1 Model categories

1.1 Localization of categories

Definition 1 (localization of a category)**.** Let $\mathcal{C}$ be a category, and $\mathcal{W}$ a set of morphisms of $\mathcal{C}$. The localization of $\mathcal{C}$ at $\mathcal{W}$ is a pair $(\mathcal{C}[\mathcal{W}^{-1}], \pi)$ where $\mathcal{C}([\mathcal{W}^{-1}])$ is a category and $\pi\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is a functor, such that the following conditions hold.

- For all $w \in \mathcal{W}$, $\pi(w)$ is an isomorphism.
- For any category $\mathcal{A}$ and functor $F\colon \mathcal{C} \to \mathcal{A}$ such that $F(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a functor $\tilde{F}\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ and a natural isomorphism $\tilde{F} \circ \pi \Rightarrow F$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"'] & \mathcal{A} \arrow[dl, Rightarrow, shorten=2mm] \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, "\exists \tilde{F}"'] & \end{tikzcd}$$

  Note that the above diagram is not required to commute.
- For every category $\mathcal{A}$, the pullback functor

$$\pi^*\colon \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{A}) \to \mathbf{Cat}(\mathcal{C}, \mathcal{A})$$

  is fully faithful.

Note that by the third axiom, the functor $\tilde{F}$ is guaranteed to be unique up to isomorphism: fully faithfulness tells us precisely that functors $\mathcal{C} \to \mathcal{A}$ and functors $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ sending $\mathcal{W}$ to isomorphisms are in bijection up to isomorphism

The idea behind the localization of a category is very similar to that of a ring. Roughly speaking, in localizing a category $\mathcal{C}$ at a set of morphisms $\mathcal{W}$, one adds just enough morphisms to $\mathcal{C}$ so that all the morphisms in $\mathcal{W}$ have inverses. Of course in adding any morphism to a category, one also has to freely add in compositions with other morphisms. For this reason, the localization of a category is in general unwieldy. Furthermore, one runs into size-theoretic issues: even if $\mathcal{C}$ is a small category, one cannot in general be sure that $\mathcal{C}[\mathcal{W}^{-1}]$ is small.

There is an explicit construction of the localization of a category, but it is extremely unweildy to work with. Fortunately, model categories provide a way of computing and working with localizations which is practical and avoids issues of size.

1.2 Model categories

Model categories provide a setting in which one can define a purely category-theoretic form of homotopy theory.

Definition 2 (model category)**.** A model category is a category $\mathcal{C}$ together with three sets of morphisms

- W, the set of *weak equivalences*
- Fib, the set of *fibrations*
- Cof, the set of *cofibrations*

subject to the following axioms.

(M1) The category $\mathcal{C}$ has all small limits and colimits.

(M2) The set W satisfies the so-called *two-out-of-three law:* if the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

commutes and two out of three of f, g, and h are weak equivalences, then the third is as well.

(M3) The sets W, Fib, and Cof are all closed under retracts: given the following diagram, if g belongs to any of the above sets, then so does f.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r] & W \arrow[r] & Z \end{tikzcd}$$

(M4) We can solve a lifting problem

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & P \end{tikzcd}$$

if all of the following conditions hold.

- The morphism i is a cofibration.
- The morphism p is a fibration.
- At least one of i and p is a weak equivalence.

(M5) Any morphism $f\colon X \to Y$ admits the following factorizations.

- We can write $f = p \circ i$, where p is a fibration and i is both a cofibration and a weak equivalence.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathsf{Cof}\cap\mathsf{W}\ni i"'] & & B \\ & X \arrow[ur, "p\in\mathsf{Fib}"'] & \end{tikzcd}$$

- We can write $f = p \circ i$, where p is both a fibration and a weak equivalence, and i is a cofibration.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathsf{Cof}\ni i"'] & & B \\ & X \arrow[ur, "p\in\mathsf{Fib}\cap\mathsf{W}"'] & \end{tikzcd}$$

To simplify terminology, we will use the following terms.

- We will call morphisms which are both weak equivalences and fibrations *trivial fibrations*:
$$\mathsf{W} \cap \mathsf{Fib} = \{\text{trivial fibrations}\}.$$
- We will call morphisms which are both weak equivalences and cofibrations *trivial cofibrations*:
$$\mathsf{W} \cap \mathsf{Cof} = \{\text{trivial cofibrations}\}.$$
- We will call an object $X \in \mathcal{C}$ *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration.
- We will call an object $Y \in \mathcal{C}$ *fibrant* if the unique morphism $X \to *$ is a fibration.

Lemma 3 (lifting lemma)**.** Let $\mathcal{C}$ be any category, and let $f = p \circ i$ be a morphism in $\mathcal{C}$.

1. If f has the left lifting property with respect to p, then f is a retract of i.
2. If f has the right lifting property with respect to i then f is a retract of p.

Proof. Suppose f has the left-lifting property with respect to p. Then the following diagram admits a lift.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

Expanding, we find the following diagram.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, dashed] \arrow[rr, bend right, "\mathrm{id}_B"'] & X \arrow[r, "p"] & B \end{tikzcd}$$

This exhibits f as a retract of i.

The second case is dual. □

The axioms for a model category are over-specified, in the sense that providing W and either Fib and Cof is sufficient to derive the other one.

Lemma 4. Let $\mathcal{C}$ be a model category. We have the following.

1. $\mathsf{Cof}_{\perp} = \mathsf{Fib} \cap \mathsf{W}$
2. $\mathsf{Cof} = {}_{\perp}(\mathsf{Fib} \cap \mathsf{W})$
3. $(\mathsf{Cof} \cap \mathsf{W})_{\perp} = \mathsf{Fib}$
4. $\mathsf{Cof} \cap \mathsf{W} = {}_{\perp}\mathsf{Fib}$

Proof. We prove 4. The proof of 1 is mutatis mutandis equivalent.

Let $f \in \mathsf{Cof} \cap \mathsf{W}$. Then by Axiom (M4), $f \in {}_{\perp}\mathsf{Fib}$.

Now suppose $f \in {}_{\perp}\mathsf{Fib}$. By Axiom (M5), we can always find a factorization $f = p \circ i$, where p is a fibration and i is a trivial cofibration.

By Axiom (M4), f has the left lifting property with respect to p, so by Lemma 3, f is a retract of i. But since by Axiom (M3) both W and Cof are closed under retracts, f is also a trivial cofibration as needed.

From 4, 3 follows immediately. The proof of 1 is precisely the same as that of 4, and 2 follows from 1 immediately. □

Corollary 5. In any model category, the sets Cof and $\mathsf{Cof} \cap \mathsf{W}$ are saturated (Definition **??**).

Proof. By Theorem **??**, any set which is of the form ${}_{\perp}\mathcal{M}$ is saturated. □

Example 6. The following are examples of model structures.

1. The category **Top** carries the so-called *Quillen model structure,* with
   - W = weak homotopy equivalences,
   - Fib = Serre fibrations,
   - $\mathsf{Cof} = {}_{\perp}(\mathsf{Fib} \cap \mathsf{W})$.
2. The category $\mathbf{Set}_{\Delta}$ has a model structure called the *Kan model structure* with
   - W = weak homotopy equivalences, i.e. Kan fibrations $X \longrightarrow Y$ such that $\pi_0(X) \longrightarrow \pi_0(Y)$ is a bijection and $\pi_n(X, x) \longrightarrow \pi_n(Y, f(x))$ is an isomorphism for all $x \in X$.
   - Fib = Kan fibrations,
   - Cof = monomorphisms.

   Note that in this case, $\mathsf{W} \cap \mathsf{Cof} = {}_{\perp}\mathsf{Fib} = \{\text{anodyne morphisms}\}$.

3. For any small ring R, the category $\mathbf{Ch}_{\geq}(R)$ of bounded-below chain complexes of small left R-modules has a model structure, called the *projective model structure,* with
   - W = Quasi-isomorphisms
   - Fib = morphisms f of complexes such that f_n is an epimorphism for all $n \geq 0$
   - Cof = morphisms f of complexes such that f_n is a monomorphism for all $n \geq 0$, and coker f_n is projective for all $n \geq 0$.
4. For any small ring R, the category $\mathbf{Ch}^{\leq}(R)$ of bounded-below cochain complexes of small left R-modules has a model structure, called the *injective model structure,* with
   - W = Quasi-isomorphisms
   - Fib = morphisms f of complexes such that f_n is an epimorphism for all $n \geq 0$, and ker f_n injective for all $n \geq 0$.
   - Cof = morphisms f of complexes such that f_n is a monomorphism for all $n \geq 0$.

Example 7. Let $\mathcal{C}$ be a category with model structure $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$. Then $\mathcal{C}^{\mathrm{op}}$ is a model category with model structure $(\mathsf{W}, \mathsf{Cof}, \mathsf{Fib})$.

1.2.1 A model structure on the category of categories

In general, proving that three sets of morphisms form the fibrations, cofibrations, and weak equivalences of a model structure is in general an immense undertaking. There are a few model structures, however, which are somewhat simple.

Consider the category I with the following objects and morphisms (excluding identity morphisms).

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

Denote the functor including $0 \hookrightarrow I$ by ι.

The following is a model structure on **Cat**.

- W = categorical equivalences
- Fib = *isofibrations*, i.e. functors with the right lifting property with respect to ι
- Cof = functors which are injective on objects

We verify the axioms one by one. This section is incomplete and a bit wrong at the moment. I'll come back to it.

Axiom M1: Limits and colimits

We already know that **Cat** has all limits and colimits.

Axiom M2: Two-out-of-three

We have the following results.

- If a functor is a weak equivalence, then its weak inverse is also a weak equivalence.
- Composition of weak equivalences gives again a weak equivalence (by the horizontal composition formula).

Thus, the same reasoning that shows that the 2/3 law holds for bona-fide isomorphisms also holds for weak equivalences.

Axiom M3: Closure under retracts

First suppose that G is an equivalence of categories. We need to show that F is also an equivalence of categories.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

Since both $S \circ R$ and $U \circ T$ compose to the identity, T and R are injective on objects and morphisms (in the on-the-nose, evil sense), and S and U are similarly surjective. Thus G is:

- **Essentially surjective:** We can map any $b \in \mathcal{B}$ to $\mathcal{D}$ with T, then lift it to something in $\mathcal{C}$ using G's weak inverse, which we'll call G^{-1}. Mapping the result of this to $\mathcal{A}$ with S gives us an object

  $$(S \circ G^{-1} \circ T)(b) \in A.$$

  The claim is that under F this maps to something isomorphic to b. Evaluating on F, we have

  $$F((S \circ G^{-1} \circ T)(b)) = (U \circ G \circ G^{-1} \circ T)(b).$$

  We have a natural isomorphism $G \circ G^{-1} \Rightarrow \mathrm{id}$, which tells us that

  $$(U \circ G \circ G^{-1} \circ T)(b) \simeq (U \circ T)(b) = b.$$

- **Fully faithful:** The logic is very similar. We can map any morphism f in $\mathcal{B}$ to a morphism in $\mathcal{D}$ with T, and (because G is fully faithful) find a unique preimage in $\mathcal{C}$ which we'll call $G^{-1}(T(f))$. Applying S gives us a morphism

  $$(S \circ G^{-1} \circ T)(f) \quad \text{in } \mathcal{A}.$$

  Then

  $$\begin{aligned} (F \circ S \circ G^{-1} \circ T)(f) &= (U \circ G \circ G^{-1} \circ T)(f) \\ &= (U \circ T)(f) \\ &= f. \end{aligned}$$

Now suppose that G is in $\mathfrak{C}$. We need to show that $F \in \mathfrak{C}$.

Applying the forgetful functor to **Set** we have the following diagram.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "t"'] & D \arrow[r, "u"'] & B \end{tikzcd}$$

The map r is injective because the identity factors through it, and g is injective by assumption. Suppose we are given a, $a' \in A$ such that $f(a) = f(a')$. Then $t(f(a)) = t(f(a'))$, so $g(r(a)) = g(r(a'))$. But g and r are both injective, so $a = a'$.

Now suppose that G is in $\mathfrak{F}$. We need to show that $F \in \mathfrak{F}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

To see this, note that we can augment any lifting problem

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[d, "F"] \\ (* \leftrightarrow \bullet) \arrow[r] & \mathcal{B} \end{tikzcd}$$

as follows; composing with R gives us a lifting problem we can solve, yielding the dashed arrow.

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[r, "R"] \arrow[d] & \mathcal{C} \arrow[r, "S"] \arrow[d] & \mathcal{A} \arrow[d] \\ (* \leftrightarrow \bullet) \arrow[r] \arrow[urr, dashed] & \mathcal{B} \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{B} \end{tikzcd}$$

We can compose this with S to get a lift to A, which is what we want.

Axiom M4: Solving lifting problems

Consider a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

where i is injective on objects and P is both an isofibration and a weak equivalence.

Pick any $d \in \mathcal{D}$. Since P is an equivalence of categories, there is some object $b \in \mathcal{B}$ such that $P(b) \simeq d$. Define a functor $I \to \mathcal{D}$ which sends $* \mapsto b$ and $\bullet \mapsto P(b)$, and a functor

$(*) \to \mathcal{B}$ which picks out b. This gives us a commuting diagram

$$\begin{tikzcd}
(*) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\
I \arrow[r] & \mathcal{D}
\end{tikzcd}$$

Since P is an isofibration, we have a lift $I \to \mathcal{B}$, which maps $\bullet$ to an object $\hat{d} \in \mathcal{B}$ such that $P(\hat{d}) = d$. Thus, P is surjective on objects.

Suppose first that c is in the image of i. Then (because i is injective on objects) we can lift c uniquely to an object of A, then map the result to $\mathcal{B}$ with F. We define this to be $\ell(c)$.

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\
\mathcal{C} \arrow[r, "G"'] \arrow[ur, dashed, "\ell"] & \mathcal{D}
\end{tikzcd}$$

This makes both triangles formed commute trivially.

If c is not mapped to by i, then we define $\ell(c)$ to be any lift $\widehat{G(c)}$. This makes the lower triangle commute by definition, and the upper triangle is irrelevant since i never hits these objects.

Since P is an equivalence of categories, the maps

$$P_{b,b'} \colon \mathcal{B}(b,b') \to \mathcal{D}(P(b),P(b')); \qquad \left[b \overset{f}{\to} b'\right] \mapsto \left[P(b) \overset{P(f)}{\to} P(b')\right]$$

are bijections, hence can be inverted. Thus we can define ℓ on morphisms $f \colon c \to c'$ in $\mathcal{C}$ by

$$\ell(f) = P^{-1}_{\ell(c),\ell(c')}(G(f)).$$

The upper triangle

$$\begin{tikzcd}
\left[a \overset{f}{\to} a'\right] \arrow[r, mapsto] \arrow[d, mapsto] & \left[F(a) \overset{F(f)}{\to} F(a')\right] \overset{!}{=} \left[\ell(i(a)) \overset{\ell(i(f))}{\to} \ell(i(b))\right] \\
\left[i(a) \overset{i(f)}{\to} i(a')\right] \arrow[ur, mapsto] &
\end{tikzcd}$$

commutes since $\ell(i(a)) = F(a)$ and $\ell(i(a')) = F(a')$, so

$$\begin{aligned}
\ell(i(f)) &= P^{-1}_{\ell(i(a)),\ell(i(a'))}(G(i(f))) \\
&= P^{-1}_{F(a),F(a')}(P(F(f))) \\
&= F(f).
\end{aligned}$$

To see that the lower triangle

$$\begin{tikzcd} & & \left[\ell(c) \overset{\ell(f)}{\to} \ell(c')\right] \arrow[d, mapsto] \\ \left[c \overset{f}{\to} c'\right] \arrow[r, mapsto] \arrow[urr, mapsto] & \left[G(c) \overset{G(f)}{\to} G(c')\right] \overset{!}{=} & \left[P(\ell(c)) \overset{P(\ell(f))}{\to} P(\ell(c'))\right] \end{tikzcd}$$

note that we have already seen that $P(\ell(c)) = G(c)$ and $P(\ell(c')) = G(c')$, and for $f : c \to c'$ we defined

$$\ell(f) = P^{-1}_{\ell(c),\ell(c')}(G(f)),$$

so

$$P(\ell(f)) = G(f).$$

Now we are in the situation where i is both injective and an equivalence of categories, and P is an isofibration.

Let $c \in \mathcal{C}$. Becase i is essentially surjective we can always pick $c' \in \mathcal{C}$ such that c' is in the image of i, i.e. such that there exists some a_c with $i(a_c) = c' \simeq c$. For each such c, pick such a c' and a_c. If c is already in the image of i, pick $c' = c$; that is, demand that $a_{i(a)} = a$.

This allows us to define, for each $c \in \mathcal{C}$ functors as follows.

$$* \longmapsto F(a_c)$$

$$(* \leftrightarrow \bullet) \longmapsto G(c') \leftrightarrow G(c)$$

Then P's isofibricity gives a functor $I \to \mathcal{B}$ making the diagram

commute, which sends $\bullet$ to some object $b_c \in \mathcal{B}$. By the commutativity, $P(b_c) = G(c)$.

Define the lift ℓ by $\ell(c) = b_c$. The upper triangle is

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \overset{!}{=} \ell(i(a)) \\ i(a) \arrow[ur, mapsto] & \end{tikzcd}$$

which commutes because $\ell(i_a) = F(a_a) = F(a)$.

The lower triangle

$$\begin{tikzcd} & \ell(c) \arrow[d, mapsto] \\ c \arrow[r, mapsto] \arrow[ur, mapsto] & G(c) \overset{!}{=} P(\ell(c)) \end{tikzcd}$$

commutes because $P(\ell(c)) = P(b_c) = G(c)$.

By the same logic as the previous problem, any morphism $f\colon c \to c'$ lifts well-definedly to a map $\hat{f}\colon a_c \to a_{c'}$. We define

$$\ell(f) = F(\hat{f}).$$

The upper triangle

$$\begin{tikzcd} f \arrow[r, mapsto] \arrow[d, mapsto] & F(f) \overset{!}{=} \ell(i(f)) \\ i(f) \arrow[ur, mapsto] & \end{tikzcd}$$

commutes since $\ell(i(f)) = f$ by definition of $\hat{f}$. The lower triangle

$$\begin{tikzcd} & \ell(f) \arrow[d, mapsto] \\ f \arrow[ur, mapsto] \arrow[r, mapsto] & P(\ell(f)) \overset{!}{=} G(f) \end{tikzcd}$$

commutes because

$$\begin{aligned} P(\ell(f)) &= P(F(\hat{f})) \\ &= G(i(\hat{f})) \\ &= G(f). \end{aligned}$$

Axiom M5: Factorization

For any functor $F\colon \mathcal{C} \to \mathcal{D}$, define a functor $G\colon \mathcal{C} \to \mathcal{L}$ on objects by

$$x \mapsto (x, Fx, \mathrm{id}_{Fx}),$$

and on morphisms by $f \mapsto f$. The functor G' sending

$$(c, d, f) \mapsto c.$$

is a weak inverse to G. As a check: these form an equivalence of categories since

$$G' \circ G\colon x \mapsto (x, Fx, \mathrm{id}_{Fx}) \mapsto x$$

and

$$G \circ G'\colon (c, d, f) \mapsto c \mapsto (c, Fc, \mathrm{id}_{Fc});$$

the map

$$G(\mathrm{id}_c)\colon (c, d, f) \to (c, Fc, \mathrm{id}_{Fc})$$

is a natural isomorphism since the naturality square

$$\begin{tikzcd} (c, d, f) \arrow[r, "G(\mathrm{id}_c)"] \arrow[d, "g"'] & (c, Fc, \mathrm{id}_{Fc}) \arrow[d, "g"] \\ (c', d', f') \arrow[r, "G(\mathrm{id}_{c'})"'] & (c', Fc', \mathrm{id}_{Fc'}) \end{tikzcd}$$

commutes.

It is also clear that G is injective on objects since $(c,d,f) \neq (c',d',f') \implies c \neq c'$, so $G \in \mathfrak{C}$.

Define also a functor $H\colon \mathcal{L} \to \mathcal{D}$ sending $(c,d,f) \mapsto d$ and $g\colon c \to c'$ to $F(g)$. Then

$$H \circ G\colon c \mapsto (c, Fc, \mathrm{id}_{Fc}) \mapsto Fc; \qquad g \mapsto g \mapsto Fg.$$

It remains only to show that $H \in \mathfrak{F}$, we need to show that it is an isofibration. Consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[rr] \arrow[dd, hook] & & \mathcal{L} \arrow[dd, "H"] \\
& & \\
(* \leftrightarrow \bullet) \arrow[rr] & & \mathcal{D}
\end{tikzcd}$$

$$\begin{tikzcd}
* \arrow[r, mapsto] \arrow[d, mapsto] & (c,d,f) \arrow[d, mapsto] \\
* \arrow[r, mapsto] & d
\end{tikzcd}$$

We need to show that given an isomorphism $r\colon d \to d'$ and an object (c,d,f) in $\mathcal{L}$, we can find an object (c',d',f') which is isomorphic to (c,d,f). We can; taking $c' = c$, and $f' = r \circ f$ gives us what we want since the following diagram commutes.

$$\begin{tikzcd}
& (c,d,f) \leftrightarrow (c,d',r \circ f) \arrow[d, mapsto] \\
(* \leftrightarrow \bullet) \arrow[ur, mapsto] \arrow[r, mapsto] & c \leftrightarrow c'
\end{tikzcd}$$

Part 2: We define G to be the canonical injection, and H to be defined on objects as follows.

$$H\colon x \mapsto \begin{cases} Fx, & x \in \mathcal{C} \\ x, & x \in \mathcal{D} \end{cases}$$

The hom-sets in $\mathcal{R}$ are already appropriate hom-sets in $\mathcal{D}$, so we don't need to do anything to define H on morphisms: we simply define it to be the identity on hom-sets.

To see that H is an equivalence of categories, note that it is essentially surjective because it is surjective 'on the nose,' and that it is fully faithful because H is the identity on hom-sets.

It is obvious that G is injective on objects.

Now consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[r, "a"] \arrow[d] & \mathcal{R} \arrow[d] \\
(* \leftrightarrow \bullet) \arrow[r, "b"] & \mathcal{D}
\end{tikzcd}$$

Define a functor $\ell \colon I \to \mathcal{R}$ sending $* \mapsto a(*)$ and $\bullet \mapsto b(\bullet)$. This makes both triangles commute: the upper triangle is

$$\begin{tikzcd} * \arrow[r, mapsto] \arrow[d, mapsto] & a(*) \\ * \arrow[ur, mapsto] & \end{tikzcd}$$

and the lower triangle is

$$a(*) \leftrightarrow b(\bullet)$$

$$* \leftrightarrow \bullet$$

1.3 The Homotopy category of a model category

In Subsection 1.3.1, we define the notion of a *left homotopy* of maps $A \to B$, denoted $\overset{l}{\sim}$, and show that for any cofibrant object A and any object B, left homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$\overbrace{A}^{\text{Cof}} \xrightarrow{f} B$$

In Subsection 1.3.2, we define a dual notion, *right homotopy*, denoted $\overset{r}{\sim}$, and show that for any object A and any fibrant object B, right homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$A \xrightarrow{f} \overbrace{B}^{\text{Fib}}$$

In Subsection 1.3.3, we show that when looking at maps from a cofibrant object to a fibrant object

$$\overbrace{A}^{\text{Cof}} \xrightarrow{f} \overbrace{B}^{\text{Fib}}$$

a left homotopy $f \overset{l}{\sim} f'$ gives rise to a right homotopy and vice versa, and thus the two notions agree. Thus, for maps from a cofibrant object to a fibrant object, we do not keep track of left- and right homotopies, calling them simply *homotopies.*

This allows us to define, for two maps between fibrant-cofibrant objects, a notion of homotopy equivalence; we say that a map $f \colon A \to B$, where A and B are fibrant-cofibrant, is a homotopy equivalence if it has a homotopy inverse g, i.e. a map g such that

$$f \circ g \sim \mathrm{id}_B, \qquad \text{and} \qquad g \circ f \sim \mathrm{id}_A.$$

Next, we show that this notion is preserved under composition: if f and g are homotopy equivalences, then so is $g \circ f$. This allows us to prove Whitehead's theorem, which tells us that any weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.

Finally, we define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category, whose objects are the fibrant-cofibrant objects of $\mathcal{C}$, and whose morphisms $X \to Y$ are the homotopy classes of morphisms $X \to Y$.

1.3.1 Left homotopy

Definition 8 (cylinder object)**.** Let $\mathcal{C}$ be a model category, and let $A \in \mathcal{C}$. A cylinder object for A is an object C together with a factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{i=(i_0,i_1)}"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

such that i is a cofibration and σ is a weak equivalence.

Example 9. In $\mathbf{Set}_\Delta$ equipped with the Kan model structure, for each simplicial set S there is a cylinder object, given by $\Delta^1 \times S$.

$$\begin{tikzcd} S \amalg S \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(s_0\times\mathrm{id},s_1\times\mathrm{id})}"'] & S \\ \Delta^1 \times S \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

In this case, i is a cofibration because it is an inclusion (hence a monomorphism as required) and π_2 is a weak equivalence (i.e. a homotopy equivalence) with homotopy inverse $s \mapsto (0, s)$.

Example 10. In the model category on categories, defined in Subsection 1.2.1, any category $\mathcal{C}$ has cylinder object $\mathcal{C} \times I$, taken with the below factorization.

$$\begin{tikzcd} \mathcal{C} \amalg \mathcal{C} \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{i=(\mathrm{id}\times\{*\},\mathrm{id}\times\{\bullet\})}"'] & \mathcal{C} \\ \mathcal{C} \times I \arrow[ur, "\pi"'] & \end{tikzcd}$$

The map i is obviously injective on objects, and the map π is an equivalence since the map $c \mapsto (c, 0)$ is a weak inverse.

Example 11. Let R be a small ring, and consider the category $\mathbf{Ch}_{\geq}(R)$ with the projective model structure of Subsection 1.2.1. Denote by $I_\bullet$ the chain complex[1]

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{(\mathrm{id},-\mathrm{id})} R \oplus R \ .$$

[1] This is, under the Dold-Kan correspondence, the chain complex corresponding to the unit interval.

Then each chain complex $A_\bullet$ has a cylinder object

$$\begin{tikzcd} A \oplus A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d] & A \\ A \otimes I \arrow[ur] & \end{tikzcd}.$$

Definition 12 (left homotopy)**.** Let $\mathcal{C}$ be a model category, and let $f, g\colon A \to B$ be two morphsims in $\mathcal{C}$. A left homotopy between f and g consists of a cylinder object C for A (Definition 8) and a map $H\colon C \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

In this case, we will write

$$f \overset{l}{\sim} g.$$

Example 13. In the projective model structure on $\mathbf{Ch}_{\geq 0}(R)$, a homotopy with respect to the

Lemma 14. Let A be a cofibrant object. If C is a cylinder object for A, then the maps i_1 and i_2 are trivial cofibrations.

Proof. Since A is cofibrant, by definition the map $\emptyset \to A$ is a cofibration. Consider the following pushout square.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \amalg A \end{tikzcd}$$

Since by Corollary 5 cofibrations are closed under pushout, the canonical injections $\iota_\alpha\colon A \to A \amalg A$, $\alpha = 1, 2$, are also cofibrations. Since $i_\alpha = i \circ \iota_\alpha$ is a composition of cofibrations, it is again a cofibration by closure under countable composition.

Since the diagram

$$\begin{tikzcd} A \arrow[d, hook, "\iota_\alpha"] \arrow[drr, "\mathrm{id}"] \arrow[dd, bend right=60, "\iota_\alpha"'] & & \\ A \amalg A \arrow[rr, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "i"] & & A \\ C \arrow[urr, "\sigma"'] & & \end{tikzcd}$$

commutes, we can write $\mathrm{id}_A = \sigma \circ i_\alpha$ (with id_A a weak equivalence because it is an isomorphsim and σ a weak equivalence by assumption), so by the two-out-of-three law i_α is a weak equivalence. $\square$

Proposition 15. If A is a cofibrant object, then left homotopy of maps $A \to B$ is an equivalence relation.

Proof. To see that left homotopy is an equivalence relation, let C be any cylinder object over A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Then the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,f)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "f\circ\sigma"'] & \end{tikzcd}$$

This means precisely that $f \circ \sigma$ functions as a left homotopy from f to itself.

Next, we show that left homotopy is symmetric. First, we note that in any category with coproducts, there is a natural isomorphism $\mathrm{swap}_A : A \amalg A \to A \amalg A$ which switches the components. Because Cof is saturated and thus contains all isomorphisms, swap_A is a cofibration. This gives us a new factorization exhibiting C as a cylinder object of A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i\circ\mathrm{swap}_A"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Let H be a left homotopy $f \overset{l}{\sim} g$ as follows.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

The diagram

$$\begin{tikzcd} A \amalg A \arrow[d, "\mathrm{swap}_A"'] \arrow[dr, "(g,f)"] & \\ A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

exhibits a left homotopy $g \overset{l}{\sim} f$.

It remains only to show that left homotopy is transitive. To this end, suppose $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$,

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "{(i_0,i_1)}"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "{(g,h)}"] \arrow[d, "{(i_0',i_1')}"'] & B \\ C' \arrow[ur, "H'"'] & \end{tikzcd}$$

where C and C' are cylinder objects for A with the following factorizations.

$$\begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i_0,i_1)}"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd} \qquad \begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i_0',i_1')}"'] & B \\ C' \arrow[ur, "\sigma'"'] & \end{tikzcd}$$

Consider the following diagram, where the top left square is a pushout square.

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"'] \arrow[ddr, bend left, "H'"] & \\ C \arrow[r, "j_0'"] \arrow[drr, bend right, "H"'] & C \coprod_A C' \arrow[dr, dashed, "\exists ! \tilde{H}"] & \\ & & B \end{tikzcd}$$

The following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,h)}"] \arrow[d, "{(j_0' \circ i_0, j_1 \circ i_0')}"'] & B \\ C \amalg_A C' \arrow[ur, "\tilde{H}"'] & \end{tikzcd}$$

It remains only to show that $C \amalg_A C'$ is a cylinder object for A. Define a map $\tilde{\sigma}$ as follows.

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"'] \arrow[ddr, bend left, "\sigma'"] & \\ C \arrow[r, "j_0'"] \arrow[drr, bend right, "\sigma"'] & C \coprod_A C' \arrow[dr, dashed, "\exists ! \tilde{\sigma}"] & \\ & & A \end{tikzcd}$$

Since $j_1 \in \mathrm{W} \cap \mathrm{Cof}$ and $\sigma' \in \mathrm{W}$, we have by Axiom (M2) that $\tilde{\sigma}$ is a weak equivalence.

Thus, to show that $C \amalg_A C'$ is a cylinder object with the factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(j_0' \circ i_0, j_1 \circ i_0')}"'] & A \\ C \amalg_A C' \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

we need only show that $(j_0' \circ i_1, j_1 \circ i_0')$ is a cofibration.

First note that the following square is a pushout square, where δ is the codiagonal.

$$\begin{tikzcd}
(A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d, "(j_0',j_1)"] \\
A \amalg A \amalg A \arrow[r] & C \amalg_A C
\end{tikzcd}$$

To see this, note maps

$$(\alpha, \beta)\colon C \amalg C' \to Z \qquad \text{and} \qquad (a, b, c)\colon A \amalg A \amalg A$$

which form a cocone satisfy $\beta \circ i_0' = b$ and $\alpha \circ i_1 = f$.

$$\begin{tikzcd}
(A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d] \arrow[ddr, bend left, "(\alpha,\beta)"] & \\
A \amalg A \amalg A \arrow[r, "k"] \arrow[drr, bend right, "(a,b,c)"'] & C \amalg_A C \arrow[dr, dashed, "\exists!"] & \\
 & & Z
\end{tikzcd}$$

Thus, there is a unique map

$$C \amalg_A C \to Z$$

making the above diagram commute.

Note that because $(i_0, i_1) \amalg (i_0', i_1')$ is a cofibration, k must be also.

Also, the map $A \amalg A \to A \amalg A \amalg A$ is a cofibration because the following diagram is a pushout.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d] \\
A \amalg A \arrow[r] & A \amalg A \amalg A
\end{tikzcd}$$

Thus, the composition

$$A \amalg A \to A \amalg A \amalg A \to C \amalg_A C'$$

is a cofibration. □

1.3.2 Right homotopy

There is a dual notion to left fibrations, called right fibrations. The theory of right fibrations is dual to that of left fibrations.

Definition 16 (path object)**.** Let B be an object of a model category. A path object for B is an object P, together with a factorization

$$\begin{tikzcd} & P \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, "(\mathrm{id},\mathrm{id})"'] & B \times B \end{tikzcd}$$

where p is a fibration and s is a weak equivalence.

Definition 17 (right homotopy)**.** A right homotopy between morphisms $f, g \colon A \longrightarrow B$ consists of a path object P and a factorization as follows.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "H"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

Lemma 18. Let B be a fibrant object. If P is a cylinder object for B, then the maps p_0 and p_1 are trivial fibrations.

Proof. Apply Lemma 14 to the opposite model category (Example 7). □

Proposition 19. Let B a fibrant object. Right homotopy of maps $f, g \colon A \longrightarrow B$ is an equivalence relation.

Proof. Apply Proposition 15 to the opposite model category (Example 7). □

1.3.3 Homotopy equivalents

Under certain conditions, namely when the domain and codomain are fibrant-cofibrant, left and right homotopy are equivalent.

Lemma 20. Suppose A is cofibrant, and suppose that $f, g \colon A \longrightarrow B$ are left homotopic. Then given any path object P for B, there is a right homotopy between f and g with underlying path object P.

Proof. Suppose we are given a left homotopy

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

and suppose that P has the following factorization

$$\begin{tikzcd} & P \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, "(\mathrm{id},\mathrm{id})"'] & B \times B \end{tikzcd}$$

Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "s \circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[r, "(f\circ\sigma,H)"'] & B \times B \end{tikzcd}$$

It commutes because

$$p \circ s \circ f = (\mathrm{id}, \mathrm{id}) \circ f = (f, f)$$

and

$$(f \circ \sigma, H) \circ i_0 = (f \circ \sigma \circ i_0, H \circ i_0) = (f \circ \mathrm{id}, f) = (f, f).$$

By definition, p is a fibration, and we have seen in Lemma 14 that i_0 is a trivial cofibration. Thus, we have a lift $h\colon C \to P$.

$$\begin{tikzcd} A \arrow[r, "s \circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[r, "(f\circ\sigma,H)"'] \arrow[ur, dashed, "h"] & B \times B \end{tikzcd}$$

Then we have

$$\begin{aligned} p \circ h \circ i_1 &= (f \circ \sigma, H) \circ i_1 \\ &= (f \circ \sigma \circ i_1, H \circ i_1) \\ &= (f, g), \end{aligned}$$

so the diagram

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "h\circ i_1"] \arrow[r, "(f,g)"'] & B \times B \end{tikzcd}$$

commutes. Thus, $h \circ i_1$ gives a right homotopy between f and g. □

Corollary 21. Let $f, g\colon A \to B$ be morphisms, and suppose that A is cofibrant and B is fibrant. Then TFAE.

1. f and g are left homotopic.
2. For every path object P for B, there is a right homotopy between f and g with path object P.

3. f and g are right homotopic.
4. For every cylinder object C for A, there is a left homotopy between f and g with cylinder object C.

Proof.

1 ⇒ 2. Lemma 20.

2 ⇒ 3. Trivial.

3 ⇒ 4. Dual to Lemma 20.

4 ⇒ 1. Trivial.

□

The above result shows that when considering morphisms between a cofibrant object and a fibrant object, there is no difference between left and right fibrations. Therefore, we will not distinguish them.

Definition 22 (homotopy equivalence)**.** Let $f\colon X \to Y$ be a morphism with X and Y fibrant-cofibrant. We say that f is a homotopy equivalence if there exists a morphism $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

Composition on the left and on the right preserves homotopy equivalence.

Lemma 23. Let $f, g\colon X \to Y$ be homotopy equivalent morphisms between fibrant-cofibrant objects, and let W and Z be fibrant-cofibrant. Then the following hold.

1. For any $h\colon Y \to Z$, $h \circ f \sim h \circ g$.
2. For any $h'\colon W \to X$, $f \circ h' \sim g \circ h'$.

Proof.

1. Suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} X \amalg X \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

Post-composing with h gives us the following

$$\begin{tikzcd} X \amalg X \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & Y \arrow[r, "h"] & Z \\ C \arrow[ur, "H"'] & & \end{tikzcd} \implies \begin{tikzcd} X \amalg X \arrow[r, "{(h\circ f,h\circ g)}"] \arrow[d, "i"'] & Z \\ C \arrow[ur, "h\circ H"'] & \end{tikzcd}$$

But the second diagram is precisely a homotopy $h \circ f \sim h \circ g$.

2. Now suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H'"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

Pre-composing with h yields the following.

$$\begin{tikzcd} & & P \arrow[d, "s"] \\ W \arrow[r, "h"] & X \arrow[ur, "H'"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd} \implies \begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H' \circ h"] \arrow[r, "(f \circ h, g \circ h)"'] & Y \times Y \end{tikzcd}$$

This is preciesly a homotopy $f \circ h \sim g \circ f$.

□

Corollary 24. Let X, Y, and Z be fibrant-cofibrant objects, and let f and g be homotopy equivalences as follows.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Then $g \circ f$ is a homotopy equivalence.

Proof. By definition, f has a homotopy inverse f' such that $f' \circ f \sim \text{id}$ and $f \circ f' \sim \text{id}$. Similarly, there is a g' such that $g' \circ g \sim \text{id}$ and $g \circ g' \sim \text{id}$. But then

$$\begin{aligned} \text{id} &\sim g' \circ g \\ &\sim f' \circ g' \circ g \\ &\sim f' \circ g' \circ g \circ f \end{aligned}$$

and

$$\begin{aligned} \text{id} &\sim f' \circ f \\ &\sim g' \circ f' \circ f \circ g. \end{aligned}$$

so $f' \circ g'$ functions as a homotopy inverse to $g \circ f$. □

Lemma 25. Let X and Y be fibrant-cofibrant objects, and $f \colon X \to Y$ a weak equivalence.

1. If f is a fibration, then it is a homotopy equivalence.
2. If f is a cofibration, then it is a homotopy equivalence.

Proof.

1. We have a lift as follows.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\text{id}"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

Thus, $f \circ g = \mathrm{id}_Y$.

Thus the diagram

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f{,}\mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] & Y \end{tikzcd}$$

commutes, because

$$\begin{aligned} f \circ (g \circ f, \mathrm{id}) &= (f \circ g \circ f, f) \\ &= (f, f) \\ &= (f \circ \sigma \circ i_1, f \circ \sigma \circ i_2) \\ &= f \circ \sigma \circ i. \end{aligned}$$

Since f is a trivial fibration and i is a cofibration, we have by Axiom (M4) a lift as follows.

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f{,}\mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

The upper triangle tells us that H is a left homotopy between $g \circ f$ and id_X.

Since $f \circ g = \mathrm{id}_Y$, the cylinder object C itself functions as a homotopy $f \circ g \sim \mathrm{id}_Y$. Thus, f is a homotopy equivalent.

2. The commutativity of the solid diagram

$$\begin{tikzcd} X \arrow[r, "s\circ f"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ Y \arrow[r, "(\mathrm{id}_Y{,}f\circ h)"'] \arrow[ur, dashed] & Y \times Y \end{tikzcd}$$

together with the fact that f is a trivial cofibration and p is a fibration imply that there is a dashed lift. This exhibits a right homotopy $f \circ h \sim \mathrm{id}_Y$. But $h \circ f = \mathrm{id}_X$, so $h \circ f \sim \mathrm{id}_X$.

So far we have shown that the result holds if f is a trivial fibration or a trivial cofibration.

Now suppose f is merely a weak equivalence. Then we can factor

$$f = g \circ h,$$

where g is a fibration and h is a trivial cofibration. But by Axiom (M2), g is also a trivial cofibration.

□

Theorem 26 (Whitehead)**.** Let $f\colon X \to Y$ be a weak equivalence, and suppose that X and Y are fibrant-cofibrant. Then f is a homotopy equivalence.

Proof. By Axiom (M5), we are allowed to factor $f = p \circ i$, where p is a fibration, i is a cofibration, and at least one of p and i is a weak equivalence. By Axiom (M2), both i and p are weak equivalences.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, swap, "\mathsf{W}\cap\mathsf{Cof}\ni i"] & & Y \\ & Z \arrow[ur, swap, "p\in\mathsf{W}\cap\mathsf{Fib}"] & \end{tikzcd}$$

By assumption, X is cofibrant, i.e. the map $\emptyset \to X$ is a cofibration. The map $i\colon X \to Z$ is also a cofibration, so their composition must also be a cofibration. But this means that the map $\emptyset \to Z$ is also a cofibration, so Z is cofibrant

Dually, the map $Y \to *$ is a fibration. Precomposing by p, we have that Z is fibrant. Thus, i and p are homotopy equivalences by Lemma 25, parts 2 and 1 respectively. But by Lemma 23, their composition is also a homotopy equivalence. □

Definition 27 (homotopy category of a model category)**.** Let $\mathcal{C}$ be a model category. The homotopy category of $\mathcal{C}$, denoted $\mathrm{Ho}(\mathcal{C})$, is defined as follows.

- The objects of $\mathrm{Ho}(\mathcal{C})$ are the fibrant-cofibrant objects of $\mathcal{C}$.
- The set of morphisms $X \to Y$ is the set of homotopy classes of $\mathcal{C}(X, Y)$.
- Composition is given by the rule

  $$[f] \circ [g] \equiv [f \circ g].$$

  This is well-defined by Lemma 23.

Example 28. Consider the category $\mathbf{Set}_\Delta$ together with the Kan model structure (Example 6). We have seen (in Example 9) that for any simplicial set S, the object $\Delta^1 \times S$ is a cylinder object for S. Following our proofs through, one finds that the homotopy theory in the model structure on $\mathbf{Set}_\Delta$ agrees with the homotopy theory on $\mathbf{Set}_\Delta$ we have worked out in Chapter **??**.

In the Kan model structure $\mathbf{Set}_\Delta$, the fibrant objects are Kan complexes, and every object is cofibrant. Therefore, the fibrant-cofibrant objects are simply Kan complexes.

The homotopy category $\mathrm{Ho}(\mathbf{Set}_\Delta)$ therefore simply has objects given by Kan complex, and morphisms given by equivalence classes of morphisms under homotopy. We will meet this category later. It is called the *homotopy category of spaces*.

1.4 Fibrant-Cofibrant replacement

Let $\mathcal{C}$ be a model category.

In Subsection 1.4.1, we give a way of assigning to any object $X \in \mathcal{C}$ a (not necessarily unique) cofibrant object QX to which it is weakly equivalent. We call any choice of such at cofibrant object a *cofibrant replacement.*

We also define the notion of cofibrant replacement on morphisms, i.e. a way of assigning

$$(f : X \to Y) \mapsto (Qf : QX \to QY).$$

Again, this assignment is not uniquely defined, so cofibrant does not yield a functor, as one might hope.

Next we next prove that for any cofibrant replacements Qf of f and Qg of g,

$$f \overset{l}{\sim} g \implies Qf \overset{l}{\sim} Qg,$$

showing that the cofibrant replacement of a morphism is uniquely defined up to homotopy if we restrict our attention to morphisms out of cofibrant objects X. In this setting, we have the further nice property that cofibrant replacement respects composition in the sense that

$$Q(g \circ f) \overset{l}{\sim} Qg \circ Qf.$$

In Subsection 1.4.2, we study the dual notion, called *fibrant replacement,* providing a method by which one can replace any object X by a fibrant object RX to which it is weakly equivalent. We then again define a notion of cofibrant replacement on morphims, i.e. a way of assigning

$$(f : X \to Y) \mapsto (Rf : RX \to RY).$$

Then, restricting to morphisms whose domain Y is fibrant, and considering morphisms up to right homotopy, we find that fibrant replacement becomes well-defined in the sense that

$$f \overset{r}{\sim} f' \implies Rf \overset{r}{\sim} Rf',$$

and preserves composition in the sense that $R(g \circ f) = Rg \circ Rf$.

In Subsection 1.4.3, we prove two preliminary lemmas, which show the following.

- The fibrant replacement of a cofibrant object is both fibrant and cofibrant.
- fibrant replacement takes left-homotopic morphisms $f, g : X \to Y$ with X cofibrant to morphisms which are right homotopic.

The first lemma implies that taking first the cofibrant and then fibrant replacement of any object X yields an object RQX which is both fibrant and cofibrant; one calls such objects *fibrant-cofibrant,* and the act of replacing an object by a weakly equivalent fibrant-cofibrant one *fibrant-cofibrant replacement.* The second lemma then implies that fibrant-cofibrant replacement gives a functor from $\mathcal{C}$ to $\mathrm{Ho}(\mathcal{C})$.

1.4.1 Cofibrant replacement

Definition 29 (cofibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a <u>cofibrant replacement</u> of X consists of a cofibrant object QX and a trivial fibration $p_X : QX \to X$, defined as follows.

- If X is cofibrant, define $QX = X$, and $p_X = \mathrm{id}_X$.
- If X is not cofibrant, pick a factorization of the unique morphism $\emptyset \to X$

$$\emptyset \to QX \xrightarrow{p_X} X$$

  where the map $\emptyset \to Qx$ is a cofibration and p_X is a trivial fibration.

For a morphism $f\colon X \to Y$, a fibrant replacement of f is any solution Rf to the following lifting problem.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\ QX \arrow[r, "f\circ p_X"'] \arrow[ur, dashed, "Qf"] & Y \end{tikzcd}$$

Example 30. Consider the category $\mathbf{Ch}_{\geq}(R)$ of bounded-below chain complexes of modules over a ring R. Let M be an R-module, considered as a chain complex concentrated in degree 0. A cofibrant replacement for M is a chain complex of projective modules P^M together with a trivial fibration $P^M \to M$. This is the same thing as a projective resolution of M.

Lemma 31. For any map $f\colon X \to Y$, any choice of cofibrant replacement $f \mapsto Qf$ respects left homotopy; that is, if $f \overset{l}{\sim} f'$, then $Qf \overset{l}{\sim} Qf'$.

Proof. Suppose we have lifts Qf and Qf' of f and f' respectively, and consider a left homotopy between them.

$$\begin{tikzcd} X \amalg X \arrow[r, "(f{,}f')"] \arrow[d, "i"'] & Y \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd} QX \amalg QX \arrow[r, "(Qf{,}Qf')"] \arrow[d, "i"'] & QY \arrow[d, "p_Y"] \\ C \arrow[r, "f\circ p_X\circ H"'] \arrow[ur, dashed] & Y \end{tikzcd}$$

The outer square commutes because

$$\begin{aligned} p_Y \circ (Qf, Qf') &= (f \circ p_X, f' \circ p_X) \\ &= f \circ p_X \circ (f, f') \\ &= f \circ p_X \circ H \circ i. \end{aligned}$$

The map i is a cofibration by definition, and p_Y is a trivial fibration. Thus, we get a dashed lift. But this is precisely a left homotopy between Qf and Qf'. $\square$

Corollary 32. We have
$$Q(g \circ f) \overset{l}{\sim} Q(g) \circ Q(f).$$

Proof. Consider the following square.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & QZ \arrow[d, "p_Z"] \\
* \arrow[r] \arrow[d] & QY \arrow[r, "g \circ p_Y"] \arrow[d, "p_Y"] & Z \arrow[d, "\mathrm{id}"] \\
QX \arrow[r, "f \circ p_X"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

The outer square is the lifting problem defining $Q(g \circ f)$, and the top right and bottom left squares define Qg and Qf respectively. Thus, $Qg \circ Qf \overset{l}{\sim} Q(g \circ f)$. □

There is a dual notion.

1.4.2 Fibrant replacement

Definition 33 (fibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a fibrant replacement of X consists of a fibrant object RX and a trivial cofibration $i_X : X \to RX$, defined as follows.

- If X is fibrant, define $RX = X$, and $i_X = \mathrm{id}_X$.
- If X is not fibrant, pick factorization of the unique morphism $X \to *$
$$X \overset{i_X}{\to} RX \to *$$
where the map $\emptyset \to Qx$ is a fibration and i_X is a trivial cofibration.

For a morphism $f : X \to Y$, a fibrant replacement of f is a solution Rf to the following lifting problem.

$$\begin{tikzcd}
X \arrow[r, "i_Y \circ f"] \arrow[d, "i_X"'] & RY \arrow[d] \\
RX \arrow[r] \arrow[ur, dashed, "Rf"] & *
\end{tikzcd}$$

Lemma 34. For any map $f : X \to Y$, fibrant replacement $f \mapsto Rf$ respects right homotopy; that is, if $f \overset{r}{\sim} f'$, then $Rf \overset{r}{\sim} Rf'$.

Proof. Dual to □

Corollary 35. We have
$$R(g \circ f) \overset{r}{\sim} R(g) \circ R(f).$$

Proof. Dual to □

1.4.3 Fibrant-cofibrant replacement

Lemma 36. Suppose $f, f' \colon X \to Y$ are left homotopic, and that X is cofibrant. Then the fibrant replacements Rf and Rf' are right homotopic.

Proof. Since f and f' are left homotopic, so are $i_Y \circ f$ and $i_Y \circ f'$. But since these are between a fibrant and a cofibrant object, they are right homotopic. Pick some right homotopy H as follows.

$$\begin{tikzcd}
& P \arrow[d, "p"] \\
X \arrow[ur, "H"] \arrow[r, "(i_Y \circ f, i_Y \circ f')"'] & RY \times RY
\end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd}
X \arrow[r, "H"] \arrow[d, "i_X"'] & R \arrow[d, "p"] \\
RX \arrow[ur, dashed] \arrow[r, "(Rf, Rf')"'] & RY \times RY
\end{tikzcd}$$

The outer square commutes because

$$p \circ H = (i_Y \circ f, i_Y \circ f') = (Rf \circ i_X, Rf' \circ i_X).$$

□

Theorem 37. There is a functor $\pi \colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, defined on objects by $X \mapsto RQX$ and on morphisms by $f \mapsto [RQf]$.

Proof. First, note that the assignment on objects is well-defined since RQX is fibrant by definition and cofibrant because the composition

$$\emptyset \to QX \xrightarrow{i_X} RQX$$

is a cofibration.

It remains to check that π respects composition. To see this, note that

$$\begin{aligned}
\pi(g \circ f) &= [RQ(g \circ f)] \\
&= [R(Qg \circ Qf)] \\
&= [RQ(g) \circ RQ(f)] \\
&= [RQ(g)] \circ [RQ(f)] \\
&= \pi(g) \circ \pi(f),
\end{aligned}$$

where the second equality follows from Corollary 32 and Lemma 36 and the third follows from Lemma 34. □

Theorem 38. The homotopy category $\mathrm{Ho}(\mathcal{C})$, together with the map $\pi\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, is the localization of $\mathcal{C}$ at $\mathcal{W} = \mathrm{W}$, the set of weak equivalences. That is, there is an equivalence of categories $\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathrm{Ho}(\mathcal{C})$.

Proof. Define a category $\overline{\mathcal{C}}$ as follows.

- $\mathrm{Obj}(\overline{\mathcal{C}}) = \mathrm{Obj}(\mathcal{C})$
- For each $X, Y \in \mathrm{Obj}(\overline{\mathcal{C}})$, $\overline{\mathcal{C}}(X,Y) = \mathrm{Ho}(\mathcal{C})(RQX, RQY)$.

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\pi"] \arrow[dr, "\bar{\pi}"'] & & \mathrm{Ho}(\mathcal{C}) \\
& \overline{\mathcal{C}} \arrow[ur, "\xi"'] &
\end{tikzcd}$$

where $\bar{\pi}$ is the identity on objects and maps $f \mapsto [RQf]$, and ξ acts on objects by taking $X \mapsto RQX$ and is the identity on morphisms.

Note that ξ is surjective on objects and fully faithful, hence an equivalence of categories. Hence, we only need to show that $\overline{\mathcal{C}}$ satisfies the universal property for the localization.

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which takes weak equivalences to isomorphisms in $\mathcal{D}$. If we can show that there exists a unique functor $\overline{F}\colon \overline{\mathcal{C}} \to \mathcal{D}$ such that $F = \overline{F} \circ \overline{\pi}$, then we are done.

Suppose $f \overset{r}{\sim} g$. Pick some path object P, and a homotopy through P.

$$\begin{tikzcd}
& P \arrow[d, "p"] \\
X \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & X \times X
\end{tikzcd}
\qquad
\begin{tikzcd}
& P \arrow[d, "p"] \\
X \arrow[ur, "H"] \arrow[r, "{(f,g)}"'] & Y \times Y
\end{tikzcd}$$

Putting these diagrams back to back and unfolding them, we have the following.

$$\begin{tikzcd}
& Y & \\
X \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "g"'] & P \arrow[u, "p_0"'] \arrow[d, "p_1"'] & Y \arrow[l, "s"'] \arrow[ul, "\mathrm{id}"'] \arrow[dl, "\mathrm{id}"] \\
& Y &
\end{tikzcd}$$

Consider the image in $\mathcal{D}$. Since p_0 and p_1 are weak equivalences, Theorem 26 implies that they are sent to isomorphisms in $\mathcal{D}$. Thus, $F(p_0) = F(s)^{-1} = F(p_1)$, so

$$\begin{aligned}
F(f) &= F(p_0) \circ F(H) \\
&= F(p_1) \circ F(H) \\
&= F(g).
\end{aligned}$$

Let $[g]\colon RQX \to RQY$ be a morphism in $\overline{\mathcal{C}}$. Pick a representative g. Consider the following commutative diagram.

$$\begin{tikzcd}
X \arrow[r, "g"] & Y \\
QX \arrow[u, "p_X"] \arrow[r, "Qg"] \arrow[d, "i_{QX}"'] & QY \arrow[u, "p_Y"'] \arrow[d, "i_{QY}"] \\
RQX \arrow[r, "RQg"] & RQY
\end{tikzcd}$$

Applying F, we find the diagram

$$\begin{tikzcd}
F(X) \arrow[r, "F(g)"] & F(Y) \\
F(QX) \arrow[u, "F(p_X)"] \arrow[r, "F(Qg)"] \arrow[d, "F(i_{QX})"'] & F(QY) \arrow[u, "F(p_Y)"'] \arrow[d, "F(i_{QY})"] \\
F(RQX) \arrow[r, "F(RQg)"] & F(RQY)
\end{tikzcd}$$

But the vertical morphisms are isomorphisms, so

$$F(RQg) = F(i_{QY}) \circ F(p_Y)^{-1} \circ F(g) \circ F(p_X) \circ F(i_{QX})^{-1}.$$

In order to have $F = \bar{F} \circ \bar{\pi}$, we must have

$$\overline{F}([RQg]) = F(RQg),$$

which specifies $\bar{F}$ completely. □

1.5 Quillen adjunctions

Proposition 39. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and consider an adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G.$$

The following are equivalent.

1. F preserves cofibrations and trivial cofibrations.
2. G preserves fibrations and trivial fibrations.

Proof. Suppose 2. holds. By Lemma 4, a morphism is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations. Thus, in order to show that F preserves cofibrations, it suffices to show that if f is a cofibration in $\mathcal{C}$, then $F(f)$ has the left lifting property with respect to all trivial fibrations.

Consider solid adjunct diagrams of the following form, with the first in $\mathcal{D}$ and the second in $\mathcal{C}$.

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d, "F(f)"'] & d \arrow[d, "g"] \\ F(c') \arrow[r] \arrow[ur, dashed] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & G(d) \arrow[d, "G(g)"] \\ c' \arrow[r] \arrow[ur, dashed] & G(d') \end{tikzcd}$$

Suppose that f is a cofibration and that F preserves cofibrations. Then the left-hand diagram has a lift for any trivial fibration g. The adjunct of this lift is a lift for the diagram on the right. This means that $G(g) \in \mathsf{Cof}_\perp = \mathsf{Fib} \cap \mathsf{W}$.

Similarly, if f is a trivial cofibration and F preserves trivial cofibrations, then the left-hand diagram has a lift for any fibration g, whose adjunct is a lift for the diagram on the RHS, implying that $G(d) \in (\mathsf{Cof} \cap \mathsf{W})_\perp = \mathsf{Fib}$.

The other statement is dual. □

Definition 40 (Quillen adjunction)**.** An adjunction of model categories satisfying the equivalent conditions in Proposition 39 is known as a Quillen adjunction.

Lemma 41 (Ken Brown's Lemma)**.** Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be a Quillen adjunction.

1. The left adjoint F preserves weak equivalences between cofibrant objects
2. The right adjoint G preserves weak equivalences between fibrant objects.

Proof. By Proposition 39. Let A and B be cofibrant objects, and $f\colon A \to B$ be a weak equivalence.

Pick for the map $(f, \mathrm{id})\colon A \amalg B \to B$ a factorization

$$A \amalg B \xrightarrow{i} C \xrightarrow{p} B,$$

where $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib} \cap \mathsf{W}$.

Consider the following diagram.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d, "i_A"] \arrow[dr, bend left, "f"] & \\ B \arrow[r, "i_B"'] & A \amalg B \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "q"'] & B \\ & C \arrow[ur, "p"'] & \end{tikzcd}$$

Because A and B are cofibrant, i_A and i_B are cofibrations. Thus, $q \circ i_A$ is a cofibration, so C is cofibrant.

Note that by commutativity, we have

$$f = p \circ q \circ i_A, \qquad \mathrm{id}_B = p \circ q \circ i_B.$$

By Axiom (M2), we have that $q \circ i_A$ and $q \circ i_B$ are weak equivalences, hence trivial cofibrations.

Applying F, we find that

$$F(f) = F(p) \circ F(q \circ i_A), \qquad F(\mathrm{id}) = F(p) \circ F(q \circ i_B).$$

We know that $F(\mathrm{id}_B) = \mathrm{id}_{F(b)}$ is a trivial cofibration, hence certainly a weak equivalence, and that $F(q \circ i_B)$ and $F(q \circ i_A)$ are a weak equivalence because F preserves trivial cofibrations. Thus $F(p)$ is a weak equivalence, implying that

$$F(f) = F(p) \circ F(q \circ i_A)$$

is a weak equivalence. □

Given a Quillen adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G,$$

consider the full subcategory of all cofibrant objects $\mathcal{C}_c \subset \mathcal{C}$. Since the full subcategory of $\mathcal{C}$ on fibrant-cofibrant objects is the same as the full subcategory of $\mathcal{C}_c$ on fibrant-cofibrant objects, there is an equivalence of categories

$$\mathcal{C}_c[\mathcal{W}^{-1}] \simeq \mathcal{C}[\mathcal{W}^{-1}].$$

We will denote any weak inverse by

$$Q \colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}_c[\mathcal{W}^{-1}].$$

Dually, we get a functor

$$R \colon \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{D}_f[\mathcal{W}^{-1}],$$

where $\mathcal{D}_f$ is the full subcategory of $\mathcal{D}$ on fibrant objects.

Definition 42 (derived functor)**.** Let

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G$$

be a Quillen adjunction.

- The left derived functor LF is the composite

$$\mathcal{C}[\mathcal{W}^{-1}] \xrightarrow{Q} \mathcal{C}_c[\mathcal{W}^{-1}] \xrightarrow{F} \mathcal{D}[\mathcal{W}^{-1}].$$

- The right derived functor RG is the composite

$$\mathcal{D}[\mathcal{W}^{-1}] \xrightarrow{R} \mathcal{D}_f[\mathcal{W}^{-1}] \xrightarrow{G} \mathcal{C}[\mathcal{W}^{-1}].$$

Proposition 43. Let

$$\mathcal{C} : F \leftrightarrow \mathcal{G} : \mathcal{D}$$

be a Quillen adjunction. Then there is an adjunction of derived functors

$$LF : \mathcal{C}[\mathcal{W}^{-1}] \leftrightarrow \mathcal{D}[\mathcal{W}^{-1}] : RG.$$

Definition 44 (Quillen equivalence)**.** A Quillen adjunction is called a Quillen equivalence if LF, or equivalently RG, is an equivalence of localized categories.

Theorem 45. The adjunction

$$|\cdot| : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Top} : \mathbf{Sing}$$

is a Quillen adjunction.