1 Cohomology

1.1 Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition **??**) which govern cohomology theories.

Definition 1 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n : \mathbf{Pair}^{\mathrm{op}} \to \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial : H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H^n(i) : H^n(X, A) \cong H^n(X \setminus U, A \setminus U)$$

   for all n.
3. **Dimension:** We have

$$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then

$$H^n(X) = \prod_\alpha H^n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

1.2 Singular cohomology

Definition 2 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet(X), G).$$

We also define relative homology:

$$S^\bullet(X,A;G) = \mathrm{Hom}(S_\bullet(X,A), G).$$

We will denote the evaluation map $\mathrm{Hom}(A,G) \otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 3. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G) \otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet) \otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha \colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned} \langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\ &= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\ &= \langle \alpha, a \rangle. \end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned} \langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\ &= \langle \alpha, a \rangle + \langle \beta, da \rangle \\ &= \langle \alpha, a \rangle \end{aligned}$$

□

Via $\otimes$-hom adjunction, we get a map

$$\kappa \colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 4 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space, and G an abelian group. There is a split exact sequence

$$0 \longrightarrow \operatorname{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X; G) \twoheadrightarrow \operatorname{Hom}(H_n(X), G) \longrightarrow 0$$

Proof. We now □

Example 5. We have seen (in Example ??) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n,\ k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \leq k \leq n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \operatorname{Ext}^1(0, \mathbb{Z}) \oplus \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \leq k \leq n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

1.3 The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \to \mathbb{Z},$$

allowing an n-cochain to eat an n-chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q-simplex, on which our cochain can act, and a singular $n - q$-simplex, which is along for the ride.

Definition 6 (front, rear face). Let $a \colon \Delta^n \to X$ be a singular n-simplex, and let $0 \leq q \leq n$.

- The <u>$(n - q)$-dimensional front face</u> of a is given by

  $$F^{n-q}(a) = \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X,$$

  where i is induced by the inclusion

  $$\{0, \ldots, n - q\} \hookrightarrow \{1, \ldots, n\}; \qquad k \mapsto k.$$

- The <u>q-dimensional rear face</u> of a is given by

$$R^q(a) = \quad \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X \ ,$$

  where r is the map

$$\{0, \ldots q\} \to \{0, \ldots, n\}; \qquad k \mapsto n - q + k.$$

Note that for $a \in S_n(X)$, we can write

$$F^{n-q}(a) = (\partial_{n-q+1} \circ \partial_{n-q+2} \circ \cdots \circ \partial_n)(a)$$

and

$$R^q(a) = (\overbrace{\partial_0 \circ \partial_0 \circ \cdots \circ \partial_0}^{q \text{ times}})(a).$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \to S_{n-q}(X); \qquad \alpha \otimes a \mapsto F^{n-q}(a)\langle a, R^q(a)\rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 7 (cap product)**.** The <u>cap product</u> is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \to S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b)\rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b)\rangle = 0$ by definition of relative cohomology.

Lemma 8. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial\alpha \otimes r).$$

Proof. We do three calculations.

$$\begin{aligned}
\partial(\alpha \frown a) &= \partial(F^{n-q}(\alpha) \otimes \langle \alpha, R(a)\rangle) \\
&= \partial(F^{n-q}(a)) \otimes \langle \alpha, R^q(a)\rangle \\
&= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q+1} \circ \cdots \circ \partial_n)(a) \otimes \langle \alpha, R^q(a)\rangle
\end{aligned}$$

$$\begin{aligned}
(\delta\alpha) \frown a &= F^q(a) \otimes \langle \delta\alpha, R^q(a)\rangle \\
&= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a)\rangle \\
&= F^{n-q}(a) \otimes \langle \alpha, \partial R^q(a)\rangle \\
&= \sum_{i=0}^{q} (-1)^i F^{n-q}(a) \otimes \langle \alpha, \partial_i \circ \partial_0^q a\rangle
\end{aligned}$$

$$\alpha \frown (\partial a) =$$

□

Proposition 9. The cup product is natural in the sense that for any $f : X \to X$ inducing a map of pairs $(X, A) \to (X, B)$, the diagram

$$\begin{tikzcd}
H^q(X, A; R) \otimes H^n(X, A; R) \arrow[rr, "\frown"] \arrow[d] & & H_{n-q}(X; R) \arrow[d] \\
H^q(X, B; R) \otimes H^n(X, B; R) \arrow[rr, "\frown"] & & H_{n-q}(X; R)
\end{tikzcd}$$

$$\begin{tikzcd}
(f^*\beta, a) \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \otimes \langle f^*\beta, R(a)\rangle \arrow[d, mapsto] \\
(\beta, f_*a) \arrow[r, mapsto] & \star
\end{tikzcd}$$

commutes, i.e.

$$f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) = F(f_*a) \otimes \langle \beta, R(f_*a)\rangle.$$

Proof.

$$\begin{aligned} f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) &= f_*(F(a) \otimes \langle f^*\beta, R(\alpha)\rangle) \\ &= f_*(F(a) \otimes \langle \beta, f_*(R(a))\rangle) \\ &= f_*(F(a) \otimes \langle \beta, R(f_*a)\rangle) \\ &= F(f_*a) \otimes \langle \beta, R(f_*a)\rangle. \end{aligned}$$

□

We can write this in a prettier way as

$$f_*(f^*\beta \frown a) = \beta \frown f_*(a).$$

Proposition 10. The cap product descends to a map on homology; that is, we get a map

$$\frown : H^q(X, A; R) \otimes H_n(X, A; R) \to H_{n-q}(X; R); \qquad [\alpha] \otimes [a \otimes r] \mapsto [F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r].$$

1.4 The cup product

Recall that we have constructed an *Eilenberg-Zilber* map

$$H_i(X) \otimes H_j(X) \to H_{i+j}(X \times X)$$

in Subsection **??**. By

1.5 Manifold orientations

Let M be an n-dimensional topological manifold. Without loss of generality, we may choose an atlas for M such that each each point $x \in M$ has a chart (U_x, ϕ_x) such that $U_x \cong \mathbb{R}^n$ and $\phi_x(x) = 0 \in \mathbb{R}^n$.

Let $x \in M$. Then

$$\begin{aligned} H_n(M, M \smallsetminus \{x\}) &\cong H_n(M \smallsetminus (M \smallsetminus U_x), M \smallsetminus \{x\} \smallsetminus (M \smallsetminus U_x)) && \text{(excision)} \\ &\cong H_n(U_x, U_x \smallsetminus \{x\}) \\ &\cong H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) && \text{(homotopy)} \\ &\cong H_{n-1}(\mathbb{S}^{n-1}) && \text{(Example ??)} \\ &\cong \mathbb{Z}. \end{aligned}$$

For any other point $y \in U_x$, an orientation at x gives an orientation at y via the composition

$$H_n(X, X \smallsetminus \{x\}) \cong H_n(X, X \smallsetminus U_x) \cong H_n(X, X \smallsetminus \{y\}).$$

We call a choice of generator of $H_n(X, X \smallsetminus \{x\})$ a *local orientation* on M at x.

We will use the following notation: for a pair X, A, we will write

$$H_n(X, A \smallsetminus X) = H_n(X \mid A).$$

Furthermore, if $A = \{x\}$, we will write $H_n(X \mid x)$ instead of $H_n(X \mid \{x\})$.

We think of $H_n(X \mid A)$ as the *local homology* of A in X, since it depends (by excision) only on the closure of A in X.

Definition 11 (orientation cover). Let M be an n-dimensional manifold. The orientation cover of M is, as a set,

$$\tilde{M} = \{\mu_x \mid x \in M,\ \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

There is a clear surjection from $\tilde{M}$ to M as follows.

$$\begin{tikzcd} \tilde{M} \arrow[d] & \mu_x \arrow[d, mapsto] \\ M & x \end{tikzcd}$$

The set $\tilde{M}$ turns out to be a topological manifold, and the surjection $\tilde{M} \to M$ turns out to be a double cover.

Definition 12 (orientation). Let M be an n-dimensional manifold. An orientation on M is a set-section σ of the bundle $\tilde{M} \to M$ satisfying the following condition: for any $x \in M$ and any $y \in U_x$, the induced local orientation at y is equal to $\sigma(y)$.

For a triple $B \subset A \subset X$ denote the canonical map on pairs $(X, X \smallsetminus A) \to (X, X \smallsetminus B)$ by $\rho_{B,A}$.

Lemma 13. Let M be a connected, orientable topological manifold of dimension m, and let $Ks\{M\}$ be a compact subset.

1. For all $q > m$, $H_q(M \mid K) = 0$.
2. If $a \in H_m(M \mid k)$, then $a = 0$ if and only if $(\rho_{x,K})_* a = 0$ for all $x \in X$.

Proof. We prove the lemma in several steps.

1. First take $M = \mathbb{R}^m$ and $K \subset M$ compact and convex, so that any $x \in K$ is a deformation retract of K. Then, since K is closed and bounded, we can find an

open ball $\mathring{\mathbb{D}}^m$ with $K \subset \mathring{\mathbb{D}}^m$. Then certainly

$$\overline{M \setminus \mathring{\mathbb{D}}^m} = M \setminus \mathbb{D}^m \subset (M \overset{\circ}{\setminus} K) = M \setminus K,$$

so by excision and deformation retractness we have

$$\begin{aligned} H_q(M, M \setminus K) &\cong H_q(M \setminus (M \setminus \mathring{\mathbb{D}}^m), (M \setminus K) \setminus (M \setminus \mathring{\mathbb{D}}^m)) \\ &= H_q(\mathbb{D}^m, \mathbb{D}^m \setminus K) \\ &\cong H_q(\mathbb{D}^m, \mathbb{D}^m \setminus \{x\}) \\ &\cong H_q(\mathbb{D}^m, \mathbb{S}^{m-1}) \\ &\cong H_{q-1}(\mathbb{S}^{m-1}). \end{aligned}$$

For $q > m$, this is 0, and for $q = m$ we note that similar reasoning provides an isomorphism

$$H_q(M, M \setminus K) \cong H_q(M, M \setminus \{x\}) \text{ for all } x \in X.$$

2. Now let $M = \mathbb{R}^n$ and $K = K_1 \cup K_2$, where both K_i are compact and convex. Note that $K_1 \cap K_2$ is also compact and convex. Consider the following relative Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & H_{q+1}(M \mid K_1 \cap K_2) \arrow[dll, "\delta"'] \\ H_q(M \mid K_1 \cup K_2) \arrow[r] & H_q(M \mid K_1) \oplus H_q(M \mid K_2) \arrow[r] & H_q(M \mid K_1 \cap K_2) \arrow[dll, "\delta"'] \\ H_{q-1}(M \mid K_1 \cup K_2) \arrow[r] & \cdots & \end{tikzcd}$$

For $q > n$, Part 1. implies that

$$H_{q+1}(M \mid K_1 \cap K_2) = H_q(M \mid K_1) = H_q(M \mid K_2) = 0,$$

so by exactness $H_q(M \mid K_1 \cup K_2) = 0$.

3. For $M = \mathbb{R}^n$, $K = \bigcup_i K_i$ for K_i compact and contractible, we get the result by induction.

4.

□

Proposition 14. Let $K \subset M$ be compact, and assume that M is connected and oriented via an orientation $x \mapsto o_x \in H_m(X \mid x)$. Then there exists a unique orientation class $o_K \in H_m(M|K)$ such that $(\rho_{x,K})_* o_K = o_x$ for all $x \in K$.

Proof. First we show uniqueness, as we will need it later. Suppose that there exist two orientation classes o_K, o'_K such that

$$(\rho_{x,K})_* o_K = (\rho_{x,K})_* o'_K = o_x.$$

Taking a difference, we find

$$(\rho_{x,K})_*(o_K - o'_K) = 0,$$

so by Lemma 13, we get uniqueness.

As a first step towards proving existence, suppose that K is contained in the domain $U_\alpha \cong \mathbb{R}^n$ of a single chart. Pick some $x \in K$, and consider the following morphisms in **Pair**

$$\begin{tikzcd}
M \smallsetminus U_\alpha \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \smallsetminus K \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \smallsetminus \{x\} \arrow[r, hook] & M
\end{tikzcd}$$

which correspond to the commuting diagram.

$$\begin{tikzcd}
H_m(M \mid U_\alpha) \arrow[r, equal] \arrow[d, "\phi"'] & H_m(M \mid x) \\
H_m(M \mid K) \arrow[ur] &
\end{tikzcd}$$

Pulling $o_x \in H_m(M \mid x)$ back to $H_m(M \mid U_\alpha)$, then mapping it to $H_m(M|K)$ with ϕ gives an orientation class $o_K \in H_m(M \mid K)$. The commutativity of the diagram ensures that $(\rho_{x,K})_* o_K = o_x$.

Now suppose that K is not contained in a single chart. We can find an open cover of K by charts, and since K is compact, there exists a finite subcover.

Suppose K is contained in two charts U_1 and U_2; the general case follows by induction. Then we can express K as a union of compacta $K = K_1 \cup K_2$, where $K_i \subset U_i$. Consider the relative Mayer-Vietoris sequence

$$0 \longrightarrow H_m(M \mid K_1 \cup K_2) \overset{i}{\hookrightarrow} H_m(M \mid K_1) \oplus H_m(M \mid K_2) \overset{\kappa}{\longrightarrow} H_m(M \mid K_1 \cap K_2) \longrightarrow \cdots$$

By our above work, we get unique orientation classes $o_{K_i} \in H_m(M|K_i)$ such that $(\rho_{x,K_i})_* = o_x$ for all $x \in K_i$, $i = 1, 2$. Applying κ, we find

$$\kappa(o_{K_1}, o_{K_2}) = \rho_{K_1 \cap K_2, K_1} o_{K_1} - \rho_{K_1 \cap K_2, K_2} o_{K_2}.$$

But by uniqueness, these must be equal, so there exists a unique o_K in $H_m(M \mid K)$ as in the theorem. □

Definition 15 (orientation class)**.** Let M be a connected, orientable, compact manifold of dimension m. An orientation class on M is an element $o_m \in H_m(M\colon \mathbb{Z})$ such that $o_x = (\rho_{x,M})_* o_m$ is an orientation.

Theorem 16. Let M be a compact, connected manifold of dimension m. The following are equivalent.

1. M is ($\mathbb{Z}$-)orientable.
2. There exists an orientation class $o_M \in H_m(M; \mathbb{Z})$.
3. $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof.

(1 ⇒ 2) Consequence of Proposition 14.

(2 ⇒ 3) Let $a \in H_m(M)$, and let $x \in M$. Then $(\rho_{x,M})_* a = k_x o_x$ for some $k_x \in \mathbb{Z}$. This defines a function $M \longrightarrow \mathbb{Z}$, $x \mapsto k_x$.

We aim to show that this function is constant. Choose a chart U_x around x, and let $y \in U_x$. Then the outside of the following diagram commutes by functoriality of H_m, and the lower triangle commutes by definition of an orientation class, so the upper triangle commutes.

$$\begin{tikzcd}
& H_n(M) \arrow[dl] \arrow[dr] & \\
H_m(M \mid x) \arrow[rr, "\phi"] \arrow[dr, equal] & & H_m(M \mid y) \arrow[dl, equal] \\
& H_m(X \mid U_x) &
\end{tikzcd}$$

This means in particular that

$$\phi(k_x o_x) = k_x \phi(o_x) = k_x o_y$$

must be equal to $k_y o_y$, so $k_x = k_y$. Thus, the function $x \mapsto k_x$ is locally constant, hence constant. Call this constant k.

Thus, $ka - o_M$ is zero when restricted to any $x \in M$, implying by Proposition 14 that $a = k o_M$. o_M generates $H_m(M)$.

We know that o_M is non-trivial because $(\rho_{x,M})_* o_M$ generates $H_m(M|x)$, and that it has infinite order because if $\ell o_M = 0$ for some $\ell \neq 0 \in \mathbb{Z}$ $H_m(M)$, then $\ell(\rho_{x,M})_* o_M = \ell o_x = 0$. This shows that $H_m(M)$ has a single generator of infinite order, i.e. is isomorphic to $\mathbb{Z}$.

(2 ⟹ 3) Pick a generator $o_M \in \mathbb{Z}$, and define an orientation by

$$o_x = (\rho_{x,M})_* o_M.$$

□

From now, we will sometimes denote an orientation class on M by $[M]$.

Definition 17 (degree)**.** Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $g\colon M \to N$ be continuous. The degree of f is the integer d such that

$$f(o_M) = d o_N.$$

Proposition 18. Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $f, g\colon M \to N$ be continuous.

1. The degree is multiplicative, i.e.

$$\deg(f \circ g) = \deg(f) \circ \deg(g)$$

2. If $\bar{M}$ is the same manifold as M but with the opposite orientation, then

$$\deg(f : M \to N) = -\deg(f : \bar{M} \to N).$$

3. If the degree of f is not trivial, then f is surjective.

Proof. All obvious. □

1.6 Cohomology with compact support

For this section, let R be a commutative ring with unit 1_R. For most of this section, we will work with cohomology over R, but notationally suppress this.

Definition 19 (singular cochains with compact support)**.** Let X be a topological space, and let R be a commutative ring with unit 1_R. The singular n-cochains with compact support are the set

$$S_c^n(X;R) = \left\{\phi\colon S_n(X) \to R \mid {}_{\phi(\sigma)=0 \text{ for all } \sigma : \Delta^n \to X \text{ with } \sigma(\Delta^n)\cap K_\phi = \emptyset}^{\text{there exists } K_\phi \subset X \text{ compact such that}}\right\}.$$

That is, singular cochains with compact support are those ϕ which ignore simplices which do not hit some compact subset K_ϕ.

Clearly $S_c^n(X) \subset S^n(X)$, and since the boundary of any simplex with compact support clearly has compact support, there is an induced chain complex structure

$$S_c^*(X).$$

However, there is a more categorical description of this chain complex coming from relative cohomology.

Let X be a topological space. Denote by $\mathcal{K}$ the set of compact subsets of X. This is clearly a poset under

$$K \leq K' \iff K \subseteq K',$$

and it is filtered because for a finite set I, $\bigcup_i K_i$ is an upper bound for $\{K_i\}_{i \in I}$.

Fixing some X, consider the functor

$$\mathcal{K}^{\mathrm{op}} \to \mathbf{Pair}; \qquad K \mapsto (X, X \setminus K).$$

We get a functor $\mathcal{K} \to \mathbf{Coch}_{\geq 0}(R\text{-}\mathbf{Mod})$ via the composition

$$\mathcal{K} \longrightarrow \mathbf{Pair}^{\mathrm{op}} \xrightarrow{S^\bullet} \mathbf{Coch}_{\geq 0}(R\text{-}\mathbf{Mod}) \ .$$

For obvious reasons, we call this functor $S^\bullet(X \mid -)$

Proposition 20. The colimit

$$\operatorname*{colim}_K S^\bullet(X \mid K)$$

agrees with cohomology with compact support $S_c^\bullet(X)$.

Proof. Let $f \in S^n(X \mid K)$. Then by definition, f is a map $S_n(X \mid K) \to R$, i.e. a map $S_n(X) \to R$ such that for any $\sigma\colon \Delta^n \to X$ such that $\sigma(\Delta^n) \cap K = \emptyset$, $f(\sigma) = 0$. Thus, $f \in S_c^n(X)$.

This inclusion gives a chain map $S^\bullet(X \mid K) \hookrightarrow S_c^\bullet(X)$, and these together induce an inclusion[1] $\operatorname{colim}_K S^\bullet(X \mid K) \to S_c^\bullet(X)$. In fact, this is a level-wise epimorphsim since any $g \in S_c^\bullet(X)$ is of the above form. □

Corollary 21. We have an isomorphism

$$H_c^n(X) \cong \operatorname*{colim}_K H^n(X \mid K).$$

Proof. Filtered colimits are exact by Corollary **??**. □

[1] This is a monomorphism because the components are monomorphisms, and by Corollary **??** filtered colimits are exact, hence preserve monomorphisms.

1.7 Poincaré duality

This section takes place over a ring R with unit 1_R. We will usually notationally suppress R.

Let M be a connected, m-dimensional manifold with R-orientation $x \mapsto o_x$, and let $K \subset M$ be a compact subset. By Proposition 14, there exists an orientation class $o_K \in H_m(M \mid K)$ compatible with the orientation o_x.

Consider the cap product

$$H^q(M \mid K) \otimes H_m(M \mid K) \xrightarrow{\frown} H_{m-q}(M) \ .$$

Sticking $o_K \in H_m(M \mid K)$ into the second slot of the cap product and setting $p = n - q$, we find a map

$$H^{m-p}(M \mid K) \xrightarrow{(-)\frown o_K} H_p(M)$$

Proposition 22. The maps

$$(-) \frown o_K \colon H^{m-p}(M|K) \to H_p(M)$$

descend to a map

$$H_c^{m-p}(M) \to H_p(M).$$

Proof. By Proposition 9, the identity $\mathrm{id}_M \colon M \to M$ makes the following diagram commute.

$$\begin{tikzcd}
H^{m-p}(M \mid L) \otimes H_m(M \mid L) \arrow[r, "(-)\frown(-)"] \arrow[d] & H_p(M) \arrow[d, "(\mathrm{id}_M)_*"] \\
H^{m-p}(M \mid K) \otimes H_m(M \mid K) \arrow[r, "(-)\frown(-)"'] & H_p(M)
\end{tikzcd}$$

Since

$$(\mathrm{id}_M)_* = \mathrm{id}_{H_p(M)},$$

this in particular means that the triangles

$$\begin{tikzcd}
H^{m-p}(M \mid L) \arrow[dr, "(-)\frown o_L"] \arrow[dd] & \\
& H_p(M) \\
H^{m-p}(M \mid K) \arrow[ur, "(-)\frown o_K"'] &
\end{tikzcd}$$

commute. But this says that the maps $(-) \frown o_K$ form a cocone under $H^{m-p}(M \mid -)$. □

Definition 23 (Poincaré duality)**.** The map defined in Proposition 22 is called the Poincaré

Duality map, and denoted

$$\mathsf{PD}\colon H_c^{m-q}(M) \to H_m(M).$$

Theorem 24 (Poincaré duality)**.** Let M be a connected manifold of dimension m with an R-orientation $x \mapsto o_x$. Then

$$\mathsf{PD}\colon H_c^{m-p}(M;R) \to H_p(M;R)$$

is an isomorphism for all $p \in \mathbb{Z}$.