1 Adjunctions

1.1 Motivating example

Note 1. This section is under heavy construction. It's not very good at the moment.

Consider the following functors:

- $\mathcal{U}\colon \mathsf{Grp} \to \mathsf{Set}$, which sends a group to its underlying set, and
- $\mathcal{F}\colon \mathsf{Set} \to \mathsf{Grp}$, which sends a set to the free group on it.

The functors $\mathcal{U}$ and $\mathcal{F}$ are dual in the following sense: $\mathcal{U}$ is the most efficient way of moving from Grp to Set since all groups are in particular sets; $\mathcal{F}$ might be thought of as providing the most efficient way of moving from Set to Grp. But how would one go about formalizing this?

Free-forgetful adjunctions are an example of an extremely powerful structure called an *adjuction.* We have encountered it before—we saw in Example **??** that there was a natural bijection between

$$\mathrm{Hom}_{\mathsf{C}}(X \times (-), B) \qquad \text{and} \qquad \mathrm{Hom}_{\mathsf{C}}(X, B^{(-)}).$$

this is a so-called *hom-set adjunction.*

1.1.1 Hom-set adjunction

The functors $\mathcal{U}$ and $\mathcal{F}$ have the following property. Let S be a set, G be a group, and let $f\colon S \to \mathcal{U}(G)$, $s \mapsto f(s)$ be a set-function. Then there is an associated group homomorphism $\tilde{f}\colon \mathcal{F}(S) \to G$, which sends $s_1 s_2 \ldots s_n \mapsto f(s_1 s_2 \ldots s_n) = f(s_1) \cdots f(s_n)$. In fact, $\tilde{f}$ is the unique homomorphism $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Similarly, for every group homomorphism $g\colon \mathcal{F}(S) \to G$, there is an associated function $S \to \mathcal{U}(G)$ given by restricting g to S. In fact, this is the unique function $\mathcal{F}(S) \to G$ such that $f(s) = \tilde{f}(s)$ for all $s \in S$.

Thus for each $f \in \mathrm{Hom}_{\mathsf{Grp}}(S, \mathcal{U}(G))$ we can construct an $\tilde{f} \in \mathrm{Hom}_{\mathsf{Set}}(\mathcal{F}(S), G)$, and vice versa.

Let us add some mathematical scaffolding to the ideas explored above. We build two functors $\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp} \to \mathsf{Set}$ as follows.

1. Our first functor maps the object $(S, G) \in \mathrm{Obj}(\mathsf{Set}^{\mathrm{op}} \times \mathsf{Grp})$ to the hom-set $\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G)$, and a morphism $(\alpha, \beta)\colon (S, G) \to (S', G')$ to a function
$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S'), G'); \qquad m \mapsto \mathcal{F}(\alpha) \circ m \circ \beta$$
2. Our second functor maps (S, G) to $\mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$, and (α, β) to
$$m \mapsto \alpha \circ m \circ \mathcal{U}(\beta).$$

We can define a natural isomorphism Φ between these functors with components

$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G)); \qquad f \to \tilde{f}.$$

1.1.2 Unit-counit adjunction

Suppose we take the natural isomorphism from the last section

$$\Phi_{S,G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), G) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(G))$$

and evaluate it at $S = \mathcal{U}(G)$. This gives us an isomorphsim

$$\Phi_{\mathcal{U}(G),G}\colon \mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(\mathcal{U}(S)), G) \to \mathrm{Hom}_{\mathsf{Set}}(\mathcal{U}(G), \mathcal{U}(G)).$$

In particular, the element $\mathrm{id}_{\mathcal{U}(G)}$ is the preimage of some group homomorphism

$$\epsilon_G\colon \mathcal{F}(\mathcal{U}(G)) \to G.$$

Taken together for all G, the ϵ_G form a natural transformation

$$\epsilon\colon \mathcal{F} \circ \mathcal{U} \to \mathrm{id}_{\mathsf{Grp}}.$$

evaluate it at $G = \mathcal{F}(S)$. This gives us an isomorphism

$$\mathrm{Hom}_{\mathsf{Grp}}(\mathcal{F}(S), \mathcal{F}(S)) \to \mathrm{Hom}_{\mathsf{Set}}(S, \mathcal{U}(\mathcal{F}(G))).$$

In particular, the identity $\mathrm{id}_{\mathcal{F}(G)}$ is mapped to some map

$$\eta_S\colon S \to \mathcal{U}(\mathcal{G}(S)).$$

In fact, taken together, the η_S form a natural transformation

$$\eta\colon \mathrm{id}_{\mathsf{Set}} \Rightarrow \mathcal{U} \circ \mathcal{G}.$$

1.2 Hom set adjunction

Definition 2 (hom-set adjunction)**.** Let C, D be categories and $\mathcal{F}$, $\mathcal{G}$ functors as follows.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"] \end{tikzcd}$$

We say that $\mathcal{F}$ is left-adjoint to $\mathcal{G}$ (or equivalently $\mathcal{G}$ is right-adjoint to $\mathcal{F}$) and write $\mathcal{F} \dashv \mathcal{G}$ if there is a natural isomorphism

$$\Phi\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-), -) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(-, \mathcal{G}(-)),$$

which fits between $\mathcal{F}$ and $\mathcal{G}$ like this.

$$\begin{tikzcd}[column sep=huge] \mathsf{C}^{\mathrm{op}} \times \mathsf{D} \arrow[r, bend left, "\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(-){,}-)"{name=U}] \arrow[r, bend right, "\mathrm{Hom}_{\mathsf{D}}(-{,}\mathcal{G}(-))"'{name=L}] & \mathsf{Set} \arrow[Rightarrow, from=U, to=L, "\Phi"] \end{tikzcd}$$

The natural isomorphsim amounts to a family of bijections

$$\Phi_{A,B}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$$

which satisfies the coherence conditions for a natural transformation.

1.3 Unit-counit adjunction

Definition 3 (unit-counit adjunction)**.** We say that two functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a unit-counit adjunction if there are two natural transformations

$$\eta\colon \mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}, \qquad \text{and} \qquad \varepsilon\colon \mathcal{F} \circ \mathcal{G} \Rightarrow \mathrm{id}_{\mathsf{D}},$$

called the *unit* and *counit* respectively, which make the following so-called *triangle diagrams*

$$\begin{tikzcd} \mathcal{F} \arrow[r, Rightarrow, "\mathcal{F}\eta"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F}\mathcal{G}\mathcal{F} \arrow[d, Rightarrow, "\varepsilon\mathcal{F}"] \\ & \mathcal{F} \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{G} \arrow[r, Rightarrow, "\eta\mathcal{G}"] \arrow[dr, Rightarrow, "\mathrm{id}_{\mathcal{G}}"'] & \mathcal{G}\mathcal{F}\mathcal{G} \arrow[d, Rightarrow, "\mathcal{G}\varepsilon"] \\ & \mathcal{G} \end{tikzcd}$$

commute.

Note 4. Using η for the unit and ϵ for the counit is very common, but not universal! Both Wikipedia and the nLab have them this way, but the Higher Categories course uses them the other way around.

The triangle diagrams take quite some explanation. The unit η is a natural transformation $\mathrm{id}_{\mathsf{C}} \Rightarrow \mathcal{G} \circ \mathcal{F}$. We can draw it like this.

$$\begin{tikzcd} \mathsf{C} \arrow[r, bend left, "\mathrm{id}_{\mathsf{C}}"] \arrow[r, bend right, "\mathcal{G}\circ\mathcal{F}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathsf{C} \end{tikzcd}.$$

Analogously, we can draw ε like this.

$$\begin{tikzcd} \mathsf{D} \arrow[r, bend left, "\mathcal{F}\circ\mathcal{G}"] \arrow[r, bend right, "\mathrm{id}_{\mathsf{D}}"'] \arrow[r, phantom, "\Downarrow\varepsilon"] & \mathsf{D} \end{tikzcd}.$$

We can arrange these artfully like so.

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rr, bend left=60, "\mathrm{id}_{\mathsf{C}}"] \arrow[rr, bend left=20, "\mathcal{G}\circ\mathcal{F}"'] \arrow[rr, phantom, bend left=40, "\Downarrow\eta"] & \mathsf{D} \arrow[r, "\mathcal{G}"] \arrow[rr, bend right=20, "\mathcal{F}\circ\mathcal{G}"] \arrow[rr, bend right=60, "\mathrm{id}_{\mathsf{D}}"'] \arrow[rr, phantom, bend right=40, "\Downarrow\varepsilon"] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \end{tikzcd}$$

Notice that we haven't actually *done* anything; this diagram is just the diagrams for the unit and counit, plus some extraneous information.

We can whisker the η on top from the right, and the ε below from the left, to get the following diagram,

$$\begin{tikzcd} \mathsf{C} \arrow[r, "\mathcal{F}"] \arrow[rrr, bend left=60, "\mathcal{F}"] \arrow[rrr, bend left=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"'] \arrow[rrr, phantom, bend left=45, "\Downarrow\mathcal{F}\eta"] \arrow[rrr, bend right=25, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}"] \arrow[rrr, bend right=60, "\mathcal{F}"'] \arrow[rrr, phantom, bend right=45, "\Downarrow\varepsilon\mathcal{F}"] & \mathsf{D} \arrow[r, "\mathcal{G}"] & \mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \end{tikzcd}$$

then consolidate to get this:

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left=60, "\mathcal{F}"] \arrow[r, "\mathcal{F}\circ\mathcal{G}\circ\mathcal{F}" description] \arrow[r, bend right=60, "\mathcal{F}"'] & \mathsf{D}
\end{tikzcd}
\qquad \Downarrow \mathcal{F}\eta, \ \Downarrow \varepsilon\mathcal{F}$$

We can then take the composition $\varepsilon\mathcal{F} \circ \mathcal{F}\eta$ to get a natural transformation $\mathcal{F} \Rightarrow \mathcal{F}$

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}", ""{name=U, below}] \arrow[r, bend right, "\mathcal{F}"', ""{name=L, above}] & \mathsf{D};
\arrow[Rightarrow, from=U, to=L, "\varepsilon\mathcal{F}\circ\mathcal{F}\eta"]
\end{tikzcd}$$

the first triangle diagram says that this must be the same as the identity natural transformation $\mathrm{id}_{\mathcal{F}}$.

The second triangle diagram is analogous.

1.4 Adjunction from universal morphism

Definition 5 (universal morphism adjunction)**.** Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. We say that $\mathcal{F}$ is a left adjoint universal morphism adjunction if for each $X \in \mathrm{Obj}(\mathsf{D})$, the category of morphisms $(\mathcal{F} \downarrow X)$(Definition **??**) has a terminal object, which we denote by $\mathcal{G}(X)$. That is, $\mathcal{F}$ is a left adjoing universal morphism functor if for each $X \in \mathrm{Obj}(\mathsf{C})$, $Y \in \mathrm{Obj}(\mathsf{D})$, and morphism $f\colon \mathcal{F}(Y) \to X$, we have the following diagram.

$$\begin{tikzcd}
\mathcal{F}(Y) \arrow[d, "\mathcal{F}(g)"'] \arrow[dr, "f"] & & Y \arrow[d, dashed, "\exists! g"] \\
\mathcal{F}(\mathcal{G}(X)) \arrow[r, "\epsilon_X"'] & X & \mathcal{G}(X)
\end{tikzcd}$$

Similarly, we say that $\mathcal{F}$ is a right adjoint universal morphism adjunction if for each $Y \in \mathrm{Obj}(\mathsf{C})$, the category of morphisms $(Y \downarrow \mathcal{G})$ (Definition **??**) has a initial object.

The above terminology is horrendous, and should probably be changed. The reason for introducing it is that it will not be around for long, since we will see that all of these definitions of left- and right adjunctions are equivalent.

1.5 Unit-counit adjunctions are hom-set adjunctions

Lemma 6. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a hom-set adjunction, they form a unit-counit adjunction.

Sketch of proof. Our plan of attack is the following.

1. We show that from a hom-set adjunction $\mathcal{F} \dashv \mathcal{G}$, we can form, for each $A \in \mathrm{Obj}(\mathsf{C})$, a map
$$\eta_A : A \to (\mathcal{G} \circ \mathcal{F})(A)$$
and for each $B \in \mathrm{Obj}(\mathsf{D})$, a map
$$\epsilon_B : (\mathcal{F} \circ \mathcal{G})(B) \to B.$$

2. We show that, component-wise, these satisfy the triangle identities.
$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_{\mathcal{F}})_A \qquad \text{and} \qquad (\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_{\mathcal{G}})_B.$$

3. We show that η_A and ϵ_B are components of two natural transformations
$$\eta : \mathrm{id}_{\mathsf{C}} \to \mathcal{G} \circ \mathcal{F} \qquad \text{and} \qquad \epsilon : \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_{\mathsf{D}}.$$

The reason for doing things in this order is that we will use the component-wise triangle identities in showing naturality. □

Proof.

1. Suppose $\mathcal{F}$ and $\mathcal{G}$ form a hom-set adjunction with natural isomorphism Φ. Then for any $A \in \mathrm{Obj}(\mathsf{C})$, we have $\mathcal{F}(A) \in \mathrm{Obj}(\mathsf{D})$, so Φ give us a bijection
$$\Phi_{A,\mathcal{F}(A)} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \to \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
We don't know much in general about $\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A))$, but the category axioms tell us that it always contains $\mathrm{id}_{\mathcal{F}(A)}$. We can use $\Phi_{A,\mathcal{F}(A)}$ to map this to
$$\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) \in \mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)).$$
Let's call $\Phi_{A,\mathcal{F}(A)}(\mathrm{id}_{\mathcal{F}(A)}) = \eta_A$.
Similarly, if $B \in \mathrm{Obj}(\mathsf{D})$, then $\mathcal{G}(B) \in \mathrm{Obj}(\mathsf{C})$, so Φ gives us a bijection
$$\Phi_{\mathcal{G}(B),B} : \mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \to \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)).$$
Since $\Phi_{\mathcal{G}(B),B}$ is a bijection, it is invertible, and we can evaluate the inverse on $\mathrm{id}_{\mathcal{G}(B)}$. Let's call
$$\Phi^{-1}_{\mathcal{G}(B),B}(\mathrm{id}_{\mathcal{G}(B)}) = \varepsilon_B.$$

2. Clearly, η_A and ε_B are completely determined by Φ and Φ^{-1} respectively. It turns out that the converse is also true; in a manner reminiscent of the proof of the Yoneda lemma, we can express Φ in terms of η, and Φ^{-1} in terms of ε, for *any* A and B. We do this using the naturality of Φ.
The naturality of Φ tells us for any $A \in \mathrm{Obj}(\mathsf{C})$, $B \in \mathrm{Obj}(\mathsf{D})$, and $g : \mathcal{F}(A) \to B$, the following diagram has to commute.
$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[r, "g\circ(-)"] \arrow[d, "\Phi_{A,\mathcal{F}(A)}"'] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \arrow[d, "\Phi_{A,B}"] \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)) \arrow[r, "\mathcal{G}(g)\circ(-)"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).
\end{tikzcd}$$

Let's start at the top left with $\mathrm{id}_{\mathcal{F}(A)}$ and see what happens. Taking the top road to the bottom right, we have $\Phi_{A,B}(g)$, and from the bottom road we have $\mathcal{G}(g) \circ \eta_A$. The diagram commutes, so we have

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A. \tag{1.1}$$

Similarly, the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}((\mathcal{F} \circ \mathcal{G})(B), B) \arrow[r, "(-)\circ\mathcal{F}(f)"] & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \\
\mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(B), \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{\mathcal{G}(B),B}"] \arrow[r, "(-)\circ f"] & \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)) \arrow[u, "\Phi^{-1}_{A,B}"']
\end{tikzcd}$$

means that, for any $f\colon A \to \mathcal{G}(B)$,

$$\Phi^{-1}_{A,B}(f) = \varepsilon_B \circ \mathcal{F}(f) \tag{1.2}$$

To show that η and ε as defined here satisfy the triangle identities, we need to show that for all $A \in \mathrm{Obj}(\mathsf{C})$ and all $B \in \mathrm{Obj}(\mathsf{D})$,

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = (\mathrm{id}_{\mathcal{F}})_A \qquad \text{and} \quad (\mathcal{G}\varepsilon)_B \circ (\eta\mathcal{G})_B = (\mathrm{id}_{\mathcal{G}})_B.$$

We have

$$(\varepsilon\mathcal{F})_A \circ (\mathcal{F}\eta)_A = \varepsilon_{\mathcal{F}(A)} \circ \mathcal{F}(\eta_A) = \Phi^{-1}_{A,\mathcal{F}(A)}(\eta_A) = \mathrm{id}_A = (\mathrm{id}_{\mathcal{F}})_A$$

and

$$(\mathcal{G}_\varepsilon)_B \circ (\eta\mathcal{G})_B = \mathcal{G}(\varepsilon_B) \circ \eta_{\mathcal{G}(B)} = \Phi_{\mathcal{G}(B),B}(\varepsilon_B) = \mathrm{id}_B = (\mathrm{id}_{\mathcal{G}})_B.$$

3. Let $g\colon A' \to A$. By the naturality of Φ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), \mathcal{F}(A)) \arrow[rrr, "(-)\circ\mathcal{F}(g)"] \arrow[ddd, "\Phi_{A,\mathcal{F}(A)}"'] & & & \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A'), \mathcal{F}(A)) \arrow[ddd, "\Phi_{A',\mathcal{F}(A)}"] \\
& \mathrm{id}_{\mathcal{F}(A)} \arrow[r, mapsto] \arrow[d, mapsto] & \mathcal{F}(g) \arrow[d, mapsto] & \\
& \eta_A \arrow[r, mapsto] & \eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g)) & \\
\mathrm{Hom}_{\mathsf{C}}(A, (\mathcal{G} \circ \mathcal{F})(A)) \arrow[rrr, "(-)\circ g"] & & & \mathrm{Hom}_{\mathsf{C}}(A', (\mathcal{G} \circ \mathcal{F})(A))
\end{tikzcd}$$

Following $\mathrm{id}_{\mathcal{F}(A)}$ around, we find the condition $\eta_A \circ g \overset{!}{=} \Phi_{A',\mathcal{F}_A}(\mathcal{F}(g))$. But by Equation 1.1, we have

$$\Phi_{A',\mathcal{F}(A)}(\mathcal{F}(g)) = (\mathcal{G} \circ \mathcal{F})(g) \circ \eta_{A'},$$

so our condition reads

$$\eta_A \circ g = (\mathcal{G} \circ \mathcal{F})(g) \circ \eta_{A'}.$$

But this says precisely that the naturality square

$$\begin{tikzcd}
(\mathcal{G} \circ \mathcal{F})(A) \arrow[r, "\eta_A"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & A \arrow[d, "f"] \\
(\mathcal{G} \circ \mathcal{F})(A') \arrow[r, "\eta_{A'}"] & A'
\end{tikzcd}$$

commutes.

□

Lemma 7. If the functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ form a unit-counit adjunction, they form a hom-set adjunction.

Proof. We have seen in the proof of the previous lemma that we can write

$$\Phi_{A,B}(g) = \mathcal{G}(g) \circ \eta_A \qquad \text{and} \qquad \Phi^{-1}_{A,B}(g) = \epsilon_B \circ \mathcal{F}(f).$$

□

Definition 8 (adjunct)**.** Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Then for each $A \in \mathrm{Obj}(\mathsf{C})$ $B \in \mathrm{Obj}(\mathsf{D})$, we have a natural isomorphism (i.e. a bijection)

$$\Phi_{A,B} : \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B)).$$

Thus, for each $f \in \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(A), B)$ there is a corresponding element $\tilde{f} \in \mathrm{Hom}_{\mathsf{C}}(A, \mathcal{G}(B))$, and vice versa. The morphism $\tilde{f}$ is called the adjunct of f, and f is called the adjunct of $\tilde{f}$.

Lemma 9. Let C and D be categories, $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}, \mathcal{G}'\colon \mathsf{D} \to \mathsf{C}$ functors,

$$\begin{tikzcd}
\mathsf{C} \arrow[r, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend right, "\mathcal{G}"'] \arrow[l, bend left, "\mathcal{G}'"]
\end{tikzcd}$$

and suppose that $\mathcal{G}$ and $\mathcal{G}'$ are both right-adjoint to $\mathcal{F}$. Then there is a natural isomorphism $\mathcal{G} \Rightarrow \mathcal{G}'$.

Proof. Since any adjunction is a hom-set adjunction, we have two isomorphisms

$$\Phi_{C,D}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \qquad \text{and} \qquad \Psi_{C,D}\colon \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D))$$

which are natural in both C and D. By Lemma **??**, we can construct the inverse natural isomorphism

$$\Phi^{-1}_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{D}}(\mathcal{F}(C), D),$$

and compose it with Ψ to get a natural isomorphsim

$$(\Psi \circ \Phi^{-1})_{C,D}\colon \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \Rightarrow \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)).$$

Thus for any morphism $f\colon D \longrightarrow E$, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C,\mathcal{G}(f))"] \arrow[d, "(\Psi\circ\Phi^{-1})_{C,D}"'] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}(E)) \arrow[d, "(\Psi\circ\Phi^{-1})_{C,E}"] \\
\mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(D)) \arrow[r, "\mathrm{Hom}_{\mathsf{C}}(C,\mathcal{G}'(f))"] & \mathrm{Hom}_{\mathsf{C}}(C, \mathcal{G}'(E))
\end{tikzcd}$$

But the full- and faithfulness of the Yoneda embedding tells us that there is a natural isomorphism with components μ_D and μ_E making the following diagram commute.

$$\begin{tikzcd}
\mathcal{G}(D) \arrow[r, "\mathcal{G}(f)"] \arrow[d, "\mu_D"'] & \mathcal{G}(E) \arrow[d, "\mu_E"] \\
\mathcal{G}'(D) \arrow[r, "\mathcal{G}'(f)"] & \mathcal{G}'(E)
\end{tikzcd}$$

□

Note 10. We do not give very many examples of adjunctions now because of the frequency with which category theory graces us with them. Howver, it is worth mentioning a specific class of adjunctions: the so-called *free-forgetful adjunctions*. There are many *free* objects in mathematics: free groups, free modules, free vector spaces, free categories, etc. These are all unified by the following property: the functors defining them are all left adjoints.

Let us take a specific example: the free vector space over a set. This takes a set S and constructs a vector space which has as a basis the elements of S.

There is a forgetful functor $\mathcal{U}\colon \mathsf{Vect}_k \longrightarrow \mathsf{Set}$ which takes any set and returns the set underlying it. There is a functor $\mathcal{F}\colon \mathsf{Set} \longrightarrow \mathsf{Vect}_k$, which takes a set and returns the free vector space on it. It turns out that there is an adjunction $\mathcal{F} \dashv \mathcal{U}$.

And this is true of any free object! (In fact by definition.) In each case, the functor giving the free object is left adjoint to a forgetful functor.

Theorem 11. Let C and D be categories and $\mathcal{F}$ and $\mathcal{G}$ functors as follows.

$$\begin{tikzcd}
\mathsf{C} \arrow[r, bend left, "\mathcal{F}"] & \mathsf{D} \arrow[l, bend left, "\mathcal{G}"]
\end{tikzcd}$$

Let $\mathcal{F} \dashv \mathcal{G}$ be an adjunction. Then $\mathcal{G}$ preserves limits, i.e. if $\mathcal{D}\colon \mathsf{J} \longrightarrow \mathsf{C}$ is a diagram and $\lim_{\leftarrow i} \mathcal{D}_i$ exists in C, then

$$\mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i) \simeq \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i).$$

Proof. We have the following chain of isomorphisms, natural in $Y \in \mathrm{Obj}(\mathsf{D})$.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G}(\lim_{\leftarrow i} \mathcal{D}_i)) &\simeq \mathrm{Hom}_{\mathsf{C}}(\mathcal{F}(Y), \lim_{\leftarrow i} \mathcal{D}_i) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{C}}(\mathcal{G}(Y), \mathcal{D}_i) && \left(\substack{\text{Hom functor commutes with} \\ \text{limits: Theorem ??}}\right) \\
&\simeq \lim_{\leftarrow i} \mathrm{Hom}_{\mathsf{D}}(Y, \mathcal{G} \circ \mathcal{D}_i) \\
&\simeq \mathrm{Hom}_{\mathsf{C}}(Y, \lim_{\leftarrow i}(\mathcal{G} \circ \mathcal{D}_i)).
\end{aligned}$$

By the Yoneda lemma, specifically Corollary ??, we have a natural isomorphism

$$\mathcal{G}(\varprojlim_i \mathcal{D}_i) \simeq \varprojlim_i (\mathcal{G} \circ \mathcal{D}_i).$$

□

Corollary 12. Any functor $\mathcal{F}$ which is a left-ajoint preserves colimits.

Proof. Dual to the proof of Theorem 11. □

1.6 Adjunctions generaize categorical equivalences

Theorem 13. Let $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ be a functor. The following are equivalent.

1. $\mathcal{F}$ is part of an equivalence of categories (Definition **??**) $(\mathcal{F}, \mathcal{G}, \eta, \epsilon)$.
2. $\mathcal{F}$ is fully faithful (Definition **??**) and essentially surjective (Definition **??**).
3. $\mathcal{F}$ is part of a unit-counit adjunction (Definition 3) $(\mathcal{F}, \mathcal{G}, \epsilon, \eta)$, where ϵ and η are isomorphisms.

Proof.

3. $\Rightarrow$ 1. Obvious.

1. $\Rightarrow$ 2. Assume $(\mathcal{F}, \mathcal{G}, \eta\colon \mathcal{F} \circ \mathcal{G} \to \mathrm{id}_G, \epsilon\colon \mathrm{id}_C \to \mathcal{G} \circ \mathcal{F})$ is an equivalence of categories.

   First we show that $\mathcal{F}$ is essentially surjective. Evaluating η at any $Y \in \mathrm{Obj}(\mathsf{D})$, we get an isomorphism $(\mathcal{F} \circ \mathcal{G})(Y) \simeq Y$. Thus, for any $Y \in \mathrm{Obj}(\mathsf{C})$, there is an isomorphism between Y and some element $\mathcal{F}(\mathcal{G}(Y))$ in the image of $\mathcal{F}$.

   Next, we show that $\mathcal{F}$ is fully faithful.

2. $\Rightarrow$ 3.

□