A Gentle Introduction to Category Theory
(Work in progress)

Angus Rush

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[r] \arrow[d] & \mathrm{Hom}(B, A) \arrow[d] \\
\mathcal{F}(A) \arrow[r] & \mathcal{F}(B)
\end{tikzcd}$$

Contents

1. Introduction

This document is a collection of notes on category theory that I took as I was learning it. My goal is to have somewhere I can write the things that I've learned so I can refer to them later, with the added benefit that someone else may find this useful as well.

I keep it up to date for another reason: I think that there's some pedagogical value in the writings of beginners. The proofs included here are not here because they're the most elegant or the cleanest; instead, they're here because they're the proofs that finally made me, a novice struggling with category theory for the first time, understand the material.

However, things written by beginners come with an obvious caveat: there are undoubtedly many mistakes.

2. Basic category theory

2.1. Categories

2.1.1. What is a category?

It is surprisingly hard to define mathematics, but people tend to agree that mathematics has something to do with studying mathematical structures.[1] Mathematicians study lots of different mathematical structures, each of which is good for different things. Groups are good for talking about symmetries, for instance, and topological spaces are good for talking about geometrical objects.

Categories are just another type of mathematical structure, but they have a different, more abstract feel to them than groups or topological spaces. One reason for this is that they are inherently meta-mathematical: they are good for talking about other mathematical objects and the relationships between them.

For this reason, category theory has developed a reputation for being abstract and difficult to learn, which is mostly undeserved. Categories are nothing special; they are mathematical objects, just like groups or topological spaces. They are simply good for different things.

Unfortunately, there are many different conventions regarding how to typeset category theory. I will do my best to stick to the conventions used at the nLab ([20]).

Definition 1 (category)**.** A category C consists of the following pieces of data.

- A class[2] $\mathrm{Obj}(\mathsf{C})$ of *objects*.
- For every two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, a class $\mathrm{Hom}(A, B)$ of *morphisms*.
- For every objects X, Y, Z, a map

$$\mathrm{Hom}(Y, Z) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Z); \qquad (f, g) \mapsto f \circ g$$

  called the *composition* of f and g, which satisfies the following properties.

  1. This composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
  2. For every $A \in \mathrm{Obj}(\mathsf{C})$, there is at least one morphism id_A, called the *identity morphism* which functions as both a left and right identity with respect to the composition of morphisms, i.e.

$$f \circ \mathrm{id}_A = f \qquad \text{amd} \qquad \mathrm{id}_A \circ g = g.$$

[1] Whatever they are.

[2] Note that if we choose to work within ZFCU (see appendix REF), one gets away with saying a *set* of objects. However, this doesn't really make much of a difference to the development of the theory. We will mostly avoid issues of size.

If ever it is potentially unclear which category we are talking about, we will add a subscript to Hom, writing for example $\mathrm{Hom}_{\mathsf{C}}(A, B)$ instead of $\mathrm{Hom}(A, B)$.

Notation 2. Following Aluffi ([10]), we will use a sans serif font to denote categories. For example C, Set.

One often thinks of the objects in a category as 'generalized sets,' and of morphisms as 'generalized functions.' It is therefore often useful to use functional notation $f: A \to B$ to describe morphisms. For example, the following two notations are equivalent:

$$f \in \mathrm{Hom}(A, B) \iff f: A \to B.$$

Here is something to be aware of: the identity morphism $\mathrm{id}_A: A \to A$ is often simply denoted by A. This is actually a good notation, and we will use it freely in later chapters. However, we will avoid it in earlier chapters since it is potentially confusing whether we are talking about A the object or A the morphism.

2.1.2. Some examples of categories

The idea of a category is very abstract, but not abstruse. The easiest way to convince oneself of this is to see some examples. The most common type of category is given by 'mathematical structures and structure-preserving maps between them,' so many of our examples will be in this vein.

Example 3 (The category of sets)**.** The prototypical category is Set, the category whose objects are sets and whose morphisms are functions between them.

In order to check that Set is indeed a category, we need to check that the axioms are satisfied. In fact, everything is more or less clear by definition.

Our category Set first needs a class of objects. Although there is no set of all sets, there certainly is a class of small sets. Therefore, we define

$$\mathrm{Obj}(\mathsf{Set}) = \text{all sets}.$$

The next thing Set needs is, for each pair of objects (i.e. sets) A and B, a class of morphisms $\mathrm{Hom}_{\mathsf{Set}}(A, B)$ between them. The morphisms between the objects in Set will consist of all set-functions between. That is,

$$\mathrm{Hom}_{\mathsf{Set}}(A, B) = \{f: A \to B \mid f \text{ is a function}\}.$$

Next one has to check that functions satisfy the necessary axioms. Given a function $f: A \to B$ and a function $g: B \to C$, we can compose them to get a function $g \circ f: A \to C$. Thus, functions can indeed be composed in the necessary way.

The composition of morphisms is associative, since composition of functions is.

Furthermore, every set A has an identity function

$$\mathrm{id}_A: A \to A; \qquad a \mapsto a.$$

which maps every element to itself.

Thus, Set is a category! That's all there is to it!

Example 4 (category with one object)**.** There is an important category called 1, which has one object and one morphism.

- The set of objects Obj(1) is the singleton $\{*\}$.
- The only morphism is the identity morphism $\mathrm{id}_* \colon * \to *$.

Here is a picture of the category 1.

$$
\begin{tikzcd}
* \arrow[loop right, "\mathrm{id}_*"]
\end{tikzcd}
$$

Example 5 (poset category)**.** Let $(P, \leq)$ be a poset. One can define from this a category $\mathcal{P}(P)$, whose objects are $\mathrm{Obj}(\mathcal{P}) = P$ and whose morphisms are

$$
\mathrm{Hom}_{\mathcal{P}}(A, B) = \begin{cases} \{*\}, & A \leq B \\ \emptyset, & \text{otherwise.} \end{cases}
$$

This immediately furnishes us with many examples of categories. For example:

- Any set
$$[n] = \{0, 1, \ldots, n\}$$
is a poset with $\leq$ inherited from the integers. We can draw these categories as follows (omitting identity arrows)

$$
[0]\colon\ 0\ , \qquad [1]\colon\ 0 \longrightarrow 1\ , \qquad [2]\colon
\begin{tikzcd}
& 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr] & & 2
\end{tikzcd}\ , \qquad [3]\colon
\begin{tikzcd}
0 \arrow[r] \arrow[d] \arrow[dr] & 1 \arrow[d] \arrow[dl] \\
2 \arrow[r] & 3
\end{tikzcd}
$$

- Any power set is a poset. The poset category of the power set of $\{0, 1\}$, denoted $\mathcal{P}(\{0, 1\})$ by mild abuse of notation, can be drawn as follows.

$$
\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] \arrow[dr] & \{0\} \arrow[d] \\
\{1\} \arrow[r] & \{0, 1\}
\end{tikzcd}
$$

Categories capture the essence of 'things and composable maps between them.' There are many mathematical structures which naturally come with a notion of map between them, usually called homomorphisms. These structure preserving maps are almost always composable, which means it is often natural to imagine such objects as living in their own categories.

Example 6. If you know what the following mathematical objects are, it is nearly effortless to check that they are categories.

- Grp, whose objects are groups and whose morphisms are group homomorphisms.
- Ab, whose objects are abelian groups and whose morphisms are group homomorphisms.
- Ring, whose objects are rings and whose morphisms are ring homomorphisms.
- R-Mod, whose objects are modules over a ring R and whose morphisms are module homomorphisms.

- Vect_k, whose objects are vector spaces over a field k and whose morphisms are linear maps.
- $\mathsf{FinVect}_k$, whose objects are finite-dimensional vector spaces over a field k and whose morphisms are linear maps.
- k-Alg, whose objects are algebras over a field k and whose morphisms are algebra homomorphisms.

Example 7. In addition to algebraic structures, categories help to talk about geometrical structures. The following are also categories.

- Top, whose objects are topological spaces and whose morphisms are continuous maps.
- Met, whose objects are metric spaces and whose morphisms are metric maps.
- Man^p, whose objects are manifolds of class C^p and whose morphisms are p-times differentiable functions.
- $\mathsf{SmoothMfd}$, whose objects are C^∞ manifolds and whose morphisms are smooth functions.

Example 8. Here is a slightly whimsical example of a category, which is different in nature to the other categories we've looked at before. This category is important when studying group representations.

Let G be a group. We are going to create a category BG which behaves like this group.

- Our category BG has only one object, called $*$.
- The set $\mathrm{Hom}_{BG}(*, *)$ is equal to the underlying set of the group G, and for $f, g \in \mathrm{Hom}_{BG}(*, *)$, the compositions $f \circ g = f \cdot g$, where $\cdot$ is the group operation in G. The identity $e \in G$ is the identity morphism id_* on $*$.

$$\begin{tikzcd} * \arrow["g", loop above] \arrow["f", loop left] \arrow["e=\mathrm{id}_*", loop right] \arrow["g\circ f", loop below] \end{tikzcd}$$

Example 9. Let X be a topological space. We can define a category $\pi_{\leq q}(X)$ whose objects are the points of X, and whose morphisms $x \longrightarrow y$ are homotopy classes of paths $x \longrightarrow y$.

Commutative diagrams

It is often helpful to visualize objects and morphisms in categories as as graphs[3] with the objects as vertices and the morphisms as edges. This makes it possible to reason about categories using graph-like diagrams. Scroll to almost any later page in this document, and you

[3]Strictly speaking, they are *multidigraphs,* i.e. directed graphs which are allowed to have more than one edge between any given pair of vertices.

will likely see a diagram or two which resembles this one (copied directly from Definition 182).

$$\begin{tikzcd}
V \arrow[drr, bend left, "t"] \arrow[dr, dashed, "\exists! h"] \arrow[ddr, bend right, "s"'] & & \\
& U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Category theorists frequently employ these diagrams, called *commutative diagrams,*[4] to make the ideas beind a proof more clear. Many results in category theory are most easily proved by drawing such a diagram, and then pointing at edges and vertices while repeating the phrase 'and then this goes here' and furrowing one's brow. This sort of proof is called a 'diagram chase,' and category theorists are very fond of them.

2.1.3. Building new categories from existing ones

One of the first things one learns about when talking about a mathematical structure is how to use existing ones to create new ones. For example, one can take the product of two groups to create a third, or take the quotient a vector space by a subspace. Categories are no exception. In this section, we will learn how to create new categories out of old categories.

The opposite category

The first way of doing this we'll examine creates out of any category its *opposite category,* where all the morphisms go the other way.

Definition 10 (opposite category)**.** Let C be a category. Its opposite category C^{op} is the category defined in the following way.

- The objects of C^{op} are the same as the objects of C:

$$\mathrm{Obj}(\mathsf{C}^{\mathrm{op}}) = \mathrm{Obj}(\mathsf{C}).$$

- The morphisms $A \to B$ in C^{op} are defined to be the morphisms $B \to A$ in C:

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathsf{C}}(B, A).$$

That is to say, the opposite category is the category one gets by formally reversing all the arrows in a category. If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, i.e. $f \colon A \to B$, then in C^{op}, $f \colon B \to A$.

This may seem like an unintererting definition, but having it around will make life a lot easier when it comes to defining functors (Section 2.2).

[4] We will define later when a diagram does or does not commute. Annoyingly, even diagrams which don't commute are often called commutative diagrams.

Product categories

One can take the Cartesian product of two sets by creating ordered pairs. One can also take the product of two categories, which works in basically the same way.

Definition 11 (product category)**.** Let C and D be categories. The product category $\mathsf{C} \times \mathsf{D}$ is the following category.

- The objects $\mathrm{Obj}(\mathsf{C} \times \mathsf{D})$ consist of all ordered pairs (C, D), where $C \in \mathrm{Obj}(\mathsf{C})$ and $D \in \mathrm{Obj}(\mathsf{D})$.
- For any two objects (C, D), and $(C', D') \in \mathrm{Obj}(\mathsf{C}\times\mathsf{D})$, the morphisms $\mathrm{Hom}_{C\times D}((C, D), (C', D'))$ are ordered pairs (f, g), where $f \in \mathrm{Hom}_{\mathsf{C}}(C, C')$ and $g \in \mathrm{Hom}_{\mathsf{D}}(D, D')$.
- Composition is taken componentwise, so that

$$(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ g_1, f_2 \circ g_2).$$

- The identity morphisms are given in the obvious way:

$$\mathrm{id}_{(C,D)} = (\mathrm{id}_C, \mathrm{id}_D).$$

Product categories are best pictured in the following way. The object $(C, D) \in \mathrm{Obj}(\mathsf{C}, \mathsf{D})$ is two objects right next to each other.

$$\begin{matrix} C \\ \\ D \end{matrix}$$

A morphism $(f, g)\colon (C, D) \to (C', D')$ is two morphisms next to each other:

$$\begin{matrix} C \xrightarrow{f} C' \\ \\ D \xrightarrow{g} D' \end{matrix}$$

Subcategories

Many mathematical objects have a notion of 'sub-object.' For example, groups can have subgroups, and vector spaces can have vector subspaces. Categories also have this notion. As with groups and vector spaces, the definition of a 'subcategory' is more or less obvious, but it is helpful to know exactly what one has to check to show that a subcollection of objects and morphisms from a category forms a subcategory.

Definition 12 (subcategory)**.** Let C be a category. A category S is a subcategory of C if the following conditions hold.

- The objects $\mathrm{Obj}(\mathsf{S})$ of S are a subcollection of the objects of C.
- For S, $T \in \mathrm{Obj}(\mathsf{S})$, the morphisms $\mathrm{Hom}_{\mathsf{S}}(S, T)$ are a subcollection of the morphisms $\mathrm{Hom}_{\mathsf{C}}(S, T)$ which satisfy the following.
  - For every $S \in \mathrm{Obj}(S)$, the identity $\mathrm{id}_S \in \mathrm{Hom}_{\mathsf{S}}(S, S)$.
  - For all $f \in \mathrm{Hom}_{\mathsf{S}}(S, T)$ and $g \in \mathrm{Hom}_{\mathsf{S}}(T, U)$, the composite $g \circ f \in \mathrm{Hom}_{\mathsf{S}}(S, U)$.

If S is a subcategory of C, we will write $\mathsf{S} \subseteq \mathsf{C}$.

One particularly important kind of subcategory occurs when one takes a subset of all objects in a subcategory, but keeps all morphisms between them. This is known as a *full subcategory.*

Definition 13 (full subcategory)**.** Let C be a category, S $\subseteq$ C a subcategory. We say that S is full in C if for every $S, T \in \mathrm{Obj}(\mathsf{S})$, $\mathrm{Hom}_{\mathsf{S}}(S,T) = \mathrm{Hom}_{\mathsf{C}}(S,T)$. That is, if there are no morphisms in $\mathrm{Hom}_{\mathsf{C}}(S,T)$ that are not in $\mathrm{Hom}_{\mathsf{S}}(S,T)$.

Example 14. Recall that Vect_k is the category of vector spaces over a field k, and $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces.

It is not difficult to see that $\mathsf{FinVect}_k \subset \mathsf{Vect}_k$: all finite dimensional vector spaces are vector spaces, and all linear maps between finite-dimensional vector spaces are maps between vector spaces. In fact, since for V and W finite-dimensional, one does not gain any maps by moving from $\mathrm{Hom}_{\mathsf{FinVect}_k}(V,W)$ to $\mathrm{Hom}_{\mathsf{Vect}_k}(V,W)$, $\mathsf{FinVect}_k$ is even a *full* subcategory of Vect_k.

2.1.4. Properties of morphisms

Category theory has many essences, one of which is as a major generalization of set theory. It is often possible to upgrade statements about functions between sets to statements about morphisms between objects in an arbitrary category. However, this is not as simple as it may sound: nowhere in the axioms in the definition of a category does it say that the objects of a category have to *be* sets, so we cannot talk about their elements.

In a sense, it's a miracle that this works at all. We are generalizing definitions from set theory, which are almost always given in terms of elements because elements are the only structure sets have, without ever mentioning the elements of the objects we're talking about. We therefore have to find definitions which we can give purely in terms objects and morphisms between them.

Isomorphism

The concept of an isomorphism exists for many mathematical entities. Two groups can be isomorphic, as can two sets or graphs. It turns out that the concept of isomorphism is best understood as a categorical one.

Definition 15 (isomorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f \in \mathrm{Hom}(A,B)$ is said to be an isomorphism if there exists a morphism $g \in \mathrm{Hom}(B,A)$ such that

$$g \circ f = \mathrm{id}_A, \qquad \text{and} \qquad f \circ g = \mathrm{id}_B.$$

$$\begin{tikzcd} A \arrow[loop left, "\mathrm{id}_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "\mathrm{id}_B"] \end{tikzcd}$$

We usually denote such a g by f^{-1}.

If we have an isomorphism $f : A \to B$, we say that A and B are isomorphic, and write $A \simeq B$. This is not an abuse of notation; it is easy to check that isomorphism is an equivalence relation.

Monomorphisms

Monomorphicity is an attempt to define a property analogous to injectivity which can be used in any category. The regular definition of injectivity, i.e.

$$f(a) = f(b) \implies a = b$$

will not do, because the objects in a category do not in general have elements. Therefore, we have to use a different property of injective functions: that they are left-cancellable. That is, if f is injective, then

$$f \circ g_1 = f \circ g_2 \implies g_1 = g_2$$

for any functions g_1 and g_2.

Definition 16 (monomorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathrm{C})$. A morphism $f \colon A \to B$ is said to be a monomorphism (or simply *mono*) if for any $Z \in \mathrm{Obj}(\mathrm{C})$ and any $g_1, g_2 \colon Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Note 17. When we wish to notationally distinguish monomorphisms, we will denote them by hooked arrows: if $f \colon A \to B$ is mono, we will write

$$A \xhookrightarrow{f} B\ .$$

This turns out to work due to the following theorem.

Theorem 18. In Set, a morphism is a monomorphism if and only if it is injective.

Proof. Suppose $f \colon A \to B$ is a monomorphism. Then for any set Z and any maps $g_1, g_2 \colon Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. In particular, take Z to be the singleton $Z = \{*\}$ and call $g_1(*) = a_1$ and $g_2(*) = a_2$. Then $(f \circ g_1)(*) = f(a_1)$ and $(f \circ g_2)(*) = f(a_2)$, so

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

But this is exactly the definition of injectivity.

Now suppose that f is injective. Then for any Z and g_1, g_2 as above,

$$(f \circ g_1)(z) = (f \circ g_2)(z) \implies g_1(z) = g_2(z) \qquad \text{for all } z \in Z.$$

But this means that $g_1 = g_2$, so f is mono. □

Example 19. In Vect_k, a morphism $L \colon V \to W$ is mono if and only if it is injective. That injectivity implies monomorphicity is obvious because any linear map is in particular a set function. To see that any linear monomorphism is injective, consider k as a vector space over itself. Then for any maps $A, B \colon k \to V$, we have

$$L \circ A = L \circ B \implies A = B.$$

In particular, for any $a, b \in V$, let $A(1) = a$ and $B(1) = b$. Then

$$\begin{aligned} L(a) &= L(b) \\ L(A(1)) &= L(B(1)) \\ (L \circ A)(1) &= (L \circ B)(1) \\ L \circ A &= L \circ B \\ A &= B \\ A(1) &= B(1) \\ a &= b. \end{aligned}$$

Epimorphisms

The notion of a surjection can also be generalized. Pleasingly, it is more clear that epimorphisms are dual to monomorphisms than that surjectivity is dual to injectivity.

Definition 20 (epimorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f : A \to B$ is said to be a epimorphism if for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all $g_1, g_2 : B \to Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd} .$$

Notation 21. We will denote epimorphisms by two-headed arrows. That is, if $f : A \to B$ is epi, we will write

$$A \xrightarrow{f} \mathrel{\mkern-14mu}\rightarrow B$$

Theorem 22. In Set, a morphism f is an epimorphism if and only if it is a surjection.

Proof. Suppose $f : A \to B$ is an epimorphism. Then for any set Z and maps $g_1, g_2 : B \to Z$, there $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. In particular, this is true if $Z = \{0, 1\}$, and g_1 and g_2 are as follows.

$$g_1 : b \mapsto 0 \quad \text{for all } b; \qquad g_2 : b \mapsto \begin{cases} 0, & b \in \mathrm{im}(f) \\ 1, & b \notin \mathrm{im}(f). \end{cases}$$

But then $g_1 \circ f = g_2 \circ f$, so $g_1 = g_2$ since f is epi. Hence, $\mathrm{im}(f) = B$, so f is surjective.

If f is surjective, then there exists a right inverse f^{-1} such that $f^{-1} \circ f = \mathrm{id}_A$. Then

$$g_1 \circ f = g_2 \circ f \implies g_1 \circ f \circ f^{-1} = g_2 \circ f \circ f^{-1} \implies g_1 = g_2.$$

□

Example 23. In Vect_k, epimorphisms are surjective linear maps.

Note 24. In Set, we have the following correspondences.

$$\text{Isomorphism} \iff \text{Bijective} \iff \text{Injective \& Surjective} \iff \text{Monic \& Epic}$$

Thus in Set a morphism is an isomorphism if and only if it is a monomorphism and an epimorphism. A very common mistake is to assume that this holds true in any category. This is wrong! A morphism can be monic and epic without being an isomorphism.

This occurrs, for example, in the category Top, whose objects are topological spaces and whose morphisms are continuous maps. This is because of the following chain of reasoning.

- In order for a continuous map f to be monic, it is certainly sufficient that it be injective, since the injectivity of a function implies that for any other functions α and β,

$$f \circ \alpha = f \circ \beta \implies \alpha = \beta.$$

  This certainly remains true if α and β are taken from the subset of set-functions which are continuous. Thus in Top,

$$\text{injective} \implies \text{monic}.$$

- Exactly analogous reasoning shows that in Top,

$$\text{surjective} \implies \text{epic}.$$

- Thus, any continuous map which is bijective is both mono and epic in Top. However, there are continuous, bijective maps which are not isomorphisms! Take, for example, the map

$$[0, 2\pi) \longrightarrow S^1 \subset \mathbb{R}^2; \qquad x \longmapsto (\cos x, \sin x).$$

  This is clearly continuous and bijective, hence both a monomorphism and an epimorphism in Top. However, it has no inverse in Top: its inverse function is discontinuous at $(1, 0) \in S^1$, so it is not a morphism $S^1 \to [0, 2\pi)$.

Smallness and local smallness

Set theory has some foundational annoyances. Among the most famous of these is Russel's paradox, which demonstrates that that not every definable collection is a set. Category theory has its own foundational issues, which for the most part we will avoid. However, there are a few important situations in which foundational questions of size play an unavoidably important role.

Recall that in our definition of a category, we said that objects and morphisms of a category need not fit inside a set; we only need them to form a 'collection.' We were cagy about what exactly we meant by this. The more precise statement is that they must form a *class,* although we will not have to worry about what a class is.

Using classes instead of sets works perfectly well. However, sets are better behaved than classes, and it is useful to have a special name for categories whose objects and/or morphisms *really do* fit into a set.

Definition 25 (small, locally small, hom-set)**.** A category C can have the following properties.

- We say that C is <u>locally small</u> if for all $A, B \in \text{Obj}(\mathsf{C})$, $\text{Hom}_{\mathsf{C}}(A, B)$ is a small set.
- We say that C is <u>small</u> if $\text{Obj}(\mathsf{C})$ is a small set and for all objects $A, B \in \text{Obj}(\mathsf{C})$, $\text{Hom}_{\mathsf{C}}(A, B)$ is a small set.

If we are working with a category which is locally small, so that $\text{Hom}_{\mathsf{C}}(A, B)$ is always a small set, we call $\text{Hom}_{\mathsf{C}}(A, B)$ the <u>hom-set</u>. (Actually, terminology is often abused, and $\text{Hom}_{\mathsf{C}}(A, B)$ is called a hom-set even if it is not a set.)

Example 26. The category Set is locally small but not small.

2.2. Functors

Category theory is as fantastically useful as it is because of the ubiquity of the notion of a structure preserving map. Many interesting examples of categories arise by considering as objects some class of mathematical entities, and as morphisms the structure-preserving maps between them.

Categories are themselves mathematical objects, so it would be hypocrytical not to look for the correct notion of a structure preserving map between them. This is called a functor.

Definition 27 (functor)**.** Let C and D be categories. A functor $\mathcal{F}$ from C to D is the following.

- It assigns to each object $X \in \mathrm{Obj}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Obj}(\mathsf{D})$.
- It assigns each morphism $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$ to a morphism $\mathcal{F}(f) \in \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ in such a way that the following conditions are satisfied.
    - It maps identities to identities, i.e.

      $$\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_{\mathcal{F}(X)} \qquad \text{for all } X \in \mathrm{Obj}(\mathsf{C}).$$

    - It respects composition:

      $$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

Notation 28. We will typeset functors with calligraphic letters, and notate them with squiggly arrows. For example, if C and D are categories and $\mathcal{F}$ is a functor from C to D, then we would write

$$\mathcal{F} \colon \mathsf{C} \rightsquigarrow \mathsf{D}.$$

Later, when we confront the idea that functors are really just morphisms in the category of categories, we will stop using the squiggly-arrow notation and denote functors with straight arrows. For now, however, it will be helpful to keep morphisms straight from functors.

2.2.1. Examples

Example 29. Let $\mathsf{1}$ be the category with one object $*$ (Example 4). Let C be any category. Then for each $X \in \mathrm{Obj}(\mathsf{C})$, we have the functor

$$\mathcal{F}_X \colon \mathsf{1} \rightsquigarrow \mathsf{C}; \qquad \mathcal{F}(*) = X, \quad \mathcal{F}(\mathrm{id}_*) = \mathrm{id}_X.$$

There is a special name for functors whose domain is a product category.

Definition 30 (bifunctor)**.** A bifunctor is a functor whose domain is a product category (Definition 11).

Example 31. The Cartesian product of sets is a bifunctor

$$\times \colon \mathsf{Set} \times \mathsf{Set} \to \mathsf{Set},$$

which sends an object (i.e. a pair of sets) (A, B) to its Cartesian product $A \times B$, and sends a morphism

$$(f, g) \colon (A, B) \to (C, D)$$

to a morphism

$$f \times g\colon A \times B \to C \times D; \qquad (a, b) \mapsto (f(a), g(b)).$$

Checking that this respects composition and sends identities to identities is not at all tricky.

Example 32. Let G be a group, and V a vector space over some field k. A *representation* ρ of G on V assigns to each $g \in G$ a linear transformation $\rho(g)\colon V \to V$ such that

$$\rho(gh) = \rho(g) \cdot \rho(h).$$

Let G be a group. Recall Example 8, in which we constructed from a group G a category BG with one object $*$, whose morphisms $\mathrm{Hom}_{BG}(*, *)$ were given by the elements of G.

Then functors $\rho\colon BG \rightsquigarrow \mathsf{Vect}_k$ are k-linear representations of G!

To see this, let us unwrap the definition. The functor ρ assigns to $* \in \mathrm{Obj}(BG)$ an object $\rho(*) = V \in \mathrm{Obj}(\mathsf{Vect})$, and to each morphism $g\colon * \to *$ a morphism $\rho(g)\colon V \to V$. Furthermore, ρ must respect composition in the sense that

$$\rho(g \cdot h) = \rho(g) \cdot \rho(h).$$

But this means precisely that ρ is a representation of G.

Example 33. Recall that Grp is the category of groups. Denote by CRing the category of commutative rings.

The following are functors $\mathsf{CRing} \rightsquigarrow \mathsf{Grp}$.

- GL_n, which assigns to each commutative ring K the group of all $n \times n$ invertible matrices with entries in K, and to each commutative ring homomorphism $f\colon K \to K'$ a map

$$\mathrm{GL}_n(f)\colon\ \mathrm{GL}_n(K) \to \mathrm{GL}_n(K'); \qquad \begin{pmatrix} a_{11} & \cdots & a_{nn} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mapsto \begin{pmatrix} f(a_{11}) & \cdots & f(a_{nn}) \\ \vdots & \ddots & \vdots \\ f(a_{n1}) & \cdots & f(a_{nn}) \end{pmatrix}.$$

- $(\,\cdot\,)^*$, which maps each commutative ring K to its group of units K^*, and each morphism $K \to K'$ to its restriction to K^*.

Example 34. Let $\mathcal{P}(\{0, 1\})$ be the poset category from Example 5. A functor $\mathcal{P}(\{0, 1\}) \rightsquigarrow \mathsf{C}$ is a commuting square in C. Similarly, a functor $\mathcal{P}(\{0, 1, 2\}) \rightsquigarrow \mathcal{C}$ is a cube whose faces commute.

Example 35. Let k be a field. There is a functor $(-)^*\colon \mathsf{Vect}_k^{\mathrm{op}} \rightsquigarrow \mathsf{Vect}_k$ which sends each vector space to its dual.

2.2.2. Properties of functors

Full, faithful, and essentially surjective

Definition 36 (full, faithful)**.** Let C and D be locally small categories (Definition 25), and let $\mathcal{F}: \mathsf{C} \rightsquigarrow \mathsf{D}$. Then $\mathcal{F}$ induces a family of set-functions

$$\mathcal{F}_{X,Y}\colon\ \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y)).$$

We say that $\mathcal{F}$ is

- full if $\mathcal{F}_{X,Y}$ is surjective for all $X, Y \in \mathrm{Obj}(\mathsf{C})$
- faithful if $\mathcal{F}_{X,Y}$ is injective for all $X, Y \in \mathrm{Obj}(\mathsf{C})$,
- fully faithful if $\mathcal{F}$ is full and faithful.

Note 37. Fullness and faithfulness are *not* the functorial analogs of surjectivity and injectivity. A functor between small categories can be full (resp. faithful) without being surjective (resp. injective) on objects. Instead, we have the following result.

Lemma 38. A fully faithful functor is injective on objects up to isomorphism. That is, if $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is a fully faithful functor and $\mathcal{F}(A) \simeq \mathcal{F}(B)$, then $A \simeq B$.

Proof. Let $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a fully faithful functor, and suppose that $\mathcal{F}(A) \simeq \mathcal{F}(B)$. Then there exist $f'\colon \mathcal{F}(A) \to \mathcal{F}(B)$ and $g'\colon \mathcal{F}(B) \to \mathcal{F}(A)$ such that $f' \circ g' = \mathrm{id}_{\mathcal{F}(B)}$ and $g' \circ f' = \mathrm{id}_{\mathcal{F}(A)}$. Because the function $\mathcal{F}_{A,B}$ is bijective it is invertible, so there is a unique morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ such that $\mathcal{F}(f) = f'$, and similarly there is a unique $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ such that $\mathcal{F}(g) = g'$.

Now,
$$\mathrm{id}_{\mathcal{F}(A)} = g' \circ f' = \mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f),$$
and since $\mathcal{F}$ is injective, we must have $g \circ f = \mathrm{id}_A$. Identical logic shows that we must also have $f \circ g = \mathrm{id}_B$. Thus $A \simeq B$. □

It now makes sense to define a different functorial version of surjectivity.

Definition 39 (essentially surjective)**.** A functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is essentially surjective if for every $A' \in \mathrm{Obj}(\mathsf{D})$, there exists $A \in \mathrm{Obj}(\mathsf{C})$ such that $A' \simeq \mathcal{F}(A)$.

Covariance and contravariance

What we have called simply a *functor*, many people would call a *covariant functor*. These people would say that there is a second kind of functor, called a *contravariant functor*, which flips arrows around. That is, applying a contravariant functor $\mathcal{F}$ to a morphism $f\colon A \to B$, one would find a morphism
$$\mathcal{F}(f)\colon \mathcal{F}(B) \to \mathcal{F}(A).$$
In order to respect compositions, these people say, a contravariant functor must obey the modified composition rule
$$\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f).$$

We are choosing not to use the notion of a contravariant functor, and for a good reason: we don't need it! Recall the so-called opposite category (Definition 10). Roughly speaking, the opposite to a category is a category in which arrows go the other way.

A contravariant functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$, then, can be thought of as a *covariant* (i.e. ordinary) functor $\mathcal{F}\colon \mathsf{C}^{\mathrm{op}} \rightsquigarrow\!\!\to \mathcal{D}$. This turns out to be much simpler and more convenient than dealing with two kinds of functors all the time.

Definition 40 (contravariant functor)**.** A contravariant functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is simply a (covariant) functor $\mathcal{F}\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{D}$.

2.3. Natural transformations

Saunders Mac Lane, one of the fathers of category theory, used to say that he invented categories so he could talk about functors, and he invented functors so he could talk about natural transformations. Indeed, arguably the first paper ever published on category theory, published by Eilenberg and Mac Lane in 1945 and titled "General Theory of Natural Equivalences" [23], was about natural transformations.

Natural transformation can be thought of at several different levels of abstraction. At their most abstract, they provide a notion of 'morphisms between functors,' and indeed we will later see that natural transformations provide the correct notion of morphisms for a category whose objects are functors. However, they were originally studied not for this reason, but because they provide a rigorous footing for the notion of canonicalness itself.

For example, one often hears the statement "The cartesian product of sets is commutative (up to isomorphism) because for any two sets A and B, $A \times B$ is isomorphic to $B \times A$." This is certainly true, but not very meaningful; any two sets with the same cardinality are isomorphic. What one is trying to say is that $A \times B$ and $B \times A$ are *canonically* isomorphic. The question is: what exactly is meant by 'canonically'?

This is a surprisingly elusive question. For any two sets A and B, the canonical isomorphism in question is the obvious map

$$\mathrm{swap}_{A,B} \colon A \times B \to B \times A; \qquad (a, b) \mapsto (b, a).$$

However, picking some other sets C and D gives is a different swap function. In what sense are these swap functions instances of the same 'swap operation?' And what does this have to do with canonicalness?

One way of formalizing the sameness comes from the fact that this swap function commutes with functional evaluation. Notice that for *any* functions $f \colon A \to C$ and $g \colon B \to D$, the following diagram commutes.

$$\begin{tikzcd}
A \times B \arrow[rrr, "{(f,g)}"] \arrow[ddd, "\mathrm{swap}_{A,B}"'] & & & C \times D \arrow[ddd, "\mathrm{swap}_{C,D}"] \\
& (a,b) \arrow[r, mapsto] \arrow[d, mapsto] & (f(a), g(b)) \arrow[d, mapsto] & \\
& (b,a) \arrow[r, mapsto] & (g(b), f(a)) & \\
B \times A \arrow[rrr, "{(g,f)}"'] & & & D \times C
\end{tikzcd}$$

That is to say, it doesn't matter whether you use first use f and g to map $A \times B$ into $C \times D$ and then use the swap isomorphism, or whether you do the swap first and then use f and g to map $B \times A$ into $D \times C$; you get the same map either way. Furthermore, this is true for *any* maps f and g.

As we have seen (in Example 31), the cartesian product of sets is a functor from the product category (Definition 11)

$$\times \colon \mathsf{Set} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}; \qquad (A, B) \mapsto A \times B.$$

There is another 'twisted' Cartesian product functor

$$\tilde{\times} \colon \mathsf{Set} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}; \qquad (A, B) \mapsto B \times A.$$

As we will see in the rest of this chapter, the above discussion means precisely that the functions $\mathrm{swap}_{A,B}$ form a natural transformation, called swap, between $\times$ and $\tilde{\times}$.

This example makes clear a common theme of natural transformations: they formalize the idea of 'the same function between different objects.' They allow us to make precise statements like '$\mathrm{swap}_{A,B}$ and $\mathrm{swap}_{C,D}$ are somehow the same, despite the fact that they cannot be equal as functions.'

Definitions and elementary examples

Definition 41 (natural transformation)**.** let C and D be categories, and let $\mathcal{F}$ and $\mathcal{G}$ be functors from C to D. A natural transformation η between $\mathcal{F}$ and $\mathcal{G}$ consists of

- for each object $A \in \mathrm{Obj}(\mathsf{C})$ a morphism $\eta_A \colon \mathcal{F}(A) \to \mathcal{G}(A)$, such that
- for all $A, B \in \mathrm{Obj}(C)$, for each morphism $f \in \mathrm{Hom}(A, B)$, the diagram

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B) \end{tikzcd}$$

  commutes.

We will use double-shafted arrows to denote natural transformations: if $\mathcal{F}$ and $\mathcal{G}$ are functors and η is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, we will write

$$\eta \colon \mathcal{F} \Rightarrow \mathcal{G}.$$

Definition 42 (natural isomorphism)**.** A natural isomorphism $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation such that each η_A is an isomorphism.

Lemma 43. Given a natural isomorphism $\eta \colon \mathcal{F} \Rightarrow \mathcal{G}$, we can construct an inverse natural isomorphism $\eta^{-1} \colon \mathcal{G} \Rightarrow \mathcal{F}$.

Proof. The natural transformation gives us for any two objects A and B and morphism $f \colon A \to B$ a naturality square

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(B) \end{tikzcd}$$

which tells us that

$$\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A.$$

Since η is a natural isomorphism, its components η_A are isomorphisms, so they have inverses η_A^{-1}. Acting on the above equation with η_A^{-1} from the right and η_B^{-1} from the left, we find

$$\mathcal{F}(f) \circ \eta_A^{-1} = \eta_B^{-1} \circ \mathcal{G}(f),$$

i.e. the following diagram commutes.

$$\begin{tikzcd}
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\eta_A^{-1}"'] & \mathcal{G}(B) \arrow[d, "\eta_B^{-1}"] \\
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B)
\end{tikzcd}$$

But this is just the naturality square for a natural isomorphism η^{-1} with components $(\eta^{-1})_A = \eta_A^{-1}$. □

Example 44 (the determinant)**.** Recall the functors GL_n and $(\cdot)^*$ from Example 33.

Denote by $\det_K M$ the determinant of a matrix M with its entries in a commutative ring K. Then the determinant is a map

$$\det_K \colon \mathrm{GL}_n(K) \to K^*.$$

Because the determinant is defined by the same formula for each K, the action of f commutes with $\det_K$: it doesn't matter whether we map the entries of M with f first and then take the determinant, or take the determinant first and then feed the result to f.

That is to say, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{GL}_n(K) \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n(f)"'] & K^* \arrow[d, "f^*"] \\
\mathrm{GL}_n(K') \arrow[r, "\det_{K'}"] & K'^*
\end{tikzcd}$$

This means that det is a natural transformation $\mathrm{GL}_n \Rightarrow (\,\cdot\,)^*$.

Example 45 (intertwiners and scattering theory)**.** Let G be a group, BG its delooping groupoid, and let ρ and $\rho' \colon \mathrm{G} \rightsquigarrow \mathrm{Vect}$ be representations (recall Example 32). Let $\rho(*) = V$ and $\rho'(*) = W$.

A natural transformation $\eta \colon \rho \Rightarrow \rho'$ is called an *intertwiner*. So what *is* an intertwiner?

The natural transformation η has only one component, $\eta_* \colon V \to W$, which is subject to the condition that for any $g \in \mathrm{Hom}_{BG}(*, *)$, the diagram below commutes.

$$\begin{tikzcd}
V \arrow[r, "\rho(g)"] \arrow[d, "\eta_*"'] & V \arrow[d, "\eta_*"] \\
W \arrow[r, "\rho'(g)"] & W
\end{tikzcd}$$

That is, an intertwiner is a linear map $V \to W$ such that for all g,

$$\eta_* \circ \rho(g) = \rho'(g) \circ \eta_*.$$

Intertwiners are extremely important in high-energy physics.

In scattering theory one models the state of the world as $T \to -\infty$ as a composite system of several incoming free particles, and as $T \to +\infty$ a composite system of outgoing free particles. Therefore, the Hilbert space of the incoming particles is

$$\mathscr{H}_{\text{in}} = \mathscr{H}_{1,\text{in}} \otimes \cdots \otimes \mathscr{H}_{m,\text{in}}$$

and that of the outgoing particles is

$$\mathscr{H}_{\text{out}} = \mathscr{H}_{1,\text{out}} \otimes \cdots \otimes \mathscr{H}_{n,\text{out}}.$$

In relativistic quantum mechanics, the Hilbert space of a free particle alone in the world is the representation space of some irreducible representation of the Poincaré group. The action of the Poincaré group on this representation space implements Poincaré transformations on the particle. For a system of several particles, the Poincaré group acts on the incoming and outgoing Hilbert spaces via the tensor product representation.

Scattering theory finds a map $U\colon \mathscr{H}_{\text{in}} \to \mathscr{H}_{\text{out}}$, which interpolates between incoming states and outgoing states. However, not any linear map will do; since we demand that our laws be Poincaré-invariant, we want our transformation to be Poincaré-equivariant. Therefore, we want scattering theory to produce for us a map $U\colon \mathscr{H}_{\text{in}} \to \mathscr{H}_{\text{out}}$ such that the diagram

$$\begin{tikzcd}
\mathscr{H}_{\text{in}} \arrow[r, "\rho_{\text{in}}(g)"] \arrow[d, "U"'] & \mathscr{H}_{\text{in}} \arrow[d, "U"] \\
\mathscr{H}_{\text{out}} \arrow[r, "\rho_{\text{out}}(g)"] & \mathscr{H}_{\text{out}}
\end{tikzcd}$$

commutes.

That is, scattering theory produces intertwiners between the incoming and outgoing representations ρ_{in} and ρ_{out} of the Poincaré group.

Vertical composition

We mentioned earlier that one could define a category whose objects were functors and whose morphisms were natural transformations. We are now going to make good on that threat. In order to do so, we need only specify how to compose natural transformations. The correct notion is known as *vertical composition.*

Lemma 46. Let C and D be categories, $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be functors, and Φ and Ψ be natural transformations as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=60, "\mathcal{F}"{name=F}] \arrow[r, squiggly, "\mathcal{G}"{name=G, description}] \arrow[r, squiggly, bend right=60, "\mathcal{H}"'{name=H}] & \mathsf{D}
\arrow[from=F, to=G, Rightarrow, "\Phi"]
\arrow[from=G, to=H, Rightarrow, "\Psi"]
\end{tikzcd}$$

This induces a natural transformation $\mathcal{F} \Rightarrow \mathcal{H}$.

Proof. For each object $A \in \text{Obj}(\mathsf{C})$, the composition $\Psi_A \circ \Phi_A$ exists and maps $\mathcal{F}(A) \to \mathcal{H}(A)$. Let's write

$$\Psi_A \circ \Phi_A = (\Psi \circ \Phi)_A.$$

We have to show that these are the components of a natural transformation, i.e. that they make the following diagram commute for all $A, B \in \mathrm{Obj}(\mathsf{C})$, all $f : A \to B$.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "(\Psi\circ\Phi)_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "(\Psi\circ\Phi)_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

We can do this by adding a middle row.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "\Phi_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

The top and bottom squares are the naturality squares for Φ and Ψ respectively. The outside square is the one we want to commute, and it manifestly does because each of the inside squares does. $\square$

Definition 47 (vertical composition)**.** The above composition $\Psi \circ \Phi$ is called vertical composition.

One of the lessons of category theory is that any time we see a collection of things and mappings between them, turning them into a category will be a good investment. We saw that functors were mappings between categories, and defined a category Cat whose objects are categories and whose morphisms are functors. We can now define a category whose objects are functors and whose morphisms are natural transformations.

Definition 48 (functor category)**.** Let C and D be categories. The functor category $\mathsf{Fun}(\mathsf{C}, \mathsf{D})$ (sometimes D^{C} or $[\mathsf{C}, \mathsf{D}]$) is the category whose objects are functors $\mathsf{C} \rightsquigarrow \mathsf{D}$, and whose morphisms are natural transformations between them. The composition is given by vertical composition.

Horizontal composition

We can also compose natural transformations in a not so obvious way.

Lemma 49. Consider the following arrangement of categories, functors, and natural transformations.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"{name=F}] \arrow[r, squiggly, bend right, "\mathcal{F}'"'{name=Fp}] & \mathsf{D} \arrow[r, squiggly, bend left, "\mathcal{G}"{name=G}] \arrow[r, squiggly, bend right, "\mathcal{G}'"'{name=Gp}] & \mathsf{E}
\arrow[Rightarrow, from=F, to=Fp, "\Phi"]
\arrow[Rightarrow, from=G, to=Gp, "\Psi"]
\end{tikzcd}$$

This induces a natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$.

Proof. By definition, Φ and Ψ make the diagrams

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(B) \arrow[d, "\Phi_B"] \\ \mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B) \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\ \mathcal{G}'(A) \arrow[r, "\mathcal{G}'(f)"'] & \mathcal{G}'(B) \end{tikzcd}$$

commute. Since functors respect composition they take commutative diagrams to commutative diagrams, so if we map everything in the first diagram to E with $\mathcal{G}$ to get another commutative diagram in E.

$$\begin{tikzcd} (\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\ (\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] & (\mathcal{G} \circ \mathcal{F}')(B) \end{tikzcd}$$

Since $\mathcal{F}'(f)\colon \mathcal{F}'(A) \to \mathcal{F}'(B)$ is a perfectly good example of a morphism in D, the natural transformation and Ψ has components $\Psi_{\mathcal{F}'(A)}$ and $\Psi_{\mathcal{F}'(B)}$ which makes the following diagram commutes.

$$\begin{tikzcd} (\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\ (\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B) \end{tikzcd}$$

Sticking these two diagrams on top of each other gives a new commutative diagram.

$$\begin{tikzcd} (\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\ (\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\ (\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B) \end{tikzcd}$$

The outside rectangle is nothing else but the commuting square for a natural transformation

$$(\Psi * \Phi)\colon \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$$

with components $(\Psi * \Phi)_A = \Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$. □

Definition 50 (horizontal composition)**.** The natural transformation $\Psi * \Phi$ defined above by by

$$(\Psi * \Phi)_A = \Psi_{\mathcal{F}(A)} \circ G(\Phi(A))$$

is called the <u>horizontal composition</u> of Φ and Ψ.

One might worry the above definition of the horizontal composition is lopsided. Why did we apply the functor $\mathcal{G}$ to the first square rather than $\mathcal{G}'$? We find the following fact reassuring.

If we had applied the functor $\mathcal{G}'$ instead of $\mathcal{G}$, we would have found the following.

$$\begin{tikzcd}
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] \arrow[d, "\mathcal{G}'(\Phi_A)"'] & (\mathcal{G}' \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}'(\Phi_B)"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

Then we could have glued to it the bottom of the following commuting square.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G} \circ \mathcal{F})(f)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\Psi_{\mathcal{F}(B)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] & (\mathcal{G}' \circ \mathcal{F})(B)
\end{tikzcd}$$

If you do this you get *another* natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$, with components $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$. Why did we use the first definition rather than this one?

It turns out that $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$ are equal. To see this, pick any $A \in \mathrm{Obj}(\mathsf{A})$. From the morphism

$$\Phi_A \colon \mathcal{F}(A) \to \mathcal{F}'(A),$$

the natural transformation Ψ gives us a commuting square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}(\Phi_A)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(A) \arrow[d, "\Psi_{\mathcal{F}'(A)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}'(\Phi_A)"] & (\mathcal{G}' \circ \mathcal{F}')(A)
\end{tikzcd};$$

the two ways of going from top left to bottom right are nothing else but $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$.

It doesn't matter whether we first compose horizontally or vertically.

Theorem 51. Suppose we have categories, functors, and natural transformations as follows.

$$\begin{tikzcd}[column sep=huge]
\mathsf{C} \arrow[r, squiggly, bend left=60, "\mathcal{F}"{name=F}] \arrow[r, squiggly, "\mathcal{G}"{name=G, description}] \arrow[r, squiggly, bend right=60, "\mathcal{H}"'{name=H}] \arrow[Rightarrow, from=F, to=G, "\Phi"] \arrow[Rightarrow, from=G, to=H, "\Psi"] & \mathsf{D} \arrow[r, squiggly, bend left=60, "\mathcal{F}'"{name=F2}] \arrow[r, squiggly, "\mathcal{G}'"{name=G2, description}] \arrow[r, squiggly, bend right=60, "\mathcal{H}'"'{name=H2}] \arrow[Rightarrow, from=F2, to=G2, "\Phi'"] \arrow[Rightarrow, from=G2, to=H2, "\Psi'"] & \mathsf{E}
\end{tikzcd}$$

Then

$$(\Psi' \circ \Phi') * (\Psi \circ \Phi) \simeq (\Psi' * \Psi) \circ (\Phi' * \Phi).$$

Proof. We have

$$[(\Psi' \circ \Phi') * (\Psi \circ \Phi)]_c =$$

□

Whiskering

We will mainly be interested in a special case of horizontal composition, in which one of the natural transformations is the identity natural transformation:

Example 52 (whiskering)**.** Consider the following assemblage of categories, functors, and natural transformations.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] \arrow[r, squiggly, bend right, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] & \mathsf{E} \end{tikzcd}$$

The horizontal composition allows us to Φ to a natural transformation $\mathcal{G}\circ\mathcal{F}$ to $\mathcal{G}\circ\mathcal{F}'$ as follows. First, augment the diagram as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] \arrow[r, squiggly, bend right, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathsf{D} \arrow[r, squiggly, bend left, "\mathcal{G}"] \arrow[r, squiggly, bend right, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_{\mathcal{G}}"] & \mathsf{E} \end{tikzcd}$$

We can then take the horizontal composition of Φ and $\mathrm{id}_{\mathcal{G}}$ to get a natural transformation from $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ with components

$$(\mathrm{id}_{\mathcal{G}} * \Phi)_A = \mathcal{G}(\Phi_A).$$

This natural transformation is called the *right whiskering* of Φ with $\mathcal{G}$, and is denoted $\mathcal{G}\Phi$. That is to say, $(\mathcal{G}\Phi)_A = \mathcal{G}(\Phi_A)$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{G}\circ\mathcal{F}"] \arrow[r, squiggly, bend right, "\mathcal{G}\circ\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\mathcal{G}\Phi"] & \mathsf{E} \end{tikzcd}$$

The reason for the name is clear: we removed a whisker from the RHS of our diagram.

We can also remove a whisker from the LHS. Given this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[r, squiggly, bend left, "\mathcal{G}"] \arrow[r, squiggly, bend right, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow\Psi"] & \mathsf{E} \end{tikzcd}$$

we can build a natural transformation (denoted $\Psi\mathcal{F}$) with components

$$(\Psi\mathcal{F})_A = \Psi_{\mathcal{F}(A)},$$

making this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{G}\circ\mathcal{F}"] \arrow[r, squiggly, bend right, "\mathcal{G}'\circ\mathcal{F}"'] \arrow[r, phantom, "\Downarrow\Psi\mathcal{F}"] & \mathsf{E} \end{tikzcd} .$$

This is called the *left whiskering* of Ψ with $\mathcal{F}$

Categorical equivalence

We would like to be able to express when two categories are the 'same.' One would think that this would be provided by the notion of an isomorphism, i.e. that two categories C and D should be thought of as the same if there are functors $\mathcal{F}$ and $\mathcal{G}$ between them such that $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathsf{D}}$ and $\mathcal{G} \circ \mathcal{F} = \mathrm{id}_{\mathsf{C}}$.

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

between them.

However, this is not really in the spirit of category theory. We have seen that functors from one category to another assemble themselves into a category whose morphisms are natural transformations. Demanding $\mathcal{F} \circ \mathcal{G} = \mathrm{id}_{\mathsf{D}}$ would be demanding the equality of objects in a category, which is not usually the right thing to do; we should never demand that two objects be equal, when some notion of isomorphism will do. Therefore, we should demand not that the composition of $\mathcal{F}$ and $\mathcal{G}$ be equal to the identity functor 'on the nose.' Instead, we should ask that they be isomorphic, in the sense that there should exist a natural isomorphism between them.

This gives us the following definition.

Definition 53 (categorical equivalence)**.** Let C and D be categories. We say that C and D are <u>equivalent</u> if there is a pair of functors

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend left, "\mathcal{G}"]
\end{tikzcd}
$$

and natural isomorphisms $\eta\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{C}}$ and $\epsilon\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$.

Note 54. The above definition of categorical equivalence is equivalent to the following: C and D are equivalent if there is a functor $\mathcal{F} : \mathsf{C} \rightsquigarrow \mathsf{D}$ which is fully faithful (Definition 36) and essentially surjective (Definition 39). That is to say, if the equivalence $\mathcal{F}$ is 'bijective up to isomorphism.'

Definition 55 (essentially small category)**.** A category C is said to be <u>essentially small</u> if it is equivalent to a small category (Definition 25).

3. Universal properties

In a first course on linear algebra, one tends to reach for a basis every time one has to prove something about a vector space. Over time, many people begin to find it more aesthetically pleasing to properties intrinsic to the vector space itself. This is a reflection of a more general subjective phenomenon: the most beautiful, deepest way of working with a structure is often to use only the information available in the category in which the structure lives.

There is a very powerful way of describing and working with structures using only category-theoretic information, called *universal properties.*

Universal properties are ubiquitous in mathematics. The reader, whether he or she is aware of it or not, has almost certainly seen and used a few examples, such as the universal properties for products and tensor algebras. They are astonishingly useful.

For instance, suppose someone approaches you on the street late at night and tells you that they're going to steal your wallet unless you can write down a map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$. You write down

$$x \mapsto (x, x^2).$$

Your would-be mugger scoffs, saying "That's not a map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$. What you have really done is write down a pair of maps $\mathbb{R} \to \mathbb{R}$!"

You, however, are as cool as a cucumber. You calmly tell the thief, "The universal property for the Cartesian product tells us that writing down a pair of maps $\mathbb{R} \to \mathbb{R}$ is the same as writing down a map $\mathbb{R} \to \mathbb{R} \times \mathbb{R}$." Your assailant leaves, dejected.

3.1. Comma categories

The natural language for formalizing universal properties comes from the notion of a comma category.

Definition 56 (comma category)**.** Let A, B, C be categories, $\mathcal{S}$ and $\mathcal{T}$ functors as follows.

$$\mathsf{A} \xrightarrow{\mathcal{S}} \mathsf{C} \xleftarrow{\mathcal{T}} \mathsf{B}$$

The comma category $(\mathcal{S} \downarrow \mathcal{T})$ is the category whose

- objects are triples (α, β, f) where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(\mathsf{B})$, and $f \in \mathrm{Hom}_{\mathsf{C}}(\mathcal{S}(\alpha), \mathcal{T}(\beta))$, and whose
- morphisms $(\alpha, \beta, f) \to (\alpha', \beta', f')$ are all pairs (g, h), where $g\colon \alpha \to \alpha'$ and $h\colon \beta \to \beta'$, such that the diagram

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[d, "f'"] \\ \mathcal{T}(\beta) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\beta') \end{tikzcd}$$

commutes.

Notation 57. We will often specify the comma category $(\mathcal{S} \downarrow \mathcal{T})$ by simply writing down the diagram

$$\mathsf{A} \xrightarrow{\mathcal{S}} \mathsf{C} \xleftarrow{\mathcal{T}} \mathsf{B}\ .$$

Let us check in some detail that a comma category really is a category. To do so, we need to check the three properties listed in Definition 1

1. We must be able to compose morphisms, i.e. we must have the following diagram.

$$\begin{tikzcd}
(\alpha, \beta, f) \arrow[r, "{(g,h)}"] \arrow[rr, bend left, "{(g',h')\circ(g,h)}"] & (\alpha', \beta', f') \arrow[r, "{(g',h')}"] & (\alpha'', \beta'', f'')
\end{tikzcd}$$

   We certainly do, since by definition, each square of the following diagram commutes,

$$\begin{tikzcd}
\mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[r, "\mathcal{S}(g')"] \arrow[d, "f'"'] & \mathcal{S}(\alpha'') \arrow[d, "f''"] \\
\mathcal{T}(\alpha) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\alpha') \arrow[r, "\mathcal{T}(h')"] & \mathcal{T}(\alpha'')
\end{tikzcd}$$

   so the square formed by taking the outside rectangle

$$\begin{tikzcd}
\mathcal{S}(\alpha) \arrow[rr, "\mathcal{S}(g')\circ\mathcal{S}(g)"] \arrow[d, "f"'] & & \mathcal{S}(\alpha'') \arrow[d, "f''"] \\
\mathcal{T}(\alpha) \arrow[rr, "\mathcal{T}(h')\circ\mathcal{T}(h)"] & & \mathcal{T}(\alpha'')
\end{tikzcd}$$

   commutes. But $\mathcal{S}$ and $\mathcal{T}$ are functors, so

$$\mathcal{S}(g') \circ \mathcal{S}(g) = \mathcal{S}(g' \circ g),$$

   and similarly for $\mathcal{T}(h' \circ h)$. Thus, the composition of morphisms is given via

$$(g', h') \circ (g, h) = (g' \circ g, h' \circ h).$$

2. We can see from this definition that associativity in $(\mathcal{S} \downarrow \mathcal{T})$ follows from associativity in the underlying categories A and B.
3. The identity morphism is the pair $(\mathrm{id}_{\mathcal{S}(\alpha)}, \mathrm{id}_{\mathcal{T}(\beta)})$. It is trivial from the definition of the composition of morphisms that this morphism functions as the identity morphism.

3.2. Slice categories

A special case of a comma category, the so-called *slice category*, occurs when $\mathsf{C} = \mathsf{A}$, $\mathcal{S}$ is the identity functor, and $\mathsf{B} = 1$, the category with one object and one morphism (Example 4).

Definition 58 (slice category)**.** A slice category is a comma category

$$\mathsf{A} \xrightarrow{\mathrm{id}_A} \mathsf{A} \xleftarrow{\mathcal{T}} 1 \; .$$

Let us unpack this prescription. Taking the definition literally, the objects in our category are triples (α, β, f), where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(1)$, and $f \in \mathrm{Hom}_{\mathsf{A}}(\mathrm{id}_{\mathsf{A}}(\alpha), \mathcal{T}(\beta))$.

There's a lot of extraneous information here, and our definition can be consolidated considerably. Since the functor $\mathcal{T}$ is given and 1 has only one object (call it $*$), the object $\mathcal{T}(*)$ (call it X) is singled out in A. We can think of $\mathcal{T}$ as $\mathcal{F}_X$ (Example 29). Similarly, since the identity morphism doesn't do anything interesting, Therefore, we can collapse the following diagram considerably.

$$\begin{tikzcd} \mathcal{F}_X(*) \arrow[r, "\mathcal{F}_X(\mathrm{id}_*)"] \arrow[d, "f"'] & \mathcal{F}_X(*) \arrow[d, "f'"] \\ \mathrm{id}_A(\alpha) \arrow[r, "\mathrm{id}_A(g)"] & \mathrm{id}_A(\alpha') \end{tikzcd}$$

The objects of a slice category therefore consist of pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and

$$f : \alpha \to X;$$

the morphisms $(\alpha, f) \to (\alpha', f')$ consist of maps $g\colon \alpha \to \alpha'$. This allows us to define a slice category more neatly.

Let A be a category, $X \in \mathrm{Obj}(\mathsf{A})$. The slice category $(\mathsf{A} \downarrow X)$ is the category whose objects are pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and $f\colon \alpha \to X$, and whose morphisms $(\alpha, f) \to (\alpha', f')$ are maps $g : \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} \alpha \arrow[rr, "g"] \arrow[dr, "f"'] & & \alpha' \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes.

One can also define a coslice category, which is what you get when you take a slice category and turn the arrows around: coslice categories are *dual* to slice categories.

Definition 59 (coslice category)**.** Let A be a category, $X \in \mathrm{Obj}(A)$. The coslice category $(X \downarrow \mathsf{A})$ is the comma category given by the diagram

$$1 \xrightarrow{\mathcal{F}_X} \mathsf{A} \xleftarrow{\mathrm{id}_A} \mathsf{A} \; .$$

The objects are morphisms $f\colon X \to \alpha$ and the morphisms are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \alpha \arrow[rr, "g"'] & & \alpha' \end{tikzcd}$$

commutes.

3.3. Initial and terminal objects

It is difficult to talk about specific objects in a category using only category-theoretic information. For example, imagine trying to explain the structure of the nth dihedral group to someone without talking about its elements.

However, not all is lost when passing to situations in which more data is not available. Certain objects in a category are special, and you can talk about them without any reference to non-categorical data.

Definition 60 (initial objects, final objects, zero objects)**.** Let C be a category, and let $A \in \mathrm{Obj}(\mathsf{C})$.

- A is said to be an initial object if for all $B \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}(A, B)$ has exactly one element, i.e. if there is exactly one arrow from A to every object in C. Initial objects are generally called $\emptyset$.
- A is said to be a terminal object if $\mathrm{Hom}(B, A)$ has exactly one element for all $B \in \mathrm{Obj}(\mathsf{C})$, i.e. if there is exactly one arrow from every object in C to A. Terminal objects are generally called 1.
- A is said to be a zero object if it is both initial and terminal. Zero objects are almost always called 0.

The names $\emptyset$ and 1 for the initial and terminal objects may seem odd, but in Set they make sense.

Example 61. In Set, there is exactly one map from any set S to any one-element set $\{*\}$. Thus $\{*\} = 1$ is a terminal object in Set.

Furthermore, it is conventional that there is exactly one map from the empty set $\emptyset$ to any set B. Thus the empty set $\emptyset$ is initial in Set.

The category Set has no zero objects.

Example 62. The trivial group is a zero object in Grp.

You may have noticed that initial, terminal, and zero objects may not be unique. For example, in Set, any singleton is terminal. This is true; however, they are unique up to unique isomorphism.

Theorem 63. Let C be a category, let I and I' be two initial objects in C. Then there exists a unique isomorphism between I and I'.

Proof. Since I is initial, there exists exactly one morphism from I to *any* object, including I itself. By Definition 1, Part 2, this morphism must be the identity morphism id_I. Similarly, the only morphism from I' to itself is the identity morphism $\mathrm{id}_{I'}$.

But since I is initial, there exists a unique morphism from I to I'; call it f. Similarly, there exists a unique morphism from I' to I; call it g. By Definition 1, Part 1, we can take the composition $g \circ f$ to get a morphism $I \to I$.

But there is only one isomorphism $I \to I$: the identity morphism! Thus

$$g \circ f = \mathrm{id}_I.$$

Similarly,

$$f \circ g = \mathrm{id}_{I'}.$$

This means that, by Definition 15, f and g are isomorphisms. They are clearly unique because of the uniqueness condition in the definition of an initial object. Thus, between any two initial objects there is a unique isomorphism.

$$\begin{tikzcd} I \arrow[r, bend left, "f"] \arrow[loop left, "\mathrm{id}_I = g\circ f"] & I' \arrow[l, bend left, "g"] \arrow[loop right, "\mathrm{id}_{I'} = f\circ g"] \end{tikzcd}$$

□

Showing that terminal and zero objects are unique up to unique isomorphism is almost exactly the same.

3.4. Morphisms to and from functors

We have succeeded in specifying certain objects (namely initial, terminal, and zero objects) in a category using only categeorical information uniquely up to a unique isomorphism. Unfortunately, initial, terminal, and zero objects are usually not very interesting; they are often trivial examples of the structures contained in the category.

However, they give us a way of defining interesting mathematical objects: we have to cook up a category where the object in question is initial or terminal. In a sense, this is the most general definition of a universal property: an object satisfies a universal property if it is initial or terminal in some category. However, we will see in Chapter 7 that there is a way of constructing interesting objects satisfying a universal property, namely partial adjunctions: these specify universal objects in the form of *morphisms to and from functors.*

Definition 64 (category of morphisms from an object to a functor)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \to \mathsf{C}$ be a functor. Further let $X \in \mathrm{Obj}(C)$. The category of morphisms $(X \downarrow \mathcal{U})$ is the following comma category (see Definition 56):

$$\mathsf{1} \xrightarrow{\mathcal{F}_X} \mathsf{C} \xleftarrow{\mathcal{U}} \mathsf{D}\ .$$

Just as for (co)slice categories (Definitions 58 and 59), there is some unpacking to be done. In fact, the unpacking is very similar to that of coslice categories. The LHS of the commutative square diagram collapses because the functor $\mathcal{F}_X$ picks out a single element X; therefore, the objects of $(X \downarrow \mathcal{U})$ are ordered pairs

$$(\alpha, f); \qquad \alpha \in \mathrm{Obj}(\mathsf{D}), \quad f\colon X \to \mathcal{U}(\alpha),$$

and the morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha') \end{tikzcd}$$

commutes.

Just as slice categories are dual to coslice categories, we can take the dual of the previous definition.

Definition 65 (category of morphisms from a functor to an object)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \to \mathsf{C}$ be a functor. The category of morphisms $(\mathcal{U} \downarrow X)$ is the comma category

$$\mathsf{D} \xrightarrow{\mathcal{U}} \mathsf{C} \xleftarrow{\mathcal{F}_X} 1 \ .$$

The objects in this category are pairs (α, f), where $\alpha \in \mathrm{Obj}(\mathsf{D})$ and $f\colon \mathcal{U}(\alpha) \to X$. The morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(\alpha') \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes.

3.5. Initial and terminal morphisms

Definition 66 (initial morphism)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \to \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. An initial morphism (called a *universal arrow* in [17]) is an initial object in the category $(X \downarrow \mathcal{U})$, i.e. the comma category which has the diagram

$$1 \xrightarrow{\mathcal{F}_X} \mathsf{C} \xleftarrow{\mathcal{U}} \mathsf{D}$$

This is not by any stretch of the imagination a transparent definition, but decoding it will be good practice.

The definition tells us that an initial morphism is an object in $(X \downarrow \mathcal{U})$, i.e. a pair (I, φ) for $I \in \mathrm{Obj}(\mathsf{D})$ and $\varphi\colon X \to \mathcal{U}(\alpha)$. But it is not just any object: it is an initial object. This means that for any other object (α, f), there exists a unique morphism $(I, \varphi) \to (\alpha, f)$. But such morphisms are simply maps $g\colon I \to \alpha$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "f"] & \\ \mathcal{U}(I) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha) \end{tikzcd}$$

commutes.

We can express this schematically via the following diagram (which is essentially the above diagram, rotated to agree with the literature).

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{U}(I) \arrow[d, "\mathcal{U}(g)"] \\ & \mathcal{U}(\alpha) \end{tikzcd} \qquad \begin{tikzcd} I \arrow[d, "\exists! g"] \\ \alpha \end{tikzcd}$$

As always, there is a dual notion.

Definition 67 (terminal morphism)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \to \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. A terminal morphism is a terminal object in the category $(\mathcal{U} \downarrow X)$.

This time "terminal object" means a pair (I, φ) such that for any other object (α, f) there is a unique morphism $g\colon \alpha \to I$ such that the diagram

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(I) \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

commutes.

Again, with the diagram helpfully rotated, we have the following.

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[d, "\mathcal{U}(g)"'] \arrow[dr, "f"] & \\ \mathcal{U}(I) \arrow[r, "\varphi"'] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} \alpha \arrow[d, "\exists! g"] \\ I \end{tikzcd}$$

3.6. Universal properties

Now is the time for an admission: despite what Wikipedia may tell you, there is no hard and fast definition of a universal property. In complete generality, an object with a universal property is just an object which is initial or terminal in some category. However, enough interesting universal properties follow the following pattern that it is worth formalizing in a definition.

Definition 68 (universal property)**.** A pair (α, f) satisfies a universal property if it satisfies either of the following criteria.

- It is initial in some category $(X \downarrow \mathcal{F})$; that is, if it is an initial morphism.
- It is terminal in some category $(\mathcal{F} \downarrow X)$; that is, if it is a terminal morphism.

Note 69. One often hand-wavily says that an object I satisfies a universal property if (I, φ) is an initial or terminal morphism. This is actually rather annoying; one has to remember that when one states a universal property in terms of a universal morphism, one is defining not only an object I but also a morphism φ, which is often left implicit.

Generally when one sees universal properties, they are not in the form 'an initial morphism in such-and-such a comma category.' Often, one has to play around with them a bit to get them into that form. In fact, this is seldom useful. In this section, we give several examples of very common universal properties as they are commonly written down, and translate them into the form above, in order to convince the likely disbelieving reader that many familiar universal properties really are of this form.

Example 70 (tensor algebra)**.** One often sees some variation of the following universal characterization of the tensor algebra, which was taken (almost) verbatim from Wikipedia. We will try to stretch it to fit our definition, following the logic through in some detail.

Let V be a vector space over a field k, and let A be an algebra over k. The tensor algebra $T(V)$ satisfies the following universal property.

Any linear transformation $f\colon V \to A$ from V to A can be uniquely extended to an algebra homomorphism $T(V) \to A$ as indicated by the following commutative diagram.

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

As it turns out, it will take rather a lot of stretching.

Let $\mathfrak{U}\colon k\text{-}\mathsf{Alg} \to \mathsf{Vect}_k$ be the forgetful functor which assigns to each algebra over a field k its underlying vector space. Pick some k-vector space V. We consider the category $(V \downarrow \mathfrak{U})$, which is given by the following diagram.

$$1 \xrightarrow{\mathcal{F}_V} \mathsf{Vect}_k \xleftarrow{\mathfrak{U}} k\text{-}\mathsf{Alg}$$

By Definition 66, the objects of $(V \downarrow \mathfrak{U})$ are pairs (A, L), where A is a k-algebra and L is a linear map $V \to \mathfrak{U}(A)$. The morphisms are algebra homomorphisms $\rho\colon A \to A'$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "L"'] \arrow[dr, "L'"] & \\ \mathfrak{U}(A) \arrow[rr, "\mathfrak{U}(\rho)"'] & & \mathfrak{U}(A') \end{tikzcd}$$

commutes. An object $(T(V), i)$ is initial if for any object (A, f) there exists a unique morphism $g\colon T(V) \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "L"'] & \mathfrak{U}(T(V)) \arrow[d, "\mathfrak{U}(g)"] \\ & \mathfrak{U}(A) \end{tikzcd} \qquad \begin{tikzcd} T(V) \arrow[d, "\exists! g"] \\ A \end{tikzcd}$$

commutes.

Thus, the pair $(i, T(V))$ is the initial object in the category $(V \downarrow \mathfrak{U})$. We called $T(V)$ the *tensor algebra* over V.

But what is i? Notice that in the Wikipedia definition above, the map i is from V to $T(V)$, but in the diagram above, it is from V to $\mathfrak{U}(T(V))$. What gives?

The answer that the diagram in Wikipedia's definition does not take place in a specific category. Instead, it implicitly treats $T(V)$ only as a vector space. But this is exactly what the functor $\mathfrak{U}$ does.

Example 71 (product)**.** Here is the universal property for a product, taken verbatim from Wikipedia ([21]).

> Let C be a category with some objects X_1 and X_2. A product of X_1 and X_2 is an object X (often denoted $X_1 \times X_2$) together with a pair of morphisms $\pi_1 \colon X \to X_1$ and $\pi_2 \colon X \to X_2$ such that for every object Y and pair of morphisms $f_1 \colon Y \to X_1$, $f_2 \colon Y \to X_2$, there exists a unique morphism $f \colon Y \to X_1 \times X_2$ such that the following diagram commutes.
>
> $$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

Consider the comma category $(\delta \downarrow (X_1, X_2))$ (where δ is the diagonal functor, see Example 78) given by the following diagram.

$$\mathsf{C} \xrightarrow{\delta} \mathsf{C} \times \mathsf{C} \xleftarrow{\mathcal{F}_{(X_1, X_2)}} \mathsf{1}$$

The objects of this category are pairs $(A, (s, t))$, where $A \in \mathrm{Obj}(\mathsf{C})$ and

$$(s, t) \colon \delta(A) = (A, A) \to (X_1, X_2).$$

The morphisms $(A, (s, t)) \to (B, (u, v))$ are morphisms $r \colon A \to B$ such that the diagram

$$\begin{tikzcd} (A, A) \arrow[rr, "(r,r)"] \arrow[dr, "(s,t)"'] & & (B, B) \arrow[dl, "(u,v)"] \\ & (X_1, X_2) & \end{tikzcd}$$

commutes.

An object $(X_1 \times X_2, (\pi_1, \pi_2))$ is final if for any other object $(Y, (f_1, f_2))$, there exists a unique morphism $f \colon Y \to X_1 \times X_2$ such that the diagram

$$\begin{tikzcd} (Y, Y) \arrow[d, "(f,f)"'] \arrow[dr, "(f_1,f_2)"] & \\ (X_1 \times X_2, X_1 \times X_2) \arrow[r, "(\pi_1,\pi_2)"'] & (X_1, X_2) \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[d, "\exists! f"] \\ X_1 \times X_2 \end{tikzcd}$$

commutes. If we re-arrange our diagram a bit, it is not too hard to see that it is equivalent to the one given above. Thus, we can say: a product of two sets X_1 and X_2 is a final object in the category $(\delta \downarrow (X_1, X_2))$

Example 72 (tensor product)**.** According to the excellent book [3], the tensor product satisfies the following universal property.

> Let V_1 and V_2 be vector spaces. Then we say that a vector space V_3 together with a bilinear map $\iota \colon V_1 \times V_2 \to V_3$ has the *universal property* provided that for any bilinear map $B \colon V_1 \times V_2 \to W$, where W is also a vector space, there exists a

unique linear map $L\colon V_3 \to W$ such that $B = L\iota$. Here is a diagram describing this 'factorization' of B through ι:

$$\begin{tikzcd}
V_1 \times V_2 \arrow[r, "\iota"] \arrow[dr, "B"'] & V_3 \arrow[d, "L"] \\
& W
\end{tikzcd}$$

It turns out that the tensor product defined in this way is neither an initial or final morphism. This is because in the category Vect_k, there is no way of making sense of a bilinear map.

3.7. Why does this structure appear?

We must admit that the categories $(X \downarrow \mathcal{F})$ and $(\mathcal{F} \downarrow X)$ defined above are not very natural, and that the definition of a universal property as either an initial object in $(X \downarrow \mathcal{F})$ or a terminal morphism in $(\mathcal{F} \downarrow X)$ is downright cryptic. Why don't we care, for example, about terminal objects in $(X \downarrow \mathcal{F})$ or initial objects in $(\mathcal{F} \downarrow X)$?

We will have to wait until Chapter 7 to answer this question properly; for now, the reader will have to content him or herself with the knowledge that these *are* the correct notions to study.

4. Cartesian closed categories

4.1. Cartesian closed categories

This section loosely follows [24].

One of the limiting aspects of category theory is that it is very difficult to talk about specifics: one is not allowed to talk about the elements of sets, or the

4.2. Products

We saw in Example 71 the definition for a categorical product.

Definition 73 (categorical product)**.** Let C be a category, and A and B in Obj(C). A product of A and B is an object X (often denoted $A \times B$) together with a pair of morphisms $\pi_1 : X \to A$ and $\pi_2 : X \to B$ such that for every object Y and pair of morphisms $f_1 : Y \to A$, $f_2 : Y \to B$, there exists a unique morphism $f : Y \to A \times B$ such that the following diagram commutes.

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ A & X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

Example 74. In the category Set, the categorical product is the Cartesian product. More specifically:

- For any two sets A and B, the product object X is $A \times B$
- The projectios π_1 and π_2 are defined by

$$\pi_1(a, b) = a, \qquad \pi_2(a, b) = b.$$

To check this, we must show that X, together with π_1 and π_2 satisfy the universal property for products.

Let Y be any set and choose functions

$$f_1 : Y \to A, \qquad \text{and} \qquad f_2 : Y \to B.$$

In some categories, we can take the product of any two objects; one generally says that such a category *has products.*

Definition 75 (category with products)**.** Let C be a category such that for every two objects $A, B \in \mathrm{Obj}(C)$, there exists an object $A \times B$ which satisfies the universal property (Example 71). Then we say that C has products.

Note 76. Sometimes people call a category with products a *Cartesian category*, but others use this terminology to mean a category with all small limits.[1] We will avoid it altogether.

Recall that in Example 31 we showed that the Cartesian product of two sets is functorial. The same turns out to be true for the categorical product.

Theorem 77. Let C be a category with products. Then the product can be extended to a bifunctor (Definition 30) $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$.

Proof. Let $X, Y \in \mathrm{Obj}(C)$. We need to check that the assignment $(X, Y) \mapsto \times(X, Y) \equiv X \times Y$ is functorial, i.e. that $\times$ assigns

- to each pair $(X, Y) \in \mathrm{Obj}(\mathsf{C} \times \mathsf{C})$ an object $X \times Y \in \mathrm{Obj}(\mathsf{C})$ and
- to each pair of morphisms $(f, g)\colon (X, Y) \to (X', Y')$ a morphism

  $$\times(f, g) = f \times g\colon X \times Y \to X' \times Y'$$

  such that

  - $\mathrm{id}_X \times \mathrm{id}_Y = \mathrm{id}_{X\times Y}$, and
  - $\times$ respects composition as follows.

$$\begin{tikzcd}
(X, Y) \arrow[r, "{(f,g)}"] \arrow[rr, bend left, "{(f',g')\circ(f,g)=(f'\circ f,g'\circ g)}"] & (X', Y') \arrow[r, "{(f',g')}"] \arrow[d, "\times"] & (X'', Y'') \\
X \times Y \arrow[r, "f\times g"] \arrow[rr, bend right, "{f'\times g'\circ f\times g=(f'\circ f)\times(g'\circ g)}"'] & X' \times Y' \arrow[r, "f'\times g'"] & X'' \times Y''
\end{tikzcd}$$

We know how $\times$ assigns objects in $\mathsf{C}\times\mathsf{C}$ to objects in C. We need to figure out how $\times$ should assign to a morphism (f, g) in $\mathsf{C}\times\mathsf{C}$ a morphism $f \times g$ in C. We do this by diagram chasing.

Suppose we are given two maps $f : X_1 \to X_2$ and $g\colon Y_1 \to Y_2$. We can view this as a morphism (f, g) in $\mathsf{C} \times \mathsf{C}$.

Recall the universal property for products: a product $X_1 \times Y_1$ is a final object in the category $(\Delta \downarrow (X_1, Y_1))$. Objects in this category can be thought of as diagrams in $\mathsf{C} \times \mathsf{C}$.

$$(X_1 \times Y_1, X_1 \times Y_1) \xrightarrow{\pi_1,\pi_2} (X_1, Y_1)\ .$$

By assumption, we can take the product of both X_1 and Y_1, and X_2 and Y_2. This gives us two

[1]We will see later that any category with both products and equalizers has all finite limits.

diagrams living in C × C, which we can put next to each other.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[r, "(\pi_1,\pi_2)"] & (X_1, Y_1) \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"] & (X_2, Y_2)
\end{tikzcd}$$

We can draw in our morphism (f, g), and take its composition with (π_1, π_2).

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[r, "(\pi_1,\pi_2)"] \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"'] & (X_1, Y_1) \arrow[d, "(f,g)"] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2, Y_2)
\end{tikzcd}$$

Now forget about the top right of the diagram. I'll erase it to make this easier.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"] & \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2, Y_2)
\end{tikzcd}$$

The universal property for products says that there exists a unique map $h\colon X_1 \times Y_1 \to X_2 \times Y_2$ such that the diagram below commutes.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[d, "(h,h)"'] \arrow[dr, "(f\circ\pi_1, g\circ\pi_2)"] & & X_1 \times Y_1 \arrow[d, "\exists! h"] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_1, X_2) & X_2 \times Y_2
\end{tikzcd}$$

And h is what we will use for the product $f \times g$.

Of course, we must also check that h behaves appropriately. Draw two copies of the diagram for the terminal object in $(\Delta \downarrow (X, Y))$ and identity arrows between them.

$$\begin{tikzcd}
(X \times Y, X \times Y) \arrow[r, "(\pi_1,\pi_2)"] \arrow[d, "(\mathrm{id}_{X\times Y}, \mathrm{id}_{X\times Y})"'] & (X, Y) \arrow[d, "(\mathrm{id}_X, \mathrm{id}_Y)"] \\
(X \times Y, X \times Y) \arrow[r, "(\pi_1,\pi_2)"'] & (X, Y)
\end{tikzcd}$$

Just as before, we compose the morphisms to and from the top right to draw a diagonal arrow, and then erase the top right object and the arrows to and from it.

$$\begin{tikzcd}
(X\times Y, X\times Y) \arrow[d, "{(\mathrm{id}_{X\times Y}, \mathrm{id}_{X\times Y})=(\mathrm{id}_X\times\mathrm{id}_Y, \mathrm{id}_X\times\mathrm{id}_Y)}"'] \arrow[dr, "{(\mathrm{id}_X\circ\pi_1, \mathrm{id}_Y\circ\pi_2)}"] & \\
(X\times Y, X\times Y) \arrow[r, "{(\pi_1,\pi_2)}"'] & (X, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X\times Y \arrow[d, "\mathrm{id}_X\times\mathrm{id}_Y=\mathrm{id}_X\times\mathrm{id}_Y"] \\
X\times Y
\end{tikzcd}$$

By definition, the arrow on the right is $\mathrm{id}_X\times\mathrm{id}_Y$. It is also $\mathrm{id}_{X\times Y}$, and by the universal property it is unique. Therefore $\mathrm{id}_{X\times Y} = \mathrm{id}_X \times \mathrm{id}_Y$.

Next, put three of these objects together. The proof of the last part is immediate.

$$\begin{tikzcd}
(X_1\times Y_1, X_1\times Y_1) \arrow[d, "{(f\times g, f\times g)}"] \arrow[dd, bend right=60, "{(f'\times g')\circ(f\times g)=(g'\circ g)\times(f'\circ f)}"'] & (X_1, Y_1) \arrow[d, "{(f,g)}"'] \arrow[dd, bend left=60, "{(g'\circ g, f'\circ f)}"] \\
(X_2\times Y_2, X_2\times Y_2) \arrow[d, "{(f'\times g', f'\times g')}"] & (X_2, Y_2) \arrow[d, "{(f',g')}"'] \\
(X_3\times Y_3, X_3\times Y_3) & (X_3, Y_3)
\end{tikzcd}$$

□

The meaning of this theorem is that in any category where you can take products of the objects, you can also take products of the morphisms.

Example 78 (diagonal functor)**.** Let C be a category, C × C the product category of C with itself. The diagonal functor $\Delta\colon \mathsf{C} \to \mathsf{C}\times\mathsf{C}$ is the functor which sends

- each object $A \in \mathrm{Obj}(\mathsf{C})$ to the pair $(A, A) \in \mathrm{Obj}(\mathsf{C}\times\mathsf{C})$, and
- each morphism $f\colon A \to B$ to the ordered pair

$$(f, f) \in \mathrm{Hom}_{\mathsf{C}\times\mathsf{C}}(A\times B, A\times B).$$

Example 79. The category Vect_k of vector spaces over a field k has the direct sum $\oplus$ as a product.

The product is, in an appropriate way, commutative.

Lemma 80. There is a natural isomorphism between the following functors $\mathsf{C}\times\mathsf{C} \to \mathsf{C}$

$$\times\colon (A, B) \to A\times B \qquad \text{and} \qquad \tilde{\times}\colon (A, B) \to B\times A.$$

Proof. We define a natural transformation $\Phi\colon \times \Rightarrow \tilde{\times}$ with components

$$\Phi_{A,B}\colon A\times B \to B\times A$$

as follows. Denote the canonical projections for the product $A\times B$ by π_A and π_B. Then (π_B, π_A) is a map $A\times B \to (B, A)$, and the universal property for products gives us a map $A\times B \to B\times A$.

We can pull the same trick to go from $B \times A$ to $A \times B$, using the pair (π_A, π_B) and the universal property. Furthermore, these maps are inverse to each other, so $\Phi_{A,B}$ is an isomorphism.

We need only check naturality, i.e. that for $f: A \to A'$, $g: B \to B'$, the following square commutes.

$$\begin{tikzcd} A \times B \arrow[r] \arrow[d] & A' \times B' \arrow[d] \\ B \times A \arrow[r] & B' \times A' \end{tikzcd}$$

□

Note 81. It is an important fact that the product is associative. We shall see this in Theorem 115, after we have developed the machinery to do it cleanly.

4.3. Coproducts

Definition 82. Let C be a category, X_1 and $X_2 \in \mathrm{Obj}(\mathrm{C})$. The coproduct of X_1 and X_2, denoted $X_1 \amalg X_2$, is the initial object in the category $((X_1, X_2) \downarrow \Delta)$. In everyday language, we have the following.

An object $X_1 \amalg X_2$ is called the coproduct of X_1 and X_2 if

1. there exist morphisms $i_1: X_1 \to X_1 \amalg X_2$ and $i_2: X_2 \to X_1 \amalg X_2$ called *canonical injections* such that
2. for any object Y and morphisms $f_1: X_1 \to Y$ and $f_2: X_2 \to Y$ there exists a unique morphism $f: X_1 \amalg X_2 \to Y$ such that $f_1 = f \circ i_1$ and $f = f_2 = f \circ i_2$, i.e. the following diagram commutes.

$$\begin{tikzcd} & Y & \\ X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \amalg X_2 \arrow[u, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"] \end{tikzcd}$$

Definition 83 (category with coproducts)**.** We say that a category C has coproducts if for all $A, B \in \mathrm{Obj}(C)$, the coproduct $A \amalg B$ is in $\mathrm{Obj}(C)$.

We have the following analog of Theorem 77.

Theorem 84. Let C be a category with coproducts. Then we have a functor $\amalg: C \times C \to C$.

Proof. We verify that $\amalg$ allows us to define canonically a coproduct of morphisms. Let X_1, X_2, Y_1, $Y_2 \in \mathrm{Obj}(C)$, and let $f: X_1 \to Y_1$ and $g: X_2 \to Y_2$. We have the following diagram.

$$\begin{tikzcd} X_1 \arrow[r, "i_{1,X}"] \arrow[d, "f"'] \arrow[dr, "i_{i,Y} \circ f"] & X_1 \amalg X_2 & X_2 \arrow[l, "i_{2,X}"'] \arrow[d, "g"] \arrow[dl, "i_{2,Y} \circ g"'] \\ Y_1 \arrow[r, "i_{1,Y}"] & Y_1 \amalg Y_2 & Y_2 \arrow[l, "i_{2,Y}"'] \end{tikzcd}$$

But by the universal property of coproducts, the diagonal morphisms induce a map $X_1 \amalg X_2 \to Y_1 \amalg Y_2$, which we define to be $f \amalg g$.

$$\begin{tikzcd}
X_1 \arrow[r, "i_{1,X}"] \arrow[dr, "i_{i,Y}\circ f"'] & X_1 \amalg X_2 \arrow[d, "f\amalg g"] & X_2 \arrow[l, "i_{2,X}"'] \arrow[dl, "i_{2,Y}\circ g"] \\
& Y_1 \amalg Y_2 &
\end{tikzcd}$$

The rest of the verification that $\amalg$ really is a functor is identical to that in Theorem 77. $\square$

Example 85. In Set, the coproduct is the disjoint union.

Example 86. In Vect_k, the coproduct $\oplus$ is the direct sum.

Example 87. In k-Alg the tensor product is the coproduct. To see this, let A, B, and R be k-algebras. The tensor product $A \otimes_k B$ has canonical injections $\iota_1 \colon A \to A \otimes B$ and $\iota_2 \colon B \to A \otimes B$ given by

$$\iota_A \colon v \to v \otimes \mathrm{id}_B \qquad \text{and} \qquad \iota_B \colon v \mapsto \mathrm{id}_A \otimes v.$$

Let $f_1 \colon A \to R$ and $f_2 \colon B \to R$ be k-algebra homomorphisms. Then there is a unique homomorphism $f \colon A \otimes B \to R$ which makes the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[dr, "f_1"'] & A \otimes B \arrow[d, "f"] & B \arrow[l, "\iota_B"'] \arrow[dl, "f_2"] \\
& R &
\end{tikzcd}$$

4.4. Biproducts

A lot of the stuff in this section was adapted and expanded from [33].

One often hears that a biproduct is an object which is both a product and a coproduct, but this is both misleading and unnecessarily vague. Objects satisfying universal properties are only defined up to isomorphism, so it doesn't make much sense to demand that products *coincide* with coproducts. However, demanding only that they be isomorphic (or even naturally isomorphic) is too weak, and does not determine a biproduct uniquely. In this section we will give the correct definition which encapsulates all of the properties of the product and the coproduct we know and love.

In order to talk about biproducts, we need to be aware of the following result, which we will consider in much more detail in Definition 10.2.

Definition 88 (zero morphism)**.** Let C be a category with zero object 0. For any two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd}
A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B
\end{tikzcd}$$

Notation 89. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}. With this notation, for all morphisms f and g we have $f \circ 0 = 0$ and $0 \circ g = 0$.

Definition 90 (category with biproducts)**.** Let C be a category with zero object 0, products $\times$ (with canonical projections π), and coproducts $\amalg$ (with natural injections ι). We say that C has biproducts if for each two objects $A_1, A_2 \in \mathrm{Obj}(\mathsf{C})$, there exists an isomorphism

$$\Phi_{A_1,A_2} \colon A_1 \amalg A_2 \to A_1 \times A_2$$

such that

$$A_i \xrightarrow{\iota_i} A_1 \amalg A_2 \xrightarrow{\Phi_{A_1,A_2}} A_1 \times A_2 \xrightarrow{\pi_j} A_j \quad = \begin{cases} \mathrm{id}_{A_1}, & i = j \\ 0, & i \neq j. \end{cases}$$

Here is a picture.

$$\begin{tikzcd} & A_1 \amalg A_2 \arrow[dd, "\Phi_{A_1,A_2}"] & \\ A_1 \arrow[ur, "\iota_1"] & & A_2 \arrow[ul, "\iota_2"'] \\ & A_1 \times A_2 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Lemma 91. The isomorphisms $\Phi_{A,B}$ are unique if they exist.

Proof. We can compose Φ_{A_1,A_2} with π_1 and π_2 to get maps $A_1 \amalg A_2 \to A_1$ and $A_1 \amalg A_2 \to A_2$. The universal property for products tells us that there is a unique map $\varphi \colon A_1 \amalg A_2 \to A_1 \times A_2$ such that $\pi_1 \circ \varphi = \pi_1 \circ \Phi_{A_1,A_2}$ and $\pi_2 \circ \varphi = \pi_2 \circ \Phi_{A_1,A_2}$; but φ is just Φ_{A_1,A_2}. Hence Φ_{A_1,A_2} is unique. $\square$

Lemma 92. In any category C with biproducts $\Phi_{A,B} \colon A \amalg B \to A \times B$, any map $A \amalg B \to A' \times B'$ is completely determined by its components $A \to A'$, $A \to B'$, $B \to A'$, and $B \to B'$. That is, given any map $f \colon A \amalg B \to A' \times B'$, we can create four maps

$$\pi_{A'} \circ f \circ \iota_A \colon A \to A', \qquad \pi_{B'} \circ f \circ \iota_A \colon A \to B', \qquad \text{etc},$$

and given four maps

$$\psi_{A,A'} \colon A \to A'; \qquad \psi_{A,B'} \colon A \to B'; \qquad \psi_{B,A'} \colon B \to A', \qquad \text{and } \psi_{B,B'} \colon B \to B',$$

we can construct a unique map $\psi \colon A \amalg B \to A' \times B'$ such that

$$\psi_{A,A'} = \pi_{A'} \circ \psi \circ \iota_A, \qquad \psi_{A,B'} = \pi_{B'} \circ \psi \circ \iota_A, \qquad \text{etc.}$$

Proof. The universal property for coproducts give allows us to turn the morphisms $\psi_{A,A'}$ and $\psi_{A,B'}$ into a map $\psi_A \colon A \to B \times B'$, the unique map such that

$$\pi_{A'} \circ \psi_A = \psi_{A,A'} \qquad \text{and} \qquad \pi_{B'} \circ \psi_A = \psi_{A,B'}.$$

The morphisms $\psi_{B,A'}$ and $\psi_{B,B'}$ give us a map ψ_B which is unique in the same way.

Now the universal property for coproducts gives us a map $\psi \colon A \amalg B \to A' \times B'$, the unique map such that $\psi \circ \iota_A = \psi_A$ and $\psi \circ \iota_B = \psi_B$. $\square$

Theorem 93. The $\Phi_{A,B}$ defined above form the components of a natural isomorphism.

Proof. Let A, B, A', and $B' \in \mathrm{Obj}(\mathsf{C})$, and let $f\colon A \to A'$ and $g\colon B \to B'$. To check that $\Phi_{A,B}$ are the components of a natural isomorphism, we have to check that the following naturality square commutes.

$$\begin{tikzcd} A \amalg B \arrow[r, "f \amalg g"] \arrow[d, "\Phi_{A,B}"'] & A' \amalg B' \arrow[d, "\Phi_{A',B'}"] \\ A \times B \arrow[r, "f \times g"'] & A' \times B' \end{tikzcd}$$

That is, we need to check that $(f \times g) \circ \Phi_{A,B} = \Phi_{A',B'} \circ (f \amalg g)$.

Both of these are morphisms $A \amalg B \to A' \times B'$, and by Lemma 92, it suffices to check that their components agree, i.e. that □

Example 94. In the category Vect_k of vector spaces over a field k, the direct sum $\oplus$ is a biproduct.

Example 95. In the category Ab of abelian groups, the direct sum $\oplus$ is both a product and a coproduct. Hence Ab has biproducts.

4.5. Exponentials

Astonishingly, we can talk about functional evaluation purely in terms of products and universal properties.

Definition 96 (exponential)**.** Let C be a category with products, A, $B \in \mathrm{Obj}(\mathsf{C})$. The exponential of A and B is an object $B^A \in \mathrm{Obj}(\mathsf{C})$ together with a morphism $\varepsilon\colon B^A \times A \to B$, called the *evaluation morphism*, which satisfies the following universal property.

> For any object $X \in \mathrm{Obj}(\mathsf{C})$ and morphism $f\colon X \times A \to B$, there exists a unique morphism $\bar{f}\colon X \to B^A$ which makes the following diagram commute.

$$\begin{tikzcd} B^A & & B^A \times A \arrow[r, "\varepsilon"] & B \\ X \arrow[u, "\exists! \bar{f}"] & & X \times A \arrow[u, "\bar{f} \times \mathrm{id}_A"] \arrow[ur, "f"'] & \end{tikzcd}$$

Example 97. In Set, the exponential object B^A is the set of functions $A \to B$, and the evaluation morphism ε is the map which assigns $(f, a) \mapsto f(a)$. Let us check that these objects indeed satisfy the universal property given.

Suppose we are given a function $f\colon X \times A \to B$. We want to construct from this a function $\bar{f}\colon X \to B^A$, i.e. a function which takes an element $x \in X$ and returns a function $\bar{f}(x)\colon A \to B$. There is a natural way to do this: fill the first slot of f with x! That is to say, define $\bar{f}(x) = f(x, -)$. It is easy to see that this makes the diagram commute:

$$\begin{tikzcd} (f(x,-), a) \arrow[r, mapsto, "\varepsilon"] & f(x, a) \\ (x, a) \arrow[u, mapsto, "\bar{f} \times \mathrm{id}_A"] \arrow[ur, mapsto, "f"'] & \end{tikzcd}$$

Furthermore, $\bar{f}$ is unique since if it sent x to any function other than f the diagram would not commute.

Example 98. One might suspect that the above generalizes to all sorts of categories. For example, one might hope that in Grp, the exponential H^G of two groups G and H would somehow capture all homomorphisms $G \to H$. This turns out to be impossible; we will see why in REF.

4.6. Cartesian closed categories

Definition 99 (cartesian closed category)**.** A category C is cartesian closed if it has

1. products: for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $A \times B \in \mathrm{Obj}(\mathsf{C})$;
2. exponentials: for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $B^A \in \mathrm{Obj}(\mathsf{C})$;
3. a terminal element (Definition 60) $1 \in \mathsf{C}$.

Cartesian closed categories have many inveresting properties. For example, they replicate many properties of the integers: we have the following familiar formulae for any objects A, B, and C in a Cartesian closed category with coproducts $+$.

1. $A \times (B + C) \simeq (A \times B) + (A \times C)$
2. $C^{A+B} \simeq C^A \times C^B$.
3. $(C^A)^B \simeq C^{A \times B}$
4. $(A \times B)^C \simeq A^C \times B^C$
5. $C^1 \simeq C$

We could prove these immediately, by writing down diagrams and appealing to the universal properties. However, we will soon have a powerful tool, the Yoneda lemma, that makes their proof completely routine.

5. Hom functors and the Yoneda lemma

5.1. The Hom functor

Given any locally small category (Definition 25) C and two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, we have thus far notated the set of all morphisms $A \to B$ by $\mathrm{Hom}_{\mathsf{C}}(A, B)$. This may have seemed an odd notation, but there was a good reason for it: $\mathrm{Hom}_{\mathsf{C}}$ is really a functor.

To be more explicit, we can view $\mathrm{Hom}_{\mathsf{C}}$ in the following way: it takes two objects, say A and B, and returns the set of morphisms $A \to B$. It's a functor $\mathsf{C} \times \mathsf{C}$ to set!

(Actually, as we'll see, this isn't quite right: it is actually a functor $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$. But this is easier to understand if it comes about in a natural way, so we'll keep writing $\mathsf{C} \times \mathsf{C}$ for the time being.)

Okay, so $\mathrm{Hom}_{\mathsf{C}}$ takes a pair of objects and returns the set of morphisms between them, but any functor has to send morphisms to morphisms. How does it do that?

A morphism in $\mathsf{C} \times \mathsf{C}$ between two objects (A', A) and (B', B) is an ordered pair (f', f) of two morphisms:

$$f : A \to B \qquad \text{and} \qquad f' : A' \to B'.$$

We can draw this as follows.

$$\begin{tikzcd} A' \arrow[r, "f'"] & B' \\ A \arrow[r, "f"] & B \end{tikzcd}$$

We have to send this to a set-function $\mathrm{Hom}_{\mathsf{C}}(A', A) \to \mathrm{Hom}_{\mathsf{C}}(B', B)$. So let m be a morphism in $\mathrm{Hom}_{\mathsf{C}}(A, A')$. We can draw this into our little diagram above like so.

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "m"'] & B' \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Notice: we want to build from m, f, and f' a morphism $B' \to B$. Suppose f' were going the other way.

$$\begin{tikzcd} A' \arrow[d, "m"'] & B' \arrow[l, "f'"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Then we could get an element of $\mathrm{Hom}_{\mathsf{C}}(B, B')$ simply by taking the composition $f \circ m \circ f'$.

$$\begin{tikzcd} A' \arrow[d, "m"'] & B' \arrow[l, "f'"'] \arrow[d, dashed, "f \circ m \circ f'"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

But we can make f' go the other way simply by making the first argument of $\mathrm{Hom}_{\mathsf{C}}$ come from the opposite category: that is, we want $\mathrm{Hom}_{\mathsf{C}}$ to be a functor $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$.

As the function $m \mapsto f \circ m \circ f'$ is the action of the functor $\mathrm{Hom}_{\mathsf{C}}$ on the morphism (f', f), it is natural to call it $\mathrm{Hom}_{\mathsf{C}}(f', f)$.

Of course, we should check that this *really is* a functor, i.e. that it treats identities and compositions correctly. This is completely routine, and you should skip down to Definition 100 if you read the last sentence with annoyance.

First, we need to show that $\mathrm{Hom}_{\mathsf{C}}(\mathrm{id}_{A'}, \mathrm{id}_A)$ is the identity function $\mathrm{id}_{\mathrm{Hom}_{\mathsf{C}}(A',A)}$. It is, since if $m \in \mathrm{Hom}_{\mathsf{C}}(A', A)$,

$$\mathrm{Hom}_{\mathsf{C}}(\mathrm{id}_{A'}, \mathrm{id}_A)\colon m \mapsto \mathrm{id}_A \circ m \circ \mathrm{id}_{A'} = m.$$

Next, we have to check that compositions work the way they're supposed to. Suppose we have objects and morphisms like this:

$$\begin{tikzcd} A' & B' \arrow[l, "f'"'] & C' \arrow[l, "g'"'] \\ A \arrow[r, "f"] & B \arrow[r, "g"] & C \end{tikzcd}$$

where the primed stuff is in C^{op} and the unprimed stuff is in C. This can be viewed as three objects (A', A), etc. in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ and two morphisms (f', f) and (g', g) between them.

Okay, so we can compose (f', f) and (g', g) to get a morphism

$$(g', g) \circ (f', f) = (f' \circ g', g \circ f)\colon (A', A) \to (C', C).$$

Note that the order of the composition in the first argument has been turned around: this is expected since the first argument lives in C^{op}. To check that $\mathrm{Hom}_{\mathsf{C}}$ handles compositions correctly, we need to verify that

$$\mathrm{Hom}_{\mathsf{C}}(g, g') \circ \mathrm{Hom}_{\mathsf{C}}(f, f') = \mathrm{Hom}_{\mathsf{C}}(f' \circ g', g \circ f).$$

Let $m \in \mathrm{Hom}_{\mathsf{C}}(A', A)$. We victoriously compute

$$\begin{aligned} [\mathrm{Hom}_{\mathsf{C}}(g, g') \circ \mathrm{Hom}_{\mathsf{C}}(f, f')](m) &= \mathrm{Hom}_{\mathsf{C}}(g, g')(f \circ m \circ f') \\ &= g \circ f \circ m \circ f' \circ g' \\ &= \mathrm{Hom}_{\mathsf{C}}(f' \circ g', g \circ f). \end{aligned}$$

Let's formalize this in a definition.

Definition 100 (hom functor)**.** Let C be a locally small category. The hom functor $\mathrm{Hom}_{\mathsf{C}}$ is the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$$

which sends the object (A', A) to the set $\mathrm{Hom}_{\mathsf{C}}(A', A)$ of morphisms $A' \to A$, and the morphism $(f', f)\colon (A', A) \to (B', B)$ to the function

$$\mathrm{Hom}_{\mathsf{C}}(f', f)\colon \mathrm{Hom}_{\mathsf{C}}(A', A) \to \mathrm{Hom}_{\mathsf{C}}(B', B); \qquad m \mapsto f \circ m \circ f'.$$

Example 101. Hom functors give us a new way of looking at the universal property for products and coproducts.

Let C be a locally small category with products, and let X, A, $B \in \mathrm{Obj}(\mathsf{C})$. Recall that the universal property for the product $A \times B$ allows us to exchange two morphisms

$$f_1 \colon X \to A \qquad \text{and} \qquad f_2 \colon X \to B$$

for a morphism

$$f \colon X \to A \times B.$$

We can also compose a morphism $g \colon X \to A \times B$ with the canonical projections

$$\pi_A \colon A \times B \to A \qquad \text{and} \qquad \pi_B \colon A \times B \to B$$

to get two morphisms

$$\pi_A \circ f \colon X \to A \qquad \text{and} \qquad \pi_B \circ f \colon X \to B.$$

This means that there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(X, A \times B) \simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B).$$

In fact this bijection is natural in X. That is to say, there is a natural bijection between the following functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$:

$$h_{A \times B} \colon X \to \mathrm{Hom}_{\mathsf{C}}(X, A \times B) \qquad \text{and} \qquad h_A \times h_B \colon X \to \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B).$$

Let's prove naturality. Let

$$\Phi_X \colon \mathrm{Hom}(X, A \times B) \to \mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B); \qquad f \mapsto (\pi_A \circ f, \pi_B \circ f)$$

be the components of the above transformation, and let $g \colon X' \to X$. Naturality follows from the fact that the diagram below commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X, A \times B) \arrow[r, "{\mathrm{Hom}(g, \mathrm{id}_{A \times B})}"] \arrow[d, "\Phi_X"'] & \mathrm{Hom}(X', A \times B) \arrow[d, "\Phi_{X'}"] \\
\mathrm{Hom}(X, A) \times \mathrm{Hom}(X, B) \arrow[r, "{\mathrm{Hom}(g, \mathrm{id}_A) \times \mathrm{Hom}(g, \mathrm{id}_B)}"] & \mathrm{Hom}(X', A) \times \mathrm{Hom}(X', B)
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & f \circ g \arrow[d, mapsto] \\
(\pi_A \circ f, \pi_B \circ g) \arrow[r, mapsto] & (\pi_A \circ f \circ g, \pi_B \circ f \circ g)
\end{tikzcd}$$

The coproduct does a similar thing: it allows us to trade two morphisms

$$A \to X \qquad \text{and} \qquad B \to X$$

for a morphism

$$A \amalg B \to X,$$

and vice versa. Similar reasoning yields a natural bijection between the following functors $\mathsf{C} \to \mathsf{Set}$:

$$h^{A \amalg B} \Rightarrow h^A \times h^B.$$

This example is really a reflection of a concept we will meet in Chapter **??**: the hom functor commutes with limits in the first slot, and turns colimits into limits in the second.

Example 102. Hom functors also give us a new way of looking at exponential objects. The universal property for exponentials is a little bit more complicated: it allows us to trade a morphism

$$X \times A \to B$$

for a morphism

$$X \to B^A,$$

and vice versa. That is to say, we have a bijection between two functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$

$$X \mapsto \mathrm{Hom}_{\mathsf{C}}(X \times A, B) \qquad \text{and} \qquad X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, B^A).$$

It is easy to check that this bijection is again natural.

This example is a reflection of yet another concept (which we will meet in Section 7), that of an adjunction.

5.2. Currying the hom functor

The hom functor as defined above is cute, but not a whole lot else. Things get interesting when we curry it.

Recall from computer science the concept of currying. Suppose we are given a function of two arguments, say $f(x, y)$. The idea of currying is this: if we like, we can view f as a family of functions of only one variable y, indexed by x:

$$h_x(y) = f(x, y).$$

We can even view h as a function which takes one argument x, and which returns a *function* h_x of one variable y.

This sets up a correspondence between functions $f: A \times B \to C$ and functions $h: A \to C^B$, where C^B is the set of functions $B \to C$. The map which replaces f by h is called *currying*. We can also go the other way (i.e. $h \mapsto f$), which is called *uncurrying*.

We have been intentionally vague about the nature of our function f, and what sort of arguments it might take; everything we have said also holds for, say, bifunctors.

In particular, it gives us two more ways to view our bifunctor $\mathrm{Hom}_{\mathsf{C}}$. We can fix any $A \in \mathrm{Obj}(\mathsf{C})$ and curry either argument. This gives us the following.

Lemma 103. Let $\mathcal{F}: \mathsf{C} \to \mathsf{D}$ be a functor between locally small categories. Then $\mathcal{F}$ induces a natural transformation with components

$$\mathcal{F}_{A,B}: \mathrm{Hom}_{\mathsf{C}}(A, B) \Rightarrow \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(B)); \qquad (f: A \to B) \mapsto (\mathcal{F}(f): \mathcal{F}(A) \to \mathcal{F}(B)).$$

Proof. The following square commutes.

$$\begin{tikzcd}
\mathrm{Hom}_C(A,B) \arrow[rrr, "\mathcal{F}_{A,B}"] \arrow[ddd, "g\circ(-)\circ f"'] & & & \mathrm{Hom}_D(\mathcal{F}(A),\mathcal{F}(B)) \arrow[ddd, "\mathcal{F}(g)\circ\mathcal{F}(-)\circ\mathcal{F}(f)"] \\
& h \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{F}(h) \arrow[d, mapsto] & \\
& g\circ h\circ f \arrow[r, mapsto] & \mathcal{F}(g\circ h\circ f)=\mathcal{F}(g)\circ\mathcal{F}(h)\circ\mathcal{F}(f) & \\
\mathrm{Hom}_C(A',B') \arrow[rrr, "\mathcal{F}_{A',B'}"] & & & \mathrm{Hom}_D(\mathcal{F}(A'),\mathcal{F}(B'))
\end{tikzcd}$$

□

Definition 104 (curried hom functor)**.** Let C be a locally small category, $A \in \mathrm{Obj}(\mathsf{C})$. We can construct from the hom functor $\mathrm{Hom}_{\mathsf{C}}$

- a functor $h^A\colon \mathsf{C} \to \mathsf{Set}$ which maps
  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(A,B)$, and
  - a morphism $f\colon B \to B'$ to a Set-function

$$h^A(f)\colon\ \mathrm{Hom}_{\mathsf{C}}(A,B) \to \mathrm{Hom}_{\mathsf{C}}(A,B'); \qquad m \mapsto f\circ m.$$

- a functor $h_A\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ which maps
  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(B,A)$, and
  - a morphism $f\colon B \to B'$ to a Set-function

$$h_A(f)\colon\ \mathrm{Hom}_{\mathsf{C}}(B',A) \to \mathrm{Hom}_{\mathsf{C}}(B,A); \qquad m \mapsto m\circ f.$$

5.3. Representable functors

Roughly speaking, a functor $\mathsf{C} \to \mathsf{Set}$ is representable if it is

Definition 105 (representable functor)**.** Let C be a category. A functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{Set}$ is representable if there is an object $A \in \mathrm{Obj}(\mathsf{C})$ and a natural isomorphism (Definition 42)

$$\eta\colon \mathcal{F} \Rightarrow \begin{cases} h^A, & \text{if } \mathcal{F} \text{ is covariant} \\ h_A, & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

Note 106. Since natural isomorphisms are invertible, we could equivalently define a representable functor with the natural isomorphism going the other way.

Example 107. Consider the forgetful functor $\mathcal{U}\colon \mathsf{Grp} \to \mathsf{Set}$ which sends a group to its underlying set, and a group homomorphism to its underlying function. This functor is represented by the group $(\mathbb{Z},+)$.

To see this, we have to check that there is a natural isomorphism η between $\mathcal{U}$ and $h^{\mathbb{Z}} = \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, -)$. That is to say, for each group G, there a Set-isomorphism (i.e. a bijection)

$$\eta_G \colon \mathcal{U}(G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$$

satisfying the naturality conditions in Definition 41.

First, let's show that there's a bijection by providing an injection in both directions. Pick some $g \in G$. Then there is a unique group homomorphism which sends $1 \mapsto g$, so we have an injection $\mathcal{U}(G) \hookrightarrow \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$.

Now suppose we are given a group homomorphism $\mathbb{Z} \to G$. This sends 1 to some element $g \in G$, and this completely determines the rest of the homomorphism. Thus, we have an injection $\mathrm{Hom}_{\mathsf{Grp}(\mathbb{Z},G)} \hookrightarrow \mathcal{U}(G)$.

All that is left is to show that η satisfies the naturality condition. Let F and H be groups, and $f \colon G \to H$ a homomorphism. We need to show that the following diagram commutes.

$$\begin{tikzcd}
\mathcal{U}(G) \arrow[r, "\mathcal{U}(f)"] \arrow[d, "\eta_G"'] & \mathcal{U}(H) \arrow[d, "\eta_H"] \\
\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \arrow[r, "h^{\mathbb{Z}}(f)"] & \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, H)
\end{tikzcd}$$

The upper path from top left to bottom right assigns to each $g \in G$ the function $\mathbb{Z} \to G$ which maps $1 \mapsto f(g)$. Walking down η_G from $\mathcal{U}(G)$ to $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$, g is mapped to the function $\mathbb{Z} \to G$ which maps $1 \mapsto g$. Walking right, we compose this function with f to get a new function $\mathbb{Z} \to H$ which sends $1 \mapsto f(g)$. This function is really the image of a homomorphism and is therefore unique, so the diagram commutes.

5.4. The Yoneda embedding

The Yoneda embedding is a very powerful tool which allows us to prove things about any locally small category by embedding that category into Set, and using the enormous amount of structure that Set has.

Definition 108 (Yoneda embedding)**.** Let C be a locally small category. The Yoneda embedding is the functor

$$\text{よ} \colon \mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \qquad A \mapsto h_A = \mathrm{Hom}_{\mathsf{C}}(-, A).$$

where $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ is the category of functors $\mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$, defined in Definition 48.

Of course, we also need to say how よ behaves on morphisms. Let $f \colon A \to A'$. Then よ(f) will be a morphism from h_A to $h_{A'}$, i.e. a natural transformation $h_A \Rightarrow h_{A'}$. We can specify how it behaves by specifying its components $(\text{よ}(f))_B$.

Plugging in definitions, we find that $(\text{よ}(f))_B$ is a map $\mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathrm{Hom}_{\mathsf{C}}(B, A')$. But we know how to get such a map—we can compose all of the maps $B \to A$ with f to get maps $B \to A'$. So よ(f), as a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-, A) \to \mathrm{Hom}_{\mathsf{C}}(-, A')$, simply composes everything in sight with f.

Theorem 109 (Yoneda lemma)**.** Let $\mathcal{F}$ be a functor from C^{op} to Set. Let $A \in \mathrm{Obj}(\mathsf{C})$. Then there is a set-isomorphism (i.e. a bijection) η between the set $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, \mathcal{F})$ of natural transformations $h_A \Rightarrow \mathcal{F}$ and the set $\mathcal{F}(A)$. Furthermore, this isomorphism is natural in A.

Note 110. A quick admission before we begin: we will be sweeping issues with size under the rug in what follows by tacitly assuming that $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, \mathcal{F})$ is a set. The reader will have to trust that everything works out in the end.

Proof. A natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ consists of a collection of Set-morphisms (that is to say, functions) $\Phi_B\colon \mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathcal{F}(A)$, one for each $B \in \mathrm{Obj}(\mathsf{C})$. We need to show that to each element of $\mathcal{F}(A)$ there corresponds exactly one Φ. We will do this by showing that any Φ is completely determined by where Φ_A sends the identity morphism $\mathrm{id}_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$, so there is exactly one natural transformation for each place Φ can send id_A.

The proof that this is the case can be illustrated by the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}(A, A) \arrow[rrr, "h_A(f)"] \arrow[ddd, "\Phi_A"'] & & & \mathrm{Hom}(B, A) \arrow[ddd, "\Phi_B"] \\
& \mathrm{id}_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& a \arrow[r, mapsto] & (\mathcal{F}f)(a) \overset{!}{=} \Phi_B(f) & \\
\mathcal{F}(A) \arrow[rrr, "\mathcal{F}(f)"'] & & & \mathcal{F}(B)
\end{tikzcd}$$

Here's what the above diagram means. The natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ has a component $\Phi_A\colon \mathrm{Hom}_{\mathsf{C}}(A, A) \to \mathcal{F}(A)$, and Φ_A has to send the identity transformation id_A *somewhere.* It can send it to any element of $\mathcal{F}(A)$; let's call $\Phi_A(\mathrm{id}_A) = a$.

Now the naturality conditions force our hand. For any $B \in \mathrm{Obj}(\mathsf{C})$ and any $f\colon B \to A$, the naturality square above means that $\Phi_B(f)$ *has to* be equal to $(\mathcal{F}f)(a)$. We get no choice in the matter.

But this completely determines Φ! So we have shown that there is exactly one natural transformation for every element of $\mathcal{F}(A)$. We are done!

Well, almost. We still have to show that the bijection we constructed above is natural. To that end, let $f\colon B \to A$. We need to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, \mathcal{F}) \arrow[r, "\eta_A"] \arrow[d, "\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(f), \mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\
\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_B, \mathcal{F}) \arrow[r, "\eta_B"] & \mathcal{F}(B)
\end{tikzcd}$$

This is notationally dense, and deserves a lot of explanation. The natural transformation $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(f), \mathcal{F})$ looks complicated, but it's really not since we know what the hom functor does: it takes every natural transformation $h_A \Rightarrow \mathcal{F}$ and pre-composes it with $\mathcal{Y}(f)$. The natural transformation η_A takes a natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ and sends it to the element $\Phi_A(\mathrm{id}_A) \in \mathcal{F}(A)$.

So, starting at the top left with a natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$, we can go to the bottom right in two ways.

1. We can head down to $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_B, \mathcal{F})$, mapping
$$\Phi \mapsto \Phi \circ \text{よ}(f),$$
then use η_B to map this to $\mathcal{F}(B)$:
$$\Phi \circ \text{よ}(f) \mapsto (\Phi \circ \text{よ}(f))_B(\mathrm{id}_B).$$
We can simplify this right away. The component of the composition of natural transformations is the composition of the components, i.e.
$$(\Phi \circ \text{よ}(f))_B(\mathrm{id}_B) = (\Phi_B \circ \text{よ}(f)_B)(\mathrm{id}_B).$$
We also know how $\text{よ}(f)$ behaves: it composes everything in sight with f.
$$\text{よ}(f)(\mathrm{id}_B) = f.$$
Thus, we have
$$(\Phi \circ \text{よ}(f))(\mathrm{id}_B) = \Phi_B(f).$$
2. We can first head to the right using η_A. This sends Φ to
$$\Phi_A(\mathrm{id}_A) \in \mathcal{F}(A).$$
We can then map this to $\mathcal{F}(B)$ with f, getting
$$\mathcal{F}(f)(\Phi_A(\mathrm{id}_A)).$$

The naturality condition is thus
$$(\mathcal{F}(f))(\Phi_A(\mathrm{id}_A)) = \Phi_B(f),$$
which we saw above was true for any natural transformation Φ. □

Lemma 111. The Yoneda embedding is fully faithful (Definition 36).

Proof. We have to show that for all $A, B \in \mathrm{Obj}(\mathsf{C})$, the map
$$\text{よ}_{A,B}\colon\ \mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$$
is a bijection.

Fix $B \in \mathrm{Obj}(\mathsf{C})$, and consider the functor h_B. By the Yoneda lemma, there is a bijection between $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$ and $h_B(A)$. By definition,
$$h_B(A) = \mathrm{Hom}_{\mathsf{C}}(A, B).$$

But $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$ is nothing else but $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$, so we are done. □

Note 112. We defined the Yoneda embedding to be the functor $A \mapsto h_A$; this is sometimes called the *contravariant Yoneda embedding*. We could also have studied to map A to h^A, called the *covariant Yoneda embedding*, in which case a slight modification of the proof of the Yoneda lemma would have told us that the map

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(B, A) \to \mathrm{Hom}_{[\mathsf{C},\mathsf{Set}]}(h^A, h^B)$$

is a natural bijection. This is often called the *covariant Yoneda lemma.*

Here is a situation in which the Yoneda lemma is commonly used.

Corollary 113. Let C be a locally small category. Suppose for all $A \in \mathrm{Obj}(\mathsf{C})$ there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A, B')$$

which is natural in A. Then $B \simeq B'$

Proof. We have a natural isomorphism $h_B \Rightarrow h_{B'}$. Since the Yoneda embedding is fully faithful, it is injective on objects up to isomorphism (Lemma 38). Thus, since h_B and $h_{B'}$ are isomorphic, so must be $B \simeq B'$. □

5.5. Applications

We have now built up enough machinery to make the proofs of a great variety of things trivial, as long as we accept a few assertions.

It is possible, for example, to prove that the product is associative in *any* locally small category, just from the fact that it is associative in Set.

Lemma 114. The Cartesian product is associative in Set. That is, there is a natural isomorphism α between $(A \times B) \times C$ and $A \times (B \times C)$ for any sets A, B, and C.

Proof. The isomorphism is given by $\alpha_{A,B,C} \colon ((a, b), c) \mapsto (a, (b, c))$. This is natural because for functions like this,

$$A \xrightarrow{f} A'$$
$$B \xrightarrow{g} B'$$
$$C \xrightarrow{h} C'$$

the following diagram commutes.

$$\begin{tikzcd}
((a,b),c) \arrow[r, mapsto, "((f,g),h)"] \arrow[d, mapsto, "\alpha_{A,B,C}"'] & ((f(a),g(b)),h(c)) \arrow[d, mapsto, "\alpha_{A',B',C'}"] \\
(a,(b,c)) \arrow[r, mapsto, "(f,(g,h))"'] & (f(a),(g(b),h(c)))
\end{tikzcd}$$

□

Theorem 115. In any locally small category C with products, there is a natural isomorphism $(A \times B) \times C \simeq A \times (B \times C)$.

Proof. We have the following string of natural isomorphisms for any $X, A, B, C \in \mathrm{Obj}(\mathsf{C})$.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(X, (A \times B) \times C) &\simeq \mathrm{Hom}_{\mathsf{C}}(X, A \times B) \times \mathrm{Hom}_{\mathsf{C}}(X, C) \\
&\simeq (\mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B)) \times \mathrm{Hom}_{\mathsf{C}}(X, C) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times (\mathrm{Hom}_{\mathsf{C}}(X, B) \times \mathrm{Hom}_{\mathsf{C}}(X, C)) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, A) \times \mathrm{Hom}_{\mathsf{C}}(X, B \times C) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, A \times (B \times C)).
\end{aligned}$$

Thus, by Corollary 113, $(A \times B) \times C \simeq A \times (B \times C)$. □

Note 116. By induction, all finite products are associative.

Theorem 117. In any category with products $\times$, coproducts $+$, initial object 0, final object 1, and exponentials, we have the following natural isomorphisms.

1. $A \times (B + C) \simeq (A \times B) + (A \times C)$
2. $C^{A+B} \simeq C^A \times C^B$.
3. $(C^A)^B \simeq C^{A \times B}$
4. $(A \times B)^C \simeq A^C \times B^C$
5. $C^0 \simeq 1$
6. $C^1 \simeq C$

Proof.

1. We have the following list of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(A \times (B + C), X) &\simeq \mathrm{Hom}_{\mathsf{C}}(B + C, X^A) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(B, X^A) \times \mathrm{Hom}_{\mathsf{C}}(C, X^A) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(B \times A, X) \times \mathrm{Hom}_{\mathsf{C}}(C \times A, X) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}((B \times A) + (C \times A), X).
\end{aligned}$$

2. We have the following list of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}(X, C^{A+B}) &\simeq \mathrm{Hom}(X \times (A + B), C) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}((X \times A) + (X \times B), C) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X \times A, C) \times \mathrm{Hom}_{\mathsf{C}}(X \times B, C) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, C^A) \times \mathrm{Hom}_{\mathsf{C}}(X, C^B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, C^A \times C^B).
\end{aligned}$$

Etc. □

6. Limits

6.1. Limits and colimits

We have seen in REF that one of the ways of viewing the universal property for products is as specifying a sort of equivalence between two different pieces of data: it says that specifying a pair of morphisms $(f: X \to A, g: X \to B)$ is the same as specifying a single morphism $(f,g): X \to A \times B$.

$$\begin{tikzcd} & A \\ X \arrow[ur, "f"] \arrow[dr, "g"'] & \\ & B \end{tikzcd} \qquad \iff \qquad X \xrightarrow{(f,g)} A \times B$$

That is, the object $A \times B$ somehow collapses the data of a pair of morphisms from an object X into objects A and B into a single datum.

The job of summarizing maps into a certain collection of objects and morphisms is done by limits.

To make this statement precise, we need to do three things:

1. We need to make precise the phrase 'a certain collection of objects and morphisms.' This is the notion of a *diagram* (Definition 118).
2. We need to come up with the correct notion of maps into such a collection from another object. This is what is known as a *cone* (Definition 120).
3. We need to decide what it means to say that another morphism 'summarizes' such a collection; that is, when providing a single morphism is equivalent to providing a cone. This happens when there exists a *universal* cone, which we call a *limit cone* (Definition 125).

The solution to the first item is the notion of a *diagram.*

Definition 118 (diagram)**.** Let J and C be categories. A diagram of type J in C is a functor

$$\mathcal{D}: \mathsf{J} \to \mathsf{C}.$$

We think of J as parametrizing the objects and morphisms in our diagram, and $\mathcal{D}$ as giving a realization of J in C.

Example 119. Consider the category J with objects and morphisms as follows.

$$\begin{tikzcd} & \bullet \arrow[d] \\ \star \arrow[r] & * \end{tikzcd}$$

A diagram of type J in a category C is a functor J → C. Such a functor picks out three objects and two morphisms in C as follows.

$$\begin{tikzcd} & y \arrow[d, "f"] \\ x \arrow[r, "g"'] & z \end{tikzcd}$$

As this example makes clear, one thinks of J as specifying the shape of the diagram in which one is interested.

The next item on our agenda is to define the notion of a cone over a diagram, i.e. a compatible collection of morphisms from an object into the diagram. Roughly, the idea is this. Given an object X and a diagram D,

$$\begin{tikzcd} & X & \\ & & \\ & \mathcal{D}(i) \arrow[dl] \arrow[dr] & \\ \mathcal{D}(j) & & \mathcal{D}(k) \end{tikzcd}$$

a cone with tip X over $\mathcal{D}$ should be, for each object $\mathcal{D}(i)$ in our diagram, a morphism $X \to \mathcal{D}(i)$.

$$\begin{tikzcd} & X \arrow[d] \arrow[ddl, bend right] \arrow[ddr, bend left] & \\ & \mathcal{D}(i) \arrow[dl] \arrow[dr] & \\ \mathcal{D}(j) & & \mathcal{D}(k) \end{tikzcd}$$

Drawing these morphisms has created, for each morphism $\mathcal{D}(f)$ in our diagram, a triangle. The natural thing to do is to demand that these triangles commute.

This really is the definition of a cone over a diagram: a collection of morphisms making some triangles commute. However, we can give a much more abstract, yet completely equivalent, definition, which will be much easier to work with.

Definition 120 (cone)**.** Let Δ be the functor $\mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ (the category of functors $\mathsf{J} \to \mathsf{C}$, see Definition 48) which assigns to each object $X \in \mathrm{Obj}(\mathsf{C})$ the constant functor $\Delta_X : \mathsf{J} \to \mathsf{C}$, i.e. the functor which maps every object of J to X and every morphism to id_X. A cone over $\mathcal{D}$ is an object in the comma category (Definition 56) $(\Delta \downarrow \mathcal{D})$ given by the diagram

$$\mathsf{C} \xrightarrow{\Delta} [\mathsf{J}, \mathsf{C}] \xleftarrow{\mathcal{D}} 1 \ .$$

The objects of this category are pairs (X, f), where $X \in \mathrm{Obj}(\mathsf{C})$ and $f : \Delta(X) \to \mathcal{D}$; that is to say, f is a natural transformation $\Delta_X \Rightarrow \mathcal{D}$.

This is an abstract definition, and it will be instructive to show that it really is equivalent to the heuristic one given above. Let $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ be a diagram, and let (X, η) be a cone over J; that is, let $\eta\colon \Delta_X \Rightarrow \mathcal{D}$ be a natural transformation.

The natural transformation η provides the data of, for each object $i \in \mathrm{Obj}(\mathsf{J})$, a morphism

$$\eta_i\colon X \to \mathcal{D}(i).$$

For each morphism $f\colon i \to j$ in J, the morphisms η_i are required to make the naturality square

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"] \arrow[d, "\eta_i"'] & X \arrow[d, "\eta_j"] \\ \mathcal{D}(i) \arrow[r, "\mathcal{D}(f)"'] & \mathcal{D}(j) \end{tikzcd}$$

commute. This is precisely the condition that the triangles described above commute.

Example 121. Let J be the category with two objects and only identity morphisms. A J-diagram in a category C is a functor $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$. This picks out a pair of objects in C.

$$\begin{pmatrix} \bullet & \star \end{pmatrix} \xrightarrow{\mathcal{D}} \begin{pmatrix} A & B \end{pmatrix}$$

Let (X, η) be a cone over $\mathcal{D}$. That is, η consists of two morphisms

$$\eta_A\colon X \to A, \qquad \eta_B\colon X \to B.$$

There are no non-trivial morphisms in J, meaning that η_A and η_B are not subject to any conditions; they are completely arbitrary.

$$\begin{tikzcd} & X \arrow[dl, "\eta_A"'] \arrow[dr, "\eta_B"] & \\ A & & B \end{tikzcd}$$

Definition 120 defines cones over a diagram as 'objects in some category.' Therefore, it makes sense to define such a category to be the category of cones over a diagram.

Definition 122 (category of cones over a diagram)**.** Let C be a category, J an index category, and $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ a diagram. The <u>category of cones over $\mathcal{D}$</u> is the category $(\Delta \downarrow \mathcal{D})$.

Definition 122 not only reiterates the definition of a cone, but even prescribes what morphisms between cones look like. Let (X, Φ) be a cone over a diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$, i.e.

- an object in the category $(\Delta \downarrow \mathcal{D})$, i.e.
- a pair (X, Φ), where $X \in \mathrm{Obj}(\mathsf{C})$ and $\Phi\colon \Delta_X \Rightarrow \mathcal{D}$ is a natural transformation, i.e.
- for each $J \in \mathrm{Obj}(\mathsf{J})$ a morphism $\Phi_J\colon X \to \mathcal{D}(J)$ such that for any other object $J' \in \mathrm{Obj}(\mathsf{J})$ and any morphism $f\colon J \to J'$ the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "\Phi_J"'] \arrow[dr, "\Phi_{J'}"] & \\ \mathcal{D}(J) \arrow[rr, "\mathcal{D}(f)"'] & & \mathcal{D}(J') \end{tikzcd}$$

Let (Y, Γ) be another cone over $\mathcal{D}$. Then a morphism $\Xi\colon (X, \Phi) \to (Y, \Gamma)$ is

- a morphism $\Xi \in \mathrm{Hom}_{(\Delta\downarrow\mathcal{D})}((X, \Phi), (Y, \Gamma))$, i.e.
- a natural transformation $\Xi\colon \Delta_X \Rightarrow \Delta_Y$ (i.e. a morphism $\Delta_X \to \Delta_Y$ in the category $[\mathsf{J}, \mathsf{C}]$) such that the diagram

$$\begin{tikzcd} \Delta_X \arrow[rr, Rightarrow, "\Xi"] \arrow[dr, Rightarrow, "\Phi"'] & & \Delta_Y \arrow[dl, Rightarrow, "\Gamma"] \\ & \mathcal{D} & \end{tikzcd}$$

  commutes, i.e.
- a morphism $\xi\colon X \to Y$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the diagram

$$\begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\Phi_J"'] & & Y \arrow[dl, "\Gamma_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

  commutes.

Cocones are the dual notion to cones. We make the following definition.

Definition 123 (cocone)**.** A cocone over a diagram $\mathcal{D}$ is an object in the comma category $(\mathcal{D} \downarrow \Delta)$.

Definition 124 (category of cocones)**.** The category of cocones over a diagram $\mathcal{D}$ is the category $(\mathcal{D} \downarrow \Delta)$.

The categorical definitions of cones and cocones allow us to define limits and colimits succinctly.

Definition 125 (limits, colimits)**.** A limit of a diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ is a final object in the category $(\Delta \downarrow \mathcal{D})$. A colimit is an initial object in the category $(\mathcal{D} \downarrow \Delta)$.

Note 126. The above definition of a limit unwraps as follows. The limit of a diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ is a cone (X, Φ) over $\mathcal{D}$ such that for any other cone (Y, Γ) over $\mathcal{D}$, there is a unique map $\xi\colon Y \to X$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the following diagram commutes.

$$\begin{tikzcd} Y \arrow[rr, "\xi"] \arrow[dr, "\Gamma_J"'] & & X \arrow[dl, "\Phi_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

Notation 127. We often denote the limit over the diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ by

$$\varprojlim \mathcal{D}.$$

Similarly, we will denote the colimit over $\mathcal{D}$ by

$$\varinjlim \mathcal{D}.$$

If there is notational confusion over which functor we are taking a (co)limit, we will add a dummy index:

$$\varprojlim_i \mathcal{D}_i \qquad \text{or} \qquad \varinjlim_i \mathcal{D}_i.$$

Example 128. Here is a definition of the product $A \times B$ equivalent to that given in Example 71: it is the limit of the following somewhat trivial diagram.

$$
\begin{tikzcd}
A \arrow[loop left, "\mathrm{id}_A"] & & B \arrow[loop right, "\mathrm{id}_B"]
\end{tikzcd}
$$

Let us unwrap this definition. We are saying that the product $A \times B$ is a cone over A and B.

$$
\begin{tikzcd}
& A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\
A & & B
\end{tikzcd}
$$

But not just any cone: a cone which is universal in the sense that any *other* cone factors through it uniquely.

$$
\begin{tikzcd}
& X \arrow[dl, "f_1"'] \arrow[d, "\exists! f"] \arrow[dr, "f_2"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}
$$

You will recognize this as precisely the diagram defining the product.

In fact, this allows us to generalize the product: the product of n objects $\prod_{i=1}^{n} A_i$ is the limit over diagram consisting of all the A_i with no morphisms between them.

Definition 129 (equalizer)**.** Let J be the category with objects and morphisms as follows. (The necessary identity arrows are omitted.)

$$
\begin{tikzcd}
J \arrow[r, shift left, "1"] \arrow[r, shift right, "2"'] & J'
\end{tikzcd}
$$

A diagram $\mathcal{D}$ of shape J in some category C looks like the following.

$$
\begin{tikzcd}
A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

The equalizer of f and g is the limit of the diagram $\mathcal{D}$; that is to say, it is an object $\mathrm{eq} \in \mathrm{Obj}(\mathrm{C})$ and a morphism $e\colon \mathrm{eq} \to A$

$$
\begin{tikzcd}
\mathrm{eq} \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

such that for any *other* object Z and morphism $i\colon Z \to A$ such that $f \circ i = g \circ i$, there is a unique morphism $e\colon Z \to \mathrm{eq}$ making the following diagram commute.

$$
\begin{tikzcd}
Z \arrow[d, dashed, "\exists! u"'] \arrow[dr, "i"] & & \\
\mathrm{eq} \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B
\end{tikzcd}
$$

6.2. Pullbacks and kernels

In what follows, C will be a category and A, B, etc. objects in Obj(C).

Definition 130 (pullback)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

A pullback of f along g (also called a pullback of g over f, sometimes notated $A \times_C B$) is a commuting square

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there is a unique morphism $h\colon V \to U$ such that the diagram

$$\begin{tikzcd} V \arrow[dr, dashed, "\exists! h"] \arrow[drr, bend left, "t"] \arrow[ddr, bend right, "s"'] & & \\ & U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ & A \arrow[r, "f"'] & C \end{tikzcd}$$

commutes.

Note 131. Here is another definition: the pullback $A \times_C B$ is the limit of the diagram

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

This might at first seem odd; after all, don't we also need an arrow $A \times_C B \to C$? But this arrow is completely determined by the commutativity conditions, so it is superfluous.

Example 132. In Set, U is given (up to unique isomorphism) by

$$U = \left\{ (a, b) \in A \times B \,\middle|\, f(a) = g(b) \right\}.$$

The morphisms p and q are given by the projections $p(a,b) = a$, $q(a,b) = b$.

To see that this really does satisfy the universal property, consider any other set V and functions $s\colon V \to A$ and $t\colon V \to B$ making the above diagram commute. Then for all $v \in V$, $f(s(v)) = t(g(v))$.

Now consider the map $V \to U$ sending v to $(s(v), t(v))$. This certainly makes the above diagram commute; furthermore, any other map from V to U would not make the diagram commute. Thus U and h together satisfy the universal property.

Example 133. Suppose $f\colon X \to Y$ is a monomorphism. Then the square

$$\begin{tikzcd} A \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & A \arrow[d, "f"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

is a pullback square.

Conversely, if the above square is a pullback square, then f is a monomorphism.

Lemma 134. Let $f\colon X \to Y$ be a monomorphism. Then any pullback of f is a monomorphism.

Proof. Suppose we have the following pullback square.

$$\begin{tikzcd} X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

Our aim is to show that p_2 is a monomorphism.

Suppose we are given an object Z and two morphisms $\alpha, \beta\colon Z \to X'$.

$$\begin{tikzcd} Z \arrow[dr, shift left, "\alpha"] \arrow[dr, shift right, "\beta"'] & & \\ & X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\ & Y' \arrow[r, "g"] & Y \end{tikzcd}$$

We can compose α and β with p_2. Suppose that these agree, i.e.

$$p_2 \circ \alpha = p_2 \circ \beta.$$

We will be done if we can show that this implies that $p_1 = p_2$.

We can compose with g to find that

$$g \circ p_2 \circ \alpha = g \circ p_2 \circ \beta.$$

but since the pullback square commutes, we can replace $g \circ p_2$ by $f \circ p_1$.

$$f \circ p_1 \circ \alpha = f \circ p_1 \circ \beta.$$

since f is a monomorphism, this implies that

$$p_1 \circ \alpha = p_1 \circ \beta.$$

Now forget α and β for a minute. We have constructed a commuting square as follows.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "p_1 \circ \alpha = p_1 \circ \beta"] \arrow[ddr, bend right, "p_2 \circ \alpha = p_2 \circ \beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

the universal property for pullbacks tells us that there is a unique morphism $Z \to X'$ making the diagram commute. But either α or β will do! So $\alpha = \beta$. $\square$

Definition 135 (kernel of a morphism)**.** Let C be a category with an initial object (Definition 60) $\emptyset$ and pullbacks. The kernel $\ker(f)$ of a morphism $f : A \to B$ is the pullback along f of the unique morphism $\emptyset \to B$.

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\iota"'] & \emptyset \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

That is to say, the kernel of f is a pair $(\ker(f), \iota)$, where $\ker(f) \in \mathrm{Obj}(\mathsf{C})$ and $\iota \colon \ker(f) \to A$ which satisfies the above universal property.

Note 136. Although the kernel of a morphism f is a pair $(\ker(f), \iota)$ as described above, we will sometimes sloppily say that the object $\ker(f)$ is the kernel of f, especially when the the morphism ι is obvious or understood. Such abuses of terminology are common; one occasionally even sees the morphism ι being called the kernel of f.

Example 137. In Vect_k, the initial object is the zero vector space $\{0\}$. For any vector spaces V and W and any linear map $f : V \to W$, the kernel of f is the pair $(\ker(f), \iota)$ where $\ker(f)$ is the vector space

$$\ker(f) = \left\{ v \in V \,\middle|\, f(v) = 0 \right\}$$

and ι is the obvious injection $\ker(f) \to V$.

Lemma 138. Let $f : A \to B$, and let $(\iota, \ker(f))$ be the kernel of f. Then ι is a monomorphism (Definition 16).

Proof. Suppose we have an object $Z \in \mathrm{Obj}(\mathsf{C})$ and two morphisms $g_1, g_2 : Z \to \ker(f)$. We have the following diagram.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Further suppose that $\iota \circ g_1 = \iota \circ g_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Now pretend that we don't know about g_1 and g_2.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1=\iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

The universal property for kernels tells us that there is a unique map $Z \to \ker(f)$ making the above diagram commute. But since g_1 and g_2 both make the diagram commute, g_1 and g_2 must be the same map, i.e. $g_1 = g_2$. $\square$

6.3. Pushouts and cokernels

Pushouts are the dual notion to pullbacks.

Definition 139 (pushouts)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \\
A &
\end{tikzcd}$$

The pushout of f along g (or g along f) is a commuting square

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \\
A \arrow[r, "p"'] & U
\end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "t"] \\
A \arrow[r, "s"'] & V
\end{tikzcd}$$

there exists a unique morphism $h\colon U \to V$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

commutes.

Note 140. As with pullbacks, we can also define a pushout as the colimit of the following diagram.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \\ A & \end{tikzcd}$$

Example 141. Let us construct the pushout in Set. If we ignore the object C and the morphisms f and g, we discover that U must satisfy the universal property of the coproduct of A and B.

$$\begin{tikzcd} & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

Let us therefore make the ansatz that $U = A \amalg B = A \sqcup B$ and see what happens when we add C, f, and g back in.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

In doing so, we find that the square A-C-B-U must also commute, i.e. we must have that $(q \circ g)(c) = (p \circ f)(c)$ for all $c \in C$. Since p and q are just inclusions, we see that

$$U = A \amalg B/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $f(c) \sim g(c)$ for all $c \in C$.

Definition 142 (cokernel of a morphism)**.** Let C be a category with terminal object 1. The cokernel of a morphism $f : A \to B$ is the pushout of f along the unique morphism $A \to 1$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\pi"] \\ 1 \arrow[r] & \operatorname{coker}(f) \end{tikzcd}$$

Example 143. In Vect_k, the terminal object is the vector space $\{0\}$. If V and W are k-vector spaces and f is a linear map $V \to W$, then $\operatorname{coker}(f)$ is

$$W/\sim,$$

where $\sim$ is the relation generated by $f(v) \sim 0$ for all $v \in V$. But this relation is exactly the one which mods out by $\operatorname{im}(f)$, so $\operatorname{coker}(f) = W/\operatorname{im}(f)$.

Lemma 144. For any morphism $f \colon A \to B$, the canonical projection $\pi \colon B \to \operatorname{coker}(f)$ is an epimorphism.

Proof. The proof is dual to the proof that the canonical injection ι is mono (Lemma 138). □

Definition 145 (normal monomorphism)**.** A monomorphism (Definition 16) $f \colon A \to B$ is normal if it the kernel of some morphism. To put it more plainly, f is normal if there exists an object C and a morphism $g \colon B \to C$ such that (A, f) is the kernel of g.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Example 146. In Vect_k, monomorphisms are injective linear maps (Example 19). If f is injective then sequence

$$\{0\} \longrightarrow V \xrightarrow{f} W \xrightarrow{\pi} W/\operatorname{im}(f), \longrightarrow \{0\}$$

is exact, and we always have that $\operatorname{im}(f) = \ker(\pi)$. Thus in Vect_k, every monomorphism is normal.

Definition 147 (conormal epimorphism)**.** An epimorphism $f \colon A \to B$ is conormal if it is the cokernel of some morphism. That is to say, if there exists an object C and a morphism $g \colon C \to A$ such that (B, f) is the cokernel of g.

Example 148. In Vect_k, epimorphisms are surjective linear maps. If $f \colon V \to W$ is a surjective linear map, then the sequence

$$\{0\} \longrightarrow \ker(f) \xrightarrow{\iota} V \xrightarrow{f} W \longrightarrow \{0\}$$

is exact. But then $\operatorname{im}(\iota) = \ker(f)$, so f is conormal. Thus in Vect_k, every epimorphism is conormal.

Note 149. To show that in our proofs that in Vect_k monomorphisms were normal and epimorphisms were conormal, we showed that monomorphisms were the kernels of their cokernels, and epimorphisms were the cokernels of their kernels. This will be a general feature of Abelian categories.

Definition 150 (binormal category)**.** A category is binormal if all monomorphisms are normal and all epimorphisms are conormal.

Example 151. As we have seen, Vect_k is binormal.

6.4. A necessary and sufficient condition for the existence of small limits

In this section we prove a simple criterion to check that a category has all small limits (i.e. limits over diagrams with a small set of objects and between each two objects a small set of morphisms). The idea is as follows.

In general, adding more objects to a diagram makes the limit over it larger, and adding more morphisms makes the limit smaller. In the extreme case in which are no morphisms (apart from the identity morphisms), the limit is simply the categorical product. As we add morphisms, roughly speaking, we have to get rid of the parts of the product which prevent the triangles they form from commuting. Thus, to make the universal cone over a diagram, we can start with the product, and cut out the bare minimum we need to make everything commute: the way to do that is with an equalizer.

The following proof was adapted from [25].

Theorem 152. Let C be a category. Then if C has all small products and binary equalizers, C has all small limits.

Proof. Let $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ be a small diagram. We want to prove that $\mathcal{D}$ has a limit; we will do this by constructing a universal cone over it, i.e.

- an object $L \in \mathrm{Obj}(\mathsf{C})$, and
- for each $j \in \mathrm{Obj}(\mathsf{J})$, a morphism $P_j\colon L \to \mathcal{D}(j)$

such that

1. for any $i, j \in \mathrm{Obj}(\mathsf{J})$ and any $\alpha\colon i \to j$ the following diagram commutes,

$$\begin{tikzcd} & L \arrow[dl, "P_i"'] \arrow[dr, "P_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   and

2. for any other object $L' \in \mathrm{Obj}(\mathsf{C})$ and family of morphisms $Q_j\colon L' \to \mathcal{D}(j)$ which make the diagrams

$$\begin{tikzcd} & L' \arrow[dl, "Q_i"'] \arrow[dr, "Q_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   commute for all i, j, and α, there is a unique morphism $f\colon L' \to L$ such that $Q_j = P_j \circ f$ for all $j \in \mathrm{Obj}(\mathsf{J})$.

Denote by $\mathrm{Mor}(\mathsf{J})$ the set of all morphisms in J. For any $\alpha \in \mathrm{Mor}(\mathsf{J})$, denote by $\mathrm{dom}(\alpha)$ the domain of α, and by $\mathrm{cod}(\alpha)$ the codomain.

Consider the following small products:

$$A = \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \qquad \text{and} \qquad B = \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha)).$$

From the universal property for products, we know that we can construct a morphism $f: A \to B$ by specifying a family of morphisms $f_\alpha: A \to \mathcal{D}(\mathrm{cod}(\alpha))$, one for each $\alpha \in \mathrm{Mor}(\mathsf{J})$. We will define two morphisms $R, S: A \to B$ in this way:

$$R_\alpha = \pi_{\mathcal{D}(\mathrm{cod}(\alpha))}; \qquad S_\alpha = \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(\mathrm{dom}(\alpha))}.$$

Now let $e: L \to A$ be the equalizer of R and S (we are guaranteed the existence of this equalizer by assumption).

$$L \xhookrightarrow{e} \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \underset{S}{\overset{R}{\rightrightarrows}} \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha))$$

Further, define $P_j: L \to \mathcal{D}(j)$ by

$$P_j = \pi_{\mathcal{D}(j)} \circ e$$

for all $j \in \mathrm{Obj}(\mathsf{J})$.

The claim is that L together with the P_j is the limit of $\mathcal{D}$. We need to verify conditions 1 and 2 on L and P_j listed above.

1. We need to show that for all $i, j \in \mathrm{Obj}(\mathsf{J})$ and all $\alpha: i \to j$, we have the equality $\mathcal{D}(\alpha) \circ P_i = P_j$. Now, for every $\alpha: i \to j$ we have

$$\begin{aligned} \mathcal{D}(\alpha) \circ P_i &= \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(i)} \circ e \\ &= S_\alpha \circ e \\ &= R_\alpha \circ e \\ &= \pi_{\mathcal{D}(j)} \circ e \\ &= P_j. \end{aligned}$$

2. We need to show that for any other $L' \in \mathrm{Obj}(\mathsf{C})$ and any other family of morphisms $Q_j: L' \to \mathcal{D}(j)$ such that for all $\alpha: i \to j$, $Q_j = \mathcal{D}(\alpha) \circ Q_i$, there is a unique morphism $h: L' \to L$ such that $Q_j = P_j \circ h$ for all $j \in \mathrm{Obj}(\mathsf{J})$. Suppose we are given such an L' and Q_j.

   The universal property for products allows us to construct from the family of morphisms Q_j a morphism $Q: L' \to A$ such that $Q_j = \pi_{\mathcal{D}(j)} \circ Q$. Now, for any $\alpha: i \to j$,

$$\begin{aligned} R_\alpha \circ Q &= \pi_{\mathcal{D}(j)} \circ Q \\ &= Q_j \\ &= \mathcal{D}(\alpha) \circ Q_i \\ &= \mathcal{D}(\alpha) \circ \pi_i \circ Q \\ &= S_\alpha \circ Q. \end{aligned}$$

   Thus, $Q: L' \to A$ equalizes R and S. But the universal property for equalizers guarantees us a unique morphism $h: L' \to L$ such that $Q = P \circ h$. We can compose both sides of this equation on the left with $\pi_{\mathcal{D}(j)}$ to find

$$\pi_{\mathcal{D}(j)} \circ Q = \pi_{\mathcal{D}(j)} \circ P \circ h,$$

   i.e.

$$Q_j = P_j \circ h$$

   as required.

$\square$

6.5. The hom functor preserves limits

Theorem 153. Let C be a locally small category. The hom functor $\mathrm{Hom}_{\mathsf{C}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$ preserves limits in the second argument, i.e. for $\mathcal{D} \colon \mathsf{J} \to \mathsf{C}$ a diagram in C we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}),$$

where the limit on the RHS is over the hom-set diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \to \mathsf{Set}.$$

Proof. Let L be the limit over the diagram $\mathcal{D}$. Then for any map $f \colon Y \to L$, there is a cone from Y to $\mathcal{D}$ by composition, and for any cone with tip Y over $\mathcal{D}$ we get a map $f \colon Y \to L$ from the universal property of limits. Thus, there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq \mathrm{Cones}(Y, \mathcal{D}),$$

which is natural in Y since the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow} \mathcal{D}) \arrow[r, "(-)\circ f"] \arrow[d] & \mathrm{Hom}_{\mathsf{C}}(Z, \lim_{\leftarrow} \mathcal{D}) \arrow[d] \\
\mathrm{Cones}(Y, \mathcal{D}) \arrow[r, "(-)\circ f"] & \mathrm{Cones}(Z, \mathcal{D})
\end{tikzcd}$$

trivially commutes. We will be done if we can show that there is also a natural isomorphism

$$\mathrm{Cones}(Y, \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}).$$

Let us understand the elements of the set $\lim_{\leftarrow} \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D})$. The diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \to \mathsf{Set}$$

maps $J \in \mathrm{Obj}(\mathsf{J})$ to $\mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J)) \in \mathrm{Obj}(\mathsf{Set})$. A universal cone over this diagram is a set S together with, for each $J_i \in \mathrm{Obj}(\mathsf{J})$, a function

$$f_i \colon S \to \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i))$$

such that for each $\alpha \in \mathrm{Hom}_{\mathsf{J}}(J_i, J_j)$, the diagram

$$\begin{tikzcd}
& S \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\
\mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i)) \arrow[rr, "\mathcal{D}(\alpha)\circ(-)"'] & & \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_j))
\end{tikzcd}$$

commutes.

Now pick any element $s \in S$. Each f_i maps this to a function $Y \to \mathcal{D}(J_i)$ which makes the diagram

$$\begin{tikzcd}
& Y \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\
\mathcal{D}(J_i) \arrow[rr, "\mathcal{D}(\alpha)"'] & & \mathcal{D}(J_j)
\end{tikzcd}$$

commute. But this is exactly an element of $\mathrm{Cones}(Y, \mathcal{D})$. Conversely, every element of $\mathrm{Cones}(Y, \mathcal{D})$ gives us an element of S, so we have a bijection

$$\mathrm{Cones}(Y, \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}),$$

which is natural as required. □

Corollary 154. The first slot of the hom functor turns colimits into limits. That is,

$$\mathrm{Hom}_{\mathsf{C}}(\varinjlim \mathcal{D}, Y) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(\mathcal{D}, Y).$$

Proof. Dual to that of Theorem 153. □

6.6. Filtered colimits and ind-objects

Definition 155 (preorder)**.** Let S be a set. A preoreder on S is a binary relation $\leq$ which is

1. reflexive ($a \leq a$)
2. transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)

A preorder is said to be directed if for any two objects a and b, there exists an 'upper bound,' i.e. an object r such that $a \leq r$ and $b \leq r$.

Filtered categories are a generalization of filtered preorders.

Definition 156 (filtered category)**.** A category J is filtered if

- for each pair of objects $J, J' \in \mathrm{Obj}(\mathsf{J})$, there exists an object K and morphisms $J \to K$ and $J' \to K$.

$$\begin{tikzcd} J \arrow[dr] & \\ & K \\ J' \arrow[ur] & \end{tikzcd}$$

  That is, every diagram with two objects and no morphisms is the base of a cocone.

- For every pair of morphisms $i, j \colon J \to J'$, there exists an object K and a morphism $f \colon J' \to K$ such that $f \circ i = f \circ j$, i.e. the following diagram commutes.

$$\begin{tikzcd} J \arrow[r, shift left, "i"] \arrow[r, shift right, "j"'] & J' \arrow[r, "f"] & K \end{tikzcd}$$

  That is, every diagram of the form $\bullet \rightrightarrows \bullet$ is the base of a cocone.

Example 157. Any poset is a filtered category.

Definition 158 (filtered colimit)**.** A filtered colimit is a colimit over a diagram $\mathcal{D} \colon \mathsf{J} \to \mathsf{C}$, where J is a filtered category.

We now define the so-called *category of inductive objects* (or simply *ind-objects*).

Definition 159 (category of ind-objects)**.** Let C be a category. We define the category of ind-objects of C, denote Ind(C) as follows.

- The objects $F \in \mathrm{Obj}(\mathsf{Ind}(\mathsf{C}))$ are defined to be filtered colimits of objects of diagrams $\mathcal{F}\colon \mathsf{D} \to \mathsf{C}$.
- For two objects $F = \lim_{\to d} \mathcal{F}_d$ and $G = \lim_{\to e} \mathcal{G}_e$, the morphisms $\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G)$ are defined to be the set

$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G) = \lim_{\leftarrow d} \lim_{\to e} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}_d, \mathcal{G}_e).$$

Note 160. There is a fully faithful embedding $\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$ which exhibits any object A as the colimit over the trivial diagram $\bullet \circlearrowleft$.

The importance of the category of ind-objects can be seen in the following example.

Example 161. Let V be an infinite-dimensional vector space over some field k. Then V can be realized as an object in the category $\mathsf{Ind}(\mathsf{FinVect})$.

To see this, let V be any vector space. Consider the poset of finite-dimensional vector subspaces of V. We can express V as the colimit of this poset. Thus, we have an embedding $\mathsf{Vect}_k \hookrightarrow \mathsf{Ind}(\mathsf{FinVect}_k)$.

In fact, there is an equivalence of categories $\mathsf{Vect}_k \simeq \mathsf{Ind}(\mathsf{FinVect}_k)$. Similarly, there is an equivalence of categories $\mathsf{SVect}_k \simeq \mathsf{Ind}(\mathsf{FinSVect}_k)$

6.7. Miscellaneous results

Theorem 162. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor.

- If C has an initial object i, then $\lim_{\leftarrow \mathcal{F}} = F(i)$.
- If C has a terminal object i, then $\lim_{\to \mathcal{F}} = F(i)$.

Proof. For each $c \in \mathrm{Obj}(\mathsf{C})$, we get a unique morphism $i \to c$. Sticking these into $\mathcal{F}$ gives us a cone $\alpha_c\colon F(i) \to F(c)$ under $\mathcal{F}(i)$.

Now let (d, η) be any other cone over $\mathcal{F}$. In particular, η has a component

$$\eta_i\colon d \to \mathcal{F}(i).$$

In order to show that α is a limit cone, we need only show that this is the only possible morphism $d \to F(i)$ such that the triangles

$$\begin{tikzcd}
& d \arrow[d, "\eta_i"] \arrow[ddl, bend right, "\eta_j"'] \\
& F(i) \arrow[dl, "\alpha_c"] \\
F(c) &
\end{tikzcd}$$

commute for all c. Let $f : d \to F(i)$ be any other such morphism. The triangle above must in particular commute when $c = i$, so $\alpha_i = F(\mathrm{id}_i) = \mathrm{id}_{F(i)}$.

$$\begin{tikzcd}
 & d \arrow[d, "f"] \arrow[ddl, bend right, "\eta_i"'] \\
 & F(i) \arrow[dl, "\mathrm{id}_{F(i)}"] \\
F(i) &
\end{tikzcd}$$

Thus, $f = \eta_i$. ◻

Theorem 163. Let C and D be categories, with D small and C locally small.

- If D is complete, then so is [C, D].
- If D is cocomplete, then so is [C, D].

Proof. The idea is as follows. A cone in a functor category is built out of natural transformations. Evaluating on a component gives a cone in the target category. A cone in a functor category is a limit cone if and only if each component is a limit cone. ◻

7. Adjunctions

Note 164. This chapter is under heavy construction. It's not very good at the moment.

7.1. Motivating example

Consider the following functors:

- $\mathcal{U}\colon \mathsf{Grp} \to \mathsf{Set}$, which sends a group to its underlying set, and
- $\mathcal{F}\colon \mathsf{Set} \to \mathsf{Grp}$, which sends a set to the free group on it.

The functor $\mathcal{U}$ is lossy; it forgets a structure, namely group multiplication. Of course, no inverse functor exists, since a set can have many different group structures on it. However, one can think of the functor $\mathcal{F}$ as a sort of weak inverse to $\mathcal{U}$; it constructs out of any set a group in a canonical way.

Many interesting mathematical structures arise in an attempt to add back in forgotten structure.

But how would one go about formalizing this?

This is an example of a free-forgetful adjunctions, a special case of an extremely powerful structure called an *adjuction*. We have encountered adjunctions before—we saw in Example 101 that there was a natural bijection between

$$\mathrm{Hom}_{\mathsf{C}}(X \times (-), B) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}(X, B^{(-)}).$$

this is a so-called *hom-set adjunction*.

7.1.1. Hom-set adjunction

The functors $\mathcal{U}$ and $\mathcal{F}$ have the following property. Let S be a set, G be a group, and let $f\colon S \to \mathcal{U}(G)$, $s \mapsto f(s)$ be a set-function. Then there is an associated group homomorphism $\tilde{f}\colon \mathcal{F}(S) \to G$, which sends $s_1 s_2 \ldots s_n \mapsto f(s_1 s_2 \ldots s_n) = f(s_1) \cdots f(s_n)$. In fact, $\tilde{f}$ is the unique homomorphism $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Similarly, for every group homomorphism $g\colon \mathcal{F}(S) \to G$, there is an associated function $S \to \mathcal{U}(G)$ given by restricting g to S. In fact, this is the unique function $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Thus for each $f \in \mathrm{Hom}_{\mathsf{Grp}}(S, \mathcal{U}(G))$ we can construct an $\tilde{f} \in \mathrm{Hom}_{\mathsf{Set}}(\mathcal{F}(S), G)$, and vice versa.

Let us add some mathematical scaffolding to the ideas explored above. We build two functors $\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp} \to \mathsf{Set}$ as follows.

1. Our first functor maps the object $(S, G) \in \mathrm{Obj}(\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp})$ to the hom-set $\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G)$, and a morphism $(\alpha, \beta)\colon (S, G) \to (S', G')$ to a function

$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S'), G'); \qquad m \mapsto \mathcal{F}(\alpha) \circ m \circ \beta$$

2. Our second functor maps (S, G) to $\mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$, and (α, β) to

$$m \mapsto \alpha \circ m \circ \mathcal{U}(\beta).$$

We can define a natural isomorphism Φ between these functors with components

$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G)); \qquad f \to \tilde{f}.$$

7.1.2. Unit-counit adjunction

Suppose we take the natural isomorphism from the last section

$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$$

and evaluate it at $S = \mathcal{U}(G)$. This gives us an isomorphsim

$$\Phi_{\mathcal{U}(G),G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(\mathcal{U}(S)), G) \to \mathrm{Hom}_{\mathsf{Set}}(\mathcal{U}(G), \mathcal{U}(G)).$$

In particular, the element $\mathrm{id}_{\mathcal{U}(G)}$ is the image of some group homomorphism

$$\epsilon_G\colon \mathcal{F}(\mathcal{U}(G)) \to G.$$

Taken together for all G, the ϵ_G form a natural transformation

$$\epsilon\colon \mathcal{F} \circ \mathcal{U} \to \mathrm{id}_{\mathsf{Grp}}.$$

Evaluating it at $G = \mathcal{F}(S)$. This gives us an isomorphism

$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), \mathcal{F}(S)) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(\mathcal{F}(G))).$$

In particular, the identity $\mathrm{id}_{\mathcal{F}(G)}$ is mapped to some map

$$\eta_S\colon S \to \mathcal{U}(\mathcal{G}(S)).$$

In fact, taken together, the η_S form a natural transformation

$$\eta\colon \mathrm{id}_{\mathsf{Set}} \Rightarrow \mathcal{U} \circ \mathcal{G}.$$

7.1.3. Adjunction from universal property

The previous discussion may be slightly odd to those who have already encountered free groups, because free groups are almost always defined in terms of a universal property rather than some sort of abstract nonsense involving functors. The universal property of the free group is as follows.

Let S be a set, and let $f: S \to G$ be any function from S to a group G. Then there is a unique group homomorphism $\tilde{f}: \mathcal{F}(S) \to G$ such that the following diagram commutes.

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "f"'] & (\mathcal{U} \circ \mathcal{F})(S) \arrow[d, "\mathcal{U}(\tilde{f})"] \\ & \mathcal{U}(G) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{F}(S) \arrow[d, "\exists! \tilde{f}"] \\ G \end{tikzcd}$$

As we saw in Section **??**, this simply says that the free group over S is initial in the comma category $(S \downarrow \mathcal{U})$.

We have written the injection $S \hookrightarrow (\mathcal{U} \circ \mathcal{F})(S)$ in the evocative notation η_S. As the reader has probably guessed, this is because taken together the η_S form a natural transformation which functions as the unit in a unit-counit adjunction.

7.2. Hom set adjunction

Definition 165 (hom-set adjunction)**.** Let C, D be categories and $\mathcal{F}$, $\mathcal{G}$ functors as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

We say that $\mathcal{F}$ is left-adjoint to $\mathcal{G}$ (or equivalently $\mathcal{G}$ is right-adjoint to $\mathcal{F}$) if there is a natural isomorphism

$$\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-)),$$

which fits between $\mathcal{F}$ and $\mathcal{G}$ like this.

$$\begin{tikzcd}[column sep=huge] \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-){,}-)", ""{name=U, below}] \arrow[r, bend right, "\mathrm{Hom}_{\mathsf{D}}(-{,}\mathcal{G}(-))"', ""{name=L, above}] & \mathsf{Set} \arrow[Rightarrow, from=U, to=L, "\Phi"] \end{tikzcd}$$

The natural isomorphsim amounts to a family of bijections

$$\Phi_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$$

which satisfies the coherence conditions for a natural transformation.

7.3. Unit-counit adjunction

Definition 166 (unit-counit adjunction)**.** We say that two functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a unit-counit adjunction if there are two natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}, \qquad \text{and} \qquad \varepsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$
\begin{tikzcd}
\mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, Rightarrow, "\varepsilon\mathcal{F}"] \\
& \mathcal{F}
\end{tikzcd}
,\qquad
\begin{tikzcd}
\mathcal{G} \arrow[r, Rightarrow, "\eta\mathcal{G}"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, Rightarrow, "\mathcal{G}\varepsilon"] \\
& \mathcal{G}
\end{tikzcd}
$$

commute.

Note 167. Using η for the unit and ϵ for the counit is very common, but not universal! Both Wikipedia and the nLab have them this way, but the Higher Categories course uses them the other way around.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $\mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$. We can draw it like this.

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathrm{id}_{\mathsf{C}}"{name=U}] \arrow[r, bend right, "\mathcal{G}\circ\mathcal{F}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\eta"] & \mathsf{C}
\end{tikzcd}
.
$$

Analogously, we can draw ε like this.

$$
\begin{tikzcd}
\mathsf{D} \arrow[r, bend left, "\mathcal{F}\circ\mathcal{G}"{name=U}] \arrow[r, bend right, "\mathrm{id}_{\mathsf{D}}"'{name=D}] \arrow[Rightarrow, from=U, to=D, "\varepsilon"] & \mathsf{D}
\end{tikzcd}
.
$$

We can arrange these artfully like so.

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rr, bend left=60, "\mathrm{id}_{\mathsf{C}}"{name=A}] \arrow[rr, bend left=25, "\mathcal{G}\circ\mathcal{F}"'{name=B}] \arrow[Rightarrow, from=A, to=B, "\eta"'] & \mathsf{D} \arrow[r, "\mathcal{G}"] \arrow[rr, bend right=25, "\mathcal{F}\circ\mathcal{G}"{name=C}] \arrow[rr, bend right=60, "\mathrm{id}_{\mathsf{D}}"'{name=E}] \arrow[Rightarrow, from=C, to=E, "\varepsilon"] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D}
\end{tikzcd}
$$

Notice that we haven't actually *done* anything; this diagram is just the diagrams for the unit and counit, plus some extraneous information.

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$
\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rrr, bend left=60, "\mathcal{F}"{name=A}] \arrow[rrr, bend left=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"'{name=B}] \arrow[Rightarrow, from=A, to=B, "\mathcal{F}\eta"'] \arrow[rrr, bend right=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"{name=C}] \arrow[rrr, bend right=60, "\mathcal{F}"'{name=E}] \arrow[Rightarrow, from=C, to=E, "\varepsilon\mathcal{F}"] & \mathsf{D} \arrow[r, "\mathcal{G}"] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D}
\end{tikzcd}
$$

then consolidate to get this:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "\mathcal{F}"{name=U}] \arrow[r, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"{name=M}] \arrow[r, bend right=60, "\mathcal{F}"'{name=L}] & \mathsf{D}
\arrow[Rightarrow, from=U, to=M, "\mathcal{F}\eta"]
\arrow[Rightarrow, from=M, to=L, "\varepsilon\mathcal{F}"]
\end{tikzcd}$$

We can then take the composition $\varepsilon\mathcal{F} \circ \mathcal{F}\eta$ to get a natural transformation $\mathcal{F} \Rightarrow \mathcal{F}$

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=40, "\mathcal{F}"{name=U}] \arrow[r, bend right=40, "\mathcal{F}"'{name=L}] & \mathsf{D};
\arrow[Rightarrow, from=U, to=L, "\varepsilon\mathcal{F}\circ\mathcal{F}\eta"]
\end{tikzcd}$$

the first triangle diagram says that this must be the same as the identity natural transformation $\mathrm{id}_{\mathcal{F}}$.

What this means in practice is that if we start with the functor $\mathcal{F}$, use the unit to get $\mathcal{F} \circ \mathcal{G}\mathcal{F}$, then use the counit to map back to $\mathcal{F}$, this should be the same as not doing anything at all.

The second triangle diagram is analogous; it tells us that starting with $\mathcal{G}$, using the counit to send this to $\mathcal{G} \circ \mathcal{F} \circ \mathcal{G}$, then going back to $\mathcal{G}$ with the unit should be the same as not doing anything.

Proposition 168. Consider a unit-counit adjunction between functors $\mathcal{F}$ and $\mathcal{G}$ as above.

1. The functor $\mathcal{F}$ is fully faithful if and only if the unit

$$\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$$

   is a natural isomorphism.

2. The functor $\mathcal{G}$ is fully faithful if and only if the counit

$$\eta\colon \mathcal{G} \circ \mathcal{F} \Rightarrow \mathrm{id}_{\mathsf{C}}$$

   is a natural isomorphism.

Proof. We prove the second part. The second part is analogous.

Suppose η is a natural isomorphism. Consider the following diagram.

$$\begin{tikzcd}
\mathsf{C}(a,b) \arrow[d, "\eta_b\circ(-)"'] \arrow[dr] & \\
\mathsf{C}\left(a, (G\circ F)(b)\right) \arrow[r, "\Phi_{a,b}"'] & \mathsf{D}(F(a), F(b))
\end{tikzcd}$$

The downward arrow is obviously a bijection, and the rightward arrow is given to us since every unit-counit adjunction is equivalently a hom-set isomorphism. Thus, their composition is also a bijection.

Now suppose that $\mathcal{G}$ is fully faithful. $\square$

7.4. Adjunction from universal property

Definition 169 (universal morphism adjunction)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. We say that $\mathcal{F}$ is a left universal morphism adjunction if for each $X \in \mathrm{Obj}(\mathsf{D})$, the category of morphisms $(\mathcal{F} \downarrow X)$ (see Definition 65) has a terminal object, which we denote by $\mathcal{G}(X)$. That is, $\mathcal{F}$ is a left adjoing universal morphism functor if for each $X \in \mathrm{Obj}(\mathsf{C})$, $Y \in \mathrm{Obj}(\mathsf{D})$, and morphism $f\colon \mathcal{F}(Y) \to X$, we have the following diagram.

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & & Y \arrow[d, dashed, "\exists! g"] \\
\mathcal{F}(\mathcal{G}(X)) \arrow[r, "\epsilon_X"'] & X & \mathcal{G}(X)
\end{tikzcd}$$

Similarly, we say that $\mathcal{F}$ is a right universal morphism adjunction if for each $Y \in \mathrm{Obj}(\mathsf{C})$, the category of morphisms $(Y \downarrow \mathcal{G})$ (Definition 64) has a initial object.

The above terminology is horrendous, and at some point I'll figure out a nicer way of doing this. probably be changed. The rationale behind introducing it is that it will not be around for long, since we will see that all of these definitions of left- and right adjunctions are equivalent.

7.5. All of these are equivalent

In this section, we will prove three main lemmas, which schematically do the following.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Universal morphism}\\ \text{adjunctions}\end{matrix}\right\} \arrow[rr, Rightarrow, bend left] & & \left\{\begin{matrix}\text{Hom-set}\\ \text{adjunctions}\end{matrix}\right\} \arrow[dl, Rightarrow, bend left] \\
& \left\{\begin{matrix}\text{Unit-counit}\\ \text{adjunctions}\end{matrix}\right\} \arrow[ul, Rightarrow, bend left] &
\end{tikzcd}$$

Note 170. It is common to typeset the statement $\mathcal{F}$ *is left-adjoint to* $\mathcal{G}$ by writing

$$\mathcal{F} \dashv \mathcal{G}.$$

This is fine, but if we want to be more explicit about the domains and codomains, we will use the notation

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}.$$

Lemma 171. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ is part of a left universal morphism adjunction, there is a functor $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ such that $\mathcal{F} \dashv \mathcal{G}$ is a hom-set adjunction.

Proof. Suppose $\mathcal{F}$ is part of a universal adjunction (Definition 169), i.e. for every object $X \in \mathrm{Obj}(\mathsf{D})$ there exists a terminal object in the category $(\mathcal{F} \downarrow X)$, which we will denote $(\mathcal{G}(X), \epsilon_X)$.[1]

[1]Of course, the objects $\mathcal{G}(X)$ are only specified up to unique isomorphism, and we have to pick representatives using the axiom of choice.

The first claim is that the assignment $X \mapsto \mathcal{G}(X)$ extends to a functor $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$. To this end, let $f\colon X \to X'$.

$$\begin{tikzcd}
(\mathcal{F}\circ\mathcal{G})(X) \arrow[r, "\epsilon_X"] & X \arrow[d, "f"] \\
(\mathcal{F}\circ\mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

Composing $f \circ \epsilon_X$, we find the following diagram.

$$\begin{tikzcd}
(\mathcal{F}\circ\mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[dr, "f\circ\epsilon_X"] & X \arrow[d, "f"] \\
(\mathcal{F}\circ\mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

The universal property now gives us a unique morphism $\mathcal{G}(X) \to \mathcal{G}(X')$, which we'll call $\mathcal{G}(f)$.

$$\begin{tikzcd}
(\mathcal{F}\circ\mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] \arrow[dr, "f\circ\epsilon_X"] & X \arrow[d, "f"] & & \mathcal{G}(X) \arrow[d, "\exists!\mathcal{G}(f)"] \\
(\mathcal{F}\circ\mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X' & & \mathcal{G}(X')
\end{tikzcd}$$

The usual manipulations (formally identical to those in Theorem 77) show that $\mathcal{G}$ respects compositions, and therefore defines a bona fide functor.

Let's take a closer look at the commuting square above.

$$\begin{tikzcd}
(\mathcal{F}\circ\mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & X \arrow[d, "f"] \\
(\mathcal{F}\circ\mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

It remains to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}}(F(-), -) \simeq \mathrm{Hom}_{\mathsf{C}}(-, G(-)).$$

To this end, let $f\colon \mathcal{F}(Y) \to X$ be a morphism in D. By the universal property for $\mathcal{G}(X)$, there exists a unique morphism $\tilde{f}\colon Y \to \mathcal{G}(X)$ making the following diagram commute.

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[rr, "\mathcal{F}(\tilde{f})"] \arrow[dr, "f"'] & & \mathcal{F}(\mathcal{G}(X)) \arrow[dl, "\epsilon_X"] \\
& X &
\end{tikzcd}$$

This assignment gives us our map $\mathrm{Hom}_D(\mathcal{F}(X), Y) \to \mathrm{Hom}_C(X, \mathcal{G}(Y))$.

On the other hand, given a map $g\colon Y \to \mathcal{G}(X)$, one can construct a map $\tilde{g}\colon \mathcal{F}(Y) \to X$ via the composition

$$\mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(\mathcal{G}(X)) \xrightarrow{\epsilon_X} X \ .$$

It remains to show that these operations are mutually inverse. Consider the following diagram.

$$\begin{tikzcd}
\mathcal{F}(X) & & \mathcal{F}(\mathcal{G}(Y)) \\
& \mathcal{F}(\mathcal{G}(Y)) &
\end{tikzcd}$$

□

Lemma 172. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a hom-set adjunction, they form a unit-counit adjunction.

Sketch of proof. Our plan of attack is the following.

1. We show that from a hom-set adjunction $\mathcal{F} \dashv \mathcal{G}$, we can form, for each $A \in \mathrm{Obj}(\mathsf{C})$, a map
$$\eta_A\colon A \to (\mathcal{G} \circ \mathcal{F})(A)$$
and for each $B \in \mathrm{Obj}(\mathsf{D})$, a map
$$\epsilon_B\colon (\mathcal{F} \circ \mathcal{G})(B) \to B.$$

2. We show that, component-wise, these satisfy the triangle identities.
$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_{\mathcal{F}})_A \qquad \text{and} \qquad (\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_{\mathcal{G}})_B.$$

3. We show that η_A and ϵ_B are components of two natural transformations
$$\eta\colon \mathrm{id}_{\mathsf{C}} \to \mathcal{G} \circ \mathcal{F} \qquad \text{and} \qquad \epsilon\colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_{\mathsf{D}}.$$

The reason for doing things in this order is that we will use the component-wise triangle identities in showing naturality. □

Proof.

1. Suppose $\mathcal{F}$ and $\mathcal{G}$ form a hom-set adjunction with natural isomorphism Φ. Then for any $A \in \mathrm{Obj}(\mathsf{C})$, we have $\mathcal{F}(A) \in \mathrm{Obj}(\mathsf{D})$, so Φ give us a bijection
$$\Phi_{A,\mathcal{F}(A)}\colon\ \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \to \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
We don't know much in general about $\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A))$, but the category axioms tell us that it always contains $\mathrm{id}_{\mathcal{F}(A)}$. We can use $\Phi_{A,\mathcal{F}(A)}$ to map this to
$$\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) \in \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
Let's call $\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) = \eta_A$.
Similarly, if $B \in \mathrm{Obj}(\mathsf{D})$, then $\mathcal{G}(B) \in \mathrm{Obj}(\mathsf{C})$, so Φ gives us a bijection
$$\Phi_{\mathcal{G}(B),B}\colon\ \mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \to \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)).$$
Since $\Phi_{\mathcal{G}(B),B}$ is a bijection, it is invertible, and we can evaluate the inverse on $\mathrm{id}_{\mathcal{G}(B)}$. Let's call
$$\Phi^{-1}_{\mathcal{G}(B),B}(\mathrm{id}_{\mathcal{G}(B)}) = \varepsilon_B.$$

2. Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a manner reminiscent of the proof of the Yoneda lemma, we can express Φ in terms of η, and Φ^{-1} in terms of ε, for *any* A and B. We do this using the naturality of Φ.

   The naturality of Φ tells us for any $A \in \mathrm{Obj}(\mathsf{C})$, $B \in \mathrm{Obj}(\mathsf{D})$, and $g\colon \mathcal{F}(A) \to B$, the following diagram has to commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[r, "g\circ(-)"] \arrow[d, "\Phi_{A,\mathcal{F}(A)}"'] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \arrow[d, "\Phi_{A,B}"] \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G}\circ\mathcal{F})(A)) \arrow[r, "\mathcal{G}(g)\circ(-)"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).
\end{tikzcd}$$

   Let's start at the top left with $\mathrm{id}_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A,B}(g)$, and from the bottom road we have $\mathcal{G}(g)\circ\eta_A$. The diagram commutes, so we have

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A. \tag{7.1}$$

   Similarly, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}((\mathcal{F}\circ\mathcal{G})(B), B) \arrow[r, "(-)\circ\mathcal{F}(f)"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \\
\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)) \arrow[r, "(-)\circ f"] \arrow[u, "\Phi^{-1}_{\mathcal{G}(B),B}"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{A,B}"']
\end{tikzcd}$$

   means that, for any $f: A \to \mathcal{G}(B)$,

$$\Phi^{-1}_{A,B}(f) = \varepsilon_B \circ \mathcal{F}(f) \tag{7.2}$$

   To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \mathrm{Obj}(\mathsf{C})$ and all $B \in \mathrm{Obj}(\mathsf{D})$,

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_{\mathcal{F}})_A \qquad \text{and} \quad (\mathcal{G}\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_{\mathcal{G}})_B.$$

   We have

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi^{-1}_{A,\mathcal{F}(A)}(\eta_A) = \mathrm{id}_A = (\mathrm{id}_{\mathcal{F}})_A$$

   and

$$(\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = \mathrm{id}_B = (\mathrm{id}_{\mathcal{G}})_B.$$

3. Let $g\colon A' \to A$. By the naturality of Φ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[rrr, "(-)\circ\mathcal{F}(g)"] \arrow[ddd, "\Phi_{A,\mathcal{F}(A)}"'] & & & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A'), \mathcal{F}(A)) \arrow[ddd, "\Phi_{A',\mathcal{F}(A)}"] \\
& \mathrm{id}_{\mathcal{F}(A)} \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{F}(g) \arrow[d, mapsto] & \\
& \eta_A \arrow[r, mapsto] & \eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g)) & \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G}\circ\mathcal{F})(A)) \arrow[rrr, "(-)\circ g"] & & & \mathrm{Hom}_{\mathsf{C}}(A', (\mathcal{G}\circ\mathcal{F})(A))
\end{tikzcd}$$

Following $\mathrm{id}_{\mathcal{F}(A)}$ around, we find the condition $\eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g))$. But by Equation 7.1, we have

$$\Phi_{A',\mathcal{F}(A)}(\mathcal{F}(g)) = (\mathcal{G} \circ \mathcal{F})(g) \circ \eta_{A'},$$

so our condition reads

$$\eta_A \circ g = (\mathcal{G} \circ \mathcal{F})(g) \circ \eta_{A'}.$$

But this says precisely that the naturality square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "\eta_A"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & A \arrow[d, "f"] \\
(\mathcal{G} \circ \mathcal{F})(A') \arrow[r, "\eta_{A'}"] & A'
\end{tikzcd}$$

commutes.

□

Lemma 173. If the functors $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G} \colon \mathsf{D} \to \mathsf{C}$ form a unit-counit adjunction, they form a hom-set adjunction.

Proof. We have seen in the proof of the previous lemma that we can write

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A \qquad \text{and} \qquad \Phi^{-1}_{A,B}(g) = \epsilon_B \circ \mathcal{F}(f).$$

□

Definition 174 (adjunct)**.** Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Then for each $A \in \mathrm{Obj}(\mathsf{C})$ $B \in \mathrm{Obj}(\mathsf{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism $\tilde{f}$ is called the <u>adjunct</u> of f, and f is called the adjunct of $\tilde{f}$.

Lemma 175. Let C and D be categories, $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}, \mathcal{G}' \colon \mathsf{D} \to \mathsf{C}$ functors,

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend right, "\mathcal{G}"'] \arrow[l, bend left, "\mathcal{G}'"]
\end{tikzcd}$$

and suppose that $\mathcal{G}$ and $\mathcal{G}'$ are both right-adjoint to $\mathcal{F}$. Then there is a natural isomorphism $\mathcal{G} \Rightarrow \mathcal{G}'$.

Proof. Since any adjunction is a hom-set adjunction, we have two isomorphisms

$$\Phi_{C,D} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \qquad \text{and} \qquad \Psi_{C,D} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D))$$

which are natural in both C and D. By Lemma 43, we can construct the inverse natural isomorphism

$$\Phi^{-1}_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D),$$

and compose it with Ψ to get a natural isomorphsim

$$(\Psi \circ \Phi^{-1})_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)).$$

Thus for any morphism $f\colon D \to E$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C,\mathcal{G}(f))"] \arrow[d, "(\Psi\circ\Phi^{-1})_{C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(E)) \arrow[d, "(\Psi\circ\Phi^{-1})_{C,E}"] \\
\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C,\mathcal{G}'(f))"] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(E))
\end{tikzcd}$$

But the full- and faithfulness of the Yoneda embedding tells us that there is a natural isomorphism with components μ_D and μ_E making the following diagram commute.

$$\begin{tikzcd}
\mathcal{G}(D) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\mu_D"'] & \mathcal{G}(E) \arrow[d, "\mu_E"] \\
\mathcal{G}'(D) \arrow[r, "\mathcal{G}'(f)"] & \mathcal{G}'(E)
\end{tikzcd}$$

□

Theorem 176. Let C and D be categories and $\mathcal{F}$ and $\mathcal{G}$ functors as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction. Then $\mathcal{G}$ preserves limits, i.e. if $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ is a diagram and $\lim_{\leftarrow i} \mathcal{D}_i$ exists in C, then

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

Proof. We have the following chain of isomorphisms, natural in $Y \in \mathrm{Obj}(\mathsf{D})$.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i)) &\simeq \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(Y), \lim_{\leftarrow i} \mathcal{D}_i) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(Y), \mathcal{D}_i) && \left(\substack{\text{Hom functor commutes with} \\ \text{limits: Theorem 153}}\right) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G} \circ \mathcal{D}_i) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i)).
\end{aligned}$$

By the Yoneda lemma, specifically Corollary 113, we have a natural isomorphism

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

□

Corollary 177. Any functor $\mathcal{F}$ which is a left-ajoint preserves colimits.

Proof. Dual to the proof of Theorem 176. □

7.6. The data of an adjunction

In the last section, we saw that the concepts of hom-set adjunction, unit-counit adjunction, and universal morphism adjunction were all instances of one meta-concept, the adjunction. This means that, when specifying an adjunction, one can choose to specify any of these, and derive the rest. One should really think of an adjunction, however, as consisting of all simultaneously.

More specifically, an adjunction

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}$$

gives rise to the following pieces of data.

1. A natural isomorphism $\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-))$.
2. A unit $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$ and counit $\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$.
3. A terminal object in each category $(\mathcal{F} \downarrow d)$.
4. An initial object in each category $(c \downarrow \mathcal{G})$.

These data are related to each other in the following way.

- $(1 \to 2)$: The counit has components $\epsilon_X = \Phi^{-1}_{\mathcal{G}(X),X}(\mathrm{id}_{\mathcal{G}(X)})$, and the unit has components $\epsilon_Y = \Phi_{Y,\mathcal{F}(Y)}(\mathrm{id}_{\mathcal{F}(Y)})$
- $(1 \to 3)$: The terminal object is $(\mathcal{G}(d), \epsilon_d)$
- $(1 \to 4)$: The initial object is $(\mathcal{F}(c), \eta_c)$
- $(2 \to 1)$: The natural transformation has components defined by $\Phi_{X,Y}(f) = \mathcal{G}(f) \circ \eta_X$, or equivalently $\Phi^{-1}_{X,Y}(g) = \epsilon_Y \circ \mathcal{F}(g)$.
- $(2 \to 3)$: The terminal object

7.7. Important examples

Example 178 (free-forgetful adjunction)**.** There are many *free* objects in mathematics: free groups, free modules, free vector spaces, free categories, etc. These are all unified by the following property: the functors defining them are all left adjoints to forgetful functors.

Let us take a specific example: the free vector space over a set. This takes a set S and constructs a vector space which has as a basis the elements of S. When defining free vector space V over a set S, one sees the following universal property:

> For any vector space W and any map $S \to W$, there exists a unique map $W \to V$ making the following diagram commute.

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & V \arrow[d, dashed, "\exists!"] \\ & W \end{tikzcd}$$

There is a forgetful functor $\mathcal{U}\colon \mathsf{Vect}_k \to \mathsf{Set}$ which takes any set and returns the set underlying it. There is a functor $\mathcal{F}\colon \mathsf{Set} \to \mathsf{Vect}_k$, which takes a set and returns the free vector space on it. It turns out that there is an adjunction $\mathcal{F} \dashv \mathcal{U}$.

And this is true of any free object! (In fact by definition.) In each case, the functor giving the free object is left adjoint to a forgetful functor.

Example 179 (limit-constant adjunction)**.** Let J and C be small categories. Denote by $\Delta\colon \mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ the functor such that for all $X \in \mathrm{Obj}(\mathsf{C})$, $\Delta(X)$ is the constant functor which sends every object of J to X and every morphism of J to id_X.

Assume that every diagram $\mathcal{F}\colon \mathsf{J} \to \mathsf{C}$ has a limit. By Defintion 125 this means that for every $\mathcal{D} \in [\mathsf{J}, \mathsf{C}]$, the comma category $(\Delta \downarrow \mathcal{D})$ has a terminal object. But this means that there is an adjoint functor $[\mathsf{J}, \mathsf{C}] \to \mathsf{C}$ which acts on objects by sending $\mathcal{D} \mapsto \lim_{\leftarrow} \mathcal{D}$. That is, we have an adjunction

$$\Delta : \mathsf{C} \leftrightarrow [\mathsf{J}, \mathsf{C}] : \varprojlim .$$

Similarly, there is an adjunction

$$\varinjlim : \mathsf{C} \leftrightarrow [\mathsf{J}, \mathsf{C}] : \Delta.$$

Keep this example in mind during the chapter on Kan extensions.

7.8. What are adjunctions for?

One of the many ways to think about adjunctions is as a relaxation of categorical equivalences. Many examples of adjunctions arise when one has a functor which forgets some information. Of course, no functor will exist which adds back in forgotten information, so there is no hope of turning such a forgetful functor into an equivalence. However, one can often find an approximate inverse as a left or right adjoint.

In this section, we will explore the relationship between adjunctions and categorical equivalences.

Theorem 180. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. The following are equivalent.

1. $\mathcal{F}$ is part of an equivalence of categories (Definition 53) $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$.
2. $\mathcal{F}$ is fully faithful (Definition 39) and essentially surjective (Definition 36).
3. $\mathcal{F}$ is part of a unit-counit adjunction (Definition 166) $(\mathcal{F}, \mathcal{G}, \epsilon, \eta)$, where ϵ and η are isomorphisms.

Proof.

3. $\Rightarrow$ 1. Obvious.

1. $\Rightarrow$ 2. Assume $(\mathcal{F}, \mathcal{G}, \eta\colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_G, \epsilon\colon \mathrm{id}_C \to \mathcal{G} \circ \mathcal{F})$ is an equivalence of categories.

   First we show that $\mathcal{F}$ is essentially surjective. Evaluating η at any $Y \in \mathrm{Obj}(\mathsf{D})$, we get an isomorphism $(\mathcal{F} \circ \mathcal{G})(Y) \simeq Y$. Thus, for any $Y \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism between Y and some element $\mathcal{F}(\mathcal{G}(Y))$ in the image of $\mathcal{F}$.

   Next, we show that $\mathcal{F}$ is fully faithful.

2. $\Rightarrow$ 3.

∎

Proposition 181. Let

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}$$

be an adjunction.

1. The functor $\mathcal{F}$ is fully faithful if and only if the unit $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$ is a natural isomorphism.
2. The functor $\mathcal{G}$ is fully faithful if and only if the counit $\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$ is a natural isomorphism

Proof. We prove the first. The second is the first applied to the adjunction

$$\mathcal{F}^{\mathrm{op}} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}^{\mathrm{op}}.$$

Suppose η is a natural isomorphism. Consider the following composition.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(a, b) \arrow[r, "\mathcal{F}_{x,x}"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(a), \mathcal{F}(b)) \arrow[r, "\Phi_{a,\mathcal{F}(b)}"] & \mathrm{Hom}_{\mathsf{C}}(a, (\mathcal{G} \circ \mathcal{F})(b)) \\
f \arrow[r, mapsto] & \mathcal{F}(f) \arrow[r, mapsto] & (\mathcal{G} \circ \mathcal{F})(f) \circ \eta_a
\end{tikzcd}$$

By naturality of η, the following square commutes.

$$\begin{tikzcd}
a \arrow[r, "f"] \arrow[d, "\eta_a"'] & b \arrow[d, "\eta_b"] \\
(\mathcal{G} \circ \mathcal{F})(a) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"'] & (\mathcal{G} \circ \mathcal{F})(b)
\end{tikzcd}$$

Thus, we can equivalently view the above composition as sending $f \mapsto \eta_b \circ f$.

The second arrow $\Phi_{a,\mathcal{F}(b)}$ is a bijection by the definition of adjunction, and their composition is a bijection since the map $f \mapsto \eta_b \circ f$ has inverse $g \mapsto \eta_b^{-1} \circ g$. Thus, by the 2-out-of-3 law for bijections, $\mathcal{F}_{x,x}$ is a bijection.

Now suppose that η is a natural isomorphism. Then both of the above maps are bijections, giving us a bijection

$$\mathrm{Hom}_{\mathsf{C}}(a, b) \to \mathrm{Hom}_{\mathsf{C}}(a, (\mathcal{G} \circ \mathcal{F})(b)); \qquad f \mapsto \eta_a \circ f.$$

Thus there exists some $g \in \mathrm{Hom}_{\mathsf{C}}((\mathcal{G} \circ \mathcal{F})(a), a)$ such that $\eta_a \circ g = \mathrm{id}_{(\mathcal{G}\circ\mathcal{F})(a)}$. Then

$$\eta_a = \eta_a \circ \mathrm{id}_b = \eta_a \circ g \circ \eta_a.$$

But since $\eta_a \circ -$ is a bijection, this implies that $g \circ \eta_a = \mathrm{id}_b$, so g is a two-sided inverse to η_a, as we were trying to show. ◻

8. Kan extensions

8.1. Motivation

Let C be a category, and $\mathcal{I}\colon S \to C$ a subcategory inclusion. Given a functor $\mathcal{F}$ from C to any category D, we can restrict $\mathcal{F}$ to S by pulling it back with $\mathcal{I}$. We denote this restriction by

$$\mathcal{I}^*(\mathcal{F}) = \mathcal{F} \circ \mathcal{I}.$$

In fact, $\mathcal{I}$ induces a functor from the category of functors $C \to D$ (Definition 48) to the category of functors $S \to D$. Recall that we denoted these categories by $[C, D]$ and $[S, D]$ respectively. What we have created is a functor

$$\mathcal{I}^*\colon [C, D] \to [S, D]$$

which acts on functors $C \to D$ by restricting their domain to the subcategory S.

Admittedly, restricting the domain of a functor is not a very good party trick, even if done in a functorial way. The more impressive thing would be to *extend* the domain of a functor along some inclusion. The problem of extending the domain of a functor is one example of the sort of problem that Kan extensions are designed to do.

We have come up against the problem of finding a weak inverse to a lossy process before. For example, it is easy to turn a group into a set via the lossy process of forgetting the group structure; this is done by a functor $\mathcal{U}\colon \mathsf{Grp} \to \mathsf{Set}$. The inverse problem, i.e. turning a set into a group, is solved (to the extent possible) by taking a left adjoint to the forgetful functor. Together, these two functors form an adjunction, an example of a more general phenomenon known as *free-forgetful adjunction* (Note 178). If we believe the mantra that adjunctions are weak inverses, we should try to solve the problem of extending the domain of a functor by searching for a functor which is adjoint to the restriction functor $\mathcal{I}^*$.

Of course, it is also interesting to consider the case where $\mathcal{I}$ is not an inclusion. In fact, we will see that by choosing $\mathcal{I}$ in different ways leads to a generalization of many different categorical concepts, such as limits.

8.2. Global Kan extensions

The most high-level notion (although, as we will see, not the most useful) of a Kan extension is called a *global Kan extension.* This is the concept discussed in Section 8.1.

Definition 182 (pullback)**.** Let $\mathcal{F}\colon C \to C'$ be a functor. For any other category D, $\mathcal{F}$ induces a functor, called the pullback by $\mathcal{F}$,

$$\mathcal{F}^*\colon [C', D] \to [C, D]$$

which sends a functor $\mathcal{G}\colon \mathsf{C}' \to \mathsf{D}$ to its precomposition $\mathcal{G} \circ \mathcal{F}$

$$\mathsf{C}' \xrightarrow{\mathcal{G}} \mathsf{D} \qquad \mapsto \qquad \begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{F}^*\mathcal{G}"] \arrow[dr, "\mathcal{F}"'] & & \mathsf{D} \\ & \mathsf{C}' \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}$$

and a natural transformation $\eta\colon \mathcal{G} \to \mathcal{G}'$ to its left whiskering (Example 52) $\eta\mathcal{F}$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{C}' \arrow[r, bend left, "\mathcal{G}"] \arrow[r, bend right, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{D} \end{tikzcd} \qquad \mapsto \qquad \begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{G}\circ\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}\circ\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\mathcal{F}^*(\eta)"] & \mathsf{D} \end{tikzcd}$$

Definition 183 (Kan extension functor)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{C}'$ be a functor, and let D be any other category. Consider the pullback functor $\mathcal{F}^*$.

- If $\mathcal{F}^*$ has right adjoint
$$\mathcal{F}_*\colon [\mathsf{C}, \mathsf{D}] \to [\mathsf{C}', \mathsf{D}].$$
We call $\mathcal{F}_*$ the right Kan extension functor along $\mathcal{F}$.
- If $\mathcal{F}^*$ has left adjoint
$$\mathcal{F}_!\colon [\mathsf{C}, \mathsf{D}] \to [\mathsf{C}', \mathsf{D}].$$
We call $\mathcal{F}_!$ the left Kan extension functor along $\mathcal{F}$.

8.3. Kan extensions capture limits and colimits

When working with Kan extensions of a functor $\mathcal{D}$ along another functor $\mathcal{F}$, it is always helpful to have a drawing of the following triangle handy.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & & \mathsf{D} \\ & \mathsf{C}' & \end{tikzcd}$$

Consider the case in which C' is the terminal category $\mathsf{1}$ (Example 4). There is a unique functor from any category $\mathsf{C} \to \mathsf{1}$, which sends every object to the unique object $* \in \mathrm{Obj}(\mathsf{1})$ and every morphism to the identity morphism. Furthermore, any functor $\mathcal{F}\colon \mathsf{1} \to \mathsf{D}$ is completely determined by where it sends the unique object $*$.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{C}"'] & & \mathsf{D} \\ & \mathsf{1} & \end{tikzcd}$$

The restriction of any $\mathcal{F}\colon \mathsf{1} \to \mathsf{C}$ along $\mathcal{C}$ is simply the constant functor $\Delta_{\mathcal{F}(*)}$. Therefore, the functor $\mathcal{C}^*\colon [\mathsf{1}, \mathsf{D}] \to [\mathsf{C}, \mathsf{D}]$ carries exactly the same data as the diagonal functor $\Delta\colon \mathsf{D} \to [\mathsf{C}, \mathsf{D}]$. Thus, $\mathcal{C}_!$ fits into the following adjunction.

$$\mathcal{C}_! : [\mathsf{C}, \mathsf{D}] \leftrightarrow \mathsf{D} : \Delta$$

But we have seen that the colimit functor is right adjoint to the diagonal functor, so we have a natural isomorphism

$$\mathcal{C}_! \simeq \varinjlim .$$

That is, the left Kan extension of any functor $\mathcal{D}$ along the terminal functor $\mathcal{C}$ is[1] the colimit of $\mathcal{D}$:

$$\mathcal{C}_! \mathcal{D} = \varinjlim \mathcal{D}.$$

Exactly analogous reasoning tells us that the right Kan extension of any functor $\mathcal{D}$ along the terminal functor $\mathcal{C}$ is the limit of $\mathcal{D}$:

$$\mathcal{C}_* \mathcal{D} = \varprojlim \mathcal{D}.$$

8.4. Local Kan extensions

By Section 7.5, we can also look at the right-adjointness of the pullback functor $\mathcal{F}^*$'s as guaranteeing that for every $\mathcal{D} \in \mathrm{Obj}([\mathsf{C}', \mathsf{D}])$, the category $(\mathcal{F}^* \downarrow \mathcal{D})$ has a terminal object. Similarly, we can look at $\mathcal{F}_!$'s left-adjointness as demonstrating that any category $(\mathcal{D} \downarrow \mathcal{F}^*)$ has an initial object. This mirrors the definition of limits and colimits as being terminal (resp. initial) objects in the category $(\Delta \downarrow \mathcal{D})$ (resp. $(\mathcal{D} \downarrow \Delta)$). As we saw in section 8.3, this is no accident; Kan extensions form a vast generalization of limits and colimits.

For now, we will specialize our discussion to the case of right Kan extensions and limits, so as not to have to keep writing down 'resp.'

Recall that we defined limits as universal cones. We have seen that there is a functor which assigns limits, and that this is the right adjoint to the diagonal functor. This gives us another way to define limits: we could first have defined the limit functor as right adjoint to the diagonal functor, and then defined limits as being the images of the limit functor. However, this would have been an unnecessarily restrictive definition, since in general some, but not all, limits exist; if we had chosen to define limits in this functorial way, in order to define limits for any diagram $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ we would have had to demand that limits exist for *every* diagram in D. Clearly this is not what we want; for example, we want to be able to take products of two objects in a category without worrying about whether all limits exist in that category.

Fortunately, this is not what we did: we first defined the categories $(\Delta \downarrow \mathcal{D})$, then defined limits as their terminal objects. Only much later did we show that under certain conditions, limits organized themselves into a functor whose left adjoint was the diagonal functor.

By comparison, we first defined right Kan extensions as right adjoints to a restriction functor, then showed that their existence was equivalent to the existence of terminal morphisms in appropriate comma categories. The analogy with limits suggests that we can define Kan extension of one functor along another even when not all extensions exist.

Definition 184 (local Kan extension)**.** Consider categories and functors as below.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & & \mathsf{D} \\ & \mathsf{C}' & \end{tikzcd}$$

[1]Strictly speaking, we should write "uniquely isomorphic to," but we don't because limits are only defined up to unique isomorphism anyway.

- A <u>right Kan extension of $\mathcal{D}$ along $\mathcal{F}$</u> is a terminal morphism in the category $(\mathcal{F}^* \downarrow \mathcal{D})$.
- A <u>left Kan extension of $\mathcal{D}$ along $\mathcal{F}$</u> is a terminal morphism in the category $(\mathcal{D} \downarrow \mathcal{F}^*)$.

This local definition will turn out to be much more useful than Definition 183. It will even to allow us to compute Kan extensions via an explicit formula in certain situations.

Example 185. In the case where $\mathsf{C}' = 1$, we saw that global left and right Kan extensions recovered the limit and colimit functors. Essentially by definition, this works for local Kan extensions as well; we have

$$\mathcal{C}_* \mathcal{D} = \varprojlim \mathcal{D}, \qquad \text{and} \qquad \mathcal{C}_! \mathcal{D} = \varinjlim \mathcal{D}.$$

We defined limits in terms of cones. The close analogy between limits and Kan extensions suggests that we should define a Kan extensionish analog of cones.

The appropriate notion is that of an *extension.* Extensions generalize (co)cones in the sense that they reduce to them in the case of Kan extensions along $\mathcal{C}$.

Definition 186 (extension)**.** Consider categories and functors as below.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & \mathsf{D} \\ & \mathsf{C}' \end{tikzcd}$$

- A <u>right extension of $\mathcal{G}$ along $\mathcal{F}$</u> is an object in the comma category $(\mathcal{F}^* \downarrow \mathcal{D})$. That is, it is a pair $(\mathcal{G}, \eta)$, where $\mathcal{G} \colon \mathsf{C}' \to \mathsf{D}$ and

$$\eta \colon \mathcal{D} \Rightarrow \mathcal{F}^* \mathcal{G}$$

- A <u>left extension of $\mathcal{G}$ along $\mathcal{F}$</u> is an object in the comma category $(\mathcal{D} \downarrow \mathcal{F}^*)$.

Example 187. We have seen that Kan extensions along the functor $\mathsf{C} \to 1$ reproduce limits and colimits. It will be instructive to study Kan extensions along functors $\mathsf{C} \to \mathsf{n}$, where n is the category with n objects and only identity morphisms, i.e. the discrete groupoid with n objects.

Consider such a functor $\mathsf{C} \to \mathsf{n}$. Denote by i the functor $1 \to \mathsf{n}$ which takes $* \in \mathrm{Obj}(1)$ to the ith object in n.

A functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{n}$ is the same thing as a decomposition

$$\mathsf{C} = \coprod_{c=1}^{n} \mathsf{C}_i,$$

where each C_i is the fiber over $i \in \mathrm{Obj}(\mathsf{n})$, together with a family of functors

$$\mathcal{D}|_{\mathsf{C}_i} \colon \mathsf{C}_i \to 1.$$

Consider right extensions of $\mathcal{D}$ along $\mathcal{F}$, i.e. objects in $(\mathcal{F}^* \downarrow \mathcal{D})$. These consist of pairs $(\eta, \mathcal{G})$, where $\eta \colon \mathcal{F}^* \mathcal{G} \to \mathcal{D}$.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & \arrow[d, phantom, "\Uparrow\eta"] & \mathsf{D} \\ & \mathsf{n} \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}$$

The functor $\mathcal{F}^*\mathcal{G}$ is really the composition $\mathcal{G} \circ \mathcal{F}$, which (by the universal property for coproducts) consists of n constant functors $\Delta^i_d \colon \mathsf{C}_i \to \mathsf{D}$, where Δ^i_d is the constant functor mapping everything in C_i to $d \in \mathrm{Obj}(\mathsf{D})$. The natural transformation η therefore consists of n cones

$$\eta^i \colon \Delta^i_d \Rightarrow \mathcal{D}|_{\mathsf{C}_i}.$$

Such a right extension is universal if any other right extension factors through it. That is, given another right extension $(\epsilon, \mathcal{H})$, there is a unique natural transformation $\xi \colon \mathcal{H} \circ \mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{F}$ making the following diagram commute.

$$\begin{tikzcd}
\mathcal{H} \circ \mathcal{F} \arrow[rr, Rightarrow, "\xi"] \arrow[dr, Rightarrow, "\epsilon"'] & & \mathcal{G} \circ \mathcal{F} \arrow[dl, Rightarrow, "\eta"] \\
& \mathcal{D} &
\end{tikzcd}$$

This says precisely that each cone must be universal. That is, we have

$$(\mathcal{F}_*\mathcal{D})(i) = \varprojlim \mathcal{D}|_{\mathsf{C}_i},$$

where $\mathsf{D}|_{\mathsf{C}_i}$ is simply $\mathcal{D}$ restricted to the fiber over i.

The same is true for left Kan extensions; the value

$$(\mathcal{F}_!\mathcal{D})(i) = \varinjlim \mathcal{D}|_{\mathsf{C}_i}.$$

We have more or less proved the following. I still need to work out size conditions, e.g. C' should almost certainly have to be small.

Proposition 188. Let C and D be categories and C' a discrete groupoid, and let $\mathcal{F} \colon \mathsf{C} \to \mathsf{C}'$ and $\mathcal{D} \colon \mathsf{C} \to \mathsf{D}$ be functors.

- Suppose the limit $\lim_{\leftarrow} \mathcal{D}|_{\mathsf{C}_i}$ exists for all $i \in \mathrm{Obj}(\mathsf{C}')$. Then a right Kan extension $\mathcal{F}_*\mathcal{D}$ exists, and is given by the formula

$$(\mathcal{F}_*\mathcal{D})(i) = \varprojlim \mathcal{D}|_{\mathsf{C}_i},$$

- Suppose the colimit $\lim_{\rightarrow} \mathcal{D}|_{\mathsf{C}_i}$ exists for all $i \in \mathrm{Obj}(\mathsf{C}')$. Then a left Kan extension $\mathcal{F}_!\mathcal{D}$ exists, and is given by the formula

$$(\mathcal{F}_!\mathcal{D})(i) = \varinjlim \mathcal{D}|_{\mathsf{C}_i}.$$

This says that a Kan extension along a functor to a discrete groupoid C' consists of a collection of (co)limits parametrized by the objects of C'.

Note 189. In our simple case, the converse also holds: any Kan extension through a discrete groupoid is given by the above formulae. This is *not* a general feature of the formula we derive! There are Kan extensions whose

8.5. A pointwise formula for Kan extensions

One would hope that we would be able to generalize the above formulae easily. However, their derivation rested crucially on the fact that for any category C' with no morphisms, any functor $\mathsf{C} \to \mathsf{C}'$ splits C into a disjoint union of its fibers, and we can easily construct (co)cones over the images of these fibers. Adding any morphisms to C' ruins this splitting, and we lose our (co)cones.

The solution to this problem is to use 'smarter' categories than the fibers of $\mathcal{F}$, over which we can still construct (co)cones in an obvious way: comma categories.

The intuition is as follows. Any morphism $f : c' \to \tilde{c}'$ we add to C' allows the fiber $\mathsf{C}_{c'}$ to map into to the fiber $\mathsf{C}_{\tilde{c}'}$. We should include in the base of our cone not just the parts of C which are mapped to the identity $\mathrm{id}_{c'}$ (i.e. the fiber over c'), but also any fibers to which $\mathsf{C}_{c'}$ has access via some morphism. We have such a category: $(c' \downarrow \mathcal{F})$.

Let $\mathcal{D} : \mathsf{C} \to \mathsf{D}$ and $\mathcal{F} : \mathsf{C} \to \mathsf{C}'$ be functors. Let $(\mathcal{G}, \eta)$ be a right extension of $\mathcal{D}$ along $\mathcal{F}$.

$$\begin{tikzcd} \mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & {} \arrow[d, phantom, "\Uparrow\eta"] & \mathsf{D} \\ & \mathsf{C}' \arrow[ur, "\mathcal{G}"'] & \end{tikzcd}$$

For any $c' \in \mathrm{Obj}(\mathsf{C}')$, denote by $(c' \downarrow \mathcal{F})$ the category of morphisms from c' down to $\mathcal{F}$ (see Definition 64), and denote by $\mathcal{U}$ the forgetful functor $(c' \downarrow \mathcal{F}) \to \mathsf{C}$. We can define a functor $\mathrm{evl}\, \mathcal{D}_{(c' \downarrow \mathcal{F})} : (c' \downarrow \mathcal{F}) \to \mathsf{D}$ as the following composition.

$$(c' \downarrow \mathcal{F}) \xrightarrow{\mathcal{U}} \mathsf{C} \xrightarrow{\mathcal{D}} \mathsf{D}$$

To each such $c' \in \mathrm{Obj}(\mathsf{C}')$, there is an associated cone ξ over $\mathrm{evl}\, \mathcal{D}_{(c' \downarrow \mathcal{F})}$, constructed as follows.

- The tip of the cone is $\mathcal{G}(c')$.
- For each $(\alpha, f) \in \mathrm{Obj}(c' \downarrow \mathcal{F})$, the morphism $\xi_{(\alpha,f)} : \mathcal{D}(c') \to \mathrm{evl}\, \mathcal{D}_{(c' \downarrow \mathcal{F})}(\alpha, f) = \mathcal{D}(\alpha)$ is given by the composition

$$\mathcal{G}(c') \xrightarrow{\mathcal{G}(f)} \mathcal{G}(\mathcal{F}(\alpha)) \xrightarrow{\eta_\alpha} \mathcal{D}(\alpha) \ .$$

Dually, considering now a *left* extension $(\mathcal{G}, \eta)$ of $\mathcal{D}$ along $\mathcal{F}$, we get a cone as follows.

- The tip is $\mathcal{G}(c')$
- For each $(\alpha, f) \in \mathrm{Obj}(\mathcal{F} \downarrow c')$, the morphism $\xi_{\alpha,f} : \mathcal{D}(\alpha) \to \mathcal{D}(c')$ is given by the composition

$$\mathcal{D}(\alpha) \xrightarrow{\eta_\alpha} (\mathcal{G} \circ \mathcal{F})(\alpha) \xrightarrow{\mathcal{G}(f)} \mathcal{G}(c')$$

Lemma 190. The cone described above really is a cone. That is, for each morphism $g : (\alpha, f) \to (\alpha', f')$ as follows

$$\begin{tikzcd} & c' \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{F}(\alpha) \arrow[rr, "\mathcal{F}(g)"'] & & \mathcal{F}(\alpha') \end{tikzcd}$$

the diagram

$$\begin{tikzcd}
 & \mathcal{G}(c') \arrow[dl, "\xi_{(\alpha,f)}"'] \arrow[dr, "\xi_{(\alpha',f')}"] & \\
\mathcal{D}(\alpha) \arrow[rr, "\mathcal{D}(g)"'] & & \mathcal{D}(\alpha')
\end{tikzcd}$$

commutes.

Proof. Consider the following diagram.

$$\begin{tikzcd}
 & \mathcal{G}(c') \arrow[dl, "\mathcal{G}(f)"'] \arrow[dr, "\mathcal{G}(f')"] & \\
(\mathcal{G} \circ \mathcal{F})(\alpha) \arrow[rr, "\mathcal{G}\circ\mathcal{F}(\alpha)"'] \arrow[d, "\eta_\alpha"'] & & \mathcal{G} \circ \mathcal{F}(\alpha') \arrow[d, "\eta_{\alpha'}"] \\
\mathcal{D}(\alpha) \arrow[rr, "\mathcal{D}(f)"'] & & \mathcal{D}(\alpha')
\end{tikzcd}$$

The upper triangle is the functor $\mathcal{D}$ applied to the definition of commutativity in a category of morphisms, and the lower square is the naturality square for η. □

Example 191. In the case that C' is a discrete groupoid, we have the following.

- The category $\mathsf{C}_{c'}$ is equivalent (in fact isomorphic) to both of the categories $(\mathcal{F} \downarrow c')$ and $(c' \downarrow \mathcal{F})$.
- The cone which constructed above over $\operatorname{evl} \mathcal{D}_{(c' \downarrow \mathcal{F})}$ agrees with the cones η^i from Example 187.
- The cocone under $\operatorname{evl} \mathcal{D}_{(\mathcal{F} \downarrow c')}$ agrees with the cocones that we would have constructed if we had more time.

That is, the (co)cones we have constructed are a more general case of those we got from considering discrete groupoids.

Lemma 192. Let $\mathcal{F}\colon \mathsf{I} \to \mathsf{J}$ be a functor. Let $f\colon j \to j'$ be a morphism in J. Then $\mathcal{F}$ induces the following functors.

- $(\mathcal{F} \downarrow j) \to (\mathcal{F} \downarrow j')$.
- $(j' \downarrow \mathcal{F}) \to (j \downarrow \mathcal{F})$.

Proof. Obvious. □

We saw above that given any extension $(\mathcal{G}, \eta)$ and any object $c' \in \operatorname{Obj}(\mathsf{C}')$, we could construct a cone. We hoped that this cone would hope turn out to be the analog of the cones over C_i from Example 187. This turns out to be the case.

Theorem 193. Consider categories and functors as follows,

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & \mathsf{D} \\
 & \mathsf{C}'
\end{tikzcd}$$

where C is small and D has all small colimits. Then the Kan extension $\mathcal{F}_!\mathcal{D}$ exists and has values

$$(\mathcal{F}_!\mathcal{D})(c') = \varinjlim \left((\mathcal{F} \downarrow c') \longrightarrow \mathsf{C} \xrightarrow{\mathcal{D}} \mathsf{D} \right)$$

Proof. We first have to define how $\mathcal{F}_!\mathcal{D}$ behaves on morphisms. To this end, let $f\colon c' \to c''$. Then by Lemma 192, f induces a ☐

Lemma 194. Let $(\mathcal{G}, \eta)$ be a right extension as below.

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & \arrow[d, phantom, "\Uparrow\eta"] & \mathsf{D} \\
& \mathsf{C}' \arrow[ur, "\mathcal{G}"'] &
\end{tikzcd}$$

If for every $c' \in \mathrm{Obj}(\mathsf{C}')$ the cone with tip $\mathcal{G}(c')$ over the functor given by the composition

$$(c' \downarrow \mathcal{F}) \to \mathsf{C} \to \mathsf{D}$$

with sides

$$\xi_{(\alpha,f)} = \left(\mathcal{G}(c') \xrightarrow{\mathcal{G}(f)} (\mathcal{G} \circ \mathcal{F})(\alpha) \xrightarrow{\eta_\alpha} \mathcal{D}(\alpha) \right)$$

is a limit cone, then $(\mathcal{G}, \eta)$ is a right Kan extension.

Proof. Let $(\mathcal{G}', \eta')$ be any other extension. We need to show that there is a unique natural transformation χ as follows.

$$\mathcal{G}' \circ \mathcal{F} \qquad\qquad \mathcal{G} \circ \mathcal{F}$$

☐

Definition 195 (limit over a comma category)**.** Consider categories and functors as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[rr, "\mathcal{D}"] \arrow[dr, "\mathcal{F}"'] & & \mathsf{D} \\
& \mathsf{C}' &
\end{tikzcd}$$

We define the notations

$$\lim_{c \in (\mathcal{F} \downarrow c')} \mathcal{D}(c) = \varprojlim \left[(\mathcal{F} \downarrow c') \to \mathsf{C} \to \mathsf{D} \right]$$

and

$$\operatorname*{colim}_{c \in (c' \downarrow \mathcal{F})} \mathcal{F}(c) = \varinjlim \left[(c' \downarrow \mathcal{F}) \to \mathsf{C} \to \mathsf{D} \right]$$

8.6. Properties of Kan extensions

Recall that our original motivation for defining Kan extensions was the problem of extending the domain of an inclusion. One would hope that this extension, restricted to the original domain, would agree with the original functor. In situations in which we are allowed to use the colimit formula, this turns out to be the case.

Theorem 196. Let C be a small category, and D locally small. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{C}'$ be fully faithful.

- If D has small limits, then the counit

  $$\epsilon\colon \mathcal{F}^* \circ \mathcal{F}_* \Rightarrow \mathrm{id}_{\mathsf{C}'}$$

  is a natural isomorphism.

- If D has small colimits, then the unit

  $$\eta\colon \mathrm{id}_{\mathsf{D}} \Rightarrow \mathcal{F}^* \circ \mathcal{F}_!$$

  is a natural isomorphism.

Proof. We need to show that each component

$$\epsilon_{\mathcal{D}}\colon (\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D}) \Rightarrow \mathrm{id}_{\mathsf{C}'}$$

is a natural isomorphism. By definition of the pullback, we have

$$(\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D}) = \mathcal{F}^*(\mathcal{F}_*\mathcal{D}) = \mathcal{F}_*\mathcal{D} \circ \mathcal{F}.$$

Consider some $c' \in \mathrm{Obj}(\mathsf{C}')$. By Theorem 193, we can compute $\mathcal{F}_*\mathcal{D}(c')$ via the limit formula

$$\mathcal{F}_*\mathcal{D}(c') = \lim_{c \in (\mathcal{F}\downarrow c')} \mathcal{D}(c).$$

In particular, if $c' = \mathcal{F}(c)$ for some c, we have

$$\mathcal{F}_*\mathcal{D}(\mathcal{F}(c)) = \lim_{c \in (\mathcal{F}\downarrow\mathcal{F}(c))} \mathcal{D}(c) = \lim\left[(\mathcal{F} \downarrow \mathcal{F}(c)) \to \mathcal{C} \to \mathcal{D}\right].$$

Since the object $(c, \mathrm{id}_{\mathcal{F}(c)})$ is initial in $(\mathcal{F} \downarrow \mathcal{F}(c))$ we have by Theorem 162 that

$$\lim\left[(\mathcal{F} \downarrow \mathcal{F}(c)) \to \mathcal{C} \to \mathcal{D}\right] = \mathcal{D}(c).$$

Thus, we have

$$(\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D})(c) \simeq \mathcal{D}(c)$$

for all $c \in \mathrm{Obj}(\mathsf{C})$ □

Corollary 197. Let C be a small category, and D locally small. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{C}'$ be fully faithful.

- If D has small limits, then $\mathcal{F}_*$ is fully faithful.
- If D has small colimits, then $\mathcal{F}_!$ fully faithful.

8.7. Yoneda extensions

One particularly important example of Kan extension is Kan extension along the Yoneda embedding.

Definition 198 (Yoneda extension)**.** One calls left Kan extensions along the Yoneda embedding (Definition 108) Yoneda extensions.

One can express any presheaf as a colimit.

Lemma 199. Let C be a locally small category. Denote by $\mathcal{Y}$ the Yoneda embedding $\mathsf{C} \to [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$. Then for any functor $\mathcal{F} \colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ we have

$$\mathcal{F} \simeq \varinjlim \left[(\mathcal{Y} \downarrow \mathcal{F}) \xrightarrow{\mathcal{U}} \mathsf{C} \xrightarrow{\mathcal{Y}} [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \right].$$

Proof. We need to show that $\mathcal{F}$ is the base of a universal cocone under $\mathcal{Y} \circ \mathcal{U}$. That $\mathcal{F}$ is the base of a cocone is obvious, since every object (α, f) comes with a map $f \colon \mathcal{Y}(\alpha) \to \mathcal{F}$ as required, and the diagram below manifestly commutes for each morphism g.

$$\begin{tikzcd}
\mathcal{Y}(\alpha) \arrow[rr, Rightarrow, "\mathcal{Y}(g)"] \arrow[dr, Rightarrow, "f"'] & & \mathcal{Y}(\alpha') \arrow[dl, Rightarrow, "f'"] \\
& \mathcal{F} &
\end{tikzcd}$$

That is, $f' \circ \mathcal{Y}(g) = f$ It remains to show that $\mathcal{F}$ is universal, i.e. that for any other cocone with tip $\mathcal{G} \in [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ and components

$$\xi_{(\alpha,f)} \colon \mathcal{Y}(c) \Rightarrow \mathcal{G}$$

making the diagrams

$$\begin{tikzcd}
\mathcal{Y}(\alpha) \arrow[rr, Rightarrow, "\mathcal{Y}(g)"] \arrow[dr, Rightarrow, "\xi_{(\alpha,f)}"'] & & \mathcal{Y}(\alpha') \arrow[dl, Rightarrow, "\xi_{(\alpha',f')}"] \\
& \mathcal{G} &
\end{tikzcd}$$

commute (i.e. so that $\xi_{(\alpha',f')} \circ \mathcal{Y}(g) = \xi_{(\alpha,f)}$), this data is enough to define a unique natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$.

But each object (α, f) also gives us, again by Yoneda, an element of $\mathcal{G}(\alpha)$. Thus, we are able to assign to each element of $\mathcal{F}(\alpha)$ a unique element of $\mathcal{G}(\alpha)$, giving us a map $\mathcal{F}(\alpha) \to \mathcal{G}(\alpha)$. This gives us a correspondence

$$\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(\alpha), \mathcal{F}) \to \mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(\alpha), \mathcal{G}),$$

natural in α. By Yoneda, we get a natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$. ☐

Lemma 200. Let I be a small category, and let C be locally small and cocomplete. Let $\mathcal{F}$ be a functor. Then the Yoneda extension $\mathcal{Y}_!\mathcal{F}$ is left adjoint to the functor

$$\mathcal{G} \colon \mathsf{C} \to [\mathsf{I}^{\mathrm{op}}, \mathsf{Set}]; \qquad c \mapsto \mathrm{Hom}_{\mathsf{C}}(F(-), c).$$

$$\begin{tikzcd}
\mathsf{I} \arrow[rr, "\mathcal{F}"] \arrow[dr, "\mathcal{Y}"'] & & \mathsf{C} \\
& {[\mathsf{I}^{\mathrm{op}}, \mathsf{Set}]} \arrow[ur, "\mathcal{Y}_!\mathcal{F}"'] &
\end{tikzcd}
\qquad
\mathcal{Y}_!\mathcal{F} : [\mathsf{I}^{\mathrm{op}}, \mathsf{Set}] \leftrightarrow \mathsf{C} : \mathcal{G}$$

Proof. Since C is cocomplete, we are entitled to use the colimit formula for left Kan extensions. That is, we can write

$$(\mathcal{Y}_!\mathcal{F})(\mathcal{H}) = \varinjlim \left[(\mathcal{Y} \downarrow \mathcal{H}) \xrightarrow{\mathcal{U}} I \xrightarrow{\mathcal{F}} \mathsf{C} \right] = \varinjlim [\mathcal{F} \circ \mathcal{U}] \,.$$

Thus, we have the following string of natural isomorphisms.

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}((\mathcal{Y}_!\mathcal{F})(\mathcal{H}), c) &\simeq \mathrm{Hom}_{\mathsf{C}}(\varinjlim[\mathcal{F} \circ \mathcal{U}], c) \\ &\simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(-), c) \circ \mathcal{U} \\ &\simeq \varprojlim \mathcal{G}(c) \circ \mathcal{U} \\ &\simeq \varprojlim \mathrm{Hom}_{[\mathsf{I}^{\mathrm{op}}, \mathsf{Set}]}(\mathcal{Y}(-), \mathcal{G}(c)) \circ \mathcal{U}. \\ &\simeq \mathrm{Hom}_{[\mathsf{I}^{\mathrm{op}}, \mathsf{Set}]}(\varinjlim \mathcal{U} \circ \mathcal{Y}, \mathcal{G}(c)) \\ &\simeq \mathrm{Hom}_{[\mathsf{I}^{\mathrm{op}}, \mathsf{Set}]}(\mathcal{H}, \mathcal{G}(c)). \end{aligned}$$

□

Corollary 201. Let I be a small category, and let C be locally small and cocomplete. Let $\mathcal{F}$ be a functor. Then the Yoneda extension $\mathcal{Y}_!\mathcal{F}$ is cocontiuous.

Theorem 202. Let C be a small category, and let D be cocomplete. Let $\mathcal{F}\colon [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{D}$ be any functor. Then $\mathcal{F}$ preserves colimits if and only if it is the left Yoneda extension of its restriction $\mathcal{Y}^*F$.

Proof. Assume that $\mathcal{F} = (\mathcal{Y}_!\mathcal{Y}^*)(\mathcal{F})$. By Lemma 200, $(\mathcal{Y}_!\mathcal{Y}^*)(\mathcal{F})$ is a left adjoint, hence preserves colimits by Corollary 177.

Now assume that $\mathcal{F}$ preserves colimits. Pick some functor $\mathcal{G}\colon \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$. By Lemma 199, we can write

$$\begin{aligned} \mathcal{F}(\mathcal{G}) &\simeq \mathcal{F}\left(\varinjlim \left[(\mathcal{Y} \downarrow \mathcal{G}) \to \mathsf{C} \xrightarrow{\mathcal{Y}} [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \right] \right) \\ &\simeq \varinjlim \left[(\mathcal{Y} \downarrow \mathcal{G}) \to \mathsf{C} \xrightarrow{\mathcal{Y}} [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \xrightarrow{\mathcal{F}} \mathsf{D} \right] \\ &\simeq \varinjlim \left[(\mathcal{Y} \downarrow \mathcal{G}) \to \mathsf{C} \xrightarrow{\mathcal{Y}^*\mathcal{F}} \mathsf{D} \right] \\ &\simeq \mathcal{Y}_!(\mathcal{Y}^*\mathcal{F})(\mathcal{G}). \end{aligned}$$

□

Corollary 203. Let C be a small category, and D locally small. There is an equivalence of categories

$$[[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \mathsf{D}]_{\text{cocontinuous}} \simeq [\mathsf{C}, \mathsf{D}].$$

That is, colimit preserving functors $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}] \to \mathsf{D}$ are equivalent to functors $\mathsf{C} \to \mathsf{D}$.

Proof. We have an adjunction

$$\mathcal{Y}_! : [\mathsf{C}, \mathsf{D}] \leftrightarrow [[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \mathsf{D}] : \mathcal{Y}^*.$$

By Corollary 201, the image of $\mathcal{Y}_!$ is completely contained in $[[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \mathsf{D}]_{\mathrm{cocontinuous}}$, so the above adjunction extends to an adjunction

$$\mathcal{Y}_! : [\mathsf{C}, \mathsf{D}] \leftrightarrow [[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \mathsf{D}]_{\mathrm{cocontinuous}} : \mathcal{Y}^*.$$

We know that the Yoneda embedding is fully faithful. Thus, by Theorem 196, we have that the unit is an isomorphism. Also, by Theorem 202, the counit is an isomorphism, so the above adjunction is in fact an equivalence of categories. □

9. Monoidal categories

9.1. Structure

9.1.1. Basic definitions

Monoidal categories

Many types of mathematical structures have an operation which combines them to form new ones. For example:

- The tensor product allows us to use two vector spaces to produce a new one, as does the direct sum.
- The Cartesian product and the disjoint union of sets produce a new set out of two given ones.

In each of these cases, there is a natural isomorphisms making this combination associative, and an identity object which acts, up to natural isomorphism, as a left and right identity. This means that each of these operations gives the collection of all objects the structure of a monoid.

A monoidal category will be a category in which the objects have the structure of a monoid, i.e. there is a suitably defined 'multiplication' (usually written $\otimes$) which is unital (with unit 1) and associative. The prototypical example of categorical multiplication is the product (see Example 71), although it is far from the only one.

When put like this, monoidal categories don't sound like complicated entities, and indeed in practice they are not. However, there is a lot of subtlety in making associativity play well with multiplication by units. Since we are in a category, demanding that our multiplication be associative 'on the nose,' i.e. that, for example,

$$(V \otimes 1) \otimes (W \otimes T) \qquad \text{and} \qquad (V \otimes W) \otimes T$$

should be literally equal, is too draconian, not to mention difficult to interpret. The natural weakening of this is to demand that these be merely isomorphic, but this is far too weak: for vector spaces, for example, this requires only that the dimension be the same. Even natural isomorphism is not quite strong enough.

The correct way of solving this problem is by demanding that certain diagrams, called *coherenece diagrams*, commute. These diagrams then force other diagrams to commute in a way that solves our problem, as Mac Lane showed in the so-called *Mac Lane's Coherence Theorem.*

Definition 204 (monoidal category)**.** A monoidal category is a category C equipped with a monoidal structure. A monoidal structure is the following:

- A bifunctor (Definition 30) $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the *tensor product*,

- An object I called the *unit object*, and
- Three natural isomorphisms (Definition 42) subject to coherence conditions expressing the fact that the tensor product
    - is associative: there is a natural isomorphism α called the *associator*, with components
    $$\alpha_{A,B,C} \colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$
    - has left and right identity: there are two natural isomorphisms λ and ρ respectively called the *left unitor* and *right unitor* with components
    $$\lambda_A \colon I \otimes A \xrightarrow{\sim} A$$
    and
    $$\rho_A \colon A \otimes I \xrightarrow{\sim} A.$$

The coherence conditions are that the following diagrams commute for all A, B, C, and $D \in \mathrm{Obj}(C)$.

- The *triangle diagram*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes \mathrm{id}_B"'] & & A \otimes (I \otimes B) \arrow[dl, "\mathrm{id}_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

- The *home plate diagram*[1] (usually the *pentagon diagram*)

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes \mathrm{id}_D"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "\mathrm{id}_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

More succinctly, a monoidal structure on a category C is a quintuple $(\otimes, 1, \alpha, \lambda, \rho)$.

Notation 205. The notation $(\otimes, 1, \alpha, \lambda, \rho)$ is prone to change. If the associator and unitors are not important, or understood from context, they are often left out. We will often say "Let $(\mathrm{C}, \otimes, 1)$ be a monoidal category."

[1]Unfortunately, the home plate, as drawn, is actually upside down.

Example 206. The simplest (though not the prototypical) example of a monoidal category is Set with the Cartesian product. We have already studied this structure in some detail in Section 4.1. We check that it satisfies the axioms in Definition 204.

- The Cartesian product on Set *is* a set-theoretic product, and can be naturally viewed, thanks to Theorem 77, as the bifunctor.
- Any set with one element $I = \{*\}$ functions as the unit object.
- For all sets A, B, C
  - The universal property of products gives us a natural isomorphism with components

$$\alpha_{A,B,C} \colon (A \times B) \times C \to A \times (B \times C); \qquad ((a, b), c) \mapsto (a, (b, c)).$$

  - Since $\{*\}$ is terminal, we get an isomorphism $\lambda_A \colon \{*\} \times A \to A$ which sends $(*, a) \mapsto a$.
  - Similarly, we get a map $\rho_A \colon A \times \{*\} \to A$ which sends $(a, *) \mapsto a$.

The pentagon and triangle diagram commute vacuously since the cartesian product is associative.

Lemma 207. The tensor product $\otimes$ of vector spaces is a bifunctor $\mathsf{Vect}_k \times \mathsf{Vect}_k \to \mathsf{Vect}_k$.

Proof. It is clear what the domain and codomain of the tensor product is, and how it behaves on objects and morphisms. The only non-trivial aspect is showing that the standard definition of the tensor product of morphisms respects composition, which is not difficult. □

Example 208. The category Vect_k is a monoidal category with

1. The bifunctor $\otimes$ is given by the tensor product.
2. The unit 1 is given by the field k regarded as a 1-dimensional vector space over itself.
3. The associator is the map which sends $(v_1 \otimes v_2) \otimes v_3$ to $v_1 \otimes (v_2 \otimes v_3)$. It is not *a priori* obvious that this is well-defined, but it is also not difficult to check.
4. The left unitor is the map which sends $(x, v) \in k \times V$ to $xv \in V$.
5. The right unitor is the map which sends

$$(v, x) \mapsto xv.$$

Definition 209 (monoidal subcategory)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. A monoidal subcategory of C is a subcategory (Definition 12) $\mathsf{S} \subseteq \mathsf{C}$ which is closed under the tensor product, and which contains 1.

Example 210. Recall that $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces over a field k. We saw in Example 14 that $\mathsf{FinVect}_k$ was a full subcategory of Vect_k.

We have just seen that Vect_k is a monoidal category with unit object $1 = k$. Since k is a one-dimensional vector space over itself, $k \in \mathrm{Obj}(\mathsf{FinVect}_k)$, and since the dimension of the tensor product of two finite-dimensional vector spaces is the product of their dimensions, the tensor product is closed in $\mathsf{FinVect}_k$. Hence $\mathsf{FinVect}_k$ is a monoidal subcategory of Vect_k.

Lemma 211. Let C be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Then the maps λ_1 and $\rho_1 \colon 1 \otimes 1 \to 1$ agree.

Proof. See [11], Lemma 3.9. ◻

Note 212. The triangle and pentagon diagram are the first in a long list of coherence diagrams designed to make categorical structures behave like their algebraic counterparts. For a monoid, multiplication is associative by fiat, and the extension of this to n-fold products follows by induction. In monoidal categories, associators are isomorphisms rather than equalities, and we have no a priori guarantee that different ways of composing associators give the same isomorphism.

Mac Lane's coherence theorem shows us that this is exactly what the coherence diagrams guarantee.

Theorem 213 (Mac Lane's coherence theorem)**.** Any two ways of freely composing unitors and associators to go from one expression to another coincide.

Monoidal functors

The following definition was taken *mutatis mutandis* from [13].

Definition 214 (monoidal functor)**.** Let C and C′ be monoidal categories. A functor $\mathcal{F} \colon C \to C'$ is lax monoidal if it is equipped with

- a natural transformation $\Phi_{X,Y} \colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$, and
- a morphism $\varphi \colon \mathrm{id}_{C'} \to \mathcal{F}(\mathrm{id}_C)$ such that
- the following diagrams commute for any $X, Y, Z \in \mathrm{Obj}(C)$.

$$\begin{tikzcd}
(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X,Y} \otimes \mathrm{id}_{\mathcal{F}(Z)}"] \arrow[d, "\alpha_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}"'] & \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X \otimes Y, Z}"] & \mathcal{F}((X \otimes Y) \otimes Z) \arrow[d, "\mathcal{F}(\alpha_{X,Y,Z})"] \\
\mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) \arrow[r, "\mathrm{id}_{\mathcal{F}(X)} \otimes \Phi_{Y,Z}"'] & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \arrow[r, "\varphi_{X, Y \otimes Z}"'] & \mathcal{F}(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes \mathcal{F}(X) \arrow[r, "\lambda_{\mathcal{F}(X)}"] \arrow[d, "\varphi \otimes \mathrm{id}_{\mathcal{F}(X)}"'] & \mathcal{F}(X) \\
\mathcal{F}(1) \otimes \mathcal{F}(X) \arrow[r, "\Phi_{1,X}"] & \mathcal{F}(1 \otimes X) \arrow[u, "\mathcal{F}(\lambda_X)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(X) \otimes 1 \arrow[r, "\rho_{\mathcal{F}(X)}"] \arrow[d, "\mathrm{id}_{\mathcal{F}(X) \otimes \varphi}"'] & \mathcal{F}(X) \\
\mathcal{F}(X) \otimes \mathcal{F}(1) \arrow[r, "\Phi_{X,1}"] & \mathcal{F}(X \otimes 1) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

If Φ is a natural isomorphism and φ is an isomorphism, then $\mathcal{F}$ is called a strong monoidal functor.

We will denote the above monoidal functor by $(\mathcal{F}, \Phi, \varphi)$.

Note 215. The above diagrams above are exactly those necessary to ensure that the monoidal structure is preserved. They do this by demanding that the associator and the unitors be $\mathcal{F}$-equivariant.

Note 216. In much of the literature, a strong monoidal functor is simply called a monoidal functor.

Monoidal natural transformations

Definition 217 (monoidal natural transformation)**.** Let (F, Φ, φ) and (G, Γ, γ) be monoidal functors. A natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is monoidal if the following diagrams commute.

$$\begin{tikzcd} \mathcal{F}(X) \otimes \mathcal{F}(Y) \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{G}(X) \otimes \mathcal{G}(Y) \arrow[d, "\Gamma_{X,Y}"] \\ \mathcal{F}(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"] & \mathcal{G}(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathcal{F}(1) \arrow[r, "\eta_1"] & \mathcal{G}(1) \end{tikzcd}$$

9.1.2. Line objects

Definition 218 (invertible object)**.** Let $(\mathrm{C}, \otimes, 1)$ be a monoidal category. An invertible object (sometimes called a *line object*) is an object $L \in \mathrm{Obj}(\mathrm{C})$ such that both of the functors $\mathrm{C} \to \mathrm{C}$

- $\ell_L : A \mapsto L \otimes A$
- $\mathrm{sr}_L : A \mapsto A \otimes L$

are categorical equivalences.

Lemma 219. If $(\mathrm{C}, \otimes, 1)$ is a monoidal category and L is an invertible object, then there is an object $L^{-1} \in \mathrm{Obj}(\mathrm{C})$, unique up to isomorphism, such that $L \otimes L^{-1} \simeq L^{-1} \otimes L \simeq 1$. Furthermore, L is invertible only if there exists such an L^{-1}.

Proof. Suppose L is invertible. Then the functor $\ell_L : A \mapsto L \otimes A$ is bijective up to isomorphism, i.e. for any $A \in \mathrm{Obj}(\mathrm{C})$, there is an object $A' \in \mathrm{Obj}(\mathrm{C})$, unique up to isomorphism, such that

$$\ell_L(A') = L \otimes A' \simeq A.$$

If this is true for *any* A, it must also be true for 1, so there exists an object L^{-1}, unique up to isomorphism, such that

$$\ell_L(L^{-1}) = L \otimes L^{-1} \simeq 1.$$

The same logic tells us that there exists some other element L'^{-1}, such that

$$\mathrm{sr}_L(L'^{-1}) = L'^{-1} \otimes L \simeq 1.$$

Now

$$1 \simeq L'^{-1} \otimes L \simeq L'^{-1} \otimes (L \otimes L^{-1}) \otimes L \simeq (L'^{-1} \otimes L) \otimes (L^{-1} \otimes L) \simeq (L'^{-1} \otimes L) \otimes 1 \simeq L'^{-1} \otimes L,$$

so L'^{-1} is also a left inverse for L. But since ℓ_L is an equivalence of categories, L only has one left inverse up to isomorphism, so L^{-1} and L'^{-1} must be isomorphic. □

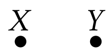
Definition 220. Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. Then the full subcategory (Definition 13) $(\mathrm{Line}(\mathsf{C}), \otimes, 1) \subseteq (\mathsf{C}, \otimes, 1)$ whose objects are the line objects in C is called the line subcategory of $(\mathsf{C}, \otimes, 1)$. Since invertibility is closed under the tensor product and 1 is invertible ($1^{-1} \simeq 1$), $\mathrm{Line}(\mathsf{C})$ is a monoidal subcategory of C.

9.1.3. Braided monoidal categories

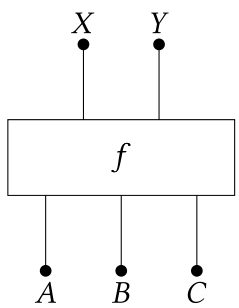
Braided monoidal categories capture the idea that we should think of morphisms between tensor products spatially, as diagrams embedded in 3-space. This sounds odd, but it turns out to be the correct way of looking at a wide class of problems.

To be slightly more precise, we can think of the objects X, Y, etc. in any monoidal category as little dots.

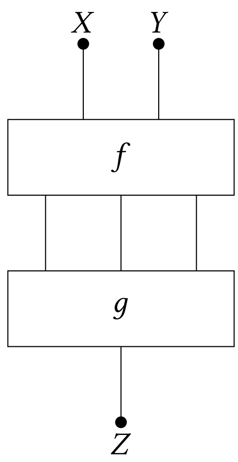
We can express the tensor product $X \otimes Y$ by putting the dots representing X and Y next to each other.



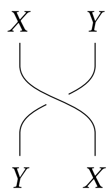
A morphism f between, say, $X \otimes Y$ and $A \otimes B \otimes C$ can be drawn as a diagram consisting of some lines and boxes.



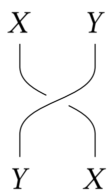
We can compose morphisms by concatenating their diagrams.



In a braided monoidal category, we require that for any two objects X and Y we have an isomorphism $\gamma_{XY} : X \otimes Y \to Y \otimes X$, which we draw like this.



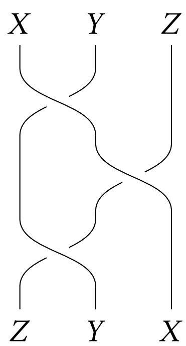
Since γ_{AB} is an isomorphism, it has an inverse γ_{AB}^{-1} (not necessarily equal to γ_{BA}!) which we draw like this.



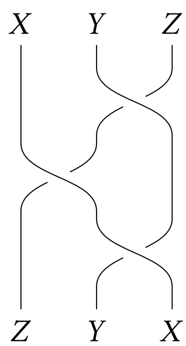
The idea of a braided monoidal category is that we want to take these pictures seriously: we want two expressions involving repeated applications of the $\gamma_{..}$ and their inverses to be equivalent if and only if the braid diagrams representing them are homotopic. Thus we want, for example,

$$\gamma_{XY} \circ \gamma_{YZ} \circ \gamma_{XY} = \gamma_{YZ} \circ \gamma_{XY} \circ \gamma_{YZ}$$

since



is homotopic to



.

A digression into the theory of braid groups would take us too far afield. The punchline is that to guarantee that all such compositions involving the γ are identified in the correct way, we must define braided monoidal categories as follows.

Definition 221 (braided monoidal category)**.** A catgory C with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$ is braided if for every two objects A and $B \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism $\gamma_{A,B} \colon A \otimes B \to B \otimes A$ such that the following *hexagon diagrams* commute.

$$\begin{tikzcd} & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{BCA}"] & \\ (A \otimes B) \otimes C \arrow[ur, "\alpha_{ABC}"] \arrow[dr, "\gamma_{AB}\otimes 1"'] & & & B \otimes (C \otimes A) \\ & (B \otimes A) \otimes C \arrow[r, "\alpha_{BAC}"'] & B \otimes (A \otimes C) \arrow[ur, "1\otimes\gamma"'] & \end{tikzcd}$$

$$\begin{tikzcd} & (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{CAB}"] & \\ A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{ABC}"] \arrow[dr, "1\otimes\gamma_{BC}"'] & & & (C \otimes A) \otimes B \\ & A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{ACB}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{AC}\otimes 1"'] & \end{tikzcd}$$

The collection of such γ form a natural isomorphism betweem the bifunctors

$$(A, B) \mapsto A \otimes B \qquad \text{and} \qquad (A, B) \mapsto B \otimes A,$$

and is called a braiding.

Definition 222 (braided monoidal functor)**.** A lax monoidal functor $(\mathcal{F}, \Phi, \phi)$ (Definition 214) is braided monoidal if it makes the following diagram commute.

$$\begin{tikzcd} \mathcal{F}(x) \otimes \mathcal{F}(y) \arrow[r, "\gamma_{\mathcal{F}(x),\mathcal{F}(y)}"] \arrow[d, "\Phi_{x,y}"'] & \mathcal{F}(y) \otimes \mathcal{F}(x) \arrow[d, "\Phi_{x,y}"] \\ \mathcal{F}(x \otimes y) \arrow[r, "\mathcal{F}(\gamma_{x,y})"] & \mathcal{F}(y \otimes x) \end{tikzcd}$$

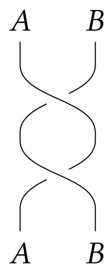
Note 223. There are no extra conditions imposed on a monoidal natural transformation to turn it into a braided natural transformation.

9.1.4. Symmetric monoidal categories

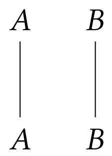
Until now, we have been calling the bifunctor $\otimes$ in Definition 204 a tensor product. This has been an abuse of terminology: in general, one defines tensor products not to be those bifunctors which come from any monoidal category, but only those which come from *symmetric* monoidal categories. We will define these shortly.

Conceptually, passing from the definition of a braided monoidal category to that of a symmetric monoidal category is rather simple. One only requires that for any two objects A and B, $\gamma_{BA} = \gamma_{AB}^{-1}$, i.e. $\gamma_{BA} \circ \gamma_{AB} = \mathrm{id}_{A \otimes B}$.

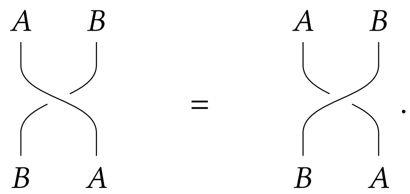
We can interpret this nicely in terms of our braid diagrams. We can draw $\gamma_{BA} \circ \gamma_{AB}$ like this.



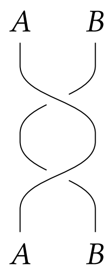
The requirement that this must be homotopic to the identity transformation



can be expressed by making the following rule: in a *symmetric* monoidal category, we don't care about the difference between undercrossings and overcrossings:



Then we can exchange the diagram representing $\gamma_{BA} \circ \gamma_{AB}$ for



which is clearly homotopic to the identity transformation on $A \otimes B$.

Definition 224 (symmetric monoidal category)**.** Let C be a braided monoidal category with braiding γ. We say that C is a symmetric monoidal category if for all $A, B \in \mathrm{Obj}(\mathrm{C})$, $\gamma_{BA} \circ \gamma_{AB} = \mathrm{id}_{A \otimes B}$. A braiding γ which satisfies such a condition is called symmetric.

Note 225. There are no extra conditions imposed on a monoidal natural transformation to turn it into a symmetric natural transformation.

9.2. Internal hom functors

We can now generalize the notion of an exponential object (Definiton 96) to any monoidal category.

9.2.1. The internal hom functor

Recall the definition of the hom functor on a locally small category C (Definition 100): it is the functor which maps two objects to the set of morphisms between them, so it is a functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}.$$

If we take $\mathsf{C} = \mathsf{Set}$, then our hom functor never really leaves Set; it is *internal* to Set. This is our first example of an *internal hom functor*. In fact, it is the prototypical internal hom functor, and we can learn a lot by studying its properties.

Let X and Y be sets. Denote the set of all functions $X \to Y$ by $[X, Y]$.

Let S be any other set, and consider a function $f: S \to [X, Y]$. For each element $s \in S$, f picks out a function $h_s: X \to Y$. But this is just a curried version of a function $S \times X \to Y$! So as we saw in Section 5.1, we have a bijection between the sets $[S, [X, Y]]$ and $[S \times X, Y]$. In fact, this is even a *natural* bijection, i.e. a natural transformation between the functors

$$[-, [-, -]] \qquad \text{and} \qquad [- \times -, -]: \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set} \to \mathsf{Set}.$$

Let's check this. First, we need to figure out how our functors act on functions. Suppose we have sets and functions like so.

$$\begin{gathered} A'' \xleftarrow{f''} B'' \\ A' \xleftarrow{f'} B' \\ A \xrightarrow{f} B \end{gathered}$$

Our functor maps

$$(A'', A', A) \mapsto [A'', [A', A]] = \mathrm{Hom}_{\mathsf{Set}}(A'', \mathrm{Hom}_{\mathsf{Set}}(A', A)),$$

so it should map (f'', f', f) to a function

$$[f'', [f', f]]: [A'', [A', A]] \to [B'', [B', B]].$$

The way to do that is by sending $m \in [A'', [A', A]]$ to

$$[f', f] \circ m \circ f''.$$

You can check that this works as advertised.

The other one's not so tough. Our functor maps an object (A'', A', A) to $[A'' \times A', A]$. We need to map (f'', f', f) to a function

$$[f'' \times f', f]: [A'' \times A', A] \to [B'' \times B', B].$$

We do that by sending $m \in [A'' \times A', A]$ to

$$f \circ m \circ (f'', f') \in [B'' \times B', B].$$

Checking that $[-, [-, -]]$ and $[- \times -, -]$ really *are* functorial would be a bit much; each is just the composition of hom functor and the Cartesian product. We will however check that there is a natural isomorphism between them, which amounts to checking that the following diagram commutes.

$$\begin{tikzcd}
{[A'' \times A', A]} \arrow[rrr, "{[f'' \times f', f]}"] \arrow[ddd, "\Phi_{[A'' \times A', A]}"'] & & & {[B'' \times B', B]} \arrow[ddd, "\Phi_{[B'' \times B', B]}"] \\
& m \arrow[r, mapsto] \arrow[d, mapsto] & f \circ m \circ (f', f'') \arrow[d, mapsto] & \\
& \Phi(m) \arrow[r, mapsto] & {[f', f] \circ \Phi(m) \circ f'' \overset{!}{=} \Phi(f \circ m \circ (f', f''))} & \\
{[A'', [A', A]]} \arrow[rrr, "{[f'', [f', f]]}"'] & & & {[B'', [B', B]]}
\end{tikzcd}$$

In other words, we have to show that

$$\Phi_{[A'' \times A', A]}(f \circ m \circ (f', f'')) = [f', f] \circ \Phi_{[B'' \times B', B]}(m) \circ f''.$$

So what is each of these? Well, $f \circ m \circ (f', f'')$ is a map $B'' \times B' \to B$, which maps (say) $(b'', b') \mapsto b$.

The natural transformation Φ tells us to curry this, i.e. turn it into a map $B'' \to [B', B]$. Not just any map, though: a map which when evaluated on b'' turns into a map which, when evaluated on b', yields b.

We know that $f \circ m \circ (f', f'') \colon (b'', b') \mapsto b$, i.e.

$$f(m(f''(b''), f'(b'))) = b.$$

If we can show that this is *also* what $[f', f] \circ \Phi_{[B'' \times B', B]}(m) \circ f''$ is equal to when evaluated on b'' and then b', we are done, since two functions are equal if they take the same value for all inputs.

Well, let's go through what this definition means. First, we take b'' and feed it to f''. Next, we let $\Phi_{[B'' \times B', B]}(m)$ act on the result, i.e. we fill the first argument of m with $f''(b'')$. What we get is the following:

$$m(f''(b''), -).$$

Then we are to precompose this with f' and stick the result into f:

$$f(m(f''(b''), f'(-))).$$

Finally, we are to evaluate this on b' to get

$$f(m(f''(b''), f'(b'))).$$

Indeed, this is equal to b, so the diagram commutes.

In other words, Φ is a natural bijection between the hom-sets $[-,[-,-]]$ and $[- \times -, -]$.

We picked this example because the collection $[A, B]$ of all functions between two sets A and B is itself a set. Therefore it makes sense to think of the hom-sets $\mathrm{Hom}_{\mathsf{C}}(A, B)$ as living within the same category as A and B. We saw in Section 4.1 that in a category with products, we could sometimes view hom-sets as exponential objects. However, we now have the technology to be even more general.

Definition 226 (internal hom functor)**.** Let $(\mathsf{C}, \otimes)$ be a monoidal category. An internal hom functor is a functor

$$[-,-]_{\mathsf{C}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{C}$$

such that for every $X \in \mathrm{Obj}(\mathsf{C})$ we have a pair of adjoint functors

$$(-) \otimes X \dashv [X, -]_{\mathsf{C}}.$$

The objects $[A, B]_{\mathsf{C}}$ are called internal hom objects.

Note 227. The reason for the long introduction to this section was that the pair of adjoint functors in Definition 226 really matches the one in Set. Recall, in Set there was a natural transformation

$$\mathrm{Hom}_{\mathsf{Set}}(S \times X, Y) \simeq \mathrm{Hom}_{\mathsf{Set}}(S, [X, Y]).$$

This means that for any set X, there is a pair of adjoint functors

$$(-) \times X \dashv [X, -],$$

which is in agreement with the statement of Definition 226.

Notation 228. The convention at the nLab is to denote the internal hom by square braces $[A, B]$, and this is for the most part what we will do. Unfortunately, we have already used this notation for the *regular* hom functor. To remedy this, we will add a subscript if the category to which the hom functor belongs is not clear: $[-,-]_{\mathsf{C}}$ for a hom functor internal to C, $[-,-]_{\mathsf{Set}}$ for the standard hom functor (or the hom functor internal to Set, which amounts to the same).

There is no universally accepted notation for the internal hom functor. One often sees it denoted by a lower-case hom: $\mathrm{hom}_{\mathsf{C}}(A, B)$. Many sources (for example DMOS [16]) distinguish the internal hom with an underline: $\underline{\mathrm{Hom}}_{\mathsf{C}}(A, B)$. Deligne typesets it with a script H: $\mathscr{H}om_{\mathsf{C}}(A, B)$.

Definition 229 (closed monoidal category)**.** A monoidal category equipped with an internal hom functor is called a closed monoidal category.

Note 230. Here is another (clearly equivalent) definition of $[X, Y]_{\mathsf{C}}$: it is the object representing (Definition 105) the functor

$$T \mapsto \mathrm{Hom}_{\mathsf{C}}(T \otimes X, Y).$$

Example 231. In many locally small categories whose objects can be thought of as "sets with extra structure," it is possible to pile structure on top of the hom sets until they themselves can be viewed as bona fide objects in their categories. It often (*but not always!*) happens that these beefed-up hom sets coincide (up to isomorphism) with the internal hom objects.

Take for example Vect_k. For any vector spaces V and W, we can turn $\mathrm{Hom}_{\mathsf{Vect}_k}(V, W)$ into a vector space by defining addition and scalar multiplication pointwise; we can then view $\mathrm{Hom}_{\mathsf{Vect}_k}(V, W)$ as belonging to $\mathrm{Obj}(\mathsf{Vect}_k)$. It turns out that this is precisely (up to isomorphism) the internal hom object $[V, W]_{\mathsf{Vect}_k}$.

To see this, we need to show that there is a natural bijection

$$\mathrm{Hom}_{\mathsf{Vect}_k}(A, \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)) \simeq \mathrm{Hom}_{\mathsf{Vect}_k}(A \otimes B, C).$$

Suppose we are given a linear map $f: A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)$. If we act with this on an element of A, we get a linear map $B \to C$. If we evaluate this on an element of B, we get an element of C. Thus, we can view f as a bilinear map $A \times B \to C$, hence as a linear map $A \otimes B \to C$.

Now suppose we are given a linear map $g: A \otimes B \to C$. By pre-composing this with the tensor product we can view this as a bilinear map $A \times B \to C$, and by currying this we get a linear map $A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)$.

For the remainder of this chapter, let $(\mathsf{C}, \otimes, 1)$ be a closed monoidal category with internal hom functor $[-, -]_{\mathsf{C}}$.

In a closed monoidal category, the adjunction between the internal hom and the tensor product even holds internally.

Lemma 232. For any $X, Y, Z \in \mathrm{Obj}(\mathsf{C})$ there is a natural isomorphism

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

Proof. Let $A \in \mathrm{Obj}(\mathsf{C})$. We have the following string of natural isomorphisms.

$$\begin{aligned} \mathrm{Hom}_{\mathsf{C}}(A, [X \otimes Y, Z]_{\mathsf{C}}) &\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes (X \otimes Y), Z) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}((A \otimes X) \otimes Y, Z) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes X, [Y, Z]_{\mathsf{C}}) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}(A, [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}). \end{aligned}$$

Since this is true for each A we have, by Corollary 113,

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

□

We saw in Lemma 103 that the any functor extends to a natural transformation between hom functors. This property extends to the internal hom.

Lemma 233. Let $(\mathsf{C}, \otimes, 1)$ be a closed symmetric monoidal category. For any $A, B, R \in \mathrm{Obj}(\mathsf{C})$, there is a natural transformation

$$[A, B]_{\mathsf{C}} \to [R \otimes A, R \otimes B]_{\mathsf{C}},$$

natural in A and B.

Proof. The following string of natural transformations guarantees it by the Yoneda lemma.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) &\simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B) \\
&\to \mathrm{Hom}_{\mathsf{C}}(R \otimes X \otimes A, R \otimes B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes (R \otimes A), R \otimes B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, [R \otimes A, R \otimes B]_{\mathsf{C}}).
\end{aligned}$$

□

9.2.2. The evaluation map

The internal hom functor gives us a way to talk about evaluating morphisms $f\colon X \to Y$ without mentioning elements of X.

Definition 234 (evaluation map)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. We have seen that the adjunction

$$(-) \otimes X \dashv [X, -]_{\mathsf{C}}$$

gives us, for any $A, X, Y \in \mathrm{Obj}(\mathsf{C})$, a natural bijection

$$\mathrm{Hom}_{\mathsf{C}}(A \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A, [X, Y]_{\mathsf{C}}).$$

In particular, with $A = [X, Y]_{\mathsf{C}}$, we have a bijection

$$\mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}} \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}}, [X, Y]_{\mathsf{C}}).$$

The adjunct (Definition 174) of $\mathrm{id}_{[X,Y]_{\mathsf{C}}} \in \mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}}, [X, Y]_{\mathsf{C}})$ is an object in $\mathrm{Hom}_{\mathsf{C}}\,([X, Y]_{\mathsf{C}} \otimes X, Y)$, denoted

$$\mathrm{ev}_{X,Y}\colon [X, Y]_{\mathsf{C}} \otimes X \to Y,$$

and called the <u>evaluation map</u>.

Example 235. As we saw in Example 206, the category Set is a monoidal category with a bifunctor given by the cartesian product. The internal hom is simply the regular hom functor

$$\mathrm{Hom}_{\mathsf{Set}}(-, -) = [-, -].$$

Let us explore the evaluation map on Set. It is the adjunct of the identity map $\mathrm{id}_{[X,Y]}$ under the adjunction

$$[[X, Y] \times X, Y] \dashv [[X, Y], [X, Y]].$$

Thus, it is a function

$$\mathrm{ev}_{X,Y}\colon [X, Y] \times X \to Y; \qquad (f, x) \mapsto \mathrm{ev}_{X,Y}(f, x).$$

So far, we don't know what $\mathrm{ev}_{X,Y}$ sends (f, x) to; we just know that we'd *like it* if it sent it to $f(x)$.

The above adjunction is given by currying: we start on the LHS with a map $\mathrm{ev}_{X,Y}$ with two arguments, and we turn it into a map which fills in only the first argument. Thus the map on the RHS adjunct to $\mathrm{ev}_{X,Y}$ is given by

$$f \mapsto \mathrm{ev}_{X,Y}(f, -).$$

If we want the map $f \mapsto \mathrm{ev}_{X,Y}(f, -)$ to be the identity map, f and $\mathrm{ev}_{X,Y}(f, -)$ must agree on all elements x, i.e.

$$f(x) = \mathrm{ev}_{X,Y}(f, x) \qquad \text{for all } x \in X.$$

Thus, the evaluation map is the map which sends $(f, x) \mapsto f(x)$.

9.2.3. The composition morphism

The evaluation map allows us to define composition of morphisms without talking about internal hom objects as if they have elements.

Definition 236 (composition morphism)**.** For $X, Y, Z \in \mathrm{Obj}(\mathsf{C})$, the composition morphism

$$\circ_{X,Y,Z} \colon [Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \to [X,Z]_{\mathsf{C}}$$

is the $(-) \otimes X \vdash [X,-]_{\mathsf{C}}$-adjunct of the composition

$$[Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \otimes X \xrightarrow{\left(\mathrm{id}_{[Y,Z]_{\mathsf{C}}}, \mathrm{ev}_{X,Y}\right)} [Y,Z]_{\mathsf{C}} \otimes Y \xrightarrow{\quad \mathrm{ev}_{Y,Z} \quad} Z \ .$$

Example 237. In Set, the composition morphism $\circ_{X,Y,Z}$ lives up to its name. Let $f \colon X \to Y$, $g \colon Y \to Z$, and $x \in X$. The above composition goes as follows.

1. The map $\left(\mathrm{id}_{[Y,Z]}, \mathrm{ev}_{X,Y}\right)$ turns the triple (g, f, x) into the pair $(g, f(x))$.
2. The map $\mathrm{ev}_{Y,Z}$ turns $(g, f(x))$ into $g(f(x)) = (g \circ f)(x)$.

The evaluation morphism $\circ_{X,Y,Z}$ is the currying of this, i.e. it sends

$$(f, g) \mapsto (f \circ g)(-).$$

9.2.4. Dual objects

Recall that for any k-vector space V, there is a dual vector space

$$V^* = \{L \colon V \to k\} \ .$$

This definition generalizes to any closed monoidal category.

Definition 238 (dual object)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. The dual object to X, denoted X^*, is defined to be the object

$$[X, 1]_{\mathsf{C}}.$$

That is to say, X^* is the internal hom object modelling the hom set of morphisms from X to the identity object 1.

Notation 239. The evaluation morphism (Definition 234) has a component

$$\mathrm{ev}_{X^*,X} \colon X^* \otimes X \to 1.$$

To clean things up a bit, we will write ev_X instead of $\mathrm{ev}_{X^*,X}$.

Notation 240. In many sources, e.g. DMOS ([16]), the dual object to X is denoted $X^\vee$ instead of X^*.

Lemma 241. There is a natural isomorphism between the functors

$$\mathrm{Hom}_{\mathsf{C}}(-, X^*) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}((-) \otimes X, 1).$$

Proof. For any $X, T \in \mathrm{Obj}(\mathsf{C})$, the definition of the internal hom $[-,-]_{\mathsf{C}}$ gives us a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(T \otimes X, 1) \simeq \mathrm{Hom}_{\mathsf{C}}(T, [X, 1]_{\mathsf{C}}) = \mathrm{Hom}_{\mathsf{C}}(T, X^*).$$

□

Theorem 242. The map $X \mapsto X^*$ can be extended to a contravariant functor.

Proof. We need to figure out how our functor should act on morphisms. We define this by analogy with the familiar setting of vector spaces. Recall that for a linear map $L\colon V \to W$, the dual map $L^t\colon W^* \to V^*$ is defined by

$$(L^t(w))(v) = w(L(v)).$$

By analogy, for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we should define the dual morphism $f^t \in \mathrm{Hom}_{\mathsf{C}}(B^*, A^*)$ by demanding that the following diagram commutes.

$$\begin{tikzcd}
h^* \otimes X \arrow[r, "f^t \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_Y \otimes f"'] & X^* \otimes X \arrow[d, "\mathrm{ev}_X"] \\
h^* \otimes Y \arrow[r, "\mathrm{ev}_Y"] & 1
\end{tikzcd}$$

To check that this is functorial, we must check that it respects compositions, i.e. that the following diagram commutes.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "(f^t \circ g^t) \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_Z \otimes (g \circ f)"'] & X^* \otimes X \arrow[d, "\mathrm{ev}_X"] \\
Z^* \otimes Z \arrow[r, "\mathrm{ev}_Z"] & 1
\end{tikzcd}$$

Let's add in some more objects and morphisms.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "g^t \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{Z^*} \otimes f"'] & h^* \otimes X \arrow[r, "f^t \otimes \mathrm{id}_X"] \arrow[d, "\mathrm{id}_{h^*} \otimes f"'] & X^* \otimes X \arrow[dd, "\mathrm{ev}_X"] \\
Z^* \otimes Y \arrow[r, "g^t \otimes \mathrm{id}_Y"] \arrow[d, "\mathrm{id}_{Z^*} \otimes g"'] & h^* \otimes Y \arrow[dr, "\mathrm{ev}_Y"] & \\
Z^* \otimes Z \arrow[rr, "\mathrm{ev}_Z"'] & & 1
\end{tikzcd}$$

We want to show that the outer square commutes. But it clearly does: that the top left square commutes is trivial, and the right and bottom 'squares' are the commutativity conditions defining f^t and g^t. □

Note 243. It's not clear to me why f^t as defined above exists and is unique.

The above is one, but not the only, way to define dual objects. We can be more general.

Definition 244 (right duality)**.** Let C be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Right duality of two objects A and $A^* \in \mathrm{Obj}(\mathsf{C})$ consists of

1. A morphism of the form
$$\mathrm{ev}_A : A^* \otimes A \to 1,$$
called the *evaluation map* (or *counit* if you're into Hopf algebras)
2. A morphism of the form
$$i_A : 1 \to A \otimes A^*,$$
called the *coevaluation* map (or *unit*)

such that the compositions

$$X \xrightarrow{i_A \otimes \mathrm{id}_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{id}_X \otimes \mathrm{ev}_X} X$$

$$X^* \xrightarrow{\mathrm{id}_{X^*} \otimes \mathrm{ev}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{ev}_X \otimes \mathrm{id}_{X^*}} X^*$$

are the identity morphism.

Definition 245 (rigid monoidal category)**.** A monoidal category $(\mathsf{C}, \otimes, 1)$ is rigid if every object has a left and right dual.

Theorem 246. Every rigid monoidal category is a closed monoidal category (i.e. has an internal hom functor, see Definition 229) with internal hom object

$$[A, B]_{\mathsf{C}} \simeq B \otimes A^*.$$

Proof. We can prove the existence of this isomorphism by showing, thanks to Corollary 113, that for any $X \in \mathrm{Obj}(\mathsf{C})$ there is an isomorphsm

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

The defining adjunction of the internal hom gives us

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Now we can map any $f \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B)$ to

$$(f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes i_A) \in \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

We will be done if we can show that the assignment

$$f \mapsto (f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes i_A)$$

is an isomorphism. We'll do this by exhibiting an inverse:

$$\mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*) \ni g \mapsto (\mathrm{id}_W \otimes \mathrm{ev}_V) \circ (g \otimes \mathrm{id}_V) \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Of course, first we should show that $(f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes i_A)$ really does map $X \to B \otimes A^*$. But it does; it does this by first acting on X with i_A:

$$X \to X \otimes A \otimes A^*$$

and then acting on the $X \otimes A$ with f and letting the A^* hang around:

$$X \otimes A \otimes A^* \to B \otimes A^*.$$

To show that

$$g \mapsto (\mathrm{id}_B \otimes \mathrm{ev}_A) \circ (g \otimes \mathrm{id}_A)$$

really is an inverse, we can shove the assignment

$$f \mapsto (f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes i_A)$$

into it and show that we get f right back out. That is to say, we need to show that

$$(\mathrm{id}_B \otimes \mathrm{ev}_A) \circ ([(f \otimes \mathrm{id}_A) \circ (\mathrm{id}_X \otimes i_A)] \otimes \mathrm{id}_A) = f.$$

This is easy to see but hard to type. Write it out. You'll need to use first of the two composition identities.

To show that the other composition yields g, you have to use the other. □

10. Abelian categories

This section draws heavily from [22].

10.1. Additive categories

Recall that Ab is the category of abelian groups.

Definition 247 (Ab-enriched category)**.** A category C is Ab-enriched if

1. for all objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition

   $$\circ\colon\ \mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{C}(A, C)$$

   is additive in each slot: for any $f_1, f_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, we must have

   $$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$

   and similarly in the second slot.

That is to say, Ab-enriched categories are categories enriched over $(\mathsf{Ab}, \otimes_{\mathbb{Z}}, 0)$.

Note 248. In any Ab-enriched category, every hom-set has at least one element—the identity element of the hom-set taken as an abelian group.

Lemma 249. Let C be an Ab-enriched category with finite products. Then the product $X_1 \times \cdots \times X_n$ satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

Proof. First, we show that initial and terminal objects coincide. Let $\emptyset$ be initial and $*$ terminal in C.

Since $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on $\emptyset$ *and* the identity element of $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, $\mathrm{id}_* = 0$ in $\mathrm{Hom}_{\mathsf{C}}(*, *)$.

Since $\emptyset$ is initial and $*$ is final, $\mathrm{Hom}_{\mathsf{C}}(\emptyset, *)$ has exactly one element f. By Note 248, we are guaranteed that $\mathrm{Hom}_{\mathsf{C}}(*, \emptyset)$ has at least one element, g. Then $g \circ f = 0 = \mathrm{id}$ and $f \circ g = 0 = \mathrm{id}$, so $f\colon \emptyset \to *$ is an isomorphism. Thus, initial and terminal objects coincide, so C has a zero object 0.

We prove the lemma for binary products. The general case is preciesly the same.

For each $X, Y \in \mathcal{C}$, we can define $i_X : X \longrightarrow X \times Y$ using the universal property for products as follows.

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}"'] \arrow[d, dashed, "i_X"] \arrow[dr, "0"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Similarly, get $i_Y : Y \longrightarrow X \times Y$.

The claim is that $X \times Y$, together with i_X and i_Y, satisfies the universal property for coproducts; that is, that we can find a unique map ϕ making the below diagram commute.

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, dashed, "\phi"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\
& Z &
\end{tikzcd}$$

In fact, $\phi = f \circ \pi_X + g \circ \pi_Y$ satisfies the universal property. It is clear that this makes the diagram commute since $\pi_X \circ i_X = \mathrm{id}$, $\pi_X \circ i_Y = 0$, and similarly for Y.

Note that, by the universal property for products, there is a unique downward map. But both $\mathrm{id}_{X \times Y}$ and $i_X \circ \pi_X + i_Y \circ \pi_Y$ make it commute, so they must be equal.

$$\begin{tikzcd}
& X \times Y \arrow[dl, "\pi_Y"'] \arrow[d] \arrow[dr, "\pi_X"] & \\
X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}$$

Now we are ready to show that ϕ is unique. Suppose we have another candidate ψ which also makes the above diagram commute. Then

$$\psi = \psi \circ \mathrm{id}_{X \times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

□

Definition 250 (additive category)**.** A category C is <u>additive</u> if it has products and is Ab-enriched.

Example 251. The category Ab of Abelian groups is an additive category. We have already seen that it has the direct sum $\oplus$ as biproduct. Given any two abelian groups A and B and morphisms $f, g : A \longrightarrow B$, we can define the sum $f + g$ via

$$(f + g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$

Then for another abelian group C and a morphism $h : B \longrightarrow C$, we have

$$[h \circ (f + g)](a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g](a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g,$$

and similarly in the other slot.

Example 252. The category Vect_k is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

Definition 253 (additive functor)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor between additive categories. We say that $\mathcal{F}$ is additive if for each $X, Y \in \mathrm{Obj}(\mathsf{C})$ the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of abelian groups.

Lemma 254. For any additive functor $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$, there exists a natural isomorphism

$$\Phi\colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

Proof. The commutativity of the following diagram is immediate.

$$\begin{tikzcd}
\mathcal{F}(X \oplus Y) \arrow[r, "\mathcal{F}(f \oplus g)"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{F}(X' \oplus Y') \arrow[d, "\Phi_{X',Y'}"] \\
\mathcal{F}(X) \oplus \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f) \oplus \mathcal{F}(g)"] & \mathcal{F}(X') \oplus \mathcal{F}(Y')
\end{tikzcd}$$

The (X, Y)-component $\Phi_{X,Y}$ is an isomorphism because □

10.2. Pre-abelian categories

Definition 255 (pre-abelian category)**.** A category C is pre-abelian if it is additive and every morphism has a kernel (Definition 135) and a cokernel (Definition 142).

Lemma 256. Pre-abelian categories have equalizers (Definition 129).

Proof. We show that in an pre-abelian category, the equalizer of f and g coincides with the kernel of $f - g$. It suffices to show that the kernel of $f - g$ satisfies the universal property for the equalizer of f and g.

Here is the diagram for the universal property of the kernel of $f - g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists ! \bar{\imath}"] \arrow[drr, bend left] \arrow[ddr, bend right, "i"'] & & \\
& \ker(f - g) \arrow[r] \arrow[d, "\iota_{f-g}"'] & 0 \arrow[d] \\
& A \arrow[r, "f-g"] & B
\end{tikzcd}$$

The universal property tells us that for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $i\colon Z \to A$ with $i \circ (f - g) = 0$ (i.e. $i \circ f = i \circ g$), there exists a unique morphism $\bar{\imath}\colon Z \to \ker(f - g)$ such that $i = \bar{\imath} \circ \iota_{f-g}$. □

Corollary 257. Every pre-abelian category has all finite limits and colimits.

Proof. By Theorem 152, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 256. The colimit case is dual. □

Recall from Section 4.4 the following definition.

Definition 258 (zero morphism)**.** Let C be a category with zero object 0. For any two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B \end{tikzcd}$$

Notation 259. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}.

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism: $f \circ 0 = 0$ and $0 \circ g = 0$.

Lemma 260. Every morphism $f\colon A \to B$ in a pre-abelian catgory has a canonical decomposition

$$A \xrightarrow{p} \mathrm{coker}(\ker(f)) \xrightarrow{\bar{f}} \ker(\mathrm{coker}(f)) \xrightarrow{i} B\ ,$$

where p is an epimorphism (Definition 20) and i is a monomorphism (Definition 16).

Proof. We start with a map

$$A \xrightarrow{f} B\ .$$

Since we are in a pre-abelian category, we are guaranteed that f has a kernel $(\ker(f), \iota)$ and a cokernel $(\mathrm{coker}(f), \pi)$. From the universality squares it is immediate that $f \circ \iota = 0$ and $\pi \circ f = 0$ This tells us that the composition $\pi \circ f \circ \iota = 0$, so the following commutes.

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & & & \mathrm{coker}(f) \\ & 0 \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

We know that π has a kernel $(\ker(\pi), i)$ and ι has a cokernel $(\mathrm{coker}(\iota), p)$, so we can add their commutativity squares as well.

$$\begin{tikzcd} & A \arrow[rr, "f"] \arrow[d, "p"] & & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & \mathrm{coker}(\iota) & & \ker(\pi) \arrow[u, "i"] \arrow[d] & \mathrm{coker}(f) \\ & 0 \arrow[u] \arrow[rr] & & 0 \arrow[ur] & \end{tikzcd}$$

If we squint hard enough, we can see the following diagram.

$$
\begin{tikzcd}
A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
& \ker(\pi) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
& B \arrow[r, "\pi"] & \operatorname{coker}(f)
\end{tikzcd}
$$

The outer square commutes because $\pi \circ f = 0$, so the universal property for $\ker(\pi)$ gives us a unique morphism $A \to \ker(\pi)$. Let's add this to our diagram, along with a morphism $0 \to \ker(\pi)$ which trivially keeps everything commutative.

$$
\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}
$$

Again, buried in the bowels of our new diagram, we find the following.

$$
\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[d] & A \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & \operatorname{coker}(\iota) & \\
& & \ker(\pi)
\end{tikzcd}
$$

And again, the universal property of cokernels gives us a unique morphism $\bar{f}\colon \operatorname{coker}(\iota) \to \ker(\pi)$.

$$
\begin{tikzcd}
& A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"'] \arrow[d] & \operatorname{coker}(f) \\
& 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}
$$

The fruit of our laborious construction is the following commuting square.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "i"] \\
\operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi)
\end{tikzcd}
$$

We have seen (Lemma 138) that i is mono, and (Lemma 144) that p is epi.

Now we abuse terminology by calling $\iota = \ker(f)$ and $\pi = \operatorname{coker}(f)$. Then we have the required decomposition. □

Note 261. The abuse of notation above is ubiquitous in the literature.

10.3. Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

Definition 262 (abelian category)**.** A pre-abelian category C is abelian if every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

Note 263. The above piecemeal definition is equivalent to the following.

A category C is *abelian* if

1. it is Ab-enriched Definition 250, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
2. it admits finite coproducts, hence (by Lemma 249) biproducts and zero objects;
3. every morphism has a kernel and a cokernel;
4. every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Example 264.

- Let R be a ring. Then the categories R-Mod and Mod-R of left- and right R-modules respectively, are abelian.
- If A is abelian, then Ch(A), the category of chain complexes in A, is abelian.
- If I is a small category, then Fun(I, A) is abelian.
  - With $I = BG$ for some group G, this gives the category of A-representations of G
  - With X a topological space and Open(X) its category of opens, taking $I = \mathrm{Open}^{\mathrm{op}}$) tells us that the category of presheaves on X, denoted Psh(X), is abelian.

Example 265. Here are two more tricky cases.

- Consider $R = \mathbb{C}[x_1, x_2, \ldots]$ the ring of polynomials in countably many variables. Consider the full subcategory
$$(R\text{-Mod})^{\text{fin gen}} \hookrightarrow R\text{-Mod}.$$
This is additive (as a full subcategory), but does not have kernels in general. For example, consider the quotient
$$f\colon R \twoheadrightarrow R/(x_1, \ldots) \overset{\text{as a vector space}}{\simeq} \mathbb{C}.$$
Both R and $R/(x_1, \ldots)$ are finitely generated R-modules. It may at first seem that there cannot be a kernel since the kernel of f is $(x_1, \ldots)$, which is not finitely generated. But this is not the kernel we are looking for—it is taken in the category of R-Mod, which satisfies a much stronger universal property than the hypothetical kernel in (R-Mod).

However, it turns out that there is no saving this construction. Suppose there was a kernel. It would have to be finitely generated, i.e. we could write

$$\ker f = R[g_1, \ldots, g_n]/(\cdots).$$

Each of the g_i are polynomials in (finitely many) variables x_i; in particular, there is a highest i such that x_i appears in one of the g_i. But then the map

For the remainder of the section, let C be an abelian category.

Lemma 266. Let $f\colon A \to B$ be a morphism in an abelian category.

- If f is mono, then $\ker f = 0$.
- If f is epi, then $\operatorname{coker} f = 0$.

Proof. Suppose f is mono. The kernel of f has the property that for any morphism $g\colon C \to A$ such that $f \circ g = 0$, there exists a unique morphism $h\colon C \to \ker f$ such that $\iota \circ h = g$. However, if f is mono, then $f \circ g = 0 \implies g = 0$, so the zero morphism also has this property. Thus, 0 is a kernel of f.

$$\begin{tikzcd} & C \arrow[dl, "\exists!0"'] \arrow[d, "g=0"'] \arrow[dr, "0"] & \\ 0 \arrow[r] & X \arrow[r, "f"'] & Y \end{tikzcd}$$

The cokernel case is dual. □

Lemma 267. Let $\mathcal{A}$ be an abelian category, and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Then f is an isomorphism if and only if f is monic and epic.

Proof. The direction iso $\implies$ monic and epic is obvious. To see the other direction, suppose that f is monic and epic. By monicness, Lemma 266 implies that $\ker f = 0$. Because f is epic, f is the cokernel of the map $0 \to X$. □

Lemma 268. In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.

Proof. □

Note 269. The above decomposition is unique up to unique isomorphism.

Definition 270 (image of a morphism)**.** Let $f\colon A \to B$ be a morphism. The object $\ker(\operatorname{coker}(f))$ is called the image of f, and is denoted $\operatorname{im}(f)$.

Lemma 271.

1. A morphism $f\colon A \to B$ is mono iff for all $Z \in \mathrm{Obj}(\mathsf{C})$ and for all $g\colon Z \to A$, $f \circ g = 0$ implies $g = 0$.
2. A morphism $f\colon A \to B$ is epi iff for all $Z \in \mathrm{Obj}(\mathsf{C})$ and for all $g\colon B \to Z$, $g \circ f = 0$ implies $g = 0$.

Proof.

1. First, suppose f is mono. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

   If the above diagram commutes, i.e. if $f \circ g = 0$, then $g = 0$, so $1 \implies 2$.

   Now suppose that for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all $g \colon Z \to A$, $f \circ g = 0$ implies $g = 0$.

   Let $g, g' \colon Z \to A$, and suppose that $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0$. But that means that $g - g' = 0$, i.e. $g = g'$. Thus, $2 \implies 1$.

2. Dual to the proof above.

□

Lemma 272. Let $f : A \to B$. We have the following.

1. The morphism f is mono iff $\ker(f) = 0$
2. The morphism f is epi iff $\mathrm{coker}(f) = 0$.

Proof.

1. We first show that if $\ker(f) = 0$, then f is mono. Suppose $\ker(f) = 0$. By the universal property of kernels, we know that for any $Z \in \mathrm{Obj}(\mathsf{C})$ and any $g \colon Z \to A$ with $f \circ g = 0$ there exists a unique map $\bar{g} \colon Z \to \ker(f)$ such that $g = \iota \circ \bar{g}$.

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists! g"] \arrow[ddr, bend right, "g"'] & & \\ & \ker(f) = 0 \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

   But then g factors through the zero object, so we must have $g = 0$. This shows that $f \circ g = 0 \implies g = 0$, and by Lemma 271 f must be mono.

   Next, we show that if f is mono, then $\ker(f) = 0$. To do this, it suffices to show that $\ker(f)$ is final, i.e. that there exists a unique morphism from every object to $\ker(f)$.

   Since $\mathrm{Hom}_{\mathsf{C}}(Z, \ker(f))$ has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms h_1 and h_2.

$$\begin{tikzcd} Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] & & \\ & \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

Our aim is to show that $h_1 = h_2$. To this end, compose each with ι.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] \arrow[ddr, bend right, "\iota\circ h_1"] \arrow[ddr, bend right=50, "\iota\circ h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Since $f \circ \iota = 0$, we have $f \circ (\iota \circ h_1) = 0$ and $f \circ (\iota \circ h_2) = 0$. But since f is mono, by Lemma 271, we must have $\iota \circ h_1 = 0 = \iota \circ h_2$. But by Lemma 138, ι is mono, so again we have

$$h_1 = h_2 = 0,$$

and we are done.

2. Dual to the proof above.

□

Lemma 273. We have the following.

1. The kernel of the zero morphism $0 : A \to B$ is the pair (A, id_A).
2. The cokernel of the zero morphism $0 : A \to B$ is the pair (B, id_B).

Proof.

1. We need only verify that the universal property is satisfied. That is, for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $h\colon Z \to A$ such that $0 \circ g = 0$, there exists a unique morphism $\bar{g}\colon Z \to \ker(0)$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\bar{g}=g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(0) = A \arrow[r, "0"] \arrow[d, "\iota=\mathrm{id}_A"'] & 0 \arrow[d] \\
& A \arrow[r, "0"] & B
\end{tikzcd}$$

But this is pretty trivial: $\bar{g} = g$.

2. Dual to above.

□

Theorem 274. All abelian categories are binormal (Definition 150). That is to say:

1. all monomorphisms are kernels
2. all epimorphisms are cokernels.

Proof.

1. Consider the following diagram taken Lemma 260, which shows the canonical factorization of any morphism f.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \mathrm{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \mathrm{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

By definition of a pre-abelian category, we know that $\bar{f}$ is an isomorphism.

□

Note 275. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial. See e.g. [40] for details.

Definition 276 (subobject, quotient object, subquotient object)**.** Let $Y \in \mathrm{Obj}(\mathsf{C})$.

1. A subobject of Y is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a monomorphism $i\colon X \hookrightarrow Y$. If X is a subobject of Y we will write $X \subseteq Y$.
2. A quotient object of Y is an object Z together with an epimorphism $p\colon Y \twoheadrightarrow Z$.
3. A subquotient object of Y is a quotient object of a subobject of Y.

Definition 277 (quotient)**.** Let $X \subseteq Y$, i.e. let there exist a monomorphism $f\colon X \hookrightarrow Y$. The quotient Y/X is the cokernel $(\mathrm{coker}(f), \pi_f)$.

Example 278. Let V be a vector space, $W \subseteq V$ a subspace. Then we have the canonical inclusion map $\iota\colon W \hookrightarrow V$, so W is a subobject of V in the sense of Definition 276.

According to Definition 277, the quotient V/W is the cokernel $(\mathrm{coker}(\iota), \pi_\iota)$ of ι. We saw in Example 143 that the cokernel of ι was $V/\mathrm{im}(\iota)$. However, $\mathrm{im}(\iota)$ is exactly W! So the categorical notion of the quotient V/W agrees with the linear algebra notion.

Definition 279 (k-linear category)**.** An abelian category Definition 262 C is k-linear if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of a k-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is k-linear.

Example 280. The category Vect_k is k-linear.

Definition 281 (k-linear functor)**.** let C and D be two k-linear categories, and $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ a functor. Suppose that for all objects $C, D \in \mathrm{Obj}(\mathsf{C})$ all morphisms $f, g\colon C \to D$, and all α, $\beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{G}(g).$$

Then we say that $\mathcal{F}$ is k-linear.

10.4. Exact sequences

Definition 282 (exact sequence)**.** A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree i if the image (Definition 270) of f_{i-1} is equal to the kernel (Definition 135) of f_i. A sequence is exact if it is exact in every degree.

Lemma 283. If a sequence

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree i, then $f_i \circ f_{i-1} = 0$.

Definition 284 (short exact sequence)**.** A short exact sequence is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Definition 285 (exact functor)**.** Let C, D be abelian categories, $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ an additive functor (Definition 253). We say that $\mathcal{F}$ is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

10.5. Length of objects

Definition 286 (simple object)**.** A nonzero object $X \in \mathrm{Obj}(\mathsf{C})$ is called simple if 0 and X are its only subobjects.

Example 287. In Vect_k, the only simple object (up to isomorphism) is k, taken as a one-dimensional vector space over itself.

Definition 288 (semisimple object)**.** An object $Y \in \mathrm{Obj}(\mathsf{C})$ is semisimple if it is isomorphic to a dirct sum of simple objects.

Example 289. In Vect_k, all finite-dimensional vector spaces are semisimple.

Definition 290 (semisimple category)**.** An abelian category C is semisimple if every object of C is semisimple.

Example 291. The category $\mathsf{FinVect}_k$ is semisimple.

Definition 292 (Jordan-Hölder series)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of X such that X_i/X_{i-1} is simple for all i is called a Jordan Hölder series for X. The integer n is called the length of the series X_i.

The importance of Jordan-Hölder series is the following.

Theorem 293 (Jordan-Hölder)**.** Let X_i and Y_i be two Jordan-Hölder series for some object $X \in \mathrm{Obj}(\mathsf{C})$. Then the length of X_i is equal to the length of Y_i, and the objects Y_i/Y_{i-1} are a reordering of X_i/X_{i-1}.

Proof. See [22], pg. 5, Theorem 1.5.4. □

Definition 294 (length)**.** The length of an object X is defined to be the length of any of its Jordan-Hölder series. This is well-defined by Theorem 293.

11. Tensor Categories

The following definition is taken almost verbatim from [11].

Definition 295 (tensor category)**.** Let k be a field. A k-tensor category A (as considered by Deligne in [30]) is an

1. essentially small (Definition 55)
2. k-linear[1] (Definition 279)
3. rigid (Definition 245)
4. symmetric (Definition 224)
5. monoidal category (Definition 204)

such that

1. the tensor product functor $\otimes\colon \mathsf{A} \times \mathsf{A} \to \mathsf{A}$ is, in both arguments separately,
   a) k-linear (Definition 281)
   b) exact (Definition 285)
2. $\mathrm{End}(1) \simeq k$, where End denotes the endomorphism ring

Example 296. Vect_k is *not* a tensor category because it is not essentially small; there is one isomorphism class of vector spaces for each cardinal, and there is no set of all cardinals. However, its subcategory $\mathsf{FinVect}_k$ is a tensor category.

Definition 297 (finite tensor category)**.** A k-tensor category A is called finite (over k) if

1. There are only finitely many simple objects in A, and each of them admits a projective presentation.
2. Each object A of A is of finite length.
3. For any two objects A, B of A, the hom-object (i.e. k-vector space) $\mathrm{Hom}_{\mathsf{A}}(A, B)$ is finite-dimensional.

Example 298. The category $\mathsf{FinVect}_k$ is finite.

1. The only simple object is k taken as a one-dimensional vector space over itself.
2. The length of a finite-dimensional vector space is simply its dimension.
3. The vector space $\mathrm{Hom}_{\mathsf{FinVect}}(V, W)$ has dimension $\dim(V)\dim(W)$.

Definition 299 (finitely $\otimes$-generated)**.** A k-tensor category A is called finitely $\otimes$-generated if there exists an object $E \in \mathrm{Obj}(\mathsf{A})$ such that every other object $X \in A$ is a subquotient

[1] Hence abelian.

(Definition 276) of a finite direct sum of tensor products of E; that is to say, if there exists a finite collection of integers n_i such that X is a subquotient of $\bigoplus_i E^{\otimes n_i}$.

$$\begin{tikzcd} & \bigoplus_i E^{\otimes n_i} \arrow[d, two heads, "\pi"] \\ X \arrow[r, hook, "\iota"] & \left(\bigoplus_i E^{\otimes n_i}\right)/Q \end{tikzcd}$$

Example 300. The category $\mathsf{FinVect}_k$ is finitely generated since any finite-dimensional vector space is isomorphic to $k^n = k \oplus \cdots \oplus k$ for some n.

Definition 301 (subexponential growth)**.** A tensor category A has subexponential growth if, for each object X there exists a natural number N_X such that

$$\operatorname{len}(X^{\otimes_n}) \leq (N_X)^n.$$

Example 302. The category $\mathsf{FinVect}_k$ has subexponential growth. For any finite-dimensional vector space V, we always have

$$\dim(V^{\otimes^n}) = (\dim(V))^n,$$

so we can take $N_V = \dim(V)$.

Theorem 303. Let A be a tensor category, and suppose that

1. every object $A \in \operatorname{Obj}(\mathsf{A})$ has a finite length
2. the dimension of every hom space $\operatorname{Hom}_{\mathsf{A}}(A, B)$ is finite over k.

Then the category $\mathsf{Ind}(\mathsf{A})$ of ind-objects of A (Definition 159) has the following properties.

1. $\mathsf{Ind}(\mathsf{A})$ is abelian (Definition 262).
2. $A \hookrightarrow \mathsf{Ind}(\mathsf{A})$ is a full subcategory (cf. Note 160).
3. The tensor product on A extends to $\mathsf{Ind}(\mathsf{A})$ via

$$\begin{aligned} X \otimes Y &\simeq (\varinjlim_i X_i) \otimes (\varinjlim_j Y_j) \\ &\simeq \varinjlim_{i,j} (X_i \otimes Y_j). \end{aligned}$$

4. The category $\mathsf{Ind}(\mathsf{A})$ fails to be a tensor category only because it is not necessarily essentially small and rigid. More specifically, an object $A \in \mathsf{Ind}(\mathsf{A})$ is dualizable if and only if it is in A.

Proof. Proposition 3.38 in [11]. □

Definition 304 (tensor functor)**.** Let $(\mathscr{A}, \otimes_A, \mathrm{id}_A)$ and $(\mathscr{B}, \otimes_B, \mathrm{id}_B)$ be k-tensor categories. A functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ is called a tensor functor if it is

1. braided (Definition 222) and
2. strong monoidal (Definition 214).

12. Internalization

One of the reasons that category theory is so useful is that it makes generalizing concepts very easy. One of the most powerful ways of doing this is known as *internalization*.

12.1. Internal groups

Definition 305 (group object)**.** Let C be a category with binary products $\times$ and a terminal object $*$. A group object in C (or a *group internal to* C) is an object $G \in \mathrm{Obj}(\mathsf{C})$ together with

- a map $e\colon * \to G$, called the *unit map*;
- a map $(-)^{-1}\colon G \to G$, called the *inverse map*; and
- a map $m\colon G \times G \to G$, called the *multiplication map*

such that

- Multiplication is associative, i.e. the following diagram commutes.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times m"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- The unit picks out the 'identity element,' i.e. the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "e \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times e"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- The inverse map behaves as an inverse, i.e. the following diagram commutes.

$$\begin{tikzcd} & G \times G \arrow[r, "(-)^{-1} \times \mathrm{id}"] & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "\delta"] \arrow[dr, "\delta"'] \arrow[rr, "\exists!"] & & * \arrow[r, "e"] & G \\ & G \times G \arrow[r, "\mathrm{id} \times (-)^{-1}"'] & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Here, $\delta \colon G \to G \times G$ is the diagonal map defined uniquely by the universal property of the product

$$\begin{tikzcd}
 & G \arrow[dl, "\mathrm{id}_G"'] \arrow[d, "\delta"] \arrow[dr, "\mathrm{id}_G"] & \\
G & G \times G \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G
\end{tikzcd} .$$

It will be convenient to represent the above data in the following way.

$$\begin{tikzcd}
G \times G \arrow[d, "m"] \\
G \arrow[loop right, "i"] \\
* \arrow[u, "e"]
\end{tikzcd}$$

13. Enrichment

Definition 306 (enriched category)**.** Let $(\mathsf{M}, \otimes, 1_{\mathsf{M}}, \alpha, \lambda, \rho)$ be a monoidal category. A M-enriched category C consists of the following data.

- A set $\mathrm{Obj}(\mathsf{C})$ of objects.
- For each pair x, y of objects, an object $\mathsf{C}(x, y) \in \mathsf{M}$ of morphisms.
- For each object $x \in \mathsf{C}$, a morphism

  $$1_{\mathsf{M}} \to \mathsf{C}(x, x)$$

  called the *unit*
- For objects x, y, and z, a morphism

  $$\mathsf{C}(x, y) \otimes \mathsf{C}(y, z) \to \mathsf{C}(x, z)$$

  called the *composition.*

This composition must be unital and associative, i.e. the diagrams

$$\begin{tikzcd}
& \mathsf{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathsf{C}(y,y) \otimes \mathsf{C}(x,y) \arrow[d] \\
& \mathsf{C}(x,y) \otimes \mathsf{C}(x,x) \arrow[r] & \mathsf{C}(x,y)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{C}(x,y) \otimes \mathsf{C}(y,z) \otimes \mathsf{C}(z,w) \arrow[r] \arrow[d] & \mathsf{C}(x,y) \otimes \mathsf{C}(y,w) \arrow[d] \\
\mathsf{C}(x,z) \otimes \mathsf{C}(z,w) \arrow[r] & \mathsf{C}(x,w)
\end{tikzcd}$$

must commute (where unitors and associators are notationally supressed).

Lemma 307. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category, and let $F\colon \mathcal{M} \to \mathcal{N}$ be a monoidal functor with data

$$\Phi_{x,y}\colon F(x) \otimes_{\mathcal{N}} F(y) \to F(x \otimes_{\mathcal{M}} y), \qquad \phi\colon 1_{\mathcal{N}} \to F(1_{\mathcal{M}}).$$

This data allows us to build from $\mathcal{C}$ a category enriched over $\mathcal{N}$.

Proof.

- We apply F to each hom-object in $\mathcal{M}$ to get a new hom-object in $\mathcal{N}$;

  $$\mathcal{C}(x, y)_{\mathcal{N}} = F(\mathcal{C}(x, y)_{\mathcal{M}}).$$

- The unit is the composition

  $$1_{\mathcal{N}} \longrightarrow F(1_{\mathcal{M}}) \longrightarrow \mathcal{F}(\mathcal{C}(x,y)_{\mathcal{M}}) = \mathcal{C}(x,y)_{\mathcal{N}}$$

- The composition is given by the obvious.

□

14. Higher categories

14.1. 2-categories

Definition 308 (2-category)**.** A 2-category is a category enriched over categories. More explicitly, a 2-category $\mathbb{C}$ consists of the following.

- A set $\mathrm{Obj}(\mathbb{C})$ of objects.
- For every two objects $x, y \in \mathrm{Obj}(\mathbb{C})$, a category $\mathbb{C}(x, y)$ of morphisms.
- For every object $x \in \mathrm{Obj}(\mathbb{C})$, an object $\mathrm{id}_x \in \mathrm{Obj}(\mathbb{C}(x, y))$.
- For every three objects x, y, z, a functor

  $$\mu\colon \mathbb{C}(y, z) \times \mathbb{C}(x, y) \to \mathbb{C}(x, z)$$

  implementing composition.

  These must make the following diagrams commute.

Associativity:
$$\begin{tikzcd} \mathbb{C}(z, w) \times \mathbb{C}(y, z) \times \mathbb{C}(x, y) \arrow[r] \arrow[d] & \mathbb{C}(z, w) \times \mathbb{C}(x, z) \arrow[d] \\ \mathbb{C}(y, w) \times \mathbb{C}(x, y) \arrow[r] & \mathbb{C}(x, w) \end{tikzcd}$$

Unitality:
$$\begin{tikzcd} \mathbb{C}(x, y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathbb{C}(y, y) \times \mathbb{C}(x, y) \arrow[d] \\ \mathbb{C}(x, y) \times \mathbb{C}(x, x) \arrow[r] & \mathbb{C}(x, y) \end{tikzcd}$$

Example 309. The category Cat is a 2-category.

- The objects Obj(Cat) are given by small categories.
- The morphisms Cat(C, D) is the category [C, D].
- The identity morphism is the identity functor. We have already seen that this is a functor.
- The composition is given by composition of functors

These satisfy the conditions:

1. Indeed, the identity functor functions as a left and right identity, and whiskering by the identity is the identity.
2. Indeed, composition of functors is associative, and vertical composition is associative, although I don't know a nice way to prove this.

A. Foundational issues

A.1. Grothendieck universes

Note 310. I want to edit this section pretty seriously as I don't really understand universes at the moment. Furthermore, since most of these notes was written in the language of proper classes, there are probably still a few latent references to them. I'll remove these references as I find them.

Set theory as formulated in ZFC has foundational annoyances. One particularly pernicious one says that not every definable collection of objects is small enough to be a set. For example, there is no set of all sets, nor set of all groups, rings, etc.

Category theory is to a large extent built upon ZFC, and needs to talk about the collection of all sets, groups, rings, and many other structures. When doing category theory, there are three common ways to deal with collections which are too large to be sets.

1. One defines the notion of a *proper class,* which is simply another word for a definable collection too large to be a set, and works with proper classes in addition to sets.
2. Following Grothendieck, one can add an additional axiom, called the *universe axiom,* to ZFC which ensures the existence a set of all sets one is allowed to use. This set of all sets one is allowed to use is called a *Grothendieck universe.*
3. Introduce categories, rather than sets, as foundational objects.

The first approach is the most common, and the third the most radical. We will use the second here.

Definition 311 (Grothendieck universe)**.** A Grothendieck universe (or simply *universe*) is a set $\mathcal{U}$ with the following properties:

U1. If $x \in \mathcal{U}$ and $y \in x$, then $y \in \mathcal{U}$. That is, if if $x \in \mathcal{U}$, then x is a subset of $\mathcal{U}$.

U2. If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$

U3. If $x \in \mathcal{U}$, then 2^X is also in $\mathcal{U}$.

U4. If $I \in \mathcal{U}$ and $\{x_i\}_{i\in U}$ with $x_i \in \mathcal{U}$, then

$$\bigcup_{i\in I} x_i \in \mathcal{U}.$$

Definition 312 (universe axiom)**.** The universe axiom says that for every set X, there is a universe which has X as an element.

These axioms imply that universes are closed under many of the properties one would hope.[1] The upshot is that all familiar constructions from everyday life can be performed within any

[1] See the below lemma.

universe $\mathcal{U}$. From now on, we fix a universe $\mathcal{U}$ such that $\mathbb{N} \in \mathcal{U}$ (which is possible by the universe axiom).

Lemma 313. Grothendieck universes have the following properties.

1. $x \in \mathcal{U} \implies \{x\} \in \mathcal{U}$
2. $y \in \mathcal{U}, x \subseteq y \implies x \in \mathcal{U}$
3. $x, y \in \mathcal{U} \implies (x, y) \equiv \{x, \{x, y\}\} \in \mathcal{U}$
4. $x, y \in \mathcal{U} \implies x \cup y, x \times y \in \mathcal{U}$.
5. $x, y \in \mathcal{U} \implies \mathrm{Maps}(x, y) \in \mathcal{U}$
6. $I \in \mathcal{U}$, $\{x_i\}_{i \in J}$ with $x_i \in \mathcal{U} \implies \prod_{i \in J} x_i, \coprod_{i \in J} x_i, \bigcap_{i \in J} x_i \in \mathcal{U}$.
7. $x \in \mathcal{U} \implies x \cup \{x\} \in \mathcal{U}$, hence $\mathbb{N} \subset \mathcal{U}$.

Proof.

1. If x in $\mathcal{U}$, then by *U2* $\{x, x\} \in \mathcal{U}$. But $\{x, x\} = \{x\}$.
2. If $y \in \mathcal{U}$, then $2^y \in \mathcal{U}$ by *U3.* Any subset of y is an element of 2^y, so *U1* implies that any subset of y belongs to $\mathcal{U}$.
3. If x and $y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$ by *U1.* Then again by *U1,* $\{x, \{x, y\}\} \in \mathcal{U}$.
4. For any $a, b \in \mathcal{U}$, $a \neq b$, we have by *U2* that $\{a, b\} \in \mathcal{U}$. Thus, binary unions follow from *U4,* which demands the existence of unions indexed by any set in $\mathcal{U}$.

   From *U2.*, if x, $y \in \mathcal{U}$ then every element of x and every element of y is in U, so ordered pairs of an element of x and an element of y are in $\mathcal{U}$. Thus, the singletons each containing an ordered pair are in $\mathcal{U}$, so the union

   $$\bigcup_{\alpha \in x} \bigcup_{\beta \in y} \{(\alpha, \beta)\}$$

   is in $\mathcal{U}$.
5. The set of all maps $x \to y$ is a subset of the set $2^{x \times y}$.
6. As a set,

   $$\prod_{i \in I} x_i = \left\{ f : I \to \bigcup_{i \in I} x_i \;\middle|\; f(i) \in x_i \text{ for all } i \in I \right\} \subset \mathrm{Maps}\left(I, \bigcup_{i \in I} x_i\right)$$

   As a set,

   $$\coprod_{i \in I} x_i = \bigcup_{i \in I} x_i \times \{i\}.$$

   Intersection is subset of union.
7. If $x \in \mathcal{U}$ then $\{x\} \in \mathcal{U}$ by 1., so by 4., $x \cup \{x\} \in \mathcal{U}$.

□

Definition 314 (small set, class, large set)**.** We make the following definitions.

- We call the elements of $\mathcal{U}$ *small sets.*

- We call subsets of $\mathcal{U}$ *classes*.
- We call everything else *Large sets*.

In general, small sets agree with the usual (i.e. ZFC) notion of sets, and classes agree with the usual notion of proper classes. In what follows, we will write *set* when we mean *small set,* unless confusion is likely to arise. Unfortunately, especially when talking about foundational issues, confusion is likely to arise.

By the universe axiom, for any set (as defined in ZFC), there is a universe which has that set as an element. From now on, we will work in some universe $\mathcal{U}$ which has $\mathbb{N}$ as an element.

Bibliography

[1] M.-L. Michelson and H.B. Lawson. *Spin Geometry*. Princeton University Press, Princeton, NJ, 1989.

[2] R. Vakil, *The Rising Sea*. http://www.math216.wordpress.com/

[3] S. B. Sontz. *Principal Bundles: The Classical Case*. Springer International Publishing, Switzerland, 2015.

[4] T. Hungerford. *Algebra*. Springer-Verlag New York, NY, 1974.

[5] J. Figuroa-O'Farril. *PG Course on Spin Geometry*. https://empg.maths.ed.ac.uk/Activities/Spin/

[6] P. Deligne et al. *Quantum Fields and Strings: a Course for Mathematicians*. American Mathematical Society, 1999.

[7] V. S. Varadarajan. *Supersymmetry for Mathematicans: An Introduction*. American Mathematical Society, 2004.

[8] The Catsters. https://www.youtube.com/channel/UC5Y9H2KDRHZZTWZJtlH4VbA

[9] A. Connes and M. Marcolli. *Noncommutative Geometry, Quantum Fields and Motives*. http://www.alainconnes.org/docs/bookwebfinal.pdf

[10] P. Aluffi. *Algebra: Chapter 0*. American Mathematical Society, 2009.

[11] *Nlab—Deligne's Theorem on Tensor Categories*. https://ncatlab.org/nlab/show/Deligne's+theorem+on+tensor+categories

[12] J. Baez. *This Week's Finds in Mathematical Physics (Week 137)*. http://math.ucr.edu/home/baez/week137.html

[13] J. Baez. *Some Definitions Everyone Should Know*. http://math.ucr.edu/home/baez/qg-fall2004/definitions.pdf

[14] *The Unapologetic Mathematician: Mac Lane's Coherence Theorem*. https://unapologetic.wordpress.com/2007/06/29/mac-lanes-coherence-theorem/

[15] D. Montgomery and L. Zippin. *Topological Transformation Groups*. University of Chicago Press, Chicago, 1955.

[16] P. Deligne, J.S. Milne, A. Ogus, and K. Shih. *Hodge Cycles, Motives, and Shimura Varieties*. Springer Verlag, 1982

[17] S. Mac Lane. *Categories for the Working Mathematician*. Springer-Verlag New York, New York, 1998.

[18] J. Baez. *An introduction to n-Categories*. https://arxiv.org/pdf/q-alg/

9705009.pdf

[19] https://groups.google.com/forum/#!topic/sci.math/7LqPFfmWGOA

[20] https://www.ncatlab.org/

[21] *Wikipedia - Product (Category Theory).* https://en.wikipedia.org/wiki/Product_(category_theory)

[22] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories.* American Mathematical Society, 2015.

[23] S. Awodey and A. Bauer. *Lecture Notes: Introduction to Categorical Logic*

[24] S. Awodey. *Category Theory Foundations.* https://www.youtube.com/watch?v=BF6kHD1DAeU&index=1&list=PLGCr8P_YncjVjwAxrifKgcQYtbZ3zuPlb

[25] S. Awodey. *Category Theory.* Oxford University Press, 2006.

[26] R. Haag. *Local Quantum Physics: Fields, Particles, Algebras.* Springer-Verlag Berlin Heidelberg New York, 1996

[27] R. Sexl and H. Urbantke. *Relativity, Groups, Particles: Special Relativity and Relativistic Symmetry in Field and Particle Physics.* Springer-Verlag Wein, 1992

[28] J. Wess and J. Bagger. *Supersymmetry and Supergravity.* Princeton University Press, Princeton NJ, 1992

[29] H J W Müller-Kirsen and Armin Wiedemann *Introduction to Supersymmetry (Second edition).* World Scientific, 2010.

[30] Pierre Deligne. *Catégories Tensorielle*, Moscow Math. Journal 2 (2002) no. 2, 227-228 https://www.math.ias.edu/files/deligne/Tensorielles.pdf

[31] I. Kolář, P. Michor, and J. Slovák. *Natural Operations in Differential Geometry.* Springer-Verlag, Berlin Heidelberg, 1993.

[32] R. Hartshorne. *Algebraic Geometry.* Springer-Verlag, New York, 1977.

[33] Q. Yuan. *Annoying Precision: A neditation on semiadditive categories.* https://qchu.wordpress.com/2012/09/14/a-meditation-on-semiadditive-categories/

[34] J. Nestruev. *Smooth Monifolds and Observables.* Springer-Verlag, New York, 2002.

[35] J. Baez and A. Lauda. *A prehistory of n-categorical physics.* https://arxiv.org/pdf/0908.2469.pdf

[36] A. Neumaier. *Elementary particles as irreducible representations.* https://www.physicsoverflow.org/21960.

[37] U. Schreiber. *Why Supersymmetry? Because of Deligne's theorem* https://www.physicsforums.com/insights/supersymmetry-delignes-theorem/

[38] G. W. Mackey. *Induced representations.* W.A. Benjamin, Inc., and Editore Boringhieri, New York, 1968.

[39] nLab: Additive category. https://ncatlab.org/nlab/show/additive+category#ProductsAreBiproducts

[40] P. Freyd. *Abelian Categories* Harper and Row, New York, 1964.

[41] J. Milne, *Basic Theory of Affine Group Schemes*, 2012. Available at www.jmilne.org/math/

[42] E. Wigner. *On Unitary Representations of the Inhomogeneous Lorentz Group.* Annals of Mathematics. Second Series, Vol. 40, No. 1 (Jan., 1939), pp. 149-204

[43] J. P. M. dos Santos. *Representation theory of symmetric groups.* https://www.math.tecnico.ulisboa.pt/~ggranja/joaopedro.pdf

[44] U. Manin. *Gauge Fields and Complex Geometry*, Springer-Verlag Berlin Heidelberg New York, 1988.

[45] Valter Moretti (https://physics.stackexchange.com/users/35354/valter-moretti). *Euler-Lagrange equations and friction forces*, URL (version: 2014-02-02): https://physics.stackexchange.com/q/96470

[46] J.S. Milne. *Lie Algebras, Algebraic Groups, and Lie Groups.* Available at www.jmilne.org/math/

[47] J. Fröhlich and F. Gabbiani. *Braid statistics in Local Quantum Theory.* Rev. Math. Phys. 02, 251 (1990).

[48] J. Binney and D. Skinner. *The Physics of Quantum Mechanics.* Available at https://www-thphys.physics.ox.ac.uk/people/JamesBinney/qb.pdf

[49] R. Feynman. *Spacetime Approach to Non-Relativistic Quantum Mechanics.* Rev. Mod. Phys. 20, 367 – Published 1 April 1948

[50] F. A. Berezin. *The Method of Second Quantization.* Nauka, Moscow, 1965. Tranlation: Academic Press, New York, 1966. (Second edition, expanded: M. K. Polivanov, ed., Nauka, Moscow, 1986.)

[51] S. Coleman and J. Mandula. *All Possible Symmetries of the S Matrix.* Physical Review, 159(5), 1967, pp. 1251–1256.

[52] R. Haag, M. Sohnius, and J. T. Łopuszański. *All possible generators of supersymmetries of the S-matrix.* Nuclear Physics B, 88: 257–274 (1975)