Category theory notes

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1 Basic category theory

1.1 Categories

There are lots of different sorts of mathematical structures, each of which is good for different things. Groups are good for talking about symmetries, for instance, and rings are good for talking about function fields.

Categories are just another type of mathematical structure, but they have a different, more abstract feel to them than groups or rings. One reason for this is that they are inherently meta-mathematical: they are good for talking about other mathematical objects, and the relationships between them.

For this reason, category theory has developed a reputation for being fairly abstract and difficult to learn, which I feel is undeserved. Categories are nothing special; they are mathematical objects, just like groups or vector spaces. They are simply good for different things.

1.1.1 Basic definitions

Unfortunately, there are many different conventions regarding how to typeset category theory. I will do my best to stick to the conventions used at the nLab ([20]).

Definition 1 (category)**.** A category C consists of the following pieces of data.

- A collection[1] $\mathrm{Obj}(\mathsf{C})$ of *objects*.
- For every two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, a collection $\mathrm{Hom}(A, B)$ of *morphisms* with the following properties.
  1. For $f \in \mathrm{Hom}(A, B)$ and $g \in \mathrm{Hom}(B, C)$, there is an associated morphism
  $$g \circ f \in \mathrm{Hom}(A, C),$$
  called the *composition* of f and g.
  2. This composition is associative: $(f \circ g) \circ h = f \circ (g \circ h)$.
  3. For every $A \in \mathrm{Obj}(\mathsf{C})$, there is at least one morphism 1_A, called the *identity morphism* which functions as both a left and right identity with respect to the composition of morphisms, i.e.
  $$f \circ 1_A = f \qquad \text{amd} \qquad 1_A \circ g = g.$$

[1] There is a reason we say that the objects and morphisms of a category are a *collection* rather than a set. It may be that there may be 'too many' objects to be contained in a set. For example, there is no set of all sets, but we will see that there is a category of sets. We will for the most part sidestep foundational questions of size, but in some cases it will be unavoidable. In particular, categories whose objects and/or morphisms *are* small enough to be contained in a set will play an especially important role.

If ever it is potentially unclear which category we are talking about, we will add a subscript to Hom, writing for example $\mathrm{Hom}_{\mathsf{C}}(A, B)$ instead of $\mathrm{Hom}(A, B)$.

Categories can be visualized as graphs, with the objects as vertices and the morphisms as edges. In fact, every category gives rise to a so-called *multidigraph,* i.e. a directed graph which can have more than one edge between any given pair of vertices in this way.

Notation 2. Following Aluffi ([10]), we will use a sans serif font to denote categories. For example C, Set.

Notation 3. The identity morphism $1_A \colon A \to A$ is often simply denoted by A. This is actually a good notation, and we will use it freely in later chapters. However, we will avoid it in earlier chapters since it is potentially confusing whether we are talking about A the object or A the morphism.

Example 4. The prototypical category is Set, the category whose objects are sets and whose morphisms are set functions.

In order to check that Set is indeed a category, we need to check that the axioms are satisfied. In fact, everything is more or less clear by definition. Sets do indeed form a collection of objects, and between every two sets there is a collection of functions. Functions can indeed be composed, and this composititon is associative. Furthermore, every set has an identity which maps every element to itself.

Example 5 (category with one object)**.** The category $\mathsf{1}$, where $\mathrm{Obj}(\mathsf{1})$ is the singleton $\{*\}$, and the only morphism is the identity morphism $\mathrm{id}_* \colon * \to *$.

Categories capture the essence of 'things and composable maps between them.' There are many mathematical structures which naturally come with a notion of map between them, usually called homomorphisms. These are almost always composable, which means it is often natural to imagine them as living in their own categories.

Example 6. If you know what the following mathematical objects are, it is nearly effortless to check that they are categories.[2]

- Grp, whose objects are groups and whose morphisms are group homomorphisms.
- Ab, whose objects are abelian groups and whose morphisms are group homomorphisms.
- Ring, whose objects are rings and whose morphisms are ring homomorphisms.
- R-Mod, whose objects are modules over a ring R and whose morphisms are module homomorphisms.
- Vect_k, whose objects are vector spaces over a field k and whose morphisms are linear maps.
- $\mathsf{FinVect}_k$, whose objects are finite-dimensional vector spaces over a field k and whose morphisms are linear maps.
- k-Alg, whose objects are algebras over a field k and whose morphisms are algebra homomorphisms.

Example 7. In addition to algebraic structures, categories help to talk about geometrical structures. The following are also categories.

[2]Look at the definition. It *really is* trivial.

- Top, whose objects are topological spaces and whose morphisms are continuous maps.
- Met, whose objects are metric spaces and whose morphisms are metric maps.
- Man^p, whose objects are manifolds of class C^p and whose morphisms are p-times differentiable functions.
- SmoothMfd, whose objects are C^∞ manifolds and whose morphisms are smooth functions.

Example 8. Here is a slightly whimsical example of a category, which is different in nature to the other categories we've looked at before. This category is important when studying group representations.

Let G be a group. We are going to create a category G which behaves like this group.

- Our category G has only one object, called $*$.
- The set $\mathrm{Hom}_{\mathsf{G}}(*, *)$ is equal to the underlying set of the group G, and for $f, g \in \mathrm{Hom}_{\mathsf{G}}(*, *)$, the compositions $f \circ g = f \cdot g$, where $\cdot$ is the group operation in G. The identity $e \in G$ is the identity morphism 1_* on $*$.

$$\begin{tikzcd} * \arrow[loop above, "g"] \arrow[loop left, "f"] \arrow[loop right, "e=1_*"] \arrow[loop below, "g\circ f"] \end{tikzcd}$$

1.1.2 Building new categories from existing ones

One of the first things one learns about when talking about a mathematical structure is how to use existing ones to create new ones. For example, one can take the product of two groups to create a third, or take the quotient a vector space by a subspace. Categories are no exception. In this section, we will learn how to create new categories out of old categories.

The first way we'll see of doing this may seem a bit trivial.

Definition 9 (opposite category)**.** Let C be a category. Its <u>opposite category</u> C^{op} is the category whose objects are the same as the objects $\mathrm{Obj}(\mathsf{C})$ and whose morphisms $f \in \mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(A, B)$ are defined to be the morphisms $\mathrm{Hom}_{\mathsf{C}}(B, A)$.

That is to say, the opposite category is the category one gets by formally reversing all the arrows in a category. If $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, i.e. $f : A \to B$, then in C^{op}, $f : B \to A$.

This may seem like an uninterersting, definition, but having it around will make life a lot easier when it comes to defining functors (Section 1.2).

One can take the Cartesian product of two sets by creating ordered pairs. One can also take the product of two categories, which works in the way one would likely expect.

Definition 10 (product category)**.** Let C and D be categories. The <u>product category</u> $\mathsf{C} \times \mathsf{D}$ is the following category.

- The objects $\mathrm{Obj}(\mathsf{C} \times \mathsf{D})$ consist of all ordered pairs (C, D), where $C \in \mathrm{Obj}(\mathsf{C})$ and $D \in \mathrm{Obj}(\mathsf{D})$.

- For any two objects (C, D), and $(C', D') \in \mathrm{Obj}(\mathsf{C}\times\mathsf{D})$, the morphisms $\mathrm{Hom}_{C\times D}((C, D), (C', D'))$ are ordered pairs (f, g), where $f \in \mathrm{Hom}_{\mathsf{C}}(C, C')$ and $g \in \mathrm{Hom}_{\mathsf{D}}(D, D')$.
- Composition is taken componentwise, so that

$$(f_1, g_1) \circ (f_2, g_2) = (f_1 \circ g_1, f_2 \circ g_2).$$

- The identity morphisms are given in the obvious way:

$$1_{(C,D)} = (1_C, 1_D).$$

Many mathematical objects have a notion of 'sub-object.' For example, groups can have subgroups, and vector spaces can have vector subspaces. Categories also have this notion. As with groups and vector spaces, the definition of a 'subcategory' is more or less obvious, but it is helpful to know exactly what one has to check to show that a subcollection of objects and morphisms from a category forms a subcategory.

Definition 11 (subcategory)**.** Let C be a category. A category S is a subcategory of C if the following conditions hold.

- The objects Obj(S) of S are a subcollection of the objects of C.
- For S, $T \in \mathrm{Obj}(\mathsf{S})$, the morphisms $\mathrm{Hom}_{\mathsf{S}}(S, T)$ are a subcollection of the morphisms $\mathrm{Hom}_{\mathsf{C}}(S, T)$ which satisfy the following.
    - For every $S \in \mathrm{Obj}(S)$, the identity $1_S \in \mathrm{Hom}_{\mathsf{S}}(S, S)$.
    - For all $f \in \mathrm{Hom}_{\mathsf{S}}(S, T)$ and $g \in \mathrm{Hom}_{\mathsf{S}}(T, U)$, the composite $g \circ f \in \mathrm{Hom}_{\mathsf{S}}(S, U)$.

If S is a subcategory of C, we will write S ⊆ C.

One particularly important kind of subcategory occurs when one takes a subset of all objects in a subcategory, but keeps all morphisms between them. This is known as a *full subcategory.*

Definition 12 (full subcategory)**.** Let C be a category, S ⊆ C a subcategory. We say that S is full in C if for every S, $T \in \mathrm{Obj}(\mathsf{S})$, $\mathrm{Hom}_{\mathsf{S}}(S, T) = \mathrm{Hom}_{\mathsf{C}}(S, T)$. That is, if there are no morphisms in $\mathrm{Hom}_{\mathsf{C}}(S, T)$ that are not in $\mathrm{Hom}_{\mathsf{S}}(S, T)$.

Example 13. Recall that Vect_k is the category of vector spaces over a field k, and $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces.

It is not difficult to see that $\mathsf{FinVect}_k \subset \mathrm{Vect}_k$: all finite dimensional vector spaces are vector spaces, and all linear maps between finite-dimensional vector spaces are maps between vector spaces. In fact, since for V and W finite-dimensional, one does not gain any maps by moving from $\mathrm{Hom}_{\mathsf{FinVect}_k}(V, W)$ to $\mathrm{Hom}_{\mathsf{Vect}_k}(V, W)$, $\mathsf{FinVect}_k$ is even a *full* subcategory of Vect_k.

1.1.3 Properties of morphisms

Category theory has many essences, one of which is as a major generalization of set theory. It is often possible to upgrade statements about functions between sets to statements about morphisms between objects in an arbitrary category. However, this is not as simple as it may sound: nowhere in the axioms in the definition of a category does it say that the objects of a category have to *be* sets, so we cannot talk about their elements.

This puts us in a rather odd position. We have to generalize definitions from set theory, which are almost always given in terms of elements because elements are the only structure sets have, without talk without ever mentioning the elements of the objects we're talking about. We therefore have to find definitions which we can give purely in terms objects and morphisms between them.

The first thing we would like to define is when two objects are isomorphic.

Definition 14 (isomorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f \in \mathrm{Hom}(A, B)$ is said to be an <u>isomorphism</u> if there exists a morphism $g \in \mathrm{Hom}(B, A)$ such that

$$g \circ f = 1_A, \qquad \text{and} \qquad f \circ g = 1_B.$$

$$\begin{tikzcd} A \arrow[loop left, "1_A"] \arrow[r, shift left, "f"] & B \arrow[l, shift left, "g"] \arrow[loop right, "1_B"] \end{tikzcd}$$

If we have an isomorphism $f : A \to B$, we say that A and B are <u>isomorphic</u>, and write $A \simeq B$. This is not an abuse of notation, as the next proposition shows.

Proposition 15. *Isomorphism is an equivalence relation.*

Next we have two rather odd-looking properties that a morphism can have.

Definition 16 (monomorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f : A \to B$ is said to be a <u>monomorphism</u> (or simply *mono*) if for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all $g_1, g_2 : Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$.

$$\begin{tikzcd} Z \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

Note 17. When we wish to notationally distinguish monomorphisms, we will denote them by hooked arrows: if $f : A \to B$ is mono, we will write

$$A \overset{f}{\hookrightarrow} B \; .$$

Theorem 18. *In* Set, *a morphism is a monomorphism if and only if it is injective.*

Proof. Suppose $f : A \to B$ is a monomorphism. Then for any set Z and any maps $g_1, g_2 : Z \to A$, $f \circ g_1 = f \circ g_2$ implies $g_1 = g_2$. In particular, take Z to be the singleton $Z = \{*\}$ and call $g_1(*) = a_1$ and $g_2(*) = a_2$. Then $(f \circ g_1)(*) = f(a_1)$ and $(f \circ g_2)(*) = f(a_2)$, so

$$f(a_1) = f(a_2) \implies a_1 = a_2.$$

But this is exactly the definition of injectivity.

Now suppose that f is injective. Then for any Z and g_1, g_2 as above,

$$(f \circ g_1)(z) = (f \circ g_2)(z) \implies g_1(z) = g_2(z) \qquad \text{for all } z \in Z.$$

But this means that $g_1 = g_2$, so f is mono. □

Example 19. In Vect_k, a morphism is mono if and only if it is injective.

Definition 20 (epimorphism)**.** Let C be a category, $A, B \in \mathrm{Obj}(\mathsf{C})$. A morphism $f : A \to B$ is said to be a epimorphism if for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all $g_1, g_2 : B \to Z$, $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$.

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, shift left, "g_1"] \arrow[r, shift right, "g_2"'] & Z \end{tikzcd} .$$

Notation 21. We will denote epimorphisms by two-headed arrows. That is, if $f : A \to B$ is epi, we will write

$$A \overset{f}{\twoheadrightarrow} B$$

Theorem 22. *In* Set*, a morphism f is an epimorphism if and only if it is a surjection.*

Proof. Suppose $f : A \to B$ is an epimorphism. Then for any set Z and maps $g_1, g_2 : B \to Z$, there $g_1 \circ f = g_2 \circ f$ implies $g_1 = g_2$. In particular, this is true if $Z = \{0, 1\}$, and g_1 and g_2 are as follows.

$$g_1 : b \mapsto 0 \quad \text{for all } b; \qquad g_2 : b \mapsto \begin{cases} 0, & b \in \mathrm{im}(f) \\ 1, & b \notin \mathrm{im}(f). \end{cases}$$

But then $g_1 \circ f = g_2 \circ f$, so $g_1 = g_2$ since f is epi. Hence, $\mathrm{im}(f) = B$, so f is surjective. □

Example 23. In Vect_k, epimorphisms are surjective linear maps.

Note 24. In Set, we have the following correspondences.

- Isomorphism $\Longleftrightarrow$ Bijective
- Bijective $\Longleftrightarrow$ Injective & Surjective
- Injective & Surjective $\Longleftrightarrow$ Mono & Epi.

Thus in Set a morphism is an isomorphism if and only if it is a monomorphism and an epimorphism. This is *not* true in general: a morphism can be monic and epic without being an isomorphism.

Take, for example, the category Top, whose objects are topological spaces and whose morphisms are continuous maps. In order for a morphism f to be monic and epic, it is necessary only that it be injective and surjective, i.e. it must have a set-theoretic inverse. However its inverse does not have to be continuous, and therefore may not be a morphism in Top.

Set theory has some foundational annoyances: not every collection of sets is small enough to be a set, for example. Category theory has its own foundational issues, which for the most part we will avoid. However, there are a few important situations in which foundational questions of size play an unavoidably important role.

Definition 25 (small, locally small, hom-set)**.** A category C can have the following properties.

- We say that C is small if Obj(C) is a set and for all objects $A, B \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is a set.
- We say that C*is*locally small if for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is a set. In this case, we call $\mathrm{Hom}_{\mathsf{C}}(A, B)$ the hom-set. (Actually, terminology is often abused, and $\mathrm{Hom}_{\mathsf{C}}(A, B)$ is called a hom-set even if it is not a set.)

1.2 Functors

Just as morphisms connect objects in the same category, functors allow us to connect different categories. In fact, functors can be viewed as morphisms in a 'category of categories,' whose objects are categories!

Definition 26 (functor)**.** Let C and D be categories. A functor $\mathcal{F}$ from C to D is the following.

- It maps each object $X \in \mathrm{Obj}(\mathsf{C})$ an object $\mathcal{F}(X) \in \mathrm{Obj}(\mathsf{D})$.
- It assigns each morphism $f \in \mathrm{Hom}(X, Y)$ to a morphism $\mathcal{F}(f)$ such that $\mathcal{F}(1_X) = 1_{\mathcal{F}(X)}$ for all X, and one of the two following properties are satisfied.
  - The morphism $f \in \mathrm{Hom}(\mathcal{F}(X), \mathcal{F}(Y))$ and if $f : X \to Y$ and $g\colon Y \to Z$, then
  $$\mathcal{F}(g \circ f) = \mathcal{F}(g) \circ \mathcal{F}(f).$$
  In this case we say that $\mathcal{F}$ is covariant.
  - The morphism $f \in \mathrm{Hom}(\mathcal{F}(Y), \mathcal{F}(X))$, and if $f\colon X \to Y$ and $g\colon Y \to Z$, then
  $$\mathcal{F}(g \circ f) = \mathcal{F}(f) \circ \mathcal{F}(g).$$
  In this case, we say that $\mathcal{F}$ is contravariant.

Notation 27. We will typeset functors with calligraphic letters using the font `eucal`, and notate them with squiggly arrows. For example, if C and D are categories and $\mathcal{F}$ is a functor from C to D, then we would write
$$\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}.$$

Note 28. One can also define a contravariant functor $\mathsf{C} \rightsquigarrow \mathsf{D}$ as a covariant functor $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{D}$. This is actually much more convenient than having two different types of functors lying around, so this is what we will do. Thus, a *functor* will always be a *covariant functor.*

Example 29. Let $\mathsf{1}$ be the category with one object $*$ (Example 5). Let C be any category. Then for each $X \in \mathrm{Obj}(\mathsf{C})$, we have the functor
$$\mathcal{F}_X\colon \mathsf{1} \rightsquigarrow \mathsf{C}; \qquad \mathcal{F}(*) = X, \quad \mathcal{F}(\mathrm{id}_*) = \mathrm{id}_X.$$

Example 30. Recall that Grp is the category of groups. Denote by CRing the category of commutative rings.

The following are functors $\mathsf{CRing} \rightsquigarrow \mathsf{Grp}$.

- GL_n, which assigns to each commutative ring K the group of all $n \times n$ invertible matrices with entries in K.
- $(\,\cdot\,)^*$, which maps each commutative ring K to its group of units K^*, and each morphism $K \to K'$ to its restriction to K^*.

One of the nice things about functors is that they map commutative diagrams to commutative diagrams. For example, if $\mathcal{F}$ is a contraviariant functor, then one might see the following.

$$\begin{tikzcd}
A \arrow[rr, "h=g\circ f"] \arrow[dr, "f"'] & & B \\
& C \arrow[ur, "g"'] &
\end{tikzcd}
\quad \overset{\mathcal{F}}{\rightsquigarrow} \quad
\begin{tikzcd}
\mathcal{F}(A) & & \mathcal{F}(B) \arrow[ll, "\mathcal{F}(h)=\mathcal{F}(f)\circ\mathcal{F}(g)"'] \arrow[dl, "\mathcal{F}(g)"] \\
& \mathcal{F}(C) \arrow[ul, "\mathcal{F}(f)"] &
\end{tikzcd}$$

Definition 31 (full, faithful)**.** Let C and D be locally small categories (Definition 25), and let $\mathcal{F} : \mathsf{C} \rightsquigarrow \mathsf{D}$. Then $\mathcal{F}$ induces a family of set-functions

$$\mathcal{F}_{X,Y} \colon \mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y)).$$

We say that $\mathcal{F}$ is

- full if $\mathcal{F}_{X,Y}$ is surjective for all $X, Y \in \mathrm{Obj}(\mathsf{C})$
- faithful if $\mathcal{F}_{X,Y}$ is injective for all $X, Y \in \mathrm{Obj}(\mathsf{C})$,
- fully faithful if $\mathcal{F}$ is full and faithful.

Note 32. Fullness and faithfulness are *not* the functorial analogs of surjectivity and injectivity. A functor between small categories can be full (faithful) without being surjective (injective) on objects. Instead, we have the following result.

Lemma 33. *A fully faithful functor is injective on objects up to isomorphism. That is, if $\mathcal{F} \colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is a fully faithful functor and $\mathcal{F}(A) \simeq \mathcal{F}(B)$, then $A \simeq B$.*

Proof. Let $\mathcal{F} \colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a fully faithful functor, and suppose that $\mathcal{F}(A) \simeq \mathcal{F}(B)$. Then there exist $f' \colon \mathcal{F}(A) \to \mathcal{F}(B)$ and $g' \colon \mathcal{F}(B) \to \mathcal{F}(A)$ such that $f' \circ g' = 1_{\mathcal{F}(B)}$ and $g' \circ f' = 1_{\mathcal{F}(A)}$. Because the function $\mathcal{F}_{A,B}$ is bijective it is invertible, so there is a unique morphism $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ such that $\mathcal{F}(f) = f'$, and similarly there is a unique $g \in \mathrm{Hom}_{\mathsf{C}}(B, A)$ such that $\mathcal{F}(g) = g'$.

Now,

$$1_{\mathcal{F}(A)} = g' \circ f' = \mathcal{F}(g) \circ \mathcal{F}(f) = \mathcal{F}(g \circ f),$$

and since $\mathcal{F}$ is injective, we must have $g \circ f = 1_A$. Identical logic shows that we must also have $f \circ g = 1_B$. Thus $A \simeq B$. □

Definition 34 (essentially surjective)**.** A functor $\mathcal{F} \colon \mathsf{C} \rightsquigarrow \mathsf{D}$ is essentially surjective if for every $A' \in \mathrm{Obj}(\mathsf{D})$, there exists $A \in \mathrm{Obj}(\mathsf{C})$ such that $A' \simeq \mathcal{F}(A)$.

Example 35 (diagonal functor)**.** Let C be a category, $\mathsf{C} \times \mathsf{C}$ the product category of C with itself. The diagonal functor $\Delta \colon \mathsf{C} \rightsquigarrow \mathsf{C} \times \mathsf{C}$ is the functor which sends

- each object $A \in \mathrm{Obj}(\mathsf{C})$ to the pair $(A, A) \in \mathrm{Obj}(\mathsf{C} \times \mathsf{C})$, and
- each morphism $f \colon A \to B$ to the ordered pair

$$(f, f) \in \mathrm{Hom}_{\mathsf{C}\times\mathsf{C}}(A \times B, A \times B).$$

Definition 36 (bifunctor)**.** A bifunctor is a functor whose domain is a product category (Definition 10).

Example 37. Recall Example 8. Let G be a group, and G the category which mimics it.

Then functors $\rho \colon \mathsf{G} \rightsquigarrow \mathsf{Vect}_k$ are k-linear representations of G!

To see this, let us unwrap the definition. The functor ρ assigns to $* \in \mathrm{Obj}(\mathsf{G})$ an object $\rho(*) = V \in \mathrm{Obj}(\mathsf{Vect})$, and to each morphism $g \colon * \to *$ a morphism $\rho(g) \colon V \to V$. This assignment sends units to units, and this composition is associative.

1.3 Natural transformations

Saunders Mac Lane, one of the fathers of category theory, used to say that he invented categories so he could talk about functors, and he invented functors so he could talk about natural transformations. Indeed, the first paper ever published on category theory, published by Eilenberg and Mac Lane in 1945, was titled "General Theory of Natural Equivalences." [23]

Natural transformations provide a notion of 'morphism between functors.' They allow us much greater freedom in talking about the relationship between two functors than equality.

Here is an example which motivates the definition of a natural transformation. Let A and B be sets. There is an isomorphism from $A \times B$ to $B \times A$ which is given by switching the order of the ordered pairs:

$$\mathrm{swap}_{A,B} \colon (a, b) \mapsto (b, a).$$

Similarly, for sets C and D, there is an isomorphism $C \times D$ to $D \times C$ which is given by switching the order of ordered pairs.

$$\mathrm{swap}_{C,D} \colon (c, d) \mapsto (d, c).$$

In some obvious intuitive sense, these isomorphisms are really the same isomorphism. However, it is not immediately obvious how to formalize this. They are not equal as functions, for instance, since the definition of a function contains the information about the image and coimage, so they cannot be equal as functions.

Here is how it is done. Notice that for *any* functions $f \colon A \to C$ and $g \colon B \to D$, the following diagram commutes.

$$\begin{tikzcd} A \times B \arrow[r, "{(f,g)}"] \arrow[d, "\mathrm{swap}_{A,B}"'] & C \times D \arrow[d, "\mathrm{swap}_{C,D}"] \\ B \times A \arrow[r, "{(g,f)}"] & D \times C \end{tikzcd}$$

That is to say, if you're trying to go from $A \times B$ to $D \times C$ and all you've got is functions $f \colon A \to B$ and $g \colon C \to D$ and the two swap isomorphisms, it doesn't matter whether you use the swap isomorphism on $A \times B$ and then use (g, f) to get to $D \times C$, or immediately go to $C \times D$ with (f, g), then use the swap isomorphism there: the result is the same. Furthermore, this is true for *any* functions f and g.

As we will see, the cartesian product of sets is a functor from the product category (Definition 10) $\mathsf{Set} \times \mathsf{Set}$ to Set which maps (A, B) to $A \times B$. There is another functor $\mathsf{Set} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}$ which sends (A, B) to $B \times A$. The above means that there is a natural isomorphism, called swap, between them.

This example makes clear another common theme of natural transformations: they formalize the idea of 'the same function' between different objects. They allow us to make precise statements like '$\mathrm{swap}_{A,B}$ and $\mathrm{swap}_{C,D}$ are somehow the same, despite the fact that they are in no way equal as functions.'

Definition 38 (natural transformation)**.** let C and D be categories, and let $\mathcal{F}$ and $\mathcal{G}$ be covariant functors from C to D. A natural transformation η between $\mathcal{F}$ and $\mathcal{G}$ consists of

- for each object $A \in \mathrm{Obj}(\mathsf{C})$ a morphism $\eta_A \colon \mathcal{F}(A) \to \mathcal{G}(A)$, such that

- for all $A, B \in \mathrm{Obj}(C)$, for each morphism $f \in \mathrm{Hom}(A, B)$, the diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"'] & \mathcal{G}(B)
\end{tikzcd}$$

  commutes.

If the functors $\mathcal{F}$ and $\mathcal{G}$ are contravariant, the changes needed to make the definition make sense are obvious: basically, some arrows need to be reversed. However, this is one of those times where it's easier, in line with Note 28, to just pretend that we have a covariant functor from the opposite category.

Definition 39 (natural isomorphism)**.** A natural isomorphism $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation such that each η_A is an isomorphism.

Notation 40. We will use double-shafted arrows to denote natural transformations: if $\mathcal{F}$ and $\mathcal{G}$ are functors and η is a natural transformation from $\mathcal{F}$ to $\mathcal{G}$, we will write

$$\eta\colon \mathcal{F} \Rightarrow \mathcal{G}.$$

Example 41. Recall the functors GL_n and $(\cdot)^*$ from Example 30.

Denote by $\det_K M$ the determinant of a matrix M with its entries in a commutative ring K. Then the determinant is a map

$$\det_K\colon \mathrm{GL}_n(K) \to K^*.$$

Because the determinant is defined by the same formula for each K, the action of f commutes with $\det_K$: it doesn't matter whether we map the entries of M with f first and then take the determinant, or take the determinant first and then feed the result to f.

That is to say, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{GL}_n(K) \arrow[r, "\det_K"] \arrow[d, "\mathrm{GL}_n(f)"'] & K^* \arrow[d, "f^*"] \\
\mathrm{GL}_n(K') \arrow[r, "\det_{K'}"] & K'^*
\end{tikzcd}$$

But this means that det is a natural transformation $\mathrm{GL}_n \Rightarrow (\,\cdot\,)^*$.

We can compose natural transformations in the obvious way.

Lemma 42. *Let* C *and* D *be categories,* $\mathcal{F}, \mathcal{G}, \mathcal{H}$ *be functors, and* Φ *and* Ψ *be natural transformations as follows.*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left=60, "\mathcal{F}"{name=F}] \arrow[r, squiggly, "\mathcal{G}"{name=G, description}] \arrow[r, squiggly, bend right=60, "\mathcal{H}"'{name=H}] & \mathsf{D}
\arrow[Rightarrow, from=F, to=G, "\Phi"]
\arrow[Rightarrow, from=G, to=H, "\Psi"]
\end{tikzcd}$$

This induces a natural transformation $\mathcal{F} \Rightarrow \mathcal{H}$.

Proof. For each object $A \in \mathrm{Obj}(\mathsf{C})$, the composition $\Psi_A \circ \Phi_A$ exists and maps $\mathcal{F}(A) \to \mathcal{H}(A)$. Let's write

$$\Psi_A \circ \Phi_A = (\Psi \circ \Phi)_A.$$

We have to show that these are the components of a natural transformation, i.e. that they make the following diagram commute for all $A, B \in \mathrm{Obj}(\mathsf{C})$, all $f : A \to B$.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "(\Psi\circ\Phi)_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "(\Psi\circ\Phi)_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

We can do this by adding a middle row.

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(\mathcal{B}) \arrow[d, "\Phi_B"] \\
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\
\mathcal{H}(A) \arrow[r, "\mathcal{H}(f)"] & \mathcal{H}(B)
\end{tikzcd}$$

The top and bottom squares are the naturality squares for Φ and Ψ respectively. The outside square is the one we want to commute, and it manifestly does because each of the inside squares does. $\square$

Definition 43 (vertical composition)**.** The above composition $\Psi \circ \Phi$ is called vertical composition.

Definition 44 (functor category)**.** Let C and D be categories. The functor category $\mathsf{Func}(\mathsf{C}, \mathsf{D})$ (sometimes D^{C} or $[\mathsf{C}, \mathsf{D}]$) is the category whose objects are functors $\mathsf{C} \rightsquigarrow \mathsf{D}$, and whose morphisms are natural transformations between them. The composition is given by vertical composition.

We can also compose natural transformations in a not so obvious way.

Lemma 45. *Consider the following arrangement of categories, functors, and natural transformations.*

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"{name=F}] \arrow[r, squiggly, bend right, "\mathcal{F}'"'{name=Fp}] & \mathsf{D} \arrow[r, squiggly, bend left, "\mathcal{G}"{name=G}] \arrow[r, squiggly, bend right, "\mathcal{G}'"'{name=Gp}] & \mathsf{E}
\arrow[Rightarrow, from=F, to=Fp, "\Phi"]
\arrow[Rightarrow, from=G, to=Gp, "\Psi"]
\end{tikzcd}$$

This induces a natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$.

Proof. By definition, Φ and Ψ make the diagrams

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(B) \arrow[d, "\Phi_B"] \\
\mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B)
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\Psi_A"'] & \mathcal{G}(B) \arrow[d, "\Psi_B"] \\
\mathcal{G}'(A) \arrow[r, "\mathcal{G}'(f)"'] & \mathcal{G}'(B)
\end{tikzcd}$$

commute. Since functors take commutative diagrams to commutative diagrams, we can map everything in the first diagram to E with $\mathcal{G}$.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] & (\mathcal{G} \circ \mathcal{F}')(B)
\end{tikzcd}$$

Since $\mathcal{F}'(f)\colon \mathcal{F}'(A) \to \mathcal{F}'(B)$ is a morphism in D and Ψ is a natural transformation, the following diagram commutes.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

Sticking these two diagrams on top of each other gives a new commutative diagram.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"] \arrow[d, "\mathcal{G}(\Phi_A)"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}(\Phi_B)"] \\
(\mathcal{G} \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}\circ\mathcal{F}')(f)"] \arrow[d, "\Psi_{\mathcal{F}'(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(B) \arrow[d, "\Psi_{\mathcal{F}'(B)}"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}'\circ\mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

The outside rectangle is nothing else but the commuting square for a natural transformation

$$(\Psi * \Phi)\colon \mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$$

with components $(\Psi * \Phi)_A = \Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$. □

Definition 46 (horizontal composition)**.** The natural transformation $\Psi * \Phi$ defined above is called the horizontal composition of Φ and Ψ.

Note 47. Really, the above definition of the horizontal composition is lopsided and ugly. Why did we apply the functor $\mathcal{G}$ to the first square rather than $\mathcal{G}'$? It becomes less so if we notice the following. The first step in our construction of $\Psi * \Phi$ was to apply the functor $\mathcal{G}$ to the commutative diagram

$$\begin{tikzcd}
\mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\Phi_A"'] & \mathcal{F}(B) \arrow[d, "\Phi_B"] \\
\mathcal{F}'(A) \arrow[r, "\mathcal{F}'(f)"'] & \mathcal{F}'(B)
\end{tikzcd} .$$

We could instead have applied the functor $\mathcal{G}'$, giving us the following.

$$\begin{tikzcd}
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] \arrow[d, "\mathcal{G}'(\Phi_A)"'] & (\mathcal{G}' \circ \mathcal{F})(B) \arrow[d, "\mathcal{G}'(\Phi_B)"] \\
(\mathcal{G}' \circ \mathcal{F}')(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F}')(f)"] & (\mathcal{G}' \circ \mathcal{F}')(B)
\end{tikzcd}$$

Then we could have glued to it the bottom of the following commuting square.

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G} \circ \mathcal{F})(f)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F})(B) \arrow[d, "\Psi_{\mathcal{F}(B)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "(\mathcal{G}' \circ \mathcal{F})(f)"] & (\mathcal{G}' \circ \mathcal{F})(B)
\end{tikzcd}$$

If you do this you get *another* natural transformation $\mathcal{G} \circ \mathcal{F} \Rightarrow \mathcal{G}' \circ \mathcal{F}'$, with components $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$. Why did we use the first definition rather than this one?

It turns out that $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$ are equal. To see this, pick any $A \in \mathrm{Obj}(\mathsf{A})$. From the morphism

$$\Phi_A \colon \mathcal{F}(A) \to \mathcal{F}'(A),$$

the natural transformation Ψ gives us a commuting square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}(\Phi_A)"] \arrow[d, "\Psi_{\mathcal{F}(A)}"'] & (\mathcal{G} \circ \mathcal{F}')(A) \arrow[d, "\Psi_{\mathcal{F}'(A)}"] \\
(\mathcal{G}' \circ \mathcal{F})(A) \arrow[r, "\mathcal{G}'(\Phi_A)"] & (\mathcal{G}' \circ \mathcal{F}')(A)
\end{tikzcd};$$

the two ways of going from top left to bottom right are nothing else but $\Psi_{\mathcal{F}(A)} \circ \mathcal{G}(\Phi_A)$ and $\mathcal{G}'(\Phi_A) \circ \Psi_{\mathcal{F}(A)}$.

Example 48 (whiskering)**.** Consider the following assemblage of categories, functors, and natural transformations.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, squiggly, "\mathcal{F}"] \arrow[r, bend right=50, squiggly, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow \Phi"] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] & \mathsf{E}
\end{tikzcd}$$

The horizontal composition allows us to Φ to a natural transformation $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ as follows. First, augment the diagram as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=50, squiggly, "\mathcal{F}"] \arrow[r, bend right=50, squiggly, "\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow \Phi"] & \mathsf{D} \arrow[r, bend left=50, squiggly, "\mathcal{G}"] \arrow[r, bend right=50, squiggly, "\mathcal{G}"'] \arrow[r, phantom, "\Downarrow 1_{\mathcal{G}}"] & \mathsf{E}
\end{tikzcd}$$

We can then take the horizontal composition of Φ and $1_{\mathcal{G}}$ to get a natural transformation from $\mathcal{G} \circ \mathcal{F}$ to $\mathcal{G} \circ \mathcal{F}'$ with components

$$(1_{\mathcal{G}} * \Phi)_A = \mathcal{G}(\Phi_A).$$

This natural transformation is called the *right whiskering* of Φ with $\mathcal{G}$, and is denoted $\mathcal{G}\Phi$. That is to say, $(\mathcal{G}\Phi)_A = \mathcal{G}(\Phi_A)$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left=50, "\mathcal{G}\circ\mathcal{F}"{name=U}] \arrow[r, squiggly, bend right=50, "\mathcal{G}\circ\mathcal{F}'"'{name=D}] & \mathsf{E} \arrow[from=U, to=D, Rightarrow, "\mathcal{G}\Phi"] \end{tikzcd}$$

The reason for the name is clear: we removed a whisker from the RHS of our diagram.

We can also remove a whisker from the LHS. Given this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[r, squiggly, bend left=50, "\mathcal{G}"{name=U}] \arrow[r, squiggly, bend right=50, "\mathcal{G}'"'{name=D}] & \mathsf{E} \arrow[from=U, to=D, Rightarrow, "\Psi"] \end{tikzcd}$$

we can build a natural transformation (denoted $\Psi\mathcal{F}$) with components

$$(\Psi\mathcal{F})_A = \Psi_{\mathcal{F}(A)},$$

making this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left=50, "\mathcal{G}\circ\mathcal{F}"{name=U}] \arrow[r, squiggly, bend right=50, "\mathcal{G}'\circ\mathcal{F}"'{name=D}] & \mathsf{E} \arrow[from=U, to=D, Rightarrow, "\Psi\mathcal{F}"] \end{tikzcd} .$$

This is called the *left whiskering* of Ψ with $\mathcal{F}$

Lemma 49. *If we have a natural isomorphism $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$, we can construct a natural isomorphism $\eta^{-1}\colon \mathcal{G} \Rightarrow \mathcal{F}$.*

Proof. The natural transformation gives us for any two objects A and B and morphism $f\colon A \to B$ a naturality square

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] \arrow[d, "\eta_A"'] & \mathcal{F}(B) \arrow[d, "\eta_B"] \\ \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] & \mathcal{G}(B) \end{tikzcd}$$

which tells us that

$$\eta_B \circ \mathcal{F}(f) = \mathcal{G}(f) \circ \eta_A.$$

Since η is a natural isomorphism, its components η_A are isomorphisms, so they have inverses η_A^{-1}. Acting on the above equation with η_A^{-1} from the right and η_B^{-1} from the left, we find

$$\mathcal{F}(f) \circ \eta_A^{-1} = \eta_B^{-1} \circ \mathcal{G}(f),$$

i.e. the following diagram commutes.

$$\begin{tikzcd} \mathcal{G}(A) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\eta_A^{-1}"'] & \mathcal{G}(B) \arrow[d, "\eta_B^{-1}"] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(f)"] & \mathcal{F}(B) \end{tikzcd}$$

But this is just the naturality square for a natural isomorphism η^{-1} with components $(\eta^{-1})_A = \eta_A^{-1}$. □

We would like to be able to express when two categories are the 'same.' The correct notion of sameness is provided by the following definition.

Definition 50 (categorical equivalence)**.** Let C and D be categories. We say that C and D are equivalent if there is a pair of functors

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, shift left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, shift left, "\mathcal{G}"] \end{tikzcd}$$

and natural isomorphisms $\eta\colon \mathcal{F}\circ\mathcal{G} \Rightarrow 1_{\mathsf{C}}$ and $\varphi\colon 1_{\mathsf{C}} \Rightarrow \mathcal{G}\circ\mathcal{F}$.

Note 51. The above definition of categorical equivalence is equivalent to the following: C and D are equivalent if there is a functor $\mathcal{F} : \mathsf{C} \rightsquigarrow \mathsf{D}$ which is fully faithful (Definition 31) and essentially surjective (Definition 34). That is to say, if the equivalence $\mathcal{F}$ is 'bijective up to isomorphism.'

Definition 52 (essentially small category)**.** A category C is said to be essentially small if it is equivalent to a small category (Definition 25).

Example 53. Let G be a group, G the category which mimics it, and let ρ and $\rho'\colon \mathsf{G} \rightsquigarrow \mathsf{Vect}$ be representations (recall Example 37.)

A natural transformation $\eta\colon \rho \Rightarrow \rho'$ is called an *intertwiner*. So what *is* an intertwiner?

Well, η has only one component, $\eta_*\colon \rho(*) \to \rho'(*)$, which is subject to the condition that for any $g \in \mathrm{Hom}_{\mathsf{G}}(*, *)$, the diagram below commutes.

$$\begin{tikzcd} \rho(*) \arrow[r, "\rho(g)"] \arrow[d, "\eta_*"'] & \rho(*) \arrow[d, "\eta_*"] \\ \rho'(*) \arrow[r, "\rho'(g)"] & \rho'(*) \end{tikzcd}$$

That is, an intertwiner is a linear map $\rho(*) \to \rho'(*)$ such that for all g,

$$\eta_* \circ \rho(g) = \rho'(g) \circ \eta_*.$$

1.4 Some special categories

1.4.1 Comma categories

Definition 54 (comma category)**.** Let A, B, C be categories, $\mathcal{S}$ and $\mathcal{T}$ functors as follows.

$$\mathsf{A} \overset{\mathcal{S}}{\rightsquigarrow} \mathsf{C} \overset{\mathcal{T}}{\leftsquigarrow} \mathsf{B}$$

The comma category $(\mathcal{S} \downarrow \mathcal{T})$ is the category whose

- objects are triples (α, β, f) where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(\mathsf{B})$, and $f \in \mathrm{Hom}_{\mathsf{C}}(\mathcal{S}(\alpha), \mathcal{T}(\beta))$, and whose

- morphisms $(\alpha, \beta, f) \to (\alpha', \beta', f')$ are all pairs (g, h), where $g\colon \alpha \to \alpha'$ and $h\colon \beta \to \beta'$, such that the diagram

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[d, "f'"] \\ \mathcal{T}(\beta) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\beta') \end{tikzcd}$$

  commutes.

Notation 55. We will often specify the comma category $(\mathcal{S} \downarrow \mathcal{T})$ by simply writing down the diagram

$$\mathsf{A} \overset{\mathcal{S}}{\rightsquigarrow} \mathsf{C} \overset{\mathcal{T}}{\leftsquigarrow} \mathsf{B} \ .$$

Let us check in some detail that a comma category really is a category. To do so, we need to check the three properties listed in Definition 1

1. We must be able to compose morphisms, i.e. we must have the following diagram.

$$\begin{tikzcd} (\alpha, \beta, f) \arrow[r, "{(g,h)}"] \arrow[rr, bend left, "{(g',h')\circ(g,h)}"] & (\alpha', \beta', f') \arrow[r, "{(g',h')}"] & (\alpha'', \beta'', f'') \end{tikzcd}$$

   We certainly do, since by definition, each square of the following diagram commutes,

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha') \arrow[r, "\mathcal{S}(g')"] \arrow[d, "f'"'] & \mathcal{S}(\alpha'') \arrow[d, "f''"] \\ \mathcal{T}(\alpha) \arrow[r, "\mathcal{T}(h)"] & \mathcal{T}(\alpha') \arrow[r, "\mathcal{T}(h')"] & \mathcal{T}(\alpha'') \end{tikzcd}$$

   so the square formed by taking the outside rectangle

$$\begin{tikzcd} \mathcal{S}(\alpha) \arrow[r, "\mathcal{S}(g')\circ\mathcal{S}(g)"] \arrow[d, "f"'] & \mathcal{S}(\alpha'') \arrow[d, "f''"] \\ \mathcal{T}(\alpha) \arrow[r, "\mathcal{T}(h')\circ\mathcal{T}(h)"] & \mathcal{T}(\alpha'') \end{tikzcd}$$

   commutes. But $\mathcal{S}$ and $\mathcal{T}$ are functors, so

$$\mathcal{S}(g') \circ \mathcal{S}(g) = \mathcal{S}(g' \circ g),$$

   and similarly for $\mathcal{T}(h' \circ h)$. Thus, the composition of morphisms is given via

$$(g', h') \circ (g, h) = (g' \circ g, h' \circ h).$$

2. We can see from this definition that associativity in $(\mathcal{S} \downarrow \mathcal{T})$ follows from associativity in the underlying categories A and B.

3. The identity morphism is the pair $(1_{\mathcal{S}(\alpha)}, 1_{\mathcal{T}(\beta)})$. It is trivial from the definition of the composition of morphisms that this morphism functions as the identity morphism.

1.4.2 Slice categories

A special case of a comma category, the so-called *slice category*, occurs when $\mathsf{C} = \mathsf{A}$, $\mathcal{S}$ is the identity functor, and $\mathsf{B} = \mathsf{1}$, the category with one object and one morphism (Example 5).

Definition 56 (slice category)**.** A slice category is a comma category

$$\mathsf{A} \overset{\mathrm{id}_A}{\rightsquigarrow} \mathsf{A} \overset{\mathcal{T}}{\leftsquigarrow} \mathsf{1} \ .$$

Let us unpack this prescription. Taking the definition literally, the objects in our category are triples (α, β, f), where $\alpha \in \mathrm{Obj}(\mathsf{A})$, $\beta \in \mathrm{Obj}(\mathsf{1})$, and $f \in \mathrm{Hom}_{\mathsf{A}}(\mathrm{id}_{\mathsf{A}}(\alpha), \mathcal{T}(\beta))$.

There's a lot of extraneous information here, and our definition can be consolidated considerably. Since the functor $\mathcal{T}$ is given and $\mathsf{1}$ has only one object (call it $*$), the object $\mathcal{T}(*)$ (call it X) is singled out in A. We can think of $\mathcal{T}$ as $\mathcal{F}_X$ (Example 29). Similarly, since the identity morphism doesn't do anything interesting, Therefore, we can collapse the following diagram considerably.

$$\begin{tikzcd}
\mathcal{F}_X(*) \arrow[r, "\mathcal{F}_X(1_*)"] \arrow[d, "f"'] & \mathcal{F}_X(*) \arrow[d, "f'"] \\
\mathrm{id}_A(\alpha) \arrow[r, "\mathrm{id}_A(g)"] & \mathrm{id}_A(\alpha')
\end{tikzcd}$$

The objects of a slice category therefore consist of pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and

$$f : \alpha \to X;$$

the morphisms $(\alpha, f) \to (\alpha', f')$ consist of maps $g\colon \alpha \to \alpha'$. This allows us to define a slice category more neatly.

Let A be a category, $X \in \mathrm{Obj}(\mathsf{A})$. The slice category $(\mathsf{A} \downarrow X)$ is the category whose objects are pairs (α, f), where $\alpha \in \mathrm{Obj}(A)$ and $f\colon \alpha \to X$, and whose morphisms $(\alpha, f) \to (\alpha', f')$ are maps $g : \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd}
\alpha \arrow[rr, "g"] \arrow[dr, "f"'] & & \alpha' \arrow[dl, "f'"] \\
& X &
\end{tikzcd}$$

commutes.

One can also define a coslice category, which is what you get when you take a slice category and turn the arrows around: coslice categories are *dual* to slice categories.

Definition 57 (coslice category)**.** Let A be a category, $X \in \mathrm{Obj}(A)$. The coslice category $(X \downarrow \mathsf{A})$ is the comma category given by the diagram

$$\mathsf{1} \overset{\mathcal{F}_X}{\rightsquigarrow} \mathsf{A} \overset{\mathrm{id}_A}{\leftsquigarrow} \mathsf{A} \ .$$

The objects are morphisms $f : X \to \alpha$ and the morphisms are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd}
& X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\
\alpha \arrow[rr, "g"'] & & \alpha'
\end{tikzcd}$$

commutes.

1.5 Universal properties

Note 58. It is assumed that the reader is familar with the basic idea of a universal property and has seen (but not necessarily understood) a few examples, such as the universal properties for products and tensor algebras.

In general, the 'nicest' definition of a structure uses only the category-theoretic information about the category in which the structure lives; the definition of a product (of sets, for example) is best given without making use of hand-wavy statements like "ordered pairs of an element here and an element there." The idea is similar to the situation in linear algebra, where it is aesthetically preferrable to avoid introducing an arbitrary basis every time one needs to prove something, instead using properties intrinsic to the vector space itself.

The easiest way to do this in general is by using the idea of a *universal property.*

Definition 59 (initial objects, final objects, zero objects)**.** Let C be a category. An object $A \in \text{Obj}(\mathsf{C})$ is said to be an

- initial object if $\text{Hom}(A, B)$ has exactly one element for all $B \in \text{Obj}(\mathsf{C})$, i.e. if there is exactly one arrow from A to every object in C.
- final object (or *terminal object*) if $\text{Hom}(B, A)$ has exactly one element for all $B \in \text{Obj}(\mathsf{C})$, i.e. if there is exactly one arrow from every object in C to A.
- zero object if it is both initial and final.

Example 60. In Set, there is exactly one map from any set S to any one-element set $\{*\}$. Thus $\{*\}$ is a terminal object in Set.

Furthermore, it is conventional that there is exactly one map from the empty set $\emptyset$ to any set B. Thus the $\emptyset$ is initial in Set.

Example 61. The trivial group is a zero object in Grp.

Theorem 62. *Let* C *be a category, let I and I' be two initial objects in* C*. Then there exists a unique isomorphism between I and I'.*

Proof. Since I is initial, there exists exactly one morphism from I to *any* object, including I itself. By Definition 1, Part 3, I must have at least one map to itself: the identity morphism 1_I. Thus, the only morphism from I to itself is the identity morphism. Similarly, the only morphism from I' to itself is also the identity morphism.

But since I is initial, there exists a unique morphism from I to I'; call it f. Similarly, there exists a unique morphism from I' to I; call it g. By Definition 1, Part 1, we can take the composition $g \circ f$ to get a morphism $I \to I$.

But there is only one isomorphism $I \to I$: the identity morphism! Thus

$$g \circ f = 1_I.$$

Similarly,

$$f \circ g = 1_{I'}.$$

This means that, by Definition 14, f and g are isomorphisms. They are clearly unique because of the uniqueness condition in the definition of an initial object. Thus, between any two initial objects there is a unique isomorphism, and we are done.

Here is a bad picture of this proof.

$$\begin{tikzcd} I \arrow[loop left, "1_I = g\circ f"] \arrow[r, bend left, "f"] & I' \arrow[l, bend left, "g"] \arrow[loop right, "1_{I'} = f\circ g"] \end{tikzcd}$$

□

Definition 63 (category of morphisms from an object to a functor)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor. Further let $X \in \mathrm{Obj}(C)$. The category of morphisms $(X \downarrow \mathcal{U})$ is the following comma category (see Definition 54):

$$1 \xrightarrow{\mathcal{F}_X} \mathsf{C} \xleftarrow{\mathcal{U}} \mathsf{D}\ .$$

Just as for (co)slice categories (Definitions 56 and 57), there is some unpacking to be done. In fact, the unpacking is very similar to that of coslice categories. The LHS of the commutative square diagram collapses because the functor $\mathcal{F}_X$ picks out a single element X; therefore, the objects of $(X \downarrow \mathcal{U})$ are ordered pairs

$$(\alpha, f); \qquad \alpha \in \mathrm{Obj}(\mathsf{D}), \quad f\colon X \to \mathcal{U}(\alpha),$$

and the morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "f'"] & \\ \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha') \end{tikzcd}$$

commutes.

Just as slice categories are dual to coslice categories, we can take the dual of the previous definition.

Definition 64 (category of morphisms from a functor to an object)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor. The category of morphisms $(\mathcal{U} \downarrow X)$ is the comma category

$$\mathsf{D} \xrightarrow{\mathcal{U}} \mathsf{C} \xleftarrow{\mathcal{F}_X} 1\ .$$

The objects in this category are pairs (α, f), where $\alpha \in \mathrm{Obj}(\mathsf{D})$ and $f\colon \mathcal{U}(\alpha) \to X$. The morphisms $(\alpha, f) \to (\alpha', f')$ are morphisms $g\colon \alpha \to \alpha'$ such that the diagram

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(\alpha') \arrow[dl, "f'"] \\ & X & \end{tikzcd}$$

commutes.

Definition 65 (initial morphism)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. An initial morphism (called a *universal arrow* in [17]) is an initial object in the category $(X \downarrow \mathcal{U})$, i.e. the comma category which has the diagram

$$1 \overset{\mathcal{F}_X}{\rightsquigarrow} \mathsf{C} \overset{\mathcal{U}}{\leftsquigarrow} \mathsf{D}$$

This is not by any stretch of the imagination a transparent definition, but decoding it will be good practice.

The definition tells us that an initial morphism is an object in $(X \downarrow \mathcal{U})$, i.e. a pair (I, φ) for $I \in \mathrm{Obj}(\mathsf{D})$ and $\varphi\colon X \to \mathcal{U}(\alpha)$. But it is not just any object: it is an initial object. This means that for any other object (α, f), there exists a unique morphism $(I, \varphi) \to (\alpha, f)$. But such morphisms are simply maps $g\colon I \to \alpha$ such that the diagram

$$\begin{tikzcd} & X \arrow[dl, "\varphi"'] \arrow[dr, "f"] & \\ \mathcal{U}(I) \arrow[rr, "\mathcal{U}(g)"'] & & \mathcal{U}(\alpha) \end{tikzcd}$$

commutes.

We can express this schematically via the following diagram (which is essentially the above diagram, rotated to agree with the literature).

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "f"'] & \mathcal{U}(I) \arrow[d, "\mathcal{U}(g)"] & & I \arrow[d, "\exists! g"] \\ & \mathcal{U}(\alpha) & & \alpha \end{tikzcd}$$

As always, there is a dual notion.

Definition 66 (terminal morphism)**.** Let C, D be categories, let $\mathcal{U}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ be a functor, and let $X \in \mathrm{Obj}(\mathsf{C})$. A terminal morphism is a terminal object in the category $(\mathcal{U} \downarrow X)$.

This time "terminal object" means a pair (I, φ) such that for any other object (α, f) there is a unique morphism $g\colon \alpha \to I$ such that the diagram

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[rr, "\mathcal{U}(g)"] \arrow[dr, "f"'] & & \mathcal{U}(I) \arrow[dl, "\varphi"] \\ & X & \end{tikzcd}$$

commutes.

Again, with the diagram helpfully rotated, we have the following.

$$\begin{tikzcd} \mathcal{U}(\alpha) \arrow[d, "\mathcal{U}(g)"'] \arrow[dr, "f"] & & & \alpha \arrow[d, "\exists! g"] \\ \mathcal{U}(I) \arrow[r, "\varphi"'] & X & & I \end{tikzcd}$$

Definition 67 (universal property)**.** There is no hard and fast definition of a universal property. In complete generality, an object with a universal property is just an object which is initial or terminal in some category. However, quite a few interesting universal properties are given in terms of initial and terminal morphisms, and it will pay to study a few examples.

Note 68. One often hand-wavily says that an object I satisfies a universal property if (I, φ) is an initial or terminal morphism. This is actually rather annoying; one has to remember that when one states a universal property in terms of a universal morphism, one is defining not only an object I but also a morphism φ, which is often left implicit.

Example 69 (tensor algebra)**.** One often sees some variation of the following universal characterization of the tensor algebra, which was taken (almost) verbatim from Wikipedia. We will try to stretch it to fit our definition, following the logic through in some detail.

> Let V be a vector space over a field k, and let A be an algebra over k. The tensor algebra $T(V)$ satisfies the following universal property.
>
> Any linear transformation $f\colon V \to A$ from V to A can be uniquely extended to an algebra homomorphism $T(V) \to A$ as indicated by the following commutative diagram.
>
> $$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "f"'] & T(V) \arrow[d, "\tilde{f}"] \\ & A \end{tikzcd}$$

As it turns out, it will take rather a lot of stretching.

Let $\mathcal{U}\colon k\text{-}\mathsf{Alg} \rightsquigarrow \mathsf{Vect}_k$ be the forgetful functor which assigns to each algebra over a field k its underlying vector space. Pick some k-vector space V. We consider the category $(V \downarrow \mathcal{U})$, which is given by the following diagram.

$$\mathsf{1} \overset{\mathcal{F}_V}{\rightsquigarrow} \mathsf{Vect}_k \overset{\mathcal{U}}{\leftsquigarrow} k\text{-}\mathsf{Alg}$$

By Definition 65, the objects of $(V \downarrow \mathcal{U})$ are pairs (A, L), where A is a k-algebra and L is a linear map $V \to \mathcal{U}(A)$. The morphisms are algebra homomorphisms $\rho\colon A \to A'$ such that the diagram

$$\begin{tikzcd} & V \arrow[dl, "L"'] \arrow[dr, "L'"] & \\ \mathcal{U}(A) \arrow[rr, "\mathcal{U}(\rho)"'] & & \mathcal{U}(A') \end{tikzcd}$$

commutes. An object $(T(V), i)$ is initial if for any object (A, f) there exists a unique morphism $g\colon T(V) \to A$ such that the diagram

$$\begin{tikzcd} V \arrow[r, "i"] \arrow[dr, "L"'] & \mathcal{U}(T(V)) \arrow[d, "\mathcal{U}(g)"] \\ & \mathcal{U}(A) \end{tikzcd} \qquad \begin{tikzcd} T(V) \arrow[d, "\exists! g"] \\ A \end{tikzcd}$$

commutes.

Thus, the pair $(i, T(V))$ is the initial object in the category $(V \downarrow \mathcal{U})$. We called $T(V)$ the *tensor algebra* over V.

But what is i? Notice that in the Wikipedia definition above, the map i is from V to $T(V)$, but in the diagram above, it is from V to $\mathcal{U}(T(V))$. What gives?

The answer that the diagram in Wikipedia's definition does not take place in a specific category. Instead, it implicitly treats $T(V)$ only as a vector space. But this is exactly what the functor $\mathcal{U}$ does.

Example 70 (tensor product)**.** According to the excellent book [3], the tensor product satisfies the following universal property.

> Let V_1 and V_2 be vector spaces. Then we say that a vector space V_3 together with a bilinear map $\iota\colon V_1 \times V_2 \to V_3$ has the *universal property* provided that for any bilinear map $B\colon V_1 \times V_2 \to W$, where W is also a vector space, there exists a unique linear map $L\colon V_3 \to W$ such that $B = L\iota$. Here is a diagram describing this 'factorization' of B through ι:

$$\begin{tikzcd} V_1 \times V_2 \arrow[r, "\iota"] \arrow[dr, "B"'] & V_3 \arrow[d, "L"] \\ & W \end{tikzcd}$$

It turns out that the tensor product defined in this way is neither an initial or final morphism. This is because in the category Vect_k, there is no way of making sense of a bilinear map.

Example 71 (categorical product)**.** Here is the universal property for a product, taken verbatim from Wikipedia ([21]).

> Let C be a category with some objects X_1 and X_2. A product of X_1 and X_2 is an object X (often denoted $X_1 \times X_2$) together with a pair of morphisms $\pi_1\colon X \to X_1$ and $\pi_2\colon X \to X_2$ such that for every object Y and pair of morphisms $f_1\colon Y \to X_1$, $f_2\colon Y \to X_2$, there exists a unique morphism $f\colon Y \to X_1 \times X_2$ such that the following diagram commutes.

$$\begin{tikzcd} & Y \arrow[dl, "f_1"'] \arrow[d, "f"] \arrow[dr, "f_2"] & \\ X_1 & X_1 \times X_2 \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X_2 \end{tikzcd}$$

Consider the comma category $(\Delta \downarrow (X_1, X_2))$ (where Δ is the diagonal functor, see Example 35) given by the following diagram.

$$\begin{tikzcd} \mathsf{C} \arrow[r, rightsquigarrow, "\Delta"] & \mathsf{C} \times \mathsf{C} & \mathsf{1} \arrow[l, leftsquigarrow, "\mathcal{F}_{(X_1, X_2)}"'] \end{tikzcd}$$

The objects of this category are pairs $(A, (s, t))$, where $A \in \mathrm{Obj}(\mathsf{C})$ and

$$(s, t)\colon \Delta(A) = (A, A) \to (X_1, X_2).$$

The morphisms $(A,(s,t)) \longrightarrow (B,(u,v))$ are morphisms $r\colon A \longrightarrow B$ such that the diagram

$$\begin{tikzcd}
(A,A) \arrow[rr, "{(r,r)}"] \arrow[dr, "{(s,t)}"'] & & (B,B) \arrow[dl, "{(u,v)}"] \\
& (X_1,X_2) &
\end{tikzcd}$$

commutes.

An object $(X_1 \times X_2,(\pi_1,\pi_2))$ is final if for any other object $(Y,(f_1,f_2))$, there exists a unique morphism $f\colon Y \longrightarrow X_1 \times X_2$ such that the diagram

$$\begin{tikzcd}
(Y,Y) \arrow[d, "{(f,f)}"'] \arrow[dr, "{(f_1,f_2)}"] & \\
(X_1 \times X_2, X_1 \times X_2) \arrow[r, "{(\pi_1,\pi_2)}"'] & (X_1,X_2)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
Y \arrow[d, "\exists! f"] \\
X_1 \times X_2
\end{tikzcd}$$

commutes. If we re-arrange our diagram a bit, it is not too hard to see that it is equivalent to the one given above. Thus, we can say: a product of two sets X_1 and X_2 is a final object in the category $(\Delta \downarrow (X_1,X_2))$

2 Cartesian closed categories

2.1 Cartesian closed categories

This section loosely follows [24].

Cartesian closed categories are the prototype for many of the structures possessed by monoidal categories.

2.2 Products

We saw in Example 71 the definition for a categorical product. In some categories, we can naturally take the product of any two objects; one generally says that such a category *has products*. We formalize that in the following.

Definition 72 (category with products)**.** Let C be a category such that for every two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, there exists an object $A \times B$ which satisfies the universal property (Example 71). Then we say that C has products.

Note 73. Sometimes people call a category with products a *Cartesian category*, but others use this terminology to mean a category with all finite limits.[1] We will avoid it altogether.

Theorem 74. *Let* C *be a category with products. Then the product can be extended to a bifunctor (Definition 36)* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$.

Proof. Let $X, Y \in \mathrm{Obj}(C)$. We need to check that the assignment $(X, Y) \mapsto \times(X, Y) \equiv X \times Y$ is functorial, i.e. that $\times$ assigns

- to each pair $(X, Y) \in \mathrm{Obj}(\mathsf{C} \times \mathsf{C})$ an object $X \times Y \in \mathrm{Obj}(\mathsf{C})$ and
- to each pair of morphisms $(f, g)\colon (X, Y) \to (X', Y')$ a morphism

$$\times(f, g) = f \times g\colon X \times Y \to X' \times Y'$$

  such that

  - $1_X \times 1_Y = 1_{X \times Y}$, and

[1] We will see later that any category with both products and equalizers has all finite limits.

- × respects composition as follows.

$$\begin{tikzcd}
(X,Y) \arrow[r, "(f,g)"] \arrow[rr, bend left, "(f',g')\circ(f,g)=(f'\circ f,g'\circ g)"] & (X',Y') \arrow[r, "(f',g')"] & (X'',Y'')
\end{tikzcd}$$

$$\rightsquigarrow \times$$

$$\begin{tikzcd}
X\times Y \arrow[r, "f\times g"] \arrow[rr, bend right, "f'\times g'\circ f\times g=(f'\circ f)\times(g'\circ g)"'] & X'\times Y' \arrow[r, "f'\times g'"] & X''\times Y''
\end{tikzcd}$$

□

We know how × assigns objects in C × C to objects in C. We need to figure out how × should assign to a morphism (f,g) in C × C a morphism $f\times g$ in C. We do this by diagram chasing.

Suppose we are given two maps $f\colon X_1\to X_2$ and $g\colon Y_1\to Y_2$. We can view this as a morphism (f,g) in C × C.

Recall the universal property for products: a product $X_1\times Y_1$ is a final object in the category $(\Delta\downarrow(X_1,Y_1))$. Objects in this category can be thought of as diagrams in C × C.

$$(X_1\times Y_1, X_1\times Y_1)\xrightarrow{\pi_1,\pi_2}(X_1,Y_1)\ .$$

By assumption, we can take the product of both X_1 and Y_1, and X_2 and Y_2. This gives us two diagrams living in C × C, which we can put next to each other.

$$\begin{tikzcd}
(X_1\times Y_1, X_1\times Y_1) \arrow[r, "(\pi_1,\pi_2)"] & (X_1,Y_1) \\
(X_2\times Y_2, X_2\times Y_2) \arrow[r, "(\rho_1,\rho_2)"] & (X_2,Y_2)
\end{tikzcd}$$

We can draw in our morphism (f,g), and take its composition with (π_1,π_2).

$$\begin{tikzcd}
(X_1\times Y_1, X_1\times Y_1) \arrow[r, "(\pi_1,\pi_2)"] \arrow[dr, "(f\circ\pi_1,g\circ\pi_2)"'] & (X_1,Y_1) \arrow[d, "(f,g)"] \\
(X_2\times Y_2, X_2\times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2,Y_2)
\end{tikzcd}$$

Now forget about the top right of the diagram. I'll erase it to make this easier.

$$\begin{tikzcd}
(X_1\times Y_1, X_1\times Y_1) \arrow[dr, "(f\circ\pi_1,g\circ\pi_2)"] & \\
(X_2\times Y_2, X_2\times Y_2) \arrow[r, "(\rho_1,\rho_2)"'] & (X_2,Y_2)
\end{tikzcd}$$

The universal property for products says that there exists a unique map $h\colon X_1 \times Y_1 \to X_2 \times Y_2$ such that the diagram below commutes.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[d, "{(h,h)}"'] \arrow[dr, "{(f\circ\pi_1, g\circ\pi_2)}"] & \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[r, "{(\rho_1,\rho_2)}"'] & (X_1, X_2)
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \times Y_1 \arrow[d, "\exists! h"] \\
X_2 \times Y_2
\end{tikzcd}$$

And h is what we will use for the product $f \times g$.

Of course, we must also check that h behaves appropriately. Draw two copies of the diagram for the terminal object in $(\Delta \downarrow (X, Y))$ and identity arrows between them.

$$\begin{tikzcd}
(X \times Y, X \times Y) \arrow[r, "{(\pi_1,\pi_2)}"] \arrow[d, "{(1_{X\times Y}, 1_{X\times Y})}"'] & (X, Y) \arrow[d, "{(1_X, 1_Y)}"] \\
(X \times Y, X \times Y) \arrow[r, "{(\pi_1,\pi_2)}"'] & (X, Y)
\end{tikzcd}$$

Just as before, we compose the morphisms to and from the top right to draw a diagonal arrow, and then erase the top right object and the arrows to and from it.

$$\begin{tikzcd}
(X \times Y, X \times Y) \arrow[d, "{(1_{X\times Y}, 1_{X\times Y}) = (1_X \times 1_Y, 1_X \times 1_Y)}"'] \arrow[dr, "{(1_X\circ\pi_1, 1_Y\circ\pi_2)}"] & \\
(X \times Y, X \times Y) \arrow[r, "{(\pi_1,\pi_2)}"'] & (X, Y)
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times Y \arrow[d, "1_X \times 1_Y = 1_X \times 1_Y"] \\
X \times Y
\end{tikzcd}$$

By definition, the arrow on the right is $1_X \times 1_Y$. It is also $1_{X\times Y}$, and by the universal property it is unique. Therefore $1_{X\times Y} = 1_X \times 1_Y$.

Next, put three of these objects together. The proof of the last part is immediate.

$$\begin{tikzcd}
(X_1 \times Y_1, X_1 \times Y_1) \arrow[d, "{(f\times g, f\times g)}"] \arrow[dd, bend right=60, "{(f'\times g')\circ(f\times g) = (g'\circ g)\times(f'\circ f)}"'] \\
(X_2 \times Y_2, X_2 \times Y_2) \arrow[d, "{(f'\times g', f'\times g')}"] \\
(X_3 \times Y_3, X_3 \times Y_3)
\end{tikzcd}
\qquad
\begin{tikzcd}
(X_1, Y_1) \arrow[d, "{(f,g)}"'] \arrow[dd, bend left=60, "{(g'\circ g, f'\circ f)}"] \\
(X_2, Y_2) \arrow[d, "{(f',g')}"'] \\
(X_3, Y_3)
\end{tikzcd}$$

The meaning of this theorem is that in any category where you can take products of the objects, you can also take products of the morphisms.

Example 75. The category Vect_k of vector spaces over a field k has the direct sum $\oplus$ as a product.

The product is, in an appropriate way, commutative.

Lemma 76. *There is a natural isomorphism between the following functors* $\mathsf{C} \times \mathsf{C} \to \mathsf{C}$

$$\times\colon (A, B) \to A \times B \qquad \textit{and} \qquad \tilde{\times}\colon (A, B) \to B \times A.$$

Proof. We define a natural transformation $\Phi\colon \times \Rightarrow \tilde{\times}$ with components

$$\Phi_{A,B}\colon A \times B \to B \times A$$

as follows. Denote the canonical projections for the product $A \times B$ by π_A and π_B. Then (π_B, π_A) is a map $A \times B \to (B, A)$, and the universal property for products gives us a map $A \times B \to B \times A$. We can pull the same trick to go from $B \times A$ to $A \times B$, using the pair (π_A, π_B) and the universal property. Furthermore, these maps are inverse to each other, so $\Phi_{A,B}$ is an isomorphism.

We need only check naturality, i.e. that for $f\colon A \to A'$, $g\colon B \to B'$, the following square commutes.

$$\begin{tikzcd}
A \times B \arrow[r] \arrow[d] & A' \times B' \arrow[d] \\
B \times A \arrow[r] & B' \times A'
\end{tikzcd}$$

□

Note 77. It is an important fact that the product is associative. We shall see this in Theorem 109, after we have developed the machinery to do it cleanly.

2.3 Coproducts

Definition 78. Let C be a category, X_1 and $X_2 \in \mathrm{Obj}(\mathsf{C})$. The coproduct of X_1 and X_2, denoted $X_1 \amalg X_2$, is the initial object in the category $((X_1, X_2) \downarrow \Delta)$. In everyday language, we have the following.

An object $X_1 \amalg X_2$ is called the coproduct of X_1 and X_2 if

1. there exist morphisms $i_1\colon X_1 \to X_1 \amalg X_2$ and $i_2\colon X_2 \to X_1 \amalg X_2$ called *canonical injections* such that
2. for any object Y and morphisms $f_1\colon X_1 \to Y$ and $f_2\colon X_2 \to Y$ there exists a unique morphism $f\colon X_1 \amalg X_2 \to Y$ such that $f_1 = f \circ i_1$ and $f = f_2 = f \circ i_2$, i.e. the following diagram commutes.

$$\begin{tikzcd}
 & Y & \\
X_1 \arrow[ur, "f_1"] \arrow[r, "i_1"'] & X_1 \amalg X_2 \arrow[u, "f"] & X_2 \arrow[ul, "f_2"'] \arrow[l, "i_2"]
\end{tikzcd}$$

Definition 79 (category with coproducts)**.** We say that a category C has coproducts if for all $A, B \in \mathrm{Obj}(C)$, the coproduct $A \amalg B$ is in $\mathrm{Obj}(C)$.

We have the following analog of Theorem 74.

Theorem 80. *Let C be a category with coproducts. Then we have a functor* $\amalg\colon C \times C \rightsquigarrow C$.

Proof. We verify that $\amalg$ allows us to define canonically a coproduct of morphisms. Let X_1, X_2, Y_1, $Y_2 \in \mathrm{Obj}(C)$, and let $f : X_1 \to Y_1$ and $g : X_2 \to Y_2$. We have the following diagram.

$$\begin{tikzcd}
X_1 \arrow[r, "i_{1,X}"] \arrow[d, "f"'] \arrow[dr, "i_{i,Y}\circ f"] & X_1 \amalg X_2 & X_2 \arrow[l, "i_{2,X}"'] \arrow[d, "g"] \arrow[dl, "i_{2,Y}\circ g"'] \\
Y_1 \arrow[r, "i_{1,Y}"] & Y_1 \amalg Y_2 & Y_2 \arrow[l, "i_{2,Y}"']
\end{tikzcd}$$

But by the universal property of coproducts, the diagonal morphisms induce a map $X_1 \amalg X_2 \to Y_1 \amalg Y_2$, which we define to be $f \amalg g$.

$$\begin{tikzcd}
X_1 \arrow[r, "i_{1,X}"] \arrow[dr, "i_{i,Y}\circ f"'] & X_1 \amalg X_2 \arrow[d, "f\amalg g"] & X_2 \arrow[l, "i_{2,X}"'] \arrow[dl, "i_{2,Y}\circ g"] \\
& Y_1 \amalg Y_2 &
\end{tikzcd}$$

The rest of the verification that $\amalg$ really is a functor is identical to that in Theorem 74. □

Example 81. In Set, the coproduct is the disjoint union.

Example 82. In Vect_k, the coproduct $\oplus$ is the direct sum.

Example 83. In k-Alg the tensor product is the coproduct. To see this, let A, B, and R be k-algebras. The tensor product $A \otimes_k B$ has canonical injections $\iota_1 : A \to A \otimes B$ and $\iota_2 : B \to A \otimes B$ given by

$$\iota_A : v \to v \otimes 1_B \qquad \text{and} \qquad \iota_B : v \mapsto 1 \otimes v.$$

Let $f_1 : A \to R$ and $f_2 : B \to R$ be k-algebra homomorphisms. Then there is a unique homomorphism $f : A \otimes B \to R$ which makes the following diagram commute.

$$\begin{tikzcd}
A \arrow[r, "\iota_A"] \arrow[dr, "f_1"'] & A \otimes B \arrow[d, "f"] & B \arrow[l, "\iota_B"'] \arrow[dl, "f_2"] \\
& R &
\end{tikzcd}$$

2.4 Biproducts

A lot of the stuff in this section was adapted and expanded from [33].

One often says that a biproduct is an object which is both a product and a coproduct, but this is both misleading and unnecessarily vague. Objects satisfying universal properties are only defined up to isomorphism, so it doesn't make much sense to demand that products *coincide* with coproducts. However, demanding only that they be isomorphic (or even naturally isomorphic) is too weak, and does not determine a biproduct uniquely. In this section we will give the correct definition which encapsulates all of the properties of the product and the coproduct we know and love.

In order to talk about biproducts, we need to be aware of the following result, which we will consider in much more detail in Definition 7.2.

Definition 84 (zero morphism)**.** Let C be a category with zero object 0. For any two objects A, $B \in \mathrm{Obj}(\mathrm{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B \end{tikzcd}$$

Notation 85. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}. With this notation, for all morphisms f and g we have $f \circ 0 = 0$ and $0 \circ g = 0$.

Definition 86 (category with biproducts)**.** Let C be a category with zero object 0, products $\times$ (with canonical projections π), and coproducts $\amalg$ (with natural injections ι). We say that C has biproducts if for each two objects $A_1, A_2 \in \mathrm{Obj}(\mathrm{C})$, there exists an isomorphism

$$\Phi_{A_1,A_2} \colon A_1 \amalg A_2 \to A_1 \times A_2$$

such that

$$A_i \xrightarrow{\iota_i} A_1 \amalg A_2 \xrightarrow{\Phi_{A_1,A_2}} A_1 \times A_2 \xrightarrow{\pi_j} A_j \;=\; \begin{cases} 1_{A_1}, & i = j \\ 0, & i \neq j. \end{cases}$$

Here is a picture.

$$\begin{tikzcd} & A_1 \amalg A_2 \arrow[dd, "\Phi_{A_1,A_2}"] & \\ A_1 \arrow[ur, "\iota_1"] & & A_2 \arrow[ul, "\iota_2"'] \\ & A_1 \times A_2 \arrow[ul, "\pi_1"] \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

Lemma 87. *The isomorphisms $\Phi_{A,B}$ are unique if they exist.*

Proof. We can compose Φ_{A_1,A_2} with π_1 and π_2 to get maps $A_1 \amalg A_2 \to A_1$ and $A_1 \amalg A_2 \to A_2$. The universal property for products tells us that there is a unique map $\varphi \colon A_1 \amalg A_2 \to A_1 \times A_2$ such that $\pi_1 \circ \varphi = \pi_1 \circ \Phi_{A_1,A_2}$ and $\pi_2 \circ \varphi = \pi_2 \circ \Phi_{A_1,A_2}$; but φ is just Φ_{A_1,A_2}. Hence Φ_{A_1,A_2} is unique. □

Lemma 88. *In any category* C *with biproducts $\Phi_{A,B} \colon A \amalg B \to A \times B$, any map $A \amalg B \to A' \times B'$ is completely determined by its components $A \to A'$, $A \to B'$, $B \to A'$, and $B \to B'$. That is, given any map $f \colon A \amalg B \to A' \times B'$, we can create four maps*

$$\pi_{A'} \circ f \circ \iota_A \colon A \to A', \qquad \pi_{B'} \circ f \circ \iota_A \colon A \to B', \qquad etc,$$

and given four maps

$$\psi_{A,A'} \colon A \to A'; \qquad \psi_{A,B'} \colon A \to B'; \qquad \psi_{B,A'} \colon B \to A', \qquad and\ \psi_{B,B'} \colon B \to B',$$

we can construct a unique map $\psi \colon A \amalg B \to A' \times B'$ such that

$$\psi_{A,A'} = \pi_{A'} \circ \psi \circ \iota_A, \qquad \psi_{A,B'} = \pi_{B'} \circ \psi \circ \iota_A, \qquad etc.$$

Proof. The universal property for coproducts give allows us to turn the morphisms $\psi_{A,A'}$ and $\psi_{A,B'}$ into a map $\psi_A\colon A \to B \times B'$, the unique map such that

$$\pi_{A'} \circ \psi_A = \psi_{A,A'} \qquad \text{and} \qquad \pi_{B'} \circ \psi_A = \psi_{A,B'}.$$

The morphisms $\psi_{B,A'}$ and $\psi_{B,B'}$ give us a map ψ_B which is unique in the same way.

Now the universal property for coproducts gives us a map $\psi\colon A \amalg B \to A' \times B'$, the unique map such that $\psi \circ \iota_A = \psi_A$ and $\psi \circ \iota_B = \psi_B$. $\square$

Theorem 89. *The $\Phi_{A,B}$ defined above form the components of a natural isomorphism.*

Proof. Let A, B, A', and $B' \in \mathrm{Obj}(\mathsf{C})$, and let $f\colon A \to A'$ and $g\colon B \to B'$. To check that $\Phi_{A,B}$ are the components of a natural isomorphism, we have to check that the following naturality square commutes.

$$\begin{tikzcd}
A \amalg B \arrow[r, "f \amalg g"] \arrow[d, "\Phi_{A,B}"'] & A' \amalg B' \arrow[d, "\Phi_{A',B'}"] \\
A \times B \arrow[r, "f \times g"'] & A' \times B'
\end{tikzcd}$$

That is, we need to check that $(f \times g) \circ \Phi_{A,B} = \Phi_{A',B'} \circ (f \amalg g)$.

Both of these are morphisms $A \amalg B \to A' \times B'$, and by Lemma 88, it suffices to check that their components agree, i.e. that $\square$

Example 90. In the category Vect_k of vector spaces over a field k, the direct sum $\oplus$ is a biproduct.

Example 91. In the category Ab of abelian groups, the direct sum $\oplus$ is both a product and a coproduct. Hence Ab has biproducts.

2.5 Exponentials

Astonishingly, we can talk about functional evaluation purely in terms of products and universal properties.

Definition 92 (exponential)**.** Let C be a category with products, $A, B \in \mathrm{Obj}(\mathsf{C})$. The exponential of A and B is an object $B^A \in \mathrm{Obj}(\mathsf{C})$ together with a morphism $\varepsilon\colon B^A \times A \to B$, called the *evaluation morphism*, which satisfies the following universal property.

> For any object $X \in \mathrm{Obj}(\mathsf{C})$ and morphism $f\colon X \times A \to B$, there exists a unique morphism $\bar{f}\colon X \to B^A$ which makes the following diagram commute.

$$\begin{tikzcd}
B^A & B^A \times A \arrow[r, "\varepsilon"] & B \\
X \arrow[u, "\exists! \bar{f}"] & X \times A \arrow[u, "\bar{f} \times 1_A"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Example 93. In Set, the exponential object B^A is the set of functions $A \to B$, and the evaluation morphism ε is the map which assigns $(f, a) \mapsto f(a)$. Let us check that these objects indeed satisfy the universal property given.

Suppose we are given a function $f : X \times A \to B$. We want to construct from this a function $\bar{f} : X \to B^A$, i.e. a function which takes an element $x \in X$ and returns a function $\bar{f}(x) \colon A \to B$. There is a natural way to do this: fill the first slot of f with x! That is to say, define $\bar{f}(x) = f(x, -)$. It is easy to see that this makes the diagram commute:

$$\begin{tikzcd}
(f(x,-),a) \arrow[r, mapsto, "\varepsilon"] & f(x,a) \\
(x,a) \arrow[u, mapsto, "\bar{f}\times 1_A"] \arrow[ur, mapsto, "f"'] &
\end{tikzcd}$$

Furthermore, $\bar{f}$ is unique since if it sent x to any function other than f the diagram would not commute.

Example 94. One might suspect that the above generalizes to all sorts of categories. For example, one might hope that in Grp, the exponential H^G of two groups G and H would somehow capture all homomorphisms $G \to H$. This turns out to be impossible; we will see why in REF.

2.6 Cartesian closed categories

Definition 95 (cartesian closed category)**.** A category C is cartesian closed if it has

1. products: for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $A \times B \in \mathrm{Obj}(\mathsf{C})$;
2. exponentials: for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $B^A \in \mathrm{Obj}(\mathsf{C})$;
3. a terminal element (Definition 59) $1 \in \mathsf{C}$.

Cartesian closed categories have many inveresting properties. For example, they replicate many properties of the integers: we have the following familiar formulae for any objects A, B, and C in a Cartesian closed category with coproducts $+$.

1. $A \times (B + C) \simeq (A \times B) + (A \times C)$
2. $C^{A+B} \simeq C^A \times C^B$.
3. $(C^A)^B \simeq C^{A\times B}$
4. $(A \times B)^C \simeq A^C \times B^C$
5. $C^1 \simeq C$

We could prove these immediately, by writing down diagrams and appealing to the universal properties. However, we will soon have a powerful tool, the Yoneda lemma, that makes their proof completely routine.

3 Hom functors and the Yoneda lemma

3.1 The Hom functor

Given any locally small category (Definition 25) C and two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, we have thus far notated the set of all morphisms $A \to B$ by $\mathrm{Hom}_{\mathsf{C}}(A, B)$. This may have seemed a slightly odd notation, but there was a good reason for it: $\mathrm{Hom}_{\mathsf{C}}$ is really a functor.

To be more explicit, we can view $\mathrm{Hom}_{\mathsf{C}}$ in the following way: it takes two objects, say A and B, and returns the set of morphisms $A \to B$. It's a functor $\mathsf{C} \times \mathsf{C}$ to set!

(Actually, as we'll see, this isn't quite right: it is actually a functor $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$. But this is easier to understand if it comes about in a natural way, so we'll keep writing $\mathsf{C} \times \mathsf{C}$ for the time being.)

That takes care of how it sends objects to objects, but any functor has to send morphisms to morphisms. How does it do that?

A morphism in $\mathsf{C} \times \mathsf{C}$ between two objects (A', A) and (B', B) is an ordered pair (f', f) of two morphisms:

$$f : A \to B \qquad \text{and} \qquad f' : A' \to B'.$$

We can draw this as follows.

$$\begin{tikzcd} A' \arrow[r, "f'"] & B' \\ A \arrow[r, "f"] & B \end{tikzcd}$$

We have to send this to a set-function $\mathrm{Hom}_{\mathsf{C}}(A', A) \to \mathrm{Hom}_{\mathsf{C}}(B', B)$. So let m be a morphism in $\mathrm{Hom}_{\mathsf{C}}(A, A')$. We can draw this into our little diagram above like so.

$$\begin{tikzcd} A' \arrow[r, "f'"] \arrow[d, "m"'] & B' \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Notice: we want to build from m, f, and f' a morphism $B' \to B$. Suppose f' were going the other way.

$$\begin{tikzcd} A' \arrow[d, "m"'] & B' \arrow[l, "f'"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Then we could get what we want simply by taking the composition $f \circ m \circ f'$.

$$\begin{tikzcd} A' \arrow[d, "m"'] & B' \arrow[l, "f'"'] \arrow[d, "f \circ m \circ f'"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

But we can make f' go the other way simply by making the first argument of Hom_C come from the opposite category: that is, we want Hom_C to be a functor $\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \to \mathsf{Set}$.

As the function $m \mapsto f \circ m \circ f'$ is the action of the functor Hom_C on the morphism (f', f), it is natural to call it $\mathrm{Hom}_\mathsf{C}(f', f)$.

Of course, we should check that this *really is* a functor, i.e. that it treats identities and compositions correctly. This is completely routine, and you should skip down to Definition 96 if you read the last sentence with annoyance.

First, we need to show that $\mathrm{Hom}_\mathsf{C}(1_{A'}, 1_A)$ is the identity function $1_{\mathrm{Hom}_\mathsf{C}(A',A)}$. It is, since if $m \in \mathrm{Hom}_\mathsf{C}(A', A)$,

$$\mathrm{Hom}_\mathsf{C}(1_{A'}, 1_A) \colon m \mapsto 1_A \circ m \circ 1_{A'} = m.$$

Next, we have to check that compositions work the way they're supposed to. Suppose we have objects and morphisms like this:

$$\begin{array}{ccccc} A' & \xleftarrow{f'} & B' & \xleftarrow{g'} & C' \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \end{array}$$

where the primed stuff is in C^{op} and the unprimed stuff is in C. This can be viewed as three objects (A', A), etc. in $\mathsf{C}^{\mathrm{op}} \times \mathsf{C}$ and two morphisms (f', f) and (g', g) between them.

Okay, so we can compose (f', f) and (g', g) to get a morphism

$$(g', g) \circ (f', f) = (f' \circ g', g \circ f) \colon (A', A) \to (C', C).$$

Note that the order of the composition in the first argument has been turned around: this is expected since the first argument lives in C^{op}. To check that Hom_C handles compositions correctly, we need to verify that

$$\mathrm{Hom}_\mathsf{C}(g, g') \circ \mathrm{Hom}_\mathsf{C}(f, f') = \mathrm{Hom}_\mathsf{C}(f' \circ g', g \circ f).$$

Let $m \in \mathrm{Hom}_\mathsf{C}(A', A)$. We victoriously compute

$$\begin{aligned} [\mathrm{Hom}_\mathsf{C}(g, g') \circ \mathrm{Hom}_\mathsf{C}(f, f')](m) &= \mathrm{Hom}_\mathsf{C}(g, g')(f \circ m \circ f') \\ &= g \circ f \circ m \circ f' \circ g' \\ &= \mathrm{Hom}_\mathsf{C}(f' \circ g', g \circ f). \end{aligned}$$

Let's formalize this in a definition.

Definition 96 (hom functor)**.** Let C be a locally small category. The hom functor Hom_C is the functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}$$

which sends the object (A', A) to the set $\mathrm{Hom}_\mathsf{C}(A', A)$ of morphisms $A' \to A$, and the morphism $(f', f) \colon (A', A) \to (B', B)$ to the function

$$\mathrm{Hom}_\mathsf{C}(f', f) \colon \mathrm{Hom}_\mathsf{C}(A', A) \to \mathrm{Hom}_\mathsf{C}(B', B); \qquad m \mapsto f \circ m \circ f'.$$

Example 97. Hom functors give us a new way of looking at the universal property for products, coproducts, and exponentials.

Let C be a locally small category with products, and let X, A, $B \in \text{Obj}(\mathsf{C})$. Recall that the universal property for the product $A \times B$ allows us to exchange two morphisms

$$f_1 : X \to A \qquad \text{and} \qquad f_2 : X \to B$$

for a morphism

$$f : X \to A \times B.$$

We can also compose a morphism $g : X \to A \times B$ with the canonical projections

$$\pi_A : A \times B \to A \qquad \text{and} \qquad \pi_B : A \times B \to B$$

to get two morphisms

$$\pi_A \circ f : X \to A \qquad \text{and} \qquad \pi_B \circ f : X \to B.$$

This means that there is a bijection

$$\text{Hom}_{\mathsf{C}}(X, A \times B) \simeq \text{Hom}_{\mathsf{C}}(X, A) \times \text{Hom}_{\mathsf{C}}(X, B).$$

In fact this bijection is natural in X. That is to say, there is a natural bijection between the following functors $\mathsf{C}^{\text{op}} \to \mathsf{Set}$:

$$h_{A \times B} : X \to \text{Hom}_{\mathsf{C}}(X, A \times B) \qquad \text{and} \qquad h_A \times h_B : X \to \text{Hom}_{\mathsf{C}}(X, A) \times \text{Hom}_{\mathsf{C}}(X, B).$$

Let's prove naturality. Let

$$\Phi_X : \text{Hom}(X, A \times B) \to \text{Hom}(X, A) \times \text{Hom}(X, B); \qquad f \mapsto (\pi_A \circ f, \pi_B \circ f)$$

be the components of the above transformation, and let $g : X' \to X$. Naturality follows from the fact that the diagram below commutes.

$$\begin{tikzcd}
\text{Hom}(X, A \times B) \arrow[r, "\text{Hom}(g, 1_{A\times B})"] \arrow[d, "\Phi_X"'] & \text{Hom}(X', A \times B) \arrow[d, "\Phi_{X'}"] \\
\text{Hom}(X, A) \times \text{Hom}(X, B) \arrow[r, "\text{Hom}(g,1_A)\times\text{Hom}(g,1_B)"] & \text{Hom}(X', A) \times \text{Hom}(X', B)
\end{tikzcd}$$

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & f \circ g \arrow[d, mapsto] \\
(\pi_A \circ f, \pi_B \circ g) \arrow[r, mapsto] & (\pi_A \circ f \circ g, \pi_B \circ f \circ g)
\end{tikzcd}$$

The coproduct does a similar thing: it allows us to trade two morphisms

$$A \to X \qquad \text{and} \qquad B \to X$$

for a morphism

$$A \amalg B \to X,$$

and vice versa. Similar reasoning yields a natural bijection between the following functors $\mathsf{C} \rightsquigarrow \mathsf{Set}$:

$$h^{A \amalg B} \Rightarrow h^A \times h^B.$$

The exponential is a little bit more complicated: it allows us to trade a morphism

$$X \times A \to B$$

for a morphism

$$X \to B^A,$$

and vice versa. That is to say, we have a bijection between two functors $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$

$$X \mapsto \mathrm{Hom}_{\mathsf{C}}(X \times A, B) \qquad \text{and} \qquad X \mapsto \mathrm{Hom}_{\mathsf{C}}(X, B^A).$$

The fact that the functors are more complicated does not hurt us: the above bijection is still natural.

The hom functor as defined above is cute, but not a whole lot else. Things get interesting when we curry it.

Recall from computer science the concept of currying. Suppose we are given a function of two arguments, say $f(x, y)$. The idea of currying is this: if we like, we can view f as a family of functions of only one variable y, indexed by x:

$$h_x(y) = f(x, y).$$

We can even view h as a function which takes one argument x, and which returns a *function* h_x of one variable y.

This sets up a correspondence between functions $f : A \times B \to C$ and functions $h : A \to C^B$, where C^B is the set of functions $B \to C$. The map which replaces f by h is called *currying*. We can also go the other way (i.e. $h \mapsto f$), which is called *uncurrying*.

We have been intentionally vague about the nature of our function f, and what sort of arguments it might take; everything we have said also holds for, say, bifunctors.

In particular, it gives us two more ways to view our bifunctor $\mathrm{Hom}_{\mathsf{C}}$. We can fix any $A \in \mathrm{Obj}(\mathsf{C})$ and curry either argument. This gives us the following.

Definition 98 (curried hom functor)**.** Let C be a locally small category, $A \in \mathrm{Obj}(\mathsf{C})$. We can construct from the hom functor $\mathrm{Hom}_{\mathsf{C}}$

- a functor $h^A : \mathsf{C} \rightsquigarrow \mathsf{Set}$ which maps
  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(A, B)$, and
  - a morphism $f : B \to B'$ to a Set-function

    $$h^A(f) : \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{\mathsf{C}}(A, B'); \qquad m \mapsto f \circ m.$$

- a functor $h_A : \mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$ which maps
  - an object $B \in \mathrm{Obj}(\mathsf{C})$ to the set $\mathrm{Hom}_{\mathsf{C}}(B, A)$, and
  - a morphism $f : B \to B'$ to a Set-function

    $$h_A(f) : \mathrm{Hom}_{\mathsf{C}}(B', A) \to \mathrm{Hom}_{\mathsf{C}}(B, A); \qquad m \mapsto m \circ f.$$

3.2 Representable functors

Roughly speaking, a functor $\mathsf{C} \rightsquigarrow \mathsf{Set}$ is representable if it is

Definition 99 (representable functor)**.** Let C be a category. A functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{Set}$ is representable if there is an object $A \in \mathrm{Obj}(\mathsf{C})$ and a natural isomorphism (Definition 39)

$$\eta\colon \mathcal{F} \Rightarrow \begin{cases} h^A, & \text{if } \mathcal{F} \text{ is covariant} \\ h_A, & \text{if } \mathcal{F} \text{ is contravariant.} \end{cases}$$

Note 100. Since natural isomorphisms are invertible, we could equivalently define a representable functor with the natural isomorphism going the other way.

Example 101. Consider the forgetful functor $\mathcal{U}\colon \mathsf{Grp} \rightsquigarrow \mathsf{Set}$ which sends a group to its underlying set, and a group homomorphism to its underlying function. This functor is represented by the group $(\mathbb{Z}, +)$.

To see this, we have to check that there is a natural isomorphism η between $\mathcal{U}$ and $h^{\mathbb{Z}} = \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, -)$. That is to say, for each group G, there a Set-isomorphism (i.e. a bijection)

$$\eta_G\colon \mathcal{U}(G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$$

satisfying the naturality conditions in Definition 38.

First, let's show that there's a bijection by providing an injection in both directions. Pick some $g \in G$. Then there is a unique group homomorphism which sends $1 \mapsto g$, so we have an injection $\mathcal{U}(G) \hookrightarrow \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$.

Now suppose we are given a group homomorphism $\mathbb{Z} \to G$. This sends 1 to some element $g \in G$, and this completely determines the rest of the homomorphism. Thus, we have an injection $\mathrm{Hom}_{\mathsf{Grp}(\mathbb{Z},G)} \hookrightarrow \mathcal{U}(G)$.

All that is left is to show that η satisfies the naturality condition. Let F and H be groups, and $f\colon G \to H$ a homomorphism. We need to show that the following diagram commutes.

$$\begin{tikzcd} \mathcal{U}(G) \arrow[r, "\mathcal{U}(f)"] \arrow[d, "\eta_G"'] & \mathcal{U}(H) \arrow[d, "\eta_H"] \\ \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G) \arrow[r, "h^{\mathbb{Z}}(f)"] & \mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, H) \end{tikzcd}$$

The upper path from top left to bottom right assigns to each $g \in G$ the function $\mathbb{Z} \to G$ which maps $1 \mapsto f(g)$. Walking down η_G from $\mathcal{U}(G)$ to $\mathrm{Hom}_{\mathsf{Grp}}(\mathbb{Z}, G)$, g is mapped to the function $\mathbb{Z} \to G$ which maps $1 \mapsto g$. Walking right, we compose this function with f to get a new function $\mathbb{Z} \to H$ which sends $1 \mapsto f(g)$. This function is really the image of a homomorphism and is therefore unique, so the diagram commutes.

3.3 The Yoneda embedding

The Yoneda embedding is a very powerful tool which allows us to prove things about any locally small category by embedding that category into Set, and using the enormous amount of structure that Set has.

Definition 102 (Yoneda embedding)**.** Let C be a locally small category. The Yoneda embedding is the functor

$$\mathcal{Y}\colon \mathsf{C} \rightsquigarrow [\mathsf{C}^{\mathrm{op}}, \mathsf{Set}], \qquad A \mapsto h_A = \mathrm{Hom}_{\mathsf{C}}(-, A).$$

where $[\mathsf{C}^{\mathrm{op}}, \mathsf{Set}]$ is the category of functors $\mathsf{C}^{\mathrm{op}} \rightsquigarrow \mathsf{Set}$, defined in Definition 44.

Of course, we also need to say how $\mathcal{Y}$ behaves on morphisms. Let $f\colon A \to A'$. Then $\mathcal{Y}(f)$ will be a morphism from h_A to $h_{A'}$, i.e. a natural transformation $h_A \Rightarrow h_{A'}$. We can specify how it behaves by specifying its components $(\mathcal{Y}(f))_B$.

Plugging in definitions, we find that $(\mathcal{Y}(f))_B$ is a map $\mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathrm{Hom}_{\mathsf{C}}(B, A')$. But we know how to get such a map—we can compose all of the maps $B \to A$ with f to get maps $B \to A'$. So $\mathcal{Y}(f)$, as a natural transformation $\mathrm{Hom}_{\mathsf{C}}(-, A) \to \mathrm{Hom}_{\mathsf{C}}(-, A')$, simply composes everything in sight with f.

Theorem 103 (Yoneda lemma)**.** *Let $\mathcal{F}$ be a functor from C^{op} to Set. Let $A \in \mathrm{Obj}(\mathsf{C})$. Then there is a set-isomorphism (i.e. a bijection) η between the set $\mathrm{Hom}_{[\mathsf{C}^{op},\mathsf{Set}]}(h_A, \mathcal{F})$ of natural transformations $h_A \Rightarrow \mathcal{F}$ and the set $\mathcal{F}(A)$. Furthermore, this isomorphism is natural in A.*

Note 104. A quick admission before we begin: we will be sweeping issues with size under the rug in what follows by tacitly assuming that $\mathrm{Hom}_{[\mathsf{C}^{op},\mathsf{Set}]}(h_A, \mathcal{F})$ is a set. The reader will have to trust that everything works out in the end.

Proof. A natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ consists of a collection of Set-morphisms (that is to say, functions) $\Phi_B\colon \mathrm{Hom}_{\mathsf{C}}(B, A) \to \mathcal{F}(A)$, one for each $B \in \mathrm{Obj}(\mathsf{C})$. We need to show that to each element of $\mathcal{F}(A)$ there corresponds exactly one Φ. We will do this by showing that any Φ is completely determined by where Φ_A sends the identity morphism $1_A \in \mathrm{Hom}_{\mathsf{C}}(A, A)$, so there is exactly one natural transformation for each place Φ can send 1_A.

The proof that this is the case can be illustrated by the following commutative diagram.

$$\begin{tikzcd}
\mathrm{Hom}(A,A) \arrow[rrr, "h_A(f)"] \arrow[ddd, "\Phi_A"'] & & & \mathrm{Hom}(B,A) \arrow[ddd, "\Phi_B"] \\
& 1_A \arrow[r, mapsto] \arrow[d, mapsto] & f \arrow[d, mapsto] & \\
& a \arrow[r, mapsto] & (\mathcal{F}f)(a) \overset{!}{=} \Phi_B(f) & \\
\mathcal{F}(A) \arrow[rrr, "\mathcal{F}(f)"'] & & & \mathcal{F}(B)
\end{tikzcd}$$

Here's what the above diagram means. The natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ has a component $\Phi_A\colon \mathrm{Hom}_{\mathsf{C}}(A, A) \to \mathcal{F}(A)$, and Φ_A has to send the identity transformation 1_A *somewhere*. It can send it to any element of $\mathcal{F}(A)$; let's call $\Phi_A(1_A) = a$.

Now the naturality conditions force our hand. For any $B \in \mathrm{Obj}(\mathsf{C})$ and any $f\colon B \to A$, the naturality square above means that $\Phi_B(f)$ *has to* be equal to $(\mathcal{F}f)(a)$. We get no choice in the matter.

But this completely determines Φ! So we have shown that there is exactly one natural transformation for every element of $\mathcal{F}(A)$. We are done!

Well, almost. We still have to show that the bijection we constructed above is natural. To that end, let $f: B \to A$. We need to show that the following square commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, \mathcal{F}) \arrow[r, "\eta_A"] \arrow[d, "\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(f),\mathcal{F})"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(f)"] \\
\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_B, \mathcal{F}) \arrow[r, "\eta_B"] & \mathcal{F}(B)
\end{tikzcd}$$

This is notationally dense, and deserves a lot of explanation. The natural transformation $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(\mathcal{Y}(f), \mathcal{F})$ looks complicated, but it's really not since we know what the hom functor does: it takes every natural transformation $h_A \Rightarrow \mathcal{F}$ and pre-composes it with $\mathcal{Y}(f)$. The natural transformation η_A takes a natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$ and sends it to the element $\Phi_A(1_A) \in \mathcal{F}(A)$.

So, starting at the top left with a natural transformation $\Phi\colon h_A \Rightarrow \mathcal{F}$, we can go to the bottom right in two ways.

1. We can head down to $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_B, \mathcal{F})$, mapping

$$\Phi \mapsto \Phi \circ \mathcal{Y}(f),$$

then use η_B to map this to $\mathcal{F}(B)$:

$$\Phi \circ \mathcal{Y}(f) \mapsto (\Phi \circ \mathcal{Y}(f))_B(1_B).$$

We can simplify this right away. The component of the composition of natural transformations is the composition of the components, i.e.

$$(\Phi \circ \mathcal{Y}(f))_B(1_B) = (\Phi_B \circ \mathcal{Y}(f)_B)(1_B).$$

We also know how $\mathcal{Y}(f)$ behaves: it composes everything in sight with f.

$$\mathcal{Y}(f)(1_B) = f.$$

Thus, we have

$$(\Phi \circ \mathcal{Y}(f))(1_B) = \Phi_B(f).$$

2. We can first head to the right using η_A. This sends Φ to

$$\Phi_A(1_A) \in \mathcal{F}(A).$$

We can then map this to $\mathcal{F}(B)$ with f, getting

$$\mathcal{F}(f)(\Phi_A(1_A)).$$

The naturality condition is thus

$$(\mathcal{F}(f))(\Phi_A(1_A)) = \Phi_B(f),$$

which we saw above was true for any natural transformation Φ. □

Lemma 105. *The Yoneda embedding is fully faithful (Definition 31).*

Proof. We have to show that for all $A, B \in \mathrm{Obj}(\mathsf{C})$, the map

$$よ_{A,B} \colon \mathrm{Hom}_{\mathsf{C}}(A, B) \to \mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$$

is a bijection.

Fix $B \in \mathrm{Obj}(\mathsf{C})$, and consider the functor h_B. By the Yoneda lemma, there is a bijection between $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$ and $h_B(A)$. By definition,

$$h_B(A) = \mathrm{Hom}_{\mathsf{C}}(A, B).$$

But $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$ is nothing else but $\mathrm{Hom}_{[\mathsf{C}^{\mathrm{op}},\mathsf{Set}]}(h_A, h_B)$, so we are done. □

Note 106. We defined the Yoneda embedding to be the functor $A \mapsto h_A$; this is sometimes called the *contravariant Yoneda embedding*. We could also have studied to map A to h^A, called the *covariant Yoneda embedding*, in which case a slight modification of the proof of the Yoneda lemma would have told us that the map

$$\mathrm{Hom}_{\mathsf{C}^{\mathrm{op}}}(B, A) \to \mathrm{Hom}_{[\mathsf{C},\mathsf{Set}]}(h^A, h^B)$$

is a natural bijection. This is often called the *covariant Yoneda lemma.*

Here is a situation in which the Yoneda lemma is commonly used.

Corollary 107. *Let* C *be a locally small category. Suppose for all* $A \in \mathrm{Obj}(\mathsf{C})$ *there is a bijection*

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A, B')$$

which is natural in A*. Then* $B \simeq B'$

Proof. We have a natural isomorphism $h_B \Rightarrow h_{B'}$. Since the Yoneda embedding is fully faithful, it is injective on objects up to isomorphism (Lemma 33). Thus, since h_B and $h_{B'}$ are isomorphic, so must be $B \simeq B'$. □

3.4 Applications

We have now built up enough machinery to make the proofs of a great variety of things trivial, as long as we accept a few assertions.

It is possible, for example, to prove that the product is associative in *any* locally small category, just from the fact that it is associative in Set.

Lemma 108. *The Cartesian product is associative in* Set*. That is, there is a natural isomorphism* α *between* $(A \times B) \times C$ *and* $A \times (B \times C)$ *for any sets* A*,* B*, and* C*.*

Proof. The isomorphism is given by $\alpha_{A,B,C} \colon ((a, b), c) \mapsto (a, (b, c))$. This is natural because for functions like this,

$$A \xrightarrow{f} A'$$

$$B \xrightarrow{g} B'$$

$$C \xrightarrow{h} C'$$

the following diagram commutes.

$$\begin{tikzcd}
((a,b),c) \arrow[r, mapsto, "((f,g),h)"] \arrow[d, mapsto, "\alpha_{A,B,C}"'] & ((f(a),g(b)),h(c)) \arrow[d, mapsto, "\alpha_{A',B',C'}"] \\
(a,(b,c)) \arrow[r, mapsto, "(f,(g,h))"'] & (f(a),(g(b),h(c)))
\end{tikzcd}$$

□

Theorem 109. *In any locally small category* C *with products, there is a natural isomorphism* $(A \times B) \times C \simeq A \times (B \times C)$.

Proof. We have the following string of natural isomorphisms for any X, A, B, $C \in \mathrm{Obj}(\mathsf{C})$.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(X,(A\times B)\times C) &\simeq \mathrm{Hom}_{\mathsf{C}}(X,A\times B)\times \mathrm{Hom}_{\mathsf{C}}(X,C)\\
&\simeq (\mathrm{Hom}_{\mathsf{C}}(X,A)\times \mathrm{Hom}_{\mathsf{C}}(X,B))\times \mathrm{Hom}_{\mathsf{C}}(X,C)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X,A)\times (\mathrm{Hom}_{\mathsf{C}}(X,B)\times \mathrm{Hom}_{\mathsf{C}}(X,C))\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X,A)\times \mathrm{Hom}_{\mathsf{C}}(X,B\times C)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X,A\times (B\times C)).
\end{aligned}$$

Thus, by Corollary 107, $(A \times B) \times C \simeq A \times (B \times C)$. □

Note 110. By induction, all finite products are associative.

Theorem 111. *In any category with products* $\times$*, coproducts* $+$*, initial object* 0*, final object* 1*, and exponentials, we have the following natural isomorphisms.*

1. $A \times (B + C) \simeq (A \times B) + (A \times C)$
2. $C^{A+B} \simeq C^A \times C^B$.
3. $(C^A)^B \simeq C^{A\times B}$
4. $(A \times B)^C \simeq A^C \times B^C$
5. $C^0 \simeq 1$
6. $C^1 \simeq C$

Proof.

1. We have the following list of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(A\times(B+C),X) &\simeq \mathrm{Hom}_{\mathsf{C}}(B+C,X^A)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(B,X^A)\times \mathrm{Hom}_{\mathsf{C}}(C,X^A)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(B\times A,X)\times \mathrm{Hom}_{\mathsf{C}}(C\times A,X)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}((B\times A)+(C\times A),X).
\end{aligned}$$

2. We have the following list of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}(X,C^{A+B}) &\simeq \mathrm{Hom}(X\times(A+B),C)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}((X\times A)+(X\times B),C)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X\times A,C)\times \mathrm{Hom}_{\mathsf{C}}(X\times B,C)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X,C^A)\times \mathrm{Hom}_{\mathsf{C}}(X,C^B)\\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X,C^A\times C^B).
\end{aligned}$$

Etc. □

4 Limits

4.1 Limits and colimits

As we have seen, one often gets categorical concepts by 'categorifying' concepts from set theory. One example of this is the notion of a *diagram*, which is the categorical generalization of an indexed family.

Definition 112 (indexed family)**.** Let J and X be sets. A family of elements in X indexed by J is a function

$$x\colon J \to X; \qquad j \mapsto x_j.$$

To categorify this, one considers a functor from one category J, called the *index category*, to another category C. Using a functor from a category instead of a function from a set allows us to index the morphisms as well as the objects.

Definition 113 (diagram)**.** Let J and C be categories. A diagram of type J in C is a (covariant) functor

$$\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}.$$

One thinks of the functor $\mathcal{D}$ embedding the index category J into C.

Definition 114 (cone)**.** Let C be a category, J an index category, and $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ be a diagram. Let 1 be the category with one object and one morphism (Example 5) and $\mathcal{F}_X$ the functor $1 \rightsquigarrow \mathsf{C}$ which picks out $X \in \mathrm{Obj}(\mathsf{C})$ (see Example 29). Let $\mathcal{K}$ be the unique functor $\mathsf{J} \rightsquigarrow 1$.

$$\begin{tikzcd}
1 \arrow[dr, squiggly, "\mathcal{F}_X"] & \\
\mathsf{J} \arrow[u, squiggly, "\mathcal{K}"] \arrow[r, squiggly, "\mathcal{D}"] & \mathsf{C}
\end{tikzcd}$$

A cone of shape J from X is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a natural transformation

$$\varepsilon\colon \mathcal{F}_X \circ \mathcal{K} \Rightarrow \mathcal{D}.$$

That is to say, a cone to J is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a family of morphisms $\Phi_A\colon X \to \mathcal{D}(A)$ (one for each $A \in \mathrm{Obj}(\mathsf{J})$) such that for all $A, B \in \mathrm{Obj}(\mathsf{J})$ and all $f\colon A \to B$ the following diagram commutes.

$$\begin{tikzcd}
& X \arrow[dl, "\Phi_A"'] \arrow[dr, "\Phi_B"] & \\
\mathcal{D}(A) \arrow[rr, "\mathcal{D}(f)"] & & \mathcal{D}(B)
\end{tikzcd}$$

Note 115. Here is an alternate definition. Let Δ be the functor $\mathsf{C} \rightsquigarrow [\mathsf{J}, \mathsf{C}]$ (the category of functors $\mathsf{J} \rightsquigarrow \mathsf{C}$, see TODO) which assigns to each object $X \in \mathrm{Obj}(\mathsf{C})$ the constant functor $\Delta_X : \mathsf{J} \rightsquigarrow \mathsf{C}$, i.e. the functor which maps every object of J to X and every morphism to 1_X. A cone over $\mathcal{D}$ is then an object in the comma category (Definition 54) $(\Delta \downarrow \mathcal{D})$ given by the diagram

$$\mathsf{C} \overset{\Delta}{\rightsquigarrow} [\mathsf{J}, \mathsf{C}] \overset{\mathcal{D}}{\leftsquigarrow} \mathbf{1} \ .$$

The objects of this category are pairs (X, f), where $X \in \mathrm{Obj}(\mathsf{C})$ and $f \colon \Delta(X) \to \mathcal{D}$; that is to say, f is a natural transformation $\Delta_X \Rightarrow \mathcal{D}$.

This allows us to make the following definition.

Definition 116 (category of cones over a diagram)**.** Let C be a category, J an index category, and $\mathcal{D} \colon \mathsf{J} \to \mathsf{C}$ a diagram. The category of cones over $\mathcal{D}$ is the category $(\Delta \downarrow \mathcal{D})$.

Note 117. The alternate definition given in Note 115 not only reiterates what cones look like, but even prescribes what morphisms between cones look like. Let (X, Φ) be a cone over a diagram $\mathcal{D} \colon \mathsf{J} \rightsquigarrow \mathsf{C}$, i.e.

- an object in the category $(\Delta \downarrow \mathcal{D})$, i.e.
- a pair (X, Φ), where $X \in \mathrm{Obj}(\mathsf{C})$ and $\Phi \colon \Delta_X \Rightarrow \mathcal{D}$ is a natural transformation, i.e.
- for each $J \in \mathrm{Obj}(\mathsf{J})$ a morphism $\Phi_J \colon X \to \mathcal{D}(J)$ such that for any other object $J' \in \mathrm{Obj}(\mathsf{J})$ and any morphism $f \colon J \to J'$ the following diagram commutes.

$$\begin{tikzcd} & X \arrow[dl, "\Phi_J"'] \arrow[dr, "\Phi_{J'}"] & \\ \mathcal{D}(J) \arrow[rr, "\mathcal{D}(f)"] & & \mathcal{D}(J') \end{tikzcd}$$

This agrees with our previous definition of a cone.

Let (Y, Γ) be another cone over $\mathcal{D}$. Then a morphism $\Xi \colon (X, \Phi) \to (Y, \Gamma)$ is

- a morphism $\Xi \in \mathrm{Hom}_{(\Delta \downarrow \mathcal{D})}((X, \Phi), (Y, \Gamma))$, i.e.
- a natural transformation $\Xi \colon \Delta_X \Rightarrow \Delta_Y$ (i.e. a morphism $\Delta_X \to \Delta_Y$ in the category $[\mathsf{J}, \mathsf{C}]$) such that the diagram

  $$\begin{tikzcd} \Delta_X \arrow[rr, Rightarrow, "\Xi"] \arrow[dr, Rightarrow, "\Phi"'] & & \Delta_Y \arrow[dl, Rightarrow, "\Gamma"] \\ & \mathcal{D} & \end{tikzcd}$$

  commutes, i.e.
- a morphism $\xi \colon X \to Y$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the diagram

  $$\begin{tikzcd} X \arrow[rr, "\xi"] \arrow[dr, "\Phi_J"'] & & Y \arrow[dl, "\Gamma_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

  commutes.

Cocones are the dual notion to cones. We make the following definition.

Definition 118 (cocone)**.** A cocone over a diagram $\mathcal{D}$ is an object in the comma category $(\mathcal{D} \downarrow \Delta)$.

Definition 119 (category of cocones)**.** The category of cocones over a diagram $\mathcal{D}$ is the category $(\mathcal{D} \downarrow \Delta)$.

The categorical definitions of cones and cocones allow us to define limits and colimits succinctly.

Definition 120 (limits, colimits)**.** A limit of a diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ is a final object in the category $(\Delta \downarrow \mathcal{D})$. A colimit is an initial object in the category $(\mathcal{D} \downarrow \Delta)$.

Note 121. The above definition of a limit unwraps as follows. The limit of a diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ is a cone (X, Φ) over $\mathcal{D}$ such that for any other cone (Y, Γ) over $\mathcal{D}$, there is a unique map $\xi\colon Y \to X$ such that for each $J \in \mathrm{Obj}(\mathsf{J})$, the following diagram commutes.

$$\begin{tikzcd} Y \arrow[rr, "\xi"] \arrow[dr, "\Gamma_J"'] & & X \arrow[dl, "\Phi_J"] \\ & \mathcal{D}(J) & \end{tikzcd}$$

Notation 122. We often denote the limit over the diagram $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ by

$$\varprojlim \mathcal{D}.$$

Similarly, we will denote the colimit over $\mathcal{D}$ by

$$\varinjlim \mathcal{D}.$$

If there is notational confusion over which functor we are taking a (co)limit, we will add a dummy index:

$$\varprojlim_i \mathcal{D}_i.$$

Example 123. Here is a definition of the product $A \times B$ equivalent to that given in Example 71: it is the limit of the following somewhat trivial diagram.

$$\begin{tikzcd} A \arrow[loop left, "1_A"] & & B \arrow[loop right, "1_B"] \end{tikzcd}$$

Let us unwrap this definition. We are saying that the product $A \times B$ is a cone over A and B

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[dr, "\pi_B"] & \\ A & & B \end{tikzcd}$$

But not just any cone: a cone which is universal in the sense that any *other* cone factors through it uniquely.

$$\begin{tikzcd} & X \arrow[dl, "f_1"'] \arrow[d, "\exists! f"] \arrow[dr, "f_2"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

In fact, this allows us to generalize the product: the product of n objects $\prod_{i=1}^{n} A_i$ is the limit over diagram consisting of all the A_i with no morphisms between them.

Definition 124 (equalizer)**.** Let J be the category with objects and morphisms as follows. (The necessary identity arrows are omitted.)

$$\begin{tikzcd} J \arrow[r, shift left, "1"] \arrow[r, shift right, "2"'] & J' \end{tikzcd}$$

A diagram $\mathcal{D}$ of shape J in some category C looks like the following.

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

The equalizer of f and g is the limit of the diagram $\mathcal{D}$; that is to say, it is an object eq $\in$ Obj(C) and a morphism $e\colon \mathrm{eq} \to A$

$$\begin{tikzcd} \mathrm{eq} \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

such that for any *other* object Z and morphism $i\colon Z \to A$ such that $f \circ i = g \circ i$, there is a unique morphism $e\colon Z \to \mathrm{eq}$ making the following diagram commute.

$$\begin{tikzcd} Z \arrow[d, dashed, "\exists! u"'] \arrow[dr, "i"] & & \\ \mathrm{eq} \arrow[r, "e"'] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$$

4.2 Pullbacks and kernels

In what follows, C will be a category and A, B, etc. objects in Obj(C).

Definition 125 (pullback)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd} & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

A pullback of f along g (also called a pullback of g over f, sometimes notated $A \times_C B$) is a commuting square

$$\begin{tikzcd} U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} V \arrow[r, "t"] \arrow[d, "s"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"'] & C \end{tikzcd}$$

there is a unique morphism $h\colon V \to U$ such that the diagram

$$\begin{tikzcd}
V \arrow[drr, bend left, "t"] \arrow[dr, dashed, "\exists! h"] \arrow[ddr, bend right, "s"'] & & \\
& U \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Note 126. Here is another definition: the pullback $A \times_C B$ is the limit of the diagram

$$\begin{tikzcd}
& B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

This might at first seem odd; after all, don't we also need an arrow $A \times_C B \to C$? But this arrow is completely determined by the commutativity conditions, so it is superfluous.

Example 127. In Set, U is given (up to unique isomorphism) by

$$U = \left\{(a, b) \in A \times B \,\middle|\, f(a) = g(b)\right\}.$$

The morphisms p and q are given by the projections $p(a, b) = a$, $q(a, b) = b$.

To see that this really does satisfy the universal property, consider any other set V and functions $s\colon V \to A$ and $t\colon V \to B$ making the above diagram commute. Then for all $v \in V$, $f(s(v)) = t(g(v))$.

Now consider the map $V \to U$ sending v to $(s(v), t(v))$. This certainly makes the above diagram commute; furthermore, any other map from V to U would not make the diagram commute. Thus U and h together satisfy the universal property.

Lemma 128. *Let $f : X \to Y$ be a monomorphism. Then any pullback of f is a monomorphism.*

Proof. Suppose we have the following pullback square.

$$\begin{tikzcd}
X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

Our aim is to show that p_2 is a monomorphism.

Suppose we are given an object Z and two morphisms $\alpha, \beta\colon Z \to X'$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "\alpha"] \arrow[dr, shift right, "\beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

We can compose α and β with p_2. Suppose that these agree, i.e.

$$p_2 \circ \alpha = p_2 \circ \beta.$$

We will be done if we can show that this implies that $p_1 = p_2$.

We can compose with g to find that

$$g \circ p_2 \circ \alpha = g \circ p_2 \circ \beta.$$

but since the pullback square commutes, we can replace $g \circ p_2$ by $f \circ p_1$.

$$f \circ p_1 \circ \alpha = f \circ p_1 \circ \beta.$$

since f is a monomorphism, this implies that

$$p_1 \circ \alpha = p_1 \circ \beta.$$

Now forget α and β for a minute. We have constructed a commuting square as follows.

$$\begin{tikzcd}
Z \arrow[drr, bend left, "p_1\circ\alpha=p_1\circ\beta"] \arrow[ddr, bend right, "p_2\circ\alpha=p_2\circ\beta"'] & & \\
& X' \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, hook, "f"] \\
& Y' \arrow[r, "g"] & Y
\end{tikzcd}$$

the universal property for pullbacks tells us that there is a unique morphism $Z \to X'$ making the diagram commute. But either α or β will do! So $\alpha = \beta$. □

Definition 129 (kernel of a morphism)**.** Let C be a category with an initial object (Definition 59) 0 and pullbacks. The kernel $\ker(f)$ of a morphism $f : A \to B$ is the pullback along f of the unique morphism $0 \to B$.

$$\begin{tikzcd}
\ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

That is to say, the kernel of f is a pair $(\ker(f), \iota)$, where $\ker(f) \in \mathrm{Obj}(\mathsf{C})$ and $\iota\colon \ker(f) \to A$ which satisfies the above universal property.

Note 130. Although the kernel of a morphism f is a pair $(\ker(f), \iota)$ as described above, we will sometimes sloppily say that the object $\ker(f)$ is the kernel of f, especially when the the morphism ι is obvious or understood. Such abuses of terminology are common; one occasionally even sees the morphism ι being called the kernel of f.

Example 131. In Vect_k, the initial object is the zero vector space $\{0\}$. For any vector spaces V and W and any linear map $f : V \to W$, the kernel of f is the pair $(\ker(f), \iota)$ where $\ker(f)$ is the vector space

$$\ker(f) = \left\{ v \in V \,\middle|\, f(v) = 0 \right\}$$

and ι is the obvious injection $\ker(f) \to V$.

Lemma 132. *Let $f : A \to B$, and let $(\iota, \ker(f))$ be the kernel of f. Then ι is a monomorphism (Definition 16).*

Proof. Suppose we have an object $Z \in \mathrm{Obj}(\mathsf{C})$ and two morphisms $g_1, g_2 : Z \to \ker(f)$. We have the following diagram.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Further suppose that $\iota \circ g_1 = \iota \circ g_2$.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "g_1"] \arrow[dr, shift right, "g_2"'] \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1 = \iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Now pretend that we don't know about g_1 and g_2.

$$\begin{tikzcd}
Z \arrow[drr, bend left] \arrow[ddr, bend right, "\iota\circ g_1 = \iota\circ g_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

The universal property for kernels tells us that there is a unique map $Z \to \ker(f)$ making the above diagram commute. But since g_1 and g_2 both make the diagram commute, g_1 and g_2 must be the same map, i.e. $g_1 = g_2$. □

4.3 Pushouts and cokernels

Pushouts are the dual notion to pullbacks.

Definition 133 (pushouts)**.** Let f, g be morphisms as follows.

$$\begin{tikzcd}
C \arrow[r, "g"] \arrow[d, "f"'] & B \\
A &
\end{tikzcd}$$

The <u>pushout of f along g</u> (or g along f) is a commuting square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \\ A \arrow[r, "p"'] & U \end{tikzcd}$$

such that for any other commuting square

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "t"] \\ A \arrow[r, "s"'] & V \end{tikzcd}$$

there exists a unique morphism $h \colon U \to V$ such that the diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

commutes.

Note 134. As with pullbacks, we can also define a pushout as the colimit of the following diagram.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \\ A & \end{tikzcd}$$

Example 135. Let us construct the pushout in Set. If we ignore the object C and the morphisms f and g, we discover that U must satisfy the universal property of the coproduct of A and B.

$$\begin{tikzcd} & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

Let us therefore make the ansatz that $U = A \amalg B = A \sqcup B$ and see what happens when we add C, f, and g back in.

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[d, "q"] \arrow[ddr, bend left, "t"] & \\ A \arrow[r, "p"'] \arrow[drr, bend right, "s"'] & U \arrow[dr, dashed, "\exists! h"] & \\ & & V \end{tikzcd}$$

In doing so, we find that the square A-C-B-U must also commute, i.e. we must have that $(q \circ g)(c) = (p \circ f)(c)$ for all $c \in C$. Since p and q are just inclusions, we see that

$$U = A \amalg B/\sim,$$

where $\sim$ is the equivalence relation generated by the relations $f(c) \sim g(c)$ for all $c \in C$.

Definition 136 (cokernel of a morphism)**.** Let C be a category with terminal object 1. The cokernel of a morphism $f\colon A \to B$ is the pushout of f along the unique morphism $A \to 1$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "\pi"] \\ 1 \arrow[r] & \operatorname{coker}(f) \end{tikzcd}$$

Example 137. In Vect_k, the terminal object is the vector space $\{0\}$. If V and W are k-vector spaces and f is a linear map $V \to W$, then $\operatorname{coker}(f)$ is

$$W/\sim,$$

where $\sim$ is the relation generated by $f(v) \sim 0$ for all $v \in V$. But this relation is exactly the one which mods out by $\operatorname{im}(f)$, so $\operatorname{coker}(f) = W/\operatorname{im}(f)$.

Lemma 138. *For any morphism $f\colon A \to B$, the canonical projection $\pi\colon B \to \operatorname{coker}(f)$ is an epimorphism.*

Proof. The proof is dual to the proof that the canonical injection ι is mono (Lemma 132). □

Definition 139 (normal monomorphism)**.** A monomorphism (Definition 16) $f\colon A \to B$ is normal if it the kernel of some morphism. To put it more plainly, f is normal if there exists an object C and a morphism $g\colon B \to C$ such that (A, f) is the kernel of g.

$$A \xrightarrow{f} B \xrightarrow{g} C$$

Example 140. In Vect_k, monomorphisms are injective linear maps (Example 19). If f is injective then sequence

$$\{0\} \longrightarrow V \xrightarrow{f} W \xrightarrow{\pi} W/\operatorname{im}(f), \longrightarrow \{0\}$$

is exact, and we always have that $\operatorname{im}(f) = \ker(\pi)$. Thus in Vect_k, every monomorphism is normal.

Definition 141 (conormal epimorphism)**.** An epimorphism $f\colon A \to B$ is conormal if it is the cokernel of some morphism. That is to say, if there exists an object C and a morphism $g\colon C \to A$ such that (B, f) is the cokernel of g.

Example 142. In Vect_k, epimorphisms are surjective linear maps. If $f\colon V \to W$ is a surjective linear map, then the sequence

$$\{0\} \longrightarrow \ker(f) \xrightarrow{\iota} V \xrightarrow{f} W \longrightarrow \{0\}$$

is exact. But then $\operatorname{im}(\iota) = \ker(f)$, so f is conormal. Thus in Vect_k, every epimorphism is conormal.

Note 143. To show that in our proofs that in Vect_k monomorphisms were normal and epimorphisms were conormal, we showed that monomorphisms were the kernels of their cokernels, and epimorphisms were the cokernels of their kernels. This will be a general feature of Abelian categories.

Definition 144 (binormal category)**.** A category is binormal if all monomorphisms are normal and all epimorphisms are conormal.

Example 145. As we have seen, Vect_k is binormal.

4.4 A necessary and sufficient condition for the existence of finite limits

In this section we prove a simple criterion to check that a category has all finite limits (i.e. limits over diagrams with a finite number of objects and a finite number of morphisms). The idea is as follows.

In general, adding more objects to a diagram makes the limit over it larger, and adding more morphisms makes the limit smaller. In the extreme case in which the only morphisms are the identity morphisms, the limit is simply the categorical product. As we add morphisms, roughly speaking, we have to get rid of the parts of the product which are preventing the necessary triangles from commuting. Thus, to make the universal cone over a diagram, we can start with the product, and cut out the bare minimum we need to make everything commute: the way to do that is with an equalizer.

The following proof was adapted from [25].

Theorem 146. *Let* C *be a category. Then if* C *has all finite products and equalizers,* C *has all finite limits.*

Proof. Let $\mathcal{D}\colon \mathsf{J} \rightsquigarrow \mathsf{C}$ be a finite diagram. We want to prove that $\mathcal{D}$ has a limit; we will do this by constructing a universal cone over it, i.e.

- an object $L \in \mathrm{Obj}(\mathsf{C})$, and
- for each $j \in \mathrm{Obj}(\mathsf{J})$, a morphism $P_j\colon L \to \mathcal{D}(j)$

such that

1. for any $i, j \in \mathrm{Obj}(\mathsf{J})$ and any $\alpha\colon i \to j$ the following diagram commutes,

$$\begin{tikzcd} & L \arrow[dl, "P_i"'] \arrow[dr, "P_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

   and

2. for any other object $L' \in \mathrm{Obj}(\mathsf{C})$ and family of morphisms $Q_j\colon L' \to \mathcal{D}(j)$ which make the diagrams

$$\begin{tikzcd} & L' \arrow[dl, "Q_i"'] \arrow[dr, "Q_j"] & \\ \mathcal{D}(i) \arrow[rr, "\mathcal{D}(\alpha)"] & & \mathcal{D}(j) \end{tikzcd}$$

commute for all i, j, and α, there is a unique morphism $f\colon L' \longrightarrow L$ such that $Q_j = P_j \circ f$ for all $j \in \mathrm{Obj}(\mathsf{J})$.

Denote by $\mathrm{Mor}(\mathsf{J})$ the set of all morphisms in J. For any $\alpha \in \mathrm{Hom}_{\mathsf{J}}(i, j) \subseteq \mathrm{Mor}(\mathsf{J})$, let $\mathrm{dom}(\alpha) = i$ and $\mathrm{cod}(\alpha) = j$.

Consider the following finite products:

$$A = \prod_{j \in \mathrm{Obj}(\mathsf{J})} \mathcal{D}(j) \qquad \text{and} \qquad B = \prod_{\alpha \in \mathrm{Mor}(\mathsf{J})} \mathcal{D}(\mathrm{cod}(\alpha)).$$

From the universal property for products, we know that we can construct a morphism $f\colon A \longrightarrow B$ by specifying a family of morphisms $f_\alpha\colon A \longrightarrow \mathcal{D}(\mathrm{cod}(\alpha))$, one for each $\alpha \in \mathrm{Mor}(\mathsf{J})$. We will define two morphisms $R, S\colon A \longrightarrow B$ in this way:

$$R_\alpha = \pi_{\mathcal{D}(\mathrm{cod}(\alpha))}; \qquad S_\alpha = \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(\mathrm{dom}(\alpha))}.$$

Now let $e\colon L \longrightarrow A$ be the equalizer of R and S (we are guaranteed the existence of this equalizer by assumption). Further, define $P_j\colon L \longrightarrow \mathcal{D}(j)$ by

$$P_j = \pi_{\mathcal{D}(j)} \circ e$$

for all $j \in \mathrm{Obj}(\mathsf{J})$.

The claim is that L together with the P_j is the limit of $\mathcal{D}$. We need to verify conditions 1 and 2 on L and P_j listed above.

1. We need to show that for all i, $j \in \mathrm{Obj}(\mathsf{J})$ and all $\alpha\colon i \longrightarrow j$, we have the equality $\mathcal{D}(\alpha) \circ P_i = P_j$. Now, for every $\alpha\colon i \longrightarrow j$ we have

$$\begin{aligned} \mathcal{D}(\alpha) \circ P_i &= \mathcal{D}(\alpha) \circ \pi_{\mathcal{D}(i)} \circ e \\ &= S_\alpha \circ e \\ &= R_\alpha \circ e \\ &= \pi_{\mathcal{D}(j)} \circ e \\ &= P_j. \end{aligned}$$

2. We need to show that for any other $L' \in \mathrm{Obj}(\mathsf{C})$ and any other family of morphisms $Q_j\colon L' \longrightarrow \mathcal{D}(j)$ such that for all $\alpha\colon i \longrightarrow j$, $Q_j = \mathcal{D}(\alpha) \circ Q_i$, there is a unique morphism $h\colon L' \longrightarrow L$ such that $Q_j = P_j \circ h$ for all $j \in \mathrm{Obj}(\mathsf{J})$. Suppose we are given such an L' and Q_j.

   The universal property for products allows us to construct from the family of morphisms Q_j a morphism $Q\colon L' \longrightarrow A$ such that $Q_j = \pi_{\mathcal{D}(j)} \circ Q$. Now, for any $\alpha\colon i \longrightarrow j$,

$$\begin{aligned} R_\alpha \circ Q &= \pi_{\mathcal{D}(j)} \circ Q \\ &= Q_j \\ &= \mathcal{D}(\alpha) \circ Q_i \\ &= \mathcal{D}(\alpha) \circ \pi_i \circ Q \\ &= S_\alpha \circ Q. \end{aligned}$$

Thus, $Q \colon L' \to A$ equalizes R and S. But the universal property for equalizers guarantees us a unique morphism $h \colon L' \to L$ such that $Q = P \circ h$. We can compose both sides of this equation on the left with $\pi_{\mathcal{D}(j)}$ to find

$$\pi_{\mathcal{D}(j)} \circ Q = \pi_{\mathcal{D}(j)} \circ P \circ h,$$

i.e.

$$Q_j = P_j \circ h$$

as required.

□

4.5 The hom functor preserves limits

Theorem 147. *Let* C *be a locally small category. The hom functor* $\mathrm{Hom}_{\mathsf{C}} \colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}$ *preserves limits in the second argument, i.e. for* $\mathcal{D} \colon \mathsf{J} \rightsquigarrow \mathsf{C}$ *a diagram in* C *we have a natural isomorphism*

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}),$$

where the limit on the RHS is over the hom-set diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \rightsquigarrow \mathsf{Set}.$$

Proof. Let L be the limit over the diagram $\mathcal{D}$. Then for any map $f \colon Y \to L$, there is a cone from Y to $\mathcal{D}$ by composition, and for any cone with tip Y over $\mathcal{D}$ we get a map $f \colon Y \to L$ from the universal property of limits. Thus, there is a bijection

$$\mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) \simeq \mathrm{Cones}(Y, \mathcal{D}),$$

which is natural in Y since the diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{C}}(Y, \varprojlim \mathcal{D}) & \xrightarrow{(-)\circ f} & \mathrm{Hom}_{\mathsf{C}}(Z, \varprojlim \mathcal{D}) \\ \downarrow & & \downarrow \\ \mathrm{Cones}(Y, \mathcal{D}) & \xrightarrow{(-)\circ f} & \mathrm{Cones}(Z, \mathcal{D}) \end{array}$$

trivially commutes. We will be done if we can show that there is also a natural isomorphism

$$\mathrm{Cones}(Y, \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}).$$

Let us understand the elements of the set $\varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D})$. The diagram

$$\mathrm{Hom}_{\mathsf{C}}(Y, -) \circ \mathcal{D} \colon \mathsf{J} \rightsquigarrow \mathsf{Set}$$

maps $J \in \mathrm{Obj}(\mathsf{J})$ to $\mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J)) \in \mathrm{Obj}(\mathsf{Set})$. A universal cone over this diagram is a set S together with, for each $J_i \in \mathrm{Obj}(\mathsf{J})$, a function

$$f_i \colon S \to \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i))$$

such that for each $\alpha \in \mathrm{Hom}_{\mathsf{J}}(J_i, J_j)$, the diagram

$$\begin{tikzcd} & S \arrow[dl, "f_i"'] \arrow[dr, "f_j"] & \\ \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_i)) \arrow[rr, "\mathcal{D}(\alpha)\circ(-)"'] & & \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}(J_j)) \end{tikzcd}$$

commutes.

Now pick any element $s \in S$. Each f_i maps this to a function $Y \to \mathcal{D}(J_i)$ which makes the diagram

$$\begin{tikzcd} & Y \arrow[dl, "f_i(s)"'] \arrow[dr, "f_j(s)"] & \\ \mathcal{D}(J_i) \arrow[rr, "\mathcal{D}(\alpha)"'] & & \mathcal{D}(J_j) \end{tikzcd}$$

commute. But this is exactly an element of $\mathrm{Cones}(Y, \mathcal{D})$. Conversely, every element of $\mathrm{Cones}(Y, \mathcal{D})$ gives us an element of S, so we have a bijection

$$\mathrm{Cones}(Y, \mathcal{D}) \simeq \varprojlim \mathrm{Hom}_{\mathsf{C}}(Y, \mathcal{D}),$$

which is natural as required. □

Corollary 148. *The first slot of the hom functor turns colimits into limits. That is,*

$$\mathrm{Hom}_{\mathsf{C}}(\varinjlim \mathcal{D}, Y) \simeq \varprojlim \mathrm{Hom}_{C}(\mathcal{D}, Y).$$

Proof. Dual to that of Theorem 147. □

4.6 Filtered colimits and ind-objects

Definition 149 (preorder)**.** Let S be a set. A preoreder on S is a binary relation $\leq$ which is

1. reflexive ($a \leq a$)
2. transitive (if $a \leq b$ and $b \leq c$, then $a \leq c$)

A preorder is said to be directed if for any two objects a and b, there exists an 'upper bound,' i.e. an object r such that $a \leq r$ and $b \leq r$.

Filtered categories are a generalization of filtered preorders.

Definition 150 (filtered category)**.** A category J is filtered if

- for each pair of objects $J, J' \in \mathrm{Obj}(\mathsf{J})$, there exists an object K and morphisms $J \to K$ and $J' \to K$.

$$\begin{tikzcd} J \arrow[dr] & \\ & K \\ J' \arrow[ur] & \end{tikzcd}$$

  That is, every diagram with two objects and no morphisms is the base of a cocone.

- For every pair of morphisms $i, j \colon J \to J'$, there exists an object K and a morphism $f \colon J' \to K$ such that $f \circ i = f \circ j$, i.e. the following diagram commutes.

$$J \underset{j}{\overset{i}{\rightrightarrows}} J' \xrightarrow{f} K$$

  That is, every diagram of the form $\bullet \rightrightarrows \bullet$ is the base of a cocone.

Definition 151 (filtered colimit)**.** A filtered colimit is a colimit over a diagram $\mathcal{D} \colon \mathsf{J} \to \mathsf{C}$, where J is a filtered category.

We now define the so-called *category of inductive objects* (or simply *ind-objects*).

Definition 152 (category of ind-objects)**.** Let C be a category. We define the category of ind-objects of C, denote $\mathsf{Ind}(\mathsf{C})$ as follows.

- The objects $F \in \mathrm{Obj}(\mathsf{Ind}(\mathsf{C}))$ are defined to be filtered colimits of objects of diagrams $\mathcal{F} \colon \mathsf{D} \rightsquigarrow \mathsf{C}$.
- For two objects $F = \lim_{\to d} \mathcal{F}_d$ and $G = \lim_{\to e} \mathcal{G}_e$, the morphisms $\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G)$ are defined to be the set

$$\mathrm{Hom}_{\mathsf{Ind}(\mathsf{C})}(F, G) = \lim_{\leftarrow d} \lim_{\to e} \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}_d, \mathcal{G}_e).$$

Note 153. There is a fully faithful embedding $\mathsf{C} \hookrightarrow \mathsf{Ind}(\mathsf{C})$ which exhibits any object A as the colimit over the trivial diagram $\bullet \circlearrowright$.

The importance of the category of ind-objects can be seen in the following example.

Example 154. Let V be an infinite-dimensional vector space over some field k. Then V can be realized as an object in the category $\mathsf{Ind}(\mathsf{FinVect})$.

In fact, there is an equivalence of categories $\mathsf{Vect}_k \simeq \mathsf{Ind}(\mathsf{FinVect}_k)$. Similarly, there is an equivalence of categories $\mathsf{SVect}_k \simeq \mathsf{Ind}(\mathsf{FinSVect}_k)$

Note 155. This is stated without proof as Example 3.39 of [11]. I haven't been able to find a real source for it.

5 Adjunctions

Consider the following functors:

- $\mathcal{U}\colon \mathsf{Grp} \rightsquigarrow \mathsf{Set}$, which sends a group to its underlying set, and
- $\mathcal{F}\colon \mathsf{Set} \rightsquigarrow \mathsf{Grp}$, which sends a set to the free group on it.

The functors $\mathcal{U}$ and $\mathcal{F}$ are dual in the following sense: $\mathcal{U}$ is the most efficient way of moving from Grp to Set since all groups are in particular sets; $\mathcal{F}$ might be thought of as providing the most efficient way of moving from Set to Grp. But how would one go about formalizing this?

Well, these functors have the following property. Let S be a set, G be a group, and let $f\colon S \to \mathcal{U}(G)$, $s \mapsto f(s)$ be a set-function. Then there is an associated group homomorphism $\tilde{f}\colon \mathcal{F}(S) \to G$, which sends $s_1s_2 \ldots s_n \mapsto f(s_1s_2 \ldots s_n) = f(s_1)\cdots f(s_n)$. In fact, $\tilde{f}$ is the unique homomorphism $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Similarly, for every group homomorphism $g\colon \mathcal{F}(S) \to G$, there is an associated function $S \to \mathcal{U}(G)$ given by restricting g to S. In fact, this is the unique function $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Thus for each $f \in \mathrm{Hom}_{\mathsf{Grp}}(S, \mathcal{U}(G))$ we can construct an $\tilde{f} \in \mathrm{Hom}_{\mathsf{Set}}(\mathcal{F}(S), G)$, and vice versa.

Let us add some mathematical scaffolding to the ideas explored above. We build two functors $\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp} \rightsquigarrow \mathsf{Set}$ as follows.

1. Our first functor maps the object $(S, G) \in \mathrm{Obj}(\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp})$ to the hom-set $\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G)$, and a morphism $(\alpha, \beta)\colon (S, G) \to (S', G')$ to a function

$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S'), G'); \qquad m \mapsto \mathcal{F}(\alpha) \circ m \circ \beta$$

2. Our second functor maps (S, G) to $\mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$, and (α, β) to

$$m \mapsto \alpha \circ m \circ \mathcal{U}(\beta).$$

We can define a natural isomorphism Φ between these functors with components

$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G)); \qquad f \to \tilde{f}.$$

This mathematical structure turns out to be a recurring theme in the study of categories, called an *adjuction*. We have encountered it before: we saw in Example 97 that there was a natural bijection between

$$\mathrm{Hom}_{\mathsf{C}}(X \times (-), B) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}(X, B^{(-)}).$$

Definition 156 (hom-set adjunction). Let C, D be categories and $\mathcal{F}$, $\mathcal{G}$ functors as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, squiggly, "\mathcal{G}"] \end{tikzcd}$$

We say that $\mathcal{F}$ is left-adjoint to $\mathcal{G}$ (or equivalently $\mathcal{G}$ is right-adjoint to $\mathcal{F}$) and write $\mathcal{F} \dashv \mathcal{G}$ if there is a natural isomorphism

$$\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-)),$$

which fits between $\mathcal{F}$ and $\mathcal{G}$ like this.

$$\begin{tikzcd} \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \arrow[r, squiggly, bend left, "\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-),-)"] \arrow[r, squiggly, bend right, "\mathrm{Hom}_{\mathsf{D}}(-,\mathcal{G}(-))"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathsf{Set} \end{tikzcd}$$

The natural isomorphsim amounts to a family of bijections

$$\Phi_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$$

which satisfies the coherence conditions for a natural transformation.

Here are two equivalent definitions which are often used.

Definition 157 (unit-counit adjunction)**.** We say that two functors $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ form a unit-counit adjunction if there are two natural transformations

$$\eta\colon 1_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}, \qquad \text{and} \qquad \varepsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow 1_{\mathsf{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$\begin{tikzcd} \mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"] \arrow[dr, Rightarrow, "1_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, Rightarrow, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{G} \arrow[r, Rightarrow, "\eta\mathcal{G}"] \arrow[dr, Rightarrow, "1_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, Rightarrow, "\mathcal{G}\varepsilon"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $1_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$. We can draw it like this.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "1_{\mathsf{C}}"] \arrow[r, squiggly, bend right, "\mathcal{G}\circ\mathcal{F}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{C} \end{tikzcd}.$$

Analogously, we can draw ε like this.

$$\begin{tikzcd} \mathsf{D} \arrow[r, squiggly, bend left, "\mathcal{F}\circ\mathcal{G}"] \arrow[r, squiggly, bend right, "1_{\mathsf{D}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathsf{D} \end{tikzcd}.$$

We can arrange these artfully like so.

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] \arrow[rr, squiggly, bend left=60, "1_{\mathsf{C}}"] \arrow[rr, squiggly, bend left=30, "\mathcal{G}\circ\mathcal{F}"'] & \mathsf{D} \arrow[r, squiggly, "\mathcal{G}"] \arrow[rr, squiggly, bend right=30, "\mathcal{F}\circ\mathcal{G}"] \arrow[rr, squiggly, bend right=60, "1_{\mathsf{D}}"'] & \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \end{tikzcd}$$

Notice that we haven't actually *done* anything; this diagram is just the diagrams for the unit and counit, plus some extraneous information.

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$\begin{tikzcd}
C \arrow[r, squiggly, "\mathcal{F}"] \arrow[rrr, squiggly, bend left=50, "\mathcal{F}"] \arrow[rrr, squiggly, bend left=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"'] \arrow[rrr, squiggly, bend right=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"] \arrow[rrr, squiggly, bend right=50, "\mathcal{F}"'] & D \arrow[r, squiggly, "\mathcal{G}"] \arrow[r, phantom, shift left=6ex, "\Downarrow \mathcal{F}\eta"] \arrow[r, phantom, shift right=6ex, "\Downarrow \varepsilon\mathcal{F}"] & C \arrow[r, squiggly, "\mathcal{F}"] & D
\end{tikzcd}$$

then consolidate to get this:

$$\begin{tikzcd}
C \arrow[r, squiggly, bend left=60, "\mathcal{F}"] \arrow[r, squiggly, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"] \arrow[r, squiggly, bend right=60, "\mathcal{F}"'] \arrow[r, phantom, shift left=3ex, "\Downarrow \mathcal{F}\eta"] \arrow[r, phantom, shift right=3ex, "\Downarrow \varepsilon\mathcal{F}"] & D
\end{tikzcd}$$

We can then take the composition $\varepsilon\mathcal{F} \circ \mathcal{F}\eta$ to get a natural transformation $\mathcal{F} \Rightarrow \mathcal{F}$

$$\begin{tikzcd}
C \arrow[r, squiggly, bend left=50, "\mathcal{F}"] \arrow[r, squiggly, bend right=50, "\mathcal{F}"'] \arrow[r, phantom, "\Downarrow \varepsilon\mathcal{F}\circ\mathcal{F}\eta"] & D;
\end{tikzcd}$$

the first triangle diagram says that this must be the same as the identity natural transformation $1_{\mathcal{F}}$.

The second triangle diagram is analogous.

Lemma 158. *The functors $\mathcal{F}$ and $\mathcal{G}$ form a unit-counit adjunction if and only if they form a hom-set adjunction.*

Proof. Suppose $\mathcal{F}$ and $\mathcal{G}$ form a hom-set adjunction with natural isomorphism Φ. Then for any $A \in \mathrm{Obj}(\mathsf{C})$, we have $\mathcal{F}(A) \in \mathrm{Obj}(\mathsf{D})$, so Φ give us a bijection

$$\Phi_{A,\mathcal{F}(A)} \colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \to \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$

We don't know much in general about $\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A))$, but the category axioms tell us that it always contains $1_{\mathcal{F}(A)}$. We can use $\Phi_{A,\mathcal{F}(A)}$ to map this to

$$\Phi_{A,\mathcal{F}(A)}(1_{\mathcal{F}(A)}) \in \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$

Let's call $\Phi_{A,\mathcal{F}(A)}(1_{\mathcal{F}(A)}) = \eta_A$.

Similarly, if $B \in \mathrm{Obj}(\mathsf{D})$, then $\mathcal{G}(B) \in \mathrm{Obj}(\mathsf{C})$, so Φ gives us a bijection

$$\Phi_{\mathcal{G}(B),B} \colon \mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \to \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)).$$

Since $\Phi_{\mathcal{G}(B),B}$ is a bijection, it is invertible, and we can evaluate the inverse on $1_{\mathcal{G}(B)}$. Let's call

$$\Phi^{-1}_{\mathcal{G}(B),B}(1_{\mathcal{G}(B)}) = \varepsilon_B.$$

Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a manner reminiscent of the proof of the Yoneda lemma, we can express $\Phi_{A,B}$ in terms of η, and $\Phi^{-1}_{A,B}$ in terms of ε, for *any* A and B. Here's how this is done.

We use the naturality of Φ. We know that for any $A \in \mathrm{Obj}(\mathsf{C})$, $B \in \mathrm{Obj}(\mathsf{D})$, and $g\colon \mathcal{F}(A) \longrightarrow B$, the following diagram has to commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[r, "g\circ(-)"] \arrow[d, "\Phi_{A,\mathcal{F}(A)}"'] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \arrow[d, "\Phi_{A,B}"] \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)) \arrow[r, "\mathcal{G}(g)\circ(-)"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).
\end{tikzcd}$$

Let's start at the top left with $1_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A,B}(g)$, and from the bottom road we have $\mathcal{G}(g) \circ \eta_A$. The diagram commutes, so we have

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A.$$

Similarly, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \arrow[r, "(-)\circ\mathcal{F}(f)"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \\
\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)) \arrow[r, "(-)\circ f"] \arrow[u, "\Phi^{-1}_{\mathcal{G}(B),B}"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{A,B}"']
\end{tikzcd}$$

means that, for any $f : A \longrightarrow \mathcal{G}(B)$,

$$\Phi^{-1}_{A,B}(f) = \varepsilon_B \circ \mathcal{F}(f)$$

To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \mathrm{Obj}(\mathsf{C})$ and all $B \in \mathrm{Obj}(\mathsf{D})$,

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (1_{\mathcal{F}})_A \qquad \text{and} \qquad (\mathcal{G}\varepsilon)_B \circ (\eta\mathcal{G})_B = (1_{\mathcal{G}})_B.$$

We have

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi^{-1}_{A,\mathcal{F}(A)}(\eta_A) = 1_A = (1_{\mathcal{F}})_A$$

and

$$(\mathcal{G}_{\varepsilon})_B \circ (\eta\mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = 1_B = (1_{\mathcal{G}})_B.$$

□

Definition 159 (adjunct)**.** Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Then for each $A \in \mathrm{Obj}(\mathsf{C})$ $B \in \mathrm{Obj}(\mathsf{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \longrightarrow \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism $\tilde{f}$ is called the adjunct of f, and f is called the adjunct of $\tilde{f}$.

Lemma 160. *Let* C *and* D *be categories,* $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ *and* $\mathcal{G}, \mathcal{G}'\colon \mathsf{D} \rightsquigarrow \mathsf{C}$ *functors,*

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend right, "\mathcal{G}"'] \arrow[l, squiggly, bend left, "\mathcal{G}'"] \end{tikzcd}$$

and suppose that $\mathcal{G}$ *and* $\mathcal{G}'$ *are both right-adjoint to* $\mathcal{F}$. *Then there is a natural isomorphism* $\mathcal{G} \Rightarrow \mathcal{G}'$.

Proof. Since any adjunction is a hom-set adjunction, we have two isomorphisms

$$\Phi_{C,D}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \qquad \text{and} \qquad \Psi_{C,D}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D))$$

which are natural in both C and D. By Lemma 49, we can construct the inverse natural isomorphism

$$\Phi^{-1}_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D),$$

and compose it with Ψ to get a natural isomorphsim

$$(\Psi \circ \Phi^{-1})_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)).$$

Thus for any morphism $f\colon D \to E$, the following diagram commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C{,}\mathcal{G}(f))"] \arrow[d, "(\Psi\circ\Phi^{-1})_{C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(E)) \arrow[d, "(\Psi\circ\Phi^{-1})_{C,E}"] \\ \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C{,}\mathcal{G}'(f))"] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(E)) \end{tikzcd}$$

But the fully faithfulness of the Yoneda embedding tells us that there exist isomorphisms μ_D and μ_E making the following diagram commute,

$$\begin{tikzcd} \mathcal{G}(D) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\mu_D"'] & \mathcal{G}(E) \arrow[d, "\mu_E"] \\ \mathcal{G}'(D) \arrow[r, "\mathcal{G}'(f)"] & \mathcal{G}'(E) \end{tikzcd}$$

and taking the collection of all such $\mu_{(-)}$ gives us a natural isomorphism $\mu\colon \mathcal{G} \Rightarrow \mathcal{G}'$. □

Note 161. We do not give very many examples of adjunctions now because of the frequency with which category theory graces us with them. Howver, it is worth mentioning a specific class of adjunctions: the so-called *free-forgetful adjunctions*. There are many *free* objects in mathematics: free groups, free modules, free vector spaces, free categories, etc. These are all unified by the following property: the functors defining them are all left adjoints.

Let us take a specific example: the free vector space over a set. This takes a set S and constructs a vector space which has as a basis the elements of S.

There is a forgetful functor $\mathcal{U}\colon \mathsf{Vect}_k \rightsquigarrow \mathsf{Set}$ which takes any set and returns the set underlying it. There is a functor $\mathcal{F}\colon \mathsf{Set} \rightsquigarrow \mathsf{Vect}_k$, which takes a set and returns the free vector space on it. It turns out that there is an adjunction $\mathcal{F} \dashv \mathcal{U}$.

And this is true of any free object! (In fact by definition.) In each case, the functor giving the free object is left adjoint to a forgetful functor.

Theorem 162. *Let* C *and* D *be categories and* $\mathcal{F}$ *and* $\mathcal{G}$ *functors as follows.*

$$\begin{tikzcd} \mathsf{C} \arrow[r, squiggly, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, squiggly, bend left, "\mathcal{G}"] \end{tikzcd}$$

Let $\mathcal{F} \dashv \mathcal{G}$ *be an adjunction. Then* $\mathcal{G}$ *preserves limits, i.e. if* $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ *is a diagram and* $\lim_{\leftarrow i} \mathcal{D}_i$ *exists in* C*, then*

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

Proof. We have the following chain of isomorphisms, natural in $Y \in \mathrm{Obj}(\mathsf{D})$.

$$\begin{aligned} \mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i)) &\simeq \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(Y), \lim_{\leftarrow i} \mathcal{D}_i) \\ &\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(Y), \mathcal{D}_i) && \left(\begin{matrix}\text{Hom functor commutes with} \\ \text{limits: Theorem 147}\end{matrix}\right) \\ &\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G} \circ \mathcal{D}_i) \\ &\simeq \mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i)). \end{aligned}$$

By the Yoneda lemma, specifically Corollary 107, we have a natural isomorphism

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

□

Corollary 163. *Any functor* $\mathcal{F}$ *which is a left-ajoint preserves colimits.*

Proof. Dual to the proof of Theorem 162. □

6 Monoidal categories

6.1 Structure

6.1.1 Basic definitions

Monoidal categories are the first ingredient in the categorification and generalization of the tensor product. Roughly speaking, the tensor product allows us to multiply two vector spaces to produce a new vector space. There are natural isomorphisms making this multiplication associative, and an 'identity vector space' given by the ground field regarded as a one-dimensional vector space over itself. This means that the tensor product gives the set of all vector spaces the structure of a monoid.

A monoidal category will be a category in which the objects have the structure of a monoid, i.e. there is a suitably defined 'multiplication' (usually written $\otimes$) which is unital (with unit 1) and associative. The prototypical example of categorical multiplication is the product (see Example 71), although it is far from the only one.

When put like this, monoidal categories don't sound like complicated entities, and indeed in practice they are not. However, there is a lot of subtlety in making associativity play well with multiplication by units. Since we are in a category, demanding that our multiplication be associative 'on the nose,' i.e. that, for example,

$$(V \otimes 1) \otimes (W \otimes T) \qquad \text{and} \qquad (V \otimes W) \otimes T$$

should be literally equal, is too draconian, not to mention difficult to interpret. The natural weakening of this is to demand that these be merely isomorphic, but this is far too weak: for vector spaces, for example, this requires only that the dimension be the same. Even natural isomorphism is not quite strong enough.

The correct way of solving this problem is by demanding that certain diagrams, called *coherenece diagrams*, commute. These diagrams then force other diagrams to commute in a way that solves our problem, as Mac Lane showed in the so-called *Mac Lane's Coherence Theorem.*

Definition 164 (monoidal category)**.** A monoidal category is a category C equipped with a monoidal structure. A monoidal structure is the following:

- A bifunctor (Definition 36) $\otimes\colon \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ called the *tensor product*,
- An object I called the *unit object*, and
- Three natural isomorphisms (Definition 39) subject to coherence conditions expressing the fact that the tensor product
  - is associative: there is a natural isomorphism α called the *associator*, with components

$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C)$$

◦ has left and right identity: there are two natural isomorphisms λ and ρ respectively called the *left unitor* and *right unitor* with components

$$\lambda_A \colon I \otimes A \xrightarrow{\sim} A$$

and

$$\rho_A \colon A \otimes I \xrightarrow{\sim} A.$$

The coherence conditions are that the following diagrams commute for all A, B, C, and $D \in \mathrm{Obj}(C)$.

- The *triangle diagram*

$$\begin{tikzcd}
(A \otimes I) \otimes B \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\rho_A \otimes 1_B"'] & & A \otimes (I \otimes B) \arrow[dl, "1_A \otimes \lambda_B"] \\
& A \otimes B &
\end{tikzcd}$$

- The *home plate diagram*[1] (usually the *pentagon diagram*)

$$\begin{tikzcd}
& (A \otimes (B \otimes C)) \otimes D \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\
((A \otimes B) \otimes C) \otimes D \arrow[ur, "\alpha_{A,B,C} \otimes 1_D"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & A \otimes ((B \otimes C) \otimes D) \arrow[d, "1_A \otimes \alpha_{B,C,D}"] \\
(A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A \otimes (B \otimes (C \otimes D))
\end{tikzcd}$$

More succinctly, a monoidal structure on a category C is a quintuple $(\otimes, 1, \alpha, \lambda, \rho)$.

Notation 165. The notation $(\otimes, 1, \alpha, \lambda, \rho)$ is prone to change. If the associator and unitors are not important, or understood from context, they are often left out. We will often say "Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category."

Example 166. The simplest (though not the prototypical) example of a monoidal category is Set with the Cartesian product. We have already studied this structure in some detail in Section 2.1. We check that it satisfies the axioms in Definition 164.

- The Cartesian product on Set *is* a set-theoretic product, and can be naturally viewed, thanks to Theorem 74, as the bifunctor.
- Any set with one element $I = \{*\}$ functions as the unit object.
- For all sets A, B, C

[1]Unfortunately, the home plate, as drawn, is actually upside down.

- The universal property of products gives us a natural isomorphism with components

$$\alpha_{A,B,C} \colon (A \times B) \times C \to A \times (B \times C); \qquad ((a,b),c) \mapsto (a,(b,c)).$$

- Since $\{*\}$ is terminal, we get an isomorphism $\lambda_A \colon \{*\} \times A \to A$ which sends $(*, a) \mapsto a$.
- Similarly, we get a map $\rho_A \colon A \times \{*\} \to A$ which sends $(a, *) \mapsto a$.

The pentagon and triangle diagram commute vacuously since the cartesian product is associative.

Lemma 167. *The tensor product* $\otimes$ *of vector spaces is a bifunctor* $\mathsf{Vect}_k \times \mathsf{Vect}_k \rightsquigarrow \mathsf{Vect}_k$.

Proof. It is clear what the domain and codomain of the tensor product is, and how it behaves on objects and morphisms. The only non-trivial aspect is showing that the standard definition of the tensor product of morphisms respects composition, which is not difficult. □

Example 168. The category Vect_k is a monoidal category with

1. The bifunctor $\otimes$ is given by the tensor product.
2. The unit 1 is given by the field k regarded as a 1-dimensional vector space over itself.
3. The associator is the map which sends $(v_1 \otimes v_2) \otimes v_3$ to $v_1 \otimes (v_2 \otimes v_3)$. It is not *a priori* obvious that this is well-defined, but it is also not difficult to check.
4. The left unitor is the map which sends $(x, v) \in k \times V$ to $xv \in V$.
5. The right unitor is the map which sends

$$(v, x) \mapsto xv.$$

Definition 169 (monoidal subcategory)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. A monoidal subcategory of C is a subcategory (Definition 11) $\mathsf{S} \subseteq \mathsf{C}$ which is closed under the tensor product, and which contains 1.

Example 170. Recall that $\mathsf{FinVect}_k$ is the category of finite dimensional vector spaces over a field k. We saw in Example 13 that $\mathsf{FinVect}_k$ was a full subcategory of Vect_k.

We have just seen that Vect_k is a monoidal category with unit object $1 = k$. Since k is a one-dimensional vector space over itself, $k \in \mathrm{Obj}(\mathsf{FinVect}_k)$, and since the dimension of the tensor product of two finite-dimensional vector spaces is the product of their dimensions, the tensor product is closed in $\mathsf{FinVect}_k$. Hence $\mathsf{FinVect}_k$ is a monoidal subcategory of Vect_k.

Lemma 171. *Let* C *be a category with monoidal structure* $(\otimes, 1, \alpha, \lambda, \rho)$. *Then the maps* λ_1 *and* $\rho_1 \colon 1 \otimes 1 \to 1$ *agree.*

Proof. See [11], Lemma 3.9. □

Note 172. This insight is due to [14].

The triangle and pentagon diagram are the first in a long list of coherence diagrams designed to pacify categorical structures into behaving like their algebraic counterparts. For a monoid, multiplication is associative by fiat, and the extension of this to n-fold products follows by a trivial application of induction. In monoidal categories, associators are isomorphisms rather than equalities, and we have no a priori guarantee that different ways of composing associators give the same isomorphism.

Mac Lane's coherence theorem shows us that this is exactly what the coherence diagrams guarantee.

Theorem 173 (Mac Lane's coherence theorem)**.** *Any two ways of freely composing unitors and associators to go from one expression to another coincide.*

The following definition was taken *mutatis mutandis* from [13].

Definition 174 (monoidal functor)**.** Let C and C′ be monoidal categories. A functor $\mathcal{F}\colon \mathrm{C} \rightsquigarrow \mathrm{C}'$ is lax monoidal if it is equipped with

- a natural transformation $\Phi_{X,Y}\colon \mathcal{F}(X) \otimes \mathcal{F}(Y) \to \mathcal{F}(X \otimes Y)$, and
- a morphism $\varphi\colon 1_{C'} \to \mathcal{F}(1_C)$ such that
- the following diagrams commute for any $X, Y, Z \in \mathrm{Obj}(\mathrm{C})$.

$$\begin{tikzcd}
(\mathcal{F}(X) \otimes \mathcal{F}(Y)) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X,Y} \otimes 1_{\mathcal{F}(Z)}"] \arrow[d, "\alpha_{\mathcal{F}(X),\mathcal{F}(Y),\mathcal{F}(Z)}"'] & \mathcal{F}(X \otimes Y) \otimes \mathcal{F}(Z) \arrow[r, "\Phi_{X \otimes Y, Z}"] & \mathcal{F}((X \otimes Y) \otimes Z) \arrow[d, "\mathcal{F}(\alpha_{X,Y,Z})"] \\
\mathcal{F}(X) \otimes (\mathcal{F}(Y) \otimes \mathcal{F}(Z)) \arrow[r, "1_{\mathcal{F}(X)} \otimes \Phi_{Y,Z}"'] & \mathcal{F}(X) \otimes \mathcal{F}(Y \otimes Z) \arrow[r, "\varphi_{X, Y \otimes Z}"'] & \mathcal{F}(X \otimes (Y \otimes Z))
\end{tikzcd}$$

$$\begin{tikzcd}
1 \otimes \mathcal{F}(X) \arrow[r, "\lambda_{\mathcal{F}(X)}"] \arrow[d, "\varphi \otimes 1_{\mathcal{F}(X)}"'] & \mathcal{F}(X) \\
\mathcal{F}(1) \otimes \mathcal{F}(X) \arrow[r, "\Phi_{1,X}"] & \mathcal{F}(1 \otimes X) \arrow[u, "\mathcal{F}(\lambda_X)"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{F}(X) \otimes 1 \arrow[r, "\rho_{\mathcal{F}(X)}"] \arrow[d, "1_{\mathcal{F}(X)} \otimes \varphi"'] & \mathcal{F}(X) \\
\mathcal{F}(X) \otimes \mathcal{F}(1) \arrow[r, "\Phi_{X,1}"] & \mathcal{F}(X \otimes 1) \arrow[u, "F(\lambda_X)"']
\end{tikzcd}$$

If Φ is a natural isomorphism and φ is an isomorphism, then $\mathcal{F}$ is called a strong monoidal functor.

We will denote the above monoidal functor by $(\mathcal{F}, \Phi, \varphi)$.

Note 175. The above diagrams above are exactly those necessary to ensure that the monoidal structure is preserved. They do this by demanding that the associator and the unitors be $\mathcal{F}$-equivariant.

Note 176. In much of the literature, a strong monoidal functor is simply called a monoidal functor.

Definition 177 (monoidal natural transformation)**.** Let (F, Φ, φ) and (G, Γ, γ) be monoidal functors. A natural transformation $\eta : \mathcal{F} \Rightarrow \mathcal{G}$ is monoidal if the following diagrams commute.

$$\begin{tikzcd} \mathcal{F}(X) \otimes \mathcal{F}(Y) \arrow[r, "\eta_X \otimes \eta_Y"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{G}(X) \otimes \mathcal{G}(Y) \arrow[d, "\Gamma_{X,Y}"] \\ \mathcal{F}(X \otimes Y) \arrow[r, "\eta_{X \otimes Y}"] & \mathcal{G}(X \otimes Y) \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[d, "\varphi"'] \arrow[dr, "\gamma"] & \\ \mathcal{F}(1) \arrow[r, "\eta_1"] & \mathcal{G}(1) \end{tikzcd}$$

6.1.2 Line objects

Definition 178 (invertible object)**.** Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. An invertible object (sometimes called a *line object*) is an object $L \in \mathrm{Obj}(\mathsf{C})$ such that both of the functors $\mathsf{C} \rightsquigarrow \mathsf{C}$

- $\ell_L : A \mapsto L \otimes A$
- $r_L : A \mapsto A \otimes L$

are categorical equivalences.

Lemma 179. *If* $(\mathsf{C}, \otimes, 1)$ *is a monoidal category and* L *is an invertible object, then there is an object* $L^{-1} \in \mathrm{Obj}(\mathsf{C})$*, unique up to isomorphism, such that* $L \otimes L^{-1} \simeq L^{-1} \otimes L \simeq 1$*. Furthermore,* L *is invertible only if there exists such an* L^{-1}*.*

Proof. Suppose L is invertible. Then the functor $\ell_L : A \mapsto L \otimes A$ is bijective up to isomorphism, i.e. for any $A \in \mathrm{Obj}(\mathsf{C})$, there is an object $A' \in \mathrm{Obj}(\mathsf{C})$, unique up to isomorphism, such that

$$\ell_L(A') = L \otimes A' \simeq A.$$

If this is true for *any* A, it must also be true for 1, so there exists an object L^{-1}, unique up to isomorphism, such that

$$\ell_L(L^{-1}) = L \otimes L^{-1} \simeq 1.$$

The same logic tells us that there exists some other element L'^{-1}, such that

$$r_L(L'^{-1}) = L'^{-1} \otimes L \simeq 1.$$

Now

$$1 \simeq L'^{-1} \otimes L \simeq L'^{-1} \otimes (L \otimes L^{-1}) \otimes L \simeq (L'^{-1} \otimes L) \otimes (L^{-1} \otimes L) \simeq (L'^{-1} \otimes L) \otimes 1 \simeq L'^{-1} \otimes L,$$

so L'^{-1} is also a left inverse for L. But since ℓ_L is an equivalence of categories, L only has one left inverse up to isomorphism, so L^{-1} and L'^{-1} must be isomorphic. □

Definition 180. Let $(\mathsf{C}, \otimes, 1)$ be a monoidal category. Then the full subcategory (Definition 12) $(\mathrm{Line}(\mathsf{C}), \otimes, 1) \subseteq (\mathsf{C}, \otimes, 1)$ whose objects are the line objects in C is called the line subcategory of $(\mathsf{C}, \otimes, 1)$. Since invertibility is closed under the tensor product and 1 is invertible ($1^{-1} \simeq 1$), $\mathrm{Line}(\mathsf{C})$ is a monoidal subcategory of C.

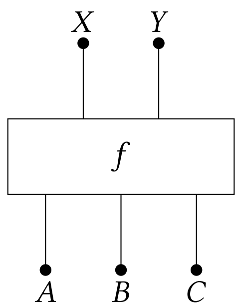
6.1.3 Braided monoidal categories

Braided monoidal categories capture the idea that we should think of morphisms between tensor products spatially, as diagrams embedded in 3-space. This sounds odd, but it turns out to be the correct way of looking at a wide class of problems.

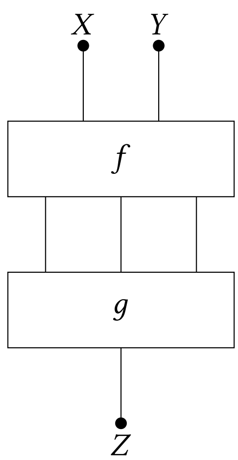
To be slightly more precise, we can think of the objects X, Y, etc. in any monoidal category as little dots.

We can express the tensor product $X \otimes Y$ by putting the dots representing X and Y next to each other.

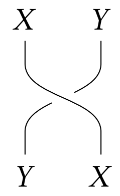
A morphism f between, say, $X \otimes Y$ and $A \otimes B \otimes C$ can be drawn as a diagram consisting of some lines and boxes.



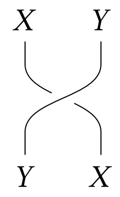
We can compose morphisms by concatenating their diagrams.



In a braided monoidal category, we require that for any two objects X and Y we have an isomorphism $\gamma_{XY} \colon X \otimes Y \to Y \otimes X$, which we draw like this.



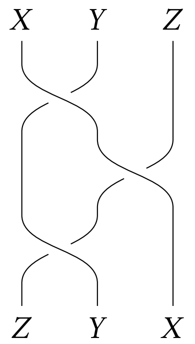
Since γ_{AB} is an isomorphism, it has an inverse γ_{AB}^{-1} (not necessarily equal to γ_{BA}!) which we draw like this.



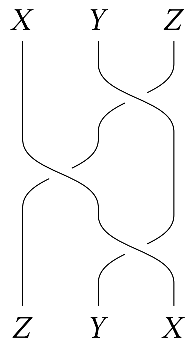
The idea of a braided monoidal category is that we want to take these pictures seriously: we want two expressions involving repeated applications of the $\gamma_{\cdot\cdot}$ and their inverses to be equivalent if and only if the braid diagrams representing them are homotopic. Thus we want, for example,

$$\gamma_{XY} \circ \gamma_{YZ} \circ \gamma_{XY} = \gamma_{YZ} \circ \gamma_{XY} \circ \gamma_{YZ}$$

since



is homotopic to



.

A digression into the theory of braid groups would take us too far afield. The punchline is that to guarantee that all such compositions involving the γ are identified in the correct way, we must define braided monoidal categories as follows.

Definition 181 (braided monoidal category)**.** A catgory C with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$ is braided if for every two objects A and $B \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism $\gamma_{A,B} \colon A \otimes B \to B \otimes A$ such that the following *hexagon diagrams* commute.

$$\begin{tikzcd}
 & A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[dr, "\alpha_{BCA}"] & \\
(A \otimes B) \otimes C \arrow[ur, "\alpha_{ABC}"] \arrow[dr, "\gamma_{AB}\otimes 1"'] & & & B \otimes (C \otimes A) \\
 & (B \otimes A) \otimes C \arrow[r, "\alpha_{BAC}"'] & B \otimes (A \otimes C) \arrow[ur, "1\otimes\gamma"'] &
\end{tikzcd}$$

$$
\begin{tikzcd}[column sep=small]
& (A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B, C}"] & C \otimes (A \otimes B) \arrow[dr, "\alpha^{-1}_{CAB}"] & \\
A \otimes (B \otimes C) \arrow[ur, "\alpha^{-1}_{ABC}"] \arrow[dr, "1\otimes\gamma_{BC}"'] & & & (C \otimes A) \otimes B \\
& A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{ACB}"'] & (A \otimes C) \otimes B \arrow[ur, "\gamma_{AC}\otimes 1"'] &
\end{tikzcd}
$$

The collection of such γ form a natural isomorphism betweem the bifunctors

$$(A, B) \mapsto A \otimes B \qquad \text{and} \qquad (A, B) \mapsto B \otimes A,$$

and is called a braiding.

Definition 182 (braided monoidal functor)**.** A lax monoidal functor $(\mathcal{F}, \Phi, \phi)$ (Definition 174) is braided monoidal if it makes the following diagram commute.

$$
\begin{tikzcd}
\mathcal{F}(x) \otimes \mathcal{F}(y) \arrow[r, "\gamma_{\mathcal{F}(x),\mathcal{F}(y)}"] \arrow[d, "\Phi_{x,y}"'] & \mathcal{F}(y) \otimes \mathcal{F}(x) \arrow[d, "\Phi_{x,y}"] \\
\mathcal{F}(x \otimes y) \arrow[r, "\mathcal{F}(\gamma_{x,y})"] & \mathcal{F}(y \otimes x)
\end{tikzcd}
$$

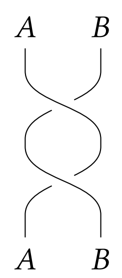
Note 183. There are no extra conditions imposed on a monoidal natural transformation to turn it into a braided natural transformation.

6.1.4 Symmetric monoidal categories

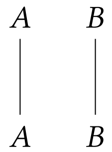
Until now, we have been calling the bifunctor $\otimes$ in Definition 164 a tensor product. This has been an abuse of terminology: in general, one defines tensor products not to be those bifunctors which come from any monoidal category, but only those which come from *symmetric* monoidal categories. We will define these shortly.

Conceptually, passing from the definition of a braided monoidal category to that of a symmetric monoidal category is rather simple. One only requires that for any two objects A and B, $\gamma_{BA} = \gamma_{AB}^{-1}$, i.e. $\gamma_{BA} \circ \gamma_{AB} = 1_{A\otimes B}$.

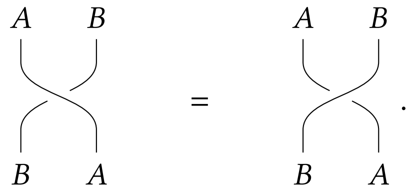
We can interpret this nicely in terms of our braid diagrams. We can draw $\gamma_{BA} \circ \gamma_{AB}$ like this.



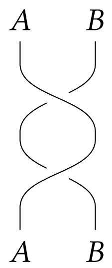
The requirement that this must be homotopic to the identity transformation



can be expressed by making the following rule: in a *symmetric* monoidal category, we don't care about the difference between undercrossings and overcrossings:



Then we can exchange the diagram representing $\gamma_{BA} \circ \gamma_{AB}$ for



which is clearly homotopic to the identity transformation on $A \otimes B$.

Definition 184 (symmetric monoidal category)**.** Let C be a braided monoidal category with braiding γ. We say that C is a symmetric monoidal category if for all $A, B \in \mathrm{Obj}(\mathsf{C})$, $\gamma_{BA} \circ \gamma_{AB} = 1_{A \otimes B}$. A braiding γ which satisfies such a condition is called symmetric.

Note 185. There are no extra conditions imposed on a monoidal natural transformation to turn it into a symmetric natural transformation.

6.2 Internal hom functors

We can now generalize the notion of an exponential object (Definiton 92) to any monoidal category.

6.2.1 The internal hom functor

Recall the definition of the hom functor on a locally small category C (Definition 96): it is the functor which maps two objects to the set of morphisms between them, so it is a functor

$$\mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{Set}.$$

If we take $\mathsf{C} = \mathsf{Set}$, then our hom functor never really leaves Set; it is *internal* to Set. This is our first example of an *internal hom functor*. In fact, it is the prototypical internal hom functor, and we can learn a lot by studying its properties.

Let X and Y be sets. Denote the set of all functions $X \to Y$ by $[X, Y]$.

Let S be any other set, and consider a function $f: S \to [X, Y]$. For each element $s \in S$, f picks out a function $h_s: X \to Y$. But this is just a curried version of a function $S \times X \to Y$! So as we saw in Section 3.1, we have a bijection between the sets $[S, [X, Y]]$ and $[S \times X, Y]$. In fact, this is even a *natural* bijection, i.e. a natural transformation between the functors

$$[-, [-, -]] \qquad \text{and} \quad [- \times -, -]: \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set}^{\mathrm{op}} \times \mathsf{Set} \rightsquigarrow \mathsf{Set}.$$

Let's check this. First, we need to figure out how our functors act on functions. Suppose we have sets and functions like so.

$$\begin{gathered} A'' \xleftarrow{f''} B'' \\ A' \xleftarrow{f'} B' \\ A \xrightarrow{f} B \end{gathered}$$

Our functor maps

$$(A'', A', A) \mapsto [A'', [A', A]] = \mathrm{Hom}_{\mathsf{Set}}(A'', \mathrm{Hom}_{\mathsf{Set}}(A', A)),$$

so it should map (f'', f', f) to a function

$$[f'', [f', f]]: [A'', [A', A]] \to [B'', [B', B]].$$

The way to do that is by sending $m \in [A'', [A', A]]$ to

$$[f', f] \circ m \circ f''.$$

You can check that this works as advertised.

The other one's not so tough. Our functor maps an object (A'', A', A) to $[A'' \times A', A]$. We need to map (f'', f', f) to a function

$$[f'' \times f', f]: [A'' \times A', A] \to [B'' \times B', B].$$

We do that by sending $m \in [A'' \times A', A]$ to

$$f \circ m \circ (f'', f') \in [B'' \times B', B].$$

Checking that $[-, [-, -]]$ and $[- \times -, -]$ really *are* functorial would be a bit much; each is just the composition of hom functor and the Cartesian product. We will however check that there is a natural isomorphism between them, which amounts to checking that the following diagram

commutes.

$$\begin{tikzcd}
{[A'' \times A', A]} \arrow[rrr, "{[f''\times f', f]}"] \arrow[ddd, "\Phi_{[A''\times A',A]}"'] & & & {[B''\times B', B]} \arrow[ddd, "\Phi_{[B''\times B',B]}"] \\
& m \arrow[r, mapsto] \arrow[d, mapsto] & f\circ m\circ(f',f'') \arrow[d, mapsto] & \\
& \Phi(m) \arrow[r, mapsto] & {[f',f]\circ\Phi(m)\circ f'' \overset{!}{=} \Phi(f\circ m\circ(f',f''))} & \\
{[A'',[A',A]]} \arrow[rrr, "{[f'',[f',f]]}"'] & & & {[B'',[B',B]]}
\end{tikzcd}$$

In other words, we have to show that

$$\Phi_{[A''\times A',A]}(f\circ m\circ(f',f'')) = [f',f]\circ\Phi_{[B''\times B',B]}(m)\circ f''.$$

So what is each of these? Well, $f\circ m\circ(f',f'')$ is a map $B''\times B' \to B$, which maps (say) $(b'',b')\mapsto b$.

The natural transformation Φ tells us to curry this, i.e. turn it into a map $B''\to[B',B]$. Not just any map, though: a map which when evaluated on b'' turns into a map which, when evaluated on b', yields b.

We know that $f\circ m\circ(f',f'')\colon(b'',b')\mapsto b$, i.e.

$$f(m(f''(b''),f'(b'))) = b.$$

If we can show that this is *also* what $[f',f]\circ\Phi_{[B''\times B',B]}(m)\circ f''$ is equal to when evaluated on b'' and then b', we are done, since two functions are equal if they take the same value for all inputs.

Well, let's go through what this definition means. First, we take b'' and feed it to f''. Next, we let $\Phi_{[B''\times B',B]}(m)$ act on the result, i.e. we fill the first argument of m with $f''(b'')$. What we get is the following:

$$m(f''(b''),-).$$

Then we are to precompose this with f' and stick the result into f:

$$f(m(f''(b''),f'(-))).$$

Finally, we are to evaluate this on b' to get

$$f(m(f''(b''),f'(b'))).$$

Indeed, this is equal to b, so the diagram commutes.

In other words, Φ is a natural bijection between the hom-sets $[-,[-,-]]$ and $[-\times-,-]$.

We picked this example because the collection $[A,B]$ of all functions between two sets A and B is itself a set. Therefore it makes sense to think of the hom-sets $\mathrm{Hom}_{\mathsf{C}}(A,B)$ as living within the same category as A and B. We saw in Section 2.1 that in a category with products, we could sometimes view hom-sets as exponential objects. However, we now have the technology to be even more general.

Definition 186 (internal hom functor)**.** Let $(\mathsf{C}, \otimes)$ be a monoidal category. An internal hom functor is a functor

$$[-,-]_{\mathsf{C}}\colon \mathsf{C}^{\mathrm{op}} \times \mathsf{C} \rightsquigarrow \mathsf{C}$$

such that for every $X \in \mathrm{Obj}(\mathsf{C})$ we have a pair of adjoint functors

$$(-) \otimes X \dashv [X,-]_{\mathsf{C}}.$$

The objects $[A,B]_{\mathsf{C}}$ are called internal hom objects.

Note 187. The reason for the long introduction to this section was that the pair of adjoint functors in Definition 186 really matches the one in Set. Recall, in Set there was a natural transformation

$$\mathrm{Hom}_{\mathsf{Set}}(S \times X, Y) \simeq \mathrm{Hom}_{\mathsf{Set}}(S, [X,Y]).$$

This means that for any set X, there is a pair of adjoint functors

$$(-) \times X \dashv [X,-],$$

which is in agreement with the statement of Definition 186.

Notation 188. The convention at the nLab is to denote the internal hom by square braces $[A,B]$, and this is for the most part what we will do. Unfortunately, we have already used this notation for the *regular* hom functor. To remedy this, we will add a subscript if the category to which the hom functor belongs is not clear: $[-,-]_{\mathsf{C}}$ for a hom functor internal to C, $[-,-]_{\mathsf{Set}}$ for the standard hom functor (or the hom functor internal to Set, which amounts to the same).

There is no universally accepted notation for the internal hom functor. One often sees it denoted by a lower-case hom: $\mathrm{hom}_{\mathsf{C}}(A,B)$. Many sources (for example DMOS [16]) distinguish the internal hom with an underline: $\underline{\mathrm{Hom}}_{\mathsf{C}}(A,B)$. Deligne typesets it with a script H: $\mathscr{H}om_{\mathsf{C}}(A,B)$.

Definition 189 (closed monoidal category)**.** A monoidal category equipped with an internal hom functor is called a closed monoidal category.

Note 190. Here is another (clearly equivalent) definition of $[X,Y]_{\mathsf{C}}$: it is the object representing (Definition 99) the functor

$$T \mapsto \mathrm{Hom}_{\mathsf{C}}(T \otimes X, Y).$$

Example 191. In many locally small categories whose objects can be thought of as "sets with extra structure," it is possible to pile structure on top of the hom sets until they themselves can be viewed as bona fide objects in their categories. It often (*but not always!*) happens that these beefed-up hom sets coincide (up to isomorphism) with the internal hom objects.

Take for example Vect_k. For any vector spaces V and W, we can turn $\mathrm{Hom}_{\mathsf{Vect}_k}(V,W)$ into a vector space by defining addition and scalar multiplication pointwise; we can then view $\mathrm{Hom}_{\mathsf{Vect}_k}(V,W)$ as belonging to $\mathrm{Obj}(\mathsf{Vect}_k)$. It turns out that this is precisely (up to isomorphism) the internal hom object $[V,W]_{\mathsf{Vect}_k}$.

To see this, we need to show that there is a natural bijection

$$\mathrm{Hom}_{\mathsf{Vect}_k}(A, \mathrm{Hom}_{\mathsf{Vect}_k}(B,C)) \simeq \mathrm{Hom}_{\mathsf{Vect}_k}(A \otimes B, C).$$

Suppose we are given a linear map $f\colon A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B,C)$. If we act with this on an element of A, we get a linear map $B \to C$. If we evaluate this on an element of B, we get an element of C. Thus, we can view f as a bilinear map $A \times B \to C$, hence as a linear map $A \otimes B \to C$.

Now suppose we are given a linear map $g\colon A \otimes B \to C$. By pre-composing this with the tensor product we can view this as a bilinear map $A \times B \to C$, and by currying this we get a linear map $A \to \mathrm{Hom}_{\mathsf{Vect}_k}(B, C)$.

For the remainder of this chapter, let $(\mathsf{C}, \otimes, 1)$ be a closed monoidal category with internal hom functor $[-,-]_{\mathsf{C}}$.

In a closed monoidal category, the adjunction between the internal hom and the tensor product even holds internally.

Lemma 192. *For any $X, Y, Z \in \mathrm{Obj}(\mathsf{C})$ there is a natural isomorphism*

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

Proof. Let $A \in \mathrm{Obj}(\mathsf{C})$. We have the following string of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(A, [X \otimes Y, Z]_{\mathsf{C}}) &\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes (X \otimes Y), Z) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}((A \otimes X) \otimes Y, Z) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(A \otimes X, [Y, Z]_{\mathsf{C}}) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(A, [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}).
\end{aligned}$$

Since this is true for each A we have, by Corollary 107,

$$[X \otimes Y, Z]_{\mathsf{C}} \xrightarrow{\sim} [X, [Y, Z]_{\mathsf{C}}]_{\mathsf{C}}.$$

□

Lemma 193. *Let $(\mathsf{C}, \otimes, 1)$ be a closed symmetric monoidal category. For any $A, B, R \in \mathrm{Obj}(\mathsf{C})$, there is a natural transformation*

$$[A, B]_{\mathsf{C}} \to [R \otimes A, R \otimes B]_{\mathsf{C}},$$

natural in A and B.

Proof. The assignment $R \otimes (-)$ is a functor, and induces a transformation of the regular hom functor

$$\mathrm{Hom}_{\mathsf{C}}(A, B) \mapsto \mathrm{Hom}_{\mathsf{C}}(R \otimes A, R \otimes B)$$

which is natural in A and B. We would like to show that the internal hom functor also has this property.

The following string of natural transformations guarantees it by the Yoneda lemma.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) &\simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes A, \otimes B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(R \otimes X \otimes A, R \otimes B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes (R \otimes A), R \otimes B) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(X, [R \otimes A, R \otimes B]_{\mathsf{C}}).
\end{aligned}$$

□

6.2.2 The evaluation map

The internal hom functor gives us a way to talk about evaluating morphisms $f\colon X \to Y$ without mentioning elements of X.

Definition 194 (evaluation map)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. We have seen that the adjunction

$$(-) \otimes X \dashv [X, -]_{\mathsf{C}}$$

gives us, for any $A, X, Y \in \mathrm{Obj}(\mathsf{C})$, a natural bijection

$$\mathrm{Hom}_{\mathsf{C}}(A \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}(A, [X, Y]_{\mathsf{C}}).$$

In particular, with $A = [X, Y]_{\mathsf{C}}$, we have a bijection

$$\mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}} \otimes X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}}, [X, Y]_{\mathsf{C}}).$$

The adjunct (Definition 159) of $1_{[X,Y]_{\mathsf{C}}} \in \mathrm{Hom}_{\mathsf{C}}([X, Y]_{\mathsf{C}}, [X, Y]_{\mathsf{C}})$ is an object in $\mathrm{Hom}_{\mathsf{C}}\,([X, Y]_{\mathsf{C}} \otimes X, Y)$, denoted

$$\mathrm{eval}_{X,Y}\colon [X, Y]_{\mathsf{C}} \otimes X \to Y,$$

and called the evaluation map.

Example 195. As we saw in Example 166, the category Set is a monoidal category with a bifunctor given by the cartesian product. The internal hom is simply the regular hom functor

$$\mathrm{Hom}_{\mathsf{Set}}(-, -) = [-, -].$$

Let us explore the evaluation map on Set. It is the adjunct of the identity map $1_{[X,Y]}$ under the adjunction

$$[[X, Y] \times X, Y] \dashv [[X, Y], [X, Y]].$$

Thus, it is a function

$$\mathrm{eval}_{X,Y}\colon [X, Y] \times X \to Y; \qquad (f, x) \mapsto \mathrm{eval}_{X,Y}(f, x).$$

So far, we don't know what $\mathrm{eval}_{X,Y}$ sends (f, x) to; we just know that we'd *like it* if it sent it to $f(x)$.

The above adjunction is given by currying: we start on the LHS with a map $\mathrm{eval}_{X,Y}$ with two arguments, and we turn it into a map which fills in only the first argument. Thus the map on the RHS adjunct to $\mathrm{eval}_{X,Y}$ is given by

$$f \mapsto \mathrm{eval}_{X,Y}(f, -).$$

If we want the map $f \mapsto \mathrm{eval}_{X,Y}(f, -)$ to be the identity map, f and $\mathrm{eval}_{X,Y}(f, -)$ must agree on all elements x, i.e.

$$f(x) = \mathrm{eval}_{X,Y}(f, x) \qquad \text{for all } x \in X.$$

Thus, the evaluation map is the map which sends $(f, x) \mapsto f(x)$.

6.2.3 The composition morphism

The evaluation map allows us to define composition of morphisms without talking about internal hom objects as if they have elements.

Definition 196 (composition morphism)**.** For $X, Y, Z \in \mathrm{Obj}(\mathsf{C})$, the composition morphism

$$\circ_{X,Y,Z} \colon [Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \to [X,Z]_{\mathsf{C}}$$

is the $(-) \otimes X \vdash [X,-]_{\mathsf{C}}$-adjunct of the composition

$$[Y,Z]_{\mathsf{C}} \otimes [X,Y]_{\mathsf{C}} \otimes X \xrightarrow{\left(1_{[Y,Z]_{\mathsf{C}}}, \mathrm{eval}_{X,Y}\right)} [Y,Z]_{\mathsf{C}} \otimes Y \xrightarrow{\quad \mathrm{eval}_{Y,Z} \quad} Z \ .$$

Example 197. In Set, the composition morphism $\circ_{X,Y,Z}$ lives up to its name. Let $f \colon X \to Y$, $g \colon Y \to Z$, and $x \in X$. The above composition goes as follows.

1. The map $\left(1_{[Y,Z]}, \mathrm{eval}_{X,Y}\right)$ turns the triple (g, f, x) into the pair $(g, f(x))$.
2. The map $\mathrm{eval}_{Y,Z}$ turns $(g, f(x))$ into $g(f(x)) = (g \circ f)(x)$.

The evaluation morphism $\circ_{X,Y,Z}$ is the currying of this, i.e. it sends

$$(f, g) \mapsto (f \circ g)(-).$$

6.2.4 Dual objects

Recall that for any k-vector space V, there is a dual vector space

$$V^* = \{L \colon V \to k\} .$$

This definition generalizes to any closed monoidal category.

Definition 198 (dual object)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. The dual object to X, denoted X^*, is defined to be the object

$$[X, 1]_{\mathsf{C}}.$$

That is to say, X^* is the internal hom object modelling the hom set of morphisms from X to the identity object 1.

Notation 199. The evaluation morphism (Definition 194) has a component

$$\mathrm{eval}_{X^*,X} \colon X^* \otimes X \to 1.$$

To clean things up a bit, we will write eval_X instead of $\mathrm{eval}_{X^*,X}$.

Notation 200. In many sources, e.g. DMOS ([16]), the dual object to X is denoted $X^\vee$ instead of X^*.

Lemma 201. *There is a natural isomorphism between the functors*

$$\mathrm{Hom}_{\mathsf{C}}(-, X^*) \qquad \textit{and} \qquad \mathrm{Hom}_{\mathsf{C}}((-) \otimes X, 1).$$

Proof. For any $X, T \in \mathrm{Obj}(\mathsf{C})$, the definition of the internal hom $[-,-]_{\mathsf{C}}$ gives us a natural isomorphism

$$\mathrm{Hom}_{\mathsf{C}}(T \otimes X, 1) \simeq \mathrm{Hom}_{\mathsf{C}}(T, [X, 1]_{\mathsf{C}}) = \mathrm{Hom}_{\mathsf{C}}(T, X^*).$$

□

Theorem 202. *The map $X \mapsto X^*$ can be extended to a contravariant functor.*

Proof. We need to figure out how our functor should act on morphisms. We define this by analogy with the familiar setting of vector spaces. Recall that for a linear map $L\colon V \to W$, the dual map $L^t\colon W^* \to V^*$ is defined by

$$(L^t(w))(v) = w(L(v)).$$

By analogy, for $f \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, we should define the dual morphism $f^t \in \mathrm{Hom}_{\mathsf{C}}(B^*, A^*)$ by demanding that the following diagram commutes.

$$\begin{tikzcd}
h^* \otimes X \arrow[r, "f^t \otimes 1_X"] \arrow[d, "1_Y \otimes f"'] & X^* \otimes X \arrow[d, "\mathrm{eval}_X"] \\
h^* \otimes Y \arrow[r, "\mathrm{eval}_Y"] & 1
\end{tikzcd}$$

To check that this is functorial, we must check that it respects compositions, i.e. that the following diagram commutes.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "(f^t \circ g^t) \otimes 1_X"] \arrow[d, "1_Z \otimes (g \circ f)"'] & X^* \otimes X \arrow[d, "\mathrm{eval}_X"] \\
Z^* \otimes Z \arrow[r, "\mathrm{eval}_Z"] & 1
\end{tikzcd}$$

Let's add in some more objects and morphisms.

$$\begin{tikzcd}
Z^* \otimes X \arrow[r, "g^t \otimes 1_X"] \arrow[d, "1_{Z^*} \otimes f"'] & h^* \otimes X \arrow[r, "f^t \otimes 1_X"] \arrow[d, "1_{h^*} \otimes f"'] & X^* \otimes X \arrow[dd, "\mathrm{eval}_X"] \\
Z^* \otimes Y \arrow[r, "g^t \otimes 1_Y"] \arrow[d, "1_{Z^*} \otimes g"'] & h^* \otimes Y \arrow[dr, "\mathrm{eval}_Y"] & \\
Z^* \otimes Z \arrow[rr, "\mathrm{eval}_Z"'] & & 1
\end{tikzcd}$$

We want to show that the outer square commutes. But it clearly does: that the top left square commutes is trivial, and the right and bottom 'squares' are the commutativity conditions defining f^t and g^t. □

Note 203. It's not clear to me why f^t as defined above exists and is unique.

The above is one, but not the only, way to define dual objects. We can be more general.

Definition 204 (right duality)**.** Let C be a category with monoidal structure $(\otimes, 1, \alpha, \lambda, \rho)$. Right duality of two objects A and $A^* \in \mathrm{Obj}(\mathsf{C})$ consists of

1. A morphism of the form
$$\mathrm{eval}_A : A^* \otimes A \to 1,$$
called the *evaluation map* (or *counit* if you're into Hopf algebras)
2. A morphism of the form
$$i_A : 1 \to A \otimes A^*,$$
called the *coevaluation* map (or *unit*)

such that the compositions

$$X \xrightarrow{i_A \otimes 1_X} (X \otimes X^*) \otimes X \xrightarrow{\alpha_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{1_X \otimes \mathrm{eval}_X} X$$

$$X^* \xrightarrow{1_{X^*} \otimes \mathrm{eval}_X} X^* \otimes (X \otimes X^*) \xrightarrow{\alpha^{-1}_{X^*,X,X^*}} (X^* \otimes X) \otimes X^* \xrightarrow{\mathrm{eval}_X \otimes 1_{X^*}} X^*$$

are the identity morphism.

Definition 205 (rigid monoidal category)**.** A monoidal category $(\mathsf{C}, \otimes, 1)$ is rigid if every object has a left and right dual.

Theorem 206. *Every rigid monoidal category is a closed monoidal category (i.e. has an internal hom functor, see Definition 189) with internal hom object*

$$[A, B]_{\mathsf{C}} \simeq B \otimes A^*.$$

Proof. We can prove the existence of this isomorphism by showing, thanks to Corollary 107, that for any $X \in \mathrm{Obj}(\mathsf{C})$ there is an isomorphsm

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

The defining adjunction of the internal hom gives us

$$\mathrm{Hom}_{\mathsf{C}}(X, [A, B]_{\mathsf{C}}) \simeq \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Now we can map any $f \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B)$ to

$$(f \otimes 1_A) \circ (1_X \otimes i_A) \in \mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*).$$

We will be done if we can show that the assignment

$$f \mapsto (f \otimes 1_A) \circ (1_X \otimes i_A)$$

is an isomorphism. We'll do this by exhibiting an inverse:

$$\mathrm{Hom}_{\mathsf{C}}(X, B \otimes A^*) \ni g \mapsto (1_W \otimes \mathrm{eval}_V) \circ (g \otimes 1_V) \in \mathrm{Hom}_{\mathsf{C}}(X \otimes A, B).$$

Of course, first we should show that $(f \otimes 1_A) \circ (1_X \otimes i_A)$ really does map $X \longrightarrow B \otimes A^*$. But it does; it does this by first acting on X with i_A:

$$X \longrightarrow X \otimes A \otimes A^*$$

and then acting on the $X \otimes A$ with f and letting the A^* hang around:

$$X \otimes A \otimes A^* \longrightarrow B \otimes A^*.$$

To show that

$$g \mapsto (1_B \otimes \mathrm{eval}_A) \circ (g \otimes 1_A)$$

really is an inverse, we can shove the assignment

$$f \mapsto (f \otimes 1_A) \circ (1_X \otimes i_A)$$

into it and show that we get f right back out. That is to say, we need to show that

$$(1_B \otimes \mathrm{eval}_A) \circ ([(f \otimes 1_A) \circ (1_X \otimes i_A)] \otimes 1_A) = f.$$

This is easy to see but hard to type. Write it out. You'll need to use first of the two composition identities.

To show that the other composition yields g, you have to use the other. $\square$

7 Abelian categories

This section draws heavily from [22].

7.1 Additive categories

Recall that Ab is the category of abelian groups.

Definition 207 (Ab-enriched category)**.** A category C is Ab-enriched if

1. for all objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of an abelian group (i.e. one can add morphisms), such that
2. the composition
$$\circ\colon \mathrm{Hom}_{\mathsf{C}}(B, C) \times \mathrm{Hom}_{\mathsf{C}}(A, B) \longrightarrow \mathrm{Hom}_{C}(A, C)$$
is additive in each slot: for any $f_1, f_2 \in \mathrm{Hom}_{\mathsf{C}}(B, C)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(A, B)$, we must have
$$(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g,$$
and similarly in the second slot.

Note 208. In any Ab-enriched category, every hom-set has at least one element—the identity element of the hom-set taken as an abelian group.

Definition 209 (endomorphism ring)**.** Let C be an Ab-enriched category, and let $A \in \mathrm{Obj}(\mathsf{C})$. The endomorphism ring of A, denoted $\mathrm{End}(A)$, is $\mathrm{Hom}_{\mathsf{C}}(A, A)$, with addition given by the abelian structure and multiplication given by composition.

Lemma 210. *In an* Ab*-enriched category* C*, a finite product is also a coproduct, and vice versa. In particular, initial objects and terminal objects coincide.*

Proof. See [39], Proposition 2.1 for details. □

Definition 211 (additive category)**.** A category C is additive if it has biproducts (Definition 86) and is Ab-enriched.

Example 212. The category Ab of Abelian groups is an additive category. We have already seen that it has the direct sum $\oplus$ as biproduct. Given any two abelian groups A and B and morphisms $f, g\colon A \to B$, we can define the sum $f + g$ via
$$(f + g)(a) = f(a) + g(a) \qquad \text{for all } a \in A.$$
Then for another abelian group C and a morphism $h\colon B \to C$, we have
$$[h \circ (f + g)]\,(a) = h(f(a) + g(a)) = h(f(a)) + h(g(a)) = [h \circ f + h \circ g]\,(a),$$

so

$$h \circ (f + g) = h \circ f + h \circ g,$$

and similarly in the other slot.

Example 213. The category Vect_k is additive. Since vector spaces are in particular abelian groups under addition, it is naturally Ab-enriched,

Definition 214 (additive functor)**.** Let $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ be a functor between additive categories. We say that $\mathcal{F}$ is additive if for each $X, Y \in \mathrm{Obj}(\mathsf{C})$ the map

$$\mathrm{Hom}_{\mathsf{C}}(X, Y) \to \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(X), \mathcal{F}(Y))$$

is a homomorphism of abelian groups.

Lemma 215. *For any additive functor $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$, there exists a natural isomorphism*

$$\Phi\colon \mathcal{F}(-) \oplus \mathcal{F}(-) \Rightarrow \mathcal{F}(- \oplus -).$$

Proof. The commutativity of the following diagram is immediate.

$$\begin{tikzcd}
\mathcal{F}(X \oplus Y) \arrow[r, "\mathcal{F}(f \oplus g)"] \arrow[d, "\Phi_{X,Y}"'] & \mathcal{F}(X' \oplus Y') \arrow[d, "\Phi_{X',Y'}"] \\
\mathcal{F}(X) \oplus \mathcal{F}(Y) \arrow[r, "\mathcal{F}(f) \oplus \mathcal{F}(g)"] & \mathcal{F}(X') \oplus \mathcal{F}(Y')
\end{tikzcd}$$

The (X, Y)-component $\Phi_{X,Y}$ is an isomorphism because □

7.2 Pre-abelian categories

Definition 216 (pre-abelian category)**.** A category C is pre-abelian if it is additive and every morphism has a kernel (Definition 129) and a cokernel (Definition 136).

Lemma 217. *Pre-abelian categories have equalizers (Definition 124).*

Proof. We show that in an pre-abelian category, the equalizer of f and g coincides with the kernel of $f - g$. It suffices to show that the kernel of $f - g$ satisfies the universal property for the equalizer of f and g.

Here is the diagram for the universal property of the kernel of $f - g$.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!\bar{\imath}"] \arrow[drr, bend left] \arrow[ddr, bend right, "i"'] & & \\
& \ker(f-g) \arrow[r] \arrow[d, "\iota_{f-g}"'] & 0 \arrow[d] \\
& A \arrow[r, "f-g"] & B
\end{tikzcd}$$

The universal property tells us that for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $i\colon Z \to A$ with $i \circ (f - g) = 0$ (i.e. $i \circ f = i \circ g$), there exists a unique morphism $\bar{\imath}\colon Z \to \ker(f-g)$ such that $i = \bar{\imath} \circ \iota_{f-g}$. □

Corollary 218. *Every pre-abelian category has all finite limits.*

Proof. By Theorem 146, a category has finite limits if and only if it has finite products and equalizers. Pre-abelian categories have finite products by definition, and equalizers by Lemma 217. □

Recall from Section 2.4 the following definition.

Definition 219 (zero morphism)**.** Let C be a category with zero object 0. For any two objects $A, B \in \mathrm{Obj}(\mathsf{C})$, the zero morphism $0_{A,B}$ is the unique morphism $A \to B$ which factors through 0.

$$\begin{tikzcd} A \arrow[r] \arrow[rr, bend left, "0_{A,B}"] & 0 \arrow[r] & B \end{tikzcd}$$

Notation 220. It will often be clear what the source and destination of the zero morphism are; in this case we will drop the subscripts, writing 0 instead of 0_{AB}.

It is easy to see that the left- or right-composition of the zero morphism with any other morphism results in the zero morphism: $f \circ 0 = 0$ and $0 \circ g = 0$.

Lemma 221. *Every morphism $f\colon A \to B$ in a pre-abelian catgory has a canonical decomposition*

$$A \xrightarrow{p} \mathrm{coker}(\ker(f)) \xrightarrow{\bar{f}} \ker(\mathrm{coker}(f)) \xrightarrow{i} B ,$$

where p is an epimorphism (Definition 20) and i is a monomorphism (Definition 16).

Proof. We start with a map

$$A \xrightarrow{f} B .$$

Since we are in a pre-abelian category, we are guaranteed that f has a kernel $(\ker(f), \iota)$ and a cokernel $(\mathrm{coker}(f), \pi)$. From the universality squares it is immediate that $f \circ \iota = 0$ and $\pi \circ f = 0$ This tells us that the composition $\pi \circ f \circ \iota = 0$, so the following commutes.

$$\begin{tikzcd} & A \arrow[r, "f"] & B \arrow[dr, "\pi"] & \\ \ker(f) \arrow[ur, "\iota"] \arrow[dr] & & & \mathrm{coker}(f) \\ & 0 \arrow[r] & 0 \arrow[ur] & \end{tikzcd}$$

We know that π has a kernel $(\ker(\pi), i)$ and ι has a cokernel $(\operatorname{coker}(\iota), p)$, so we can add their commutativity squares as well.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

If we squint hard enough, we can see the following diagram.

$$\begin{tikzcd}
A \arrow[drr, bend left] \arrow[ddr, bend right, "f"'] & & \\
 & \ker(\pi) \arrow[r] \arrow[d, "i"] & 0 \arrow[d] \\
 & B \arrow[r, "\pi"] & \operatorname{coker}(f)
\end{tikzcd}$$

The outer square commutes because $\pi \circ f = 0$, so the universal property for $\ker(\pi)$ gives us a unique morphism $A \to \ker(\pi)$. Let's add this to our diagram, along with a morphism $0 \to \ker(\pi)$ which trivially keeps everything commutative.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

Again, buried in the bowels of our new diagram, we find the following.

$$\begin{tikzcd}
\ker(f) \arrow[r, "\iota"] \arrow[d] & A \arrow[d, "\pi"] \arrow[ddr, bend left] & \\
0 \arrow[r] \arrow[drr, bend right] & \operatorname{coker}(\iota) & \\
 & & \ker(\pi)
\end{tikzcd}$$

And again, the universal property of cokernels gives us a unique morphism $\bar{f} \colon \operatorname{coker}(\iota) \to \ker(\pi)$.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[r] \arrow[ur] & 0 \arrow[ur] &
\end{tikzcd}$$

The fruit of our laborious construction is the following commuting square.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "p"'] & B \arrow[d, "i"] \\ \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \end{tikzcd}$$

We have seen (Lemma 132) that i is mono, and (Lemma 138) that p is epi.

Now we abuse terminology by calling $\iota = \ker(f)$ and $\pi = \operatorname{coker}(f)$. Then we have the required decomposition. □

Note 222. The abuse of notation above is ubiquitous in the literature.

7.3 Abelian categories

This section is under very heavy construction. Don't trust anything you read here.

Definition 223 (abelian category)**.** A pre-abelian category C is abelian if for each morphism f, the canonical morphism guaranteed by Lemma 221

$$\bar{f} : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$$

is an isomorphism.

Note 224. The above piecemeal definition is equivalent to the following.

A category C is *abelian* if

1. it is Ab-enriched Definition 211, i.e. each hom-set has the structure of an abelian group and composition is bilinear;
2. it admits finite coproducts, hence (by Lemma 210) biproducts and zero objects;
3. every morphism has a kernel and a cokernel;
4. for every morphism, f, the canonical morphism $\bar{f} : \operatorname{coker}(\ker(f)) \to \ker(\operatorname{coker}(f))$ is an isomorphism.

For the remainder of the section, let C be an abelian category.

Lemma 225. *In an abelian category, every morphism decomposes into the composition of an epimorphism and a monomorphism.*

Proof. For any morphism f, bracketing the decomposition $f = i \circ \bar{f} \circ p$ as

$$i \circ (\bar{f} \circ p).$$

gives such a composition. □

Note 226. The above decomposition is unique up to unique isomorphism.

Definition 227 (image of a morphism)**.** Let $f : A \to B$ be a morphism. The object $\ker(\operatorname{coker}(f))$ is called the image of f, and is denoted $\operatorname{im}(f)$.

Lemma 228.

1. *A morphism $f\colon A \longrightarrow B$ is mono iff for all $Z \in \mathrm{Obj}(\mathsf{C})$ and for all $g\colon Z \longrightarrow A$, $f \circ g = 0$ implies $g = 0$.*
2. *A morphism $f\colon A \longrightarrow B$ is epi iff for all $Z \in \mathrm{Obj}(\mathsf{C})$ and for all $g\colon B \longrightarrow Z$, $g \circ f = 0$ implies $g = 0$.*

Proof.

1. First, suppose f is mono. Consider the following diagram.

$$\begin{tikzcd} Z \arrow[r, shift left, "g"] \arrow[r, shift right, "0"'] & A \arrow[r, "f"] & B \end{tikzcd}$$

   If the above diagram commutes, i.e. if $f \circ g = 0$, then $g = 0$, so $1 \implies 2$.

   Now suppose that for all $Z \in \mathrm{Obj}(\mathsf{C})$ and all $g\colon Z \longrightarrow A$, $f \circ g = 0$ implies $g = 0$.

   Let $g, g'\colon Z \longrightarrow A$, and suppose that $f \circ g = f \circ g'$. Then $f \circ (g - g') = 0$. But that means that $g - g' = 0$, i.e. $g = g'$. Thus, $2 \implies 1$.

2. Dual to the proof above.

□

Lemma 229. *Let $f\colon A \longrightarrow B$. We have the following.*

1. *The morphism f is mono iff $\ker(f) = 0$*
2. *The morphism f is epi iff $\mathrm{coker}(f) = 0$.*

Proof.

1. We first show that if $\ker(f) = 0$, then f is mono. Suppose $\ker(f) = 0$. By the universal property of kernels, we know that for any $Z \in \mathrm{Obj}(\mathsf{C})$ and any $g\colon Z \longrightarrow A$ with $f \circ g = 0$ there exists a unique map $\bar{g}\colon Z \longrightarrow \ker(f)$ such that $g = \iota \circ \bar{g}$.

$$\begin{tikzcd} Z \arrow[dr, dashed, "\exists ! \bar{g}"] \arrow[ddr, bend right, "g"'] & & \\ & \ker(f) = 0 \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ & A \arrow[r, "f"] & B \end{tikzcd}$$

   But then g factors through the zero object, so we must have $g = 0$. This shows that $f \circ g = 0 \implies g = 0$, and by Lemma 228 f must be mono.

   Next, we show that if f is mono, then $\ker(f) = 0$. To do this, it suffices to show that $\ker(f)$ is final, i.e. that there exists a unique morphism from every object to $\ker(f)$.

   Since $\mathrm{Hom}_{\mathsf{C}}(Z, \ker(f))$ has the structure of an abelian group, it must contain at least one element. Suppose it contains two morphisms h_1 and h_2.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Our aim is to show that $h_1 = h_2$. To this end, compose each with ι.

$$\begin{tikzcd}
Z \arrow[dr, shift left, "h_1"] \arrow[dr, shift right, "h_2"'] \arrow[ddr, bend right, shift left, "\iota\circ h_1"] \arrow[ddr, bend right=40, shift right, "\iota\circ h_2"'] & & \\
& \ker(f) \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\
& A \arrow[r, "f"] & B
\end{tikzcd}$$

Since $f \circ \iota = 0$, we have $f \circ (\iota \circ h_1) = 0$ and $f \circ (\iota \circ h_2) = 0$. But since f is mono, by Lemma 228, we must have $\iota \circ h_1 = 0 = \iota \circ h_2$. But by Lemma 132, ι is mono, so again we have

$$h_1 = h_2 = 0,$$

and we are done.

2. Dual to the proof above.

□

Lemma 230. *We have the following.*

1. *The kernel of the zero morphism* $0 : A \longrightarrow B$ *is the pair* $(A, 1_A)$.
2. *The cokernel of the zero morphism* $0 : A \longrightarrow B$ *is the pair* $(B, 1_B)$.

Proof.

1. We need only verify that the universal property is satisfied. That is, for any object $Z \in \mathrm{Obj}(\mathsf{C})$ and any morphism $h \colon Z \longrightarrow A$ such that $0 \circ g = 0$, there exists a unique morphism $\bar{g} \colon Z \longrightarrow \ker(0)$ such that the following diagram commutes.

$$\begin{tikzcd}
Z \arrow[dr, dashed, "\bar{g}=g"] \arrow[ddr, bend right, "g"'] & & \\
& \ker(0) = A \arrow[r, "0"] \arrow[d, "\iota=1_A"'] & 0 \arrow[d] \\
& A \arrow[r, "0"] & B
\end{tikzcd}$$

But this is pretty trivial: $\bar{g} = g$.

2. Dual to above.

□

Theorem 231. *All abelian categories are binormal (Definition 144). That is to say:*

1. *all monomorphisms are kernels*
2. *all epimorphisms are cokernels.*

Proof.

1. Consider the following diagram taken Lemma 221, which shows the canonical factorization of any morphism f.

$$\begin{tikzcd}
 & A \arrow[r, "f"] \arrow[d, "p"] \arrow[dr] & B \arrow[dr, "\pi"] & \\
\ker(f) \arrow[ur, "\iota"] \arrow[dr] & \operatorname{coker}(\iota) \arrow[r, "\bar{f}"] & \ker(\pi) \arrow[u, "i"] \arrow[d] & \operatorname{coker}(f) \\
 & 0 \arrow[u] \arrow[ur] \arrow[r] & 0 \arrow[ur] &
\end{tikzcd}$$

   By definition of a pre-abelian category, we know that $\bar{f}$ is an isomorphism.

□

Note 232. The above theorem is actually an equivalent definition of an abelian category, but the proof of equivalence is far from trivial. See e.g. [40] for details.

Definition 233 (subobject, quotient object, subquotient object)**.** Let $Y \in \mathrm{Obj}(\mathsf{C})$.

1. A subobject of Y is an object $X \in \mathrm{Obj}(\mathsf{C})$ together with a monomorphism $i\colon X \hookrightarrow Y$. If X is a subobject of Y we will write $X \subseteq Y$.
2. A quotient object of Y is an object Z together with an epimorphism $p\colon Y \twoheadrightarrow Z$.
3. A subquotient object of Y is a quotient object of a subobject of Y.

Definition 234 (quotient)**.** Let $X \subseteq Y$, i.e. let there exist a monomorphism $f\colon X \hookrightarrow Y$. The quotient Y/X is the cokernel $(\operatorname{coker}(f), \pi_f)$.

Example 235. Let V be a vector space, $W \subseteq V$ a subspace. Then we have the canonical inclusion map $\iota\colon W \hookrightarrow V$, so W is a subobject of V in the sense of Definition 233.

According to Definition 234, the quotient V/W is the cokernel $(\operatorname{coker}(\iota), \pi_\iota)$ of ι. We saw in Example 137 that the cokernel of ι was $V/\operatorname{im}(\iota)$. However, $\operatorname{im}(\iota)$ is exactly W! So the categorical notion of the quotient V/W agrees with the linear algebra notion.

Definition 236 (k-linear category)**.** An abelian category Definition 223 C is k-linear if for all $A, B \in \mathrm{Obj}(\mathsf{C})$ the hom-set $\mathrm{Hom}_{\mathsf{C}}(A, B)$ has the structure of a k-vector space whose additive structure is the abelian structure, and for which the composition of morphisms is k-linear.

Example 237. The category Vect_k is k-linear.

Definition 238 (k-linear functor)**.** let C and D be two k-linear categories, and $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ a functor. Suppose that for all objects $C, D \in \mathrm{Obj}(\mathsf{C})$ all morphisms $f, g\colon C \to D$, and all α, $\beta \in k$, we have

$$\mathcal{F}(\alpha f + \beta g) = \alpha\mathcal{F}(f) + \beta\mathcal{G}(g).$$

Then we say that $\mathcal{F}$ is k-linear.

7.4 Exact sequences

Definition 239 (exact sequence)**.** A sequence of morphisms

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is called exact in degree i if the image (Definition 227) of f_{i-1} is equal to the kernel (Definition 129) of f_i. A sequence is exact if it is exact in every degree.

Lemma 240. *If a sequence*

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

is exact in degree i, then $f_i \circ f_{i-1} = 0$.

Definition 241 (short exact sequence)**.** A short exact sequence is an exact sequence of the following form.

$$0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$$

Definition 242 (exact functor)**.** Let C, D be abelian categories, $\mathcal{F}\colon \mathsf{C} \rightsquigarrow \mathsf{D}$ a functor. We say that $\mathcal{F}$ is

- left exact if it preserves biproducts and kernels
- right exact if it preserves biproducts and cokernels
- exact if it is both left exact and right exact.

7.5 Length of objects

Definition 243 (simple object)**.** A nonzero object $X \in \mathrm{Obj}(\mathsf{C})$ is called simple if 0 and X are its only subobjects.

Example 244. In Vect_k, the only simple object (up to isomorphism) is k, taken as a one-dimensional vector space over itself.

Definition 245 (semisimple object)**.** An object $Y \in \mathrm{Obj}(\mathsf{C})$ is semisimple if it is isomorphic to a dirct sum of simple objects.

Example 246. In Vect_k, all finite-dimensional vector spaces are semisimple.

Definition 247 (semisimple category)**.** An abelian category C is semisimple if every object of C is semisimple.

Example 248. The category $\mathsf{FinVect}_k$ is semisimple.

Definition 249 (Jordan-Hölder series)**.** Let $X \in \mathrm{Obj}(\mathsf{C})$. A filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

of X such that X_i/X_{i-1} is simple for all i is called a Jordan Hölder series for X. The integer n is called the length of the series X_i.

The importance of Jordan-Hölder series is the following.

Theorem 250 (Jordan-Hölder)**.** *Let X_i and Y_i be two Jordan-Hölder series for some object $X \in \mathrm{Obj}(C)$. Then the length of X_i is equal to the length of Y_i, and the objects Y_i/Y_{i-1} are a reordering of X_i/X_{i-1}.*

Proof. See [22], pg. 5, Theorem 1.5.4. □

Definition 251 (length)**.** The length of an object X is defined to be the length of any of its Jordan-Hölder series. This is well-defined by Theorem 250.

8 Tensor Categories

The following definition is taken almost verbatim from [11].

Definition 252 (tensor category)**.** Let k be a field. A k-tensor category A (as considered by Deligne in [30]) is an

1. essentially small (Definition 52)
2. k-linear[1] (Definition 236)
3. rigid (Definition 205)
4. symmetric (Definition 184)
5. monoidal category (Definition 164)

such that

1. the tensor product functor $\otimes\colon \mathsf{A} \times \mathsf{A} \rightsquigarrow \mathsf{A}$ is, in both arguments separately,
   a) k-linear (Definition 238)
   b) exact (Definition 242)
2. $\mathrm{End}(1) \simeq k$, where End denotes the endomorphism ring (Definition 209).

Example 253. Vect_k is *not* a tensor category because it is not essentially small; there is one isomorphism class of vector spaces for each cardinal, and there is no set of all cardinals. However, its subcategory $\mathsf{FinVect}_k$ is a tensor category.

Definition 254 (finite tensor category)**.** A k-tensor category A is called finite (over k) if

1. There are only finitely many simple objects in A, and each of them admits a projective presentation.
2. Each object A of A is of finite length.
3. For any two objects A, B of A, the hom-object (i.e. k-vector space) $\mathrm{Hom}_{\mathsf{A}}(A, B)$ is finite-dimensional.

Example 255. The category $\mathsf{FinVect}_k$ is finite.

1. The only simple object is k taken as a one-dimensional vector space over itself.
2. The length of a finite-dimensional vector space is simply its dimension.
3. The vector space $\mathrm{Hom}_{\mathsf{FinVect}}(V, W)$ has dimension $\dim(V)\dim(W)$.

Definition 256 (finitely $\otimes$-generated)**.** A k-tensor category A is called finitely $\otimes$-generated if there exists an object $E \in \mathrm{Obj}(\mathsf{A})$ such that every other object $X \in A$ is a subquotient

[1] Hence abelian.

(Definition 233) of a finite direct sum of tensor products of E; that is to say, if there exists a finite collection of integers n_i such that X is a subquotient of $\bigoplus_i E^{\otimes n_i}$.

$$\begin{tikzcd} & \bigoplus_i E^{\otimes n_i} \arrow[d, two heads, "\pi"] \\ X \arrow[r, hook, "\iota"] & \left(\bigoplus_i E^{\otimes n_i}\right)/Q \end{tikzcd}$$

Example 257. The category $\mathsf{FinVect}_k$ is finitely generated since any finite-dimensional vector space is isomorphic to $k^n = k \oplus \cdots \oplus k$ for some n.

Definition 258 (subexponential growth)**.** A tensor category A has subexponential growth if, for each object X there exists a natural number N_X such that

$$\mathrm{len}(X^{\otimes n}) \leq (N_X)^n.$$

Example 259. The category $\mathsf{FinVect}_k$ has subexponential growth. For any finite-dimensional vector space V, we always have

$$\dim(V^{\otimes^n}) = (\dim(V))^n,$$

so we can take $N_V = \dim(V)$.

Theorem 260. *Let A be a tensor category, and suppose that*

1. *every object $A \in \mathrm{Obj}(\mathsf{A})$ has a finite length*
2. *the dimension of every hom space $\mathrm{Hom}_{\mathsf{A}}(A, B)$ is finite over k.*

Then the category $\mathsf{Ind}(\mathsf{A})$ of ind-objects of A (Definition 152) has the following properties.

1. *$\mathsf{Ind}(\mathsf{A})$ is abelian (Definition 223).*
2. *$A \hookrightarrow \mathsf{Ind}(\mathsf{A})$ is a full subcategory (cf. Note 153).*
3. *The tensor product on A extends to $\mathsf{Ind}(\mathsf{A})$ via*

$$\begin{aligned} X \otimes Y &\simeq (\varinjlim_i X_i) \otimes (\varinjlim_j Y_j) \\ &\simeq \varinjlim_{i,j} (X_i \otimes Y_j). \end{aligned}$$

4. *The category $\mathsf{Ind}(\mathsf{A})$ fails only to be a tensor category because it is not necessarily essentially small and rigid. More specifically, an object $A \in \mathsf{Ind}(\mathsf{A})$ is dualizable if and only if it is in A.*

Proof. Proposition 3.38 in [11]. □

Definition 261 (tensor functor)**.** Let $(\mathscr{A}, \otimes_A, 1_A)$ and $(\mathscr{B}, \otimes_B, 1_B)$ be k-tensor categories. A functor $\mathcal{F}\colon \mathscr{A} \to \mathscr{B}$ is called a tensor functor if it is

1. braided (Definition 182) and
2. strong monoidal (Definition 174).

9 Internalization

One of the reasons that category theory is so useful is that it makes generalizing concepts very easy. One of the most powerful ways of doing this is known as *internalization.*

9.1 Internal groups

Definition 262 (group object)**.** Let C be a category with binary products $\times$ and a terminal object $*$. A group object in C (or a *group internal to* C) is an object $G \in \mathrm{Obj}(\mathsf{C})$ together with

- a map $e\colon * \to G$, called the *unit map*;
- a map $(-)^{-1}\colon G \to G$, called the *inverse map*; and
- a map $m\colon G \times G \to G$, called the *multiplication map*

such that

- Multiplication is associative, i.e. the following diagram commutes.

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mathrm{id}_G \times m"] \arrow[d, "m \times \mathrm{id}_G"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- The unit picks out the 'identity element,' i.e. the following diagram commutes.

$$\begin{tikzcd} G \arrow[r, "e \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times e"'] & G \times G \arrow[d, "m"] \\ G \times G \arrow[r, "m"'] & G \end{tikzcd}$$

- The inverse map behaves as an inverse, i.e. the following diagram commutes.

$$\begin{tikzcd} & G \times G \arrow[rr, "(-)^{-1} \times \mathrm{id}"] & & G \times G \arrow[dr, "m"] & \\ G \arrow[ur, "\delta"] \arrow[dr, "\delta"'] \arrow[rr, "\exists!"] & & * \arrow[rr, "e"] & & G \\ & G \times G \arrow[rr, "\mathrm{id} \times (-)^{-1}"'] & & G \times G \arrow[ur, "m"'] & \end{tikzcd}$$

Here, $\delta\colon G \to G \times G$ is the diagonal map defined uniquely by the universal property of the product

$$\begin{tikzcd}
 & G \arrow[dl, "\mathrm{id}_G"'] \arrow[d, "\delta"] \arrow[dr, "\mathrm{id}_G"] & \\
G & G \times G \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & G
\end{tikzcd} .$$

It will be convenient to represent the above data in the following way.

$$\begin{tikzcd}
G \times G \arrow[d, "m"] \\
G \arrow[loop right, "i"] \\
* \arrow[u, "e"]
\end{tikzcd}$$

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