1 Kan extensions

1.1 Global Kan extensions

Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{C}'$ be a functor. For any other category D, $\mathcal{F}$ induces a functor

$$\mathcal{F}^*\colon [\mathsf{C}', \mathsf{D}] \to [\mathsf{C}, \mathsf{D}]$$

which sends a functor $\mathcal{G}\colon \mathsf{C}' \to \mathsf{D}$ to its precomposition $\mathcal{G} \circ \mathcal{F}$

$$\begin{tikzcd} \mathsf{C}' \arrow[r, "\mathcal{G}"] & \mathsf{D} & \mapsto & \mathsf{C} \arrow[r, "\mathcal{F}"'] \arrow[rr, bend left, "\mathcal{F}^*(\mathcal{G})"] & \mathsf{C}' \arrow[r, "\mathcal{G}"'] & \mathsf{D} \end{tikzcd}$$

and a natural transformation $\eta\colon \mathcal{G} \to \mathcal{G}'$ to its left whiskering (Example **??**) $\eta\mathcal{F}$.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{C}' \arrow[r, bend left, "\mathcal{G}"] \arrow[r, bend right, "\mathcal{G}'"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{D} & \mapsto & \mathsf{C} \arrow[r, bend left, "\mathcal{G}\circ\mathcal{F}"] \arrow[r, bend right, "\mathcal{G}\circ\mathcal{F}'"'] \arrow[r, phantom, "\Downarrow\mathcal{F}^*(\eta)"] & \mathsf{D} \end{tikzcd}$$

Definition 1 (Kan extension). Suppose the functor $\mathcal{F}^*$ defined above has a left adjoint

$$\mathcal{F}_!\colon [\mathsf{C}, \mathsf{D}] \to [\mathsf{C}', \mathsf{D}].$$

and right adjoint

$$\mathcal{F}_*\colon [\mathsf{C}, \mathsf{D}] \to [\mathsf{C}', \mathsf{D}].$$

- We call $\mathcal{F}_!$ the left Kan extension along $\mathcal{F}$. For $\mathcal{G} \in [\mathsf{C}, \mathsf{D}]$, we call $\mathcal{F}_!(\mathcal{G})$ the left Kan extension of $\mathcal{G}$ along $\mathcal{F}$.
- We call $\mathcal{F}_*$ the right Kan extension along $\mathcal{F}$, and for any $\mathcal{G} \in [\mathsf{C}, \mathsf{D}]$, we call $\mathcal{F}_*(\mathcal{G})$ the right Kan extension of $\mathcal{G}$ along $\mathcal{F}$.

Example 2. Consider the case in which C' is the terminal category 1 (Example **??**). There is a unique functor from any category $\mathsf{C} \to 1$, which sends every object to 1 and every morphism to the identity morphism. Furthermore, any functor $\mathcal{F}\colon 1 \to \mathsf{D}$ is completely determined by where it sends the unique object $*$, as $\mathcal{F}(\mathrm{id}_*) = \mathrm{id}_{\mathcal{F}(\mathrm{id}(*))}$.

Now fix categories C, and consider functors $\mathcal{F}\colon \mathsf{C} \to 1$ and $\mathcal{F}_D\colon 1 \to \mathsf{D}$.