1 Abelian categories

1.1 Basics

In this chapter we recall some basic results.

A category is **Ab**-*enriched* if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$. In an **Ab**-enriched category with finite products, products agree with coproducts, and we call both direct sums. We call an **Ab**-enriched category with finite direct sums an *additive* category, and we call a functor between additive categories such that the maps

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

are homomorphisms of abelian groups an *additive functor.* A *pre-abelian* category is an additive category such that every morphism has a kernel and a cokernel, and an *abelian category* is a pre-abelian category such that every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

This immediately implies the following results.

- Kernels are monic and cokernels are epic
- Since the equalizer of f and g is the kernel of $f - g$, abelian categories have all finite limits, and dually colimits.

As the following proposition shows, additive functors from additive categories preserve direct sums.

Lemma 1. Let A and B be objects in an additive category $\mathcal{A}$, and let X be an object in $\mathcal{A}$ equipped with morphisms as follows,

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_A"'] & \\
B \arrow[ur, "i_B"'] & & B
\end{tikzcd}$$

such that the following equations hold.

$$
\begin{aligned}
i_A \circ \pi_A + i_B \circ \pi_B &= \mathrm{id}_X \\
\pi_A \circ i_B = 0 &= \pi_B \circ i_A \\
\pi_A \circ i_A &= \mathrm{id}_A \\
\pi_B \circ i_B &= \mathrm{id}_B
\end{aligned}
$$

Then the πs and is exhibit $X \simeq A \oplus B$.

Corollary 2. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(a \oplus b) \simeq F(a) \oplus F(b).$$

Once it is known that a category $\mathcal{A}$ is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is abelian, as we will see in
- For any small category I, the category $\mathbf{Fun}(I, \mathcal{A})$ of I-diagrams in $\mathcal{A}$ is abelian.

1.2 Embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

In abelian categories, one can defines kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero obect: f is a kernel for g if and only the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pullback.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category $\mathbf{Set}_*$ of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category $\mathcal{A}$, suppose one could find an embedding $\mathcal{H}\colon \mathcal{A} \to \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \to B \to C$ in $\mathcal{A}$, mapped it into $\mathbf{Set}_*$ finding a sequence $\mathcal{H}(A) \to \mathcal{H}(B) \to \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \to B \to C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in $\mathbf{Set}_*$ using $\mathcal{H}$ and check exactness there. Effectively, one could check exactness of a sequence in $\mathcal{A}$ by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking the that the image of the diagram under our functor $\mathcal{H}$ commutes.

In fact, for $\mathcal{A}$ small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to $\mathbf{Set}_*$.

Definition 3 (category of contravariant epimorphisms)**.** Let $\mathcal{A}$ be a small abelian category. Define a category $\mathcal{A}_{\twoheadleftarrow}$ with $\mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow}) = \mathrm{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}_{\twoheadleftarrow}}(X, Y) = \{f\colon Y \to X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity functor is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A\colon \mathcal{A}_{\twoheadleftarrow} \to \mathbf{Set}_*; \qquad Z \mapsto \mathrm{Hom}_{\mathcal{A}}(Z, A)$$

and sends a morphism $f\colon Z_1 \to Z_2$ in $\mathcal{A}_{\twoheadleftarrow}$ (which is to say, an epimorphism $\tilde{f}\colon Z_2 \twoheadrightarrow Z_1$ in $\mathcal{A}$) to the map

$$\mathcal{H}_A(f)\colon \mathrm{Hom}_{\mathcal{A}}(Z_1, A) \to \mathrm{Hom}_{\mathcal{A}}(Z_2, A); \quad (\alpha\colon Z_1 \to A) \mapsto (\alpha \circ \tilde{f}\colon Z_2 \to A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 4 (member functor)**.** Let $\mathcal{A}$ be a small abelian category. We define a functor $\mathcal{M}\colon \mathcal{A} \to \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \operatorname{colim} \mathcal{H}_A.$$

On morphisms, functorality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(\mathcal{A})$ have a much simpler interpretation.

Proposition 5. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \operatorname{Hom}(X, A)/\sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\ X' \arrow[r, "g'"'] & A \end{tikzcd}$$

Proof. The colimit can be computed using the following coequalizer.

$$\coprod_{\substack{f \in \operatorname{Morph}(\mathcal{A}_{\twoheadleftarrow}) \\ \tilde{f}\colon Y \to X}} \operatorname{Hom}_{\mathcal{A}}(X, A) \overset{\mathrm{id}}{\underset{-\circ \tilde{f}}{\rightrightarrows}} \coprod_{Z \in \operatorname{Obj}(\mathcal{A}_{\twoheadleftarrow})} \operatorname{Hom}(Z, A) \xrightarrow{\mathrm{coeq}} \mathcal{M}(A)$$

On elements, we have the following.

$$\begin{tikzcd} & (g\colon X \to A) \arrow[d, equal, "!"] \\ (g\colon X \to A) \arrow[ur, mapsto, "\mathrm{id}"] \arrow[r, mapsto, "-\circ\tilde{f}"'] & (g \circ \tilde{f}\colon Y \to A) \end{tikzcd}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \operatorname{Obj}(\mathcal{A}_{\twoheadleftarrow})} \operatorname{Hom}(Z, A)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(g\colon X \to A) R (g'\colon X' \to A) \iff \exists f\colon X' \twoheadrightarrow X \text{ such that } g' = g \circ \tilde{f}.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g\colon X \to A) \sim (g'\colon X' \to A) \iff \exists f\colon Z \twoheadrightarrow X, f'\colon Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\ X' \arrow[r, "g'"'] & A \end{tikzcd}$$

□

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphsims into A modulo the above relation, and for a morphism $f : A \to B$ in $\mathcal{A}$, $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f) : [g] \mapsto [g \circ f].$$

Lemma 6. Let $f : A \to B$ be a morphism in a small abelian category $\mathcal{A}$ such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms making the following diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "g"] \arrow[d, two heads] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. □

Now we introduce some load-lightening notation: we write $\hat{(-)} = \mathcal{M}(-)$.

Lemma 7. Let $f : A \to B$ be a morphism in a small abelian category $\mathcal{A}$. Then $f = 0$ if and only if $\hat{f} : \hat{A} \to \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g, so $\hat{f} = 0$.

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\mathrm{id}_A]) = [0]$. But

$$\hat{f}([\mathrm{id}_A]) = [\mathrm{id}_A \circ f] = [f].$$

By Lemma 6, $[f] = [0]$ implies $f = 0$. □

Corollary 8. A diagram commutes in $\mathcal{A}$ if and only if its image in $\mathbf{Set}_*$ under $\mathcal{M}$ commutes.

Proof. A diagram commutes in $\mathcal{A}$ if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. □

Lemma 9. Let $f: A \to B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

1.3 Chain complexes

Definition 10 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex in $\mathcal{A}$ is…The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms of chain complexes, i.e.…

We also define $\mathbf{Ch}^{+}(\mathcal{A})$ to be the category of bounded-below chain complexes, $\mathbf{Ch}^{-}(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Proposition 11. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the **Ab**-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd}
 & Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\
C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet
\end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned}
(d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\
&= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\
&= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q,
\end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd}
 & Q_\bullet \arrow[dl, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\
\ker(f_\bullet) \arrow[r, "\iota_\bullet"] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet
\end{tikzcd}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{1.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi \colon Q_n \to \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 1.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned} \iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n \end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

Definition 12 (exact sequence)**.** Given a morphism $f \colon A \to B$, the image $\operatorname{im} f = \ker \operatorname{coker} f$ and the coimage is $\operatorname{coker} \ker f$. From this data, we build the following commuting diagram.

$$\begin{tikzcd} & \operatorname{im} f \arrow[d, "i"] \arrow[dr, "0"] & \\ A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\ & \operatorname{coker} f \arrow[ur] & \end{tikzcd}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a morphism $\phi \colon \operatorname{im} f \to \ker g$. We say that the above is exact if ϕ is an isomorphism. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

Definition 13 (homology)**.** Let $(C_\bullet, d)$ be a complex. The nth homology of $C_\bullet$ is the cokernel of the map $\phi \colon \operatorname{im} d_{n+1} \to \ker d_n$ described in Definition 12, denoted $H_n(C_\bullet)$.

It is useful to have the following picture in mind.

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[r, "\phi"] \arrow[dr, "i"'] & \ker g \arrow[r] \arrow[d] & H_n \\
C_{n+1} \arrow[rr, "f"] & & C_n \arrow[r, "g"] \arrow[d, "p"] & C_{n-1} \\
 & & \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

Proposition 14. Homology extends to a family of additive functors

$$H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d^C_n \to \ker d^D_n$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. □

We can also go the other way, by defining a map

$$\iota_n \colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\begin{tikzcd}
{} \arrow[r, dotted] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r, dotted] & {}
\end{tikzcd}$$

concentrated in degree n.

Example 15. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex which is exact except in degree zero. Then $H_0(C_\bullet) = \operatorname{coker} f$.

Similarly, if

$$0 \longrightarrow C_{-1} \xrightarrow{f} C_{-2} \longrightarrow \cdots$$

is a chain complex whose non-trivial homology is in degree 0, then $H_0(C_\bullet) = \ker f$.

Definition 16 (quasi-isomorphism)**.** Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a <u>quasi-isomorphism</u> if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Quasi-isomorphisms are a fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope quasi-isomorphisms themselves were invertible. Unfortunately, we are not so lucky. Consider the following

By Example 15, we

Definition 17 (resolution)**.** Let A be an object in an abelian category. A resolution of A consists of a chain complex $C_\bullet$ together with a quasi-isomorphism $C_\bullet \to \iota(A)$.

1.4 Diagram lemmas

1.4.1 The splitting lemma

Lemma 18 (Splitting lemma)**.** Consider the following solid exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[l, dashed, bend left, "\pi_A"] \arrow[r, "\pi_C"] & C \arrow[l, dashed, bend left, "i_C"] \arrow[r] & 0 \end{tikzcd}$$

The following are equivalent.

- There exists a morphism $\pi_A : B \to A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
- There exists a morphism $i_C : C \to B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
- B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

1.4.2 The snake lemma

Lemma 19. Let $f : B \twoheadrightarrow C$ be an epimorphism, and let $g : D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{tikzcd} \ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "g"] \\ \ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "g"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K also functions as the kernel of f'. Of course, in order for this statement to make sense we need a map $i\colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$\begin{tikzcd}
K \arrow[dr, dashed, "\exists ! i"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v\colon Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.

$$\begin{tikzcd}
& Q \arrow[dl, dashed, "\exists ! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists !"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{elements of pullback}\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{matrix}\text{elements of } B\\ \text{which map to } C\end{matrix}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\begin{matrix}\text{elements of } B\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 20 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed, rounded corners] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd} \tag{1.2}$$

The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\operatorname{coker} g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 1.2 the following diagram.

$$\begin{tikzcd}
 & & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}$$

Take a pullback and a pushout, and using Lemma 19 (and its dual), we find the following.

$$\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}$$

Consider the map

$$\delta_0 = \quad S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \ .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \ \operatorname{coker} f \to \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 21 (snake lemma II)**.** Given the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

we get a exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

1.4.3 The long exact sequence on homology

Corollary 22. Given an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta"] \\
H_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. For each n, we have a diagram of the following form.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d^A_{n+1}"'] & B_{n+1} \arrow[r] \arrow[d, "d^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma (Theorem 20) gives us, for each n, two exact sequences as follows.

$$0 \longrightarrow \ker d^A_{n+1} \longrightarrow \ker d^B_{n+1} \longrightarrow \ker d^C_{n+1}$$

and

$$\operatorname{coker} d^A_n \longrightarrow \operatorname{coker} d^B_n \longrightarrow \operatorname{coker} d^C_n \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{tikzcd}
 & \operatorname{coker} f_{n+1} & \operatorname{coker} g_{n+1} & \operatorname{coker} h_{n+1} & 0 \\
0 & \ker f_n & \ker g_n & \ker h_n &
\end{tikzcd}$$

Recall that for each n, we get a map

$$\phi\colon \operatorname{im} f_{n+1} \to \ker f_n$$

□

Lemma 23. Let $\mathcal{A}$ be an abelian category, and let f be a morphism of short exact sequences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{tikzcd}
0 \arrow[r] & A'_\bullet \arrow[r, "\alpha_\bullet"] \arrow[d, "f'_\bullet"] & A_\bullet \arrow[r, "\alpha'_\bullet"] \arrow[d, "f_\bullet"] & A''_\bullet \arrow[r] \arrow[d, "f''_\bullet"] & 0 \\
0 \arrow[r] & B'_\bullet \arrow[r, "\beta_\bullet"'] & B_\bullet \arrow[r, "\beta'_\bullet"'] & B''_\bullet \arrow[r] & 0
\end{tikzcd}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha')"] \arrow[d, "H_n(f)"'] & H_n(A'') \arrow[r, "\delta"] \arrow[d, "H_n(f'')"] & H_{n-1}(A') \arrow[r, "H_{n-1}(\alpha)"] \arrow[d, "H_n(f')"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_n(f)"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "H_n(\beta')"] & H_n(B'') \arrow[r, "\delta"] & H_{n-1}(B') \arrow[r, "H_{n-1}(\beta)"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, the connecting morphism δ is natural.

1.4.4 The five lemma

Lemma 24 (four lemmas)**.** Let

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, two heads] & E \arrow[d, hook] \\
B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then

Theorem 25 (five lemma)**.**

1.4.5 The nine lemma

Theorem 26 (nine lemma)**.**

2 Homological algebra in abelian categories

2.1 Exactness in abelian categories

Definition 27 (exact functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

  is exact

- We call F exact if it is both left exact and right exact.

Example 28. We showed in Lemma **??** that both hom functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ are left exact.

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 29. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\mathrm{im}\, f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\mathrm{im}\, \iota = \ker g \subset \mathrm{im}\, f$. □

Proposition 30. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
   - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.
   - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.
3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

Any functor G which is a right adjoint preserves limits. By the above reasoning, G certainly preserves zero objects and kernels. Thus, G preserves the first two conditions, which means precisely that G is left exact.

Dually, any functor F which is a left adjoint preserves colimits, hence zero objects and cokernels. Thus, F preserves the last two conditions, which means that F is right exact. □

Note that the previous theorem makes no demand that F be additive, which may seem surprising; after all, by Definition 27, only exact funcotors are allowed the honor of being called additive. However, it turns out that a functor which preserves either finite limits or finite colimits is always additive. To see this, note that by applying any functor which is either left or right exact to a split exact sequence gives a split exact sequence.

This is often useful when trying to check the exactness of a functor which is a left or right adjoint, by the following proposition.

Lemma 31. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

Corollary 32. $0 \longrightarrow \operatorname{Hom}(X, A) \xrightarrow{f_*} \operatorname{Hom}(X, B) \xrightarrow{g_*} \operatorname{Hom}(X, C)$

is an exact sequence of abelian groups.

Proof. The hom functor □

2.2 Chain Homotopies

Definition 33 (chain homotopy)**.** Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A <u>chain homotopy</u>

Lemma 34. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 35. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 36. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

Definition 37 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

Lemma 38. The homotopy category $\mathcal{K}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

In fact, the homotopy category satisfies the following universal property.

Proposition 39. Let $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{C}$ be an additive functor such that $\mathcal{F}(f)$ is an isomorphism for all quasi-isomorphisms f. Then there exists a unique functor $\mathcal{K}(\mathcal{A}) \to \mathcal{C}$ making the following diagram commute.

Proposition 40. Let $\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$ be an additive functor which takes quasi-isomorphisms to isomorphisms. Then for homotopic morphisms $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, we have $F(f) = F(g)$.

2.3 Projectives and injectives

Definition 41 (projective, injective)**.** An object P in an abelian category is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A, it is very easy to see that an object P in an abelian category $\mathcal{A}$ is projective if for every epimorphism $f : B \to C$ and every morphism $p : P \to C$, there exists a morphism $\tilde{p} : P \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
& P \arrow[d, "p"] \arrow[dl, dashed, "\exists \tilde{p}"'] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g : A \to B$ and every morphism $q : A \to Q$ there exists a morphism $\tilde{q} : B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists \tilde{q}"] \\
Q &
\end{tikzcd}$$

Definition 42 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Proposition 43. Let

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0
\end{tikzcd}$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd}
& & & P \arrow[d, "\mathrm{id}"] & \\
0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0
\end{tikzcd}$$

By definition, we get a morphism $P \to B$ making the triangle commute.

$$\begin{tikzcd}
& & & P \arrow[d, "\mathrm{id}"] \arrow[dl, "\exists g"'] & \\
0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0
\end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma 18). □

Corollary 44. Let F be an additive functor, and let

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0
\end{tikzcd}$$

be an exact sequence with P projective. Then F applied to the sequence gives an exact sequence.

Proof. Applying F to a split sequence gives a split sequence, and all split sequences are exact. □

Corollary 45. The category R-**Mod** has enough projectives.

Proof. By Proposition **??**, free modules are projective, and every module is a quotient of a free module. □

Proposition 46. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Dually, any abelian category with enough injectives has injective resolutions.

Proof. We provide running commentary on the following diagram. □

Definition 47 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \to \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \to Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

Lemma 48. Let $f\colon P_\bullet \to \iota(M)$ be a projective resolution. Then $f\colon P_0 \to M$ is an epimorphism.

Proof. We know that $H_0(f)\colon H_0(P) \to H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(m)) \simeq \iota_M$, and that □

In fact, we can say more.

Lemma 49. Let $f_\bullet\colon P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots\cdots\ P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Theorem 50 (extended horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "\tilde{g}"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, given a morphism $M \to N$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd}
 & \vdots \arrow[d, phantom] & & \vdots \arrow[d, phantom] & \\
0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f \circ p'_0"'] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since by Lemma 48 p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P'_0 \oplus P''_0 \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P''_1 to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r] \arrow[d] & \ker p \arrow[r] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0 \arrow[r] & P_0'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Proposition 51. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P_\bullet' \to M'$$

be projective resolutions. Then for every morphism $f\colon M \to M'$ there exists a lift $\tilde{f}\colon P_\bullet \to P_\bullet'$ making the diagram

$$\begin{tikzcd}
P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P_\bullet' \arrow[d] \\
\iota M \arrow[r, "\iota f"'] & \iota M'
\end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Corollary 52. For any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}\mathcal{A})$.

2.4 Mapping cones

Definition 53 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the <u>k-shifted chain complex</u> $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 54 (mapping cone)**.** Let $f\colon C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\operatorname{cone}(f)$ as follows.

- For each n, define

$$\operatorname{cone}(f)_n = C_{n-1} \oplus D_n.$$

- Define $d_n^{\operatorname{cone}(f)}$ by

$$d_n^{\operatorname{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$

which is shorthand for

$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1}x_{n-1}).$$

Lemma 55. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow C_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} D_\bullet \longrightarrow 0\ .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 56. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. By Corollary 22, we get a long exact sequence

$$\cdots \to H_n(X) \xrightarrow{\delta} H_n(Y) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see (although I'm missing a sign somewhere); picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0, f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. □

Proposition 57. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$. Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g \colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots \\
\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
 & & P_1 \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists!"] \arrow[d, "r"] \arrow[dr, "0"] & \\
 & K \arrow[dr, hookrightarrow] & & \\
C_2 \arrow[ur, twoheadrightarrow] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\mathrm{cone}(f)$ is exact, which means that $\iota\colon P \to \mathrm{cone}(f)$ is homotopic to the zero morphism.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d_{i+1}^P"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d_i^P"] \arrow[dl, dashed, "\tilde{h}_i"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d_{i+1}^{\mathrm{cone}(f)}"'] & C_{i-1} \oplus P_i \arrow[r, "d_i^{\mathrm{cone}(f)}"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}$$

That is, there exist $\tilde{h}_i \colon P_i \to \mathrm{cone}(f)_{i+1}$ such that

$$d_{i+1}^{\mathrm{cone}\, f} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \tag{2.1}$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i \colon P_i \to C_i \qquad \text{and} \qquad \gamma_i \colon P_{i-1} \to P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet \colon P_\bullet \to C_\bullet$. Indeed, writing Equation 2.1 in components, we find the following.

$$\begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$
$$\begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that[1]

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

2.5 Localization at weak equivalences: a love story

We can embed any abelian category $\mathcal{A}$ into the corresponding category $\mathbf{Ch}(\mathcal{A})$ of chain complexes via inclusion into degree zero. This is obviously a lossless procedure, in the sense that we recover $\mathcal{A}$ by restricting to degree 0.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

Equivalently, we recover our category A by taking zeroth homology.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "H_0"] \end{tikzcd}$$

[1] Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

However, we notice that there are much better-behaved embeddings of $\mathcal{A}$ into $\mathbf{Ch}(\mathcal{A})$ such that we recover $\mathcal{A}$ when taking zeroth homology. For example, taking projective resolutions of objects and lifting morphisms using Proposition 51 will do the trick, although this is not in general well-defined; one can take many different projective resolutions, and lift morphisms in many different ways. There is no reason to expect these to assemble themselves functorially.

However, if we were to modify $\mathbf{Ch}(\mathcal{A})$ in such a way that all quasi-isomorphisms became bona-fide isomorphisms, then we would have this functoriality, thanks to Proposition 40.

Thus we find ourselves in a common situation: we have a category $\mathbf{Ch}(\mathcal{A})$, and identified a collection of morphisms inside this category which we would like to view as weak equivalences, namely the quasi-isomorphisms. In an ideal world, we would simply promote quasi-isomorphisms to bona fide isomorphisms. That is, we would like to form the localization

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}(\mathcal{A}) \twoheadrightarrow \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. In fact, this is universal in the sense that every other additive functor sending quasi-isomorphisms to isomorphisms factors through it. In particular, the functor

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}^{-1}\}]$$

factors through it.

But now we are in a really wonderful position, as long as $\mathcal{A}$ has enough projectives: we can (by REF we have already proved) find a projective resolution functor

$$P\colon \mathcal{A} \to \mathcal{K}^{+}(\mathcal{A}).$$

In fact, this functor is an equivalence of categories.

2.6 Homological δ-functors and derived functors

We have noticed that we can

Definition 58 (derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the left derived functor to be the cohomological δ-functor $\{L_nF\}$ defined by

$$L_nF = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

  where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the right derived functor to be the cohomological δ-functor

$$R^nF(X) = H^n \circ F \circ Q$$

  where Q is an injective resolution functor.

It may seem that we still have something left to prove; after all, we have not shown that the result of our composition is independant of the projective/injective resolution functor we used. However,

Because $\{H_n\}$ is a homological δ-functor, it is trivial that L_nF is a homological delta-functor. Note that it really does extend F in the following sense.

Proposition 59. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 49, the following sequence is exact.

$$P_2 \longrightarrow P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_2) \longrightarrow F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But

$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$

Similarly, on morphisms □

Definition 60 (homological δ-functor)**.** A homological δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n\colon \mathcal{A} \to \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n \colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

3. For every morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

and every n, the diagram

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

This seems like an inelegant definition, and indeed it is.

Definition 61 (cohomological δ-functors)**.** Dual to Definition 60.

Example 62. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors,* come from homology.

2.7 Double complexes

Definition 63 (double complex)**.** Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d_{i,j} \colon M_{i,j} \to M_{i-1,j}, \qquad \delta_{i,j} \colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $d^2 = 0$
- $\delta^2 = 0$
- $d\delta + \delta d = 0$

We say that $M_{\bullet,\bullet}$ is first-quadrant if $M_{i,j} = 0$ for $i, j < 0$.

That is, for M a double complex the following diagram anticommutes rather than commutes.

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} & M_{1,0} \arrow[l] & M_{2,0} \arrow[l] & M_{3,0} \arrow[l]
\end{tikzcd}$$

Given any complex of complexes

$$\Big(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \Big) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

one can construct a double complex by multiplying every other differential by -1.

Definition 64 (total complex)**.** Let $M_{\bullet,\bullet}$ be a first-quadrant[2] double complex. The total complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

[2] This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

Lemma 65. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Theorem 66. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex, without loss of generality with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$\begin{tikzcd}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} \arrow[d] & = & \mathrm{Tot}(M)_3 \arrow[d] \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} \arrow[d] & = & \mathrm{Tot}(M)_2 \arrow[d] \\
M_{1,0} \oplus M_{0,1} \arrow[d] & = & \mathrm{Tot}(M)_1 \arrow[d] \\
M_{0,0} \arrow[d] & = & \mathrm{Tot}(M)_0 \arrow[d] \\
0 & & 0
\end{tikzcd}$$

is exact.

The construction is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we construct the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0} m_{2,0} + \delta_{1,1} m_{1,1} = 0, \qquad d_{11} m_{1,1} + \delta_{0,2} m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}} n = m$.

We make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d \delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - d n_{2,1}) = 0.$$

This means that $m_{1,1} - d n_{2,1} \in \ker \delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - d n_{2,1}$. Thus

$$d \delta n_{1,2} = d m_{1,1} = -\delta m_{0,2}.$$

But

$$d \delta n_{1,2} = -\delta d n_{1,2},$$

so

$$\delta(m_{0,2} - d n_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - d n_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - d n_{1,2}.$$

Then

$$\begin{aligned} d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d + \delta) n_{2,1} + (d + \delta) n_{1,2} + (d + \delta) n_{0,3} \\ &= 0 + 0 + m_{2,0} + d n_{2,1} + (m_{1,1} - d n_{2,1}) + d n_{1,2} + (m_{0,2} - d n_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. □

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain

  $$C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$

  hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;

- As a morphism between double complexes, hence between totalizations

$$\mathrm{Tot}(\alpha)_\bullet\colon\ \mathrm{Tot}(C)_\bullet \to \mathrm{Tot}(D)_\bullet.$$

Lemma 67. These two points of view agree in the sense that

$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 68. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet
\end{tikzcd}$$

be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$

is exact. By Theorem 66, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 67 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 56) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 69. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now Theorem 68 gives us the result we want. □

2.8 Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let $\mathcal{A}$ be an abelian category and let

$$- \otimes - : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{H},$$

be an additive functor, where $\mathcal{H}$ is some abelian category equipped with an exact functor to **Ab**. Further suppose that

Definition 70 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$(C \otimes D)_\bullet = \mathrm{Tot}(C_\bullet \otimes D_\bullet),$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 71 (internal hom)**.** Let $\mathcal{A}$ be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We have

2.9 The Künneth formula

This section takes place in R-**Mod**, where R is a PID. For example, everything we are saying holds in **Ab**.

Lemma 72. Let $C_\bullet$ be a chain complex of free R-modules with trivial differential, and let $C'_\bullet$ be an arbitrary chain complex. Then there is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \cong H_n(C_\bullet \otimes C'_\bullet).$$

Proof. We write

$$Z_p = Z_p(C_\bullet), \qquad B_p = B_p(C_\bullet), \qquad Z'_p = Z_p(C'_\bullet), \qquad B'_p = B_p(C'_\bullet),$$

The sequence

$$0 \longrightarrow Z'_q \hookrightarrow C'_q \twoheadrightarrow B'_{q-1} \longrightarrow 0$$

is exact by definition, and since $Z_p = C_p$ is by assumption free, the sequence

By definition, $Z_p = C_p$ is free. Thus the sequence

$$0 \longrightarrow Z_p \otimes Z'_q \hookrightarrow Z_p \otimes C'_q \twoheadrightarrow Z_p \otimes B'_{q-1} \longrightarrow 0$$

is exact.

The differential of the tensor product complex $C_\bullet \otimes C'_\bullet$ is

$$\begin{aligned} d_n^{\mathrm{tot}} &= \sum_{i=0}^{n} d_i^C \otimes \mathrm{id}_{C_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C'} \\ &= \sum_{i=0}^{n} (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C}. \end{aligned}$$

Since C_i is by assumption free, tensoring with it doesn't change homology, and we have

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

and

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

□