g_i$ . But then the map

For the remainder of the section, let C be an abelian category.

**Lemma 20.** Let  $\mathcal{A}$  be an abelian category, and let  $f: A \to B$  be a morphism in  $\mathcal{A}$ . Then f is an isomorphism if and only if f is monic and epic.

*Proof.* The direction iso  $\Longrightarrow$  monic and epic is obvious. To see the other direction, suppose that f is monic and epic.

**Lemma 21.** In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.

*Note* 22. The above decomposition is unique up to unique isomorphism.

**Definition 23** (image of a morphism). Let  $f: A \to B$  be a morphism. The object ker(coker(f)) is called the image of f, and is denoted im(f).

### Lemma 24.

- 1. A morphism  $f: A \to B$  is mono iff for all  $Z \in \text{Obj}(C)$  and for all  $g: Z \to A$ ,  $f \circ g = 0$  implies g = 0.
- 2. A morphism  $f: A \to B$  is epi iff for all  $Z \in \text{Obj}(\mathbb{C})$  and for all  $g: B \to Z$ ,  $g \circ f = 0$  implies g = 0.

Proof.

1. First, suppose f is mono. Consider the following diagram.

$$Z \xrightarrow{g} A \xrightarrow{f} B$$

If the above diagram commutes, i.e. if  $f \circ q = 0$ , then q = 0, so  $1 \implies 2$ .

Now suppose that for all  $Z \in \text{Obj}(C)$  and all  $g: Z \to A$ ,  $f \circ g = 0$  implies g = 0.

Let  $g, g': Z \to A$ , and suppose that  $f \circ g = f \circ g'$ . Then  $f \circ (g - g') = 0$ . But that means that g - g' = 0, i.e. g = g'. Thus, g = g'. Thus, g = g'.

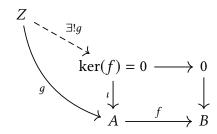
2. Dual to the proof above.

**Lemma 25.** Let  $f: A \to B$ . We have the following.

- 1. The morphism f is mono iff ker(f) = 0
- 2. The morphism f is epi iff coker(f) = 0.

Proof.

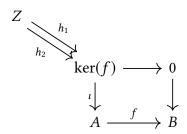
1. We first show that if  $\ker(f) = 0$ , then f is mono. Suppose  $\ker(f) = 0$ . By the universal property of kernels, we know that for any  $Z \in \mathrm{Obj}(C)$  and any  $g \colon Z \to A$  with  $f \circ g = 0$  there exists a unique map  $\bar{g} \colon Z \to \ker(f)$  such that  $g = \iota \circ \bar{g}$ .



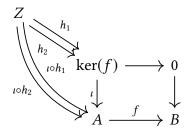
But then g factors through the zero object, so we must have g = 0. This shows that  $f \circ g = 0 \implies g = 0$ , and by Lemma 24 f must be mono.

Next, we show that if f is mono, then ker(f) = 0. To do this, it suffices to show that ker(f) is final, i.e. that there exists a unique morphism from every object to ker(f).

Since  $\operatorname{Hom}_{\mathbb{C}}(Z, \ker(f))$  has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms  $h_1$  and  $h_2$ .



Our aim is to show that  $h_1 = h_2$ . To this end, compose each with  $\iota$ .



Since  $f \circ \iota = 0$ , we have  $f \circ (\iota \circ h_1) = 0$  and  $f \circ (\iota \circ h_2) = 0$ . But since f is mono, by Lemma 24, we must have  $\iota \circ h_1 = 0 = \iota \circ h_2$ . But by Lemma ??,  $\iota$  is mono, so again we have

$$h_1=h_2=0,$$

and we are done.

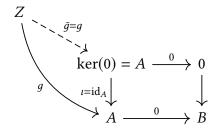
2. Dual to the proof above.

**Lemma 26.** We have the following.

- 1. The kernel of the zero morphism  $0: A \to B$  is the pair  $(A, id_A)$ .
- 2. The cokernel of the zero morphism  $0: A \to B$  is the pair  $(B, id_B)$ .

Proof.

1. We need only verify that the universal property is satisfied. That is, for any object  $Z \in \text{Obj}(\mathbb{C})$  and any morphism  $h \colon Z \to A$  such that  $0 \circ g = 0$ , there exists a unique morphism  $\bar{q} \colon Z \to \ker(0)$  such that the following diagram commutes.



But this is pretty trivial:  $\bar{q} = q$ .

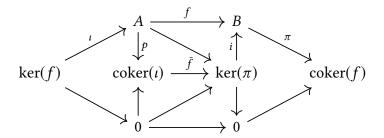
2. Dual to above.

**Theorem 27.** All abelian categories are binormal (Definition ??). That is to say:

- 1. all monomorphisms are kernels
- 2. all epimorphisms are cokernels.

### Proof.

1. Consider the following diagram taken Lemma 14, which shows the canonical factorization of any morphism f.



By definition of a pre-abelian category, we know that  $\bar{f}$  is an isomorphism.

*Note* 28. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial. See e.g. [?] for details.

**Definition 29** (subobject, quotient object, subquotient object). Let  $Y \in Obj(C)$ .

- 1. A <u>subobject</u> of *Y* is an object  $X \in \text{Obj}(C)$  together with a monomorphism  $i: X \hookrightarrow Y$ . If *X* is a subobject of *Y* we will write  $X \subseteq Y$ .
- 2. A quotient object of Y is an object Z together with an epimorphism  $p: Y \rightarrow Z$ .
- 3. A subquotient object of *Y* is a quotient object of a subobject of *Y*.

**Definition 30** (quotient). Let  $X \subseteq Y$ , i.e. let there exist a monomorphism  $f: X \hookrightarrow Y$ . The quotient Y/X is the cokernel (coker(f),  $\pi_f$ ).

**Example 31.** Let V be a vector space,  $W \subseteq V$  a subspace. Then we have the canonical inclusion map  $\iota \colon W \hookrightarrow V$ , so W is a subobject of V in the sense of Definition 29.

According to Definition 30, the quotient V/W is the cokernel ( $\operatorname{coker}(\iota), \pi_{\iota}$ ) of  $\iota$ . We saw in Example ?? that the cokernel of  $\iota$  was  $V/\operatorname{im}(\iota)$ . However,  $\operatorname{im}(\iota)$  is exactly W! So the categorical notion of the quotient V/W agrees with the linear algebra notion.

**Definition 32** (k-linear category). An abelian category Definition 16 C is k-linear if for all  $A, B \in \text{Obj}(C)$  the hom-set  $\text{Hom}_C(A, B)$  has the structure of a k-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is k-linear.

**Example 33.** The category  $Vect_k$  is k-linear.

**Definition 34** (k-linear functor). let C and D be two k-linear categories, and  $\mathcal{F} \colon C \to D$  a functor. Suppose that for all objects C,  $D \in Obj(C)$  all morphisms f,  $g \colon C \to D$ , and all  $\alpha$ ,  $\beta \in k$ , we have

$$\mathfrak{F}(\alpha f + \beta g) = \alpha \mathfrak{F}(f) + \beta \mathfrak{G}(g).$$

Then we say that  $\mathcal{F}$  is k-linear.

### 1.4 Exact sequences

**Definition 35** (exact sequence). A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree i if the image (Definition 23) of  $f_{i-1}$  is equal to the kernel (Definition ??) of  $f_i$ . A sequence is exact if it is exact in every degree.

Lemma 36. If a sequence

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree *i*, then  $f_i \circ f_{i-1} = 0$ .

**Definition 37** (short exact sequence). A <u>short exact sequence</u> is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

**Definition 38** (exact functor). Let C, D be abelian categories,  $\mathcal{F}: C \to D$  a functor. We say that  $\mathcal{F}$  is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

### 1.5 Length of objects

**Definition 39** (simple object). A nonzero object  $X \in \text{Obj}(C)$  is called <u>simple</u> if 0 and X are its only subobjects.

**Example 40.** In Vect<sub>k</sub>, the only simple object (up to isomorphism) is k, taken as a one-dimensional vector space over itself.

**Definition 41** (semisimple object). An object  $Y \in \text{Obj}(C)$  is <u>semisimple</u> if it is isomorphic to a direct sum of simple objects.

**Example 42.** In Vect $_k$ , all finite-dimensional vector spaces are semisimple.

**Definition 43** (semisimple category). An abelian category C is <u>semisimple</u> if every object of C is semisimple.

**Example 44.** The category FinVect $_k$  is semisimple.

**Definition 45** (Jordan-Hölder series). Let  $X \in \text{Obj}(C)$ . A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of X such that  $X_i/X_{i-1}$  is simple for all i is called a <u>Jordan Hölder series</u> for X. The integer n is called the length of the series  $X_i$ .

The importance of Jordan-Hölder series is the following.

**Theorem 46** (Jordan-Hölder). Let  $X_i$  and  $Y_i$  be two Jordan-Hölder series for some object  $X \in \text{Obj}(\mathbb{C})$ . Then the length of  $X_i$  is equal to the length of  $Y_i$ , and the objects  $Y_i/Y_{i-1}$  are a reordering of  $X_i/X_{i-1}$ .

Proof. See [?], pg. 5, Theorem 1.5.4.

**Definition 47** (length). The <u>length</u> of an object X is defined to be the length of any of its Jordan-Hölder series. This is well-defined by Theorem 46.