1 Basic applications

1.1 The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \operatorname{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by Proposition **??**, is right exact. Thus, we may form the left derived functor.

Definition 1 (Tor functor)**.** The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\operatorname{Tor}_i^R(-, N).$$

Example 2. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free resolution of R/rR. Thus we can calculate

$$\operatorname{Tor}_i^R(R/rR, N) \simeq H_i\left(0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\operatorname{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & n = 0 \\ {}_rN, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

Proposition 3. The functor Tor is balanced; that is,

$$L_i \operatorname{Hom}(M, -)(N) \simeq L_i \operatorname{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 \arrow[d, dashed] & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
M \otimes_R N & P_0 \otimes_R N \arrow[l, dashed] & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

1.2 Ext and extensions

We know that $\operatorname{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 4 (Ext functor)**.** Let R be a ring, and $N \in R\text{-}\mathbf{Mod}$ an R-module. The right derived functors

$$R^i \operatorname{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i \operatorname{Hom}(N, -) = \operatorname{Ext}^i_R(N, -).$$

An argument very similar to that given for Tor in Proposition 3 shows that Ext is balanced; that is, when computing

$$\operatorname{Ext}^i_R(M, N),$$

it doesn't matter whether we take an injective resolution of N or a projective resolution of M; we will get the same result either way. Rather than spelling this out in detail, we will wait for Chapter **??**, where a spectral sequences argument gives the result almost immediately.

There turns out to be a deep connection between the Ext groups and extensions.

Definition 5 (extension)**.** Let A and B be R-modules. An extension of A by B is a short exact sequence

$$\xi\colon \qquad 0 \longrightarrow B \hookrightarrow X \twoheadrightarrow A \longrightarrow 0\ .$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd} \xi\colon & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\mathrm{id}_B"] & X \arrow[r, two heads] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ \xi'\colon & 0 \arrow[r] & B \arrow[r, hook] & X' \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}.$$

The five lemma (Theorem **??**) implies that φ is an isomorphism.

Let ξ be an extension of A by B. Applying the Ext functor we find an associated long exact sequence on cohomology.

$$\begin{tikzcd} \operatorname{Hom}(A, B) \arrow[r] & \operatorname{Hom}(X, B) \arrow[r] & \operatorname{Hom}(B, B) \arrow[dll, "\delta"] \\ \operatorname{Ext}^1(A, B) \arrow[r] & \operatorname{Ext}^1(X, B) \arrow[r] & \cdots \end{tikzcd}$$

Definition 6 (obstruction class)**.** The image of id_B under the connecting homomorphism $\delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A, B)$ is known as the obstruction class of ξ. For any extension ξ, we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 7. An extension ξ is split if and only if the correspoding obstruction class is zero.

Proof. We are given an extension

$$\xi : \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] & 0 \end{tikzcd} .$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$\mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(B, B) \longrightarrow \mathrm{Ext}^1(A, B)$$

If $\delta : \mathrm{id} \to 0$, then there exists $a \in \mathrm{Hom}(X, B)$ such that $a \circ f = \mathrm{id}$. Thus, by the splitting lemma (Lemma **??**), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a : A \to X$ such that $a \circ f = \mathrm{id}_B$.

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[l, bend left, "a"] \arrow[r] & 0 \end{tikzcd} .$$

Since $\mathrm{Ext}^1(-, B)$ is additive, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(X, B) \arrow[l, bend left, "\mathrm{Ext}^1(a,B)"] \arrow[r, two heads, "\mathrm{Ext}^1(f,B)"] & \mathrm{Ext}^1(B, B) \arrow[r] & 0 \end{tikzcd}$$

is split exact. Thus, $\mathrm{Ext}^1(g, B)$ is a monomorphism, so it has trivial kernel.

$$\begin{tikzcd} & & & \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \cdots \\ \mathrm{Hom}(A, B) \arrow[r] & \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[ur, "\delta"] & & \end{tikzcd}$$

Exactness then forces $\delta = 0$. □

Proposition 7 explains the name *obstruction class:* $\Theta(\xi)$ is the obstruction to the splitness of ξ. The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\mathrm{Ext}^1(A, B)$ parametrizes extensions of A by B.

Theorem 8. There is a natural bijection

$$\left\{\substack{\text{Extension classes of}\\ \text{extensions of } A \text{ by } B}\right\} \xrightarrow{\cong} \operatorname{Ext}^1(A, B); \qquad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ, the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Hom}(X, B) \arrow[r] & \operatorname{Hom}(B, B) \arrow[r, "\delta"] & \operatorname{Ext}^1(A, B) \\
\operatorname{Hom}(X, B) \arrow[r] \arrow[u, "\mathrm{id}"] & \operatorname{Hom}(B, B) \arrow[r, "\delta"'] \arrow[u, "\mathrm{id}"] & \operatorname{Ext}^1(A, B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ. Since R-**Mod** has enough projectives, we can find a projective P which surjects onto A. Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] & P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}$$

We construct from such an exact sequence a map which takes an element of $\operatorname{Ext}^1(A, B)$ and gives an extension of A by B. First applying $\operatorname{Ext}^*(-, B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\operatorname{Hom}(K, B) \xrightarrow{\delta} \operatorname{Ext}^1(A, B) \longrightarrow \operatorname{Ext}^1(P, B)$$

Since P is projective, the group $\operatorname{Ext}^1(P, B)$ vanishes, so $\delta\colon \operatorname{Hom}(K, B) \to \operatorname{Ext}^1(A, B)$ is surjective. Thus, given any $e \in \operatorname{Ext}^1(A, B)$, we can find a (not unique!) lift to $\phi \in \operatorname{Hom}(K, B)$. Take the pushout in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads] \arrow[d, "i_P"] & A \arrow[r] & 0 \\
& B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P & &
\end{tikzcd}$$

By the dual to Lemma **??**, the cokernel of i_B is equal to the cokernel of ψ, so we get an induced short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}$$

It remains to show that Ψ as defined is independent of the choice of ϕ, and that it really is an inverse to Φ. To this end, pick a different $\tilde{\phi} \in \mathrm{Hom}(N, B)$ such that $\delta(\tilde{\phi}) = e$. Then there exists some $\rho \in \mathrm{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$.

This gives us a new morphism of exact sequences as follows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\tilde{\phi}"'] & P \arrow[r, two heads] \arrow[d, "\tilde{i}_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\tilde{\phi}} P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd} \tag{1.1}$$

Consider the following outer square.

$$\begin{tikzcd}
K \arrow[r, "\psi"] \arrow[d, "\phi"'] & P \arrow[d, "i_P"] \arrow[ddr, bend left, "\tilde{i}_P - \tilde{i}_B \circ \rho"] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "\tilde{i}_B"'] & B \amalg_N^{\phi} P \arrow[dr, dashed, "\exists ! \alpha"] & \\
& & B \amalg_N^{\tilde{\phi}} P
\end{tikzcd}$$

Using the commutativity of the left-hand square in Diagram 1.1, we find

$$\begin{aligned}
(\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
&= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
&= \tilde{i}_B \circ \phi,
\end{aligned}$$

so this outer square commutes, giving us a unique morphism α. □

1.3 Quiver representations

Definition 9 (quiver). A quiver Q consists of the following data.

- A set Q_0 of *vertices*
- A set Q_1 of *arrows*
- A pair of maps

  $$s, t \colon Q_1 \to Q_0$$

  called the *source* and *target* maps respectively.

1.4 Group (co)homology

1.4.1 The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 10 (normalized chain complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2}d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1}d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} NA_{n-1} \xrightarrow{(-1)^{n-1}d_{n-1}} \cdots$$

Definition 11 (Moore complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n-chains A_n and differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
\partial^2 &= \left(\sum_{j=0}^{n-1}(-1)^j d_j\right) \circ \left(\sum_{i=0}^{n}(-1)^i d_i\right) \\
&= \sum\sum(-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\leq j<i\leq n} (-1)^{i+j} d_j \circ d_i + \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\leq j<i\leq n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0\leq i\leq j\leq n-1} (-1)^{i+j} d_j \circ d_i \\
&= 0.
\end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A, unless this is confusing. In this case, we will denote it by MA.

Definition 12 (alternating face maps chain modulo degeneracies)**.** Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial\colon A_n \to A_{n-1}$ takes degerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain modulo degeneracies.[1]

Lemma 13. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \xhookrightarrow{i} A \xrightarrow{p} A/D(A)$$

Proof. We need only show that the following diagram commutes.

$$\begin{tikzcd}
NA_n \arrow[r, hook] \arrow[d, "(-1)^n d_n"'] & A_n \arrow[r, two heads] \arrow[d, "\partial"] & A_n/DA_n \arrow[d, "{[\partial]}"] \\
NA_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, two heads] & A_{n-1}/DA_{n-1}
\end{tikzcd}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n, and the right-hand square commutes trivially. □

Theorem 14. The composite

$$p \circ i\colon NA \to A/D(A)$$

is an isomorphism of chain complexes.

Let $f\colon [m] \to [n]$ be a morphism in the simplex category Δ. By functoriality, f induces a map

$$N\Delta^n \to N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0\colon [n] \to [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]\colon \Delta \to \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

and on morphisms by

$$f\colon [m] \to [n] \mapsto$$

[1] The nLab is responsible for this terminology.

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \tag{1.2}$$

Theorem 15. The adjunction in Equation 1.2 is an equivalence of categories

Proof. later, if I have time. □

Also later if time, Dold-Puppe correspondence: this works not only for **Ab**, but for any abelian category.

1.4.2 The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta\colon \mathrm{id}_\mathcal{C} \to RL$ and counit $\varepsilon\colon LR \to \mathrm{id}_\mathcal{D}$.

Recall that this data gives a comonad in $\mathcal{D}$, i.e. a comonoid internal to the category $\mathrm{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$LRLR \xLeftarrow{L\eta R} LR \xRightarrow{\varepsilon} \mathrm{id}_\mathcal{D}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category $\mathcal{C}$ and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M}\colon \Delta \to \mathcal{C}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, dashed, "\exists! \tilde{M}"] \\ & \mathcal{C} \end{tikzcd}$$

In particular, with $\mathcal{C} = \mathrm{End}(\mathcal{D})^{\mathrm{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \to \mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, "\exists!"] \\ & \mathrm{End}(\mathcal{D})^{\mathrm{op}} \end{tikzcd}$$

Equivalently, this gives a functor

$$B\colon \Delta_+^{\mathrm{op}} \longrightarrow \mathrm{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\mathrm{ev}_d\colon\ \mathrm{End}(\mathcal{D}) \longrightarrow \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in $\mathcal{D}$.

$$S_d = \ \Delta_+^{\mathrm{op}} \xrightarrow{B} \mathrm{End}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{D}$$

Definition 16 (bar construction)**.** The composition $\mathrm{ev}_d \circ B$ is called the bar construction.

In the case that the category $\mathcal{D}$ is abelian, we can form the Moore complex (Definition 11) of the bar construction.

Definition 17 (bar complex)**.** Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ the bar construction of b.

The bar complex Bar_b of b is the Moore complex of the bar construction of b.

Example 18. Consider the functor

$$U : \mathbb{Z}G\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$$

which forgets multiplication. This has left adjoint

$$\mathbb{Z}G \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \longrightarrow \mathbb{Z}G\text{-}\mathbf{Mod},$$

which assigns to an abelian group A the left $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta\colon \mathrm{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_Z -)$ and counit $\varepsilon\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components

$$\eta_A\colon A \longrightarrow U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \qquad a \longmapsto 1 \otimes a$$

and

$$\varepsilon_M\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \longrightarrow M; \qquad g \otimes m \longmapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z, we find the following comonad in $\mathbb{Z}G$-$\mathbf{Mod}$, where $\nabla =$

$\mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{tikzcd} Z \circ Z \\ Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\ \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \end{tikzcd}$$

This gives us a simplicial object in End($\mathbb{Z}G$-**Mod**).

$$\begin{tikzcd} \cdots\cdots\cdots \; Z \circ Z \circ Z \arrow[r, Rightarrow, shift left=6, "ZZ\varepsilon"] \arrow[r, Rightarrow, shift left=2] \arrow[r, Rightarrow, shift right=2] \arrow[r, Rightarrow, shift right=6, "\varepsilon ZZ"'] & Z \circ Z \arrow[l, Rightarrow, shift left=4] \arrow[l, Rightarrow] \arrow[l, Rightarrow, shift right=4] \arrow[r, Rightarrow, shift left=3, "Z\varepsilon"] \arrow[r, Rightarrow, shift right=3, "\varepsilon Z"'] & Z \arrow[l, Rightarrow, "\nabla"'] \arrow[r, Rightarrow] & \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \end{tikzcd}$$

Evaluating on a specific $\mathbb{Z}G$-module M then gives a simplicial object in $\mathbb{Z}G$-**Mod**.

We continue in the special case of $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$-module. In this case,

$$Z^{\circ n}(Z) = \overbrace{\mathbb{Z}G \otimes_Z \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G}^{n \text{ times}} \otimes_{\mathbb{Z}} \mathbb{Z}.$$

We are justified in suppressing the last copy of $\mathbb{Z}$.

The face maps d_i remove the ith copy of Z; on a generator, we have the action of $d_i \colon Z^{n+1}\mathbb{Z} \to Z^n\mathbb{Z}$ given by

$$d_i \colon x \otimes x_1 \otimes \cdots \otimes x_n \otimes a \mapsto \begin{cases} xx_1 \otimes \cdots \otimes x_n, & i = 0 \\ x \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, & 0 < i < n+1 \\ x \otimes \cdots \otimes x_{n-1}, & i = n+1. \end{cases}$$

It is traditional to use the notation

$$x \otimes x_1 \otimes \cdots \otimes x_n \otimes a = x[x_1 | \cdots | x_n].$$

This notation is the origin of the name *bar complex.*

1.4.3 Group cohomology

One would like to be able to apply the techniques of homological algebra to the study of groups. Unfortunately, the category **Grp** is not very nicely behaved; in particular it is not abelian.

Out of any group G, one can form the group ring $\mathbb{Z}G$. Categorically speaking, the category of rings is arguably even worse then that of groups, so one might question the sanity of such a construction. However, one can study groups using homological techiques by studying modules over the group ring $\mathbb{Z}G$; as a category of modules, it is sure to be abelian.

Given a group G, a <u>G-module</u> is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-$\mathbf{Mod}$.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in $\mathbf{Ab}$, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the equivalence $G\text{-}\mathbf{Mod} \simeq \mathbb{Z}G\text{-}\mathbf{Mod}$, we know that G-$\mathbf{Mod}$ has enough projectives.

Note that there is a canonical functor $\mathrm{triv}\colon \mathbf{Ab} \to \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 19. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \mathrm{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \mathrm{Hom}_{G\text{-}\mathbf{Mod}}(M, \mathrm{triv}\, A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \to A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \to A$ such that $\hat{\alpha}(g \cdot m - g) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \to A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{1.3}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \to \mathrm{triv}\, A$. In fact, G-linear maps $M \to \mathrm{triv}\, A$ are precisely those satisfying Equation 1.3, showing that this is really an isomorphism.

The other case is similar. $\square$

Definition 20 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Note that we actually have a fairly good handle on invariants and coinvariants.

Lemma 21. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\textbf{-Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula $\mathbb{Z}$ is taken to be a trivial right $\mathbb{Z}G$-module, and in the second a trivial left $\mathbb{Z}G$-module.

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

This tells us immediately that the invariants functor $(-)^G$ is left exact, and the coinvariant functor $(-)_G$ is right. We could have discovered this by explicit investigation, and formed the right and left derived hom functors. However, this would have required picking resolutions for each object under consideration, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[2]

Example 22. Let $T \cong \mathbb{Z}$ be an infinite cyclic group with a single generator t. Then

$$\mathbb{Z}T \cong \mathbb{Z}[t, t^{-1}],$$

and we have an exact sequence

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \longrightarrow \mathbb{Z} \longrightarrow 0$$

giving a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}T$-module.

In general, our life is not so easy. Fortunately, we still have, for any group G, a general construction of a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}G$-module, known as the *bar construction.*

[2] In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

Definition 23 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module

$$B_n = \bigoplus_{(g_i)\in(G\smallsetminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$

where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to $(g_1,\ldots,g_n)$. The differential is defined by

$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

This is in fact a chain complex; the verification of this is identical to the verification of the simplicial identities. First, we note that we can write

$$d_n = d_0 + \sum_{i=1}^{n-1}(-1)^i d_i + (-1)^n d_n,$$

where the d_i satisfy the simplicial identities

$$d_i d_j = d_{j-1}d_i, \qquad i < j.$$

Thus, we have

$$\begin{aligned}
d_n \circ d_{n-1} &= \left(\sum_{j=0}^{n-1}(-1)^j d^j\right) \circ \left(\sum_{i=0}^{n}(-1)^i d^i\right) \\
&= \sum\sum(-1)^{i+j}d^j \circ d^i \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}d^j \circ d^i + \sum_{0\le i\le j\le n-1}(-1)^{i+j}d^j \circ d^i \\
&= \sum_{0\le j<i\le n}(-1)^{i+j}d^{i-1} \circ d^j + \sum_{0\le i\le j\le n-1}(-1)^{i+j}d^j \circ d^i \\
&= 0.
\end{aligned}$$

Proposition 24. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-

linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
+
\begin{tikzcd}
& B_n \arrow[dl, dashed, "h_n"] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}$$

gives us in one direction

$$g[g_1|\cdots|g_n] \longmapsto \begin{aligned} & gg_1[g_2|\cdots|g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i g[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ & + (-1)^n g[g_1|\cdots|g_{n-1}] \end{aligned}$$

$$\longmapsto \begin{aligned} & [gg_1|\cdots|g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i [gg_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ & + (-1)^n [gg_1|\cdots|g_n] \end{aligned}$$

and in the other

$$\begin{tikzcd} & g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\ {[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{l} g[g_1|\cdots|g_n] \\ -[gg_1|g_2|\cdots|g_n] \\ -\sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ -(-1)^n[gg_1|\cdots|g_{n-1}] \end{array} \end{tikzcd}.$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. □

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G, A) = H^i\left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{(d_2)^*} \cdots\right)$$

and

$$H_i(G, A) = H_i\left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0\right)$$

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \to G$-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
   - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.
   - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived

functors $R^i(-)^G$.

4. We denote
$$L_i(-)_G = H_i(G,-), \qquad R^i(-)^G = H^i(G,-),$$
and call them *group homology* and *group cohomology* respectively.

5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have
$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$
and thus that
$$H_i(G,A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$
This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 25. Let G be a group, and consider $\mathbb{Z}$ as a free $\mathbb{Z}G$-module. Let us compute $H_1(G,\mathbb{Z})$ and $H^1(G,\mathbb{Z})$.

To compute $H_1(G,\mathbb{Z})$, consider the sequence

$$\begin{tikzcd}
\cdots \arrow[r, "d_2\otimes\mathrm{id}_{\mathbb{Z}}"] & \bigoplus_{g\in G\smallsetminus\{1\}}[g]\mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r, "d_1\otimes\mathrm{id}_{\mathbb{Z}}"] \arrow[d, equal] & {[\cdot]}\mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \bigoplus [g]\mathbb{Z} \arrow[r] & {[\cdot]}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The differential $d_1 \otimes \mathrm{id}_{\mathbb{Z}}$ acts on $[g]$ by sending it to
$$[\cdot]g - [\cdot] = [\cdot] - [\cdot],$$
i.e. everything is a cycle. The differential d_2 sends
$$[g|h] \mapsto [g]h - [gh] + [h] = [g] + [h] - [gh],$$
i.e. the terms $[gh]$ and $[g]+[h]$ are identitfied. Thus, $H_1(G,\mathbb{Z})$ is simply the abelianization of G.

To compute $H^1(G,\mathbb{Z})$, consider the sequence
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot], \mathbb{Z}) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}\Big(\bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}G[g], \mathbb{Z}\Big) \xrightarrow{(d_2)^*} \cdots .$$
Notice immediately that since the hom functor preserves direct sums in the first slot,

we have

$$\begin{aligned}\mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z}\right) &\cong \bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\mathrm{Hom}_{\mathbb{Z}G}\left(\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z}\right)\\ &\cong \bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\mathbb{Z}[g_1|\cdots|g_n].\end{aligned}$$

This notation deserves some explanation; recall that the symbols $[g_1|\cdots|g_n]$ are merely visual aids, reminding us where we are and how the differential acts. The differential

$$(d_n)^*\colon [g_1|\cdots|g_n].$$

In the low cases we have the sequence