1 Simplicial homotopy

1.1 Homotopy of edges

We will often want to consider edges in a simplicial set up to some notion of homotopy.

Definition 1 (homotopy of edges)**.** Let $f\colon x \to y$ and $g\colon x \to y$ be edges in some simplicial set K. We say that f and g are homotopic if there exists a square as follows.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 v"'] \arrow[dr] & w \arrow[d, "s_0 v"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Unfortunately, this is in general not an equivalence relation since it is not symmetric. However, it is symmetric if we restrict our attention to Kan complexes.

Theorem 2. For a Kan complex, homotopy of edges is an equivalence relation.

Proof. First, we show reflexivity. For any edge f, the simplicial identities (Theorem **??**) ensure that the degenerate simplex $\sigma_f = s_1 f$ satisfies

$$d_1\sigma_f = f, \qquad d_2\sigma_f = f, \qquad \text{and} \qquad d_0\sigma_f = s_0 d_0 f.$$

To see that it is symmetric, create from a 2-simplex

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

the 1-skeleton of a 2-simplex whose front and back are as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "f"'] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "f"'] & y \end{tikzcd}$$

This gives us the boundaries of four 1-simplices. Each of these can be filled except for the bottom simplex of the first diagram. Doing so gives us a 1-horn, which we can fill to find a 2-simplex as follows.

$$\sigma' = \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & & y \end{tikzcd}$$

This says precisely that $g \sim f$.

To check transitivity, suppose we are given two simplices as follows.

$$\begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "f"] \arrow[rr, "g"'] & & y \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} & y \arrow[dr, "s_0 y"] & \\ x \arrow[ur, "g"] \arrow[rr, "h"'] & & y \end{tikzcd}$$

We can create a horn $\Lambda^3_2 \to X$ as follows.

$$\begin{tikzcd} y \arrow[r, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[ur, "g"] \arrow[r, "h"'] & y \end{tikzcd} \qquad \begin{tikzcd} y \arrow[r, "s_0 y"] \arrow[dr, "s_0 y"] & y \arrow[d, "s_0 y"] \\ x \arrow[u, "f"] \arrow[r, "h"'] & y \end{tikzcd}$$

Filling this gives what we want. $\square$

Theorem 3. Two edges f and g in a Kan complex K are homotopic if and only if there exists a 2-simplex filling the following boundary.

$$\begin{tikzcd} & w \arrow[dr, "s_0 w"] & \\ v \arrow[ur, "f"] \arrow[rr, "g"'] & & w \end{tikzcd}$$

Proof. Suppose we have a 2-simplex exhibiting f and g as homotopic. We can form a square by attaching the 2-simplex $s_0 g$ as to the bottom.

$$\begin{tikzcd} v \arrow[r, "f"] \arrow[d, "s_0 w"'] \arrow[dr, "g"] & w \arrow[d, "s_0 v"] \\ v \arrow[r, "g"'] & w \end{tikzcd}$$

Given a square as above, call the bottom 2-simplex η and the upper simplex μ. Create the horn

$$\begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] \arrow[dr, "d_1 \eta"] & v \arrow[d, "g"] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd} \qquad \begin{tikzcd} v \arrow[r, "s_0 v"] \arrow[d, "g"'] & v \arrow[d, "g"] \arrow[dl, "g"'] \\ w \arrow[r, "s_0 w"'] & w \end{tikzcd}$$

where the top left simplex is η, the top right is $s_0 g$, the bottom right is $s_1 g$, and the bottom left is unfilled. Filling it, we find that $g \sim d_1 \eta$. But μ exhibits $f \sim d_1 \eta$, so $f \sim g$. $\square$

1.2 Higher homotopy

Definition 4 (homotopy)**.** Let $f, g \colon B \to K$ be simplicial sets. A homotopy from f to g is a morphism

$$H \colon \Delta^1 \times B \to K$$

such that the restriction of H to $\{0\} \times B$ is f, and the restriction of H to 1 is g.

In this case, we will write

$$f \overset{H}{\simeq} g$$

To put it another way, H is a homotopy above if the pullback along the inclusion $B \to \Delta^1 \times B$ sending $\sigma \mapsto (0, \sigma)$ is equal to f, and similarly for g.

$$\begin{tikzcd}
\Delta^0 \times B \arrow[d, hook, "d^1\times\mathrm{id}"'] \arrow[dr, "f"] & \\
\Delta^1 \times B \arrow[r, "H"] & K \\
\Delta^0 \times B \arrow[u, hook', "d^0\times\mathrm{id}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Note that chasing elements around, this immediately implies that, for all σ,

$$H(0, \sigma) = f(\sigma) \qquad \text{and} \qquad H(1, \sigma) = g(\sigma).$$

Definition 5 (relative homotopy)**.** Suppose we are further given a monomorphism $i\colon A \to B$ making the following diagram commute.

$$\begin{tikzcd}
& B \arrow[dr, "f"] & \\
A \arrow[ur, "i"] \arrow[dr, "i"'] & & K \\
& B \arrow[ur, "g"'] &
\end{tikzcd}$$

Denote $f \circ i = g \circ i = u$. Then we say that H is a homotopy relative to A if the diagram

$$\begin{tikzcd}
\Delta^1 \times B \arrow[r, "H"] & K \\
\Delta^1 \times A \arrow[u, "\mathrm{id}\times i"] \arrow[r, "\pi_A"'] & A \arrow[u, "u"']
\end{tikzcd}$$

commutes. We will write

$$f \overset{H}{\simeq} g \quad (\text{rel. } A).$$

The intuitive meaning of this diagram is that H leaves $A \subset B$ fixed. Indeed, again chasing elements around, we find that

$$H(e, \sigma) = f(\sigma) = g(\sigma)$$

for all $\sigma \in A$ and $e = 0, 1$.

We would like to use homotopy to define an equivalence relation on morphisms. However, in a general simplicial set this is impossible. Consider the maps

$$\iota_0,\ \iota_1\colon \Delta^0 \to \Delta^1.$$

which send the point to the zeroth and first vertices. The morphisms ι_0 and ι_1 satisfy

$$\iota_0 \simeq \iota_1,$$

since $H = \mathrm{id}_{\Delta^1}$ makes the following diagram commute.

$$\begin{tikzcd}
\Delta^0 \times \Delta^0 = \Delta^0 \arrow[d, "d^1\times\mathrm{id}"'] \arrow[dr, "\iota_0"] & \\
\Delta^1 \times \Delta^0 = \Delta^1 \arrow[r, "H=\mathrm{id}_{\Delta^1}"] & \Delta^1 \\
\Delta^0 \times \Delta^0 = \Delta^0 \arrow[u, "d^0\times\mathrm{id}"] \arrow[ur, "\iota_1"'] &
\end{tikzcd}$$

However, there is no morphism $H\colon \Delta^1 \to \Delta^1$ which sends $0 \mapsto 1$ and $1 \mapsto 0$, so there does not exist any H such that

$$\iota_1 \overset{H}{\simeq} \iota_0.$$

The way out of this is to specialize to Kan complexes.

Proposition 6. Let $i\colon A \to B$ be a monomorphism of simplicial sets, let K be a Kan complex, and let $u\colon A \to K$ be a morphism. Then the relation *is homotopic relative to A* defines an equivalence relation on the set

$$\{f\colon B \to K \mid f \circ i = u\}.$$

Proof. By Note **??**, we can think of u as a map $\Delta^0 \to \mathrm{Maps}(A, K)$ which picks out u as a zero-simplex.

Consider the pullback square

$$\begin{tikzcd}
\mathrm{Maps}(B,K)^u \arrow[r] \arrow[d] & \mathrm{Maps}(B,K) \arrow[d, "\mathrm{Maps}(i,\mathrm{id}_K)"] \\
\Delta^0 \arrow[r, "u"'] & \mathrm{Maps}(A,K)
\end{tikzcd}$$

By Corollary **??**, the map $\mathrm{Maps}(i, \mathrm{id}_K)$ is a Kan fibration. Thus, since Kan fibrations are saturated (hence closed under pullback), we have that the induced map $\mathrm{Maps}(B,K)^u \to \{u\}$ is also a Kan fibration. But by Corollary **??**, this means that $\mathrm{Maps}(B,K)^u$ is a Kan complex.

The elements of $\mathrm{Maps}(B,K)^u_0$ are ordered pairs $(u, f) \in \{u\} \times \mathbf{Set}_\Delta(B, K)$ such that $u = i \circ f$. That is, $\mathrm{Maps}(B,K)^u_0$ consists of those morphisms $B \to K$ which agree with u when restricted to A along i.

Thus, showing that homotopy between maps $B \to K$ relative to A is an equivalence relation is the same as showing homotopy between vertices of the Kan complex $\mathrm{Maps}(B,K)^u$ is an equivalence relation. I'll add this near the front of the notes at some point. □

1.3 Homotopy groups

Definition 7 (homotopy classes)**.** Let K be a Kan complex. Define the zeroth homotopy group of K, denoted $\pi_0(K)$, to be the set of homotopy classes of vertices of K. This is well-defined by Proposition 6.

For $n \geq 1$, define the nth homotopy group of K at v, denoted $\pi_n(K, v)$, to be the set of homotopy classes of n-simplices $\alpha \colon \Delta^n \to K$ relative to the inclusion $\partial\Delta^n \hookrightarrow \Delta^n$, such that the following diagram commutes.

$$\begin{tikzcd} \Delta^n \arrow[r, "\alpha"] & X \\ \partial\Delta^n \arrow[u, hook] \arrow[r] & \Delta^0 \arrow[u, hook, "v"'] \end{tikzcd}$$

This diagram says precisely that the boundary of Δ^n is collapsed to the point v.

Definition 8. Let $\alpha, \beta \colon \Delta^n \to K$ be n-simplices whose boundaries are trivial. Denote their equivalence classes by $[\alpha]$ and $[\beta]$ respective. Define their composition, denoted $[\alpha] \cdot [\beta]$ as follows. First, define an inner $(n+1)$-horn with faces

$$(\overbrace{\{v\}, \cdots, \{v\}}^{n-1}, \alpha, -, \beta).$$

This has a filler σ such that

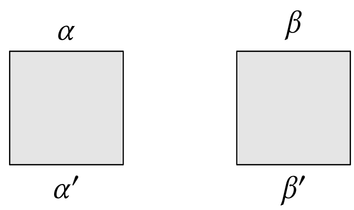
$$\partial\sigma = (\{v\}, \cdots, \{v\}, \alpha, d_n\sigma, \beta)$$

Define $[\alpha] \cdot [\beta] = [d_n\sigma]$.

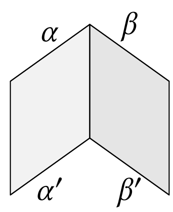
It remains to check that this composition is well-defined.

Proposition 9. The composition defined above is well-defined, i.e. if $\alpha' \overset{H}{\simeq} \alpha$ and $\beta' \overset{H'}{\simeq} \beta$, then $[\alpha'] \cdot [\beta'] = [\alpha] \cdot [\beta]$.

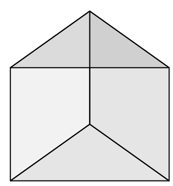
Sketch of proof. We have homotopies between the n-simplices we want to multiply,



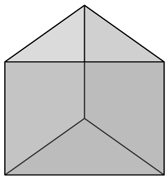
which we extend to a homotopy between the horns defining our multiplication law.



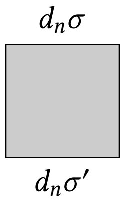
These horns have fillers, giving us the result of our multiplication.



Using the general theory of anodyne morphisms, we can extend our homotopy between the horns to a homotopy between the full filled simplices.



Restricting to the filled faces gives us a homotopy between the result of multiplication.



□

Proof. By definition, we have an $(n+1)$-simplices σ and σ' such that

$$\partial\sigma = (\{v\}, \dots, \{v\}, \alpha, d_n\sigma, \beta)$$

and

$$\partial\sigma' = (\{v\}, \dots, \{v\}, \alpha', d_n\sigma', \beta').$$

We need to show that $d_n\sigma \simeq d_n\sigma'$.

First, we construct a homotopy between the horns

$$\eta = (\{v\}, \dots, \{v\}, \alpha, -, \beta) \qquad \text{and} \qquad \eta' = (\{v\}, \dots, \{v\}, \alpha', -, \beta').$$

By the distributive law for products and coproducts

$$A \times (B \amalg C) \simeq (A \times B) \amalg (A \times C),$$

a homotopy between horns $\Lambda^{n+1}_i \to K$ consists of a homotopy for each face. Therefore, we can choose our homotopy H'' to have components $(\mathrm{id}_v, \dots, \mathrm{id}_v, H, -, H')$. This gives us a map $\Delta^1 \times \Lambda^{n+1}_n \to K$. Our fillers give us two maps $\sigma, \sigma' \colon \Delta^{n+1} \to K$, which we can view as a map $\partial\Delta^1 \times \Delta^{n+1} \to K$. This gives us the following commutative diagram

$$\begin{tikzcd}
\partial\Delta^1 \times \Lambda^{n+1}_n \arrow[r, hook] \arrow[d, hook] & \partial\Delta^1 \times \Delta^{n+1} \arrow[d, "{(\sigma,\sigma')}"] \\
\Delta^1 \times \Lambda^{n+1}_n \arrow[r, "H''"] & K
\end{tikzcd}$$

This gives us a map

$$h \colon \Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \to K.$$

Note that by Example ??, we get an anodyne extension

$$\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1} \hookrightarrow \Delta^1 \times \Delta^n.$$

This means that (by Corollary ??) we can find a lift for the following diagram.

$$\begin{tikzcd}
\Delta^1 \times \Lambda^{n+1}_n \coprod_{\partial\Delta^1 \times \Lambda^{n+1}_n} \partial\Delta^1 \times \Delta^{n+1}. \arrow[r, "h"] \arrow[d, hook] & K \\
\Delta^1 \times \Delta^{n+1} \arrow[ur, dashed, "\exists I"'] &
\end{tikzcd}$$

Restricting I to the nth face of Δ^{n+1} gives us a homotopy

$$\Delta^1 \times \Delta^n \to K$$

as required. □

Theorem 10. Let K be a Kan complex, and let $v \in K_0$ be a vertex of K. Then the multiplication from Definition 8 makes $\pi_n(K, v)$ into a group for $n \geq 1$.

Proof. First we check associativity. Let $\alpha, \beta, \gamma \colon \Delta^n \to K$ be n-simplices with trivial boundary, representing elements $[\alpha], [\beta], [\gamma] \in \pi_n(K, v)$. By definition, we get an $(n+1)$-simplex σ_{n-1} such that

$$\partial\sigma_{n-1} = (v, v, \ldots, v, \alpha, d_n\sigma_{n-1}, \beta).$$

giving us the result of the multiplication $[\alpha] \cdot [\beta] = [d_n\sigma_{n-1}]$.

Similarly, we have some σ_{n+2} such that

$$\partial\sigma_{n+2} = (v, v, \ldots, v, \beta, d_n\sigma_{n+2}, \gamma)$$

so $[\beta] \cdot [\gamma] = [d_n\sigma_{n+2}]$.

Form from these a simplex σ_{n+1} with boundary

$$\partial\sigma_{n+1} = (v, v, \ldots, v, d_n\sigma_{n-1}, d_n\sigma_{n+1}, \gamma)$$

giving us the result of the multiplication $([\alpha] \cdot [\beta]) \cdot [\gamma] = [d_n\sigma_{n-1}] \cdot [\gamma] = [d_n\sigma_{n+1}]$.

These simplices fit together to give a horn

$$\Lambda^{n+2}_n \to K$$

with boundary

$$(v, v, \ldots, v, \sigma_{n-1}, -, \sigma_{n+1}, \sigma_{n+2}).$$

Filling this gives an $(n+1)$-simplex σ with boundary

$$\partial\sigma = (v, v, \ldots, v, \sigma_{n-1}, d_n\sigma, \sigma_{n+1}, \sigma_{n+2}).$$

Using the simplicial identities, we find

$$\begin{aligned}
\partial d_n\sigma &= (d_0d_n\sigma, d_1d_n\sigma, \ldots, d_{n-2}d_n\sigma, d_{n-1}d_n\sigma, d_nd_n\sigma, d_{n+1}d_n\sigma) \\
&= (d_{n-1}d_0\sigma, d_{n-1}d_1\sigma, \ldots, d_{n-1}d_{n-2}\sigma, d_{n-1}d_{n-1}\sigma, d_nd_{n+1}\sigma, d_nd_{n+2}\sigma) \\
&= (v, v, \ldots, v, \alpha, d_n\sigma_{n+1}, d_n\sigma_{n+2}).
\end{aligned}$$

This is an n-simplex exhibiting $[\alpha] \cdot [d_n \sigma_{n-2}] = [d_n \sigma_{n+1}]$.

Thus, we have

$$\begin{aligned}([\alpha] \cdot [\beta]) \cdot [\gamma] &= [d_n \sigma_{n-1}] \cdot [\gamma] \\ &= [d_n \sigma_{n+1}] \\ &= [\alpha] \cdot [d_n \sigma_{n+2}] \\ &= [\alpha] \cdot ([\beta] \cdot [\gamma]).\end{aligned}$$

The degenerate n-simplex v functions as a unit on the left because there exists an $(n+1)$-horn $s_n \omega$ with

$$\begin{aligned}\partial(s_n \omega) &= (d_0 s_n \omega, \dots, d_{n-1} s_n \omega, d_n s_n \omega, d_{n+1} s_n \omega) \\ &= (s_{n-1} d_0 \omega, \dots, s_{n-1} d_{n-1} \omega, \omega, \omega) \\ &= (v, \dots, v, v, \omega, \omega),\end{aligned}$$

so $[e] \cdot [\omega] = [\omega]$.

It functions as a unit on the right because of $s_{n-2}\omega$.

Furthermore. □

We would like $\pi_0(K)$ and $\pi_n(K, v)$ to be the simplicial analogs of the topological notions of homotopy groups. Recall that we can turn any simplicial set into a topological space via the geometric realization functor $|\cdot|$. Therefore, one would hope that the simplicial homotopy groups would agree with the homotopy groups of the geometrical realization. At least for π_o this turns out to be the case.

Theorem 11. For any Kan complex K, we have $\pi_0(K) \simeq \pi_0(K)$ as sets.

Proof. Define a map $K_0 \to |K|$ sending $v \to |v|$.

Suppose $v \sim v'$ in K. Then there is an edge f between v and k. Applying $|\cdot|$, we find an edge $|v| \to |v'|$. Thus, the map above respects equivalence classes, and descends to a map $\pi_0(K) \to \pi_0(|K|)$.

Now some homotopy stuff I don't understand. □

Theorem 12. Let $p\colon K \to S$ be a Kan fibration. Let v be a vertex of K, and let $w = p(v)$. Let F be the fiber over w given by the following pullback square.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & K \arrow[d, "p"] \\ \{w\} \arrow[r, hook] & S \end{tikzcd}$$

There is the following long exact sequence.

$$\cdots \longrightarrow \pi_{n+1}(K, v) \longrightarrow \pi_n(F, v) \longrightarrow \pi_n(K, v) \longrightarrow \pi_n(S, w) \longrightarrow \pi_{n-1}(F, v) \longrightarrow \cdots$$