1 Higher categories

1.1 2-categories

Definition 1 (2-category)**.** A 2-category is a category enriched over categories. More explicitly, a 2-category $\mathbb{C}$ consists of the following.

- A set $\mathrm{Obj}(\mathbb{C})$ of objects.
- For every two objects $x, y \in \mathrm{Obj}(\mathbb{C})$, a category $\mathbb{C}(x, y)$ of morphisms.
- For every object $x \in \mathrm{Obj}(\mathbb{C})$, an object $\mathrm{id}_x \in \mathrm{Obj}(\mathbb{C}(x, y))$.
- For every three objects x, y, z, a functor

  $$\mu\colon \mathbb{C}(y, z) \times \mathbb{C}(x, y) \to \mathbb{C}(x, z)$$

  implementing composition.

  These must make the following diagrams commute.

  Associativity:

  $$\begin{tikzcd} \mathbb{C}(z, w) \times \mathbb{C}(y, z) \times \mathbb{C}(x, y) \arrow[r] \arrow[d] & \mathbb{C}(z, w) \times \mathbb{C}(x, z) \arrow[d] \\ \mathbb{C}(y, w) \times \mathbb{C}(x, y) \arrow[r] & \mathbb{C}(x, w) \end{tikzcd}$$

  Unitality:

  $$\begin{tikzcd} \mathbb{C}(x, y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathbb{C}(y, y) \times \mathbb{C}(x, y) \arrow[d] \\ \mathbb{C}(x, y) \times \mathbb{C}(x, x) \arrow[r] & \mathbb{C}(x, y) \end{tikzcd}$$

Example 2. The category Cat is a 2-category.

- The objects $\mathrm{Obj}(\mathrm{Cat})$ are given by small categories.
- The morphisms $\mathrm{Cat}(\mathrm{C}, \mathrm{D})$ is the category $[\mathrm{C}, \mathrm{D}]$.
- The identity morphism is the identity functor. We have already seen that this is a functor.
- The composition is given by composition of functors. We must show that this is functorial, i.e. that

These satisfy the conditions:

1. Indeed, the identity functor functions as a left and right identity, and whiskering by the identity is the identity.
2. Indeed, composition of functors is associative, and vertical composition is associative, although I don't know a nice way to prove this.