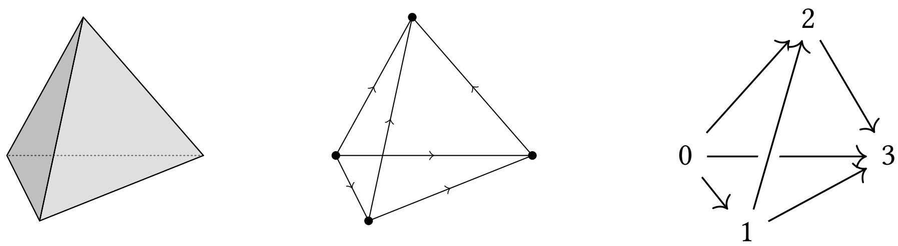
Notes on Higher Categories
(Work in progress)

Angus Rush



Contents

1 Introduction

These notes grew out of a course given by Prof. Tobias Dyckerhoff at the University of Hamburg. However, much of the presentation and many of the proofs are heavily adapted. In the process I have undoubtedly introduced many mistakes, all of which are entirely my own.

These notes are under very heavy construction at the moment, and the results in them should not be taken as fact.

1.1 Axiomatic framwork

These notes are phrased in terms of **ZFCU**, i.e. the Zermelo-Fraenkel axioms together with the following axioms.

- **C**: the axiom of choice.
- **U**: Grothendieck's universe axiom.

An introduction to category theory in the language of Grothendieck universes can be found in the corresponding notes on ordinary category theory.[1]

1.2 Notational inconsistencies

- The unit and counit are sometimes called ϵ and η respectively, and sometimes the other way around. I'm going through and fixing this.
- For functors $F, G\colon \mathcal{C} \to \mathcal{D}$, I sometimes denote the comma category by $(F \downarrow G)$ and sometimes by $\mathcal{C}_{/G}$. I think the first notaion is much better (at least in the context of ordinary categories), but the second seems to be uniquitous in the infinity categorical setting. I haven't decided in which direction to resolve this particular conflict.

[1] https://github.com/angusrush/notes-public/blob/master/category_theory/notes.pdf

2 Category-theoretic preliminaries

The theory of infinity categories rests on the shoulders of the theory of ordinary category theory. In this chapter, we recall some category-theoretic results that we will use.

2.1 Formulae for limits and Kan extensions

Fact 1. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be locally small. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{D}$ has small products and equalizers, then $\lim F$ is given by the following equalizer,

  $$\lim F \hookrightarrow \prod_{c\in\mathcal{C}} F(c) \overset{R}{\underset{S}{\rightrightarrows}} \prod_{\alpha\in\mathrm{Mor}(\mathcal{C})} F(\mathrm{codom}(\alpha))$$

  where the morphisms R and S are defined by their components

  $$R_\alpha = \pi_{F(\mathrm{codom}(\alpha))}\colon \prod_{c\in\mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

  and

  $$S_\alpha = F(\alpha) \circ \pi_{\mathrm{dom}(\alpha)}\colon \prod_{c\in\mathcal{C}} F(c) \to F(\mathrm{codom}(\alpha))$$

- If $\mathcal{D}$ has small coproducts and coequalizers, then $\operatorname{colim} F$ is given by the following coequalizer,

  $$\coprod_{\alpha\in\mathrm{Mor}(\mathcal{C})} F(\mathrm{dom}(\alpha)) \overset{A}{\underset{B}{\rightrightarrows}} \coprod_{c\in\mathcal{C}} F(c) \twoheadrightarrow \operatorname{colim} F$$

  where the morphisms A and B are defined by their components

  $$A_\alpha = \iota_{F(\mathrm{dom}(\alpha))}\colon F(\mathrm{dom}(\alpha)) \to \coprod_{c\in\mathcal{C}} F(c)$$

  and

  $$B_\alpha = \iota_{\mathrm{codom}(\alpha)} \circ F(\alpha)\colon F(\mathrm{dom}(\alpha)) \to \coprod_{c\in\mathcal{C}} F(c)$$

Fact 2. Consider categories and functors as follows,

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "D"] \arrow[dr, "F"'] & & \mathcal{D} \\ & \mathcal{C}' & \end{tikzcd}$$

where $\mathcal{C}$ is small.

- If $\mathcal{D}$ has all small colimits, then the left Kan extension $F_! D$ exists and has values

$$(F_! D)(c') = \operatorname{colim}\left[\ (c' \downarrow F) \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

- If $\mathcal{D}$ has all small limits, then the right Kan extension $F_* D$ exists and has values

$$(F_* D)(c') = \lim\left[\ (F \downarrow c') \longrightarrow \mathcal{C} \xrightarrow{D} \mathcal{D}\ \right].$$

2.2 Limits in functor categories

Theorem 3. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

- If $\mathcal{C}$ has an initial object i, then $\lim F = F(i)$.
- If $\mathcal{C}$ has a terminal object i, then $\operatorname{colim} F = F(i)$.

Proof. We prove the first; the second is dual.

For each $c \in \mathcal{C}$, we get a unique morphism $i \to c$. Sticking these into F gives us a cone $\alpha_c\colon F(i) \to F(c)$ under $F(i)$.

Now let (d, ϵ) be any other cone over F. In particular, ϵ has a component

$$\epsilon_i\colon d \to F(i).$$

In order to show that α is a limit cone, we need only show that this is the only possible morphism $d \to F(i)$ such that the triangles

$$\begin{tikzcd}
 & d \arrow[d, "\epsilon_i"] \arrow[dl, bend right, "\epsilon_j"'] \\
 & F(i) \arrow[dl, "\alpha_c"] \\
F(c) &
\end{tikzcd}$$

commute for all c. Let $f\colon d \to F(i)$ be any other such morphism. The triangle above must in particular commute when $c = i$, so $\alpha_i = F(\mathrm{id}_i) = \mathrm{id}_{F(i)}$.

$$\begin{tikzcd}
 & d \arrow[d, "f"] \arrow[dl, bend right, "\epsilon_i"'] \\
 & F(i) \arrow[dl, "\mathrm{id}_{F(i)}"] \\
F(i) &
\end{tikzcd}$$

Thus, $f = \epsilon_i$. □

Theorem 4. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, with $\mathcal{D}$ small and $\mathcal{C}$ locally small.

- If $\mathcal{D}$ is complete, then so is $[\mathcal{C}, \mathcal{D}]$.
- If $\mathcal{D}$ is cocomplete, then so is $[\mathcal{C}, \mathcal{D}]$.

Sketch of proof. A cone in a functor category is built out of natural transformations. Evaluating on a component gives a cone in the target category. A cone in a functor category is a limit cone if and only if each component is a limit cone. □

Corollary 5. Limits and colimits are computed pointwise.

Corollary 6. A morphism $\eta\colon F \to G$ in a functor category is a monomorphism (resp. epimorphism) if and only if each component η_x is a monomorphism (resp. epimorphism).

Proof. A morphism $\eta\colon F \to G$ is a monomorphism if and only if the square

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & F \arrow[d, "\eta"] \\ F \arrow[r, "\eta"'] & G \end{tikzcd}$$

is a pullback square. But by Corollary 5, the above square is a pullback square if and only if its components are pullback squares, which in turn is true if and only if each component is a monomorphism.

The epimorphism case is dual. □

2.3 Properties of Kan extensions

Theorem 7. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful.

- If $\mathcal{D}$ has small limits, then the counit
$$\eta\colon F^* \circ F_* \Rightarrow \mathrm{id}_{\mathcal{C}'}$$
is a natural isomorphism.
- If $\mathcal{D}$ has small colimits, then the unit
$$\epsilon\colon \mathrm{id}_{\mathcal{D}} \Rightarrow F^* \circ F_!$$
is a natural isomorphism.

Proof. We need to show that each component

$$\eta_G\colon (F^* \circ F_*)(G) \Rightarrow \mathrm{id}_{\mathcal{C}'}$$

is a natural isomorphism. By definition of the pullback, we have

$$(F^* \circ F_*)(G) = F^*(F_* G) = F_* G \circ F.$$

Consider some $c' \in \mathcal{C}'$. By the limit formula for Kan extensions, we can express $F_* G(c')$ as

$$F_* G(c') = \lim_{c \in (F/c')} G(c).$$

In particular, if $c' = F(c)$ for some c, we have

$$F_*G(F(c)) = \lim_{c \in (F/F(c))} G(c) = \lim\left[(F/F(c)) \to \mathcal{C} \to \mathcal{D}\right].$$

Since the object $(c, \mathrm{id}_{F(c)})$ is initial in $(F/F(c))$ we have by Theorem 3 that

$$\lim\left[(F/F(c)) \to \mathcal{C} \to \mathcal{D}\right] = G(c).$$

Thus, we have

$$(F^* \circ F_*)(G)(c) \simeq G(c)$$

for all $c \in \mathcal{C}$ □

Corollary 8. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. Let $F\colon \mathcal{C} \to \mathcal{C}'$ be fully faithful.

- If $\mathcal{D}$ has small limits, then F_* is fully faithful.
- If $\mathcal{D}$ has small colimits, then $F_!$ fully faithful.

2.4 Yoneda extensions

Definition 9 (Yoneda extension)**.** One calls left Kan extensions along the Yoneda embedding Yoneda extensions.

It is often helpful to think of the objects in a category $\mathbf{Set}_{\mathcal{C}}$ as 'generalized objects' of $\mathcal{C}$. This is possible becuase the Yoneda embedding is fully faithful, so one can reasonably think of $\mathcal{C}$ as sitting inside $\mathbf{Set}_{\mathcal{C}}$ as a full subcategory. One then imagines the objects of $\mathbf{Set}_{\mathcal{C}}$ as built by gluing together objects of $\mathcal{C}$. This point of view comes from the following result.

Lemma 10. Let $\mathcal{C}$ be a locally small category. Denote by $\mathcal{Y}$ the Yoneda embedding $\mathcal{C} \to \mathbf{Set}_{\mathcal{C}}$. Then for any functor $F \in \mathbf{Set}_{\mathcal{C}}$ we have the following expression.

$$F \simeq \operatorname{colim}\left[(\mathcal{Y}/F) \xrightarrow{\mathcal{U}} \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right].$$

Proof. By Theorem 7, the fully faithfulness of the Yoneda embedding, and the definition of the restriction functor, we have that

$$F \simeq (\mathcal{Y}^* \circ \mathcal{Y}_!)(F) \simeq (\mathcal{Y}_! \mathcal{Y})(F).$$

The functor $\mathcal{Y}_! \mathcal{Y}$ comes from the following Kan extension.

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "\mathcal{Y}"] \arrow[dr, "\mathcal{Y}"'] & & \mathbf{Set}_{\mathcal{C}} \\ & \mathbf{Set}_{\mathcal{C}} \arrow[ur, "\mathcal{Y}_!\mathcal{Y}"'] & \end{tikzcd}$$

The colimit formula for left Kan extensions (Fact 2) then tells us that

$$F \simeq \operatorname{colim}\left[(\mathcal{Y}/F) \xrightarrow{\mathcal{U}} \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right]$$

as required. □

Lemma 11. Let $\mathcal{J}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let F be a functor as below. Then the Yoneda extension $\mathcal{Y}_!F$ is left adjoint to the functor

$$G\colon \mathcal{C} \longrightarrow \mathbf{Set}_{\mathcal{J}}; \qquad c \mapsto \mathcal{C}(F(-), c).$$

$$\begin{tikzcd} \mathcal{J} \arrow[rr, "F"] \arrow[dr, "\mathcal{Y}"'] & & \mathcal{C} \\ & \mathbf{Set}_{\mathcal{J}} \arrow[ur, "\mathcal{Y}_!F"'] & \end{tikzcd} \qquad \mathcal{Y}_!F : \mathbf{Set}_{\mathcal{J}} \leftrightarrow \mathcal{C} : G$$

Proof. Since $\mathcal{C}$ is cocomplete, we are entitled to use the colimit formula for left Kan extensions. That is, we can write

$$(\mathcal{Y}_!F)(H) = \varinjlim \left[(\mathcal{Y}/H) \xrightarrow{\mathcal{U}} \mathcal{J} \xrightarrow{F} \mathcal{C}\right] = \varinjlim\,[F \circ \mathcal{U}]\,.$$

Thus, we have the following string of natural isomorphisms.

$$\begin{aligned}
\mathcal{C}((\mathcal{Y}_!F)(H), c) &\simeq \mathcal{C}(\varinjlim[F \circ \mathcal{U}], c) \\
&\simeq \varprojlim \mathcal{C}(F(-), c) \circ \mathcal{U} \\
&\simeq \varprojlim G(c) \circ \mathcal{U} \\
&\simeq \varprojlim \mathbf{Set}_{\mathcal{J}}(\mathcal{Y}(-), G(c)) \circ \mathcal{U}. \\
&\simeq \mathbf{Set}_{\mathcal{J}}(\varinjlim \mathcal{Y} \circ \mathcal{U}, G(c)) \\
&\simeq \mathbf{Set}_{\mathcal{J}}(H, G(c)).
\end{aligned}$$

□

Corollary 12. Let $\mathcal{J}$ be a small category, and let $\mathcal{C}$ be locally small and cocomplete. Let $F\colon \mathcal{J} \longrightarrow \mathcal{C}$. Then the Yoneda extension $\mathcal{Y}_!F$ preserves colimits.

Proof. By Lemma 11, $\mathcal{Y}_!F$ is a left adjoint. □

Theorem 13. Let $\mathcal{C}$ be a small category, and let $\mathcal{D}$ be cocomplete. Let $F\colon \mathbf{Set}_{\mathcal{C}} \longrightarrow \mathcal{D}$ be any functor. Then F preserves colimits if and only if it is the left Yoneda extension of its restriction $\mathcal{Y}^*F\colon \mathcal{C} \longrightarrow \mathcal{D}$, i.e. if $F = (\mathcal{Y}_!\mathcal{Y}^*)(F)$.

Proof. Assume that $F = (\mathcal{Y}_!\mathcal{Y}^*)(F)$. By Lemma 11, $(\mathcal{Y}_!\mathcal{Y}^*)(F)$ is a left adjoint, hence preserves colimits.

Now assume that F preserves colimits. Pick some functor $G\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$. By Lemma 10, we can write

$$\begin{aligned}
F(G) &\simeq F\left(\varinjlim \left[(\mathcal{Y}/G) \longrightarrow \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}}\right]\right) \\
&\simeq \varinjlim \left[(\mathcal{Y}/G) \longrightarrow \mathcal{C} \xrightarrow{\mathcal{Y}} \mathbf{Set}_{\mathcal{C}} \xrightarrow{F} \mathcal{D}\right] \\
&\simeq \varinjlim \left[(\mathcal{Y}/G) \longrightarrow \mathcal{C} \xrightarrow{\mathcal{Y}^*F} \mathcal{D}\right] \\
&\simeq \mathcal{Y}_!(\mathcal{Y}^*F)(G).
\end{aligned}$$

□

Corollary 14. Let $\mathcal{C}$ be a small category, and $\mathcal{D}$ locally small. There is an equivalence of categories
$$[\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} \simeq [\mathcal{C}, \mathcal{D}].$$
That is, colimit preserving functors $\mathbf{Set}_{\mathcal{C}} \to \mathcal{D}$ are equivalent to functors $\mathcal{C} \to \mathcal{D}$ via restriction along the Yoneda embedding.

Proof. We have an adjunction
$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathbf{Set}_{\mathcal{C}}, \mathcal{D}] : \mathcal{Y}^*.$$
By Corollary 12, the image of $\mathcal{Y}_!$ is completely contained in $[\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}}$, so the above adjunction extends to an adjunction
$$\mathcal{Y}_! : [\mathcal{C}, \mathcal{D}] \leftrightarrow [\mathbf{Set}_{\mathcal{C}}, \mathcal{D}]_{\text{cocontinuous}} : \mathcal{Y}^*.$$
We know that the Yoneda embedding is fully faithful. Thus, by Theorem 7, we have that the unit is an isomorphism. Also, by Theorem 13, the counit is an isomorphism, so the above adjunction is in fact an equivalence of categories. □

2.5 Enriched category theory

Definition 15 (enriched category)**.** Let $(\mathcal{M}, \otimes, 1_{\mathcal{M}}, \alpha, \lambda, \rho)$ be a monoidal category. A $\mathcal{M}$-enriched category $\mathcal{C}$ consists of the following data.

- A set $\mathrm{ob}(\mathcal{C})$ of objects.
- For each pair x, y of objects, an object $\mathcal{C}(x, y) \in \mathcal{M}$ of morphisms.
- For each object $x \in \mathcal{C}$, a morphism
$$1_{\mathcal{M}} \to \mathcal{C}(x, x)$$
called the *unit*
- For objects x, y, and z, a morphism
$$\mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \to \mathcal{C}(x, z)$$
called the *composition.*

This composition must be unital and associative, i.e. the diagrams
$$\begin{tikzcd}
\mathcal{C}(x,y) \arrow[r] \arrow[d] \arrow[dr, "\mathrm{id}"] & \mathcal{C}(y,y) \otimes \mathcal{C}(x,y) \arrow[d] \\
\mathcal{C}(x,y) \otimes \mathcal{C}(x,x) \arrow[r] & \mathcal{C}(x,y)
\end{tikzcd}$$
and
$$\begin{tikzcd}
\mathcal{C}(x,y) \otimes \mathcal{C}(y,z) \otimes \mathcal{C}(z,w) \arrow[r] \arrow[d] & \mathcal{C}(x,y) \otimes \mathcal{C}(y,w) \arrow[d] \\
\mathcal{C}(x,z) \otimes \mathcal{C}(z,w) \arrow[r] & \mathcal{C}(x,w)
\end{tikzcd}$$
must commute (where unitors and associators are notationally supressed).

Lemma 16. Let $\mathcal{C}$ be an $\mathcal{M}$-enriched category, and let $F\colon \mathcal{M} \to \mathcal{N}$ be a monoidal functor with data

$$\Phi_{x,y}\colon F(x) \otimes_{\mathcal{N}} F(y) \to F(x \otimes_{\mathcal{M}} y), \qquad \phi\colon 1_{\mathcal{N}} \to F(1_{\mathcal{M}}).$$

This data allows us to build from $\mathcal{C}$ a category enriched over $\mathcal{N}$.

Proof.

- We apply F to each hom-object in $\mathcal{M}$ to get a new hom-object in $\mathcal{N}$;

$$\mathcal{C}(x,y)_{\mathcal{N}} = F(\mathcal{C}(x,y)_{\mathcal{M}}).$$

- The unit is the composition

$$1_{\mathcal{N}} \longrightarrow F(1_{\mathcal{M}}) \longrightarrow F(\mathcal{C}(x,y)_{\mathcal{M}}) = = = \mathcal{C}(x,y)_{\mathcal{N}}$$

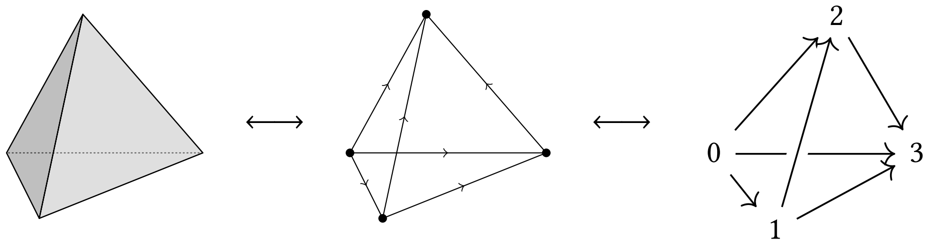
- The composition is given by the obvious.

$\square$

3 Simplicial sets

3.1 Simplices as topological spaces and categories

One of the main themes in the study of higher category theory is the interplay between topology, simplicial geometry, and category theory.



The fundamental topological object we will study is the *geometrical n-simplex.*

Definition 17 (geometrical n-simplex)**.** For any $n \geq 0$, the geometrical n-simplex, denoted $|\Delta^n|$, is the set (together with the subspace topology)

$$|\Delta^n| = \left\{ (t_0, t_1, \dots, t_n) \in \mathbb{R}^n \mid \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \text{ for all } i \right\}.$$

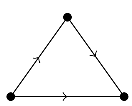
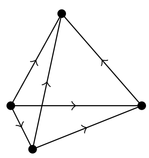
In simplicial geometry, one replaces these topological models of simplices by combinatorial models of simplices. These live in a category.

Definition 18 (simplex category)**.** Denote by Δ the category whose objects are linearly ordered sets

$$[n] = \{0, \dots, n\}$$

and whose morphisms $n \to m$ are weakly monotonic[1] maps.

The objects $[n]$ of Δ are to be interpreted as simplices with ordered vertices. For example,

$[0] = \bullet$, $\quad [1] = \bullet \longrightarrow \bullet$, $\quad [2] =$  , $\quad [3] =$ 

We can connect our simplex category to geometric simplices using a functor.

[1]i.e. nondecreasing in the sense that $i \geq j \implies f(i) \geq f(j)$

Definition 19 (realization functor)**.** The realization functor

$$\rho\colon \Delta \to \mathbf{Top}$$

is defined on objects by $[n] \mapsto |\Delta^n|$ and on morphisms $[f]\colon [m] \to [n]$ by

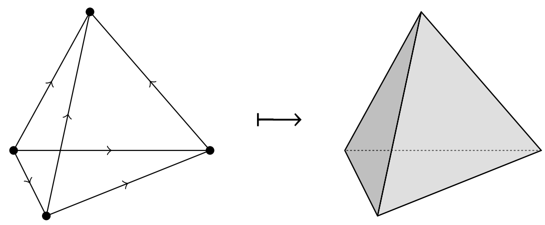
$$f_*\colon |\Delta^m| \to |\Delta^n|; \qquad (t_0, \ldots, t_m) \mapsto (s_1, \ldots, s_m),$$

where

$$s_i = \begin{cases} 0, & f^{-1}(i) = \emptyset \\ \sum_{j \in f^{-1}(i)} t_j & \text{otherwise.} \end{cases}$$

That is, ρ assigns to each combinatorial n-simplex its topological counterpart.

ρ :



One can also think of these combinatorial simplices categorically. Again, this connection is given by a functor.

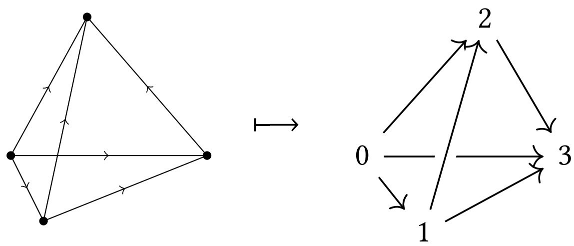
Definition 20 (categorification functor)**.** The categorification functor is the functor

$$\chi\colon \Delta \to \mathbf{Cat}$$

which sends the object $[n]$ to the poset category on $[n]$. Note that functors $[m] \to [n]$ are the same thing as weakly monotonic maps $[m] \to [n]$.

That is, χ assigns to each combinatorial n-simplex its categorical counterpart.

χ:



These categories and functors may seem too simple to be of any real use, but one can build astonishingly complicated structures out of them.

3.2 Simplicial sets

A simplicial set is thought of as a collection of standard simplices of various degrees, glued together along their faces. That is, a simplicial set consists of, for each $n \in \mathbb{N}$, a collection of simplices of degree n, together with rules for how to glue them together. This data can be specified efficiently in the following way.

Definition 21 (simplicial set)**.** A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathbf{Set}$. The category of simplicial sets is the category of such functors, i.e.

$$\mathbf{Cat}(\Delta^{\mathrm{op}}, \mathbf{Set}) = \mathbf{Set}_{\Delta} .$$

As discussed briefly in Section 2.4, one thinks of the image under the Yoneda embedding of the object $[n]$ as an n-simplex, called the *standard n-simplex,* and more general simplicial sets as being built by gluing standard simplices together.

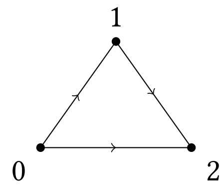
Specifically, for any simplicial set K we should interpret $K([n])$ as the set of n-simplices contained in K. The gluing conditions come from functoriality. We will investigate this in Section 3.3.

Definition 22 (standard simplex)**.** Let $n \geq 0$. The simplicial set

$$\Delta(-, [n]) = よ([n])$$

is denoted Δ^n, and called the *standard n-simplex.*

The claim is that the standard n-simplex can be interpreted geometrically, as precisely the sort of picture we have been drawing. To see this, it will be helpful to explicitly analyze some cases in which n is small. Take $n = 2$. To justify the name, the standard 2-simplex had better look something like the 2-simplex we know and love.



Specifically, we should interpret $\Delta^2([n])$ as the set of n-simplices contained in Δ^2. Naïvely, we would expect three 0-simplices, three 1-simplices, and one 2-simplex.

However, the only way to find out what we actually have is to begin calculating. First, we calculate $\Delta^2([0]) = \mathbf{Set}_{\Delta}([0], [2])$:

$$\Delta^2([0]) = \{0 \mapsto 0, 0 \mapsto 1, 0 \mapsto 2\}.$$

To avoid formulae whose lengths quickly get out of hand, we will drop the functional notation, denoting the above by $\{0, 1, 2\}$.

This is good. We said that $\Delta^2([0])$ should be interpreted as the set of 0-simplices, and indeed $\Delta^2([0])$ has three elements, which represent in an obvious way the vertices of the familiar 2-simplex. Now calculating $\Delta^2([1])$, we find

$$\Delta^2([1]) = \{\{0, 1\} \mapsto \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 0\}, \{1, 1\}, \{2, 2\}\}.$$

This is not so good; we can interpret the three edges $\{0, 1\}$, $\{0, 2\}$, and $\{1, 2\}$ as the edges between 0 and 1, 0 and 2, and 1 and 2 that we were expecting, but we also have three more. We interpret these as 'degenerate edges,' which are all bunched up at the vertices. This is one difference between simplicial complexes and simplicial sets: simplicial sets can have degenerate edges.

Continuing, we find
$$\Delta^2([2]) = \{\{0,0,0\},\{0,0,1\},\{0,0,2\},\ldots\}.$$

We recognize these as degenerate faces. The only nondegenerate face will be $\{0,1,2\}$.

Note that contrary to what you might expect, $\Delta^n([k])$ is not empty for $k > n$; it is simply filled with an enormous number of degenerate simplices.[2]

Example 23. Denote the power set of a set X by $\mathcal{P}(X)$. Let $\mathcal{J} \subseteq \mathcal{P}(\{0,1,\ldots,n\})$. Then there is a simplicial set $\Delta^{\mathcal{J}}$ whose morphisms are defined to be the subset

$$\Delta^{\mathcal{J}}([n]) \subseteq \Delta([m],[n])$$

consisting of those f such that $\operatorname{im}(f) \subset J$ for some $J \in \mathcal{J}$.

Here are some specific examples of $\mathcal{J}$ which will be useful to us.

- If $\mathcal{J} = \mathcal{P}(\{0,\ldots,n\})$, then
$$\Delta^{\mathcal{J}}([n]) = \Delta^n,$$
the standard n-simplex.
- If
$$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \smallsetminus \{\{0,1,\ldots,n\}\},$$
then the simplicial set we obtain is called the *boundary* of Δ^n, and denoted $\partial\Delta^n$.

  The sets $\Delta^n([i])$ and $\partial\Delta^n([i])$ are the same for $i < n$.
- Fix some i such that $0 \leq i \leq n$. Then with
$$\mathcal{J} = \mathcal{P}(\{0,1,\ldots,n\}) \smallsetminus \{\{0,1,\ldots,n\},\{0,1,\ldots,i-1,i+1,\ldots,n\}\},$$
we obtain the *ith horn* of Δ^n, and denoted Λ^n_i.

  For example, the 2-simplices in the horn Λ^2_1 consists of the subset of monotone functions $[2] \longrightarrow [2]$ whose range lies entirely within one of the sets $\{0,1,2,01,12,22\}$. That is
$$\{000,001,011,111,112,122,222\}.$$
  We can draw Λ^2_i, for $i = 0, 1, 2$, as follows.

$$\Lambda^2_0 = \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}, \qquad \Lambda^2_1 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd}, \qquad \Lambda^2_2 = \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[rr] & & 2 \end{tikzcd}$$

Definition 24 (inner, outer horn)**.** If $i = 0$ or $i = n$, then we say that Δ^n_i is an <u>outer horn</u>. Otherwise, if $0 < i < n$, we say that Λ^n_i is an <u>inner horn</u>.

Definition 25 (simplicial complex)**.** A set $\mathcal{K} \subset \mathcal{P}(\{0,1,\ldots,n\})$ is called an <u>simplicial complex</u> if for every $\mathcal{J} \in \mathcal{K}$ and $I \subseteq \mathcal{J}$ with $I \neq \emptyset$, $I \in \mathcal{K}$.

[2] To be precise, with $\binom{n+k+1}{k}$ simplices.

Example 26. Let $\mathcal{K}$ be any simplicial complex. We can interpret $\mathcal{K}$ as a simplicial set by associating

$$\mathcal{K} \mapsto \Delta^{\mathcal{K}}.$$

What would have been called the simplices of $\mathcal{K}$ correspond to the nondegenerate elements of the simplicial complex. Note however that there are many simplicial sets which are not obtained from simplicial complexes in this way.

Simplicial sets are functors, but the functorial notation is often unnecessarily burdensome. Therefore, it is standard to denote the n-simplices of a simplicial set K by K_n rather than $K([n])$. This is the notation that we will use from now on unless it will be confusing.

3.3 What sort of data does a simplicial set give us?

Let K be a simplicial set. We have seen that evaluating K on the object $[n]$ gives a set, which we interpret as the set of n-simplices making up K. We have claimed that the functoriality of K provides instructions for how these simplices should be glued together. In this section we will investigate this claim.

First, we note that we can understand how K acts on morphisms $\phi\colon [n] \to [m]$ by understanding how it behaves on simpler functions, known as the *face* and *degeneracy functions.*

Definition 27 (degeneracy function)**.** Let $n > 0$. For every i with $0 \leq i \leq n$, there is a function

$$\sigma_k\colon [n] \to [n-1]; \qquad j \mapsto \begin{cases} j, & j \leq k \\ j-1, & j > k \end{cases}$$

called the <u>ith degeneracy function</u>.

$$\sigma_k \quad = \quad \begin{tikzcd} \vdots & \vdots \\ k-2 \arrow[r, mapsto] & k-2 \\ k-1 \arrow[r, mapsto] & k-1 \\ k \arrow[r, mapsto] & k \\ k+1 \arrow[ur, mapsto] & k+1 \\ k+2 \arrow[ur, mapsto] & k+2 \\ \vdots & \vdots \end{tikzcd}$$

Definition 28 (face function)**.** Let $n \geq 0$. For any k with $0 \leq k < n$, there is a function

$$\partial_i\colon [n-1] \to [n]; \qquad j \to \begin{cases} j, & j < i \\ j+1, & j \geq i \end{cases}$$

called the <u>ith face function</u>.

$$\sigma_k \quad = \quad \begin{tikzcd}[row sep=small] \vdots & \vdots \\ k-2 \arrow[r, mapsto] & k-2 \\ k-1 \arrow[r, mapsto] & k-1 \\ k \arrow[r, mapsto] & k \\ k+1 \arrow[dr, mapsto] & k+1 \\ k+2 \arrow[dr, mapsto] & k+2 \\ \vdots & \vdots \end{tikzcd}$$

If we were being pedantic, we would denote these functions σ_k^n and ∂_i^n instead of σ_k and ∂_i, but keeping track of these indices turns out to be more confusing than elucidating, so no one does.

The face and degeneracy functions are interesting because they generate all morphisms; that is, any morphism $\phi\colon [n] \to [m]$ can be written as a composition of face and degeneracy functions.

Proposition 29. Let $\phi\colon [n] \to [m]$ be a weakly monotonic function. Then ϕ can be written as a composition of face and degeneracy maps.

Sketch of proof. Since ϕ is by definition nondecreasing, we can specify it by, starting from 0, a sequence $\square$

Definition 30 (degeneracy map)**.** Let $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. Corresponding to the degeneracy function $\sigma_k\colon [n] \to [n-1]$ there is a corresponding map

$$s_k\colon X_{n-1} \to X_n.$$

This map is also known as the kth degeneracy map.

Definition 31 (face map)**.** Let $X\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. Corresponding to the face function $\partial_i\colon [n] \to [n-1]$ (see Definition 28), there is a map

$$d_i\colon X_n \to X_{n-1}.$$

This map is also known as the ith face map.

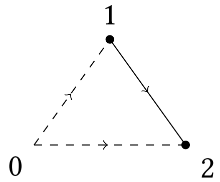
Example 32. Consider the simplicial set Δ^2, and the map

$$d_0\colon \Delta^2_1 \to \Delta^2_0.$$

Let $f \in \Delta^2_1 = \Delta([1], [2])$. For example, take

$$f\colon \quad \begin{tikzcd}[row sep=small] 0 \arrow[dr, mapsto] & 0 \\ 1 \arrow[dr, mapsto] & 1 \\ & 2 \end{tikzcd}$$

corresponding to the solid edge.



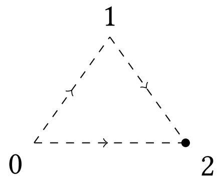
Which element of Δ^2_0 does d_0 send this to? We can draw ∂_0 like this.

$$\partial_0 : \begin{tikzcd} 0 \arrow[dr, tail] & 0 \\ & 1 \end{tikzcd}$$

The map d_0 acts by precomposing with ∂_0.

$$d_0 f = f \circ \partial_0 = \begin{tikzcd} 0 \arrow[dr, tail] & 0 \arrow[dr, tail] & 0 \\ & 1 \arrow[dr, tail] & 1 \\ & & 2 \end{tikzcd} = \begin{tikzcd} 0 \arrow[ddr, tail] & 0 \\ & 1 \\ & 2 \end{tikzcd}$$

Therefore, d_0 takes the edge between 0 and 2 to the vertex 2.



Similarly, d_0 takes the edge between 0 and 1 to the vertex 1, and the full 2-simplex to the edge between 1 and 2.

This is a general feature: the ith face d_i map deletes the ith vertex.

Theorem 33 (simplicial identites)**.** The face maps and degeneracy maps satisfy the following conditions, known as the *simplicial identities.*

$$\begin{aligned} d_i \circ d_j &= d_{j-1} \circ d_i, && i < j \\ d_i \circ s_j &= s_{j-1} \circ d_i, && i < j \\ d_j \circ s_j &= 1 = d_{j+1} \circ s_j \\ d_i \circ s_j &= s_j \circ d_{i-1}, && i > j+1 \\ s_i \circ s_j &= s_{j+1} \circ s_i, && i \leq j. \end{aligned}$$

Corollary 34. We have immediately that

$$d_i \circ d_j = d_j \circ d_{i+1}, \qquad i + 1 > j.$$

Fact 35. The definition of a simplicial set is equivalent to the definition the following data.

- The sets X_n, $n \geq 0$.
- The face and degeneracy maps satisfying the simplicial identities.

3.4 Basic properties of the category of simplicial sets

The category of simplicial sets is a category of functors whose codomain is **Set**. Any functor category inherits a lot of structure from it codomain category. In the case of $\mathbf{Set}_\Delta$, the category **Set** has an enormous amount of structure. In this section we list some of the structure $\mathbf{Set}_\Delta$ inherits from **Set**.

3.4.1 Existence of limits and colimits

Proposition 36. The category $\mathbf{Set}_\Delta$ has small limits and colimits, which are computed pointwise.

Proof. The category **Set** has small limits and colimits, which implies by Theorem 4 that any functor category $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$ does.

Limits and colimits are computed pointwise thanks to Corollary 6. □

Example 37 (products)**.** By Corollary 5, we can compute this limit pointwise; that is

$$(X \times Y)_n = X_n \times Y_n.$$

Let us understand this this by studying $\Delta^1 \times \Delta^1$ in some detail.

The 0-simplices of $\Delta^1 \times \Delta^1$ are ordered pairs $\Delta^1_0 \times \Delta^1_0 = \{0, 1\} \times \{0, 1\}$. We can visualize this like so.

$$\begin{tikzcd}
(0,1) \bullet & \bullet\ (1,1) \\
(0,0) \bullet & \bullet\ (1,0)
\end{tikzcd}$$

Similarly, the 1-simplices $(a, b) \to (c, d)$ are pairs of a 1-simplex $a \to b$ and a 1-simplex $c \to d$. There are 3 1-simplices in Δ^1_1: $\{a = \{0, 0\}, b = \{0, 1\}, c = \{1, 1\}\}$. Thus, we have the following 1-simplices in $\Delta^1 \times \Delta^1$ (where we do not draw degenerate simplices).

$$\begin{tikzcd}
(0,1) \arrow[r] & (1,1) \\
(0,0) \arrow[u] \arrow[ur] \arrow[r] & (1,0) \arrow[u]
\end{tikzcd}$$

Thus, as one might expect, the product of two intervals is a square.

The fact that limits and colimits are computed pointwise implies immediately that morphisms of simplicial sets are computed pointwise.

Lemma 38. We have the following.

1. A morphism of simplicial sets is a monomorphism if and only if for each n, the component f_n is a monomorphism (i.e. injection) of sets.

2. A morphism of simplicial sets is a epimorphism if and only if for each n, the component f_n is a epimorphism (i.e. surjection) of sets.

Proof. Consequence of Corollary 6 □

3.4.2 Cartesian closure

Note 39. At the moment, the order of a lot of Cartesian products in this section is a bit out of whack, since I'm straddling two different conventions. This is okay for now because $(\mathbf{Set}_\Delta, \times)$ is a symmetric monoidal category.

Thanks to Proposition 36, the category $\mathbf{Set}_\Delta$ has products. It turns out that it also has exponential objects (i.e. an internal hom with respect to the Cartesian product). This makes it into a *Cartesian closed category.*

Before we prove that such an internal hom exists, let us assume that $\mathbf{Set}_\Delta$ admits an internal hom $[A, B]$, and see where we end up.

Lemma 40. Let B, C be simplicial sets. Should it exist, the internal hom $[B, C]$ must be given, up to natural isomorphism, by the functor

$$\mathbf{Set}_\Delta(B \times \Delta^\bullet, C),$$

where $\Delta^\bullet$ denotes the Yoneda embedding $\Delta \hookrightarrow \mathbf{Set}_\Delta$.

Proof. By definition, $[B, C]$ needs to be part of a hom-set adjunction

$$\mathbf{Set}_\Delta(A \times B, C) \simeq \mathbf{Set}_\Delta(A, [B, C]).$$

In particular, when $A = \Delta^n$, we have that

$$\mathbf{Set}_\Delta(\Delta^n \times B, C) \simeq \mathbf{Set}_\Delta(\Delta^n, [B, C]).$$

However, by the Yoneda lemma, there is an isomorphism

$$\mathbf{Set}_\Delta(\Delta^n, [B, C]) \simeq [B, C]_n,$$

natural in n. Thus the internal hom, should it exist, must be given by a functor

$$[B, C] \simeq \mathbf{Set}_\Delta(B \times \Delta^\bullet, C).$$

□

Theorem 41. For simplicial sets B and C, the simplicial set $\mathrm{Maps}(B, C)$ defined by the formula

$$\mathrm{Maps}(B, C) = \mathbf{Set}_\Delta(B \times \Delta^\bullet, C),$$

functions as the internal hom $[B, C]$. Specifically, for each simplicial set A, there is a natural bijection

$$\mathbf{Set}_\Delta(A \times B, C) \simeq \mathbf{Set}_\Delta(A, \mathrm{Maps}(B, C)).$$

Proof. First, we define a family of morphisms which turn out to be the components of the counit. Consider a map

$$\mathrm{ev}_B\colon B \times \mathrm{Maps}(B, C) \longrightarrow C$$

defined level-wise by

$$(\mathrm{ev}_B)_n\colon B_n \times \mathbf{Set}_\Delta(\Delta^n \times B, C) \to C_n \qquad (b, f) \mapsto f(\mathrm{id}_{\Delta^n}, b).$$

We will be done if we can show that there is a bijection between morphisms f and $\tilde{f}$ as follows,

$$\begin{tikzcd} B \times A \arrow[rr, "\mathrm{id}_B \times f"] \arrow[dr, "\tilde{f}"'] & & B \times \mathrm{Maps}(B, C) \arrow[dl, "\mathrm{ev}_B"] \\ & C & \end{tikzcd}$$

i.e. that for each $f\colon A \to \mathrm{Maps}(B, C)$ there is a unique map $\tilde{f}\colon A \times B \to C$ making the above diagram commute. The direction $f \mapsto \tilde{f}$ is obvious, following immediately from the above composition.

Now suppose that we are given a map $\tilde{f}$. Let us construct f. The condition that the diagram commute tells us, level-wise, that

$$\tilde{f}_n(b, a) \overset{!}{=} (\mathrm{ev}_B)_n(b, f_n(a)) = f_n(a)(\mathrm{id}_{\Delta^n}, b).$$

Thus, we have no choice in how f behaves when its first slot is filled with an identity map. □

Note 42. We have

$$\mathrm{Maps}(K, S)_0 = \mathbf{Set}_\Delta(K \times \Delta^0, S) = \mathbf{Set}_\Delta(K, S),$$

i.e. maps $K \to S$ correspond to vertices of $\mathrm{Maps}(K, S)$.

3.5 Simplicial sets from cosimplicial objects

So far, all the concrete examples of simplicial sets we have seen have been of the form $\Delta^{\mathcal{K}}$ for some $\mathcal{K} \subset \mathcal{P}(\{0, \ldots, n\})$. These examples are useful, but not very interesting in their own right.

In this section, we define two ways of creating a simplicial set out of existing mathematical data; in one case, a category, and in the other, a topological space. We will do this via *cosimplicial objects*, i.e. functors

$$\alpha\colon \Delta \to \mathcal{C}$$

where $\mathcal{C}$ is some locally small category.

The idea is as follows. Given any object C in $\mathcal{C}$, we can define a corresponding simplicial set $\tilde{C}$ by first Yoneda embedding, and then composing the resulting hom functor with α:

$$C \mapsto \tilde{C} = \mathcal{C}(\alpha(-), C).$$

That $\tilde{C}$ is a simplicial set follows immediately from the contravariance of the first slot of the hom functor.

The n-simplices of $\tilde{C}$ are then given by maps of the $\mathcal{C}$-model of the n-simplex into C:

$$\tilde{C}_n = \mathcal{C}(\alpha([n]), C).$$

In fact, by the functoriality of hom this construction gives us a functor $\mathcal{C} \to \mathbf{Set}_\Delta$ taking $C \mapsto \tilde{C}$.

The alert reader will remember that we have already defined two cosimplicial objects: the *realization functor* ρ (Definition 19) and the *categorification functor* χ (Definition 20). We said that they would provide a connection between the theory of simplicial sets and the theory of topological spaces and categories respectively. Indeed, these are the cosimplicial objects we will use.

The power of defining simplicial sets via cosimplicial objects comes from Lemma 11, which provides us immediately with a right adjoint. This right adjoint provides a weak inverse, i.e. a way of constructing from any simplicial set K an object of the category $\mathcal{C}$.

3.5.1 Nerves

Definition 43 (nerve of a category)**.** Let $\mathcal{C}$ be a small category. By composing the Yoneda embedding with the categorification functor (Definition 20)

$$\chi\colon \Delta \to \mathbf{Cat},$$

we obtain a simplicial set called the nerve of $\mathcal{C}$ and denoted $N(\mathcal{C})$. That is,

$$N(\mathcal{C}) = \mathbf{Fun}(\chi(-), \mathcal{C}).$$

Let us examine this construction in some detail. We have

$$N(\mathcal{C})_0 = \mathbf{Fun}([0], \mathcal{C}).$$

Since $[0]$ is nothing else but the one-object category, $\mathbf{Fun}([0], \mathcal{C})$ simply picks out the objects of $\mathcal{C}$. That is, the zero-simplices of $N(\mathcal{C})$ simply consist of the objects of $\mathcal{C}$.

Similarly, the category $[1]$ consists of two object and a morphism between them, so the functors in $\mathbf{Fun}([1], \mathcal{C})$ pick out diagrams of the form

$$X_0 \longrightarrow X_1$$

which is tantamount to picking out morphisms in $\mathcal{C}$. Functors in $\mathbf{Fun}([2], \mathcal{C})$ pick out commuting triangles.

$$\begin{tikzcd} & X_1 \arrow[dr] & \\ X_0 \arrow[ur] \arrow[rr] & & X_2 \end{tikzcd}$$

For $n \geq 2$, $N(\mathcal{C})([n])$ picks out chains of morphisms,

$$X_0 \longrightarrow X_1 \longrightarrow X_2 \longrightarrow \cdots \longrightarrow X_n ,$$

together with all possible compositions. One should picture these as forming an n-dimensional simplex.

To summarize, one pictures the construction of the nerve of a category $\mathcal{C}$ in the following way.

1. One imagines for each object of the category $\mathcal{C}$ a zero-simplex, i.e. a little dot.
2. One imagines for each morphism between objects x and y an edge between the corresponding zero-simplices. So far, we have constructed a multidigraph, i.e. a directed graph with multiple edges between vertices.
3. Our multidigraph has lots of little triangles formed by morphisms. We fill in each of the commuting triangles with a 2-simplex.
4. Now we proceed inductively. Given the set of $(n-1)$-simplices of $N(\mathcal{C})$, we add an n-simplex whenever we have all the $(n-1)$-simplices forming its boundary.

Note that the n-simplices for $n > 2$ don't tell us very much; we add them whenever we have their boundaries. This is known as *2-coskeletality,* and is a reflection of the fact that categories only have objects and morphisms, and no higher data. We will meet the notion of coskeletality in Section 3.7.

Proposition 44. The nerve is left is right adjoint to the functor $よ_!\chi$, where $\chi\colon \Delta \to \mathbf{Cat}$ is the categorification functor (Definition 20), which assigns $[n]$ to itself considered as a poset category, and $よ_!$ denotes the Yoneda extension.

$$よ_!\chi : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Cat} : N$$

Proof. The nerve is of the form $\mathcal{C} \mapsto \mathbf{Set}_\Delta(\chi(-), \mathcal{C})$. The result follows from Lemma 11. □

The functor $よ_!\chi$ provides a way of constructing, from any simplicial set, a category. We will not say too much about it now. Rather, we will come back to it later, when we have the tools to understand it properly.

3.5.2 Singular sets

Definition 45 (singular set of a topological space)**.** Let X be a topological space. The functor

$$\mathbf{Top}(\rho(-), X)\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$$

is called the singular set of X, and denoted $\mathbf{Sing}(X)$.

Unlike the nerve, which creates a reasonably faithful simplicial likeness of any category, the singular set creates a grotesque caricature. Consider, for example, the geometric 2-simplex $|\Delta^2|$. We would hope that the singular set of $|\Delta^2|$ would be something like Δ^2, but alas, it is a monstrosity. Rather than three 0-simplices, there are uncountably many, one for each continuous map of the point $|\Delta^0|$ into $|\Delta^2|$. Similarly, there is a 1-simplex for each continuous map of the interval into $|\Delta^2|$.

However, things are not as hopeless as they may appear. We will later see that these extra simplices are in some sense superficial, and that the simplicial set $\mathbf{Sing}(|\Delta^2|)$ is, in an appropriate sense, equivalent to Δ^2.

By Lemma 11, the singular set functor is right adjoint to the Yoneda extension of the realization functor ρ, which we call the *geometric realization.*

Definition 46 (geometric realization)**.** Let K be an simplicial set. The geometric realization of K, denoted $|K|$, is the image

$$|K| = (\text{よ}_!\rho)(K).$$

We now have the tools to study the geometric realization in some detail. This will be the content of Section 3.6.

3.5.3 A colimit formula for simplicial sets

There is one more cosimplicial object we have met, which may seem almost too simple to count: the Yoneda embedding $\text{よ}\colon \Delta \to \mathbf{Set}_\Delta$ itself! Applying the same procedure as above, we find the functor

$$\mathbf{Set}_\Delta \to \mathbf{Set}_\Delta; \qquad K \mapsto \mathbf{Set}_\Delta(\text{よ}(-), K).$$

This is (by the Yoneda lemma) simply the identity functor.

However, Lemma 11 now implies a very useful formula.

Proposition 47. Let K be any simplicial set. Then K is the colimit of the functor

$$(\Delta \downarrow K) \to \Delta \xrightarrow{\text{よ}} \mathbf{Set}_\Delta\,.$$

Proof. Lemma 10. □

3.6 Geometric realization

In Subsection 3.5.2, we saw that we could turn any topological space X into a simplicial set using the singular set functor (Definition 45)

$$\mathbf{Sing}\colon \mathbf{Top} \to \mathbf{Set}_\Delta; \qquad \mathbf{Sing}(X) = \mathbf{Top}(\rho(-), X).$$

We then saw that **Sing** had a left adjoint, the geometric realization functor, which provides a weak inverse to **Sing**. The geometric realization had the following formula.

$$|K| = (\text{よ}_!\rho)(K).$$

As we will see in this section, the geometric realiztion takes a simplicial set K to a topological space which is created by assigning to each n-simplex in K to an actual n-simplex in **Top**, and gluing these together in the appropriate way.

The goal of this section is to derive and understand the following formula for the geometric realization.

Theorem 48. Let $K\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. The geometric realization of K can be computed by the formula

$$|K| = \left(\coprod_{n\geq 0} K_n \times |\Delta^n|\right)/\sim,$$

where $\sim$ is the equivalence relation generated by

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f : [m] \to [n].$$

Proof. By the colimit formula for left Kan extensions (Fact 2), we can write

$$|K| = \operatorname{colim}\left[(\Delta \downarrow K) \to \Delta \xrightarrow{\rho} \mathbf{Top}\right].$$

Let f be a morphism $[m] \to [n]$. Then by the contravariance of any simplicial set K, f induces a morphism $f^*\colon K_n \to K_m$. Similarly, by the covariance of the geometric realization functor $\rho\colon \Delta \to \mathbf{Top}$ (defined in Definition 19), f induces a map $f_*\colon |\Delta^m| \to |\Delta^n|$.

By the coequalizer formula for colimits (Fact 1), we can express the above colimit as the following coequalizer.

$$\coprod_{\substack{f \in \mathrm{Morph}(\Delta/K) \\ f\colon [m] \to [n]}} |\Delta^m| \overset{A}{\underset{B}{\rightrightarrows}} \coprod_{\substack{([n],\alpha) \in \mathrm{ob}(\Delta/K) \\ \alpha : \Delta^n \to K}} |\Delta^n| \longrightarrow \mathrm{coeq}$$

where A and B are defined by their components

$$A_f = \iota_{|\Delta^m|}, \qquad B_f = \iota_{|\Delta^n|} \circ f_*.$$

Recall that any morphism $g \in \mathrm{Morph}(\Delta \downarrow K)$ is defined by making the following diagram commute.

$$g\colon [m] \to [n]; \qquad \begin{tikzcd} \Delta^m \arrow[rr, "\Delta^g"] \arrow[dr, "f"'] & & \Delta^n \arrow[dl, "f'"] \\ & K & \end{tikzcd}$$

This means that such a morphism contains two pieces of data: a morphism $g\colon [m] \to [n]$, and a morphism $f'\colon \Delta^n \to K$. The other morphism f must be the composite $f' \circ \Delta^g$.

This means that we are really taking the coproduct over two things: morphisms $[m] \to [n]$, and morphisms $\Delta^n \to K$. But by the Yoneda lemma

$$\mathbf{Set}_\Delta(\Delta^n, K) \simeq K_n,$$

so we write the first coproduct as

$$\coprod_{f\colon [m] \to [n]} \left(\coprod_{x \in K_n} |\Delta^m| \right)$$

which by abuse of notation is simply

$$\coprod_{f\colon [m] \to [n]} K_n \times |\Delta^m|.$$

□

Let $K\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ be a simplicial set. The geometric realization of K is the topological space

$$|K| = \left(\coprod_{n \geq 0} K_n \times |\Delta^n| \right) /\sim,$$

where $\sim$ is the equivalence relation generated by

$$(f^*(x), y) \sim (x, f_*(y)), \qquad \text{for all } f\colon [m] \to [n].$$

To define $|\cdot|$ on morphisms, let $f \in \mathbf{Set}_\Delta(K, K')$. Then f is, for each $n \in \mathbb{N}$, a map $f_n\colon K_n \to K'_n$ which commutes with all the face and degeneracy maps, so we can simply define

$$|f|[(a, b)] = [(f_n(a), b)].$$

Lemma 49. Let K be a simplicial set. We have the following.

1. K is a colimit of the functor

$$F\colon \Delta/K \to \mathbf{Set}_\Delta; \qquad (\Delta^n \to K) \mapsto \Delta^n.$$

2. $|K|$ is a colimit of the functor

$$|F|\colon \Delta/K \to \mathbf{Top}; \qquad (\Delta^n \to K) \mapsto |\Delta^n|.$$

Proof. The first part is a direct consequence of Lemma 10.

The second part comes from the Kan extension

$$\begin{tikzcd} \Delta \arrow[rr, "\rho"] \arrow[dr, "y"'] & & \mathbf{Top} \\ & \mathbf{Set}_\Delta \arrow[ur, "|\cdot|"'] & \end{tikzcd}$$

□

We can explicitly compute the colimit of any diagram $G\colon I \to \mathcal{C}$ as a coequalizer between coproducts.[3]

$$\coprod_{\substack{f \in \mathrm{Morph}(I) \\ f\colon i \to j}} G(i) \overset{\iota_{G(i)}}{\underset{\iota_{G(j)} \circ G(f)}{\rightrightarrows}} \coprod_{i \in I} G(i)$$

With G our functor $|F|$, we find the first line of the below diagram.

$$\begin{tikzcd} \coprod\limits_{\substack{f \in \mathrm{Morph}(\Delta/K) \\ f\colon [m] \to [n]}} |\Delta^m| \arrow[r, shift left, "\iota_{|\Delta^m|}"] \arrow[r, shift right, "\iota_{|\Delta^n|} \circ f_*"'] \arrow[d, "\alpha"] & \coprod\limits_{\substack{([n], \alpha) \in \mathrm{ob}(\Delta/K) \\ \alpha : \Delta^n \to K}} |\Delta^n| \arrow[r] \arrow[d, "\beta"] & \mathrm{coeq} \arrow[d] \\ \coprod\limits_{f\colon [m] \to [n]} K_n \times |\Delta^m| \arrow[r, shift left, "f^* \times \mathrm{id}"] \arrow[r, shift right, "\mathrm{id} \times f_*"'] & \coprod\limits_{n \geq 0} K_n \times |\Delta^n| \arrow[r] & |K| \end{tikzcd}$$

[3] The notation here needs some explanation. By the universal property for coproducts, providing a map out of a coproduct is equivalent to providing a map out of each summand. The labels on the morphisms we are drawing are really to be understood summand-by-summand, but the map we mean is the one guaranteed to us by the universal property, which is a bona-fide morphism between the objects.

Note that we have not really done anything: this is simply a re-indexing of the coproduct.

Similarly, in the second coproduct we pick up one copy of $|\Delta^n|$ for every $\alpha\colon \Delta^n \to K$. Again, the Yoneda lemma tells us this means we pick up one copy of $|\Delta^n|$ for every element of K_n, so we can write our coproduct as

$$\coprod_{[n]\in\Delta}\left(\coprod_{x\in K_n}|\Delta^n|\right) = \coprod_{n\in\mathbb{N}} K_n \times |\Delta^n|.$$

Again, this is only a re-indexing. We have still to figure out how to write the maps from our coequalizer in this new notation.

Recall that we really have one map for each summand in the first coproduct, which we turn in to a single map using the universal property for coproducts. Take the copy of $|\Delta^m|$ corresponding to the morphism $\Delta^m \xrightarrow{\Delta^g} \Delta^n \xrightarrow{f'} K$. This is mapped to the copy of $|\Delta^m|$ corresponding to the domain of Δ^g

Recall that we really have one map for each summand in the first coproduct, which we turn in to a single map using the universal property for coproducts. Take the copy of $|\Delta^m|$ corresponding to the morphism $\Delta^m \xrightarrow{\Delta^g} \Delta^n \xrightarrow{f'} K$. This is mapped to the copy of $|\Delta^m|$ corresponding to the object $([m], f)$ using the canonical injection $\iota_{|\Delta^m|}$.

To finish.

Example 50. Let $n \geq 0$ and let $\mathcal{K} \subset \mathcal{P}(\{0,\dots,n\})$ be an abstract simplicial complex. Then by Lemma 49,

$$|\Delta^{\mathcal{K}}| \simeq \operatorname{colim}\left(\Delta/\Delta^{\mathcal{K}} \to \mathbf{Top}\right).$$

The functor over which we are taking the colimit acts on objects by

$$\left(\Delta^n \xrightarrow{\ f\ } \Delta^{\mathcal{K}}\right) \mapsto |\Delta^n|.$$

But objects (i.e. natural transformations) $\Delta^n \to \Delta^{\mathcal{K}}$ are in natural bijection (by the Yoneda lemma) to elements of $\Delta^{\mathcal{K}}_n$. Therefore, we can equally view our colimit as being indexed by $n \in \mathbb{N}$ and $J \in \Delta^{\mathcal{K}}_n$.

$$\simeq \operatorname*{colim}_{\substack{n\in\mathbb{N}\\ J\in\Delta^{\mathcal{K}}_n}} |\Delta^n| \simeq \coprod_{\substack{n\in\mathbb{N}\\ J\in\Delta^{\mathcal{K}}_n}} |\Delta^n|.$$

Specific: do Λ^2_1 at some point.

The adjunction

$$|\cdot| : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Top} : \mathbf{Sing}$$

gives us a way of translating between the language of simplicial sets and the language of simplices in topological spaces.

3.7 Skeletons and coskeletons

For any $n \geq 0$, denote by $\iota_{\leq n}\colon \Delta_{\leq n} \hookrightarrow \Delta$ the full subcategory inclusion on objects $[0], [1], \dots, [n]$. We can restrict any simplicial set along this functor, giving its so-called *n-truncation*. This extends to a so-called *n-truncation functor*.

Definition 51 (truncation)**.** Let $n \geq 0$. The n-truncation functor is the pullback

$$\iota_{\leq n}^{*} = \mathrm{tr}_{\leq n} \colon \mathbf{Set}_{\Delta} \to \mathbf{Set}_{\Delta_{\leq n}}.$$

One visualizes the n-truncation of a simplicial set as forgetting simplices of degree greater than n.

As usual, this is a forgetful functor, so left and right adjoints (should they exist) provide algoritmic ways of attempting to solve the impossible problem of recovering the lost simplices.

Definition 52 (skeleton, coskeleton)**.** Let $n \geq 0$.

- The left adjoint to the n-truncation functor is called the n-skeleton functor, and denoted

$$\mathbf{sk}_{\leq n} \colon \mathbf{Set}_{\Delta_{\leq n}} \to \mathbf{Set}_{\Delta}\,.$$

- The right adjoint to the n-truncation functor is called the n-coskeleton functor, and denoted

$$\mathbf{sk}_{\leq n} \colon \mathbf{Set}_{\Delta_{\leq n}} \to \mathbf{Set}_{\Delta}\,.$$

Note that the skeleton and coskeleton functors are simply the left and right Kan extension functors, i.e.

$$\mathbf{sk}_{\leq n} = (\iota_{\leq n})_{!}, \qquad \mathbf{cosk}_{\leq n} = (\iota_{\leq n})_{*}.$$

For any simplicial set K, we can define the n-skeleton of K to be the simplicial set built by first forgetting simplices of degree greater than n, then adding them back in using the n-skeleton functor. Since the skeleton functor is a Kan extension, the n-skeleton of a simplicial set is the left Kan extension along the restriction. That is,

$$\mathrm{sk}_{\leq n}(K) = (\mathbf{sk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Similarly, we define its n-coskeleton to be

$$\mathrm{cosk}_{\leq n}(K) = (\mathbf{cosk}_{\leq n} \circ \mathrm{tr}_{\leq n})(K).$$

Example 53. Let K be any simplicial set. Let us compute the n-skeleton of K. By definition of the left Kan extension

$$\begin{tikzcd} \Delta_{\leq n}^{\mathrm{op}} \arrow[rr, "K\circ\iota_{\leq n}^{\mathrm{op}}"] \arrow[dr, "\iota_{\leq n}^{\mathrm{op}}"'] & & \mathbf{Set} \\ & \Delta^{\mathrm{op}} \arrow[ur, "(\iota_{\leq n}^{\mathrm{op}})_{!}(K\circ\iota_{\leq n}^{\mathrm{op}})"'] & \end{tikzcd}$$

we have, by the colimit formula, the following expression for the m-simplices of $\mathrm{sk}_{\leq n}K$.

$$(\mathrm{sk}_{\leq n}K)_m = \mathrm{colim}\left[(\iota_{\leq n}^{\mathrm{op}} \downarrow [m]) \to \Delta_{\leq n}^{\mathrm{op}} \to \mathbf{Set}\right]$$

When $m \leq n$, then the category $(\iota_{\leq n}^{\mathrm{op}} \downarrow [m])$ has the terminal object $([m], \mathrm{id}_{[m]})$, so the colimit is

$$(K \circ \mathrm{tr}_{\leq n})([m]) = K_m.$$

Thus, the n-skeleton of K agrees with K on simplices of degree less than or equal to n.

Now let $m \geq n$. In this case, the category $(\mathrm{tr}_{\leq n} \downarrow [m])$ does not have a terminal object.

3.8 Kan complexes

Definition 54 (Kan complex)**.** A Kan complex is a simplicial set such that for every $n > 0$ and every $0 \leq i \leq n$, for every map

$$\Lambda^n_i \xrightarrow{f} K$$

there exists a map $\bar{f}$ (possibly not unique!) making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & K \\ \Delta^n \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

The above condition is known as the *horn-filling condition,* and $\bar{f}$ is called the *horn filler.* Roughly, the horn-filling condition says that any horn in K can be completed to a simplex.

A priori, it is not clear that this condition is interesting. However, we will now show that every singular set (i.e. simplicial set of the form $\mathbf{Sing}(X)$ for some topological space X) is a Kan complex. If we will in the end be interested in translating between topological spaces and simplicial sets, it will be useful to have a necessary condition for a simplicial complex to be a singular set.

Theorem 55. For any topological space X, the set $\mathbf{Sing}(X)$ is a Kan complex.

Proof. The injection

$$j\colon |\Lambda^n_i| \to |\Delta^n|$$

admits a continuous right inverse

$$p\colon |\Delta^n| \to |\Lambda^n_i|.$$

That is,

$$p \circ j = \mathrm{id}_{|\Lambda^n_i|}.$$

By applying the functor $|\cdot|$, we can take any diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r, "f"] \arrow[d, hook] & \mathbf{Sing}(X) \\ \Delta^n & \end{tikzcd}$$

to the diagram

$$\begin{tikzcd} |\Delta^n_i| \arrow[r, "|f|=f'"] \arrow[d, "j"'] & |\mathbf{Sing}(X)| \\ |\Delta^n| & \end{tikzcd},$$

now in **Top**. The counit gives us a morphism $\eta_X\colon |\mathbf{Sing}(X)| \to X$

$$\begin{tikzcd} |\Lambda^n_i| \arrow[r, "\eta_X \circ f'"] \arrow[d, "j"'] & X \\ |\Delta^n| & \end{tikzcd}.$$

But the map j has a left inverse, which we can compose with f' as follow.

$$\begin{tikzcd}
{|\Lambda^n_i|} \arrow[r, "f'"] \arrow[d, "j"] & X \\
{|\Delta^n|} \arrow[u, bend left, "p"] \arrow[ur, "f\circ p"'] &
\end{tikzcd}$$

Applying the **Sing** functor again, we find

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "\epsilon_{\Lambda^n_i}"] \arrow[d, hook] & \mathbf{Sing}(|\Lambda^n_i|) \arrow[r] \arrow[d] & \mathbf{Sing}(X) \\
\Delta^n \arrow[r, "\epsilon_{\Delta^n}"] & \mathbf{Sing}(|\Delta^n|) \arrow[ur] &
\end{tikzcd} .$$

The square commutes by the naturality of ϵ. The triangle commutes because it is the image of a commuting triangle. Thus, taking the composition gives us a morphism $\Delta^n \to \mathbf{Sing}(X)$. □

Proposition 56. Let $\mathcal{C}$ be a small category.

1. All inner horns of $N(\mathcal{C})$ have a filler; that is, any morphism

$$\Lambda^n_i \xrightarrow{f} N(\mathcal{C})$$

where $0 < i < n$ can be extended to a commuting diagram

$$\begin{tikzcd}
\Lambda^n_i \arrow[r, "f"] \arrow[d] & N(\mathcal{C}) \\
\Delta^n \arrow[ur, "\bar{f}"'] &
\end{tikzcd} .$$

2. The simplicial set $N(\mathcal{C})$ has *all* horn fillers, i.e. is a Kan complex, if and only if $\mathcal{C}$ is a groupoid. Furthermore, this horn filler is unique for all $n \geq 2$.

Proof. There are no inner horns in degrees 0 or 1, so we start with a $(2, 1)$-horn.

Consider a horn

$$\Lambda^2_1 \longrightarrow N(\mathcal{C}) .$$

To build from this a map $\Delta^2 \to N(\mathcal{C})$, we have to fill in images of everything in Δ^2 missing from Λ^2_1, i.e. the dashed 1-simplex and the full 2-simplex.

$$\begin{tikzcd}
& 1 \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, dashed] & & 2
\end{tikzcd}$$

When we map the horn Λ^2_1 into $N(\mathcal{C})$, we find two morphisms as follows.

$$\begin{tikzcd}
& a \arrow[dr, "g"] & \\
b \arrow[ur, "f"] & & c
\end{tikzcd}$$

A map of a 2-simplex into $N(\mathcal{C})$ consists of a commuting triangle in $\mathcal{C}$. Such a map making the diagram

$$\begin{tikzcd} \Lambda^2_1 \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^2 \arrow[ur, dashed] & \end{tikzcd}$$

□

3.9 Kan fibrations

Definition 57 (Kan fibration)**.** Let $p\colon K \to S$ be a morphism of simplicial sets. We say that p is a Kan fibration if for every i with $0 \le i \le n$ and every diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

there exists a (not necessarily unique!) morphism $\Delta^n \to K$ making the diagram

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "\exists"] & S \end{tikzcd}$$

commute.

This is a sort of a relative version of horn filling. Note that if we take $S = *$, the constant simplicial set (which is the terminal object in the category of simplicial sets), we recover something resembling notion of a horn filler. More precisely:

Corollary 58. A Kan fibration $K \to \Delta^0$ is the same as a Kan complex.

Proof. The bottom right triangle

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \Delta^0 \end{tikzcd}$$

contains no information. □

4 Lifting problems

One often encounters problems of the following form.

> Given some commuting square
>
> $$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$
>
> construct a dashed arrow making everything commute.
>
> $$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

These problems are known as lifting problems. They turn out to be very important. In this chapter, we develop some powerful machines which enable us to easily solve lifting problems.

In Section 4.1, we define the notion of a *saturated set,* which is a set of morphisms closed under certain operations. We see that these operations are closed under intersection, and thus one can define the *saturated hull* of any set of morphisms to be the smallest saturated hull containing them.

We then define the notion of a *lifting property;* roughly, a morphism f has the left lifting property with respect to a set $\mathcal{S}$ of morphisms if for all $s \in \mathcal{S}$, and all commuting with f on the left and some $s \in \mathcal{S}$ on the right, one can find a dashed arrow as follows.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & A \arrow[d, "s"] \\ Y \arrow[r] \arrow[ur, dashed] & B \end{tikzcd}$$

Similarly, a morphism g has the *right lifting property* with respect to $\mathcal{S}$ if one can always find a dashed arrow as follows.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "s"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

For any set $\mathcal{S}$ of morphisms, we denote the set of all morphisms which have the left lifting property with respect to it by ${}_\perp\mathcal{S}$, and the set of all morphisms with the right lifting property with respect to it by $\mathcal{S}_\perp$. We then show that for any set $\mathcal{M}$, the set ${}_\perp\mathcal{M}$ is saturated, immediately implying that the saturated hull of any set $\mathcal{S}$ is contained in ${}_\perp(\mathcal{S}_\perp)$.

In Section 4.2, we define the notion of a compact object. Roughly speaking, a compact object is an object which satisfies a sort of finiteness condition.

In Section 4.3, we prove a technical lemma called the *small object argument.* This lemma provides a factorization for morphisms whose domain is compact. This factorization immediately implies that for sets of morphisms whsoe domain is compact, we have the equality $\overline{M} = {}_{\perp}(\mathcal{M}_{\perp})$.

In Section 4.4, we define the set of anodyne morphisms to be the set of all morphisms with the left lifting property with respect to Kan fibrations, and show that the set of

4.1 Saturated sets

Denote by $\mathcal{C}$ a category with small colimits.

Definition 59 (saturated)**.** A subset $\mathcal{S} \subset \mathrm{Morph}(\mathcal{C})$ is said to be saturated if the following conditions hold.

1. $\mathcal{S}$ contains all isomorphisms.
2. $\mathcal{S}$ is closed under pushouts, i.e. for any pushout square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & A' \arrow[d, "i'"] \\ B \arrow[r] & B' \end{tikzcd}$$

   with $i \in \mathcal{S}$, then $i' \in \mathcal{S}$.

3. $\mathcal{S}$ is closed under retracts, i.e. for any commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r] \arrow[d, "i"] & A \arrow[d, "i'"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r] & B \end{tikzcd}$$

   with $i \in \mathcal{S}$, we also have $i' \in \mathcal{S}$.

4. $\mathcal{S}$ is closed under countable composition, i.e. for any chain of morphisms indexed by $\mathbb{N}$

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \cdots$$

   the map

$$A_0 \to \operatorname*{colim}_{n\in\mathbb{N}} A_n = A_\infty$$

   is in $\mathcal{S}$.

5. $\mathcal{S}$ is closed under small coproducts: if the family of morphisms

$$\{i_j \colon A_j \to B_j\}_{j\in J} \subset \mathcal{S}$$

is in $\mathcal{S}$, where J is a small set, then the morphism $\coprod_j i_j$ defined using the universal property of coproducts

$$\textstyle\coprod_{j\in J} A_j \xrightarrow{\coprod i_j} \coprod_{j\in J} B_j$$

is in $\mathcal{J}$.

Saturatedness is a very powerful property. We will generally make use of it by proving that a certain collection of morphisms in a category is saturated, allowing us to use the closure properties.

Example 60. The set

$$\{f : X \to Y \mid f \text{ is mono}\} \subset \mathrm{Morph}(\mathbf{Set}_\Delta)$$

is saturated.

1. Clearly, every isomorphism is a monomorphism.
2. Since coproducts are computed pointwise (Corollary 5) it suffices to check that every pushout of monos is mono in **Set**.
3. Consider simplicial sets and maps as follows, with i a monomorphism.

$$\begin{tikzcd} A \arrow[r, "f"'] \arrow[d, "j"'] \arrow[rr, bend left, "\mathrm{id}"] & A' \arrow[r, "g"'] \arrow[d, "i"] & A \arrow[d, "j"] \\ B \arrow[r, "k"] \arrow[rr, bend right, "\mathrm{id}"'] & B' \arrow[r, "h"] & B \end{tikzcd}$$

   Since $g \circ f = \mathrm{id}$, we must have that f is a monomorphism, so $i \circ f$ is a monomorphism. Thus, $k \circ j$ is a monomorphism, so j must be a monomorphism.
4. Again, it suffices to check that countable composition respects monomorphisms in **Set**. Again, this is immediate from the universal property.
5. We can check this in **Set**. Again it is immediate from the universal property.

Definition 61 (saturated hull)**.** Let $\mathcal{M}$ be any set of morphisms in $\mathcal{C}$. We define the saturated hull $\overline{M}$ of $\mathcal{M}$ to be the smallest saturated set containing $\mathcal{M}$, i.e. the intersection of all saturated sets containing $\mathcal{M}$.

Note 62. This always exists because the set of all morphisms of $\mathcal{C}$ is a saturated set, and because the intersection of saturated sets is again saturated (because properties of the form 'closure under x' are respected by intersection).

Example 63. Consider the set $\mathcal{M}$ of boundary fillings

$$\mathcal{M} = \{\partial\Delta^n \hookrightarrow \Delta^n \mid n \geq 0\}.$$

The saturated hull $\overline{\mathcal{M}}$ is the set of monomorphisms.

Note that by Example 60, the set of all monomorphisms is saturated.

Consider some monomorphism $f\colon X \to Y$. Our goal is to construct f from boundary inclusions using pushouts, retracts, countable composition, and coproducts.

Take the coproduct

$$X^0 \equiv X \amalg \left(\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \Delta^0 \right) = X \amalg \left\{ \substack{\text{0-skeleton of } Y \text{ which} \\ \text{is not hit by } f} \right\}.$$

Since $\partial\Delta^0 = \emptyset$, we can build a map $g^0\colon X \hookrightarrow X^0$ as a coproduct of boundary fillings.

$$X = X \amalg \left(\textstyle\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \partial\Delta^0 \right) \xhookrightarrow{g^0} X \amalg \left(\textstyle\coprod_{\Delta^0 \in Y_0 \smallsetminus f_0(X_0)} \Delta^0 \right).$$

Since saturated sets are closed under coproducts, $g^0 \in \overline{\mathcal{M}}$.

We get a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g^0"'] & & Y \\ & X^0 \arrow[ur, "f^0"'] & \end{tikzcd}$$

where g^0 is simply the inclusion, and f^0 takes X to $f(X)$ and the simplices of Y we added to X to themselves.

To summarize: we have a factorization $f^0\colon X^0 \to Y$ which is an isomorphism on 0-simplices. It is trivially an isomorphism on all 0-cells (i.e. boundaries of 0-simplices), i.e. inclusions $\partial\Delta^0 = \emptyset \hookrightarrow X \to Y$. In fact, since $\partial\Delta^1 = \Delta^0 \amalg \Delta^0$, it is an isomorphism on all 1-cells (i.e. boundaries of 1-simplices). Thus, we have the following solid maps. We call the pushout X^1.

$$\begin{tikzcd} \coprod_{\Delta^1 \in Y_1 \smallsetminus f_1(X_1)} \partial\Delta^1 \arrow[r, hook] \arrow[d, hook] & X^0 \arrow[d, dashed, "g^1"] \\ \coprod_{\Delta^1 \in Y_1 \smallsetminus f_1(X_1)} \Delta^1 \arrow[r, dashed] & X^1 \end{tikzcd}$$

Note that since g^1 is a coproduct of boundary inclusions, it is in $\overline{\mathcal{M}}$.

Schematically, we have

$$X^1 = X \amalg \left\{ \substack{\text{1-skeleton of } Y \text{ which} \\ \text{is not hit by } f} \right\}.$$

Thus, we have a commutative diagram

$$\begin{tikzcd} X \arrow[rrr, "f"] \arrow[dr, "g^0"'] & & & Y \\ & X^0 \arrow[urr, "f^0"] \arrow[dr, "g^1"'] & & \\ & & X^1 \arrow[uur, "f^1"'] & \end{tikzcd}$$

where $g^1 \in \overline{\mathcal{M}}$ and f^1 is a monomorphism and an isomorphism on 1-simplices.

We continue this process, constructing sets X^2, X^3, etc., at each step adding another level of the skeleton via boundary inclusions and pushouts. At the kth step, we have a simplicial set X^k and a factorization

$$\begin{tikzcd}
X \arrow[rrrr, "f"] \arrow[dr, dotted, no head] & & & & Y \\
& X^{k-1} \arrow[urrr, "f^{k-1}"] \arrow[dr, "g^k"'] & & & \\
& & X^k \arrow[uurr, "f^k"'] & &
\end{tikzcd}$$

where $g^k \in \overline{\mathcal{M}}$ and f^k is a monomorphism and an isomorphism on k-simplices.

By construction, we have

$$\operatorname*{colim}_{n \in \mathbb{N}} X^n \equiv X^\infty = Y.$$

The countable composition of the gs is equal to f, and we get a factorization

$$\begin{tikzcd}
X \arrow[rr, "f"] \arrow[dr, "g^\infty"'] & & Y \\
& X^\infty \arrow[ur, equal] &
\end{tikzcd}$$

where g^∞ is the countable composition of elements of $\overline{\mathcal{M}}$, and hence is in $\overline{\mathcal{M}}$.

Definition 64 (lifting property)**.** Let $i\colon A \to B$ and $p\colon K \to S$ be morphisms in $\mathcal{C}$. Suppose that for every commuting square

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\
B \arrow[r] & S
\end{tikzcd}$$

there exists a morphism $B \to K$ making the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & K \arrow[d, "p"] \\
B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

commute. Then we say either of the following, which are equivalent.

- The morphism i has the left lifting property with respect to p.
- The morphism p hs the right lifting property with respect to i.

For any set of morphisms $\mathcal{M}$ we define two sets, ${}_\perp\mathcal{M}$ and $M_\perp$ as follows.

$$\begin{aligned}
{}_\perp\mathcal{M} &= \{f \mid f \text{ has the left lifting property with respect to all morphisms in } \mathcal{M}\} \\
\mathcal{M}_\perp &= \{g \mid g \text{ has the right lifting property with respect to all morphisms in } \mathcal{M}\}
\end{aligned}$$

Example 65. Let $\mathcal{C} = \mathbf{Set}_\Delta$. Let

$$\mathcal{M} = \{f : \Lambda^n_i \to \Delta^n \mid 0 \le i \le n \text{ and } n > 0\}.$$

That is, $\mathcal{M}$ is the set of all horn inclusions of the n-simplex. Then what is $\mathcal{M}_\perp$?

Any morphism $g \in \mathcal{M}_\perp$ needs to satisfy the following property: for any commuting square as follows

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, "f"'] & K \arrow[d, "g"] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

there exists some morphism $\Delta^n \to K$ making the following diagram commute.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, "f"'] & K \arrow[d, "g"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

Checking Definition 57, we see that this says precisely that g is a Kan fibration. That is, $\mathcal{M}_\perp$ is the set of all Kan fibrations.

Theorem 66. Let $\mathcal{M}$ be any set of morphisms in $\mathcal{C}$. Then ${}_\perp\mathcal{M}$ is saturated.

Proof. We need to check conditions 1–5 in Definition 59.

1. Let $f : K \to S$ be an isomorphism. We need to check that f has the left lifting property. Suppose we are given a commuting square as follows.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & K \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

We can define a morphism $h : B \to K$ by taking $h = \alpha \circ f^{-1}$.

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"] & K \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "h"] \arrow[u, bend left, "f^{-1}"] & S \end{tikzcd}$$

2. Suppose $f \in {}_\perp\mathcal{M}$, and let $\tilde{f}$ be any pushout.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "\tilde{f}"] \\ B \arrow[r] & B' \end{tikzcd}$$

We need to show that $\tilde{f} \in {}_\perp\mathcal{M}$. To this end, consider a lifting problem as follows, with $m \in \mathcal{M}$.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B' \arrow[r] & Y \end{tikzcd}$$

Stacking this square next to the pushout square above, we find the following.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B \arrow[r] & B' \arrow[r] & Y \end{tikzcd}$$

We can solve the outer lifting problem.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[r] \arrow[d, "\tilde{f}"'] & X \arrow[d, "m"] \\ B \arrow[r] \arrow[urr, dashed] & B' \arrow[r] & Y \end{tikzcd}$$

This gives us a cocone under the pushout, which gives us a unique dotted morphism, which is the lift we are after.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A' \arrow[d, "\tilde{f}"] \arrow[ddr, bend left] & \\ B \arrow[r] \arrow[drr, dashed, bend right] & B' \arrow[dr, dotted] & \\ & & X \end{tikzcd}$$

3. Suppose $i \in {}_{\perp}\mathcal{M}$, and consider a commuting square as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[d, "i'"] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \end{tikzcd} \tag{4.1}$$

We need to show that i' has the left lifting property. To this end, consider a commuting square as follows.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] & K \arrow[d] \\ B' \arrow[r] & S \end{tikzcd}$$

Sticking this onto the right of Equation 4.1, we obtain a new diagram.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d] \\ B' \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \arrow[r] & S \end{tikzcd} \tag{4.2}$$

The commuting square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & K \arrow[d] \\ B \arrow[r, "\alpha"] & S \end{tikzcd}$$

is a lifting problem we can solve, giving us a morphism $B \to K$ making the diagram commute. Adding this onto our original diagram and pre-composing it with α gives us the morphism required.

$$\begin{tikzcd} A' \arrow[r] \arrow[d, "i'"'] \arrow[rr, bend left, "\mathrm{id}"] & A \arrow[r] \arrow[d, "i"] & A' \arrow[r] \arrow[d, "i'"] & K \arrow[d] \\ B' \arrow[r] \arrow[rrru, dashed] \arrow[rr, bend right, "\mathrm{id}"'] & B \arrow[r] & B' \arrow[r] & S \end{tikzcd} \tag{4.3}$$

4. Consider a chain of morphisms

$$A_0 \xrightarrow{i_0} A_1 \xrightarrow{i_1} A_2 \xrightarrow{i_2} A_3 \xrightarrow{i_3} \cdots$$

with $i_k \in {}_{\perp}\mathcal{M}$ for all k. We need to show that the map $A_0 \to A_\infty$ induced by the colimit has the left lifting property.

Consider a commuting square as follows.

$$\begin{tikzcd} A_0 \arrow[r] \arrow[d, "i_0"] \arrow[dddd, bend right] & K \arrow[dddd] \\ A_1 \arrow[d, "i_1"] & \\ A_2 \arrow[d, "i_2"] & \\ A_3 \arrow[d, dotted] & \\ A_\infty \arrow[r] & S \end{tikzcd}$$

Suppose we have a map $A_k \to K$ which makes all squares and triangles formed commute. This gives us a commuting square as below.

$$\begin{tikzcd}
A_0 \arrow[r, dashed] \arrow[d, dashed, "i_0"] \arrow[dddd, dashed, bend right] & K \arrow[dddd] \\
A_1 \arrow[d, dotted, no head] & \\
A_k \arrow[uur] \arrow[d, "i_k"] & \\
A_{k+1} \arrow[d, dotted, no head] \arrow[dr] & \\
A_\infty \arrow[r, dashed] & S
\end{tikzcd}$$

Because i_k has the left lifting property, we get a map $A_{k+1} \to K$ making the diagram commute.

$$\begin{tikzcd}
A_0 \arrow[r] \arrow[d, "i_0"] \arrow[dddd, bend right] & K \arrow[dddd] \\
A_1 \arrow[d, dotted, no head] & \\
A_k \arrow[uur, dashed] \arrow[d, "i_k"] & \\
A_{k+1} \arrow[d, dotted, no head] \arrow[dr, dashed] \arrow[uuur, dashed] & \\
A_\infty \arrow[r] & S
\end{tikzcd}$$

By induction, this gives us a collection of maps $A_k \to K$, one for each $k \in \mathbb{N}$, which make the triangles

$$\begin{tikzcd}
A_k \arrow[rr, "i_k"] \arrow[dr] & & A_{k+1} \arrow[dl] \\
& K &
\end{tikzcd}$$

commute. That is, we get a cocone under K. But this gives us a map $A_\infty \to K$ as required.

5. Similar.

$\square$

Corollary 67. Let $\mathcal{M}$ be a set of morphisms in $\mathcal{C}$. Then

$$\overline{\mathcal{M}} \subset {}_\perp(\mathcal{M}_\perp).$$

Proof. The set $\mathcal{M}$ clearly has the left lifting property for the set of all morphisms which have the right lifting property for $\mathcal{M}$, so $\mathcal{M} \subset {}_\perp(\mathcal{M}_\perp)$. Since ${}_\perp(\mathcal{M}_\perp)$ is a saturated set, it contains $\overline{\mathcal{M}}$. $\square$

It turns out that under some conditions, we have an equality

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

4.2 Compact objects

Definition 68 (filtered poset)**.** A poset $(I, \leq)$ is called filtered if every finite subset of I has an upper bound.

Example 69. Let X be a set, and consider the poset of finite subsets $\mathcal{P}_{\text{fin}}(X)$. This is a filtered poset since every finite collection $X_i \in \mathcal{P}_{\text{fin}}(X)$ is contained in the set $\bigcup_i X_i$, which is again finite.

Definition 70 (compact object)**.** An object $X \in \mathcal{C}$ is said to be compact if for every diagram

$$\phi\colon I \to \mathcal{C}; \qquad i \mapsto Y_i$$

with I a filtered poset, the criteria below are satisfied.

Denote by Y_∞ the colimit of ϕ, and write the colimit maps

$$\eta_i\colon A_i \to Y_\infty.$$

1. There exists some $i \in I$ and $f_i\colon X \to Y_i$ making the diagram

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

   commute.

2. Given maps $f_i\colon X \to Y_i$ and $f_j\colon X \to Y_j$ making the diagrams

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d, "\eta_i"] \\ & Y_\infty \end{tikzcd}$$

   commute, there exist k with $k \geq i, j$ such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[d, "f_j"'] & Y_i \arrow[d] \\ Y_j \arrow[r] & Y_k \end{tikzcd}.$$

Lemma 71. Suppose that $\mathcal{C}$ is locally small. Then $X \in \mathcal{C}$ is compact only if the contravariant Yoneda embedding

$$\tilde{\mathcal{Y}}\colon X \mapsto \mathcal{C}(X, -)$$

commutes with small colimits.

Proof. Suppose that $\tilde{\mathcal{Y}}$ commutes with small filtered colimits. Then there is a bijection

$$\mathcal{C}(X, Y_\infty) \simeq \mathcal{C}(X, \operatorname*{colim}_{i \in I} Y_i) \simeq \operatorname*{colim}_{i \in I} \mathcal{C}(X, Y_i). \tag{4.4}$$

Since each inclusion $Y_i \hookrightarrow Y_j$ gives an inclusion

$$\mathcal{C}(X, Y_i) \hookrightarrow \mathcal{C}(X, Y_j),$$

we can think of $\operatorname{colim}_{i\in I} \mathcal{C}(X, Y_i)$ as some sort of limit (in the analysis-y sense)

$$\text{“}\lim_{i\to\infty} \mathcal{C}(X, Y_i)\text{”}.$$

Given a morphism $X \to Y_\infty$, the bijection Equation 4.4 gives us an element of the colimit $\operatorname{colim}_{i\in I} \mathcal{C}(X, Y_i)$

To be continued… □

Example 72. A set $X \in \mathbf{Set}$ is compact if and only if X is finite.

Suppose X is compact. Consider the poset $\mathcal{P}_{\text{fin}}(X)$ from Example 69. The colimit over this poset is X since every element of X is contained in some finite subset, so any map out of X is completely determined by maps out of its finite subsets which agree on intersections. We can view this as a a diagram in **Set** via the inclusion map.

Now consider the identity map $\mathrm{id}_X \colon X \to \operatorname{colim} X_i$. Since X is compact, there is some set X_i such that the following diagram commutes.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "\mathrm{id}_X"'] & X_i \arrow[d] \\ & X \end{tikzcd}$$

By definition, the set X_i is finite. Since f_i must be injective, X must also be finite.

Now suppose that X is finite. Let

$$\phi\colon I \to \mathbf{Set}$$

with I a filtered poset be a diagram, and consider a map $f\colon X \to \operatorname{colim}\phi$. In set, we have the formula

$$Y_\infty \equiv \operatorname{colim}\phi = \left(\coprod_{i\in I} Y_i\right)/\sim\ .$$

Denote by ι_i the map $Y_i \to Y_\infty$.

For every $x \in X$, there is some (certainly not unique!) $i_x \in \mathbb{N}$ such that Y_{i_x} contains an element e such that $\iota_{i_x}(e) = f(x)$. The set

$$\{Y_{i_x}\}_{x\in X}$$

is finite, so it has an upper bound Y_i. Therefore, each of element of x is represented by some element of Y_i, so there is a factorization as follows.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d] \\ & Y_\infty \end{tikzcd}$$

Now suppose that we have maps $f_i \colon X \to Y_i$ and $f_j \colon X \to Y_j$ making the following diagrams commute.

$$\begin{tikzcd} X \arrow[r, "f_i"] \arrow[dr, "f"'] & Y_i \arrow[d] \\ & Y_\infty \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "f_j"] \arrow[dr, "f"'] & Y_j \arrow[d] \\ & Y_\infty \end{tikzcd}$$

Example 73. In $\mathbf{Set}_\Delta$, the compact objects are precisely those simplicial sets with finitely many nondegenerate simplices.

4.3 The Small Object Argument

The small object argument is the following statement.

Lemma 74. Let $\mathcal{C}$ be a locally small category with small colimits. Let $\mathcal{M}$ be a small set of morphisms in $\mathcal{C}$ such that for each $i \colon A \to B \in \mathcal{M}$, the domain A of i is compact. Then every morphism $f : X \to Y$ in $\mathcal{C}$ admits a factorization

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \\ & Z \arrow[ur, "g"'] & \end{tikzcd}$$

with $h \in \overline{\mathcal{M}}$ and $g \in \mathcal{M}_\perp$.

Proof. Let $f : X \to Y$ be a morphism in $\mathcal{C}$. Any commutative diagram of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

is certainly more than specified by the data of the objects A and B, the morphsim $i \colon A \to B$, the map $A_j \to X$, and the map $B \to Y$. Therefore, the collection of all such commuting diagrams is indexed by a small set.[1] Denote this set by J. We can write the set of all commuting squares as follows.

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] & X \arrow[d, "f"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}_{j \in J}$$

Form the coproducts

$$A = \coprod_{j \in J} A_j \qquad \text{and} \qquad B = \coprod_{j \in J} B_j.$$

By the universal property for coproducts, specifying a map out of a coproduct is the same as specifying a map from each summand. Taking $\iota_{B_j} \circ i_j : A \to B_j$ gives us a map $\amalg_j i_j : A \to B$.

[1] I believe this, but I don't understand it.

Note that the following diagram commutes.

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X \arrow[d, "f"] \\ \coprod_{j \in J} B_j \arrow[r] & Y \end{tikzcd}$$

No consider the pushout.

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X \arrow[d, "h_1"'] \arrow[ddr, bend left, "f"] & \\ \coprod_{j \in J} B_j \arrow[r] \arrow[drr, bend right] & X_1 \arrow[dr, "g_1"] & \\ & & Y \end{tikzcd}$$

Note that this gives us a factorization $f = g_1 \circ h_1$, but it's not the one we want: while we have that $h_1 \in \overline{\mathcal{M}}$ since it's the pushout of something in $\overline{\mathcal{M}}$, we do not yet have that $g_1 \in \mathcal{M}_\perp$. We need to keep looking.

We do this whole process again, considering the set of diagrams

$$\left\{ \begin{tikzcd} A_j \arrow[r] \arrow[d, "i_j"'] & X_1 \arrow[d, "g_1"] \\ B_j \arrow[r] & Y \end{tikzcd} \right\}$$

Take the pushout as before.

$$\begin{tikzcd} \coprod_{j \in J} A_j \arrow[r] \arrow[d, "\amalg_j i_j"'] & X_1 \arrow[d, "h_2"'] \arrow[ddr, bend left, "g_1"] & \\ \coprod_{j \in J} B_j \arrow[r] \arrow[drr, bend right] & X_2 \arrow[dr, "g_2"] & \\ & & Y \end{tikzcd}$$

Repeating this process, we get a chain of morphisms $h_i \colon X_{i-1} \to X_i$, each of which is in $\overline{\mathcal{M}}$.

$$X \xrightarrow{h_1} X_1 \xrightarrow{h_2} X_2 \xrightarrow{h_3} X_3 \cdots\cdots\cdots X_\infty$$

We can take the countable composition without leaving $\overline{\mathcal{M}}$, giving us a map $h \colon X \to X_\infty$.

For each X_i, we also constructed a map $g_i \colon X_i \to Y$. The universal property for colimits allows us to turn this into a map $g \colon X_\infty \to Y$. It turns out that this map is in $\mathcal{M}_\perp$!

In order to see that, consider the following lifting problem, with $i \in \mathcal{M}$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X_\infty \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

By assumption, Y is compact, so any map $Y \to X_\infty$ factors through some X_j.

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr] \arrow[dd, "i"'] & & X_\infty \arrow[dd, "g"] \\
& X_j \arrow[ur] \arrow[dr] & \\
B \arrow[rr] & & Y
\end{tikzcd} \tag{4.5}$$

The lower-left square should look familiar.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_j \arrow[d, "g_j"] \\
B \arrow[r] & Y
\end{tikzcd}$$

This is one of the diagrams that we took the coproduct of in the jth step above. The pushout gave us the following commuting diagram.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "i"'] & X_j \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & X_{j+1} \arrow[dr] & \\
& & Y
\end{tikzcd}$$

Nestling this diagram into Equation 4.5, we find

$$\begin{tikzcd}
A \arrow[rrr] \arrow[dr] \arrow[ddd, "i"'] & & & X_\infty \arrow[ddd, "g"] \\
& X_j \arrow[urr] \arrow[dr] & & \\
& & X_{j+1} \arrow[uur] \arrow[dr] & \\
B \arrow[rrr] \arrow[urr] & & & Y
\end{tikzcd}$$

This gives us a map $B \to X_{j+1} \to X_\infty$ which makes the diagram commute. We have solved the lifting problem! $\square$

The main use of the small object argument is to prove the following.

Corollary 75. Let $\mathcal{C}$ be a locally small category with small colimits, and let $\mathcal{M}$ be a small set of morphisms in $\mathcal{C}$ such that for each $i\colon A \to B \in \mathcal{M}$, the domain A of i is compact as in Lemma 74. Then

$$\overline{\mathcal{M}} = {}_\perp(\mathcal{M}_\perp).$$

Proof. We have $\overline{\mathcal{M}} \subseteq {}_\perp(\mathcal{M}_\perp)$ by Corollary 67.

Consider some morphism $f \in {}_\perp(\mathcal{M}_\perp)$. Then by Lemma 74, f has a factorization

$$f = g \circ h,$$

with $g \in \mathcal{M}_\perp$ and $h \in \overline{\mathcal{M}}$. We can write this as a commuting square.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "{}_\perp(\mathcal{M}_\perp)\ni f"'] & Z \arrow[d, "g\in\mathcal{M}_\perp"] \\ Y \arrow[r, "\mathrm{id}_Y"'] & Y \end{tikzcd}$$

Because f has the left lifting property with respect to g, we get a morphism $r\colon Y \to Z$ making the diagram commute.

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\ Y \arrow[r, "\mathrm{id}_Y"'] \arrow[ur, dashed, "r"] & Y \end{tikzcd}$$

Unfolding this, we find the following diagram.

$$\begin{tikzcd} X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "f"'] \arrow[rr, bend left, "\mathrm{id}_X"] & X \arrow[r, "\mathrm{id}_X"'] \arrow[d, "h"'] & X \arrow[d, "f"'] \\ Y \arrow[r, "r"] \arrow[rr, bend right, "\mathrm{id}_Y"'] & Z \arrow[r, "g"] & Y \end{tikzcd}$$

This says precisely that f is a retract of h, so $f \in \overline{\mathcal{M}}$. □

4.4 Anodyne morphisms

Definition 76 (anodyne morphism)**.** A morphism of simplicial sets is called anodyne if it has the left lefting property with respect to all Kan fibrations (Definition 57).

Example 77. By definition, horn fillings $\Lambda^n_i \hookrightarrow \Delta^n$ are anodyne.

Lemma 78. The set of anodyne morphisms is the saturated hull of the set of horn inclusions.

Proof. We have the following string of equalities.

$$\begin{aligned} {}_\perp\{\text{Kan fibrations}\} &= {}_\perp(\{\text{horn fillings}\}_\perp) && \text{(By Example 65)} \\ &= \overline{\{\text{horn fillings}\}} && \text{(By Corollary 67)} \end{aligned}$$

where the last equality holds because horns have finitely many non-degenerate simplices, and hence are compact by Example 73. □

Corollary 79. The set of all anodyne morphisms is saturated.

Definition 80 (smash product)**.** Let $f : A \to B$ and $g : A' \to B'$ be morphisms of simplicial sets. Consider the following diagram.

$$\begin{tikzcd}
A \times B \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d, "\mathrm{id}\times g"] \\
A \times B' \arrow[r, "f\times\mathrm{id}"'] & A' \times B'
\end{tikzcd}$$

Taking the pushout, we find a morphism

$$f \wedge g : A' \times B \coprod_{A\times B} A \times B' \to A' \times B',$$

called the <u>smash product</u> of f and g.

$$\begin{tikzcd}
A \times B \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}\times g"] & \\
A \times B' \arrow[r, hook] \arrow[drr, bend right, "f\times\mathrm{id}"'] & A' \times B \coprod_{A\times B} A \times B' \arrow[dr, dashed, "f\wedge g"] & \\
& & A' \times B'
\end{tikzcd}$$

Lemma 81. Let f, g, and h be morphisms of simplicial sets. We have the following.

1. If f and g are monic, then $f \wedge g$ is monic.
2. There is a natural isomorphism $(f \wedge g) \wedge h \simeq f \wedge (g \wedge h)$

Proof.

1. The pushout square defining the smash product takes place in $\mathbf{Set}_\Delta$. By Corollary 38, it suffices to check that each component is monic. Therefore, consider the defining diagram now as a map of sets.

$$\begin{tikzcd}
A \times B \arrow[r, "f\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times g"'] & A' \times B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}\times g"] & \\
A \times B' \arrow[r, hook] \arrow[drr, bend right, "f\times\mathrm{id}"'] & A' \times B \coprod_{A\times B} A \times B' \arrow[dr, dashed, "f\wedge g"] & \\
& & A' \times B'
\end{tikzcd}$$

In the context of sets, we are allowed to think of f and g as including $A \subset A'$ and $B \subset B'$ respectively. The pushout $A' \times B \coprod_{A\times B} A \times B'$ is then simply the union

$$(A' \times B) \cup (A \times B') \subset A' \times B'.$$

The wedge product is simply the above inclusion, which is manifestly injective.

2. Draw a cube.

□

We will need this horrible lemma several times.

Lemma 82. The following lifting problems are equivalent.

$$
\begin{tikzcd}
A' \times B \coprod_{A \times B} A \times B' \arrow[r] \arrow[d] & X \arrow[d] \\
A' \times B' \arrow[r] \arrow[ur, dashed] & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r] \arrow[d] & \mathrm{Maps}(A', X) \arrow[d] \\
B' \arrow[r] \arrow[ur, dashed] & \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y)
\end{tikzcd}
$$

Proof. First, consider the first. By expanding the pushout into its cocone and replacing maps out of it with maps out of the cocone, we find the following diagram (where we have now labelled maps).

$$
\begin{tikzcd}
A \times B \arrow[r, "\alpha \times \mathrm{id}_B"] \arrow[d, "\mathrm{id}_A \times \beta"'] & A' \times B \arrow[dr, "a"] \arrow[d, "\mathrm{id}_{A'} \times \beta"'] & \\
A \times B' \arrow[rr, "b"'] \arrow[dr, "\alpha \times \mathrm{id}_B"'] & & X \arrow[d, "c"] \\
& A' \times B' \arrow[r, "d"'] \arrow[ur, dashed, "\ell"] & Y
\end{tikzcd}
$$

Re-arranging to avoid overlaps, we have

$$
\begin{tikzcd}
& A \times B \arrow[dl, "\alpha \times \mathrm{id}_B"'] \arrow[dr, "\mathrm{id}_A \times \beta"] & \\
A' \times B \arrow[dr, "\mathrm{id}_{A'} \times \beta"] \arrow[ddr, bend right, "a"'] & & A \times B' \arrow[dl, "\alpha \times \mathrm{id}_{B'}"'] \arrow[ddl, bend left, "b"] \\
& A' \times B' \arrow[d, "d"] & \\
& Y & \\
& X \arrow[u, "c"] \arrow[uu, dashed, bend right, "\ell"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& B' \arrow[ddl, bend right, "\tilde{b}"'] \arrow[ddr, bend left, "\tilde{d}"] \arrow[dd, dashed, bend left, "\tilde{\ell}"] & \\
& B \arrow[u, "\beta"] \arrow[d, "\tilde{a}"] & \\
& \mathrm{Maps}(A', X) \arrow[dl, "{[\alpha, \mathrm{id}_X]}"'] \arrow[dr, "{[\mathrm{id}_{A'}, c]}"] & \\
\mathrm{Maps}(A, X) \arrow[dr, "{[\mathrm{id}_A, c]}"'] & & \mathrm{Maps}(A', Y) \arrow[dl, "{[\alpha, \mathrm{id}_Y]}"] \\
& \mathrm{Maps}(A, Y) &
\end{tikzcd}
$$

The top square commutes trivially.

This diagram is made of four commuting squares.

1. $a \circ (\alpha \times \mathrm{id}_B) = c \circ (\mathrm{id}_A \times \beta)$
2. $b \circ c = d \circ (\alpha \times \mathrm{id}_{B'})$
3. $b \circ a = d \circ (\mathrm{id}_{A'} \times \beta)$

Taking adjuncts, define

$$\tilde{a} \colon B \to [A', X], \qquad \tilde{c} \colon B' \to [A, X], \qquad \tilde{d} \colon B' \to [A', X].$$

Similarly, expanding the product in the second diagram into its cocone ◻

Lemma 83. Fix some morphism $i\colon A \to A'$ in $\mathbf{Set}_\Delta$. The set

$$\mathcal{S} = \{f \mid i \wedge f \text{ is anodyne}\}$$

is saturated.

Proof. Let $f\colon B \to B'$. By definition, $i \wedge f$ is anodyne if and only if every lifting problem

$$\begin{tikzcd} A' \times B \coprod_{A\times B} A \times B' \arrow[r] \arrow[d, "i\wedge f"'] & X \arrow[d, "g"] \\ A' \times B' \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where g is a Kan fibration has a solution. However, by Lemma 82, we may equivalently say that $i \wedge f$ is anodyne if and only if every lifting problem of the following form has a solution.

$$\begin{tikzcd} B \arrow[r] \arrow[d, "f"'] & \mathrm{Maps}(A', X) \arrow[d] \\ B' \arrow[r] \arrow[ur, dashed] & \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y) \end{tikzcd}$$

Defining

$$\mathcal{R} = \{\mathrm{Maps}(A', X) \to \mathrm{Maps}(A, X) \underset{\mathrm{Maps}(A,Y)}{\times} \mathrm{Maps}(A', Y) \mid X \to Y \text{ Kan fibration}\},$$

we see that

$$\{f \mid f \wedge i \text{ anodyne}\} = {}_\perp\mathcal{R}.$$

Thus by Theorem 66 $\{f \mid f \wedge i \text{ anodyne}\}$ is saturated. ◻

Theorem 84. Consider the following sets of morphisms.

$$\mathcal{M}_1 = \left\{\Lambda^n_i \to \Delta^n \mid n > 0,\ 0 \le i \le n\right\}$$

$$\mathcal{M}_2 = \left\{ \{e\} \times \Delta^n \coprod_{\{e\}\times\partial\Delta^n} \Delta^1 \times \partial\Delta^n \to \Delta^1 \times \Delta^n \;\middle|\; n \ge 0,\ e = 0,\ 1 \right\}$$

$$\mathcal{M}_3 = \left\{ \{e\} \times S \coprod_{\{e\}\times K} \Delta^1 \times K \to \Delta^1 \times S \;\middle|\; K \to S \text{ monic},\ e = 0,\ 1 \right\}$$

Each of these sets generate the set of all anodyne morphisms in the sense that

$$\overline{\mathcal{M}_1} = \overline{\mathcal{M}_2} = \overline{\mathcal{M}_3} = \{\text{anodyne morphisms}\}.$$

Sketch of proof. We will show that

$$\mathcal{M}_2 \subset \overline{\mathcal{M}_1}, \qquad \mathcal{M}_1 \subset \overline{\mathcal{M}_3}, \qquad \text{and} \qquad \overline{\mathcal{M}_2} = \overline{\mathcal{M}_3}.$$

This will imply that

$$\overline{\mathcal{M}_2} \subset \overline{\mathcal{M}_1} \subset \overline{\mathcal{M}_3} \subset \overline{\mathcal{M}_2}.$$

◻

This theorem gives us

Corollary 85. Let f be anodyne, and let $g\colon X \to Y$ be a monomorphism. Then $f \wedge g$ is anodyne.

Proof. □

Example 86. Any horn filling $\Lambda^n_i \hookrightarrow \Delta^n$ is anodyne, and boundary fillings $\partial\Delta^m \to \Delta^m$ are mono, so pushouts

$$\partial\Delta^m \times \Delta^n \coprod_{\partial\Delta^m \times \Lambda^n_i} \Delta^m \times \Lambda^n_i \to \Delta^m \times \Delta^n$$

are anodyne.

4.5 A technical lemma of great importance

Lemma 87. Let $i\colon A \to B$ be a monomorphism, and let $p\colon K \to S$ be a Kan fibration (Definition 57). Consider the following pushout.

$$\begin{tikzcd}
\mathrm{Maps}(B,K) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& \mathrm{Maps}(A,K) \times_{\mathrm{Maps}(A,S)} \mathrm{Maps}(B,S) \arrow[r] \arrow[d] & \mathrm{Maps}(A,K) \arrow[d] \\
& \mathrm{Maps}(B,S) \arrow[r] & \mathrm{Maps}(A,S)
\end{tikzcd}$$

The dashed arrow is a Kan fibration.

Proof. We need to show that the dashed arrow solves right lifting problems of the following form,

$$\begin{tikzcd}
A' \arrow[r] \arrow[d, "j"'] & \mathrm{Maps}(B,K) \arrow[d, dashed] \\
B' \arrow[r] \arrow[ur, dotted] & \mathrm{Maps}(A,K) \times_{\mathrm{Maps}(B,K)} \mathrm{Maps}(B,S)
\end{tikzcd}$$

with $j\colon A' \to B'$ a horn filling. It will actually be easier to solve this problem in the generality where j is any anodyne morphism; this will certainly suffice because by Theorem 84, all horn fillings are anodyne.

By Lemma 82, this is equivalent to the following lifting problem.

$$\begin{tikzcd}
A' \times B \coprod_{A' \times A} B' \times A \arrow[r] \arrow[d] & K \arrow[d, "p"] \\
B' \times B \arrow[r] \arrow[ur, dashed] & S
\end{tikzcd}$$

□

Corollary 88. Let K be a Kan complex, and let B be a simplicial set. Then the simplicial set $\mathrm{Maps}(B, K)$ is a Kan complex.

Proof. Take $A = \emptyset$ and $S = *$. Then we have that the map

$$\mathrm{Maps}(B, K) \longrightarrow \mathrm{Maps}(\emptyset, K) \underset{\mathrm{Maps}(\emptyset, *)}{\times} \mathrm{Maps}(B, *) = *$$

is a Kan fibration. But by Corollary 58, this means that $\mathrm{Maps}(B, K)$ is a Kan complex. □

Obj(C)

Corollary 89. For $i \colon A \to B$ a monomorphism of simplicial sets and K a Kan complex, the map

$$\mathrm{Maps}(i, \mathrm{id}_K) \colon \mathrm{Maps}(B, K) \longrightarrow \mathrm{Maps}(A, K)$$

is a Kan fibration.

Proof. Setting $S = *$, we find that the map

$$\mathrm{Maps}(B, K) \longrightarrow \mathrm{Maps}(A, K) \underset{\mathrm{Maps}(A, *)}{\times} \mathrm{Maps}(B, *) = \mathrm{Maps}(A, K)$$

is a Kan fibration, which is what we are after. □

5 Simplicial homotopy

5.1 Homotopy of vertices

Definition 90 (homotopy of vertices)**.** Let S be a simplicial set. We say that $x, y \in K_0$ are homotopic if there is an edge $f : x \to y$.

Theorem 91. If $S = K$ is a Kan complex, the above notion is an equivalence relation.

Proof. We have that x is homotopic to x because of the edge $s_0 x$.

Given $f : x \to y$ and $g : y \to z$, we can fill the following horn.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed] & & z \end{tikzcd}$$

This gives a homotopy from x to z.

Similarly, suppose we have a homotopy $x \to y$. Then filling the horn

$$\begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "s_0 x"'] & & x \end{tikzcd}$$

gives us a homotopy $y \to x$. □

Thus, we can think of partitioning a Kan complex into disconnected components.

5.2 Homotopy of edges

We will often want to consider edges in a simplicial set up to some notion of homotopy.

Definition 92 (homotopy of edges)**.** Let $f : x \to y$ and $g : x \to y$ be edges in some simplicial set K. We say that f and g are homotopic if there exists a 2-simplex of the following form.

$$\begin{tikzcd} & w \arrow[dr, "s_0 w"] & \\ v \arrow[ur, "f"] \arrow[rr, "g"'] & & w \end{tikzcd}$$

Unfortunately, this is in general not an equivalence relation since it is not symmetric. It becomes one if we restrict our attention to Kan complexes. However, it turns out we will not have to make any use of outer horn fillers. This will be important later on.

Theorem 93. For any set with inner horn fillers, that is fillers $\Lambda_i^n \to \Delta^n$ for $0 < i < n$, homotopy of edges is an equivalence relation.

In particular, it is an equivalence for any Kan complex K.

Proof. First, we show reflexivity. For any edge f, the simplicial identities (Theorem 33) ensure that the degenerate simplex $\sigma_f = s_1 f$ satisfies

$$d_1\sigma_f = f, \qquad d_2\sigma_f = f, \qquad \text{and} \qquad d_0\sigma_f = s_0 d_0 f.$$

To see that it is symmetric, create from a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

the 1-skeleton of a 2-simplex whose front and back are as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "f"'] & y \end{tikzcd}$$

This gives us the boundaries of four 1-simplices. Each of these can be filled except for the bottom simplex of the first diagram. Doing so gives us a 1-horn, which we can fill to find a 2-simplex as follows.

$$\sigma' = \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This says precisely that $g \sim f$.

To check transitivity, suppose we are given two simplices as follows.

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & y \end{tikzcd}$$

We can create a horn $\Lambda_2^3 \to X$ as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "h"'] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "h"'] & y \end{tikzcd}$$

Filling this gives what we want. □

Theorem 94. Two edges f and g in a Kan complex K are homotopic in the above sense if and only if there exists a square of the following form.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 v"'] \arrow[dr] & w \arrow[d, "s_0 w"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Proof. Suppose we have a 2-simplex exhibiting f and g as homotopic. We can form a square by attaching the 2-simplex $s_0 g$ as to the bottom.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 v"'] \arrow[dr, "g"] & w \arrow[d, "s_0 w"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Given a square as above, call the bottom 2-simplex η and the upper simplex μ. Create the horn

$$\begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] \arrow[dr, "d_1\eta"] & v \arrow[d, "g"] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] & v \arrow[d, "g"] \arrow[dl, "g"'] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd}$$

where the top left simplex is η, the top right is $s_0 g$, the bottom right is $s_1 g$, and the bottom left is unfilled. Filling it, we find that $g \sim d_1\eta$. But μ exhibits $f \sim d_1\eta$, so $f \sim g$. □

5.3 Homotopy of maps

Definition 95 (homotopy)**.** Let $f, g\colon B \to K$ be simplicial sets. A homotopy from f to g is a morphism

$$H\colon \Delta^1 \times B \to K$$

such that the restriction of H to $\{0\} \times B$ is f, and the restriction of H to $\{1\} \times B$ is g.

In this case, we will write

$$f \overset{H}{\simeq} g$$

To put it another way, H is a homotopy above if the pullback along the inclusion $B \to \Delta^1 \times B$ sending $\sigma \mapsto (0, \sigma)$ is equal to f, and similarly for g.

$$\begin{tikzcd} \Delta^0 \times B \arrow[d, hook, "\{0\}\times\mathrm{id}"'] \arrow[dr, "f"] & \\ \Delta^1 \times B \arrow[r, "H"] & K \\ \Delta^0 \times B \arrow[u, hook, "\{1\}\times\mathrm{id}"] \arrow[ur, "g"'] & \end{tikzcd}$$

Note that chasing elements around, this immediately implies that, for all σ,

$$H(0, \sigma) = f(\sigma) \qquad \text{and} \qquad H(1, \sigma) = g(\sigma).$$

Definition 96 (relative homotopy)**.** Suppose we are further given a monomorphism $i\colon A \to B$ making the following diagram commute.

$$\begin{tikzcd} & B \arrow[dr, "f"] & \\ A \arrow[ur, "i"] \arrow[dr, "i"'] & & K \\ & B \arrow[ur, "g"'] & \end{tikzcd}$$

Denote $f \circ i = g \circ i = u$. Then we say that H is a homotopy relative to A if the diagram

$$\begin{tikzcd} \Delta^1 \times B \arrow[r, "H"] & K \\ \Delta^1 \times A \arrow[u, "\mathrm{id}\times i"] \arrow[r, "\pi_A"'] & A \arrow[u, "u"'] \end{tikzcd}$$

commutes. We will write

$$f \overset{H}{\simeq} g \quad \text{(rel. } A\text{)}.$$

The intuitive meaning of this diagram is that H leaves $A \subset B$ fixed. Indeed, again chasing elements around, we find that

$$H(e, \sigma) = f(\sigma) = g(\sigma)$$

for all $\sigma \in A$ and $e = 0, 1$.

We would like to use homotopy to define an equivalence relation on maps of simplicial sets. However, in a general simplicial set this is impossible. However, we saw that even in the case that f and g are inclusions of a vertex, this is impossible without control of outer horns. Consider the maps

$$\iota_0,\ \iota_1\colon \Delta^0 \to \Delta^1.$$

which send the point to the zeroth and first vertices. The morphisms ι_0 and ι_1 satisfy

$$\iota_0 \simeq \iota_1,$$

since $H = \mathrm{id}_{\Delta^1}$ makes the following diagram commute.

$$\begin{tikzcd} \Delta^0 \times \Delta^0 = \Delta^0 \arrow[d, "d^1\times\mathrm{id}"'] \arrow[dr, "\iota_0"] & \\ \Delta^1 \times \Delta^0 = \Delta^1 \arrow[r, "H=\mathrm{id}_{\Delta^1}"] & \Delta^1 \\ \Delta^0 \times \Delta^0 = \Delta^0 \arrow[u, "d^0\times\mathrm{id}"] \arrow[ur, "\iota_1"'] & \end{tikzcd}$$

However, there is no morphism $H\colon \Delta^1 \to \Delta^1$ which sends $0 \mapsto 1$ and $1 \mapsto 0$, so there does not exist any H such that

$$\iota_1 \overset{H}{\simeq} \iota_0.$$

The way out of this is to specialize to Kan complexes.

Proposition 97. Let $i \colon A \to B$ be a monomorphism of simplicial sets, let K be a Kan complex, and let $u \colon A \to K$ be a morphism. Then the relation *is homotopic relative to A* defines an equivalence relation on the set

$$\{f : B \to K \mid f \circ i = u\}.$$

Proof. By Note 42, we can think of u as a map $\Delta^0 \to \mathrm{Maps}(A, K)$ which picks out u as a zero-simplex.

Consider the pullback square

$$\begin{tikzcd} \mathrm{Maps}(B,K)^u \arrow[r] \arrow[d] & \mathrm{Maps}(B,K) \arrow[d, "\mathrm{Maps}(i,\mathrm{id}_K)"] \\ \Delta^0 \arrow[r, "u"'] & \mathrm{Maps}(A,K) \end{tikzcd}$$

By Corollary 89, the map $\mathrm{Maps}(i, \mathrm{id}_K)$ is a Kan fibration. Thus, since Kan fibrations are saturated (hence closed under pullback), we have that the induced map $\mathrm{Maps}(B, K)^u \to \{u\}$ is also a Kan fibration. But by Corollary 58, this means that $\mathrm{Maps}(B, K)^u$ is a Kan complex.

The elements of $\mathrm{Maps}(B, K)^u_0$ are ordered pairs $(u, f) \in \{u\} \times \mathbf{Set}_\Delta(B, K)$ such that $u = i \circ f$. That is, $\mathrm{Maps}(B, K)^u_0$ consists of those morphisms $B \to K$ which agree with u when restricted to A along i.

Thus, showing that homotopy between maps $B \to K$ relative to A is an equivalence relation is the same as showing homotopy between vertices of the Kan complex $\mathrm{Maps}(B, K)^u$ is an equivalence relation, which is true by Theorem 93. □

5.4 Homotopy groups

Definition 98 (homotopy group)**.** Let K be a Kan complex. Define the zeroth homotopy group of K, denoted $\pi_0(K)$, to be the set of homotopy classes of vertices of K. This is well-defined by Theorem 91.

For $n \geq 1$, define the nth homotopy group of K at v, denoted $\pi_n(K, v)$, to be the set of homotopy classes of n-simplices $\alpha \colon \Delta^n \to K$ relative to the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, such that the following diagram commutes.

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha"] & X \\ \partial\Delta^n \arrow[u, hook] \arrow[r] & \Delta^0 \arrow[u, hook, "v"'] \end{tikzcd}$$

This diagram says precisely that the boundary of Δ^n is collapsed to the point v.

Definition 99. Let $\alpha, \beta \colon \Delta^n \to K$ be n-simplices whose boundaries are trivial. Denote their equivalence classes by $[\alpha]$ and $[\beta]$ respective. Define their composition, denoted $[\alpha] \cdot [\beta]$ as follows. First, define an inner $(n + 1)$-horn with faces

$$(\overbrace{\{v\}, \cdots, \{v\}}^{n-1}, \alpha, -, \beta).$$

This has a filler σ such that

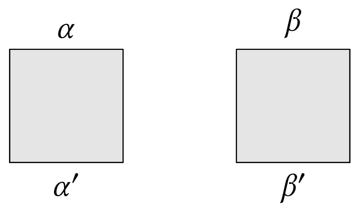
$$\partial\sigma = (\{v\}, \cdots, \{v\}, \alpha, d_n\sigma, \beta)$$

Define $[\alpha] \cdot [\beta] = [d_n\sigma]$.

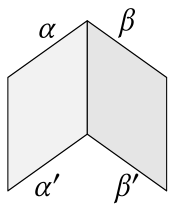
It remains to check that this composition is well-defined.

Proposition 100. The composition defined above is well-defined, i.e. if $\alpha' \overset{H}{\simeq} \alpha$ and $\beta' \overset{H'}{\simeq} \beta$, then $[\alpha'] \cdot [\beta'] = [\alpha] \cdot [\beta]$.

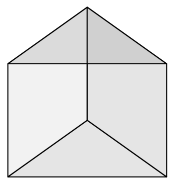
Sketch of proof. We have homotopies between the n-simplices we want to multiply,



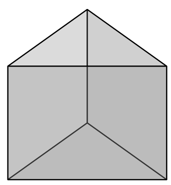
which we extend to a homotopy between the horns defining our multiplication law.



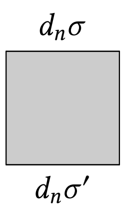
These horns have fillers, giving us the result of our multiplication.



Using the general theory of anodyne morphisms, we can extend our homotopy between the horns to a homotopy between the full filled simplices.



Restricting to the filled faces gives us a homotopy between the result of multiplication.



□

Proof. By definition, we have an $(n+1)$-simplices σ and σ' such that

$$\partial\sigma = (\{v\}, \ldots, \{v\}, \alpha, d_n\sigma, \beta)$$

and

$$\partial\sigma' = (\{v\}, \ldots, \{v\}, \alpha', d_n\sigma', \beta').$$

We need to show that $d_n\sigma \simeq d_n\sigma'$.

First, we construct a homotopy between the horns

$$\eta = (\{v\}, \ldots, \{v\}, \alpha, -, \beta) \qquad \text{and} \qquad \eta' = (\{v\}, \ldots, \{v\}, \alpha', -, \beta').$$

By the distributive law for products and coproducts

$$A \times (B \amalg C) \simeq (A \times B) \amalg (A \times C),$$

a homotopy between horns $\Lambda^{n+1}_i \to K$ consists of a homotopy for each face. Therefore, we can choose our homotopy H'' to have components $(\mathrm{id}_v, \ldots, \mathrm{id}_v, H, -, H')$. This gives us a map $\Delta^1 \times \Lambda^{n+1}_n \to K$. Our fillers give us two maps $\sigma, \sigma' \colon \Delta^{n+1} \to K$, which we can view as a map $\partial\Delta^1 \times \Delta^{n+1} \to K$. This gives us the following commutative diagram

$$\begin{tikzcd}
\partial\Delta^1 \times \Lambda^{n+1}_n \arrow[r, hook] \arrow[d, hook] & \partial\Delta^1 \times \Delta^{n+1} \arrow[d, "{(\sigma,\sigma')}"] \\
\Delta^1 \times \Lambda^{n+1}_n \arrow[r, "H''"] & K
\end{tikzcd}$$

This gives us a map

$$h \colon \Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \to K.$$

Note that by Example 86, we get an anodyne extension

$$\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^n.$$

This means that (by Corollary 58) we can find a lift for the following diagram.

$$\begin{tikzcd}
\Delta^1 \times \Lambda^{n+1}_n \amalg_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1}. \arrow[r, "h"] \arrow[d, hook] & K \\
\Delta^1 \times \Delta^{n+1} \arrow[ur, dashed, "\exists I"'] &
\end{tikzcd}$$

Restricting I to the nth face of Δ^{n+1} gives us a homotopy

$$\Delta^1 \times \Delta^n \to K$$

as required. $\square$

Theorem 101. Let K be a Kan complex, and let $v \in K_0$ be a vertex of K. Then the multiplication from Definition 99 makes $\pi_n(K, v)$ into a group for $n \geq 1$.

Proof. First we check associativity. Let $\alpha, \beta, \gamma \colon \Delta^n \to K$ be n-simplices with trivial boundary, representing elements $[\alpha]$, $[\beta]$, $[\gamma] \in \pi_n(K, v)$. By definition, we get an $(n+1)$-simplex σ_{n-1} such that

$$\partial\sigma_{n-1} = (v, v, \ldots, v, \alpha, d_n\sigma_{n-1}, \beta).$$

giving us the result of the multiplication $[\alpha] \cdot [\beta] = [d_n\sigma_{n-1}]$.

Similarly, we have some σ_{n+2} such that

$$\partial\sigma_{n+2} = (v, v, \ldots, v, \beta, d_n\sigma_{n+2}, \gamma)$$

so $[\beta] \cdot [\gamma] = [d_n\sigma_{n+2}]$.

Form from these a simplex σ_{n+1} with boundary

$$\partial\sigma_{n+1} = (v, v, \ldots, v, d_n\sigma_{n-1}, d_n\sigma_{n+1}, \gamma)$$

giving us the result of the multiplication $([\alpha] \cdot [\beta]) \cdot [\gamma] = [d_n\sigma_{n-1}] \cdot [\gamma] = [d_n\sigma_{n+1}]$.

These simplices fit together to give a horn

$$\Lambda^{n+2}_n \to K$$

with boundary

$$(v, v, \ldots, v, \sigma_{n-1}, -, \sigma_{n+1}, \sigma_{n+2}).$$

Filling this gives an $(n+1)$-simplex σ with boundary

$$\partial\sigma = (v, v, \ldots, v, \sigma_{n-1}, d_n\sigma, \sigma_{n+1}, \sigma_{n+2}).$$

Using the simplicial identities, we find

$$\begin{aligned}
\partial d_n\sigma &= (d_0d_n\sigma, d_1d_n\sigma, \ldots, d_{n-2}d_n\sigma, d_{n-1}d_n\sigma, d_nd_n\sigma, d_{n+1}d_n\sigma) \\
&= (d_{n-1}d_0\sigma, d_{n-1}d_1\sigma, \ldots, d_{n-1}d_{n-2}\sigma, d_{n-1}d_{n-1}\sigma, d_nd_{n+1}\sigma, d_nd_{n+2}\sigma) \\
&= (v, v, \ldots, v, \alpha, d_n\sigma_{n+1}, d_n\sigma_{n+2}).
\end{aligned}$$

This is an n-simplex exhibiting $[\alpha] \cdot [d_n\sigma_{n-2}] = [d_n\sigma_{n+1}]$.

Thus, we have

$$\begin{aligned}
([\alpha] \cdot [\beta]) \cdot [\gamma] &= [d_n\sigma_{n-1}] \cdot [\gamma] \\
&= [d_n\sigma_{n+1}] \\
&= [\alpha] \cdot [d_n\sigma_{n+2}] \\
&= [\alpha] \cdot ([\beta] \cdot [\gamma]).
\end{aligned}$$

The degenerate n-simplex v functions as a unit on the left because there exists an $(n+1)$-horn $s_n\omega$ with

$$\begin{aligned}
\partial(s_n\omega) &= (d_0s_n\omega, \ldots, d_{n-1}s_n\omega, d_ns_n\omega, d_{n+1}s_n\omega) \\
&= (s_{n-1}d_0\omega, \ldots, s_{n-1}d_{n-1}\omega, \omega, \omega) \\
&= (v, \ldots, v, v, \omega, \omega),
\end{aligned}$$

so $[e] \cdot [\omega] = [\omega]$.

It functions as a unit on the right because of $s_{n-2}\omega$.

Furthermore. □

We would like $\pi_0(K)$ and $\pi_n(K, v)$ to be the simplicial analogs of the topological notions of homotopy groups. Recall that we can turn any simplicial set into a topological space via the geometric realization functor $|\cdot|$. Therefore, one would hope that the simplicial homotopy groups would agree with the homotopy groups of the geometrical realization. At least for π_o this turns out to be the case.

Theorem 102. For any Kan complex K, we have $\pi_0(K) \simeq \pi_0(K)$ as sets.

Proof. Define a map $K_0 \to |K|$ sending $v \to |v|$.

Suppose $v \sim v'$ in K. Then there is an edge f between v and k. Applying $|\cdot|$, we find an edge $|v| \to |v'|$. Thus, the map above respects equivalence classes, and descends to a map $\pi_0(K) \to \pi_0(|K|)$.

Now some homotopy stuff I don't understand. □

Theorem 103. Let $p\colon K \to S$ be a Kan fibration. Let v be a vertex of K, and let $w = p(v)$. Let F be the fiber over w given by the following pullback equare.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \{w\} \arrow[r, hook] & S \end{tikzcd}$$

There is the following long exact sequence.

$$\cdots \longrightarrow \pi_{n+1}(K, v) \longrightarrow \pi_n(F, v) \longrightarrow \pi_n(K, v) \longrightarrow \pi_n(S, w) \longrightarrow \pi_{n-1}(F, v) \longrightarrow \cdots$$

6 Model categories

6.1 Localization of categories

Definition 104 (localization of a category)**.** Let $\mathcal{C}$ be a category, and $\mathcal{W}$ a set of morphisms of $\mathcal{C}$. The localization of $\mathcal{C}$ at $\mathcal{W}$ is a pair $(\mathcal{C}[\mathcal{W}^{-1}], \pi)$ where $\mathcal{C}([\mathcal{W}^{-1}])$ is a category and $\pi\colon \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ is a functor, such that the following conditions hold.

- For all $w \in \mathcal{W}$, $\pi(w)$ is an isomorphism.
- For any category $\mathcal{A}$ and functor $F\colon \mathcal{C} \to \mathcal{A}$ such that $F(w)$ is an isomorphism for all $w \in \mathcal{W}$, there exists a functor $\tilde{F}\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ and a natural isomorphism $\tilde{F} \circ \pi \Rightarrow F$.

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Swarrow"] & \mathcal{A} \\ \mathcal{C}[\mathcal{W}^{-1}] \arrow[ur, "\exists \tilde{F}"'] & \end{tikzcd}$$

  Note that the above diagram is not required to commute.

- For every category $\mathcal{A}$, the pullback functor

$$\pi^*\colon \mathbf{Cat}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{A}) \to \mathbf{Cat}(\mathcal{C}, \mathcal{A})$$

  is fully faithful.

Note that by the third axiom, the functor $\tilde{F}$ is guaranteed to be unique up to isomorphism; fully faithfulness tells us precisely that functors $\mathcal{C} \to \mathcal{A}$ and functors $\mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{A}$ sending $\mathcal{W}$ to isomorphisms are in bijection up to isomorphism

The idea behind the localization of a category is very similar to that of a ring. Roughly speaking, in localizing a category $\mathcal{C}$ at a set of morphisms $\mathcal{W}$, one adds just enough morphisms to $\mathcal{C}$ so that all the morphisms in $\mathcal{W}$ have inverses. Of course in adding any morphism to a category, one also has to freely add in compositions with other morphisms. For this reason, the localization of a category is in general unwieldy. Furthermore, one runs into size-theoretic issues: even if $\mathcal{C}$ is a small category, one cannot in general be sure that $\mathcal{C}[\mathcal{W}^{-1}]$ is small.

There is an explicit construction of the localization of a category, but it is extremely unweildy to work with. Fortunately, model categories provide a way of computing and working with localizations which is practical and avoids issues of size.

6.2 Model categories

Model categories provide a setting in which one can define a purely category-theoretic form of homotopy theory.

Definition 105 (model category)**.** A model category is a category $\mathcal{C}$ together with three sets of morphisms

- W, the set of *weak equivalences*
- Fib, the set of *fibrations*
- Cof, the set of *cofibrations*

subject to the following axioms.

(M1) The category $\mathcal{C}$ has all small limits and colimits.

(M2) The set W satisfies the so-called *two-out-of-three law:* if the diagram

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "h"'] & & Z \\ & Y \arrow[ur, "f"'] & \end{tikzcd}$$

commutes and two out of three of f, g, and h are weak equivalences, then the third is as well.

(M3) The sets W, Fib, and Cof are all closed under retracts: given the following diagram, if g belongs to any of the above sets, then so does f.

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & X \arrow[d, "f"] \\ Z \arrow[r] & W \arrow[r] & Z \end{tikzcd}$$

(M4) We can solve a lifting problem

$$\begin{tikzcd} U \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ V \arrow[r] & P \end{tikzcd}$$

if all of the following conditions hold.

- The morphism i is a cofibration.
- The morphism p is a fibration.
- At least one of i and p is a weak equivalence.

(M5) Any morphism $f\colon X \to Y$ admits the following factorizations.

- We can write $f = p \circ i$, where p is a fibration and i is both a cofibration and a weak equivalence.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathrm{Cof}\cap\mathrm{W}\ni i"'] & & B \\ & X \arrow[ur, "p\in\mathrm{Fib}"'] & \end{tikzcd}$$

- We can write $f = p \circ i$, where p is both a fibration and a weak equivalence, and i

is a cofibration.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\mathsf{Cof} \ni i"'] & & B \\ & X \arrow[ur, "p \in \mathsf{Fib} \cap \mathsf{W}"'] & \end{tikzcd}$$

To simplify terminology, we will use the following terms.

- We will call morphisms which are both weak equivalences and fibrations *trivial fibrations*:
$$\mathsf{W} \cap \mathsf{Fib} = \{\text{trivial fibrations}\}.$$
- We will call morphisms which are both weak equivalences and cofibrations *trivial cofibrations*:
$$\mathsf{W} \cap \mathsf{Cof} = \{\text{trivial cofibrations}\}.$$
- We will call an object $X \in \mathcal{C}$ *cofibrant* if the unique morphism $\emptyset \to X$ is a cofibration.
- We will call an object $Y \in \mathcal{C}$ *fibrant* if the unique morphism $X \to *$ is a fibration.

Lemma 106 (lifting lemma)**.** Let $\mathcal{C}$ be any category, and let $f = p \circ i$ be a morphism in $\mathcal{C}$.

1. If f has the left lifting property with respect to p, then f is a retract of i.
2. If f has the right lifting property with respect to i then f is a retract of p.

Proof. Suppose f has the left-lifting property with respect to p. Then the following diagram admits a lift.

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "f"'] & X \arrow[d, "p"] \\ B \arrow[r, equal] \arrow[ur, dashed] & B \end{tikzcd}$$

Expanding, we find the following diagram.

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "f"'] & A \arrow[r, equal] \arrow[d, "i"] & A \arrow[d, "f"] \\ B \arrow[r, dashed] \arrow[rr, bend right, "\mathrm{id}_B"'] & X \arrow[r, "p"] & B \end{tikzcd}$$

This exhibits f as a retract of i.

The second case is dual. □

The axioms for a model category are over-specified, in the sense that providing W and either Fib and Cof is sufficient to derive the other one.

Lemma 107. Let $\mathcal{C}$ be a model category. We have the following.

1. $\mathsf{Cof}_\perp = \mathsf{Fib} \cap \mathsf{W}$
2. $\mathsf{Cof} = {}_\perp(\mathsf{Fib} \cap \mathsf{W})$
3. $(\mathsf{Cof} \cap \mathsf{W})_\perp = \mathsf{Fib}$

4. $\mathsf{Cof} \cap \mathsf{W} = {}_\perp\mathsf{Fib}$

Proof. We prove 4. The proof of 1 is mutatis mutandis equivalent.

Let $f \in \mathsf{Cof} \cap \mathsf{W}$. Then by Axiom (M4), $f \in {}_\perp\mathsf{Fib}$.

Now suppose $f \in {}_\perp\mathsf{Fib}$. By Axiom (M5), we can always find a factorization $f = p \circ i$, where p is a fibration and i is a trivial cofibration.

By Axiom (M4), f has the left lifting property with respect to p, so by Lemma 106, f is a retract of i. But since by Axiom (M3) both W and Cof are closed under retracts, f is also a trivial cofibration as needed.

From 4, 3 follows immediately. The proof of 1 is precisely the same as that of 4, and 2 follows from 1 immediately. □

Corollary 108. In any model category, the sets Cof and $\mathsf{Cof} \cap \mathsf{W}$ are saturated (Definition 59).

Proof. By Theorem 66, any set which is of the form ${}_\perp\mathcal{M}$ is saturated. □

Example 109. The following are examples of model structures.

1. The category **Top** carries the so-called *Quillen model structure,* with
    - W = weak homotopy equivalences,
    - Fib = Serre fibrations,
    - $\mathsf{Cof} = {}_\perp(\mathsf{Fib} \cap \mathsf{W})$.
2. The category $\mathbf{Set}_\Delta$ has a model structure called the *Kan model structure* with
    - W = weak homotopy equivalences, i.e. Kan fibrations $X \longrightarrow Y$ such that $\pi_0(X)$ and $\pi_n(X, x)$ for all $n \geq 1$ and $x \in X$.
    - Fib = Kan fibrations,
    - Cof = monomorphisms.

   Note that in this case, $\mathsf{W} \cap \mathsf{Cof} = {}_\perp\mathsf{Fib} = \{\text{anodyne morphisms}\}$.
3. For any small ring R, the category $\mathbf{Ch}(R)$ of chain complexes of small left R-modules has a model structure, called the *projective model structure,* with
    - W = Quasi-isomorphisms
    - Fib = morphisms f of complexes such that f_n is a surjection for all n
    - $\mathsf{Cof} = {}_\perp(\mathsf{Fib} \cap \mathsf{W})$.

Example 110. Let $\mathcal{C}$ be a category with model structure $(\mathsf{W}, \mathsf{Fib}, \mathsf{Cof})$. Then $\mathcal{C}^{\mathrm{op}}$ is a model category with model structure $(\mathsf{W}, \mathsf{Cof}, \mathsf{Fib})$.

6.2.1 A model structure on the category of categories

In general, proving that three sets of morphisms form the fibrations, cofibrations, and weak equivalences of a model structure is in general an immense undertaking. There are a few model structures, however, which are somewhat simple.

Consider the category I with the following objects and morphisms (excluding identity morphisms).

$$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd}$$

Denote the functor including $0 \hookrightarrow I$ by ι.

The following is a model structure on **Cat**.

- W = categorical equivalences
- Fib = *isofibrations*, i.e. functors with the right lifting property with respect to ι
- Cof = functors which are injective on objects

We verify the axioms one by one. This section is incomplete and a bit wrong at the moment. I'll come back to it.

Axiom M1: Limits and colimits

We already know that **Cat** has all limits and colimits.

Axiom M2: Two-out-of-three

We have the following results.

- If a functor is a weak equivalence, then its weak inverse is also a weak equivalence.
- Composition of weak equivalences gives again a weak equivalence (by the horizontal composition formula).

Thus, the same reasoning that shows that the 2/3 law holds for bona-fide isomorphisms also holds for weak equivalences.

Axiom M3: Closure under retracts

First suppose that G is an equivalence of categories. We need to show that F is also an equivalence of categories.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

Since both $S \circ R$ and $U \circ T$ compose to the identity, T and R are injective on objects and morphisms (in the on-the-nose, evil sense), and S and U are similarly surjective. Thus G is:

- **Essentially surjective:** We can map any $b \in \mathcal{B}$ to $\mathcal{D}$ with T, then lift it to something in $\mathcal{C}$ using G's weak inverse, which we'll call G^{-1}. Mapping the result of this to $\mathcal{A}$ with S gives us an object

  $$(S \circ G^{-1} \circ T)(b) \in A.$$

  The claim is that under F this maps to something isomorphic to b. Evaluating on F, we have

  $$F((S \circ G^{-1} \circ T)(b)) = (U \circ G \circ G^{-1} \circ T)(b).$$

We have a natural isomorphism $G \circ G^{-1} \Rightarrow \mathrm{id}$, which tells us that

$$(U \circ G \circ G^{-1} \circ T)(b) \simeq (U \circ T)(b) = b.$$

- **Fully faithful:** The logic is very similar. We can map any morphism f in $\mathcal{B}$ to a morphism in $\mathcal{D}$ with T, and (because G is fully faithful) find a unique preimage in $\mathcal{C}$ which we'll call $G^{-1}(T(f))$. Applying S gives us a morphism

$$(S \circ G^{-1} \circ T)(f) \quad \text{in } \mathcal{A}.$$

Then

$$\begin{aligned}(F \circ S \circ G^{-1} \circ T)(f) &= (U \circ G \circ G^{-1} \circ T)(f) \\ &= (U \circ T)(f) \\ &= f.\end{aligned}$$

Now suppose that G is in $\mathfrak{C}$. We need to show that $F \in \mathfrak{C}$.

Applying the forgetful functor to **Set** we have the following diagram.

$$\begin{tikzcd} A \arrow[r, "r"] \arrow[d, "f"'] & C \arrow[r, "s"] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r, "t"'] & D \arrow[r, "u"'] & B \end{tikzcd}$$

The map r is injective because the identity factors through it, and g is injective by assumption. Suppose we are given $a, a' \in A$ such that $f(a) = f(a')$. Then $t(f(a)) = t(f(a'))$, so $g(r(a)) = g(r(a'))$. But g and r are both injective, so $a = a'$.

Now suppose that G is in $\mathfrak{F}$. We need to show that $F \in \mathfrak{F}$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "R"] \arrow[d, "F"'] & \mathcal{C} \arrow[r, "S"] \arrow[d, "G"] & \mathcal{A} \arrow[d, "F"] \\ \mathcal{B} \arrow[r, "T"'] & \mathcal{D} \arrow[r, "U"'] & \mathcal{B} \end{tikzcd}$$

To see this, note that we can augment any lifting problem

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[d, "F"] \\ (* \leftrightarrow \bullet) \arrow[r] & \mathcal{B} \end{tikzcd}$$

as follows; composing with R gives us a lifting problem we can solve, yielding the dashed arrow.

$$\begin{tikzcd} (*) \arrow[r] \arrow[d, hook] & \mathcal{A} \arrow[r, "R"] \arrow[d] & \mathcal{C} \arrow[r, "S"] \arrow[d] & \mathcal{A} \arrow[d] \\ (* \leftrightarrow \bullet) \arrow[r] \arrow[urr, dashed] & \mathcal{B} \arrow[r] & \mathcal{D} \arrow[r] & \mathcal{B} \end{tikzcd}$$

We can compose this with S to get a lift to A, which is what we want.

Axiom M4: Solving lifting problems

Consider a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "G"'] & \mathcal{D} \end{tikzcd}$$

where i is injective on objects and P is both an isofibration and a weak equivalence.

Pick any $d \in \mathcal{D}$. Since P is an equivalence of categories, there is some object $b \in \mathcal{B}$ such that $P(b) \simeq d$. Define a functor $I \to \mathcal{D}$ which sends $* \mapsto b$ and $\bullet \mapsto P(b)$, and a functor $(*) \to \mathcal{B}$ which picks out b. This gives us a commuting diagram

$$\begin{tikzcd} (*) \arrow[r] \arrow[d] & \mathcal{B} \arrow[d] \\ I \arrow[r] & \mathcal{D} \end{tikzcd}$$

Since P is an isofibration, we have a lift $I \to \mathcal{B}$, which maps $\bullet$ to an object $\hat{d} \in \mathcal{B}$ such that $P(\hat{d}) = d$. Thus, P is surjective on objects.

Suppose first that c is in the image of i. Then (because i is injective on objects) we can lift c uniquely to an object of A, then map the result to $\mathcal{B}$ with F. We define this to be $\ell(c)$.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "i"'] & \mathcal{B} \arrow[d, "P"] \\ \mathcal{C} \arrow[r, "G"'] \arrow[ur, dashed, "\ell"] & \mathcal{D} \end{tikzcd}$$

This makes both triangles formed commute trivially.

If c is not mapped to by i, then we define $\ell(c)$ to be any lift $\widehat{G(c)}$. This makes the lower triangle commute by definition, and the upper triangle is irrelevant since i never hits these objects.

Since P is an equivalence of categories, the maps

$$P_{b,b'} \colon \mathcal{B}(b, b') \to \mathcal{D}(P(b), P(b')); \qquad \left[b \xrightarrow{f} b'\right] \mapsto \left[P(b) \overset{P(f)}{\to} P(b')\right]$$

are bijections, hence can be inverted. Thus we can define ℓ on morphisms $f \colon c \to c'$ in $\mathcal{C}$ by

$$\ell(f) = P^{-1}_{\ell(c), \ell(c')}(G(f)).$$

The upper triangle

$$\begin{tikzcd} \left[a \overset{f}{\to} a'\right] \arrow[r, mapsto] \arrow[d, mapsto] & \left[F(a) \overset{F(f)}{\to} F(a')\right] \overset{!}{=} \left[\ell(i(a)) \overset{\ell(i(f))}{\to} \ell(i(b))\right] \\ \left[i(a) \overset{i(f)}{\to} i(a')\right] \arrow[ur, mapsto] & \end{tikzcd}$$

commutes since $\ell(i(a)) = F(a)$ and $\ell(i(a')) = F(a')$, so

$$\begin{aligned} \ell(i(f)) &= P^{-1}_{\ell(i(a)),\ell(i(a'))}(G(i(f))) \\ &= P^{-1}_{F(a),F(a')}(P(F(f))) \\ &= F(f). \end{aligned}$$

To see that the lower triangle

$$\begin{tikzcd} & & \left[\ell(c) \xrightarrow{\ell(f)} \ell(c')\right] \arrow[d, mapsto] \\ \left[c \xrightarrow{f} c'\right] \arrow[r, mapsto] \arrow[urr, mapsto] & \left[G(c) \xrightarrow{G(f)} G(c')\right] \overset{!}{=} & \left[P(\ell(c)) \xrightarrow{P(\ell(f))} P(\ell(c'))\right] \end{tikzcd}$$

note that we have already seen that $P(\ell(c)) = G(c)$ and $P(\ell(c')) = G(c')$, and for $f : c \to c'$ we defined

$$\ell(f) = P^{-1}_{\ell(c),\ell(c')}(G(f)),$$

so

$$P(\ell(f)) = G(f).$$

Now we are in the situation where i is both injective and an equivalence of categories, and P is an isofibration.

Let $c \in \mathcal{C}$. Becase i is essentially surjective we can always pick $c' \in \mathcal{C}$ such that c' is in the image of i, i.e. such that there exists some a_c with $i(a_c) = c' \simeq c$. For each such c, pick such a c' and a_c. If c is already in the image of i, pick $c' = c$; that is, demand that $a_{i(a)} = a$.

This allows us to define, for each $c \in \mathcal{C}$ functors as follows.

$$* \longmapsto F(a_c)$$

$$(* \leftrightarrow \bullet) \longmapsto G(c') \leftrightarrow G(c)$$

Then P's isofibricity gives a functor $I \to \mathcal{B}$ making the diagram

commute, which sends $\bullet$ to some object $b_c \in \mathcal{B}$. By the commutativity, $P(b_c) = G(c)$.

Define the lift ℓ by $\ell(c) = b_c$. The upper triangle is

$$\begin{tikzcd} a \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \overset{!}{=} \ell(i(a)) \\ i(a) \arrow[ur, mapsto] & \end{tikzcd}$$

which commutes because $\ell(i_a) = F(a_a) = F(a)$.

The lower triangle

$$\begin{tikzcd} & \ell(c) \arrow[d, mapsto] \\ c \arrow[ur, mapsto] \arrow[r, mapsto] & G(c) \overset{!}{=} P(\ell(c)) \end{tikzcd}$$

commutes because $P(\ell(c)) = P(b_c) = G(c)$.

By the same logic as the previous problem, any morphism $f\colon c \to c'$ lifts well-definedly to a map $\hat{f}\colon a_c \to a_{c'}$. We define

$$\ell(f) = F(\hat{f}).$$

The upper triangle

$$\begin{tikzcd}
f \arrow[r, mapsto] \arrow[d, mapsto] & F(f) \overset{!}{=} \ell(i(f)) \\
i(f) \arrow[ur, mapsto] &
\end{tikzcd}$$

commutes since $\ell(i(f)) = f$ by definition of $\hat{f}$. The lower triangle

$$\begin{tikzcd}
 & \ell(f) \arrow[d, mapsto] \\
f \arrow[ur, mapsto] \arrow[r, mapsto] & P(\ell(f)) \overset{!}{=} G(f)
\end{tikzcd}$$

commutes because

$$\begin{aligned} P(\ell(f)) &= P(F(\hat{f})) \\ &= G(i(\hat{f})) \\ &= G(f). \end{aligned}$$

Axiom M5: Factorization

For any functor $F\colon \mathcal{C} \to \mathcal{D}$, define a functor $G\colon \mathcal{C} \to \mathcal{L}$ on objects by

$$x \mapsto (x, Fx, \mathrm{id}_{Fx}),$$

and on morphisms by $f \mapsto f$. The functor G' sending

$$(c, d, f) \mapsto c.$$

is a weak inverse to G. As a check: these form an equivalence of categories since

$$G' \circ G\colon x \mapsto (x, Fx, \mathrm{id}_{Fx}) \mapsto x$$

and

$$G \circ G'\colon (c, d, f) \mapsto c \mapsto (c, Fc, \mathrm{id}_{Fc});$$

the map

$$G(\mathrm{id}_c)\colon (c, d, f) \to (c, Fc, \mathrm{id}_{Fc})$$

is a natural isomorphism since the naturality square

$$\begin{tikzcd}
(c, d, f) \arrow[r, "G(\mathrm{id}_c)"] \arrow[d, "g"'] & (c, Fc, \mathrm{id}_{Fc}) \arrow[d, "g"] \\
(c', d', f') \arrow[r, "G(\mathrm{id}_{c'})"'] & (c', Fc', \mathrm{id}_{Fc'})
\end{tikzcd}$$

commutes.

It is also clear that G is injective on objects since $(c,d,f) \neq (c',d',f') \implies c \neq c'$, so $G \in \mathfrak{C}$.

Define also a functor $H\colon \mathcal{L} \to \mathcal{D}$ sending $(c,d,f) \mapsto d$ and $g\colon c \to c'$ to $F(g)$. Then

$$H \circ G\colon c \mapsto (c, Fc, \mathrm{id}_{Fc}) \mapsto Fc; \qquad g \mapsto g \mapsto Fg.$$

It remains only to show that $H \in \mathfrak{F}$, we need to show that it is an isofibration. Consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[rrr] \arrow[ddd, hook] & & & \mathcal{L} \arrow[ddd, "H"] \\
& * \arrow[r, mapsto] \arrow[d, mapsto] & (c,d,f) \arrow[d, mapsto] & \\
& * \arrow[r, mapsto] & d & \\
(* \leftrightarrow \bullet) \arrow[rrr] & & & \mathcal{D}
\end{tikzcd}$$

We need to show that given an isomorphism $r\colon d \to d'$ and an object (c,d,f) in $\mathcal{L}$, we can find an object (c',d',f') which is isomorphic to (c,d,f). We can; taking $c' = c$, and $f' = r \circ f$ gives us what we want since the following diagram commutes.

$$\begin{tikzcd}
& (c,d,f) \leftrightarrow (c,d',r \circ f) \arrow[d, mapsto] \\
(* \leftrightarrow \bullet) \arrow[ur, mapsto] \arrow[r, mapsto] & c \leftrightarrow c'
\end{tikzcd}$$

Part 2: We define G to be the canonical injection, and H to be defined on objects as follows.

$$H\colon x \mapsto \begin{cases} Fx, & x \in \mathcal{C} \\ x, & x \in \mathcal{D} \end{cases}$$

The hom-sets in $\mathcal{R}$ are already appropriate hom-sets in $\mathcal{D}$, so we don't need to do anything to define H on morphisms: we simply define it to be the identity on hom-sets.

To see that H is an equivalence of categories, note that it is essentially surjective because it is surjective 'on the nose,' and that it is fully faithful because H is the identity on hom-sets.

It is obvious that G is injective on objects.

Now consider the following diagram.

$$\begin{tikzcd}
(*) \arrow[r, "a"] \arrow[d] & \mathcal{R} \arrow[d] \\
(* \leftrightarrow \bullet) \arrow[r, "b"] & \mathcal{D}
\end{tikzcd}$$

Define a functor $\ell\colon I \to \mathcal{R}$ sending $* \mapsto a(*)$ and $\bullet \mapsto b(\bullet)$. This makes both triangles commute: the upper triangle is

$$\begin{tikzcd}
* \arrow[r, mapsto] \arrow[d, mapsto] & a(*) \\
* \arrow[ur, mapsto] &
\end{tikzcd}$$

and the lower triangle is

$$a(*) \leftrightarrow b(\bullet)$$

$$* \leftrightarrow \bullet$$

6.3 The Homotopy category of a model category

In Subsection 6.3.1, we define the notion of a *left homotopy* of maps $A \to B$, denoted $\overset{l}{\sim}$, and show that for any cofibrant object A and any object B, left homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$\overbrace{A}^{\mathsf{Cof}} \xrightarrow{f} B$$

In Subsection 6.3.2, we define a dual notion, *right homotopy*, denoted $\overset{r}{\sim}$, and show that for any object A and any fibrant object B, right homotopy yields an equivalence relation on $\mathcal{C}(A, B)$.

$$A \xrightarrow{f} \overbrace{B}^{\mathsf{Fib}}$$

In Subsection 6.3.3, we show that when looking at maps from a cofibrant object to a fibrant object

$$\overbrace{A}^{\mathsf{Cof}} \xrightarrow{f} \overbrace{B}^{\mathsf{Fib}}$$

a left homotopy $f \overset{l}{\sim} f'$ gives rise to a right homotopy and vice versa, and thus the two notions agree. Thus, for maps from a cofibrant object to a fibrant object, we do not keep track of left- and right homotopies, calling them simply *homotopies.*

This allows us to define, for two maps between fibrant-cofibrant objects, a notion of homotopy equivalence; we say that a map $f: A \to B$, where A and B are fibrant-cofibrant, is a homotopy equivalence if it has a homotopy inverse g, i.e. a map g such that

$$f \circ g \sim \mathrm{id}_B, \qquad \text{and} \qquad g \circ f \sim \mathrm{id}_A.$$

Next, we show that this notion is preserved under composition: if f and g are homotopy equivalences, then so is $g \circ f$. This allows us to prove Whitehead's theorem, which tells us that any weak equivalence between fibrant-cofibrant objects is a homotopy equivalence.

Finally, we define the homotopy category $\mathrm{Ho}(\mathcal{C})$ of a model category, whose objects are the fibrant-cofibrant objects of $\mathcal{C}$, and whose morphisms $X \to Y$ are the homotopy classes of morphisms $X \to Y$.

6.3.1 Left homotopy

Definition 111 (cylinder object)**.** Let $\mathcal{C}$ be a model category, and let $A \in \mathcal{C}$. A cylinder object for A is an object C together with a factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

such that i is a cofibration and σ is a weak equivalence.

Example 112. In $\mathbf{Set}_\Delta$ equipped with the Kan model structure, the object $\Delta^1 \times S$ always functions as a cylinder object over S.

$$\begin{tikzcd} S \amalg S \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "(s_0\times\mathrm{id},s_1\times\mathrm{id})"'] & S \\ \Delta^1 \times S \arrow[ur, "\pi_2"'] & \end{tikzcd}$$

In this case, i is a cofibration because it is an inclusion (hence a monomorphism as required) and π_2 is a weak equivalence (i.e. a homotopy equivalence) because

Example 113. In the model category on categories, defined in Subsection 6.2.1, any category $\mathcal{C}$ has cylinder object $\mathcal{C} \times I$, taken with the below factorization.

$$\begin{tikzcd} \mathcal{C} \amalg \mathcal{C} \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(\mathrm{id}\times\{*\},\mathrm{id}\times\{\bullet\})"'] & \mathcal{C} \\ \mathcal{C} \times I \arrow[ur, "\pi"'] & \end{tikzcd}$$

The map i is obviously injective on objects, and the map π is an equivalence since the map $c \mapsto (c, 0)$ functions as a weak inverse.

Definition 114 (left homotopy)**.** Let $\mathcal{C}$ be a model category, and let $f, g \colon A \to B$ be two morphsims in $\mathcal{C}$. A left homotopy between f and g consists of a cylinder object C for A (Definition 111) and a map $H \colon C \to B$ such that the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,g)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

In this case, we will write

$$f \overset{l}{\sim} g.$$

Lemma 115. Let A be a cofibrant object. If C is a cylinder object for A, then the maps i_1 and i_2 are trivial cofibrations.

Proof. Since A is cofibrant, by definition the map $\emptyset \to A$ is a cofibration. Consider the following pushout square.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & A \amalg A \end{tikzcd}$$

Since by Corollary 108 cofibrations are closed under pushout, the canonical injections $\iota_\alpha : A \to A \amalg A$, $\alpha = 1, 2$, are also cofibrations. Since $i_\alpha = i \circ \iota_\alpha$ is a composition of cofibrations, it is again a cofibration by closure under countable composition.

Since the diagram

$$\begin{tikzcd} A \arrow[d, hook, "\iota_\alpha"] \arrow[dr, "\mathrm{id}"] \arrow[dd, bend right=60, "\iota_\alpha"'] & \\ A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i"] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

commutes, we can write $\mathrm{id}_A = \sigma \circ i_\alpha$ (with id_A a weak equivalence because it is an isomorphsim and σ a weak equivalence by assumption), so by the two-out-of-three law i_α is a weak equivalence. □

Proposition 116. If A is a cofibrant object, then left homotopy of maps $A \to B$ is an equivalence relation.

Proof. To see that left homotopy is an equivalence relation, let C be any cylinder object over A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i=(i_0,i_1)"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Then the following diagram commutes.

$$\begin{tikzcd} A \amalg A \arrow[r, "(f,f)"] \arrow[d, "i"'] & B \\ C \arrow[ur, "f\circ\sigma"'] & \end{tikzcd}$$

This means precisely that $f \circ \sigma$ functions as a left homotopy from f to itself.

Next, we show that left homotopy is symmetric. First, we note that in any category with coproducts, there is a natural isomorphism $\mathrm{swap}_A : A \amalg A \to A \amalg A$ which switches the components. Because Cof is saturated and thus contains all isomorphisms, swap_A is a cofibration. This gives us a new factorization exhibiting C as a cylinder object of A.

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "i\circ\mathrm{swap}_A"'] & A \\ C \arrow[ur, "\sigma"'] & \end{tikzcd}$$

Let H be a left homotopy $f \overset{l}{\sim} g$ as follows.

$$\begin{tikzcd}
A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & B \\
C \arrow[ur, "H"'] &
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
A \amalg A \arrow[d, "\mathrm{swap}_A"'] \arrow[dr, "{(g,f)}"] & \\
A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & B \\
C \arrow[ur, "H"'] &
\end{tikzcd}$$

exhibits a left homotopy $g \overset{l}{\sim} f$.

It remains only to show that left homotopy is transitive. To this end, suppose $f \overset{l}{\sim} g$ and $g \overset{l}{\sim} h$,

$$\begin{tikzcd}
A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "{(i_0,i_1)}"'] & B \\
C \arrow[ur, "H"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \amalg A \arrow[r, "{(g,h)}"] \arrow[d, "{(i_0',i_1')}"'] & B \\
C' \arrow[ur, "H'"'] &
\end{tikzcd}$$

where C and C' are cylinder objects for A with the following factorizations.

$$\begin{tikzcd}
A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i_0,i_1)}"'] & A \\
C \arrow[ur, "\sigma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \amalg A \arrow[r, "{(\mathrm{id},\mathrm{id})}"] \arrow[d, "{(i_0',i_1')}"'] & B \\
C' \arrow[ur, "\sigma'"'] &
\end{tikzcd}$$

Consider the following diagram, where the top left square is a pushout square.

$$\begin{tikzcd}
A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"'] \arrow[ddr, bend left, "H'"] & \\
C \arrow[r, "j_0'"] \arrow[drr, bend right, "H"'] & C \coprod_A C' \arrow[dr, dashed, "\exists! \tilde{H}"] & \\
 & & B
\end{tikzcd}$$

The following diagram commutes.

$$\begin{tikzcd}
A \amalg A \arrow[r, "{(f,h)}"] \arrow[d, "{(j_0' \circ i_0, j_1 \circ i_0')}"'] & B \\
C \amalg_A C' \arrow[ur, "\tilde{H}"'] &
\end{tikzcd}$$

It remains only to show that $C \amalg_A C'$ is a cylinder object for A. Define a map $\tilde{\sigma}$ as follows.

$$\begin{tikzcd} A \arrow[r, "i_0'"] \arrow[d, "i_1"'] & C' \arrow[d, "j_1"] \arrow[ddr, bend left, "\sigma'"] & \\ C \arrow[r, "j_0'"] \arrow[drr, bend right, "\sigma"'] & C \coprod_A C' \arrow[dr, dashed, "\exists! \tilde{\sigma}"] & \\ & & A \end{tikzcd}$$

Since $j_1 \in \mathrm{W} \cap \mathrm{Cof}$ and $\sigma' \in \mathrm{W}$, we have by Axiom (M2) that $\tilde{\sigma}$ is a weak equivalence.

Thus, to show that $C \amalg_A C'$ is a cylinder object with the factorization

$$\begin{tikzcd} A \amalg A \arrow[r, "(\mathrm{id},\mathrm{id})"] \arrow[d, "(j_0' \circ i_0, j_1 \circ i_0')"'] & A \\ C \amalg_A C' \arrow[ur, "\tilde{\sigma}"'] & \end{tikzcd}$$

we need only show that $(j_0' \circ i_1, j_1 \circ i_0')$ is a cofibration.

First note that the following square is a pushout square, where δ is the codiagonal.

$$\begin{tikzcd} (A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d, "(j_0',j_1)"] \\ A \amalg A \amalg A \arrow[r] & C \amalg_A C \end{tikzcd}$$

To see this, note maps

$$(\alpha, \beta) \colon C \amalg C' \to Z \qquad \text{and} \qquad (a, b, c) \colon A \amalg A \amalg A$$

which form a cocone satisfy $\beta \circ i_0' = b$ and $\alpha \circ i_1 = f$.

$$\begin{tikzcd} (A \amalg A) \amalg (A \amalg A) \arrow[r, "(i_0,i_1)\amalg(i_0',i_1')"] \arrow[d, "(\mathrm{id},\delta,\mathrm{id})"'] & C \amalg C' \arrow[d] \arrow[ddr, bend left, "(\alpha,\beta)"] & \\ A \amalg A \amalg A \arrow[r, "k"] \arrow[drr, bend right, "(a,b,c)"'] & C \amalg_A C \arrow[dr, dashed, "\exists!"] & \\ & & Z \end{tikzcd}$$

Thus, there is a unique map

$$C \amalg_A C \to Z$$

making the above diagram commute.

Note that because $(i_0, i_1) \amalg (i'_0, i'_1)$ is a cofibration, k must be also.

Also, the map $A \amalg A \to A \amalg A \amalg A$ is a cofibration because the following diagram is a pushout.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d] \\
A \amalg A \arrow[r] & A \amalg A \amalg A
\end{tikzcd}$$

Thus, the composition

$$A \amalg A \to A \amalg A \amalg A \to C \amalg_A C'$$

is a cofibration. □

6.3.2 Right homotopy

There is a dual notion to left fibrations, called right fibrations. The theory of right fibrations is dual to that of left fibrations.

Definition 117 (path object)**.** Let B be an object of a model category. A path object for B is an object P, together with a factorization

$$\begin{tikzcd}
 & P \arrow[d, "p"] \\
B \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & B \times B
\end{tikzcd}$$

where p is a fibration and s is a weak equivalence.

Definition 118 (right homotopy)**.** A right homotopy between morphisms $f, g\colon A \to B$ consists of a path object P and a factorization as follows.

$$\begin{tikzcd}
 & P \arrow[d, "p"] \\
A \arrow[ur, "H"] \arrow[r, "{(f,g)}"'] & B \times B
\end{tikzcd}$$

Lemma 119. Let B be a fibrant object. If P is a cylinder object for B, then the maps p_0 and p_1 are trivial fibrations.

Proof. Apply Lemma 115 to the opposite model category (Example 110). □

Proposition 120. Let B a fibrant object. Right homotopy of maps $f, g\colon A \to B$ is an equivalence relation.

Proof. Apply Proposition 116 to the opposite model category (Example 110). □

6.3.3 Homotopy equivalents

Under certain conditions, namely when the domain and codomain are fibrant-cofibrant, left and right homotopy are equivalent.

Lemma 121. Suppose A is cofibrant, and suppose that $f, g\colon A \to B$ are left homotopic. Then given any path object P for B, there is a right homotopy between f and g with underlying path object P.

Proof. Suppose we are given a left homotopy

$$\begin{tikzcd} A \amalg A \arrow[r, "{(f,g)}"] \arrow[d, "i"'] & B \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

and suppose that P has the following factorization

$$\begin{tikzcd} & P \arrow[d, "p"] \\ B \arrow[ur, "s"] \arrow[r, "{(\mathrm{id},\mathrm{id})}"'] & B \times B \end{tikzcd}$$

Consider the following diagram.

$$\begin{tikzcd} A \arrow[r, "s \circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[r, "{(f\circ\sigma,H)}"'] & B \times B \end{tikzcd}$$

It commutes because

$$p \circ s \circ f = (\mathrm{id}, \mathrm{id}) \circ f = (f, f)$$

and

$$(f \circ \sigma, H) \circ i_0 = (f \circ \sigma \circ i_0, H \circ i_0) = (f \circ \mathrm{id}, f) = (f, f).$$

By definition, p is a fibration, and we have seen in Lemma 115 that i_0 is a trivial cofibration. Thus, we have a lift $h\colon C \to P$.

$$\begin{tikzcd} A \arrow[r, "s \circ f"] \arrow[d, "i_0"'] & P \arrow[d, "p"] \\ C \arrow[ur, dashed, "h"] \arrow[r, "{(f\circ\sigma,H)}"'] & B \times B \end{tikzcd}$$

Then we have

$$\begin{aligned} p \circ h \circ i_1 &= (f \circ \sigma, H) \circ i_1 \\ &= (f \circ \sigma \circ i_1, H \circ i_1) \\ &= (f, g), \end{aligned}$$

so the diagram

$$\begin{tikzcd} & P \arrow[d, "p"] \\ A \arrow[ur, "h\circ i_1"] \arrow[r, "(f,g)"'] & B\times B \end{tikzcd}$$

commutes. Thus, $h \circ i_1$ gives a right homotopy between f and g. □

Corollary 122. Let $f, g\colon A \to B$ be morphisms, and suppose that A is cofibrant and B is fibrant. Then TFAE.

1. f and g are left homotopic.
2. For every path object P for B, there is a right homotopy between f and g with path object P.
3. f and g are right homotopic.
4. For every cylinder object C for A, there is a left homotopy between f and g with cylinder object C.

Proof.

$1 \Rightarrow 2$. Lemma 121.

$2 \Rightarrow 3$. Trivial.

$3 \Rightarrow 4$. Dual to Lemma 121.

$4 \Rightarrow 1$. Trivial.

□

The above result shows that when considering morphisms between a cofibrant object and a fibrant object, there is no difference between left and right fibrations. Therefore, we will not distinguish them.

Definition 123 (homotopy equivalence)**.** Let $f\colon X \to Y$ be a morphism with X and Y fibrant-cofibrant. We say that f is a homotopy equivalence if there exists a morphism $g\colon Y \to X$ such that $g \circ f \sim \mathrm{id}_X$ and $f \circ g \sim \mathrm{id}_Y$.

Composition on the left and on the right preserves homotopy equivalence.

Lemma 124. Let $f, g\colon X \to Y$ be homotopy equivalent morphisms between fibrant-cofibrant objects, and let W and Z be fibrant-cofibrant. Then the following hold.

1. For any $h\colon Y \to Z$, $h \circ f \sim h \circ g$.
2. For any $h'\colon W \to X$, $f \circ h' \sim g \circ h'$.

Proof.

1. Suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} X \amalg X \arrow[r, "(f,g)"] \arrow[d, "i"'] & Y \\ C \arrow[ur, "H"'] & \end{tikzcd}$$

Post-composing with h gives us the following

$$\begin{tikzcd} X \amalg X \arrow[r, "(f{,}g)"] \arrow[d, "i"'] & Y \arrow[r, "h"] & Z \\ C \arrow[ur, "H"'] & & \end{tikzcd} \implies \begin{tikzcd} X \amalg X \arrow[r, "(h\circ f{,}h\circ g)"] \arrow[d, "i"'] & Z \\ C \arrow[ur, "h\circ H"'] & \end{tikzcd}$$

But the second diagram is precisely a homotopy $h \circ f \sim h \circ g$.

2. Now suppose our homotopy is given by the following factorization.

$$\begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H'"] \arrow[r, "(f{,}g)"'] & Y \times Y \end{tikzcd}$$

Pre-composing with h yields the following.

$$\begin{tikzcd} & & P \arrow[d, "s"] \\ W \arrow[r, "h"] & X \arrow[ur, "H'"] \arrow[r, "(f{,}g)"'] & Y \times Y \end{tikzcd} \implies \begin{tikzcd} & P \arrow[d, "s"] \\ X \arrow[ur, "H'\circ h"] \arrow[r, "(f\circ h{,}g\circ h)"'] & Y \times Y \end{tikzcd}$$

This is preciesly a homotopy $f \circ h \sim g \circ f$.

□

Corollary 125. Let X, Y, and Z be fibrant-cofibrant objects, and let f and g be homotopy equivalences as follows.

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

Then $g \circ f$ is a homotopy equivalence.

Proof. By definition, f has a homotopy inverse f' such that $f' \circ f \sim \mathrm{id}$ and $f \circ f' \sim \mathrm{id}$. Similarly, there is a g' such that $g' \circ g \sim \mathrm{id}$ and $g \circ g' \sim \mathrm{id}$. But then

$$\begin{aligned} \mathrm{id} &\sim g' \circ g \\ &\sim f' \circ g' \circ g \\ &\sim f' \circ g' \circ g \circ f \end{aligned}$$

and

$$\begin{aligned} \mathrm{id} &\sim f' \circ f \\ &\sim g' \circ f' \circ f \circ g. \end{aligned}$$

so $f' \circ g'$ functions as a homotopy inverse to $g \circ f$. □

Lemma 126. Let X and Y be fibrant-cofibrant objects, and $f\colon X \to Y$ a weak equivalence.

1. If f is a fibration, then it is a homotopy equivalence.
2. If f is a cofibration, then it is a homotopy equivalence.

Proof.

1. We have a lift as follows.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "g"] & Y \end{tikzcd}$$

   Thus, $f \circ g = \mathrm{id}_Y$.

   Thus the diagram

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f, \mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] & Y \end{tikzcd}$$

   commutes, because

$$\begin{aligned} f \circ (g \circ f, \mathrm{id}) &= (f \circ g \circ f, f) \\ &= (f, f) \\ &= (f \circ \sigma \circ i_1, f \circ \sigma \circ i_2) \\ &= f \circ \sigma \circ i. \end{aligned}$$

   Since f is a trivial fibration and i is a cofibration, we have by Axiom (M4) a lift as follows.

$$\begin{tikzcd} X \amalg X \arrow[r, "(g\circ f, \mathrm{id})"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ C \arrow[r, "f\circ\sigma"'] \arrow[ur, dashed, "H"] & Y \end{tikzcd}$$

   The upper triangle tells us that H is a left homotopy between $g \circ f$ and id_X.

   Since $f \circ g = \mathrm{id}_Y$, the cylinder object C itself functions as a homotopy $f \circ g \sim \mathrm{id}_Y$. Thus, f is a homotopy equivalent.

2. The commutativity of the solid diagram

$$\begin{tikzcd} X \arrow[r, "s\circ f"] \arrow[d, "f"'] & P \arrow[d, "p"] \\ Y \arrow[r, "(\mathrm{id}_Y, f\circ h)"'] \arrow[ur, dashed] & Y \times Y \end{tikzcd}$$

   together with the fact that f is a trivial cofibration and p is a fibration imply that there is a dashed lift. This exhibits a right homotopy $f \circ h \sim \mathrm{id}_Y$. But $h \circ f = \mathrm{id}_X$, so $h \circ f \sim \mathrm{id}_X$.

   So far we have shown that the result holds if f is a trivial fibration or a trivial cofibration.

   Now suppose f is merely a weak equivalence. Then we can factor

$$f = g \circ h,$$

where g is a fibration and h is a trivial cofibration. But by Axiom (M2), g is also a trivial cofibration.

□

Theorem 127 (Whitehead)**.** Let $f : X \to Y$ be a weak equivalence, and suppose that X and Y are fibrant-cofibrant. Then f is a homotopy equivalence.

Proof. By Axiom (M5), we are allowed to factor $f = p \circ i$, where p is a fibration, i is a cofibration, and at least one of p and i is a weak equivalence. By Axiom (M2), both i and p are weak equivalences.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\mathsf{W}\cap\mathsf{Cof}\ni i"'] & & Y \\ & Z \arrow[ur, "p\in\mathsf{W}\cap\mathsf{Fib}"'] & \end{tikzcd}$$

By assumption, X is cofibrant, i.e. the map $\emptyset \to X$ is a cofibration. The map $i : X \to Z$ is also a cofibration, so their composition must also be a cofibration. But this means that the map $\emptyset \to Z$ is also a cofibration, so Z is cofibrant

Dually, the map $Y \to *$ is a fibration. Precomposing by p, we have that Z is fibrant. Thus, i and p are homotopy equivalences by Lemma 126, parts 2 and 1 respectively. But by Lemma 124, their composition is also a homotopy equivalence. □

Definition 128 (homotopy category of a model category)**.** Let $\mathcal{C}$ be a model category. The homotopy category of $\mathcal{C}$, denoted $\mathrm{Ho}(\mathcal{C})$, is defined as follows.

- The objects of $\mathrm{Ho}(\mathcal{C})$ are the fibrant-cofibrant objects of $\mathcal{C}$.
- The set of morphisms $X \to Y$ is the set of homotopy classes of $\mathcal{C}(X, Y)$.
- Composition is given by the rule

$$[f] \circ [g] \equiv [f \circ g].$$

  This is well-defined by Lemma 124.

Example 129. Consider the category $\mathbf{Set}_\Delta$ together with the Kan model structure (Example 109). We have seen (in Example 112) that for any simplicial set S, the object $\Delta^1 \times S$ functions as a cylinder object for S, implying that the homotopy theory in the model structure on $\mathbf{Set}_\Delta$ agrees with the standard homotopy theory on $\mathbf{Set}_\Delta$.

In the Kan model structure $\mathbf{Set}_\Delta$, the fibrant objects are Kan complexes, and every object is cofibrant. Therefore, the fibrant-cofibrant objects are simply Kan complexes.

The homotopy category $\mathrm{Ho}(\mathbf{Set}_\Delta)$ therefore is simply

$$\{(\text{Kan complexes})\}/(\text{homotopy equivalences}).$$

We will meet this category later. It is called the *homotopy category of spaces.*

6.4 Fibrant-Cofibrant replacement

Let $\mathcal{C}$ be a model category.

In Subsection 6.4.1, we give a way of assigning to any object $X \in \mathcal{C}$ a (not necessarily unique) cofibrant object QX to which it is weakly equivalent. We call any choice of such at cofibrant object a *cofibrant replacement.*

We also define the notion of cofibrant replacement on morphisms, i.e. a way of assigning

$$(f\colon X \to Y) \mapsto (Qf\colon QX \to QY).$$

Again, this assignment is not uniquely defined.

Next we next prove that for any cofibrant replacements Qf of f and Qg of g,

$$f \overset{l}{\sim} g \implies Qf \overset{l}{\sim} Qg,$$

showing that the cofibrant replacement of a morphism is uniquely defined if we restrict our attention to morphisms out of cofibrant objects X, and consider morphisms up to left homotopy. In this setting, we have the further nice property that cofibrant replacement respects composition in the sense that

$$Q(g \circ f) \overset{l}{\sim} Qg \circ Qf.$$

In Subsection 6.4.2, we study the dual notion, called *fibrant replacement,* providing a method by which one can replace any object X by a fibrant object RX to which it is weakly equivalent.

We again define a notion of cofibrant replacement on morphims, i.e. a way of assigning

$$(f\colon X \to Y) \mapsto (Rf\colon RX \to RY).$$

Then, restricting to morphisms whose domain Y is fibrant, and considering morphisms up to right homotopy, we find that fibrant replacement becomes well-defined in the sense that

$$f \overset{r}{\sim} f' \implies Rf \overset{r}{\sim} Rf',$$

and preserves composition in the sense that $R(g \circ f) = Rg \circ Rf$.

In Subsection 6.4.3, we prove two preliminary lemmas, which show the following.

- The fibrant replacement of a cofibrant object is fibrant-cofibrant.
- fibrant replacement takes left-homotopic morphisms $f, g\colon X \to Y$ with X cofibrant to morphisms which are right homotopic.

The first implies that taking first the cofibrant and then fibrant replacement of any object X yields an object RQX which is fibrant-cofibrant. The second then implies that fibrant-cofibrant replacement gives a functor from $\mathcal{C}$ to $\mathrm{Ho}(\mathcal{C})$.

6.4.1 Cofibrant replacement

Definition 130 (cofibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a cofibrant replacement of X consists of a cofibrant object QX and a trivial fibration $p_X\colon QX \to X$, defined as follows.

- If X is cofibrant, define $QX = X$, and $p_X = \mathrm{id}_X$.

- If X is not cofibrant, pick a factorization of the unique morphism $\emptyset \to X$

$$\emptyset \to QX \xrightarrow{p_X} X$$

where the map $\emptyset \to Qx$ is a cofibration and p_X is a trivial fibration.

For a morphism $f: X \to Y$, a fibrant replacement of f is any solution Rf to the following lifting problem.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & QY \arrow[d, "p_Y"] \\
QX \arrow[r, "f\circ p_X"'] \arrow[ur, dashed, "Qf"] & Y
\end{tikzcd}$$

Lemma 131. For any map $f: X \to Y$, any choice of cofibrant replacement $f \mapsto Qf$ respects left homotopy; that is, if $f \overset{l}{\sim} f'$, then $Qf \overset{l}{\sim} Qf'$.

Proof. Suppose we have lifts Qf and Qf' of f and f' respectively, and consider a left homotopy between them.

$$\begin{tikzcd}
X \amalg X \arrow[r, "{(f,f')}"] \arrow[d, "i"'] & Y \\
C \arrow[ur, "H"'] &
\end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd}
QX \amalg QX \arrow[r, "{(Qf,Qf')}"] \arrow[d, "i"'] & QY \arrow[d, "p_Y"] \\
C \arrow[r, "f\circ p_X\circ H"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

The outer square commutes because

$$\begin{aligned}
p_Y \circ (Qf, Qf') &= (f \circ p_X, f' \circ p_X) \\
&= f \circ p_X \circ (f, f') \\
&= f \circ p_X \circ H \circ i.
\end{aligned}$$

The map i is a cofibration by definition, and p_Y is a trivial fibration. Thus, we get a dashed lift. But this is precisely a left homotopy between Qf and Qf'. □

Corollary 132. We have

$$Q(g \circ f) \overset{l}{\sim} Q(g) \circ Q(f).$$

Proof. Consider the following square.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & * \arrow[r] \arrow[d] & QZ \arrow[d, "p_Z"] \\
* \arrow[r] \arrow[d] & QY \arrow[r, "g\circ p_Y"] \arrow[d, "p_Y"] & Z \arrow[d, "\mathrm{id}"] \\
QX \arrow[r, "f\circ p_X"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

The outer square is the lifting problem defining $Q(g \circ f)$, and the top right and bottom left squares define Qg and Qf respectively. Thus, $Qg \circ Qf \overset{l}{\sim} Q(g \circ f)$. □

There is a dual notion.

6.4.2 Fibrant replacement

Definition 133 (fibrant replacement)**.** Let $\mathcal{C}$ be a model category. For each $X \in \mathcal{C}$, a fibrant replacement of X consists of a fibrant object RX and a trivial cofibration $i_X : X \to RX$, defined as follows.

- If X is fibrant, define $RX = X$, and $i_X = \mathrm{id}_X$.
- If X is not fibrant, pick factorization of the unique morphism $X \to *$

  $$X \xrightarrow{i_X} RX \to *$$

  where the map $\emptyset \to Qx$ is a fibration and i_X is a trivial cofibration.

For a morphism $f : X \to Y$, a fibrant replacement of f is a solution Rf to the following lifting problem.

$$\begin{tikzcd} X \arrow[r, "i_Y \circ f"] \arrow[d, "i_X"'] & RY \arrow[d] \\ RX \arrow[r] \arrow[ur, dashed, "Rf"] & * \end{tikzcd}$$

Lemma 134. For any map $f : X \to Y$, fibrant replacement $f \mapsto Rf$ respects right homotopy; that is, if $f \overset{r}{\sim} f'$, then $Rf \overset{r}{\sim} Rf'$.

Proof. Dual to □

Corollary 135. We have

$$R(g \circ f) \overset{r}{\sim} R(g) \circ R(f).$$

Proof. Dual to □

6.4.3 Fibrant-cofibrant replacement

Lemma 136. Suppose $f, f' : X \to Y$ are left homotopic, and that X is cofibrant. Then the fibrant replacements Rf and Rf' are right homotopic.

Proof. Since f and f' are left homotopic, so are $i_Y \circ f$ and $i_Y \circ f'$. But since these are between a fibrant and a cofibrant object, they are right homotopic. Pick some right homotopy H as follows.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "H"] \arrow[r, "(i_Y \circ f, i_Y \circ f')"'] & RY \times RY \end{tikzcd}$$

Consider the following solid lifting problem.

$$\begin{tikzcd}
X \arrow[r, "H"] \arrow[d, "i_X"'] & R \arrow[d, "p"] \\
RX \arrow[r, "{(Rf,Rf')}"'] \arrow[ur, dashed] & RY \times RY
\end{tikzcd}$$

The outer square commutes because

$$p \circ H = (i_Y \circ f, i_Y \circ f') = (Rf \circ i_X, Rf' \circ i_X).$$

□

Theorem 137. There is a functor $\pi\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, defined on objects by $X \mapsto RQX$ and on morphisms by $f \mapsto [RQf]$.

Proof. First, note that the assignment on objects is well-defined since RQX is fibrant by definition and cofibrant because the composition

$$\emptyset \to QX \xrightarrow{i_X} RQX$$

is a cofibration.

It remains to check that π respects composition. To see this, note that

$$\begin{aligned}
\pi(g \circ f) &= [RQ(g \circ f)] \\
&= [R(Qg \circ Qf)] \\
&= [RQ(g) \circ RQ(f)] \\
&= [RQ(g)] \circ [RQ(f)] \\
&= \pi(g) \circ \pi(f),
\end{aligned}$$

where the second equality follows from Corollary 132 and Lemma 136 and the third follows from Lemma 134. □

Theorem 138. The homotopy category $\mathrm{Ho}(\mathcal{C})$, together with the map $\pi\colon \mathcal{C} \to \mathrm{Ho}(\mathcal{C})$, is the localization of $\mathcal{C}$ at $\mathcal{W} = \mathrm{W}$, the set of weak equivalences. That is, there is an equivalence of categories $\mathcal{C}[\mathcal{W}^{-1}] \simeq \mathrm{Ho}(\mathcal{C})$.

Proof. Define a category $\overline{\mathcal{C}}$ as follows.

- $\mathrm{Obj}(\overline{\mathcal{C}}) = \mathrm{Obj}(\mathcal{C})$
- For each $X, Y \in \mathrm{Obj}(\overline{\mathcal{C}})$, $\overline{\mathcal{C}}(X, Y) = \mathrm{Ho}(\mathcal{C})(RQX, RQY)$.

We have the following commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "\pi"] \arrow[dr, "\bar{\pi}"'] & & \mathrm{Ho}(\mathcal{C}) \\
& \overline{\mathcal{C}} \arrow[ur, "\xi"'] &
\end{tikzcd}$$

where $\bar{\pi}$ is the identity on objects and maps $f \mapsto [RQf]$, and ξ acts on objects by taking $X \mapsto RQX$ and is the identity on morphisms.

Note that ξ is surjective on objects and fully faithful, hence an equivalence of categories. Hence, we only need to show that $\overline{\mathcal{C}}$ satisfies the universal property for the localization.

Suppose that $F\colon \mathcal{C} \to \mathcal{D}$ is a functor which takes weak equivalences to isomorphisms in $\mathcal{D}$. If we can show that there exists a unique functor $\overline{F}\colon \overline{\mathcal{C}} \to \mathcal{D}$ such that $F = \overline{F} \circ \overline{\pi}$, then we are done.[1]

Suppose $f \overset{r}{\sim} g$. Pick some path object P, and a homotopy through P.

$$\begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "s"] \arrow[r, "(\mathrm{id},\mathrm{id})"'] & X \times X \end{tikzcd} \qquad \begin{tikzcd} & P \arrow[d, "p"] \\ X \arrow[ur, "H"] \arrow[r, "(f,g)"'] & Y \times Y \end{tikzcd}$$

Putting these diagrams back to back and unfolding them, we have the following.

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] \arrow[r, "H"] \arrow[dr, "g"'] & P \arrow[u, "p_0"'] \arrow[d, "p_1"'] & Y \arrow[ul, "\mathrm{id}"'] \arrow[l, "s"'] \arrow[dl, "\mathrm{id}"] \\ & Y & \end{tikzcd}$$

Consider the image in $\mathcal{D}$. Since p_0 and p_1 are weak equivalences, Theorem 127 implies that they are sent to isomorphisms in $\mathcal{D}$. Thus, $F(p_0) = F(s)^{-1} = F(p_1)$, so

$$\begin{aligned} F(f) &= F(p_0) \circ F(H) \\ &= F(p_1) \circ F(H) \\ &= F(g). \end{aligned}$$

Let $[g]\colon RQX \to RQY$ be a morphism in $\overline{\mathcal{C}}$. Pick a representative g. Consider the following commutative diagram.

$$\begin{tikzcd} X \arrow[r, "g"] & Y \\ QX \arrow[u, "p_X"] \arrow[r, "Qg"] \arrow[d, "i_{QX}"'] & QY \arrow[u, "p_Y"'] \arrow[d, "i_{QY}"] \\ RQX \arrow[r, "RQg"] & RQY \end{tikzcd}$$

Applying F, we find the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(g)"] & F(Y) \\ F(QX) \arrow[u, "F(p_X)"] \arrow[r, "F(Qg)"] \arrow[d, "F(i_{QX})"'] & F(QY) \arrow[u, "F(p_Y)"'] \arrow[d, "F(i_{QY})"] \\ F(RQX) \arrow[r, "F(RQg)"] & F(RQY) \end{tikzcd}$$

[1]Why?

But the vertical morphisms are isomorphisms, so

$$F(RQg) = F(i_{QY}) \circ F(p_Y)^{-1} \circ F(g) \circ F(p_X) \circ F(i_{QX})^{-1}.$$

In order to have $F = \bar{F} \circ \bar{\pi}$, we must have

$$\overline{F}([RQg]) = F(RQg),$$

which specifies $\bar{F}$ completely. □

6.5 Quillen adjunctions

Proposition 139. Let $\mathcal{C}$ and $\mathcal{D}$ be model categories, and consider an adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G.$$

The following are equivalent.

1. F preserves cofibrations and trivial cofibrations.
2. G preserves fibrations and trivial fibrations.

Proof. By Lemma 107, a morphism is a cofibration if and only if it has the left lifting property with respect to all trivial fibrations. Thus, in order to show that F preserves cofibrations, it suffices to show that if f is a cofibration in $\mathcal{C}$, then $F(f)$ has the left lifting property with respect to all trivial fibrations.

Consider solid adjunct diagrams of the following form, with the first in $\mathcal{D}$ and the second in $\mathcal{C}$.

$$\begin{tikzcd} F(c) \arrow[r] \arrow[d, "F(f)"'] & d \arrow[d, "g"] \\ F(c') \arrow[r] \arrow[ur, dashed] & d' \end{tikzcd} \qquad \begin{tikzcd} c \arrow[r] \arrow[d, "f"'] & G(d) \arrow[d, "G(g)"] \\ c' \arrow[r] \arrow[ur, dashed] & G(d') \end{tikzcd}$$

Suppose that f is a cofibration and that F preserves cofibrations. Then the left-hand diagram has a lift for any trivial fibration g. The adjunct of this lift is a lift for the diagram on the right. This means that $G(g) \in \mathsf{Cof}_{\perp} = \mathsf{Fib} \cap \mathsf{W}$.

Similarly, if f is a trivial cofibration and F preserves trivial cofibrations, then the left-hand diagram has a lift for any fibration g, whose adjunct is a lift for the diagram on the RHS, implying that $G(d) \in (\mathsf{Cof} \cap \mathsf{W})_{\perp} = \mathsf{Fib}$.

The other statement is dual. □

Definition 140 (Quillen adjunction)**.** An adjunction of model categories satisfying the equivalent conditions in Proposition 139 is known as a Quillen adjunction.

Lemma 141 (Ken Brown's Lemma)**.** Let $F : \mathcal{C} \leftrightarrow \mathcal{D} : G$ be a Quillen adjunction.

1. The left adjoint F preserves weak equivalences between cofibrant objects
2. The right adjoint G preserves weak equivalences between fibrant objects.

Proof. By Proposition 139. Let A and B be cofibrant objects, and $f: A \to B$ be a weak equivalence.

Pick for the map $(f, \mathrm{id})\colon A \amalg B \to B$ a factorization

$$A \amalg B \xrightarrow{i} C \xrightarrow{p} B,$$

where $i \in \mathsf{Cof}$ and $p \in \mathsf{Fib} \cap \mathsf{W}$.

Consider the following diagram.

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & A \arrow[d, "i_A"] \arrow[dr, bend left, "f"] & \\
B \arrow[r, "i_B"'] & A \amalg B \arrow[r, "{(f,\mathrm{id})}"] \arrow[d, "q"'] & B \\
& C \arrow[ur, "p"'] &
\end{tikzcd}$$

Because A and B are cofibrant, i_A and i_B are cofibrations. Thus, $q \circ i_A$ is a cofibration, so C is cofibrant.

Note that by commutativity, we have

$$f = p \circ q \circ i_A, \qquad \mathrm{id}_B = p \circ q \circ i_B.$$

By Axiom (M2), we have that $q \circ i_A$ and $q \circ i_B$ are weak equivalences, hence trivial cofibrations.

Applying F, we find that

$$F(f) = F(p) \circ F(q \circ i_A), \qquad F(\mathrm{id}) = F(p) \circ F(q \circ i_B).$$

We know that $F(\mathrm{id}_B) = \mathrm{id}_{F(b)}$ is a trivial cofibration, hence certainly a weak equivalence, and that $F(q \circ i_B)$ and $F(q \circ i_A)$ are a weak equivalence because F preserves trivial cofibrations. Thus $F(p)$ is a weak equivalence, implying that

$$F(f) = F(p) \circ F(q \circ i_A)$$

is a weak equivalence. □

Given a Quillen adjunction

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G,$$

consider the full subcategory of all cofibrant objects $\mathcal{C}_c \subset \mathcal{C}$. Since the full subcategory of $\mathcal{C}$ on fibrant-cofibrant objects is the same as the full subcategory of $\mathcal{C}_c$ on fibrant-cofibrant objects, there is an equivalence

$$\mathcal{C}_c[\mathcal{W}^{-1}] \simeq \mathcal{C}[\mathcal{W}^{-1}].$$

We will denote any inverse by

$$Q\colon \mathcal{C}[\mathcal{W}^{-1}] \to \mathcal{C}_c[\mathcal{W}^{-1}].$$

Dually, we get a functor

$$R\colon \mathcal{D}[\mathcal{W}^{-1}] \to \mathcal{D}_f[\mathcal{W}^{-1}],$$

where $\mathcal{D}_f$ is the full subcategory of $\mathcal{D}$ on fibrant objects.

Definition 142 (derived functor). Let

$$F : \mathcal{C} \leftrightarrow \mathcal{D} : G$$

be a Quillen adjunction.

- The left derived functor LF is the composite

$$\mathcal{C}[\mathcal{W}^{-1}] \xrightarrow{Q} \mathcal{C}_c[\mathcal{W}^{-1}] \xrightarrow{F} \mathcal{D}[\mathcal{W}^{-1}].$$

- The right derived functor RG is the composite

$$\mathcal{D}[\mathcal{W}^{-1}] \xrightarrow{R} \mathcal{D}_f[\mathcal{W}^{-1}] \xrightarrow{G} \mathcal{C}[\mathcal{W}^{-1}].$$

Proposition 143. Let

$$\mathcal{C} : F \leftrightarrow \mathcal{G} : \mathcal{D}$$

be a Quillen adjunction. Then there is an adjunction of derived functors

Definition 144 (Quillen equivalence). A Quillen adjunction is called a Quillen equivalence if LF, or equivalently RG, is an equivalence of localized categories.

Theorem 145. The adjunction

$$|\cdot| : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Top} : \mathbf{Sing}$$

is a Quillen adjunction.

7 Infinity categories

Note 146. I will edit this chapter heavily. It is in desperate need of a good de-clawing.

Lemma 147. Let K be a simplicial set. Then K is the nerve of some category if and only if K has unique inner horn fillers. That is, the following are equivalent.

- There exists a category $\mathcal{C}$ and an isomorphism $K \simeq N(\mathcal{C})$.
- Every lifting problem

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Proof. First, suppose that $K = N(\mathcal{C})$ for some category $\mathcal{C}$. Then □

7.1 Basic definitition

Definition 148 (infinity category)**.** A simplicial set $\mathcal{C}$ is called an <u>∞-category</u> if each inner horn

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd} \qquad 0 < i < n$$

has a filler.

Example 149. For any category $\mathcal{C}$, we have seen in Proposition 56 the nerve $N(\mathcal{C})$ has *unique* innner horn fillers, thus certainly has inner horn fillers.

Example 150. For any topological space X, the singular set $\mathbf{Sing}(x)$ has *all* horn fillers, hence certainly inner horn fillers.

In the previous chapter, we mainly worked with Kan complexes, so we used the Kan model structure on $\mathbf{Set}_\Delta$. There is, however, a different model structure on $\mathbf{Set}_\Delta$, which is better suited to infinity-categorical thinking.

7.2 The homotopy category of an infinity category

We would like to be able to view infinity-categories as a generalization of categories. This means that we would like to be able to throw away data from an infinity category to get a regular category. This throwing-away process creates from any infinity category $\mathcal{C}$ the so-called *homotopy category* $h\mathcal{C}$. The homotopy category inherits a lot of information from the infinity category from whence it came.

Definition 151 (homotopy category)**.** Let $\mathcal{C}$ be an ∞-category. Define a category $h\mathcal{C}$, called the homotopy category of $\mathcal{C}$, as follows.

- The objects $\mathrm{ob}(\mathcal{C})$ are the vertices $\mathcal{C}_0$.
- For objects x and y, the set $\mathcal{C}(x, y)$ consists of all edges f, g with $d_0 f = d_0 g = y$ and $d_1 f = d_1 g = x$, modulo the equivalence relation

$$f \sim g \iff \exists \sigma \in \mathcal{C}_2 \quad \text{with} \quad d_1\sigma = g, \quad d_2\sigma = f, \quad \text{and} \quad d_0\sigma = s_0 y.$$

$$\sigma = \begin{tikzcd} & y \arrow[dr, "s_0y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

- We have already seen in Theorem 93 that this is an equivalence relation.
- Composition is induced by inner horn fillings $\Lambda^2_1 \to \Delta^2$; we define the composition $[f] \circ [g]$ to be $[h]$, where h is some edge given to us by filling the following solid horn.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "h"'] & & z \end{tikzcd}$$

  To see that this is well-defined, i.e. respects equivalence classes, consider two fillings as follows.

$$\sigma = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}, \qquad \sigma' = \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h'"'] & & z \end{tikzcd}.$$

  We need to show that $h \sim h'$. We get this by filling the following horn.

$$\begin{tikzcd} y \arrow[r, "g"] & z \arrow[d, "s_0z"] \\ x \arrow[u, "f"] \arrow[ur, "h"] \arrow[r, "h'"'] & z \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "g"] \arrow[dr, "f"] & z \arrow[d, "s_0z"] \\ x \arrow[u, "f"] \arrow[r, "h'"'] & z \end{tikzcd}$$

  First suppose that $g_1 \sim g_2$. Then filling the horn

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h_2"'] \arrow[dr, "h_1"] & y \arrow[d, "g_1"] \\ z \arrow[r, "s_0z"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h_2"'] & y \arrow[d, "g_1"] \arrow[dl, "g_2"'] \\ z \arrow[r, "s_0z"'] & z \end{tikzcd}$$

  tells us that $h_1 \sim h_2$.

Now suppose that $f_1 \sim f_2$. In this case we prove something stronger, i.e. that if h is a filler for $g \circ f_1$, then it is also a filler for $g \circ f_2$. We find this from the following commutative diagram.

$$\begin{tikzcd} x \arrow[r, "f_1"] \arrow[d, "f_2"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\ z \arrow[r, "g"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "h_2"'] & y \arrow[d, "g_1"] \arrow[dl, "g_2"'] \\ z \arrow[r, "s_0 z"'] & z \end{tikzcd}$$

This gives us a well-defined composition law. For any $x \in \mathcal{C}_0$, the identity on x is $s_0 x$; it is a left and right identity because the solid diagrams

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "f"'] & & y \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "s_0 x"] \arrow[rr, dashed, "g"'] & & y \end{tikzcd}$$

can be filled by $s_1 f$ and $s_0 g$ respectively.

To see that composition is associative consider the horn

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g\circ f"'] & y \arrow[d, "h\circ g"] \arrow[dl, "g"'] \\ z \arrow[r, "h"'] & w \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "g\circ f"'] \arrow[dr, "(h\circ g)\circ f"] & y \arrow[d, "h\circ g"] \\ z \arrow[r, "h"'] & w \end{tikzcd}$$

where the bottom triangle on the right is unfilled. Filling it gives a homotopy

$$[(f \circ g) \circ h] = [f \circ (g \circ h)].$$

Lemma 152. The following are properties of $\mathrm{h}\mathcal{C}$.

1. Let $\mathcal{C}$ be a small category. Then $\mathrm{h}N(\mathcal{C}) \simeq \mathcal{C}$.
2. For every topological space X,

$$\mathrm{h}\mathbf{Sing}(X) \simeq \pi_{\geq 1}(X)$$

3. A simplicial set X is the nerve of a category if and only if it has unique inner horn fillers.
4. For every Kan complex K, the infinity-category $\mathrm{h}K$ is a groupoid.

Proof.

1. The 0-simplices of $N(\mathcal{C})$ are the objects of $\mathcal{C}$, and these are defined to be the objects of $\mathrm{h}N(\mathcal{C})$.

   The 1-simplices of $N(\mathcal{C})$ are precisely the morphisms of $\mathcal{C}$. In particular, identity morphisms correspond to degenerate simplices. In passing to $\mathrm{h}N(\mathcal{C})$, two morphisms are identified if there is a 2-simplex between them in the sense described above. But this says precisely that the only identifications are of the form $f \circ \mathrm{id} = f$ for all f.

   Simplices of degree 2 or higher are discarded.

2. The zero-simplices of both of these are the points of X, and the morphisms of both of these correspond to paths modulo homotopy.
3. Let f be an edge. By filling an outer horn, we find a left inverse g. By filling again, we find a left inverse h to this. But h has to be f, so we are done.

□

Theorem 153. Denote by **qCat** the full subcategory of $\mathbf{Set}_\Delta$ on ∞-categories.

1. There is a functor $\mathrm{h}\colon \mathbf{qCat} \mapsto \mathbf{Cat}$ which sends $\mathcal{C} \to \mathrm{h}\mathcal{C}$.
2. There is an adjunction
$$\mathrm{h} : \mathbf{qCat} \leftrightarrow \mathbf{Cat} : N,$$
where N denotes the nerve (Definition 43).

Proof.

1. Let $f\colon X \to Y$ be a map of ∞-categories. We need to define a map $\mathrm{h}f\colon \mathrm{h}X \to \mathrm{h}Y$. On objects, this is easy, since $f_0\colon X_0 \to Y_0$. On morphisms, we get a map $f_1\colon X_1 \to Y_1$ which sends morphisms $x \to y$ to morphisms $f_0(x) \to f_0(y)$, as can be checked by substituting d_0 and d_1 into the downward arrows below.
$$\begin{tikzcd} X_1 \arrow[r, "f_1"] \arrow[d] & Y_1 \arrow[d] \\ X_0 \arrow[r, "f_0"'] & Y_0 \end{tikzcd}$$
We still have to check that the map f_1 respects equivalence classes. To this end, suppose $g \sim g'\colon x \to y$, i.e. there is a simplex as follows.
$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "g'"'] & & y \end{tikzcd}$$
Then we have a simplex as follows
$$\begin{tikzcd} & f_0(y) \arrow[dr, "s_0(f_0(y))=f_1(s_0(y))"] & \\ f_0(x) \arrow[ur, "f_1(g)"] \arrow[rr, "f_1(g')"'] & & f_0(y) \end{tikzcd}$$
so $f_1(f) \sim f_1(g)$.

To see functoriality of h, suppose that if $F\colon X \to Y$ and $G\colon Y \to Z$ are morphisms of simplicial sets. We need to check that $\mathrm{h}(G \circ F) = \mathrm{h}G \circ \mathrm{h}F$. This is clearly true on objects, and on morphisms we have
$$\mathrm{h}(G \circ F)([f]) = [(G \circ F)(f)] = [G(F(f))] = \mathrm{h}G([F(f)]) = (\mathrm{h}G \circ \mathrm{h}F)([f]).$$
2. We exhibit a unit-counit adjunction. We first define a natural transformation ϵ with components
$$\epsilon_X\colon X \to N(\mathrm{h}X).$$

□

Note that we have made good on our promise, made in Section 3.5, to give a partial description of the left adjoint to the nerve N given by the Yoneda extension $\mathcal{Y}_!\chi$; when restricted to ∞-categories, it is the given by the functor h.

7.3 The coherent nerve

7.3.1 Topologically enriched categories

Definition 154 (compactly generated Hausdorff space)**.** A topological space X is known as a compactly generated Hausdorff space if for every subset $U \subset X$, the following are equivalent.

- U is closed.
- For each compact subset $C \subset X$, $C \cap U$ is closed

We denote the full subcategory of **Top** on compactly generated Hausdorff spaces by **CGHaus**.

Our reason for working in the category **CGHaus** rather than **Top** is a techical one: the category **Top** is not Cartesian closed, and therefore is not a good candidate for an enriching category. The other reason is that the geometric realization of any simplicial set is a compactly generated Hausdorff, and therefore we are entitled to

Definition 155 (topological category)**.** A category enriched over **CGHaus** (with the Cartesian monoidal structure) is known as a topological category.

Example 156. Consider the category $\mathcal{C}$, whose objects are CW complexes, and whose hom-spaces $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ consist of the set of continuous functions $X \to Y$ taken with the compact-open topology.

7.3.2 Simplicially enriched categories

Definition 157 (simplicial category)**.** A category enriched over $\mathbf{Set}_\Delta$ (with the Cartesian monoidal structure) is called a simplicial category. Denote the category whose objects are simplicial categories and whose morphisms are enriched functors by $\mathbf{Cat}_\Delta$.

Theorem 158. The category

7.3.3 Topological and simplicial enrichment

Lemma 159. The functor $\mathbf{Sing}\colon \mathbf{CGHaus} \to \mathbf{Set}_\Delta$ is strong monoidal.

Proof. To see this, we build a map

$$\Phi_{X,Y}\colon \mathbf{Sing}(X \times Y) \to \mathbf{Sing}(X) \times \mathbf{Sing}(Y)$$

as follows.

$$\begin{aligned}\mathbf{Sing}(X \times Y) &\simeq \mathbf{Top}(|\Delta^\bullet|, X \times Y)\\ &\simeq \mathbf{Top}(|\Delta^\bullet|, X) \times \mathbf{Top}(|\Delta^\bullet|, Y)\\ &\simeq \mathbf{Sing}(X) \times \mathbf{Sing}(Y).\end{aligned}$$

The singular set $\mathbf{Sing}(|\Delta^0|) \simeq \Delta^0$, so we don't have any choice in defining ϕ. □

Note 160. This notation is potentially confusing! We are *not* talking about the category of functors $\Delta^{\mathrm{op}} \to \mathbf{Cat}$.

Definition 161 (singular nerve)**.** Let $\mathcal{T}$ be a topological category. By Lemma 16, we can use $\mathbf{Sing}\colon \mathbf{Top} \to \mathbf{Set}_\Delta$ to turn $\mathcal{T}$ into a simplicial category. We will call this the singular nerve of $\mathcal{T}$ and denote it by $\mathbf{Sing}(\mathcal{T})$.

Definition 162 (topological nerve)**.** Let $\mathcal{T}$ be a topological category. The topological nerve of $\mathcal{T}$, denoted $N_{\mathrm{top}}(\mathcal{T})$, is the infinity category

$$N_\Delta \circ \mathbf{Sing}(\mathcal{T}).$$

Joke 163. How do you get a simplicially enriched category from a topologically enriched category? It's easy—just sing!

7.3.4 Thickening

Definition 164 (thickening)**.** We define a cosimplicial object called the *thickening* to be the functor

$$\mathfrak{C}\colon \Delta \to \mathbf{Cat}_\Delta$$

defined as follows.

It sends objects $[n]$ to the simplicially enriched category $\mathfrak{C}[\Delta^n]$ defined as follows.

- $\mathrm{Obj}(\mathfrak{C}[\Delta^n]) = \{0, \ldots, n\}$.
- For $i, j \in \{0, \ldots, n\}$
$$\mathfrak{C}[\Delta^n](i, j) = N(P_{i,j}),$$
where $P_{i,j}$ is the poset of subsets of $\{i, \ldots, j\}$ containing i and j.
- To define composition, we need a morphism of simplicial sets
$$N(P_{i,j}) \times N(P_{j,k}) \to N(P_{i,k}).$$
We have a multiplication $P_{i,j} \times P_{j,k} \to P_{i,k}$ given by $(I, J) \mapsto I \cup J$. This induces the map we need.

A morphism $f\colon [m] \to [n]$ induces a functor $\mathfrak{C}[\Delta^m] \to \mathfrak{C}[\Delta^n]$ on objects by $i \mapsto f(i)$, and on hom-simplicial-sets $N(P_{i,j})$ as follows. On the underlying posets, we send

$$P_{i,j} \supset \{i, k_1, \ldots, k_{r-1}, j\} \mapsto \{f(i), f(k_1), \ldots, f(k_{r-1}), f(j)\} \subset P_{f(i),f(j)}.$$

This extends to a functor of poset categories, hence a morphism of nerves.

Taking the Yoneda extension of the thickening, we find a functor $\mathfrak{C}\colon \mathbf{Set}_\Delta \to \mathbf{Cat}_\Delta$, called the thickening.

By the usual theory of cosimplicial objects (explained in Section 3.5) we have the following results.

Lemma 165. The thickening $\mathfrak{C}$ preserves colimits.

Definition 166 (simplicial nerve)**.** Let $\mathcal{C}$ be a simplicial category. The simplicial nerve of $\mathcal{C}$, denoted N_Δ, is the functor

$$N_\Delta(\mathcal{C}) = \mathbf{Cat}_\Delta(\mathfrak{C}(-), \mathcal{C})\colon \Delta^{\mathrm{op}} \to \mathbf{Set}_\Delta\,.$$

By Theorem 11, we have an adjunction

$$\text{よ}_!\mathfrak{C} : \mathbf{Set}_\Delta \leftrightarrow \mathbf{Cat}_\Delta : N_\Delta.$$

Proposition 167. Let $\mathcal{C}$ be a simplicial category such that for all x, y in $\mathcal{C}$, $\mathcal{C}(x, y)$ is a Kan complex. Then $N_\Delta(\mathcal{C})$ is an ∞-category.

Proof. We need to show that we can solve the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d, hook] & N_\Delta(\mathcal{C}) \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

Passing to adjuncts, we find the following lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i] \arrow[r] \arrow[d, hook] & \mathcal{C} \\ \mathfrak{C}[\Delta^n] \arrow[ur, dashed] & \end{tikzcd}$$

We can view the simplicial category $\mathfrak{C}[\Lambda^n_i]$ as a subcategory of $\mathfrak{C}[\Delta^n]$, where the only data missing[1] is the mapping space

$$\mathfrak{C}[\Delta^n](0, n).$$

Therefore, a solution to the above lifting problem need only fill in this information. This means we need only solve the corresponding lifting problem.

$$\begin{tikzcd} \mathfrak{C}[\Lambda^n_i](0, n) \arrow[r] \arrow[d, hook] & \mathcal{C}(0, n) \\ \mathfrak{C}[\Delta^n](0, n) \arrow[ur, dashed] & \end{tikzcd}$$

∎

Example 168. Let $\mathbf{Kan} \subset \mathbf{Set}_\Delta$ be the full subcategory on Kan complexes. We may consider $\mathbf{Kan}$ to be simplicially enriched by taking $\mathbf{Kan}(K, S) = \mathrm{Maps}(K, S)$. We have seen in Corollary 88 that each mapping space is a Kan complex. Thus, the simplicial nerve $N_\Delta(\mathbf{Kan})$ is an ∞-category.

Definition 169 (category of spaces)**.** The ∞-category of spaces is the category

$$\mathcal{S} = N_\Delta(\mathbf{Kan}).$$

[1]Why?

7.4 Mapping spaces

Definition 170 (mapping spaces of a simplicial set)**.** Let S be any simplicial set. For any x, $y \in S_0$, define the mapping space

$$\mathrm{Maps}_S(x, y)$$

to be the object

$$\mathrm{h}\,(\mathfrak{C}[S](x, y))$$

in the homotopy category of spaces $\mathcal{H}$.

7.5 Infinity-groupoids

Definition 171 (equivalence)**.** A morphism $f\colon x \to y$ in an infinity category is said to be an equivalence if the corresponding morphism in the homotopy category (Example 151) is an isomorphism.

Definition 172 (infinity-groupoid)**.** An ∞-category is said to be an ∞-groupoid if every morphism is an equivalence.

Theorem 173. The following are equivalent.

1. $\mathcal{C}$ is a Kan complex.
2. $\mathcal{C}$ is an infinity-groupoid.

Proof. The direction $1 \implies 2$ is obvious by Lemma 152. We will see why the other direction holds later. □

Definition 174 (maximal infinity groupoid)**.** Let $\mathcal{C}$ be an infinity category. The maximal infinity groupoid in $\mathcal{C}$, denoted $\mathcal{C}^{\simeq} \subset \mathcal{C}$, is the simplicial subset of $\mathcal{C}$ consisting of simplices all of whose edges are equivalences.

7.6 Functors between infinity categories

In this section, we build up the notions necessary to work with functors between infinity-categories.

In Subsection 7.6.1, we define functors between infinity-categories.

In Subsection 7.6.3, we define a monoidal product called the *join* on the category of categories, and show that it reproduces cones and cocones. We then generalize the construction to the category of simpicial sets, and show[2] that the two notions are compatible in the sense that the nerve is a monoidal functor between them; that is we have a natural isomorphism

$$N(\mathcal{C} \star \mathcal{D}) \simeq N(\mathcal{C}) \star N(\mathcal{D}).$$

In Subsection 7.6.2, we generalize the notion of the *opposite category* to the context of simplicial sets, and note that it defines an *opposite functor* on the category of simplicial sets, when one extends simplex category Δ in an appropriate way.

[2]Well, if there's time.

In Subsection 7.6.4, we see that joins allow us to define the notion of a cone in an infinity-categorical context. We also define the infinity-categorical generalization of comma categories, and note that there is a duality between the join and the construction of over- and undercategories. We then show that over- and undercategories are 'topologically' very simple—they are contractible Kan complexes.

In Subsection 7.6.5 and Subsection 7.6.6, we define the notion of initial (resp. terminal) object in an infinity-category $\mathcal{C}$. We set up our definitions in such a way that initial objects in an infinity category are initia objects in the homotopy category, and they are unique in the sense that their 'convex hull' is a contractible Kan complex.

In Subsection 7.6.7, we define when an object in an infinity category is a limit or colimit of a functor.

7.6.1 Functors and diagrams

Definition 175 (infinity-functor)**.** An infinity-functor (hereafter simply called *functor*) between infinity-categories $\mathcal{C}$ and $\mathcal{D}$ is simply a map $\mathcal{C} \longrightarrow \mathcal{D}$ of simplicial sets.

Here we list some theorems which we will prove later, after we have some more advanced notions.

Proposition 176. Let K be a simplicial set and let $\mathcal{C}$ be an infinity category. Then the simplicial set $\operatorname{Maps}(K, \mathcal{C})$ is an infinity-category.

Proof. Later. □

Definition 177 (homotopy coherent diagram)**.** Let $\mathcal{T}$ be a topological category and I an ordinary category. Let $\mathcal{C} = N_{\text{top}}(\mathcal{T})$.

7.6.2 Opposite categories

Henceforth, we have been considering the domain of simplicial sets to be the category of functors $\Delta^{\text{op}} \longrightarrow \mathbf{Set}$. However, we have sometimes relaxed this when it was convenient to do so. For example, the n-simplex Δ^n was defined by

$$\Delta^n = \Delta(-, [n]).$$

We have notated the subsimplex missing the 0th vertex by $\Delta^{\{1,\dots,n\}}$, despite the fact that $\{1, \dots, n\}$ is not an object in Δ, so it is not clear how to make sense of this; in particular, it does not make any sense to write

$$\Delta^{\{1,\dots,n\}} = \Delta(-, \{1, \dots, n\}).$$

What we are really doing is extending Δ to the category of finite, non-empty linearly ordered sets. We could do this because the inclusion functor $\Delta \hookrightarrow \mathbf{FinLinOrd}$ is an equivalence, so passing from Δ to $\mathbf{FinLinOrd}$ is just a shift of perspective.

On $\mathbf{FinLinOrd}$, there is a map Swap, which reverses the linear order. That is, it sends the set

$$\{0, \dots, n\} \mapsto \{n, \dots, 0\}.$$

Let us try to understand the behavior of Swap. Any map

$$f\colon \{0, \ldots, n\} \to \{0, \ldots, m\}$$

can be drawn diagrammatically as follows

$$\begin{tikzcd}
0 \arrow[dr, mapsto] & 0 \\
1 \arrow[dr, mapsto] & 1 \\
\vdots & \vdots \\
n \arrow[ur, mapsto] & n \\
& \vdots \\
& m
\end{tikzcd}$$

The condition that the map f be weakly monotonic is equivalent to the demand that none of the arrows in the above diagram cross. The swap operation turns the above diagram on its head, taking uncrossed arrows to uncrossed arrows. Thus, morphisms $f\colon [n] \to [m]$ are taken to morphisms $f^{\mathrm{op}}\colon [\bar{n}] \to [\bar{m}]$. Furthermore, composition is respected because composition is done by stacking diagrams left to right.

This discussion means that Swap is an endofunctor on **FinLinOrd**, and further that it is an equivalence of categories, since $\mathrm{Swap}^2 = \mathrm{id}$.[3]

Let $\mathcal{C}$ and $\mathcal{D}$ be regular categories. Any endofunctor $E\colon \mathcal{C} \to \mathcal{C}$ gives an endofunctor on the functor category $[\mathcal{C}, \mathcal{D}]$ as follows.

- On objects F, it is defined via the pullback.

$$F \mapsto F \circ E$$

- On morphisms η, it is defined via whiskering.

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "E"] & \mathcal{C} \arrow[r, bend left, "F", ""{name=U, below}] \arrow[r, bend right, "F'"', ""{name=D, above}] \arrow[Rightarrow, from=U, to=D, "\eta"] & \mathcal{D} & \mapsto & \mathcal{C} \arrow[r, bend left, "F \circ E", ""{name=U2, below}] \arrow[r, bend right, "F \circ E'"', ""{name=D2, above}] \arrow[Rightarrow, from=U2, to=D2, "E^*(\eta)"] & \mathcal{D}
\end{tikzcd}$$

Definition 178 (opposite functor)**.** We call the pullback by Swap the opposite functor, and denote it by $(-)^{\mathrm{op}}$.

Theorem 179. For any simplicial sets K and S, we have

$$(K \star S)^{\mathrm{op}} = S^{\mathrm{op}} \star K^{\mathrm{op}}.$$

Proof. □

[3]We are now solidly in evil territory. I wonder if there is a way out.

7.6.3 Joins

Definition 180 (join of categories)**.** Let $\mathcal{C}$ and $\mathcal{C}'$ be ordinary categories. The join of $\mathcal{C}$ and $\mathcal{C}'$, denoted $\mathcal{C} \star \mathcal{C}'$, is the category with objects

$$\mathrm{ob}(\mathcal{C} \star \mathcal{C}) = \mathrm{ob}(\mathcal{C}) \amalg \mathrm{ob}(\mathcal{C}')$$

and morphisms

$$(\mathcal{C} \star \mathcal{C}')(x, y) = \begin{cases} \mathcal{C}(x, y), & x, y \in \mathcal{C} \\ \mathcal{C}'(x, y), & x, y \in \mathcal{C}' \\ \{*\}, & x \in \mathcal{C}, y \in \mathcal{C}' \\ \emptyset, & x \in \mathcal{C}', y \in \mathcal{C} \end{cases}.$$

Example 181. Here is an alternative definition for the category of cones over a functor $F\colon \mathcal{C} \to \mathcal{D}$, given in terms of the join: it is the category

$$\mathbf{Fun}_F(\Delta^0 \star \mathcal{C}, \mathcal{D}),$$

i.e. the subcategory of $\mathbf{Fun}(\Delta^0 \star \mathcal{C}, \mathcal{D})$ whose objects are functors which agree with F when restricted to $\mathcal{C}$, and whose morphisms are natural transformations

Example 182. We have

$$[m] \star [n] = [m + n + 1].$$

This operation makes $\hat{\Delta}$ into a monoidal category, where $\hat{\Delta}$ includes the empty ordinal, denoted $[-1]$.

Our next goal is to extend $\star$ from Δ to $\mathbf{Set}_\Delta$. One might hope that this could be accomplished by Yoneda extension, and indeed one would be right. The general case of this is known as *Day convolution.*

For any simplicial set K, extend the domain of K to $\hat{\Delta}$ by defining $K([-1]) = \emptyset$. Given any simplicial sets K and S, we can define in this way a functor

$$K \bar{\otimes} S\colon \hat{\Delta}^{\mathrm{op}} \times \hat{\Delta}^{\mathrm{op}} \to \mathbf{Set}; \qquad ([m], [n]) \mapsto K([m]) \times S([n]).$$

Definition 183 (join of simplicial sets)**.** Let K and S be simplicial sets. The join $K \star S$ of K and S is given by the following left Kan extension

$$\begin{tikzcd} \hat{\Delta}^{\mathrm{op}} \times \hat{\Delta}^{\mathrm{op}} \arrow[rr, "K \bar{\otimes} S"] \arrow[dr, "\star^{\mathrm{op}}"'] & & \mathbf{Set} \\ & (\hat{\Delta})^{\mathrm{op}} \arrow[ur, "K \star S"'] & \end{tikzcd}$$

Theorem 184. We can compute the join by the following formula.

$$(K \star K')_n = \coprod_{[i] \oplus [j] \simeq [n]} K_i \times K'_j,$$

and which acts on a morphism $\phi\colon [m] \to [n]$ by sending a simplex

$$\sigma = (\alpha \in K_i, \beta \in K_j, [i] \oplus [j] \simeq [n]) \in (K \star K')_n$$

to a simplex

$$\phi^* \sigma = (\psi_1^* \alpha, \psi_2^* \beta, \phi^{-1}([i]) \oplus \phi^{-1}([j]) \simeq [m]) \in (K \star K')_m,$$

where

$$\psi_1 = \phi|_{\phi^{-1}([i])}, \qquad \text{and} \qquad \phi_2 = \phi|_{\phi^{-1}([j])}.$$

This is functorial since

Proposition 185. The join of two ∞-categories is again an ∞-category.

Proposition 186. The join is functorial in each argument separately.

Proof. Given a map $f\colon K \to K'$ of simplicial sets, we need to specify a map

$$(f, \mathrm{id}_S)\colon K \star S \to K' \star S.$$

By the universal property of coproducts, we can do this by defining maps

$$K_i \times S_j \to K_i' \times S_j$$

for each $[i] \oplus [j] \simeq [n]$. We have a map $f_i\colon K_i \to K_i'$, which gives a map $f_i \times \mathrm{id}$ as required. We need only check that this is a morphism of simplicial sets, i.e. makes the following square commute for each $\phi\colon [n] \to [m]$.

$$\begin{tikzcd}
(K \star S)_m \arrow[r] \arrow[d] & (K \star S)_n \arrow[d] \\
(K' \star S)_m \arrow[r] & (K' \star S)_n
\end{tikzcd}$$

Let $\sigma = (\alpha, \beta, [i] \oplus [j] \simeq [n]) \in (K \star S)_m$. We follow it around.

$$\begin{tikzcd}
(\alpha, \beta) \arrow[r, mapsto] \arrow[d, mapsto] & (\psi_i^* \alpha, \psi_j^* \beta) \arrow[d, mapsto] \\
(f_i(\alpha), \beta) \arrow[r, mapsto] & (f_{\phi^{-1}([i])} \psi_i^* \alpha \overset{!}{=} \psi_i^*(f_i \alpha), \beta)
\end{tikzcd}$$

The condition that $f_{\phi^{-1}([i])} \psi_i^* \alpha = \psi_i^*(f_i \alpha)$ is preciesely the condition that the following diagram commute, which it does by naturality of f.

$$\begin{tikzcd}
K_i \arrow[r, "\psi_i^*"] \arrow[d, "f_i"'] & K_{\phi^{-1}([i])} \arrow[d, "f_{\phi^{-1}([i])}"] \\
K_i' \arrow[r, "\psi_i^*"'] & K'_{\phi^{-1}([i])}
\end{tikzcd}$$

Functoriality in the second argument is exactly analogous. $\square$

7.6.4 Cones and cocones

Definition 187 (left cone, right cone)**.** let K be a simplicial set, and let $x \in K$.

- The left cone of x is the simplicial set
$$x^{\triangleleft} = \Delta^0 \star x.$$
- The right cone of x is the simplicial set
$$x^{\triangleright} = x \star \Delta^0.$$

Definition 188 (overcategory, undercategory)**.** Let K and S be simplicial sets and let $p\colon K \to S$ be a morphism.

- The overcategory $S_{p/}$ is the category whose n-simplices are
$$(S_{p/})_n = \{\text{maps } f\colon K \star \Delta^n \to \mathcal{C} \text{ satisfying } F|_K = p\}.$$
- The undercategory $S_{/p}$ is the category whose n-simplices are
$$(S_{/p})_n = \{\text{maps } f\colon \Delta^n \star K \to \mathcal{C} \text{ satisfying } F|_K = p\}.$$

Theorem 189. There is a pair of almost adjoint functors
$$K \star - : \mathbf{Set}_{\Delta} \leftrightarrow (\mathbf{Set}_{\Delta})_{/p} : /p$$
More explicitly, given a map $p\colon X \to \mathcal{C}$, there is a natural bijection
$$\mathrm{Hom}_p(K \star X, \mathcal{C}) \simeq \mathrm{Hom}(K, \mathcal{C}_{/p}),$$
where Hom_p denotes the set of maps $f\colon K \star X \to \mathcal{C}$ such that $f|_K = p$.

Corollary 190. A map

For regular categories, it is possible to define colimits as limits in opposite categories; the two notions are equivalent. In infinity categories, this is also the case.

In the Kan model structure, trivial fibrations are Kan fibrations which are also homotopy equivalences. However, we have immediately that $\mathsf{W} \cap \mathsf{Fib} = \mathsf{Cof}_{\perp}$, i.e. trivial Kan fibrations are precisely those morphisms of simplicial sets that have the right lifting property with respect to monomorphisms. However, since Cof is saturated, this is equivalent to having the left lifting property with respect to boundary inclusions.

Lemma 191. Let $\mathcal{C}$ be a Kan complex, and $x \in \mathcal{C}$. Then $\mathcal{C}_{/x}$ and $\mathcal{C}_{x/}$ are contractible Kan complexes.

Proof. We need to show that we can solve lifting problems of the following form.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & \mathcal{C}_{/x} \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \{*\} \end{tikzcd}$$

Consider □

7.6.5 Initial objects

Definition 192 (initial object)**.** Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We say that x is an initial object if the map

$$\mathcal{C}_{x/} \to \mathcal{C}; \qquad \Delta^0 \star \Delta^n \to \Delta^n$$

is a trivial Kan fibration.

We would like initial/terminal objects to be objects that have an edge to each other object, and we would like this edge to be unique in some sense. Demanding uniqueness on the nose, however, would go agains the spirit of the theory of infinity categories.

We have, however, the following.

Proposition 193. Let $\mathcal{C}$ be an ∞-category, and let x be an initial object in $\mathcal{C}$. Then for each other object $y \in \mathcal{C}$, there is an edge $x \to y$.

Furthermore, this edge is unique up to homotopy.

Proof. Consider the following solid lifting problem.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^0 \arrow[r, "y"] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

The left-hand arrow is a boundary inclusion $\partial\Delta^0 \to \Delta^0$, so we can find a dashed lift, giving us a zero-simplex in $\mathcal{C}_{x/}$. But by definition, a zero simplex in $\mathcal{C}_{x/}$ is a 1-simplex $\sigma\colon \Delta^0 \star \Delta^0 \to \mathcal{C}$ such that $\sigma|_{\text{first factor}} = x$. Furthermore, the commutativity of the lower triangle tells us that $\sigma|_{\text{second factor}} = y$, i.e. σ is an edge $x \to y$.

Now suppose we have two edges $f, g\colon x \to y$. These give us the following solid lifting problem.

$$\begin{tikzcd} \partial\Delta^1 \arrow[r, "{(f,g)}"] \arrow[d, hook] & \mathcal{C}_{x/} \arrow[d] \\ \Delta^1 \arrow[r, "\mathrm{id}_y"'] \arrow[ur, dashed] & \mathcal{C} \end{tikzcd}$$

Again, this is a boundary inclusion, giving us a dashed lift, i.e. a 2-simplex α in $\mathcal{C}$ whose 0th vertex is x. The commutativity of the upper square tells us that $d_2\alpha = f$ and $d_0\alpha = g$, and the commutativity of the lower square tells us that $d_1\alpha = \mathrm{id}_y$. Thus, we have the following simplex in $\mathcal{C}$.

$$\alpha = \begin{tikzcd} & y \arrow[dr, "\mathrm{id}_y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

But this provides precisely a homotopy $f \to g$. □

Corollary 194. Let $\mathcal{C}$ be an ∞-category. If x is an initial object in $\mathcal{C}$, then x is also initial in $h\mathcal{C}$.

One might expect that these conditions were equivalent. However, that is not the case; an object can be initial in $h\mathcal{C}$ without being initial in $\mathcal{C}$.

Example 195. Take any simply connected space X such that there exists a point $x \in X$ with $\pi_2(X, x)$ non-trivial. Then there is a 2-simplex $\alpha\colon \Delta^2 \to \mathbf{Sing}(X)$ with boundary $\partial\alpha = x$ which is not homotopy equivalent to x.

Since a 2-simplex with its zeroth vertex x is equivalently a map $\Delta^1 \to \mathcal{C}_{x/}$, this data organizes into the following lifting problem.

$$\begin{tikzcd} \partial\Delta^2 \arrow[r, "{(x,x,\alpha)}"] \arrow[d] & \mathbf{Sing}(X)_{x/} \arrow[d] \\ \Delta^2 \arrow[r, "x"'] \arrow[ur, dashed] & \mathbf{Sing}(X) \end{tikzcd}$$

The solution to this lifting problem would give us a homotopy from α to x. However, by assumption, there is no such homotopy. Thus, x is not initial in $\mathbf{Sing}(X)$. However, every point $x \in X$ is initial in $h\mathbf{Sing}(X)$ because $h\mathbf{Sing}(X) \simeq \pi_{\geq 1}(X)$, and there is exactly one homotopy class of paths between any two points of X because X is simply connected.

Suppose we are given any two initial objects in an infinity category. We have seen that we can find an edge between them, and that given two edge between them, we can find a homotopy. In turn, given homotopies, we can find a higher homotopy between them, etc. One might therefore think of the set of all initial objects as forming the vertices of some polyhedron, and the filling conditions above telling us that the polyhedron in question has no gaps; that is, that it is homotopy equivalent to the point. This intuition turns out to be correct.

Theorem 196. Let $\mathcal{C}$ be a simplicial set with at least one initial object, and let $\mathcal{C}'$ be the simplicial subset consisting of all simplices all of whose vertices are initial objects in $\mathcal{C}$. Then $\mathcal{C}'$ is a contractible Kan complex.

Proof. The statment that $\mathcal{C}'$ is contractible means that the map $\mathcal{C}' \to *$ is a weak equivalence. The fact that it is a Kan complex means precisely that the map $\mathcal{C}' \to *$ is a fibration. Therefore, we need to check that the map $\mathcal{C}' \to *$ is a trivial fibration, i.e. has the right lifting property with respect to all boundary fillings.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d, hook] & \mathcal{C}' \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

For $n = 0$, the lifting problem demands that there exists at least one initial object, which is true by assumption.

For $n \geq 1$, consider an n-boundary $\beta\colon \partial\Delta^n \to \mathcal{C}'$. Denote by x the 0th vertex; by assumption, this is an initial object. By forgetting the 0th face, we can consider this as a boundary inclusion

$$\partial\Delta^{\{1,\ldots,n\}} \to \mathcal{C}_{x/}$$

making the following diagram commute.

$$
\begin{tikzcd}
\partial\Delta^{\{1,\ldots,n\}} \arrow[r] \arrow[d] & \mathcal{C}_{x/} \arrow[d] \\
\Delta^{\{1,\ldots,n\}} \arrow[r, "d_0\beta"'] & \mathcal{C}
\end{tikzcd}
$$

However, we can find a lift, i.e. a map $\Delta^{\{1,\ldots,n\}} \to \mathcal{C}_{x/}$, i.e. an n-simplex in $\mathcal{C}$ which fills β. □

7.6.6 Terminal objects

Definition 197 (terminal object)**.** Let $\mathcal{C}$ be an ∞-category and $x \in \mathcal{C}$ an object. We say that x is a terminal object if the map

$$\mathcal{C}_{/x} \to \mathcal{C}; \qquad \Delta^n \star \Delta^0 \mapsto \Delta^n$$

is a trivial Kan fibration.

The theory of terminal objects is completely dual to that of initial objects. For that reason, we list only results here.

Let $\mathcal{C}$ be an infinity category.

- If y is a terminal object in $\mathcal{C}$, for each object $x \in \mathcal{C}$ there is an edge $x \to y$, which is unique up to homotopy.
- These homotopies are in turn unique up to homotopy.
- Terminal objects in an ∞-category are always terminal in the homotopy category; however, the converse is not true.
- The simplicial subset of all simplices whose vertices are terminal is a contractible Kan complex.

7.6.7 limits and colimits

Definition 198 (limit cone)**.** Let $\mathcal{C}$ be an infinity category, K a simplicial set, and $p\colon K \to \mathcal{C}$ a diagram.

- A colimit cone for p is an initial object in the category $\mathcal{C}_{p/}$.
- A limit cone for p is a terminal object in the category $\mathcal{C}_{/p}$.

Suppose that I and $\mathcal{D}$ are regular categories, and let $F\colon I \to \mathcal{D}$ be a functor. Passing to nerves, we get a functor $N(F)$ between infinity categories. Let us try and understand what a limit cone for $N(F)$ means in this context.

First, let us understand the zero-simplices of $N(\mathcal{D}_{/F})$. By definition, zero-simplices are given by maps

$$\sigma\colon \Delta^0 \star N(I) \to N(\mathcal{D}) \qquad \text{such that } \sigma|_{N(I)} = N(F).$$

By the monoidality of the nerve with respect to the join, such a map is the nerve of a functor

$$\tilde{\sigma}\colon \{*\} \star I \to \mathcal{D} \qquad \text{such that } \tilde{\sigma}|_I = F.$$

But this is precisely a natural transformation from the constant functor $\sigma|_{\{*\}} \to \mathcal{D}$ to F.

Similarly, a 1-simplex τ is a map

$$\tau \colon \Delta^1 \star N(I) \to N(\mathcal{D}) \qquad \text{such that } \tau|_{N(I)} = N(F).$$

This is the nerve of a functor

$$\tilde{\tau} \colon \{\bullet \to *\} \star I \to \mathcal{D} \qquad \text{such that } \tilde{\tau}|_I = F$$

Such functors are precisely morphisms of cones.

Corollary 199. Limit and colimit cones are unique up to contractible choice.

Proof. Theorem 200. □

Theorem 200. Let S be a simpicial set, $\mathcal{C}$ an infinity category, and $p \colon K \to C$ a map of simplicial sets.

1. There is an equivalence of categories

$$(\mathcal{C}_{/p})^{\mathrm{op}} \simeq (\mathcal{C}^{\mathrm{op}})_{p^{\mathrm{op}}/}.$$

2. The limit of p in $\mathcal{C}$ is the colimit of p^{op} in $\mathcal{C}^{\mathrm{op}}$.

Proof.

1. First we show the result on simplices. The n-simplices of $(\mathcal{C}^{\mathrm{op}})_{p^{\mathrm{op}}/}$ are

$$\{f \colon K^{\mathrm{op}} \star \Delta^n \to \mathcal{C}^{\mathrm{op}} \mid f|_{K^{\mathrm{op}}} = p^{\mathrm{op}}\} \simeq \{f^{\mathrm{op}} \colon (\Delta^n)^{\mathrm{op}} \star K \to \mathcal{C} | f^{\mathrm{op}}|_K = p\}$$

   But these are precisely the n-simplices of $(\mathcal{C}_{/p})^{\mathrm{op}}$. Thus, the map on n-simplices is given by $f \mapsto f^{\mathrm{op}}$.

   Let $f \colon [m] \to [n]$. We need to check that the following naturality square commutes.

$$\begin{tikzcd}
(\otimes\mathcal{C}^{\mathrm{op}}_{p^{\mathrm{op}}/})_n \arrow[rrr, "\mathrm{op}"] \arrow[ddd] & & & (\mathcal{C}_{/p})^{\mathrm{op}}_n \arrow[ddd] \\
& f \arrow[r, mapsto] \arrow[d, mapsto] & f^{\mathrm{op}} \arrow[d, mapsto] & \\
& f \circ (\mathrm{id}_{K^{\mathrm{op}}} \star \phi) \arrow[r, mapsto] & f^{\mathrm{op}} \circ (\phi^{\mathrm{op}} \star \mathrm{id}_K) & \\
(\otimes\mathcal{C}^{\mathrm{op}}_{p^{\mathrm{op}}/})_m \arrow[rrr, "\mathrm{op}"'] & & & (\mathcal{C}_{/p})^{\mathrm{op}}_m
\end{tikzcd}$$

   But it does.

2. Let x be a colimit of p^{op} in $\mathcal{C}^{\mathrm{op}}$. Then the map

$$\mathcal{C}^{\mathrm{op}}_{x/} \to \mathcal{C}^{\mathrm{op}}$$

   is a trivial Kan fibration. Now consider the map

□

8 The Grothendieck construction

8.1 The fibration zoo

One thinks of fibrations as a vast generalization of the notion of the projection $V \to M$ of a fiber bundle. In a fiber bundle, each fiber is isomorphic. Different notions of fibration relax this condition. For example

In Subsection 8.1.1, we study functors from one category $\mathcal{C}$ to another category $\mathcal{D}$, and define when such functors exhibit $\mathcal{C}$ as being *(co)fibered in groupoids* over $\mathcal{D}$. Roughly, (co)fibration in groupoids is a regularity condition on the fibers of F: it says that the fibers are groupoids, and that we can lift morphims between objects of $\mathcal{D}$ to functors between the fibers over them in a pseudofunctorial way. In the case of cofibration, this lifting is covariant; in the case of fibration, it is contravariant.

We then provide a generalization, the notion of a *coCartesian fibration.* A coCartesian fibration is one whose

8.1.1 Cofibrancy in groupoids

Definition 201 (cofibrant in groupoids)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that f exhibits $\mathcal{C}$ as cofibrant in groupoids over $\mathcal{D}$ if the following conditions are satisfied.

1. For every $c \in \mathcal{C}$ and every morphism
$$\eta\colon F(c) \to d \qquad \text{in } \mathcal{D}$$
there exists a morphism
$$\tilde{\eta}\colon c \to \tilde{d} \qquad \text{in } \mathcal{C}$$
such that $F(\tilde{\eta}) = \eta$.
2. For every morphism $\eta\colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map
$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c), F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$
is a bijection.

Let us unravel what the second condition means. The second set is the following pullback.

$$\begin{tikzcd}
* \arrow[r] \arrow[d] & \mathcal{D}(F(c'), F(c'')) \arrow[d, "\eta"] \\
\mathcal{C}(c, c'') \arrow[r, "F"'] & \mathcal{D}(F(c), F(c''))
\end{tikzcd}$$

Its elements are pairs

$$(f, \bar{g}), \qquad \text{where} \quad f : c \to c'' \quad \text{and} \quad \bar{g} \colon F(c') \to F(c'')$$

such that

$$F(f) = \bar{g} \circ F(\eta).$$

The bijection above is defined as follows. Starting on the LHS with a morphism $h \colon c' \to c''$, build the following commutative diagram.

$$\begin{tikzcd} & c' \arrow[dr, "h"] & \\ c \arrow[ur, "\eta"] \arrow[rr, "h\circ\eta"'] & & c'' \end{tikzcd}$$

We can read off a map $f = h \circ \eta \colon c \to c''$ and a map $\bar{g} = F(h) \colon F(c') \to F(c'')$, and indeed $F(f) = \bar{g} \circ F(\eta)$, giving us an element of the RHS.

The fact that this is a bijection means that the above operation is invertible. Fixing a morphism $\eta \colon c \to c'$ and an object c'', whenever we have the following data:

- A morphism $g \colon c \to c''$ as follows

$$\begin{tikzcd} & c' & \\ c \arrow[ur, "\eta"] \arrow[rr, "\bar{g}"'] & & c'' \end{tikzcd}$$

- A map $\bar{h} \colon F(c') \to F(c'')$ making the following diagram commute

$$\begin{tikzcd} & F(c') \arrow[dr, "\bar{h}"] & \\ F(c) \arrow[ur, "F(\eta)"] \arrow[rr, "F(g)"'] & & F(c'') \end{tikzcd}$$

we can lift $\bar{h}$ to a morphism h in $\mathcal{C}$ making the following diagram commute.

$$\begin{tikzcd} & c' \arrow[dr, "h"] & \\ c \arrow[ur, "\eta"] \arrow[rr, "g"'] & & c'' \end{tikzcd}$$

Lemma 202. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f) = \mathrm{id}$. Then f is an isomorphism.

Proof. Let $f : c \to c'$ in $\mathcal{C}_d$ be a morphism such that $F(f) = \mathrm{id}_d$. Then we can form the following commutative diagram.

$$\begin{tikzcd} & c' & \\ c \arrow[ur, "f"] \arrow[rr, "\mathrm{id}"'] & & c \end{tikzcd}$$

When mapped to $\mathcal{D}$, we find the following very simple diagram, with a dashed filler.

$$\begin{tikzcd} & d \arrow[dr, dashed, "\mathrm{id}"] & \\ d \arrow[ur, "\mathrm{id}"] \arrow[rr, "\mathrm{id}"'] & & d \end{tikzcd}$$

Lifting this gives us a left inverse $\tilde{f}$ to f. Pulling the same trick with $\tilde{f}$ gives us a left inverse $\hat{f}$ to $\tilde{f}$. Then

$$\begin{aligned} f \circ \tilde{f} &= \mathrm{id} \\ \hat{f} \circ f \circ \tilde{f} &= \hat{f} \\ \hat{f} &= f, \end{aligned}$$

so $\tilde{f}$ is both a left and a right inverse for f, i.e. f is an isomorphism. □

Corollary 203. Let $F\colon \mathcal{C} \to \mathcal{D}$ exhibit $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Then the fibers of F are groupoids.

Proof. The morphisms in the fiber $\mathcal{C}_d$ are precisely those which are mapped to the identity id_d. □

Lemma 204. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Let f be a morphism in $\mathcal{C}$ such that $F(f)$ is an equivalence. Then f is an equivalence.

Definition 205 (coCartesian morphism)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor of small categories. A morphism $f\colon c \to c'$ is said to be F-coCartesian if, for every $c'' \in \mathcal{C}$, the map

$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \times_{\mathcal{D}(F(c), F(c''))} \mathcal{D}(F(c'), F(c''))$$

is a bijection.

Definition 206 (coCartesian fibration)**.** A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be a coCartesian fibration if, for each $x \in \mathcal{C}$ and every morphism $\tilde{f}\colon F(x) \to y$ in $\mathcal{D}$, there exists a coCartesian morphism $f\colon x \to \tilde{y}$ such that $F(f) = \tilde{f}$.

The reader has probably guessed the next sentence: "There is a dual notion."

Definition 207 (fibrant in groupoids)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that f exhibits $\mathcal{C}$ as fibrant in groupoids over $\mathcal{D}$ if the following conditions are satisfied.

1. For every $c' \in \mathcal{C}$ and every morphism
$$\eta\colon d \to F(c') \qquad \text{in } \mathcal{D}$$
there exists a morphism
$$\tilde{\eta}\colon \tilde{d} \to c' \qquad \text{in } \mathcal{C}$$
such that $F(\tilde{\eta}) = \eta$.
2. For every morphism $\eta\colon c \to c'$ in $\mathcal{C}$ and every object $c'' \in \mathcal{C}$, the map
$$\mathcal{C}(c', c'') \to \mathcal{C}(c, c'') \underset{\mathcal{D}(F(c), F(c''))}{\times} \mathcal{D}(F(c'), F(c'')).$$
is a bijection.

8.1.2 Fibrations in infinity categories

Let $f: K \to S$ be a morphism of simplicial sets.

We have already met Kan fibrations. Kan fibrations have the following straightforward generalizations.

- The morphism f is a Kan fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 \leq i \leq n$.
- The morphism f is a left fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 \leq i < n$.
- The morphism f is a right fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i \leq n$.
- The morphism f is an inner fibration if it has the right lifting property with respect to all horn inclusion $\Lambda^n_i \to \Delta^n$ with $0 < i < n$.

We will use the following terminology.

- The morphism f is anodyne if it has the left-lifting property with respect to all Kan fibrations.
- The morphism f is left anodyne if it has the left-lifting property with respect to all left fibrations.
- The morphism f is right anodyne if it has the left-lifting property with respect to all right fibrations.
- The morphism f is inner anodyne if it has the left-lifting property with respect to all inner fibrations.

We now define the following notions.

Definition 208 (Cartesian fibration)**.** Let $f: K \to S$ be a morphism of simplicial sets. The morphism f is a Cartesian fibration if the following conditions are satisfied.

- The morphism f is an inner fibration
- For every $h: x \to y$ in S and $\tilde{y} \in K_0$ there is a f-Cartesian morphism $\tilde{h}: \tilde{x} \to \tilde{y}$ such that $f(\tilde{h}) = h$.
- A f-Cartesian edge in K is an edge $f: x \to y$ such that the map
$$K_{/f} \to K_{/y} \underset{S_{/f(y)}}{\times} S_{/p(f)}$$
is a trivial Kan fibration.

Theorem 209. Let f be a trivial fibration. Then f is a Kan fibration.

Proof. Let f be a trivial fibration, and consider the following lifting problem.

$$\begin{tikzcd} \Lambda^n_i \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] & S \end{tikzcd}$$

Restricting to the face opposite i, we find the lifting problem

$$\begin{tikzcd} \partial\Delta^{\{1,\dots,n\}} \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^{\{1,\dots,n\}} \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed solution. This allows us to turn our map $\Delta^n_i \to K$ to a map $\partial\Delta^n \to K$ making the folowing diagram commute.

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & K \arrow[d] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

which has a dashed lift. □

Note 210. A map $f\colon X \to Y$ is a left fibration if and only if $f^{\mathrm{op}}\colon X^{\mathrm{op}} \to Y^{\mathrm{op}}$ is a a right fibration. Thus, it suffices to study left fibrations; the theory of right fibrations is dual.

Lemma 211. Let $p\colon \mathcal{C} \to \mathcal{D}$ be a left fibration of infinity categories, and let $\tilde{f}\colon \bar{x} \to p(y)$ be an equivalence in $\mathcal{D}$, and $y \in \mathcal{C}_0$. Then there exists $f\colon x \to y$ so that $p(f) = \bar{f}$.

Lemma 212. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Then the induced map

$$S_{p/} \to X_{\bar{p}/} \underset{S_{\bar{p}/}}{\times} S_{r/}$$

is an inner fibration. If q is a left fibration, then the map is a left fibration.

Corollary 213. Let $\mathcal{C}$ be an infinity category, and let $p\colon K \to \mathcal{C}$ be a diagram in $\mathcal{C}$. Then the map $\mathcal{C}_{/p} \to \mathcal{C}$ is a left fibration, and $\mathcal{C}_{/p}$ is an infinity category.

Proof. Take $X = \mathcal{C}$, $S = \Delta^0$, and $\bar{K} = \emptyset$. This tells us that the map

$$\mathcal{C}_{p/} \to \mathcal{C}$$

is a left fibration. Thus, the composition

$$\mathcal{C}_{p/} \to \mathcal{C} \to *$$

is the composition of a left fibration and an inner fibration, so is itself an inner fibration. □

Corollary 214. Let $f\colon x \to y$ be a morphism in an ∞-category $\mathcal{C}$. Then f is an equivalence if and only if for every $n \geq 2$ and left horn $h\colon \Lambda^n_0 \to \mathcal{C}$, with $h|_{\{0,1\}} = f$, h admits a filler.

Theorem 215. The following are equivalent for an ∞-category $\mathcal{C}$.

- The homotopy category $\mathrm{h}\mathcal{C}$ is a groupoid.
- Every left horn in $\mathcal{C}$ has a filler.
- Every right horn has a filler.
- $\mathcal{C}$ is a Kan complex.

Corollary 216. All fibers of a left fibration are Kan complexes.

8.2 The classical Grothendieck construction

Definition 217 (Grothendieck construction)**.** Let $\mathcal{D}$ be a small category, and let $\chi\colon \mathcal{D} \to \mathbf{Grpd}$. The Grothendieck construction consists of the following data the following data.

- A category $\mathcal{C}_\chi$
- A functor $\pi\colon \mathcal{C}_\chi \to \mathcal{D}$

The category $\mathcal{C}_\chi$ is as follows.

- **Objects:** Pairs (d, x), where $d \in \mathcal{D}$ and $x \in \chi(d)$.
- **Morphisms:** A morphism $(d, x) \to (d', x')$ consists of
    - A morphism $f\colon d \to d'$ in $\mathcal{D}$
    - An isomorphism $\alpha\colon \chi(f)(x) \simeq x'$

The functor π is the forgetful functor to $\mathcal{D}$.

The Grothendieck construction can be understood in the following way. The functor χ gives us a way of associating a groupoid to each object $d \in \mathcal{D}$. Each of these groupoids is itself a category, and has its own objects. The idea of the Grothendieck construction is to throw all of these objects together into one big category. To keep track of where each object x comes from, we think of x as ordered pairs (d, x), where the $x \in \chi(\mathcal{D})$. The functor π simply remembers where each x came from.

For two objects x and $\tilde{x}$ from the same $\chi(d)$, the morphisms $(d, x) \to (d, \tilde{x})$ are simply the morphisms between x and $\tilde{x}$. For an object x' from a different groupoid $\chi(d')$, the morphisms $(d, x) \to (d', x')$ come from translating, using χ and a morphism $d \to d'$, morphisms in $\chi(d)$ into morphisms in $\chi(d')$.

Example 218. Let G and H be a groups, and $\mathbf{B}G$ and $\mathbf{B}H$ their delooping groupoids. Let $\phi\colon G \to H$ be a group homomorphism, or equivalently a functor $\mathbf{B}G \to \mathbf{B}H$. This can also be taken to be a functor $\mathbf{B}G \to \mathbf{Grpd}$.

The Grothendieck construction consists of a category with one object, whose morphisms consist of an element $g \in G$

There is also a contravariant version of the Grothendieck construction, consisting of the following.

Definition 219 (contravariant Grothendieck construction)**.** For any functor $\mathcal{D}^{\mathrm{op}} \to \mathrm{Grpd}$, the contravariant Grothendieck construction consists of the following.

1. A category $\mathcal{C}^\chi$
2. A functor $\pi\colon \mathcal{C}^\chi \to \mathcal{D}$.

The category $\mathcal{C}^\chi$ is as follows.

- The objects of $\mathcal{C}^\chi$ are be pairs (d, x), with $d \in \mathcal{D}$ and $x \in \chi(d)$.
- The morphisms $(d', x') \to (d, x)$ are pairs (f, α), where $f\colon d' \to d$ and $\alpha\colon x' \to F(f)(x)$.

There is the following relationship between the covariant and contravariant Grothendieck constructions. $\mathcal{C}^\chi = \mathcal{C}_\chi^{\mathrm{op}}$.

Example 220. Let $X\colon \mathcal{D}^{\mathrm{op}} \to \mathbf{Set}$ be a diagram of sets, which we interpret as discrete groupoids. Then there is an equivalence of categories between the contravariant construction and the slice category $\mathcal{D}/X$.

The objects of $\mathcal{D}/X$ are pairs (d, α), where $d \in \mathcal{D}$ and $\alpha\colon \mathrm{Hom}_{\mathcal{D}}(-, d) \Rightarrow X$. The morphisms $(d, \alpha) \to (d', \beta)$ are morphisms $f\colon d \to d'$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(-, d) \arrow[rr, Rightarrow, "f\circ-"] \arrow[dr, Rightarrow, "\alpha"'] & & \mathrm{Hom}_{\mathcal{D}}(-, d') \arrow[dl, Rightarrow, "\beta"] \\ & X & \end{tikzcd} \tag{8.1}$$

commutes.

Consider any such morphism f. The above diagram commutes if and only if it commutes component-wise, i.e. if for each c, the diagram below commutes.

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(c, d) \arrow[rr, "f\circ-"] \arrow[dr, "\alpha_c"'] & & \mathrm{Hom}_{\mathcal{D}}(c, d') \arrow[dl, "\beta_c"] \\ & X(c) & \end{tikzcd} \tag{8.2}$$

In particular, it commutes when $c = d$. Following the identity id_d around, we find that morphisms f making the above diagram commute satisfy the following equality.

$$\begin{tikzcd} \mathrm{id}_d \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \arrow[dl, mapsto] \\ & \alpha_d(\mathrm{id}_d) = \beta_d(f) & \end{tikzcd}$$

By the naturality of the Yoneda embedding, we have that

$$\beta_d(f) = X(f)(\beta_{d'}(\mathrm{id}_{d'})),$$

so

$$\alpha_d(\mathrm{id}_d) = X(f)(\beta_{d'}(\mathrm{id}_{d'})). \tag{8.3}$$

We can specify a functor $\mathcal{D}/X \to \mathcal{C}^X$ on objects by sending

$$(d, \alpha) \mapsto (d, \alpha_b(\mathrm{id}_b)),$$

and on morphisms $f\colon (d, \alpha) \to (d', \beta)$ commute by sending

$$f \mapsto (f, \mathrm{id}_{X(f)(\beta_{d'}(\mathrm{id}_{d'}))}).$$

We are allowed to use the above identity map preciesly because it is an isomorphism

$$\alpha_d(\mathrm{id}_d) \to X(f)(\beta_{d'}(d'))$$

as shown in Equation 8.3.

The fully faithfulness of the Yoneda embedding implies that the map between the objects of $\mathcal{D}/X$ and the objects of $\mathcal{C}^X$ is a bijection. We will now construct a functor $\mathcal{C}^X \to \mathcal{D}/X$, defined on objects to be the inverse of the one defined above, i.e. sending a pair (d, x) to the pair (d, α^x), where α^x is the natural transformation uniquely specified by $\alpha^x_d(\mathrm{id}_d) \mapsto x \in X(d)$.

Our inverse functor acts on any morphism (f, ϕ) by forgetting ϕ. It only remains to check that this provides a well-defined map of morphisms, as f has to make Diagram 8.1 commute. We can check that this is so by following a general morphism $g\colon c \to d$ around Diagram 8.2, finding the following condition.

$$\begin{tikzcd} g \arrow[rr, mapsto] \arrow[dr, mapsto] & & f \circ g \arrow[dl, mapsto] \\ & \alpha_c(g) \overset{!}{=} \beta_c(f \circ g) & \end{tikzcd}$$

By naturality of the Yoneda embedding, we have

$$\alpha_c(g) = (Xg)(\alpha_d(\mathrm{id}_d)), \qquad \beta_c(f \circ g) = X(f \circ g)(\beta_{d'}(\mathrm{id}_{d'})) = (Xg \circ Xf)(\beta_{d'}(\mathrm{id}_{d'})),$$

so our condition equivalently reads

$$Xg(\alpha_d(\mathrm{id}_d)) = Xg((Xf)(\beta_{d'}(\mathrm{id}_{d'}))).$$

This is true because by definition of f,

$$\alpha_d(\mathrm{id}_d) = (Xf)(\beta_{d'}(\mathrm{id}_{d'})).$$

The two functors defined in this way are manifestly each other's inverses, hence provide an isomorphism of categories, which is stronger than what we are looking for.

8.2.1 When can we invert the classical Grothendieck construction?

Suppose we are given a functor $F\colon \mathcal{C} \to \mathcal{D}$ exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. By Lemma 226, each of the fibers $\mathcal{C}_d$ is a groupoid. We may hope that we could reverse the Grothendieck construction by defining a functor $\mathcal{D} \to \mathbf{Grpd}$ defined on objects by $d \mapsto \mathcal{C}_d$.

We almost can. For any morphism $f\colon d \to d'$ in $\mathcal{D}$, we can build functor $f_!\colon \mathcal{C}_d \to \mathcal{C}_{d'}$ as follows.

Let $f\colon d \to d'$, and let $c \in \mathcal{C}_d$. By property 2 in Definition 201, we can find an object $c' \in \mathcal{C}_{d'}$ and a morphism $\bar{f}\colon c \to c'$ in $\mathcal{C}$ such that $F(\bar{f}) = f$. Define $f_!(c) = c'$, and $f_!()$

The above discussion means that the Grothendieck construction defines the following correspondence.

$$\left\{ \begin{matrix} F\colon \mathcal{C} \to \mathcal{D} \text{ exhibits } \mathcal{C} \\ \text{as cofibered in groupoids} \end{matrix} \right\} \overset{\simeq}{\longleftrightarrow} \left\{ \begin{matrix} \text{pseudofunctors} \\ \mathcal{D} \to \mathbf{Grpd} \end{matrix} \right\}$$

8.2.2 Fibrance and cofibrance in groupoids

Example 221. The Grothendieck construction

$$\pi\colon \mathcal{C}_\chi \to \mathcal{D}$$

exhibits $\mathcal{D}$ as cofibered over groupoids. We simply transcribe the definition, making the necessary changes as we go along.

In ordered for π to exhibit $\mathcal{C}_\chi$ as cofibered in groupoids over $\mathcal{D}$, we have to check the following.

1. For every $(d, x) \in \mathcal{C}_\chi$ and every morphism

$$\eta \colon d \to d' \qquad \text{in } \mathcal{D}$$

there exists a lift

$$\tilde{\eta} = (f, \alpha) \colon (d, x) \to (d', x') \qquad \text{in } \mathcal{C}_\chi$$

such that

$$\pi(\tilde{\eta}) = f \colon d \to d' = \eta.$$

2. For every morphism $\eta \colon (x, d) \to (x', d')$ in $\mathcal{C}$ and every object $(x'', d'') \in \mathcal{C}$, the map

$$\mathcal{C}_\chi((d', x') \to (d'', x'')) \to \mathcal{C}_\chi((d, x), (d'', x'')) \underset{\mathcal{D}(d, d'')}{\times} \mathcal{D}(d', d'')$$

is a bijection.

What do these mean?

1. We need to construct from our map $\eta \colon d \to d'$ an isomorphism

$$\alpha \colon \chi(f)(x) \simeq x'$$

for some x'. We can pick $x' = \chi(f)(x)$. Then $\tilde{\eta} = (f, \alpha)$ does the job. Indeed, $\pi(f, \alpha) = f$ as required.

2. Draw a picture. Given the data of
   - A morphism

Example 222. The contravariant Grothendieck construction exhibits $\mathcal{C}^\chi \to \mathcal{D}$ as fibered in groupoids over $\mathcal{D}$.

To see this, we need to make the following checks.

- **Check:** For every $(d, x) \in \mathcal{C}^\chi$, and every morphism $\eta \colon d' \to d$ in $\mathcal{D}$, we should be able to find some object $x' \in \chi(d')$ and a morphism $(f, \alpha) \colon (d', x') \to (d, x)$.

  Solution: We can pick $f = \eta$, $x' = \chi(f)(x)$, and $\alpha = \mathrm{id}_{x'}$.

- **Check:** Fixing any diagram

$$\begin{tikzcd} & (d', x') \arrow[d, "{(g,\alpha)}"] \\ (d'', x'') \arrow[r, "{(\eta,\beta)}"'] & (d, x) \end{tikzcd}$$

  in $\mathcal{C}^\chi$, for any morphism $f \colon d'' \to d'$ making the below left diagram commute, we should be able to find a morphism $(f, \gamma) \colon (d'', x'') \to (d', x')$ making the below right diagram commute.

$$\begin{tikzcd} & d' \arrow[dr, "g"] & \\ d'' \arrow[ur, dashed, "f"] \arrow[rr, "\eta"'] & & d \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} & (d', x') \arrow[dr, "{(g,\alpha)}"] & \\ (d'', x'') \arrow[ur, dashed, "{(f,\gamma)}"] \arrow[rr, "{(\eta,\beta)}"'] & & (d, x) \end{tikzcd}$$

Solution: We need a morphism $\gamma : x'' \to \chi(f)(x')$ making the above diagram commute. Using the data available to us, we can create the following diagram in $\chi(d'')$.

$$\begin{tikzcd} & \chi(f)(x') \arrow[dr, "\chi(f)(\alpha)"] & \\ x'' \arrow[rr, "\beta"'] & & \chi(\eta)(x) \end{tikzcd}$$

Note that we are entitled to draw $\chi(f)(\alpha)$ terminating at $\chi(\eta)(x)$ because

$$\chi(f)(\chi(g)(x)) = \chi(g \circ f)(x) = \chi(\eta)(x).$$

Because $\chi(d'')$ is a groupoid, we get the morphism γ via

$$\gamma = (\chi(f)(a))^{-1} \circ \beta.$$

8.3 The starred smash

Definition 223 (starred smash)**.** Let $f : A \to A'$, $g : B \to B'$ be two morphisms of simplicial sets. Define a morphism $f \overset{\star}{\wedge} g$ using the universal property for pushouts as follows.

$$\begin{tikzcd} A \star B \arrow[r, "f\star\mathrm{id}"] \arrow[d, "\mathrm{id}\star g"'] & A' \star B \arrow[d, hook] \arrow[ddr, bend left, "\mathrm{id}\star g"] & \\ A \star B' \arrow[r, hook] \arrow[drr, bend right, "f\star\mathrm{id}"'] & A' \star B \coprod_{A \star B} A \star B' \arrow[dr, dashed, "f \overset{\star}{\wedge} g"] & \\ & & A' \star B' \end{tikzcd}$$

Fact 224. We have the following facts.

1. If f is right anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are inner anodyne.
2. If f is left anodyne, then $f \overset{\star}{\wedge} g$ and $f \wedge g$ are left anodyne.
3. If f is anodyne, then $f \overset{\star}{\wedge} g$ is a Kan fibration.

We have the following 'perpendicular statements' (whatever that means).

Fact 225. Consider morphisms of simplicial sets as follows

$$\bar{K} \hookrightarrow K \xrightarrow{p} X \xrightarrow{q} S$$

where q is an inner fibration. Let $r = p \circ q$, $\bar{p} = p|_{\bar{K}}$, $\bar{r} = r|_{\bar{K}}$. Denote the

$$S_{/p} \to X_{/\bar{p}} \underset{S_{/\bar{p}}}{\times} S_{/r}.$$

8.3.1 Cofibration in groupoids

We would like to find the correct ∞-categorical notion of being cofibered in groupoids.

Lemma 226. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor exhibiting $\mathcal{C}$ as cofibered in groupoids over $\mathcal{D}$. Then for each $d \in \mathcal{D}$, the fiber $\mathcal{C}_d$ defined by the pullback below is a groupoid.

$$\begin{tikzcd} \mathcal{C}_d \arrow[r] \arrow[d] & \mathcal{C} \arrow[d, "F"] \\ \{d\} \arrow[r, hook] & \mathcal{D} \end{tikzcd}$$

Proof. Every morphism in $\mathcal{C}_d$ is by definition mapped to id_d, so by □

Theorem 227. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F exhibits $\mathcal{C}$ as cofibered in groupoids over D if and only if $N(F)\colon N(\mathcal{C}) \to N(\mathcal{D})$ is a left fibration.

Proof. Suppose F is a left fibration. We examine some horn filling conditions in small degrees. First, consider a horn filling

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\ \Delta^1 \arrow[r] & N(\mathcal{D}) \end{tikzcd}$$

The commutativity conditions tell us that we have the following solid diagrams in $\mathcal{C}$ and $\mathcal{D}$ respectively.

$$x \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

A dashed lift

$$\begin{tikzcd} \Lambda^1_0 \arrow[r] \arrow[d] & N(\mathcal{C}) \arrow[d] \\ \Delta^1 \arrow[r] \arrow[ur, dashed] & N(\mathcal{D}) \end{tikzcd}$$

gives us the following dashed data.

$$x \overset{f}{\dashrightarrow} y \qquad \xrightarrow{F} \qquad F(x) \xrightarrow{\bar{f}} \bar{y}$$

Thus, the first condition of Definition 201 is satisfied.

Similarly, a horn filling problem $\Lambda^2_0 \to \Delta^2$ gives us the following solid data, while a lift gives us the dashed data.

$$\begin{tikzcd} & y \arrow[dr, dashed, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd} \qquad \begin{tikzcd} & F(y) \arrow[dr, "\bar{g}"] & \\ F(x) \arrow[ur, "F(f)"] \arrow[rr, "F(h)"'] & & F(z) \end{tikzcd}$$

This says that there exists a lift as required, but not that it is unique.

Now consider two possible lifts $\bar{g}$ and $\bar{g}'$, arranged into a $(3,0)$-horn as follows.

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, "h"] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] & z \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "f"] \arrow[d, "f"'] & y \arrow[d, "g"] \\ y \arrow[r, "g'"'] \arrow[ur, "\mathrm{id}_y"] & z \end{tikzcd}$$

These correspond to a 3-simplex in $\mathcal{D}$ as follows.

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] \arrow[dr, "F(h)"] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] & F(z) \end{tikzcd} \qquad \begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "F(f)"'] & F(y) \arrow[d, "\bar{g}'"] \\ F(y) \arrow[r, "\bar{g}"'] \arrow[ur, "\mathrm{id}_{F(y)}"] & F(z) \end{tikzcd}$$

The lifting properties say that we can fill the above horn, telling us that $g = g'$, i.e. that the lift is unique. □

8.4 The infinity-categorical Grothendieck construction

Definition 228 (fiber over a point)**.** Let $f\colon X \to S$ be a map of simplicial sets, and let $s \in S$. The fiber over s, denoted X_s is the following pullback.

$$\begin{tikzcd} X_s \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \{s\} \arrow[r, hook] & S \end{tikzcd}$$

Theorem 229. Let $p\colon X \to S$ be a left fibration of simplicial sets. Then the assignment

$$s \mapsto X_s$$

extends to a functor $\mathrm{h}S \to \mathrm{h}\mathcal{S}$, where $\mathcal{S}$ is the category of spaces (Definition 169).

Proof. By Corollary 216, each fiber X_s is a Kan complex.

Now, let $f\colon s \to s'$ be any morphism in S. Consider the following lifting problem.

$$\begin{tikzcd} \{0\} \times X_s \arrow[rr, hook] \arrow[d, hook] & & X \arrow[d, "p"] \\ \Delta^1 \times X_s \arrow[r] \arrow[urr, dashed, "Lf"] & \Delta^1 \arrow[r, "f"] & S \end{tikzcd}$$

The left-hand morphism is left anodyne because it is the smash product

$$(\{0\} \hookrightarrow \Delta^1) \overset{\star}{\wedge} (\emptyset \hookrightarrow X_s),$$

so we can find a dashed lift. This gives us a map $Lf\colon \Delta^1 \times X_s \to X$ such that $Lf_{\{0\}} = \mathrm{id}_{X_s}$. Restricting to $\{1\} \times X_s$, we get a map $Lf_{\{1\}\times X_s}\colon \{1\} \times X_s \to X_{s'}$, giving us a map

$$\tilde{f}\colon X_s \to X_{s'}.$$

The functor $\mathrm{h}S \to \mathrm{h}\mathcal{S}$ sends $[f] \mapsto [\tilde{f}]$. It remains to check that this is well-defined, i.e. respects □

8.5 More model structures

Let S be a simplicial set.

- Denote by $(\mathbf{Set}_\Delta)_{/S}$ the category of simplicial sets over S.
- Denote by ${\mathbf{Set}_\Delta}^{\mathfrak{C}(S)}$ the category of functors of simplicial categories from $\mathfrak{C}(S) \to \mathbf{Set}_\Delta$.

Theorem 230. There is a Quillen equivalence

$$(\mathbf{Set}_\Delta)_{/S} \leftrightarrow {\mathbf{Set}_\Delta}^{\mathfrak{C}(S)}$$

where $(\mathbf{Set}_\Delta)_{/S}$ is taken to have the covariant model structure, and ${\mathbf{Set}_\Delta}^{\mathfrak{C}(S)}$ is taken to have the projective model structure.

8.6 Some correspondences

$$\left\{\begin{matrix}\text{functors } \mathcal{C}^{\chi} \to \mathcal{D} \\ \text{fibered in groupoids}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pseudofunctors} \\ \mathcal{D} \to \mathbf{Grpd}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{functors } \mathcal{C} \to \mathcal{D} \\ \text{cofibered in groupoids}\end{matrix}\right\}$$

$$\left\{\begin{matrix}\text{coCartesian fibrations} \\ \mathcal{C}^{\chi} \to \mathcal{D}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{pseudofunctors} \\ \mathcal{D} \to \mathbf{Grpd}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{coCartesian fibrations} \\ \mathcal{C}^{\chi} \to \mathcal{D}\end{matrix}\right\}$$