1 Left fibrations and the Grothendieck construction

1.1 Invitation: the classical Grothendieck construction

1.1.1 The Grothendieck construction

Definition 1 (Grothendieck construction)**.** Let $\mathcal{D}$ be a small category, and let $\chi\colon \mathcal{D} \to \mathbf{Grpd}$. The Grothendieck construction consists of the following data the following data.

- A category $\mathcal{C}_\chi$
- A functor $\pi\colon \mathcal{C}_\chi \to \mathcal{D}$

The category $\mathcal{C}_\chi$ is as follows.

- **Objects:** Pairs (d, x), where $d \in \mathcal{D}$ and $x \in \chi(d)$.
- **Morphisms:** A morphism $(d, x) \to (d', x')$ consists of
  - A morphism $f\colon d \to d'$ in $\mathcal{D}$
  - An isomorphism $\alpha\colon \chi(f)(x) \simeq x'$

The functor π is the forgetful functor to $\mathcal{D}$.

The Grothendieck construction can be understood in the following way. The functor χ gives us a way of associating a groupoid to each object $d \in \mathcal{D}$. Each of these groupoids is itself a category, and has its own objects. The idea of the Grothendieck construction is to throw all of these objects together into one big category. To keep track of where each object x comes from, we think of x as ordered pairs (d, x), where the $x \in \chi(\mathcal{D})$. The functor π simply remembers where each x came from.

For two objects x and $\tilde{x}$ from the same $\chi(d)$, the morphisms $(d, x) \to (d, \tilde{x})$ are simply the morphisms between x and $\tilde{x}$. For an object x' from a different groupoid $\chi(d')$, the morphisms $(d, x) \to (d', x')$ come from translating, using χ and a morphism $d \to d'$, morphisms in $\chi(d)$ into morphisms in $\chi(d')$.

Example 2. Let G and H be a groups, and $\mathbf{B}G$ and $\mathbf{B}H$ their delooping groupoids. Let $\phi\colon G \to H$ be a group homomorphism, or equivalently a functor $\mathbf{B}G \to \mathbf{B}H$. This can also be taken to be a functor $\mathbf{B}G \to \mathbf{Grpd}$.

The Grothendieck construction consists of a category with one object, whose morphisms consist of an element $g \in G$

There is also a contravariant version of the Grothendieck construction, consisting of the following.

Definition 3 (contravariant Grothendieck construction)**.** For any functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Grpd}$, the contravariant Grothendieck construction consists of the following.

1. A category $\mathcal{C}^{\chi}$
2. A functor $\pi \colon \mathcal{C}^{\chi} \to \mathcal{D}$.

The category $\mathcal{C}^{\chi}$ is as follows.

- The objects of $\mathcal{C}^{\chi}$ are be pairs (d, x), with $d \in \mathcal{D}$ and $x \in \chi(d)$.
- The morphisms $(d', x') \to (d, x)$ are pairs (f, α), where $f \colon d' \to d$ and $\alpha \colon x' \to F(f)(x)$.

There is the following relationship between the covariant and contravariant Grothendieck constructions. $\mathcal{C}^{\chi} = \mathcal{C}_{\chi}^{\mathrm{op}}$.

Example 4. Let $X \colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ be a diagram of sets, which we interpret as discrete groupoids. Then there is an equivalence of categories between the contravariant construction and the slice category $\mathcal{D}/X$.

The objects of $\mathcal{D}/X$ are pairs (d, α), where $d \in \mathcal{D}$ and $\alpha \colon \mathrm{Hom}_{\mathcal{D}}(-, d) \Rightarrow X$. The morphisms $(d, \alpha) \to (d', \beta)$ are morphisms $f \colon d \to d'$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(-, d) \arrow[rr, Rightarrow, "f\circ-"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathrm{Hom}_{\mathcal{D}}(-, d') \arrow[dl, Rightarrow, "\beta"] \\ & X & \end{tikzcd} \tag{1.1}$$

commutes.

Consider any such morphism f. The above diagram commutes if and only if it commutes component-wise, i.e. if for each c, the diagram below commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(c, d) \arrow[rr, "f\circ-"] \arrow[dr, "\alpha_c"'] & & \mathrm{Hom}_{\mathcal{D}}(c, d') \arrow[dl, "\beta_c"] \\ & X(c) & \end{tikzcd} \tag{1.2}$$

In particular, it commutes when $c = d$. Following the identity id_d around, we find that morphisms f making the above diagram commute satisfy the following equality.

$$\begin{tikzcd} \mathrm{id}_d \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \arrow[dl, mapsto] \\ & \alpha_d(\mathrm{id}_d) = \beta_d(f) & \end{tikzcd}$$

By the naturality of the Yoneda embedding, we have that

$$\beta_d(f) = X(f)(\beta_{d'}(\mathrm{id}_{d'})),$$

so

$$\alpha_d(\mathrm{id}_d) = X(f)(\beta_{d'}(\mathrm{id}_{d'})). \tag{1.3}$$

We can specify a functor $\mathcal{D}/X \to \mathcal{C}^X$ on objects by sending

$$(d, \alpha) \mapsto (d, \alpha_b(\mathrm{id}_b)),$$

and on morphisms $f\colon (d,\alpha) \longrightarrow (d',\beta)$ commute by sending

$$f \longmapsto (f, \mathrm{id}_{X(f)(\beta_{d'}(\mathrm{id}_{d'}))}).$$

We are allowed to use the above identity map preciesly because it is an isomorphism

$$\alpha_d(\mathrm{id}_d) \longrightarrow X(f)(\beta_{d'}(d'))$$

as shown in Equation 1.3.

The fully faithfulness of the Yoneda embedding implies that the map between the objects of $\mathcal{D}/X$ and the objects of $\mathcal{C}^X$ is a bijection. We will now construct a functor $\mathcal{C}^X \longrightarrow \mathcal{D}/X$, defined on objects to be the inverse of the one defined above, i.e. sending a pair (d,x) to the pair (d,α^x), where α^x is the natural transformation uniquely specified by $\alpha^x_d(\mathrm{id}_d) \mapsto x \in X(d)$.

Our inverse functor acts on any morphism (f,ϕ) by forgetting ϕ. It only remains to check that this provides a well-defined map of morphisms, as f has to make Diagram 1.1 commute. We can check that this is so by following a general morphism $g\colon c \longrightarrow d$ around Diagram 1.2, finding the following condition.

$$\begin{tikzcd}
g \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \circ g \arrow[dl, mapsto] \\
& \alpha_c(g) \overset{!}{=} \beta_c(f \circ g) &
\end{tikzcd}$$

By naturality of the Yoneda embedding, we have

$$\alpha_c(g) = (Xg)(\alpha_d(\mathrm{id}_d)), \qquad \beta_c(f \circ g) = X(f \circ g)(\beta_{d'}(\mathrm{id}_{d'})) = (Xg \circ Xf)(\beta_{d'}(\mathrm{id}_{d'})),$$

so our condition equivalently reads

$$Xg(\alpha_d(\mathrm{id}_d)) = Xg((Xf)(\beta_{d'}(\mathrm{id}_{d'}))).$$

This is true because by definition of f,

$$\alpha_d(\mathrm{id}_d) = (Xf)(\beta_{d'}(\mathrm{id}_{d'})).$$

The two functors defined in this way are manifestly each other's inverses, hence provide an isomorphism of categories, which is stronger than what we are looking for.

1.1.2 Fibrance and cofibrance in groupoids

Definition 5 (cofibrant in groupoids)**.** Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. We say that f exhibits $\mathcal{C}$ as cofibrant in groupoids over $\mathcal{D}$ if the following conditions are satisfied.

1. For every $c \in \mathcal{C}$ and every morphism

$$\eta\colon F(c) \longrightarrow d \qquad \text{in } \mathcal{D}$$

there exists a morphism

$$\tilde{\eta}\colon c \longrightarrow \tilde{d} \qquad \text{in } \mathcal{C}$$

such that $F(\tilde{\eta}) = \eta$.

2. For every morphism $\eta\colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map
$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c),F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$
is a bijection.

Let us unravel what the second condition means. The second set is the following pullback.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathcal{D}(F(c'), F(c'')) \arrow[d, "\eta"] \\
\mathcal{C}(c, c'') \arrow[r, "F"'] & \mathcal{D}(F(c), F(c''))
\end{tikzcd}$$

Its elements are pairs
$$(f, \bar{g}), \qquad \text{where} \quad f\colon c \to c'' \quad \text{and} \quad \bar{g}\colon F(c') \to F(c'')$$
such that
$$F(f) = \bar{g} \circ F(\eta).$$

The bijection above is defined as follows. Starting on the LHS with a morphism $h\colon c' \to c''$, build the following commutative diagram.

$$\begin{tikzcd}
& c' \arrow[dr, "h"] & \\
c \arrow[ur, "\eta"] \arrow[rr, "h\circ\eta"'] & & c''
\end{tikzcd}$$

We can read off a map $f = h \circ \eta\colon c \to c''$ and a map $\bar{g} = F(h)\colon F(c') \to F(c'')$, and indeed $F(f) = \bar{g} \circ F(\eta)$, giving us an element of the RHS.

The fact that this is a bijection means that the above operation is invertible. Fixing a morphism $\eta\colon c \to c'$ and an object c'', whenever we have the following data:

- A morphism $g\colon c \to c''$ as follows

$$\begin{tikzcd}
& c' & \\
c \arrow[ur, "\eta"] \arrow[rr, "\bar{g}"'] & & c''
\end{tikzcd}$$

- A map $\bar{h}\colon F(c') \to F(c'')$ making the following diagram commute

$$\begin{tikzcd}
& F(c') \arrow[dr, "\bar{h}"] & \\
F(c) \arrow[ur, "F(\eta)"] \arrow[rr, "F(g)"'] & & F(c'')
\end{tikzcd}$$

we can lift $\bar{h}$ to a morphism h in $\mathcal{C}$ making the following diagram commute.

$$\begin{tikzcd}
& c' \arrow[dr, "h"] & \\
c \arrow[ur, "\eta"] \arrow[rr, "g"'] & & c''
\end{tikzcd}$$

Example 6. The Grothendieck construction

$$\pi\colon \mathcal{C}_\chi \to \mathcal{D}$$

exhibits $\mathcal{D}$ as cofibered over groupoids. We simply transcribe the definition, making the necessary changes as we go along.

In ordered for π to exhibit $\mathcal{C}_\chi$ as cofibered in groupoids over $\mathcal{D}$, we have to check the following.

1. For every $(d, x) \in \mathcal{C}_\chi$ and every morphism

$$\eta\colon d \to d' \qquad \text{in } \mathcal{D}$$

   there exists a lift

$$\tilde{\eta} = (f, \alpha)\colon (d, x) \to (d', x') \qquad \text{in } \mathcal{C}_\chi$$

   such that

$$\pi(\tilde{\eta}) = f : d \to d' = \eta.$$

2. For every morphism $\eta\colon (x, d) \to (x', d')$ in $\mathcal{C}$ and every object $(x'', d'') \in \mathcal{C}$, the map

$$\mathcal{C}_\chi((d', x') \to (d'', x'')) \to \mathcal{C}_\chi((d, x), (d'', x'')) \underset{\mathcal{D}(d,d'')}{\times} \mathcal{D}(d', d'')$$

   is a bijection.

What do these mean?

1. We need to construct from our map $\eta\colon d \to d'$ an isomorphism

$$\alpha\colon \chi(f)(x) \simeq x'$$

   for some x'. We can pick $x' = \chi(f)(x)$. Then $\tilde{\eta} = (f, \alpha)$ does the job. Indeed, $\pi(f, \alpha) = f$ as required.

2. Draw a picture. Given the data of
   - A morphism

Definition 7 (fibrant in groupoids)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that f exhibits $\mathcal{C}$ as fibrant in groupoids over $\mathcal{D}$ if the following conditions are satisfied.

1. For every $c' \in \mathcal{C}$ and every morphism

$$\eta\colon d \to F(c') \qquad \text{in } \mathcal{D}$$

   there exists a morphism

$$\tilde{\eta}\colon \tilde{d} \to c' \qquad \text{in } \mathcal{C}$$

   such that $F(\tilde{\eta}) = \eta$.

2. For every morphism $\eta\colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c),F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$

   is a bijection.

Example 8. The contravariant Grothendieck construction exhibits $\mathcal{C}^{\chi} \to \mathcal{D}$ as fibered in groupoids over $\mathcal{D}$.

To see this, we need to make the following checks.

- **Check:** For every $(d, x) \in \mathcal{C}^{\chi}$, and every morphism $\eta \colon d' \to d$ in $\mathcal{D}$, we should be able to find some object $x' \in \chi(d')$ and a morphism $(f, \alpha) \colon (d', x') \to (d, x)$.

  Solution: We can pick $f = \eta$, $x' = \chi(f)(x)$, and $\alpha = \mathrm{id}_{x'}$.

- **Check:** Fixing any diagram

$$\begin{tikzcd}
 & (d', x') \arrow[d, "{(g,\alpha)}"] \\
(d'', x'') \arrow[r, "{(\eta,\beta)}"'] & (d, x)
\end{tikzcd}$$

  in $\mathcal{C}^{\chi}$, for any morphism $f \colon d'' \to d'$ making the below left diagram commute, we should be able to find a morphism $(f, \gamma) \colon (d'', x'') \to (d', x')$ making the below right diagram commute.

$$\begin{tikzcd}
 & d' \arrow[dr, "g"] & \\
d'' \arrow[ur, dashed, "f"] \arrow[rr, "\eta"'] & & d
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
 & (d', x') \arrow[dr, "{(g,\alpha)}"] & \\
(d'', x'') \arrow[ur, dashed, "{(f,\gamma)}"] \arrow[rr, "{(\eta,\beta)}"'] & & (d, x)
\end{tikzcd}$$

  Solution: We need a morphism $\gamma : x'' \to \chi(f)(x')$ making the above diagram commute.

  Using the data available to us, we can create the following diagram in $\chi(d'')$.

$$\begin{tikzcd}
 & \chi(f)(x') \arrow[d, "\chi(f)(\alpha)"] \\
x'' \arrow[r, "\beta"'] & \chi(\eta)(x)
\end{tikzcd}$$

  Note that we are entitled to draw $\chi(f)(\alpha)$ terminating at $\chi(\eta)(x)$ because

$$\chi(f)(\chi(g)(x)) = \chi(g \circ f)(x) = \chi(\eta)(x).$$

  Because $\chi(d'')$ is a groupoid, we get the morphism γ via

$$\gamma = (\chi(f)(a))^{-1} \circ \beta.$$

Lemma 9. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f) = \mathrm{id}$. Then f is an isomorphism.

Proof. Let $f \colon c \to c'$ in $\mathcal{C}_d$ be a morphism such that $F(f) = \mathrm{id}_d$. Then we can form the following commutative diagram.

$$\begin{tikzcd}
 & c' \\
c \arrow[ur, "f"] \arrow[r, "\mathrm{id}"'] & c
\end{tikzcd}$$

When mapped to $\mathcal{D}$, we find the following very simple diagram, with a dashed filler.

$$\begin{tikzcd} & d \arrow[dr, dashed, "\mathrm{id}"] & \\ d \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & d \end{tikzcd}$$

Lifting this gives us a left inverse $\tilde{f}$ to f. Pulling the same trick with $\tilde{f}$ gives us a left inverse $\hat{f}$ to $\tilde{f}$. Then

$$\begin{aligned} f \circ \tilde{f} &= \mathrm{id} \\ \hat{f} \circ f \circ \tilde{f} &= \hat{f} \\ \hat{f} &= f, \end{aligned}$$

so $\tilde{f}$ is both a left and a right inverse for f, i.e. f is an isomorphism. □

Lemma 10. Let $F: \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f)$ is an equivalence. Then f is an equivalence.

1.1.3 Cartesian and CoCartesian fibrations

Definition 11 (coCartesian morphism)**.** Let $F: \mathcal{C} \to \mathcal{D}$ be a functor of small categories. A morphism $f: c \to c'$ is said to be F-coCartesian if, for every $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \times_{\mathcal{D}(F(c), F(c''))} \mathcal{D}(F(c'), F(c''))$$

is a bijection.

Definition 12 (coCartesian fibration)**.** A functor $F: \mathcal{C} \to \mathcal{D}$ is said to be a coCartesian fibration if, for each $x \in \mathcal{C}$, there

1.1.4 When can we invert the classical Grothendieck construction?

Suppose we are given a functor $F: \mathcal{C} \to \mathcal{D}$ exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. By Lemma 24, each of the fibers $\mathcal{C}_d$ is a groupoid. We may hope that we could reverse the Grothendieck construction by defining a functor $\mathcal{D} \to \mathbf{Grpd}$ defined on objects by $d \mapsto \mathcal{C}_d$.

We almost can. For any morphism $f: d \to d'$ in $\mathcal{D}$, we can build functor $f_!: \mathcal{C}_d \to \mathcal{C}_{d'}$ as follows.

Let $f: d \to d'$, and let $c \in \mathcal{C}_d$. By property 2 in Definition 5, we can find an object $c' \in \mathcal{C}_{d'}$ and a morphism $\bar{f}: c \to c'$ in $\mathcal{C}$ such that $F(\bar{f}) = f$. Define $f_!(c) = c'$, and $f_!()$

The above discussion means that the Grothendieck construction defines the following correspondence.

$$\left\{ \begin{matrix} F: \mathcal{C} \to \mathcal{D} \text{ exhibits } \mathcal{C} \\ \text{as cofibered in groupoids} \end{matrix} \right\} \xleftrightarrow{\simeq} \left\{ \begin{matrix} \text{pseudofunctors} \\ \mathcal{D} \to \mathbf{Grpd} \end{matrix} \right\}$$

1.2 The fibration zoo

The menagerie of notions of fibrations is something that we can no longer bandy about. In order to have some hope of keeping them straight, we keep a list here. Some of these types of fibrations we have met before; some of them are new.

1.2.1 Fibrations in ordinary category theory

1.2.2 Fibrations in infinity categories

Let $f\colon K \to S$ be a morphism of simplicial sets.

- The morphism f is a left fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 \leq i < n$.
- The morphism f is a right fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i \leq n$.
- The morphism f is an inner fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i < n$.

We will use the following terminology.

- The morphism f is left anodyne if it has the left-lifting property with respect to all left fibrations.
- The morphism f is right anodyne if it has the left-lifting property with respect to all right fibrations.
- The morphism f is inner anodyne if it has the left-lifting property with respect to all inner fibrations.

Theorem 13. Let f be a trivial fibration. Then f is a Kan fibration.

Proof. Let f be a trivial fibration, and consider the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

Restricting to the face opposite i, we find the lifting problem

$$\begin{tikzcd} \partial\Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed solution. This allows us to turn our map $\Lambda^n_i \to K$ to a map $\partial\Delta^n \to K$ making the folowing diagram commute.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed lift. ◻

Note 14. A map $f\colon X \to Y$ is a left fibration if and only if $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a a right fibration. Thus, it suffices to study left fibrations; the theory of right fibrations is dual.

Lemma 15. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration of infinity categories, and let $\tilde{f}: \bar{x} \to p(y)$ be an equivalence in $\mathcal{D}$, and $y \in \mathcal{C}_0$. Then there exists $f: x \to y$ so that $p(f) = \bar{f}$.

Lemma 16. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Then the induced map

$$S_{p/} \to X_{\bar{p}/} \underset{S_{\bar{p}/}}{\times} S_{r/}$$

is an inner fibration. If q is a left fibration, then the map is a left fibration.

Corollary 17. Let $\mathcal{C}$ be an infinty category, and let $p\colon K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the map $\mathcal{C}_{/p} \to \mathcal{C}$ is a left fibration, and $\mathcal{C}_{/p}$ is an infinity category.

Proof. Take $X = \mathcal{C}$, $S = \Delta^0$, and $\bar{K} = \emptyset$. ◻

Corollary 18. Let $f: x \to y$ be a morphism in an ∞-category $\mathcal{C}$. Then f is an equivalence if and only if for every $n \geq 2$ and left horn $h\colon \Lambda^n_0 \to \mathcal{C}$, with $h|_{\{0,1\}} = f$, h admits a filler.

Theorem 19. The following are equivalent for an ∞-category $\mathcal{C}$.

- The homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid.
- Every left horn in $\mathcal{C}$ has a filler.
- Every right horn has a filler.
- $\mathcal{C}$ is a Kan complex.

Corollary 20. All fibers of a left fibration are Kan complexes.

1.3 The starred smash

Definition 21 (starred smash). Let $f\colon A \to A'$, $g\colon B \to B'$ be two morphisms of simplicial sets. Define a morphism $f \overset{\star}{\wedge} g$ using the universal property for pushouts as follows.

$$\begin{tikzcd}
A \star B \arrow[r, "f\star\mathrm{id}"] \arrow[d, "\mathrm{id}\star g"'] & A' \star B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}\star g"] & \\
A \star B' \arrow[r, hook] \arrow[drr, bend right, "f\star\mathrm{id}"'] & A' \star B \coprod_{A\star B} A \star B' \arrow[dr, dashed, "f \overset{\star}{\wedge} g"] & \\
& & A' \star B'
\end{tikzcd}$$

Fact 22. We have the following facts.

1. If f is right anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are inner anodyne.
2. If f is left anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are left anodyne.
3. If f is anodyne, then $f \overset{\star}{\wedge} g$ is a Kan fibration.

We have the following 'perpendicular statements' (whatever that means).

Fact 23. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Denote the

$$S_{/p} \to X_{/\bar{p}} \underset{S_{/\bar{p}}}{\times} S_{/r}.$$

1.3.1 Cofibration in groupoids

We would like to find the correct ∞-categorical notion of being cofibered in groupoids.

Lemma 24. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Then for each $d \in \mathcal{D}$, the fiber $\mathcal{C}_d$ defined by the pullback below is a groupoid.

$$\begin{tikzcd}
\mathcal{C}_d \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\
\{d\} \arrow[r, hook] & \mathcal{D}
\end{tikzcd}$$

Proof. Every morphism in $\mathcal{C}_d$ is by definition mapped to id_d, so by □

Theorem 25. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F exhibits $\mathcal{C}$ as cofibered in groupoids over D if and only if $N(F) \colon N(\mathcal{C}) \to N(\mathcal{D})$ is a left fibration.

Proof. Suppose F is a left fibration. We examine some horn filling conditions in small degrees. First, consider a horn filling

$$\begin{tikzcd}
\Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\
\Delta^1 \arrow[r] & N(\mathcal{D})
\end{tikzcd}$$

The commutativity conditions tell us that we have the following solid diagrams in $\mathcal{C}$ and $\mathcal{D}$ respectively.

$$x \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

A dashed lift

$$\begin{tikzcd}
\Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\
\Delta^1 \arrow[r] \arrow[ur, dashed] & N(\mathcal{D})
\end{tikzcd}$$

gives us the following dashed data.

$$x \overset{f}{\dashrightarrow} y \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

Thus, the first condition of Definition 5 is satisfied.

Similarly, a horn filling problem $\Lambda^2_0 \to \Delta^2$ gives us the following solid data, while a lift gives us the dashed data.

$$\begin{tikzcd} & y \arrow[dr, dashed, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \quad \begin{tikzcd} & F(y) \arrow[dr, "\bar{g}"] & \\ F(x) \arrow[ur, "F(f)"] \arrow[rr, "F(h)"'] & & F(z) \end{tikzcd}$$

This says that there exists a lift as required, but not that it is unique.

Now consider two possible lifts $\bar{g}$ and $\bar{g}'$, arranged into a $(3,0)$-horn as follows.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] \arrow[ur, "\mathrm{id}_y"] & z \end{tikzcd}$$

These correspond to a 3-simplex in $\mathcal{D}$ as follows.

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] \arrow[dr, "F(h)"] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] & F(z) \end{tikzcd} \quad \begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] \arrow[ur, "\mathrm{id}_{F(y)}"] & F(z) \end{tikzcd}$$

The lifting properties say that we can fill the above horn, telling us that $g = g'$, i.e. that the lift is unique. □

1.4 The infinity-categorical Grothendieck construction

Definition 26 (fiber over a point)**.** Let $f\colon X \to S$ be a map of simplicial sets, and let $s \in S$. The fiber over s, denoted X_s is the following pullback.

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r, hook] & S \end{tikzcd}$$

Theorem 27. Let $p\colon X \to S$ be a left fibration of simplicial sets. Then the assignment

$$s \mapsto X_s$$

extends to a functor $\mathrm{h}S \to \mathrm{h}\mathcal{S}$, where $\mathcal{S}$ is the category of spaces (Definition **??**).

Proof. By Corollary 20, each fiber X_s is a Kan complex.

Now, let $f : s \to s'$ be any morphism in S. Consider the following lifting problem.

$$\begin{tikzcd}
\{0\} \times X_s \arrow[rr, hook] \arrow[d, hook] & & X \arrow[d, "p"] \\
\Delta^1 \times X_s \arrow[r] \arrow[urr, dashed, "Lf"] & \Delta^1 \arrow[r, "f"] & S
\end{tikzcd}$$

The left-hand morphism is left anodyne because it is the smash product

$$(\{0\} \hookrightarrow \Delta^1) \overset{\star}{\wedge} (\emptyset \hookrightarrow X_s),$$

so we can find a dashed lift. This gives us a map $Lf : \Delta^1 \times X_s \to X$ such that $Lf_{\{0\}} = \mathrm{id}_{X_s}$. Restricting to $\{1\} \times X_s$, we get a map $Lf_{\{1\}\times X_s} : \{1\} \times X_s \to X_{s'}$, giving us a map

$$\tilde{f} : X_s \to X_{s'}.$$

The functor $\mathrm{h}S \to \mathrm{h}\mathbb{S}$ sends $[f] \mapsto [\tilde{f}]$. It remains to check that this is well-defined, i.e. respects □