# 1 Abelian categories

This section draws heavily from [?].

## 1.1 Additive categories

Recall that Ab is the category of abelian groups.

**Definition 1** (Ab-enriched category)**.** A category C is Ab-enriched if

1. for all objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition
$$\circ \colon \mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{C}(A, C)$$
is additive in each slot: for any $f_1, f_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, we must have
$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$
and similarly in the second slot.

*Note* 2. In any Ab-enriched category, every hom-set has at least one element—the identity element of the hom-set taken as an abelian group.

**Lemma 3.** Let C be an Ab-enriched category with finite products. Then the product $X_1 \times \cdots \times X_n$ satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

*Proof.* First, we show that initial and terminal objects coincide. Let $\emptyset$ be initial and $*$ terminal in C.

Since $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on $\emptyset$ *and* the identity element of $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, $\mathrm{id}_* = 0$ in $\mathrm{Hom}_{\mathsf{C}}(*, *)$.

Since $\emptyset$ is initial and $*$ is final, $\mathrm{Hom}_{\mathsf{C}}(\emptyset, *)$ has exactly one element $f$. By Note 2, we are guaranteed that $\mathrm{Hom}_{\mathsf{C}}(*, \emptyset)$ has at least one element, $g$. Then $g \circ f = 0 = \mathrm{id}$ and $f \circ g = 0 = \mathrm{id}$, so $f \colon \emptyset \to *$ is an isomorphism. Thus, initial and terminal objects coincide, so C has a zero object 0.

We prove the lemma for binary products. The general case is preciesly the same.

For each $X, Y \in \mathcal{C}$, we can define $i_X \colon X \to X \times Y$ using the universal property for products as follows.

$$\begin{tikzcd}
 & X \arrow[dl, "\mathrm{id}"'] \arrow[d, dashed, "i_X"] \arrow[dr, "0"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Similarly, get $i_Y \colon Y \to X \times Y$.

The claim is that $X \times Y$, together with $i_X$ and $i_Y$, satisfies the universal property for coproducts; that is, that we can find a unique map $\phi$ making the below diagram commute.

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, dashed, "\phi"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

In fact, $\phi = f \circ \pi_X + g \circ \pi_Y$ satisfies the universal property. It is clear that this makes the diagram commute since $\pi_X \circ i_X = \mathrm{id}$, $\pi_X \circ i_Y = 0$, and similarly for $Y$.

Note that, by the universal property for products, there is a unique downward map. But both $\mathrm{id}_{X \times Y}$ and $i_X \circ \pi_X + i_Y \circ \pi_Y$ make it commute, so they must be equal.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_Y"'] \arrow[d] \arrow[dr, "\pi_X"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Now we are ready to show that $\phi$ is unique. Suppose we have another candidate $\psi$ which also makes the above diagram commute. Then

$$\psi = \psi \circ \mathrm{id}_{X \times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

□

**Definition 4** (additive category)**.** A category $\mathsf{C}$ is additive if it has products and is $\mathsf{Ab}$-enriched.

**Example 5.** The category $\mathsf{Ab}$ of Abelian groups is an additive category. We have already seen that it has the direct sum $\oplus$ as biproduct. Given any two abelian groups $A$ and $B$ and morphisms $f, g \colon A \to B$, we can define the sum $f + g$ via

$$(f + g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$

Then for another abelian group $C$ and a morphism $h \colon B \to C$, we have

$$[h \circ (f + g)](a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g](a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g,$$

and similarly in the other slot.

**Example 6.** The category $\mathsf{Vect}_k$ is additive. Since vector spaces are in particular abelian groups under addition, it is naturally $\mathsf{Ab}$-enriched,

**Definition 7** (additive functor)**.** Let $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ be a functor between additive categories. We say that $\mathcal{F}$ is additive if for each $X, Y \in \mathrm{Obj}(\mathsf{C})$ the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of abelian groups.

**Lemma 8.** For any additive functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$, there exists a natural isomorphism

$$\Phi\colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

*Proof.* The commutativity of the following diagram is immediate.

$$\begin{tikzcd}
\mathcal{F}(X \oplus Y) \arrow[r, "\mathcal{F}(f \oplus g)"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{F}(X' \oplus Y') \arrow[d, "\Phi_{X',Y'}"] \\
\mathcal{F}(X) \oplus \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f) \oplus \mathcal{F}(g)"] & \mathcal{F}(X') \oplus \mathcal{F}(Y')
\end{tikzcd}$$

The $(X, Y)$-component $\Phi_{X,Y}$ is an isomorphism because ☐

## 1.2 Pre-abelian categories

**Definition 9** (pre-abelian category)**.** A category $\mathsf{C}$ is pre-abelian if it is additive and every morphism has a kernel (Definition **??**) and a cokernel (Definition **??**).

**Lemma 10.** Pre-abelian categories have equalizers (Definition **??**).

*Proof.* We show that in an pre-abelian category, the equalizer of $f$ and $g$ coincides with the kernel of $f - g$. It suffices to show that the kernel of $f - g$ satisfies the universal property for the equalizer of $f$ and $g$.

Here is the diagram for the universal property of the kernel of $f - g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\bar{\imath}"] \arrow[drr, bend left] \arrow[ddr, bend right, "i"'] & & \\
& \ker(f - g) \arrow[r] \arrow[d, "\iota_{f-g}"'] & 0 \arrow[d] \\
& A \arrow[r, "f-g"] & B
\end{tikzcd}$$

The universal property tells us that for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $i\colon Z \to A$ with $i \circ (f - g) = 0$ (i.e. $i \circ f = i \circ g$), there exists a unique morphism $\bar{\imath}\colon Z \to \ker(f - g)$ such that $i = \bar{\imath} \circ \iota_{f-g}$. ☐

**Corollary 11.** Every pre-abelian category has all finite limits and colimits.

*Proof.* By Theorem **??**, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 10. The colimit case is dual. ☐

Recall from Section **??** the following definition.

**Definition 12** (zero morphism)**.** Let C be a category with zero object 0. For any two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B \end{tikzcd}$$

*Notation* 13. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of $0_{AB}$.

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism: $f \circ 0 = 0$ and $0 \circ g = 0$.

**Lemma 14.** *Every morphism $f\colon A \to B$ in a pre-abelian catgory has a canonical decomposition*

$$\begin{tikzcd} A \arrow[r, "p"] & \mathrm{coker}(\ker(f)) \arrow[r, "\bar{f}"] & \ker(\mathrm{coker}(f)) \arrow[r, "i"] & B \end{tikzcd},$$

*where $p$ is an epimorphism (Definition* **??***) and $i$ is a monomorphism (Definition* **??***).*

*Proof.* We start with a map

$$\begin{tikzcd} A \arrow[r, "f"] & B \end{tikzcd}.$$

Since we are in a pre-abelian category, we are guaranteed that $f$ has a kernel $(\ker(f), \iota)$ and a cokernel $(\mathrm{coker}(f), \pi)$. From the universality squares it is immediate that $f \circ \iota = 0$ and $\pi \circ f = 0$ This tells us that the composition $\pi \circ f \circ \iota = 0$, so the following commutes.

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & & & \mathrm{coker}(f) \\ & 0 \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

We know that $\pi$ has a kernel $(\ker(\pi), i)$ and $\iota$ has a cokernel $(\mathrm{coker}(\iota), p)$, so we can add their commutativity squares as well.

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[d, "p"] & & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & \mathrm{coker}(\iota) & & \ker(\pi) \arrow[u, "i"] \arrow[d] & \mathrm{coker}(f) \\ & 0 \arrow[u] \arrow[rr] & & 0 \arrow[ur] & \end{tikzcd}$$

If we squint hard enough, we can see the following diagram.

$$\begin{tikzcd} A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\ & \ker(\pi) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\ & B \arrow[r, "\pi"] & \mathrm{coker}(f) \end{tikzcd}$$

The outer square commutes because $\pi \circ f = 0$, so the universal property for $\ker(\pi)$ gives us a unique morphism $A \to \ker(\pi)$. Let's add this to our diagram, along with a morphism $0 \to \ker(\pi)$ which trivially keeps everything commutative.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

Again, buried in the bowels of our new diagram, we find the following.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[d] & A \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & \operatorname{coker}(\iota) & \\
& & \ker(\pi)
\end{tikzcd}$$

And again, the universal property of cokernels gives us a unique morphism $\bar{f}\colon \operatorname{coker}(\iota) \to \ker(\pi)$.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

The fruit of our laborious construction is the following commuting square.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "i"] \\
\operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi)
\end{tikzcd}$$

We have seen (Lemma **??**) that $i$ is mono, and (Lemma **??**) that $p$ is epi.

Now we abuse terminology by calling $\iota = \ker(f)$ and $\pi = \operatorname{coker}(f)$. Then we have the required decomposition. □

*Note* 15. The abuse of notation above is ubiquitous in the literature.

## 1.3 Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

**Definition 16** (abelian category)**.** A pre-abelian category C is abelian if every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

*Note* 17. The above piecemeal definition is equivalent to the following.

A category C is *abelian* if

1. it is Ab-enriched Definition 4, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
2. it admits finite coproducts, hence (by Lemma 3) biproducts and zero objects;
3. every morphism has a kernel and a cokernel;
4. every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

**Example 18.**

- Let $R$ be a ring. Then the categories $R$-Mod and Mod-$R$ of left- and right $R$-modules respectively, are abelian.
- If A is abelian, then Ch(A), the category of chain complexes in A, is abelian.
- If $I$ is a small category, then $\mathrm{Fun}(I, \mathsf{A})$ is abelian.
  - With $I = BG$ for some group $G$, this gives the category of A-representations of $G$
  - With $X$ a topological space and $\mathrm{Open}(X)$ its category of opens, taking $I = \mathrm{Open}^{\mathrm{op}}$) tells us that the category of presheaves on $X$, denoted $\mathrm{Psh}(X)$, is abelian.

**Example 19.** Here are two more tricky cases.

- Consider $R = \mathbb{C}[x_1, x_2, \ldots]$ the ring of polynomials in countably many variables. Consider the full subcategory
$$(R\text{-Mod})^{\text{fin gen}} \hookrightarrow R\text{-Mod}.$$
  This is additive (as a full subcategory), but does not have kernels in general. For example, consider the quotient
$$f \colon R \twoheadrightarrow R/(x_1, \ldots) \overset{\text{as a vector space}}{\simeq} \mathbb{C}.$$
  Both $R$ and $R/(x_1, \ldots)$ are finitely generated $R$-modules. It may at first seem that there cannot be a kernel since the kernel of $f$ is $(x_1, \ldots)$, which is not finitely generated. But this is not the kernel we are looking for—it is taken in the category of $R$-Mod, which satisfies a much stronger universal property than the hypothetical kernel in ($R$-Mod).

  However, it turns out that there is no saving this construction. Suppose there was a kernel. It would have to be finitely generated, i.e. we could write
$$\ker f = R[g_1, \ldots, g_n]/(\cdots).$$
  Each of the $g_i$ are polynomials in (finitely many) variables $x_i$; in particular, there is a highest $i$ such that $x_i$ appears in one of the $g_i$. But then the map

For the remainder of the section, let C be an abelian category.

**Lemma 20.** Let $\mathcal{A}$ be an abelian category, and let $f \colon A \to B$ be a morphism in $\mathcal{A}$. Then $f$ is an isomorphism if and only if $f$ is monic and epic.

*Proof.* The direction iso $\implies$ monic and epic is obvious. To see the other direction, suppose that $f$ is monic and epic. $\square$

**Lemma 21.** In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.

*Proof.* $\square$

*Note* 22. The above decomposition is unique up to unique isomorphism.

**Definition 23** (image of a morphism)**.** Let $f: A \to B$ be a morphism. The object $\ker(\operatorname{coker}(f))$ is called the image of $f$, and is denoted $\operatorname{im}(f)$.

**Lemma 24.**

1. A morphism $f: A \to B$ is mono iff for all $Z \in \mathrm{Obj}(\mathrm{C})$ and for all $g: Z \to A$, $f \circ g = 0$ implies $g = 0$.
2. A morphism $f: A \to B$ is epi iff for all $Z \in \mathrm{Obj}(\mathrm{C})$ and for all $g: B \to Z$, $g \circ f = 0$ implies $g = 0$.

*Proof.*

1. First, suppose $f$ is mono. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

   If the above diagram commutes, i.e. if $f \circ g = 0$, then $g = 0$, so $1 \implies 2$.

   Now suppose that for all $Z \in \mathrm{Obj}(\mathrm{C})$ and all $g: Z \to A$, $f \circ g = 0$ implies $g = 0$.

   Let $g, g': Z \to A$, and suppose that $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0$. But that means that $g - g' = 0$, i.e. $g = g'$. Thus, $2 \implies 1$.

2. Dual to the proof above.

$\square$

**Lemma 25.** Let $f: A \to B$. We have the following.

1. The morphism $f$ is mono iff $\ker(f) = 0$
2. The morphism $f$ is epi iff $\operatorname{coker}(f) = 0$.

*Proof.*

1. We first show that if $\ker(f) = 0$, then $f$ is mono. Suppose $\ker(f) = 0$. By the universal property of kernels, we know that for any $Z \in \mathrm{Obj}(\mathrm{C})$ and any $g: Z \to A$ with $f \circ g = 0$ there exists a unique map $\bar{g}: Z \to \ker(f)$ such that $g = \iota \circ \bar{g}$.

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists! \bar{g}"] \arrow[ddr, bend right, "g"'] & & \\ & \ker(f) = 0 \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

But then $g$ factors through the zero object, so we must have $g = 0$. This shows that $f \circ g = 0 \implies g = 0$, and by Lemma 24 $f$ must be mono.

Next, we show that if $f$ is mono, then $\ker(f) = 0$. To do this, it suffices to show that $\ker(f)$ is final, i.e. that there exists a unique morphism from every object to $\ker(f)$.

Since $\mathrm{Hom}_{\mathrm{C}}(Z, \ker(f))$ has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms $h_1$ and $h_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Our aim is to show that $h_1 = h_2$. To this end, compose each with $\iota$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] \arrow[ddr, bend right, shift left, "\iota\circ h_1"] \arrow[ddr, bend right, shift right, "\iota\circ h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Since $f \circ \iota = 0$, we have $f \circ (\iota \circ h_1) = 0$ and $f \circ (\iota \circ h_2) = 0$. But since $f$ is mono, by Lemma 24, we must have $\iota \circ h_1 = 0 = \iota \circ h_2$. But by Lemma ??, $\iota$ is mono, so again we have

$$h_1 = h_2 = 0,$$

and we are done.

2. Dual to the proof above.

$\square$

**Lemma 26.** We have the following.

1. The kernel of the zero morphism $0 : A \to B$ is the pair $(A, \mathrm{id}_A)$.
2. The cokernel of the zero morphism $0 : A \to B$ is the pair $(B, \mathrm{id}_B)$.

*Proof.*

1. We need only verify that the universal property is satisfied. That is, for any object $Z \in \mathrm{Obj}(\mathrm{C})$ and any morphism $h \colon Z \to A$ such that $0 \circ g = 0$, there exists a unique morphism $\bar{g} \colon Z \to \ker(0)$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\bar{g}=g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(0) = A \arrow[r, "0"] \arrow[d, "\iota=\mathrm{id}_A"'] & 0 \arrow[d] \\
& A \arrow[r, "0"] & B
\end{tikzcd}$$

But this is pretty trivial: $\bar{g} = g$.

2. Dual to above.

□

**Theorem 27.** All abelian categories are binormal (Definition ??). That is to say:

1. all monomorphisms are kernels
2. all epimorphisms are cokernels.

*Proof.*

1. Consider the following diagram taken Lemma 14, which shows the canonical factorization of any morphism $f$.

$$\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

By definition of a pre-abelian category, we know that $\bar{f}$ is an isomorphism.

□

*Note* 28. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial. See e.g. [?] for details.

**Definition 29** (subobject, quotient object, subquotient object)**.** Let $Y \in \mathrm{Obj}(\mathsf{C})$.

1. A <u>subobject</u> of $Y$ is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a monomorphism $i\colon X \hookrightarrow Y$. If $X$ is a subobject of $Y$ we will write $X \subseteq Y$.
2. A <u>quotient object</u> of $Y$ is an object $Z$ together with an epimorphism $p\colon Y \twoheadrightarrow Z$.
3. A <u>subquotient object</u> of $Y$ is a quotient object of a subobject of $Y$.

**Definition 30** (quotient)**.** Let $X \subseteq Y$, i.e. let there exist a monomorphism $f\colon X \hookrightarrow Y$. The <u>quotient</u> $Y/X$ is the cokernel $(\operatorname{coker}(f), \pi_f)$.

**Example 31.** Let $V$ be a vector space, $W \subseteq V$ a subspace. Then we have the canonical inclusion map $\iota\colon W \hookrightarrow V$, so $W$ is a subobject of $V$ in the sense of Definition 29.

According to Definition 30, the quotient $V/W$ is the cokernel $(\operatorname{coker}(\iota), \pi_\iota)$ of $\iota$. We saw in Example **??** that the cokernel of $\iota$ was $V/\mathrm{im}(\iota)$. However, $\mathrm{im}(\iota)$ is exactly $W$! So the categorical notion of the quotient $V/W$ agrees with the linear algebra notion.

**Definition 32** ($k$-linear category)**.** An abelian category Definition 16 $\mathsf{C}$ is <u>$k$-linear</u> if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of a $k$-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is $k$-linear.

**Example 33.** The category $\mathsf{Vect}_k$ is $k$-linear.

**Definition 34** ($k$-linear functor)**.** let C and D be two $k$-linear categories, and $\mathcal{F}: \mathsf{C} \to \mathsf{D}$ a functor. Suppose that for all objects $C, D \in \mathrm{Obj}(\mathsf{C})$ all morphisms $f, g: C \to D$, and all $\alpha$, $\beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{G}(g).$$

Then we say that $\mathcal{F}$ is $k$-linear.

## 1.4 Exact sequences

**Definition 35** (exact sequence)**.** A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree $i$ if the image (Definition 23) of $f_{i-1}$ is equal to the kernel (Definition ??) of $f_i$. A sequence is exact if it is exact in every degree.

**Lemma 36.** If a sequence

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree $i$, then $f_i \circ f_{i-1} = 0$.

**Definition 37** (short exact sequence)**.** A short exact sequence is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

**Definition 38** (exact functor)**.** Let C, D be abelian categories, $\mathcal{F}: \mathsf{C} \to \mathsf{D}$ a functor. We say that $\mathcal{F}$ is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

## 1.5 Length of objects

**Definition 39** (simple object)**.** A nonzero object $X \in \mathrm{Obj}(\mathsf{C})$ is called simple if $0$ and $X$ are its only subobjects.

**Example 40.** In $\mathsf{Vect}_k$, the only simple object (up to isomorphism) is $k$, taken as a one-dimensional vector space over itself.

**Definition 41** (semisimple object)**.** An object $Y \in \mathrm{Obj}(\mathsf{C})$ is semisimple if it is isomorphic to a dirct sum of simple objects.

**Example 42.** In $\mathsf{Vect}_k$, all finite-dimensional vector spaces are semisimple.

**Definition 43** (semisimple category)**.** An abelian category C is semisimple if every object of C is semisimple.

**Example 44.** The category $\mathsf{FinVect}_k$ is semisimple.

**Definition 45** (Jordan-Hölder series)**.** Let $X \in \mathrm{Obj}(\mathrm{C})$. A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of $X$ such that $X_i/X_{i-1}$ is simple for all $i$ is called a Jordan Hölder series for $X$. The integer $n$ is called the length of the series $X_i$.

The importance of Jordan-Hölder series is the following.

**Theorem 46** (Jordan-Hölder)**.** Let $X_i$ and $Y_i$ be two Jordan-Hölder series for some object $X \in \mathrm{Obj}(\mathrm{C})$. Then the length of $X_i$ is equal to the length of $Y_i$, and the objects $Y_i/Y_{i-1}$ are a reordering of $X_i/X_{i-1}$.

*Proof.* See [**?**], pg. 5, Theorem 1.5.4. □

**Definition 47** (length)**.** The length of an object $X$ is defined to be the length of any of its Jordan-Hölder series. This is well-defined by Theorem 46.