1 Kan extensions

1.1 Motivation

Let C be a category, and $\mathcal{I}\colon S \to C$ a subcategory inclusion. Given a functor $\mathcal{F}$ from C to any category D, we can restrict $\mathcal{F}$ to S by pulling it back with $\mathcal{I}$. We denote this restriction by

$$\mathcal{I}^*(\mathcal{F}) = \mathcal{F} \circ \mathcal{I}.$$

In fact, $\mathcal{I}$ induces a functor from the category of functors $C \to D$ (Definition **??**) to the category of functors $S \to D$. Recall that we denoted these categories by $[C, D]$ and $[S, D]$ respectively. What we have created is a functor

$$\mathcal{I}^*\colon [C, D] \to [S, D]$$

which acts on functors $C \to D$ by restricting their domain to the subcategory S.

Admittedly, restricting the domain of a functor is not a very good party trick, even if done in a functorial way. The more impressive thing would be to *extend* the domain of a functor along some inclusion. The problem of extending the domain of a functor is one example of the sort of problem that Kan extensions are designed to do.

We have come up against the problem of finding a weak inverse to a lossy process before. For example, it is easy to turn a group into a set via the lossy process of forgetting the group structure; this is done by a functor $\mathcal{U}\colon \mathrm{Grp} \to \mathrm{Set}$. The inverse problem, i.e. turning a set into a group, is solved (to the extent possible) by taking a left adjoint to the forgetful functor. Together, these two functors form an adjunction, an example of a more general phenomenon known as *free-forgetful adjunction* (Note **??**). If we believe the mantra that adjunctions are weak inverses, we should try to solve the problem of extending the domain of a functor by searching for a functor which is adjoint to the restriction functor $\mathcal{I}^*$.

Of course, it is also interesting to consider the case where $\mathcal{I}$ is not an inclusion. In fact, we will see that by choosing $\mathcal{I}$ in different ways leads to a generalization of many different categorical concepts, such as limits.

1.2 Global Kan extensions

The most high-level notion (although, as we will see, not the most useful) of a Kan extension is called a *global Kan extension.* This is the concept discussed in .

Definition 1 (pullback)**.** Let $\mathcal{F}\colon C \to C'$ be a functor. For any other category D, $\mathcal{F}$ induces a functor, called the <u>pullback by $\mathcal{F}$</u>,

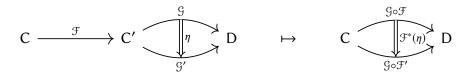$$\mathcal{F}^*\colon [C', D] \to [C, D]$$

which sends a functor $\mathcal{G}\colon C' \to D$ to its precomposition $\mathcal{G} \circ \mathcal{F}$

$$C' \xrightarrow{\mathcal{G}} D \qquad \mapsto \qquad \begin{array}{c} C \xrightarrow{\mathcal{F}^*\mathcal{G}} D \\ {}_{\mathcal{F}}\searrow \quad \nearrow_{\mathcal{G}} \\ C' \end{array}$$

and a natural transformation $\eta\colon \mathcal{G} \to \mathcal{G}'$ to its left whiskering (Example **??**) $\eta\mathcal{F}$.

$$C \xrightarrow{\mathcal{F}} C' \underset{\mathcal{G}'}{\overset{\mathcal{G}}{\Longrightarrow}}{\Downarrow}_{\eta}\, D \qquad \mapsto \qquad C \underset{\mathcal{G}\circ\mathcal{F}'}{\overset{\mathcal{G}\circ\mathcal{F}}{\Longrightarrow}}{\Downarrow}_{\mathcal{F}^*(\eta)}\, D$$

Definition 2 (Kan extension functor)**.** Let $\mathcal{F}\colon C \to C'$ be a functor, and let D be any other category. Consider the pullback functor $\mathcal{F}^*$.

- If $\mathcal{F}^*$ has right adjoint
$$\mathcal{F}_*\colon [C, D] \to [C', D].$$
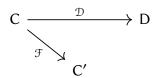We call $\mathcal{F}_*$ the <u>right Kan extension functor</u> along $\mathcal{F}$.

- If $\mathcal{F}^*$ has left adjoint
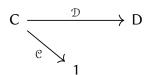$$\mathcal{F}_!\colon [C, D] \to [C', D].$$
We call $\mathcal{F}_!$ the <u>left Kan extension functor</u> along $\mathcal{F}$.

1.3 Kan extensions capture limits and colimits

When working with Kan extensions of a functor $\mathcal{D}$ along another functor $\mathcal{F}$, it is always helpful to have a drawing of the following triangle handy.

$$\begin{array}{c} C \xrightarrow{\quad\mathcal{D}\quad} D \\ {}_{\mathcal{F}}\searrow \\ \qquad C' \end{array}$$

Consider the case in which C' is the terminal category 1 (Example **??**). There is a unique functor from any category $C \to 1$, which sends every object to the unique object $* \in \mathrm{Obj}(1)$ and every morphism to the identity morphism. Furthermore, any functor $\mathcal{F}\colon 1 \to D$ is completely determined by where it sends the unique object $*$.

$$\begin{array}{c} C \xrightarrow{\quad\mathcal{D}\quad} D \\ {}_{\mathcal{C}}\searrow \\ \qquad 1 \end{array}$$

The restriction of any $\mathcal{F}\colon 1 \to C$ along $\mathcal{C}$ is simply the constant functor $\Delta_{\mathcal{F}(*)}$. Therefore, the functor $\mathcal{C}^*\colon [1, D] \to [C, D]$ carries exactly the same data as the diagonal functor $\Delta\colon D \to [C, D]$. Thus, $\mathcal{C}_!$ fits into the following adjunction.

$$\mathcal{C}_! : [C, D] \leftrightarrow D : \Delta$$

But we have seen that the colimit functor is right adjoint to the diagonal functor, so we have a natural isomorphism

$$\mathcal{C}_! \simeq \varinjlim.$$

That is, the left Kan extension of any functor $\mathcal{D}$ along the terminal functor $\mathcal{C}$ is[1] the colimit of $\mathcal{D}$:

$$\mathcal{C}_! \mathcal{D} = \varinjlim \mathcal{D}.$$

Exactly analogous reasoning tells us that the right Kan extension of any functor $\mathcal{D}$ along the terminal functor $\mathcal{C}$ is the limit of $\mathcal{D}$:

$$\mathcal{C}_* \mathcal{D} = \varprojlim \mathcal{D}.$$

1.4 Local Kan extensions

By Section **??**, we can also look at the right-adjointness of the pullback functor $\mathcal{F}^*$'s as guaranteeing that for every $\mathcal{D} \in \mathrm{Obj}([C', D])$, the category $(\mathcal{F}^* \downarrow \mathcal{D})$ has a terminal object. Similarly, we can look at $\mathcal{F}_!$'s left-adjointness as demonstrating that any category $(\mathcal{D} \downarrow \mathcal{F}^*)$ has an initial object. This mirrors the definition of limits and colimits as being terminal (resp. initial) objects in the category $(\Delta \downarrow \mathcal{D})$ (resp. $(\mathcal{D} \downarrow \Delta)$). As we saw in section 1.3, this is no accident; Kan extensions form a vast generalization of limits and colimits.
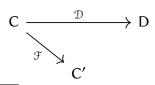
For now, we will specialize our discussion to the case of right Kan extensions and limits, so as not to have to keep writing down 'resp.'

Recall that we defined limits as universal cones. We have seen that there is a functor which assigns limits, and that this is the right adjoint to the diagonal functor. This gives us another way to define limits: we could first have defined the limit functor as right adjoint to the diagonal functor, and then defined limits as being the images of the limit functor. However, this would have been an unnecessarily restrictive definition, since in general some, but not all, limits exist; if we had chosen to define limits in this functorial way, in order to define limits for any diagram $\mathcal{D} \colon J \to C$ we would have had to demand that limits exist for *every* diagram in D. Clearly this is not what we want; for example, we want to be able to take products of two objects in a category without worrying about whether all limits exist in that category.

Fortunately, this is not what we did: we first defined the categories $(\Delta \downarrow \mathcal{D})$, then defined limits as their terminal objects. Only much later did we show that under certain conditions, limits organized themselves into a functor whose left adjoint was the diagonal functor.

By comparison, we first defined right Kan extensions as right adjoints to a restriction functor, then showed that their existence was equivalent to the existence of terminal morphisms in appropriate comma categories. The analogy with limits suggests that we can define Kan extension of one functor along another even when not all extensions exist.

Definition 3 (local Kan extension)**.** Consider categories and functors as below.

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[1]Strictly speaking, we should write "uniquely isomorphic to," but we don't because limits are only defined up to unique isomorphism anyway.

- A right Kan extension of $\mathcal{D}$ along $\mathcal{F}$ is a terminal morphism in the category $(\mathcal{F}^* \downarrow \mathcal{D})$.

- A left Kan extension of $\mathcal{D}$ along $\mathcal{F}$ is a terminal morphism in the category $(\mathcal{D} \downarrow \mathcal{F}^*)$.

This local definition will turn out to be much more useful than Definition 2. It will even to allow us to compute Kan extensions via an explicit formula in certain situations.

Example 4. In the case where $C' = 1$, we saw that global left and right Kan extensions recovered the limit and colimit functors. Essentially by definition, this works for local Kan extensions as well; we have

$$\mathcal{C}_* \mathcal{D} = \varprojlim \mathcal{D}, \qquad \text{and} \qquad \mathcal{C}_! \mathcal{D} = \varinjlim \mathcal{D}.$$

We defined limits in terms of cones. The close analogy between limits and Kan extensions suggests that we should define a Kan extensionish analog of cones.

The appropriate notion is that of an *extension*. Extensions generalize (co)cones in the sense that they reduce to them in the case of Kan extensions along $\mathcal{C}$.

Definition 5 (extension)**.** Consider categories and functors as below.

$$C \xrightarrow{\quad \mathcal{D} \quad} D$$
$$\downarrow^{\mathcal{F}}$$
$$C'$$

- A right extension of $\mathcal{G}$ along $\mathcal{F}$ is an object in the comma category $(\mathcal{F}^* \downarrow \mathcal{D})$. That is, it is a pair $(\mathcal{G}, \eta)$, where $\mathcal{G} \colon C' \to D$ and

$$\eta \colon \mathcal{D} \Rightarrow \mathcal{F}^* \mathcal{G}$$

- A left extension of $\mathcal{G}$ along $\mathcal{F}$ is an object in the comma category $(\mathcal{D} \downarrow \mathcal{F}^*)$.

Example 6. We have seen that Kan extensions along the functor $C \to 1$ reproduce limits and colimits. It will be instructive to study Kan extensions along functors $C \to n$, where n is the category with n objects and only identity morphisms, i.e. the discrete groupoid with n objects.

Consider such a functor $C \to n$. Denote by i the functor $1 \to n$ which takes $* \in \mathrm{Obj}(1)$ to the ith object in n.
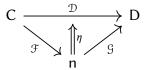
A functor $\mathcal{F} \colon C \to n$ is the same thing as a decomposition

$$C = \coprod_{c=1}^{n} C_i,$$

where each C_i is the fiber over $i \in \mathrm{Obj}(n)$, together with a family of functors

$$\mathcal{D}|_{C_i} \colon C_i \to 1.$$

Consider right extensions of $\mathcal{D}$ along $\mathcal{F}$, i.e. objects in $(\mathcal{F}^* \downarrow \mathcal{D})$. These consist of pairs $(\eta, \mathcal{G})$, where $\eta \colon \mathcal{F}^* \mathcal{G} \to \mathcal{D}$.

$$C \xrightarrow{\quad \mathcal{D} \quad} D$$
$$\downarrow^{\mathcal{F}} \quad \Uparrow^{\eta} \quad \nearrow^{\mathcal{G}}$$
$$n$$

The functor $\mathcal{F}^*\mathcal{G}$ is really the composition $\mathcal{G} \circ \mathcal{F}$, which (by the universal property for coproducts) consists of n constant functors $\Delta_d^i \colon C_i \to D$, where Δ_d^i is the constant functor mapping everything in C_i to $d \in \mathrm{Obj}(D)$. The natural transformation η therefore consists of n cones

$$\eta^i \colon \Delta_d^i \Rightarrow \mathcal{D}|_{C_i}.$$

Such a right extension is universal if any other right extension factors through it. That is, given another right extension $(\epsilon, \mathcal{H})$, there is a unique natural transformation $\xi \colon \mathcal{H} \circ \mathcal{F} \Rightarrow \mathcal{G} \circ \mathcal{F}$ making the following diagram commute.

$$\mathcal{H} \circ \mathcal{F} \overset{\xi}{\Longrightarrow} \mathcal{G} \circ \mathcal{F}$$
$$\epsilon \searrow \quad \swarrow \eta$$
$$\mathcal{D}$$

This says precisely that each cone must be universal. That is, we have

$$(\mathcal{F}_*\mathcal{D})(i) = \varprojlim \mathcal{D}|_{C_i},$$

where $D|_{C_i}$ is simply $\mathcal{D}$ restricted to the fiber over i.

The same is true for left Kan extensions; the value

$$(\mathcal{F}_!\mathcal{D})(i) = \varinjlim \mathcal{D}|_{C_i}.$$

We have more or less proved the following. I still need to work out size conditions, e.g. C' should almost certainly have to be small.

Proposition 7. Let C and D be categories and C' a discrete groupoid, and let $\mathcal{F} \colon C \to C'$ and $\mathcal{D} \colon C \to D$ be functors.

- Suppose the limit $\varprojlim \mathcal{D}|_{C_i}$ exists for all $i \in \mathrm{Obj}(C')$. Then a right Kan extension $\mathcal{F}_*\mathcal{D}$ exists, and is given by the formula

$$(\mathcal{F}_*\mathcal{D})(i) = \varprojlim \mathcal{D}|_{C_i},$$

- Suppose the colimit $\varinjlim \mathcal{D}|_{C_i}$ exists for all $i \in \mathrm{Obj}(C')$. Then a left Kan extension $\mathcal{F}_!\mathcal{D}$ exists, and is given by the formula

$$(\mathcal{F}_!\mathcal{D})(i) = \varinjlim \mathcal{D}|_{C_i}.$$

This says that a Kan extension along a functor to a discrete groupoid C' consists of a collection of (co)limits parametrized by the objects of C'.

Note 8. In our simple case, the converse also holds: any Kan extension through a discrete groupoid is given by the above formulae. This is *not* a general feature of the formula we derive! There are Kan extensions whose

1.5 A pointwise formula for Kan extensions

One would hope that we would be able to generalize the above formulae easily. However, their derivation rested crucially on the fact that for any category C′ with no morphisms, any functor C → C′ splits C into a disjoint union of its fibers, and we can easily construct (co)cones over the images of these fibers. Adding any morphisms to C′ ruins this splitting, and we lose our (co)cones.

The solution to this problem is to use 'smarter' categories than the fibers of $\mathcal{F}$, over which we can still construct (co)cones in an obvious way: comma categories.

The intuition is as follows. Any morphism $f\colon c' \to \tilde{c}'$ we add to C′ allows the fiber $C_{c'}$ to map into to the fiber $C_{\tilde{c}'}$. We should include in the base of our cone not just the parts of C which are mapped to the identity $\mathrm{id}_{c'}$ (i.e. the fiber over c'), but also any fibers to which $C_{c'}$ has access via some morphism. We have such a category: $(c' \downarrow \mathcal{F})$.

Let $\mathcal{D}\colon C \to D$ and $\mathcal{F}\colon C \to C'$ be functors. Let $(\mathcal{G}, \eta)$ be a right extension of $\mathcal{D}$ along $\mathcal{F}$.

$$
\begin{array}{ccc}
C & \xrightarrow{\;\mathcal{D}\;} & D \\
& \underset{\mathcal{F}}{\searrow} \;\; \Uparrow \eta \;\; \nearrow_{\mathcal{G}} & \\
& C' &
\end{array}
$$

For any $c' \in \mathrm{Obj}(C')$, denote by $(c' \downarrow \mathcal{F})$ the category of morphisms from c' down to $\mathcal{F}$ (see Definition ??), and denote by $\mathcal{U}$ the forgetful functor $(c' \downarrow \mathcal{F}) \to C$. We can define a functor $\mathcal{D}|_{(c'\downarrow\mathcal{F})}\colon (c' \downarrow \mathcal{F}) \to D$ as the following composition.

$$
(c' \downarrow \mathcal{F}) \xrightarrow{\;\mathcal{U}\;} C \xrightarrow{\;\mathcal{D}\;} D
$$

To each such $c' \in \mathrm{Obj}(C')$, there is an associated cone ξ over $\mathcal{D}|_{(c'\downarrow\mathcal{F})}$, constructed as follows.

- The tip of the cone is $\mathcal{G}(c')$.
- For each $(\alpha, f) \in \mathrm{Obj}(c' \downarrow \mathcal{F})$, the morphism $\xi_{(\alpha,f)}\colon \mathcal{D}(c') \to \mathcal{D}|_{(c'\downarrow\mathcal{F})}(\alpha, f) = \mathcal{D}(\alpha)$ is given by the composition

$$
\mathcal{G}(c') \xrightarrow{\;\mathcal{G}(f)\;} \mathcal{G}(\mathcal{F}(\alpha)) \xrightarrow{\;\eta_\alpha\;} \mathcal{D}(\alpha) \;.
$$

Dually, considering now a *left* extension $(\mathcal{G}, \eta)$ of $\mathcal{D}$ along $\mathcal{F}$, we get a cone as follows.
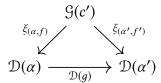
- The tip is $\mathcal{G}(c')$
- For each $(\alpha, f) \in \mathrm{Obj}(\mathcal{F} \downarrow c')$, the morphism $\xi_{\alpha,f}\colon \mathcal{D}(\alpha) \to \mathcal{D}(c')$ is given by the composition

$$
\mathcal{D}(\alpha) \xrightarrow{\;\eta_\alpha\;} (\mathcal{G} \circ \mathcal{F})(\alpha) \xrightarrow{\;\mathcal{G}(f)\;} \mathcal{G}(c')
$$

Lemma 9. The cone described above really is a cone. That is, for each morphism $g\colon (\alpha, f) \to (\alpha', f')$ as follows
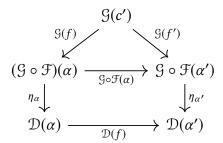
$$
\begin{array}{ccc}
& c' & \\
{}^{f}\swarrow & & \searrow^{f'} \\
\mathcal{F}(\alpha) & \xrightarrow[\mathcal{F}(g)]{} & \mathcal{F}(\alpha')
\end{array}
$$

the diagram

$$
\begin{array}{ccc}
 & \mathcal{G}(c') & \\
\xi_{(\alpha,f)}\swarrow & & \searrow\xi_{(\alpha',f')} \\
\mathcal{D}(\alpha) & \xrightarrow[\mathcal{D}(g)]{} & \mathcal{D}(\alpha')
\end{array}
$$

commutes.

Proof. Consider the following diagram.

$$
\begin{array}{ccc}
 & \mathcal{G}(c') & \\
\mathcal{G}(f)\swarrow & & \searrow\mathcal{G}(f') \\
(\mathcal{G}\circ\mathcal{F})(\alpha) & \xrightarrow[\mathcal{G}\circ\mathcal{F}(\alpha)]{} & \mathcal{G}\circ\mathcal{F}(\alpha') \\
\eta_\alpha\downarrow & & \downarrow\eta_{\alpha'} \\
\mathcal{D}(\alpha) & \xrightarrow[\mathcal{D}(f)]{} & \mathcal{D}(\alpha')
\end{array}
$$

The upper triangle is the functor $\mathcal{D}$ applied to the definition of commutativity in a category of morphisms, and the lower square is the naturality square for η. □

Example 10. In the case that C' is a discrete groupoid, we have the following.

- The category $C_{c'}$ is equivalent (in fact isomorphic) to both of the categories $(\mathcal{F}\downarrow c')$ and $(c'\downarrow\mathcal{F})$.

- The cone which constructed above over $\mathcal{D}|_{(c'\downarrow\mathcal{F})}$ agrees with the cones η^i from Example 6.

- The cocone under $\mathcal{D}|_{(\mathcal{F}\downarrow c')}$ agrees with the cocones that we would have constructed if we had more time.

That is, the (co)cones we have constructed are a more general case of those we got from considering discrete groupoids.
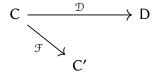
Lemma 11. Let $\mathcal{F}\colon \mathsf{I}\to\mathsf{J}$ be a functor. Let $f\colon j\to j'$ be a morphism in J. Then $\mathcal{F}$ induces the following functors.

- $(\mathcal{F}\downarrow j)\to(\mathcal{F}\downarrow j')$.

- $(j'\downarrow\mathcal{F})\to(j\downarrow\mathcal{F})$.

Proof. Obvious. □

We saw above that given any extension $(\mathcal{G},\eta)$ and any object $c'\in\mathrm{Obj}(\mathsf{C}')$, we could construct a cone. We hoped that this cone would hope turn out to be the analog of the cones over C_i from Example 6. This turns out to be the case.

Theorem 12. Consider categories and functors as follows,

$$
\begin{array}{ccc}
\mathsf{C} & \xrightarrow{\ \mathcal{D}\ } & \mathsf{D} \\
{}_{\mathcal{F}}\searrow & & \\
 & \mathsf{C}' &
\end{array}
$$

where C is small and D has all small colimits. Then the Kan extension $\mathcal{F}_!\mathcal{D}$ exists and has values

$$(\mathcal{F}_!\mathcal{D})(c') = \varinjlim \left((\mathcal{F} \downarrow c') \longrightarrow C \xrightarrow{\mathcal{D}} D \right)$$

Proof. We first have to define how $\mathcal{F}_!\mathcal{D}$ behaves on morphisms. To this end, let $f \colon c' \to c''$. Then by [Lemma 11](), f induces a $\qquad\qquad\qquad\qquad$ □

Lemma 13. Let $(\mathcal{G}, \eta)$ be a right extension as below.

If for every $c' \in \mathrm{Obj}(C')$ the cone with tip $\mathcal{G}(c')$ over the functor given by the composition

$$(c' \downarrow \mathcal{F}) \to C \to D$$

with sides

$$\xi_{(\alpha, f)} = \left(\mathcal{G}(c') \xrightarrow{\mathcal{G}(f)} (\mathcal{G} \circ \mathcal{F})(\alpha) \xrightarrow{\eta_\alpha} \mathcal{D}(\alpha) \right)$$

is a limit cone, then $(\mathcal{G}, \eta)$ is a right Kan extension.

Proof. Let $(\mathcal{G}', \eta')$ be any other extension. We need to show that there is a unique natural transformation χ as follows.

$$\mathcal{G}' \circ \mathcal{F} \qquad\qquad\qquad \mathcal{G} \circ \mathcal{F}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad □$$

Definition 14 (limit over a comma category). Consider categories and functors as follows.

We define the notations

$$\varprojlim_{c \in (\mathcal{F} \downarrow c')} \mathcal{D}(c) = \varprojlim \left[(\mathcal{F} \downarrow c') \to C \to D \right]$$

and

$$\operatorname*{colim}_{c \in (c' \downarrow \mathcal{F})} \mathcal{F}(c) = \varinjlim \left[(c' \downarrow \mathcal{F}) \to C \to D \right]$$

1.6 Properties of Kan extensions

Recall that our original motivation for defining Kan extensions was the problem of extending the domain of an inclusion. One would hope that this extension, restricted to the original domain, would agree with the original functor. In situations in which we are allowed to use the colimit formula, this turns out to be the case.

Theorem 15. Let C be a small category, and D locally small. Let $\mathcal{F}\colon C \to C'$ be fully faithful.

- If D has small limits, then the counit

$$\epsilon\colon \mathcal{F}^* \circ \mathcal{F}_* \Rightarrow \mathrm{id}_{C'}$$

  is a natural isomorphism.
- If D has small colimits, then the unit

$$\eta\colon \mathrm{id}_D \Rightarrow \mathcal{F}^* \circ \mathcal{F}_!$$

  is a natural isomorphism.

Proof. We need to show that each component

$$\epsilon_{\mathcal{D}}\colon (\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D}) \Rightarrow \mathrm{id}_{C'}$$

is a natural isomorphism. By definition of the pullback, we have

$$(\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D}) = \mathcal{F}^*(\mathcal{F}_* \mathcal{D}) = \mathcal{F}_* \mathcal{D} \circ \mathcal{F}.$$

Consider some $c' \in \mathrm{Obj}(C')$. By [Theorem 12](#), we can compute $\mathcal{F}_* \mathcal{D}(c')$ via the limit formula

$$\mathcal{F}_* \mathcal{D}(c') = \lim_{c \in (\mathcal{F} \downarrow c')} \mathcal{D}(c).$$

In particular, if $c' = \mathcal{F}(c)$ for some c, we have

$$\mathcal{F}_* \mathcal{D}(\mathcal{F}(c)) = \lim_{c \in (\mathcal{F} \downarrow \mathcal{F}(c))} \mathcal{D}(c) = \lim\left[(\mathcal{F} \downarrow \mathcal{F}(c)) \to C \to \mathcal{D}\right].$$

Since the object $(c, \mathrm{id}_{\mathcal{F}(c)})$ is initial in $(\mathcal{F} \downarrow \mathcal{F}(c))$ we have by Theorem ?? that

$$\lim\left[(\mathcal{F} \downarrow \mathcal{F}(c)) \to C \to \mathcal{D}\right] = \mathcal{D}(c).$$

Thus, we have

$$(\mathcal{F}^* \circ \mathcal{F}_*)(\mathcal{D})(c) \simeq \mathcal{D}(c)$$

for all $c \in \mathrm{Obj}(C)$ $\qquad\square$

Corollary 16. Let C be a small category, and D locally small. Let $\mathcal{F}\colon C \to C'$ be fully faithful.

- If D has small limits, then $\mathcal{F}_*$ is fully faithful.
- If D has small colimits, then $\mathcal{F}_!$ fully faithful.

1.7 Yoneda extensions

One particularly important example of Kan extension is Kan extension along the Yoneda embedding.

Definition 17 (Yoneda extension)**.** One calls left Kan extensions along the Yoneda embedding (Definition **??**) <u>Yoneda extensions</u>.

One can express any presheaf as a colimit.

Lemma 18. Let C be a locally small category. Denote by $\mathcal{Y}$ the Yoneda embedding $C \to [C^{op}, Set]$. Then for any functor $\mathcal{F}\colon C^{op} \to Set$ we have

$$\mathcal{F} \simeq \varinjlim \left[(\mathcal{Y} \downarrow \mathcal{F}) \xrightarrow{\mathcal{U}} C \xrightarrow{\mathcal{Y}} [C^{op}, Set] \right].$$

Proof. We need to show that $\mathcal{F}$ is the base of a universal cocone under $\mathcal{Y} \circ \mathcal{U}$. That $\mathcal{F}$ is the base of a cocone is obvious, since every object (α, f) comes with a map $f\colon \mathcal{Y}(\alpha) \to \mathcal{F}$ as required, and the diagram below manifestly commutes for each morphism g.

$$\mathcal{Y}(\alpha) \xrightarrow{\ \ \mathcal{Y}(g)\ \ } \mathcal{Y}(\alpha')$$
$$f \searrow \quad \swarrow f'$$
$$\mathcal{F}$$

That is, $f' \circ \mathcal{Y}(g) = f$ It remains to show that $\mathcal{F}$ is universal, i.e. that for any other cocone with tip $\mathcal{G} \in [C^{op}, Set]$ and components

$$\xi_{(\alpha,f)}\colon \mathcal{Y}(c) \Rightarrow \mathcal{G}$$

making the diagrams

$$\mathcal{Y}(\alpha) \xrightarrow{\ \ \mathcal{Y}(g)\ \ } \mathcal{Y}(\alpha')$$
$$\xi_{(\alpha,f)} \searrow \quad \swarrow \xi_{(\alpha',f')}$$
$$\mathcal{G}$$

commute (i.e. so that $\xi_{(\alpha',f')} \circ \mathcal{Y}(g) = \xi_{(\alpha,f)}$), this data is enough to define a unique natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$.

But each object (α, f) also gives us, again by Yoneda, an element of $\mathcal{G}(\alpha)$. Thus, we are able to assign to each element of $\mathcal{F}(\alpha)$ a unique element of $\mathcal{G}(\alpha)$, giving us a map $\mathcal{F}(\alpha) \to \mathcal{G}(\alpha)$. This gives us a correspondence

$$\mathrm{Hom}_{[C^{op},Set]}(\mathcal{Y}(\alpha), \mathcal{F}) \to \mathrm{Hom}_{[C^{op},Set]}(\mathcal{Y}(\alpha), \mathcal{G}),$$

natural in α. By Yoneda, we get a natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$. $\square$

Lemma 19. Let I be a small category, and let C be locally small and cocomplete. Let $\mathcal{F}$ be a functor. Then the Yoneda extension $\mathcal{Y}_!\mathcal{F}$ is left adjoint to the functor

$$\mathcal{G}\colon C \to [I^{op}, Set]; \qquad c \mapsto \mathrm{Hom}_C(F(-), c).$$

$$I \xrightarrow{\ \ \mathcal{F}\ \ } C \qquad\qquad \mathcal{Y}_!\mathcal{F} : [I^{op}, Set] \leftrightarrow C : \mathcal{G}$$
$$\mathcal{Y} \searrow \quad \nearrow \mathcal{Y}_!\mathcal{F}$$
$$[I^{op}, Set]$$

Proof. Since C is cocomplete, we are entitled to use the colimit formula for left Kan extensions. That is, we can write

$$(\mathcal{Y}_!\mathcal{F})(\mathcal{H}) = \varinjlim\left[(\mathcal{Y}\downarrow\mathcal{H}) \xrightarrow{\mathcal{U}} I \xrightarrow{\mathcal{F}} C\right] = \varinjlim\left[\mathcal{F}\circ\mathcal{U}\right].$$

Thus, we have the following string of natural isomorphisms.

$$\begin{aligned}
\mathrm{Hom}_C((\mathcal{Y}_!\mathcal{F})(\mathcal{H}), c) &\simeq \mathrm{Hom}_C(\varinjlim[\mathcal{F}\circ\mathcal{U}], c) \\
&\simeq \varprojlim \mathrm{Hom}_C(\mathcal{F}(-), c)\circ\mathcal{U} \\
&\simeq \varprojlim \mathcal{G}(c)\circ\mathcal{U} \\
&\simeq \varprojlim \mathrm{Hom}_{[I^{\mathrm{op}},\mathrm{Set}]}(\mathcal{Y}(-), \mathcal{G}(c))\circ\mathcal{U}. \\
&\simeq \mathrm{Hom}_{[I^{\mathrm{op}},\mathrm{Set}]}(\varinjlim\mathcal{U}\circ\mathcal{Y}, \mathcal{G}(c)) \\
&\simeq \mathrm{Hom}_{[I^{\mathrm{op}},\mathrm{Set}]}(\mathcal{H}, \mathcal{G}(c)).
\end{aligned}$$

$\square$

Corollary 20. Let I be a small category, and let C be locally small and cocomplete. Let $\mathcal{F}$ be a functor. Then the Yoneda extension $\mathcal{Y}_!\mathcal{F}$ is cocontiuous.

Theorem 21. Let C be a small category, and let D be cocomplete. Let $\mathcal{F}\colon [C^{\mathrm{op}}, \mathrm{Set}] \to D$ be any functor. Then $\mathcal{F}$ preserves colimits if and only if it is the left Yoneda extension of its restriction $\mathcal{Y}^*F$.

Proof. Assume that $\mathcal{F} = (\mathcal{Y}_!\mathcal{Y}^*)(\mathcal{F})$. By [Lemma 19](#), $(\mathcal{Y}_!\mathcal{Y}^*)(\mathcal{F})$ is a left adjoint, hence preserves colimits by Corollary **??**.

Now assume that $\mathcal{F}$ preserves colimits. Pick some functor $\mathcal{G}\colon C^{\mathrm{op}} \to \mathrm{Set}$. By [Lemma 18](#), we can write

$$\begin{aligned}
\mathcal{F}(\mathcal{G}) &\simeq \mathcal{F}\left(\varinjlim\left[(\mathcal{Y}\downarrow\mathcal{G}) \to C \xrightarrow{\mathcal{Y}} [C^{\mathrm{op}}, \mathrm{Set}]\right]\right) \\
&\simeq \varinjlim\left[(\mathcal{Y}\downarrow\mathcal{G}) \to C \xrightarrow{\mathcal{Y}} [C^{\mathrm{op}}, \mathrm{Set}] \xrightarrow{\mathcal{F}} D\right] \\
&\simeq \varinjlim\left[(\mathcal{Y}\downarrow\mathcal{G}) \to C \xrightarrow{\mathcal{Y}^*\mathcal{F}} D\right] \\
&\simeq \mathcal{Y}_!(\mathcal{Y}^*\mathcal{F})(\mathcal{G}).
\end{aligned}$$

$\square$

Corollary 22. Let C be a small category, and D locally small. There is an equivalence of categories

$$[[C^{\mathrm{op}}, \mathrm{Set}], D]_{\mathrm{cocontinuous}} \simeq [C, D].$$

That is, colimit preserving functors $[C^{\mathrm{op}}, \mathrm{Set}] \to D$ are equivalent to functors $C \to D$.

Proof. We have an adjunction

$$\mathcal{Y}_! : [C, D] \leftrightarrow [[C^{\mathrm{op}}, \mathrm{Set}], D] : \mathcal{Y}^*.$$

By Corollary 20, the image of $\mathcal{Y}_!$ is completely contained in $[[C^{op}, Set], D]_{cocontinuous}$, so the above adjunction extends to an adjunction

$$\mathcal{Y}_! : [C, D] \leftrightarrow [[C^{op}, Set], D]_{cocontinuous} : \mathcal{Y}^*.$$

We know that the Yoneda embedding is fully faithful. Thus, by Theorem 15, we have that the unit is an isomorphism. Also, by Theorem 21, the counit is an isomorphism, so the above adjunction is in fact an equivalence of categories. □