1 Adjunctions

Note 1. This chapter is under heavy construction. It's not very good at the moment.

1.1 Hom set adjunction

Definition 2 (hom-set adjunction)**.** Let C, D be categories and $\mathcal{F}$, $\mathcal{G}$ functors as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

We say that <u>$\mathcal{F}$ is left-adjoint to $\mathcal{G}$</u> (or equivalently $\mathcal{G}$ is right-adjoint to $\mathcal{F}$) if there is a natural isomorphism

$$\Phi\colon \operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \operatorname{Hom}_{\mathsf{C}}(-, \mathcal{G}(-)),$$

which fits between $\mathcal{F}$ and $\mathcal{G}$ like this.

$$\begin{tikzcd} \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \arrow[r, bend left, "\operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(-),-)"] \arrow[r, bend right, "\operatorname{Hom}_{\mathsf{D}}(-,\mathcal{G}(-))"'] \arrow[r, phantom, "\Downarrow\Phi"] & \mathsf{Set} \end{tikzcd}$$

The natural isomorphsim amounts to a family of bijections

$$\Phi_{A,B}\colon \operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \operatorname{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$$

which satisfies the coherence conditions for a natural transformation.

1.2 Unit-counit adjunction

Definition 3 (unit-counit adjunction)**.** We say that two functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a <u>unit-counit adjunction</u> if there are two natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}, \qquad \text{and} \qquad \varepsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$\begin{tikzcd} \mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, Rightarrow, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{G} \arrow[r, Rightarrow, "\eta\mathcal{G}"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, Rightarrow, "\mathcal{G}\varepsilon"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

Note 4. Using η for the unit and ϵ for the counit is very common, but not universal! Both Wikipedia and the nLab have them this way, but the Higher Categories course uses them the other way around.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $\mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$. We can draw it like this.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathrm{id}_{\mathsf{C}}"] \arrow[r, bend right, "\mathcal{G}\circ\mathcal{F}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{C} \end{tikzcd} .$$

Analogously, we can draw ε like this.

$$\begin{tikzcd} \mathsf{D} \arrow[r, bend left, "\mathcal{F}\circ\mathcal{G}"] \arrow[r, bend right, "\mathrm{id}_{\mathsf{D}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathsf{D} \end{tikzcd} .$$

We can arrange these artfully like so.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rr, bend left=60, "\mathrm{id}_{\mathsf{C}}"] \arrow[rr, bend left=25, "\mathcal{G}\circ\mathcal{F}"'] & \mathsf{D} \arrow[r, "\mathcal{G}"] \arrow[rr, bend right=25, "\mathcal{F}\circ\mathcal{G}"] \arrow[rr, bend right=60, "\mathrm{id}_{\mathsf{D}}"'] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \end{tikzcd}$$

Notice that we haven't actually *done* anything; this diagram is just the diagrams for the unit and counit, plus some extraneous information.

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rrr, bend left=60, "\mathcal{F}"] \arrow[rrr, bend left=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"'] \arrow[rrr, bend right=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"] \arrow[rrr, bend right=60, "\mathcal{F}"'] & \mathsf{D} \arrow[r, "\mathcal{G}"] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \end{tikzcd}$$

$$\mathcal{F}\eta \qquad \varepsilon\mathcal{F}$$

then consolidate to get this:

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}" description] \arrow[r, bend left=60, "\mathcal{F}"] \arrow[r, bend right=60, "\mathcal{F}"'] & \mathsf{D} \end{tikzcd}$$

$$\Downarrow\mathcal{F}\eta \qquad \Downarrow\varepsilon\mathcal{F}$$

We can then take the composition $\varepsilon\mathcal{F} \circ \mathcal{F}\eta$ to get a natural transformation $\mathcal{F} \Rightarrow \mathcal{F}$

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] \arrow[r, bend right, "\mathcal{F}"'] \arrow[r, phantom, "\Downarrow\varepsilon\mathcal{F}\circ\mathcal{F}\eta"] & \mathsf{D}; \end{tikzcd}$$

the first triangle diagram says that this must be the same as the identity natural transformation $\mathrm{id}_{\mathcal{F}}$.

What this means in practice is that if we start with the functor $\mathcal{F}$, use the unit to get $\mathcal{F} \circ \mathcal{G}\mathcal{F}$, then use the counit to map back to $\mathcal{F}$, this should be the same as not doing anything at all.

The second triangle diagram is analogous; it tells us that starting with $\mathcal{G}$, using the counit to send this to $\mathcal{G} \circ \mathcal{F} \circ \mathcal{G}$, then going back to $\mathcal{G}$ with the unit should be the same as not doing anything.

Proposition 5. Consider a unit-counit adjunction between functors $\mathcal{F}$ and $\mathcal{G}$ as above.

1. The functor $\mathcal{F}$ is fully faithful if and only if the unit

   $$\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$$

   is a natural isomorphism.
2. The functor $\mathcal{G}$ is fully faithful if and only if the counit

   $$\eta\colon \mathcal{G} \circ \mathcal{F} \Rightarrow \mathrm{id}_{\mathsf{C}}$$

   is a natural isomorphism.

Proof. We prove the second part. The second part is analogous.

Suppose η is a natural isomorphism. Consider the following diagram.

$$\begin{tikzcd} \mathsf{C}(a, b) \arrow[d, "\eta_b \circ (-)"'] \arrow[dr] & \\ \mathsf{C}\left(a, (G \circ F)(b)\right) \arrow[r, "\Phi_{a,b}"'] & \mathsf{D}(F(a), F(b)) \end{tikzcd}$$

The downward arrow is obviously a bijection, and the rightward arrow is given to us since every unit-counit adjunction is equivalently a hom-set isomorphism. Thus, their composition is also a bijection.

Now suppose that $\mathcal{G}$ is fully faithful. □

1.3 Adjunction from universal property

Definition 6 (universal morphism adjunction)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. We say that $\mathcal{F}$ is a left universal morphism adjunction if for each $X \in \mathrm{Obj}(\mathsf{D})$, the category of morphisms $(\mathcal{F} \downarrow X)$ (see Definition **??**) has a terminal object, which we denote by $\mathcal{G}(X)$. That is, $\mathcal{F}$ is a left adjoing universal morphism functor if for each $X \in \mathrm{Obj}(\mathsf{C})$, $Y \in \mathrm{Obj}(\mathsf{D})$, and morphism $f \colon \mathcal{F}(Y) \to X$, we have the following diagram.

$$\begin{tikzcd} \mathcal{F}(Y) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & & & Y \arrow[d, dashed, "\exists! g"] \\ \mathcal{F}(\mathcal{G}(X)) \arrow[r, "\epsilon_X"'] & X & & \mathcal{G}(X) \end{tikzcd}$$

Similarly, we say that $\mathcal{F}$ is a <u>right universal morphism adjunction</u> if for each $Y \in \mathrm{Obj}(\mathsf{C})$, the category of morphisms $(Y \downarrow \mathcal{G})$ (Definition **??**) has a initial object.

The above terminology is horrendous, and at some point I'll figure out a nicer way of doing this. probably be changed. The rationale behind introducing it is that it will not be around for long, since we will see that all of these definitions of left- and right adjunctions are equivalent.

1.4 All of these are equivalent

In this section, we will prove three main lemmas, which schematically do the following.

$$\begin{tikzcd} \left\{\begin{array}{c}\text{Universal morphism}\\ \text{adjunctions}\end{array}\right\} \arrow[rr, Rightarrow, bend left] & & \left\{\begin{array}{c}\text{Hom-set}\\ \text{adjunctions}\end{array}\right\} \arrow[dl, Rightarrow, bend left] \\ & \left\{\begin{array}{c}\text{Unit-counit}\\ \text{adjunctions}\end{array}\right\} \arrow[ul, Rightarrow, bend left] & \end{tikzcd}$$

Note 7. It is common to typeset the statement $\mathcal{F}$ *is left-adjoint to* $\mathcal{G}$ by writing

$$\mathcal{F} \dashv \mathcal{G}.$$

This is fine, but if we want to be more explicit about the domains and codomains, we will use the notation

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}.$$

Lemma 8. If the functors $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ is part of a left universal morphism adjunction, there is a functor $\mathcal{G} \colon \mathsf{D} \to \mathsf{C}$ such that $\mathcal{F} \dashv \mathcal{G}$ is a hom-set adjunction.

Proof. Suppose $\mathcal{F}$ is part of a universal adjunction (Definition 6), i.e. for every object $X \in \mathrm{Obj}(\mathsf{D})$ there exists a terminal object in the category $(\mathcal{F} \downarrow X)$, which we will denote $(\mathcal{G}(X), \epsilon_X)$.[1] The first claim is that the assignment $X \mapsto \mathcal{G}(X)$ extends to a functor

[1]Of course, the objects $\mathcal{G}(X)$ are only specified up to unique isomorphism, and we have to pick representatives using the axiom of choice.

$\mathcal{G} \colon \mathsf{D} \to \mathsf{C}$. To this end, let $f : X \to X'$.

$$\begin{tikzcd}
(\mathcal{F} \circ \mathcal{G})(X) \arrow[r, "\epsilon_X"] & X \arrow[d, "f"] \\
(\mathcal{F} \circ \mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

Composing $f \circ \epsilon_X$, we find the following diagram.

$$\begin{tikzcd}
(\mathcal{F} \circ \mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[dr, "f \circ \epsilon_X"] & X \arrow[d, "f"] \\
(\mathcal{F} \circ \mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

The universal property now gives us a unique morphism $\mathcal{G}(X) \to \mathcal{G}(X')$, which we'll call $\mathcal{G}(f)$.

$$\begin{tikzcd}
(\mathcal{F} \circ \mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] \arrow[dr, "f \circ \epsilon_X"] & X \arrow[d, "f"] & \mathcal{G}(X) \arrow[d, "\exists ! \mathcal{G}(f)"] \\
(\mathcal{F} \circ \mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X' & \mathcal{G}(X')
\end{tikzcd}$$

The usual manipulations (formally identical to those in Theorem **??**) show that $\mathcal{G}$ respects compositions, and therefore defines a bona fide functor.

Let's take a closer look at the commuting square above.

$$\begin{tikzcd}
(\mathcal{F} \circ \mathcal{G})(X) \arrow[r, "\epsilon_X"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & X \arrow[d, "f"] \\
(\mathcal{F} \circ \mathcal{G})(X') \arrow[r, "\epsilon_{X'}"'] & X'
\end{tikzcd}$$

It remains to show that there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{D}}(F(-), -) \simeq \mathrm{Hom}_{\mathsf{C}}(-, G(-)).$$

To this end, let $f : \mathcal{F}(Y) \to X$ be a morphism in D. By the universal property for $\mathcal{G}(X)$, there exists a unique morphism $\tilde{f} \colon Y \to \mathcal{G}(X)$ making the following diagram commute.

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[rr, "\mathcal{F}(\tilde{f})"] \arrow[dr, "f"'] & & \mathcal{F}(\mathcal{G}(X)) \arrow[dl, "\epsilon_X"] \\
& X &
\end{tikzcd}$$

This assignment gives us our map $\mathrm{Hom}_D(\mathcal{F}(X), Y) \to \mathrm{Hom}_C(X, \mathcal{G}(Y))$.

On the other hand, given a map $g \colon Y \to \mathcal{G}(X)$, one can construct a map $\tilde{g} \colon \mathcal{F}(Y) \to X$ via the composition

$$\mathcal{F}(Y) \xrightarrow{\mathcal{F}(g)} \mathcal{F}(\mathcal{G}(X)) \xrightarrow{\epsilon_X} X \ .$$

It remains to show that these operations are mutually inverse. Consider the following diagram.

$$\begin{tikzcd}
\mathcal{F}(X) & & \mathcal{F}(\mathcal{G}(Y)) \\
& \mathcal{F}(\mathcal{G}(Y)) &
\end{tikzcd}$$

□

Lemma 9. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a hom-set adjunction, they form a unit-counit adjunction.

Sketch of proof. Our plan of attack is the following.

1. We show that from a hom-set adjunction $\mathcal{F} \dashv \mathcal{G}$, we can form, for each $A \in \mathrm{Obj}(\mathsf{C})$, a map
$$\eta_A\colon A \to (\mathcal{G} \circ \mathcal{F})(A)$$
and for each $B \in \mathrm{Obj}(\mathsf{D})$, a map
$$\epsilon_B\colon (\mathcal{F} \circ \mathcal{G})(B) \to B.$$
2. We show that, component-wise, these satisfy the triangle identities.
$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_\mathcal{F})_A \qquad \text{and} \qquad (\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_\mathcal{G})_B.$$
3. We show that η_A and ϵ_B are components of two natural transformations
$$\eta\colon \mathrm{id}_\mathsf{C} \to \mathcal{G} \circ \mathcal{F} \qquad \text{and} \qquad \epsilon\colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_\mathsf{D}.$$

The reason for doing things in this order is that we will use the component-wise triangle identities in showing naturality. □

Proof.

1. Suppose $\mathcal{F}$ and $\mathcal{G}$ form a hom-set adjunction with natural isomorphism Φ. Then for any $A \in \mathrm{Obj}(\mathsf{C})$, we have $\mathcal{F}(A) \in \mathrm{Obj}(\mathsf{D})$, so Φ give us a bijection
$$\Phi_{A,\mathcal{F}(A)}\colon\ \mathrm{Hom}_\mathsf{D}(\mathcal{F}(A), \mathcal{F}(A)) \to \mathrm{Hom}_\mathsf{C}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
We don't know much in general about $\mathrm{Hom}_\mathsf{D}(\mathcal{F}(A), \mathcal{F}(A))$, but the category axioms tell us that it always contains $\mathrm{id}_{\mathcal{F}(A)}$. We can use $\Phi_{A,\mathcal{F}(A)}$ to map this to
$$\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) \in \mathrm{Hom}_\mathsf{C}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
Let's call $\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) = \eta_A$.
Similarly, if $B \in \mathrm{Obj}(\mathsf{D})$, then $\mathcal{G}(B) \in \mathrm{Obj}(\mathsf{C})$, so Φ gives us a bijection
$$\Phi_{\mathcal{G}(B),B}\colon\ \mathrm{Hom}_\mathsf{D}((\mathcal{F} \circ \mathcal{G})(B), B) \to \mathrm{Hom}_\mathsf{C}(\mathcal{G}(B), \mathcal{G}(B)).$$
Since $\Phi_{\mathcal{G}(B),B}$ is a bijection, it is invertible, and we can evaluate the inverse on $\mathrm{id}_{\mathcal{G}(B)}$. Let's call
$$\Phi^{-1}_{\mathcal{G}(B),B}(\mathrm{id}_{\mathcal{G}(B)}) = \varepsilon_B.$$

2. Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a manner reminiscent of the proof of the Yoneda lemma, we can express Φ in terms of η, and Φ^{-1} in terms of ε, for *any A* and B. We do this using the naturality of Φ.

   The naturality of Φ tells us for any $A \in \mathrm{Obj}(\mathsf{C})$, $B \in \mathrm{Obj}(\mathsf{D})$, and $g\colon \mathcal{F}(A) \to B$, the following diagram has to commute.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[r, "g\circ(-)"] \arrow[d, "\Phi_{A,\mathcal{F}(A)}"'] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \arrow[d, "\Phi_{A,B}"] \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G}\circ\mathcal{F})(A)) \arrow[r, "\mathcal{G}(g)\circ(-)"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).
\end{tikzcd}$$

   Let's start at the top left with $\mathrm{id}_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A,B}(g)$, and from the bottom road we have $\mathcal{G}(g) \circ \eta_A$. The diagram commutes, so we have

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A. \tag{1.1}$$

   Similarly, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}((\mathcal{F}\circ\mathcal{G})(B), B) \arrow[r, "(-)\circ\mathcal{F}(f)"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \\
\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)) \arrow[r, "(-)\circ f"] \arrow[u, "\Phi^{-1}_{\mathcal{G}(B),B}"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{A,B}"']
\end{tikzcd}$$

   means that, for any $f : A \to \mathcal{G}(B)$,

$$\Phi^{-1}_{A,B}(f) = \varepsilon_B \circ \mathcal{F}(f) \tag{1.2}$$

   To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \mathrm{Obj}(\mathsf{C})$ and all $B \in \mathrm{Obj}(\mathsf{D})$,

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_{\mathcal{F}})_A \qquad \text{and} \quad (\mathcal{G}\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_{\mathcal{G}})_B.$$

   We have

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi^{-1}_{A,\mathcal{F}(A)}(\eta_A) = \mathrm{id}_A = (\mathrm{id}_{\mathcal{F}})_A$$

   and

$$(\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = \mathrm{id}_B = (\mathrm{id}_{\mathcal{G}})_B.$$

3. Let $g\colon A' \to A$. By the naturality of Φ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[rr, "(-)\circ\mathcal{F}(g)"] \arrow[dd, "\Phi_{A,\mathcal{F}(A)}"'] & & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A'), \mathcal{F}(A)) \arrow[dd, "\Phi_{A',\mathcal{F}(A)}"] \\
& \begin{tikzcd} \mathrm{id}_{\mathcal{F}(A)} \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{F}(g) \arrow[d, mapsto] \\ \eta_A \arrow[r, mapsto] & \eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g)) \end{tikzcd} & \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G}\circ\mathcal{F})(A)) \arrow[rr, "(-)\circ g"] & & \mathrm{Hom}_{\mathsf{C}}(A', (\mathcal{G}\circ\mathcal{F})(A))
\end{tikzcd}$$

Following $\mathrm{id}_{\mathcal{F}(A)}$ around, we find the condition $\eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g))$. But by Equation 1.1, we have

$$\Phi_{A',\mathcal{F}(A)}(\mathcal{F}(g)) = (\mathcal{G}\circ\mathcal{F})(g) \circ \eta_{A'},$$

so our condition reads

$$\eta_A \circ g = (\mathcal{G}\circ\mathcal{F})(g) \circ \eta_{A'}.$$

But this says precisely that the naturality square

$$\begin{tikzcd}
(\mathcal{G}\circ\mathcal{F})(A) \arrow[r, "\eta_A"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & A \arrow[d, "f"] \\
(\mathcal{G}\circ\mathcal{F})(A') \arrow[r, "\eta_{A'}"] & A'
\end{tikzcd}$$

commutes.

$\square$

Lemma 10. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a unit-counit adjunction, they form a hom-set adjunction.

Proof. We have seen in the proof of the previous lemma that we can write

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A \qquad \text{and} \qquad \Phi^{-1}_{A,B}(g) = \epsilon_B \circ \mathcal{F}(f).$$

$\square$

Definition 11 (adjunct)**.** Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Then for each $A \in \mathrm{Obj}(\mathsf{C})$ $B \in \mathrm{Obj}(\mathsf{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism $\tilde{f}$ is called the adjunct of f, and f is called the adjunct of $\tilde{f}$.

Lemma 12. Let C and D be categories, $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}, \mathcal{G}'\colon \mathsf{D} \to \mathsf{C}$ functors,

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend right, "\mathcal{G}"'] \arrow[l, bend left, "\mathcal{G}'"] \end{tikzcd}$$

and suppose that $\mathcal{G}$ and $\mathcal{G}'$ are both right-adjoint to $\mathcal{F}$. Then there is a natural isomorphism $\mathcal{G} \Rightarrow \mathcal{G}'$.

Proof. Since any adjunction is a hom-set adjunction, we have two isomorphisms

$$\Phi_{C,D}\colon \operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \qquad \text{and} \qquad \Psi_{C,D}\colon \operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D))$$

which are natural in both C and D. By Lemma **??**, we can construct the inverse natural isomorphism

$$\Phi^{-1}_{C,D}\colon \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \operatorname{Hom}_{\mathsf{D}}(\mathcal{F}(C), D),$$

and compose it with Ψ to get a natural isomorphsim

$$(\Psi \circ \Phi^{-1})_{C,D}\colon \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)).$$

Thus for any morphism $f\colon D \to E$, the following diagram commutes.

$$\begin{tikzcd} \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \arrow[r, "\operatorname{Hom}_{\mathsf{C}}(C,\mathcal{G}(f))"] \arrow[d, "(\Psi\circ\Phi^{-1})_{C,D}"'] & \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}(E)) \arrow[d, "(\Psi\circ\Phi^{-1})_{C,E}"] \\ \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)) \arrow[r, "\operatorname{Hom}_{\mathsf{C}}(C,\mathcal{G}'(f))"] & \operatorname{Hom}_{\mathsf{C}}(C, \mathcal{G}'(E)) \end{tikzcd}$$

But the full- and faithfulness of the Yoneda embedding tells us that there is a natural isomorphism with components μ_D and μ_E making the following diagram commute.

$$\begin{tikzcd} \mathcal{G}(D) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\mu_D"'] & \mathcal{G}(E) \arrow[d, "\mu_E"] \\ \mathcal{G}'(D) \arrow[r, "\mathcal{G}'(f)"] & \mathcal{G}'(E) \end{tikzcd}$$

□

Theorem 13. Let C and D be categories and $\mathcal{F}$ and $\mathcal{G}$ functors as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction. Then $\mathcal{G}$ preserves limits, i.e. if $\mathcal{D}\colon \mathsf{J} \to \mathsf{C}$ is a diagram and $\lim_{\leftarrow i} \mathcal{D}_i$ exists in C, then

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

Proof. We have the following chain of isomorphisms, natural in $Y \in \mathrm{Obj}(\mathsf{D})$.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i)) &\simeq \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(Y), \lim_{\leftarrow i} \mathcal{D}_i) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(Y), \mathcal{D}_i) && \left(\substack{\text{Hom functor commutes with} \\ \text{limits: Theorem ??}}\right) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G} \circ \mathcal{D}_i) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow i} (\mathcal{G} \circ \mathcal{D}_i)).
\end{aligned}$$

By the Yoneda lemma, specifically Corollary **??**, we have a natural isomorphism

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i} (\mathcal{G} \circ \mathcal{D}_i).$$

□

Corollary 14. Any functor $\mathcal{F}$ which is a left-ajoint preserves colimits.

Proof. Dual to the proof of Theorem 13. □

1.5 The data of an adjunction

In the last section, we saw that the concepts of hom-set adjunction, unit-counit adjunction, and universal morphism adjunction were all instances of one meta-concept, the adjunction. This means that, when specifying an adjunction, one can choose to specify any of these, and derive the rest. One should really think of an adjunction, however, as consisting of all simultaneously.

More specifically, an adjunction

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}$$

gives rise to the following pieces of data.

1. A natural isomorphism $\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-))$.
2. A unit $\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$ and counit $\epsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$.
3. A terminal object in each category $(\mathcal{F} \downarrow d)$.
4. An initial object in each category $(c \downarrow \mathcal{G})$.

These data are related to each other in the following way.

- $(1 \longrightarrow 2)$: The counit has components $\epsilon_X = \Phi^{-1}_{\mathcal{G}(X),X}(\mathrm{id}_{\mathcal{G}(X)})$, and the unit has components $\epsilon_Y = \Phi_{Y,\mathcal{F}(Y)}(\mathrm{id}_{\mathcal{F}(Y)})$
- $(1 \longrightarrow 3)$: The terminal object is $(\mathcal{G}(d), \epsilon_d)$
- $(1 \longrightarrow 4)$: The initial object is $(\mathcal{F}(c), \eta_c)$
- $(2 \longrightarrow 1)$: The natural transformation has components defined by $\Phi_{X,Y}(f) = \mathcal{G}(f) \circ \eta_X$, or equivalently $\Phi^{-1}_{X,Y}(g) = \epsilon_Y \circ \mathcal{F}(g)$.
- $(2 \longrightarrow 3)$: The terminal object

1.6 Important examples

Example 15 (free-forgetful adjunction)**.** There are many *free* objects in mathematics: free groups, free modules, free vector spaces, free categories, etc. These are all unified by the following property: the functors defining them are all left adjoints to forgetful functors.

Let us take a specific example: the free vector space over a set. This takes a set S and constructs a vector space which has as a basis the elements of S. When defining free vector space V over a set S, one sees the following universal property:

> For any vector space W and any map $S \to W$, there exists a unique map $W \to V$ making the following diagram commute.

$$\begin{tikzcd} S \arrow[r] \arrow[dr] & V \arrow[d, dashed, "\exists!"] \\ & W \end{tikzcd}$$

There is a forgetful functor $\mathcal{U}\colon \mathsf{Vect}_k \to \mathsf{Set}$ which takes any set and returns the set underlying it. There is a functor $\mathcal{F}\colon \mathsf{Set} \to \mathsf{Vect}_k$, which takes a set and returns the free vector space on it. It turns out that there is an adjunction $\mathcal{F} \dashv \mathcal{U}$.

And this is true of any free object! (In fact by definition.) In each case, the functor giving the free object is left adjoint to a forgetful functor.

Example 16 (limit-constant adjunction)**.** Let J and C be small categories. Denote by $\Delta\colon \mathsf{C} \to [\mathsf{J}, \mathsf{C}]$ the functor such that for all $X \in \mathrm{Obj}(\mathsf{C})$, $\Delta(X)$ is the constant functor which sends every object of J to X and every morphism of J to id_X.

Assume that every diagram $\mathcal{F}\colon \mathsf{J} \to \mathsf{C}$ has a limit. By Defintion **??** this means that for every $\mathcal{D} \in [\mathsf{J}, \mathsf{C}]$, the comma category $(\Delta \downarrow \mathcal{D})$ has a terminal object. But this means that there is an adjoint functor $[\mathsf{J}, \mathsf{C}] \to \mathsf{C}$ which acts on objects by sending $\mathcal{D} \mapsto \lim_{\leftarrow} \mathcal{D}$. That is, we have an adjunction

$$\Delta : \mathsf{C} \leftrightarrow [\mathsf{J}, \mathsf{C}] : \varprojlim .$$

Similarly, there is an adjunction

$$\varinjlim : \mathsf{C} \leftrightarrow [\mathsf{J}, \mathsf{C}] : \Delta.$$

Keep this example in mind during the chapter on Kan extensions.

1.7 What are adjunctions for?

One of the many ways to think about adjunctions is as a relaxation of categorical equivalences. Many examples of adjunctions arise when one has a functor which forgets some information. Of course, no functor will exist which adds back in forgotten information, so there is no hope of turning such a forgetful functor into an equivalence. However, one can often find an approximate inverse as a left or right adjoint.

In this section, we will explore the relationship between adjunctions and categorical equivalences.

Theorem 17. Let $\mathcal{F} \colon \mathsf{C} \to \mathsf{D}$ be a functor. The following are equivalent.

1. $\mathcal{F}$ is part of an equivalence of categories (Definition **??**) $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$.
2. $\mathcal{F}$ is fully faithful (Definition **??**) and essentially surjective (Definition **??**).
3. $\mathcal{F}$ is part of a unit-counit adjunction (Definition 3) $(\mathcal{F}, \mathcal{G}, \epsilon, \eta)$, where ϵ and η are isomorphisms.

Proof.

3. $\Rightarrow$ 1. Obvious.

1. $\Rightarrow$ 2. Assume $(\mathcal{F}, \mathcal{G}, \eta \colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_G, \epsilon \colon \mathrm{id}_C \to \mathcal{G} \circ \mathcal{F})$ is an equivalence of categories.

   First we show that $\mathcal{F}$ is essentially surjective. Evaluating η at any $Y \in \mathrm{Obj}(\mathsf{D})$, we get an isomorphism $(\mathcal{F} \circ \mathcal{G})(Y) \simeq Y$. Thus, for any $Y \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism between Y and some element $\mathcal{F}(\mathcal{G}(Y))$ in the image of $\mathcal{F}$.

   Next, we show that $\mathcal{F}$ is fully faithful.

2. $\Rightarrow$ 3.

$\square$

Proposition 18. Let

$$\mathcal{F} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}$$

be an adjunction.

1. The functor $\mathcal{F}$ is fully faithful if and only if the unit $\eta \colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$ is a natural isomorphism.
2. The functor $\mathcal{G}$ is fully faithful if and only if the counit $\epsilon \colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}}$ is a natural isomorphism

Proof. We prove the first. The second is the first applied to the adjunction

$$\mathcal{F}^{\mathrm{op}} : \mathsf{C} \leftrightarrow \mathsf{D} : \mathcal{G}^{\mathrm{op}}.$$

Suppose η is a natural isomorphism. Consider the following composition.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(a, b) \arrow[r, "\mathcal{F}_{x,x}"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(a), \mathcal{F}(b)) \arrow[r, "\Phi_{a,\mathcal{F}(b)}"] & \mathrm{Hom}_{\mathsf{C}}(a, (\mathcal{G} \circ \mathcal{F})(b)) \\
f \arrow[r, mapsto] & \mathcal{F}(f) \arrow[r, mapsto] & (\mathcal{G} \circ \mathcal{F})(f) \circ \eta_a
\end{tikzcd}$$

By naturality of η, the following square commutes.

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "\eta_a"'] & b \arrow[d, "\eta_b"] \\ (\mathcal{G} \circ \mathcal{F})(a) \arrow[r, "(\mathcal{G}\circ\mathcal{F})(f)"'] & (\mathcal{G} \circ \mathcal{F})(b) \end{tikzcd}$$

Thus, we can equivalently view the above composition as sending $f \mapsto \eta_b \circ f$.

The second arrow $\Phi_{a,\mathcal{F}(b)}$ is a bijection by the definition of adjunction, and their composition is a bijection since the map $f \mapsto \eta_b \circ f$ has inverse $g \mapsto \eta_b^{-1} \circ g$. Thus, by the 2-out-of-3 law for bijections, $\mathcal{F}_{x,x}$ is a bijection.

Now suppose that η is a natural isomorphism. Then both of the above maps are bijections, giving us a bijection

$$\mathrm{Hom}_{\mathsf{C}}(a, b) \to \mathrm{Hom}_{\mathsf{C}}(a, (\mathcal{G} \circ \mathcal{F})(b)); \qquad f \mapsto \eta_a \circ f.$$

Thus there exists some $g \in \mathrm{Hom}_{\mathsf{C}}((\mathcal{G} \circ \mathcal{F})(a), a)$ such that $\eta_a \circ g = \mathrm{id}_{(\mathcal{G}\circ\mathcal{F})(a)}$. Then

$$\eta_a = \eta_a \circ \mathrm{id}_b = \eta_a \circ g \circ \eta_a.$$

But since $\eta_a \circ -$ is a bijection, this implies that $g \circ \eta_a = \mathrm{id}_b$, so g is a two-sided inverse to η_a, as we were trying to show. $\square$