1 Kan extensions

1.1 Global Kan extensions

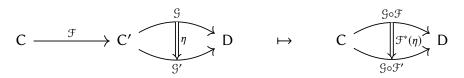
Let $\mathcal{F} \colon C \to C'$ be a functor. For any other category D, \mathcal{F} induces a functor

$$\mathfrak{F}^* \colon [C', D] \to [C, D]$$

which sends a functor $\mathcal{G} \colon C' \to D$ to its precomposition $\mathcal{G} \circ \mathcal{F}$

$$C' \xrightarrow{g} D \qquad \mapsto \qquad C \xrightarrow{\mathcal{F}^*(\mathfrak{G})} D$$

and a natural transformation $\eta: \mathcal{G} \to \mathcal{G}'$ to its left whiskering (Example ??) $\eta \mathcal{F}$.



Definition 1 (Kan extension). Suppose the functor \mathcal{F}^* defined above has a left adjoint

$$\mathcal{F}_1: [C, D] \rightarrow [C', D].$$

and right adjoint

$$\mathcal{F}_* \colon [\mathsf{C},\mathsf{D}] \to [\mathsf{C}',\mathsf{D}].$$

- We call $\mathcal{F}_!$ the <u>left Kan extension</u> along \mathcal{F} . For $\mathcal{G} \in [C, D]$, we call $\mathcal{F}_!(\mathcal{G})$ the <u>left Kan</u> extension of \mathcal{G} along \mathcal{F} .
- We call \mathcal{F}_* the <u>right Kan extension</u> along \mathcal{F} , and for any $\mathcal{G} \in [C, D]$, we call $\mathcal{F}_*(\mathcal{G})$ the right Kan extension of \mathcal{G} along \mathcal{F} .

Example 2. Consider the case in which C' is the terminal category 1 (Example ??). There is a unique functor from any category $C \to 1$, which sends every object to 1 and every morphism to the identity morphism. Furthermore, any functor $\mathcal{F} \colon 1 \to D$ is completely determined by where it sends the unique object *, as $\mathcal{F}(\mathrm{id}_*) = \mathrm{id}_{\mathcal{F}(\mathrm{id}(*))}$.

Now fix categories C, and consider functors $\mathcal{F}\colon C\to 1$ and $\mathcal{F}_D\colon 1\to D$.