1 Tor and ext

1.1 The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by Proposition **??**, is right exact. Thus, we may form the left derived functor.

Definition 1 (Tor functor)**.** The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 2. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free resolution of R/rR. Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i\left(0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & n = 0 \\ {}_rN, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

Proposition 3. The functor Tor is balanced; that is,

$$L_i \mathrm{Hom}(M, -)(N) \simeq L_i \mathrm{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 \arrow[d, dashed] & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
M \otimes_R N & P_0 \otimes_R N \arrow[l, dashed] & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

∎

1.2 Ext and extensions

We know that $\mathrm{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 4 (Ext functor)**.** Let R be a ring, and $N \in R\text{-}\mathbf{Mod}$ an R-module. The right derived functors

$$R^i \,\mathrm{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i \,\mathrm{Hom}(N, -) = \mathrm{Ext}^i_R(N, -).$$

An argument very similar to that given for Tor in Proposition 3 shows that Ext is balanced; that is, when computing $\mathrm{Ext}^i_R(M, N)$, it doesn't matter whether we take an injective resolution of N or a projective resolution of M; we will get the same result either way. Rather than spelling this out in detail, we will wait for Chapter **??**, where a spectral sequences argument gives the result almost immediately.

There turns out to be a deep connection between the Ext groups and extensions.

Definition 5 (extension)**.** Let A and B be R-modules. An extension of A by B is a short exact sequence

$$\xi\colon \qquad 0 \longrightarrow B \hookrightarrow X \twoheadrightarrow A \longrightarrow 0\ .$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd}
\xi\colon & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\mathrm{id}_B"] & X \arrow[r, two heads] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
\xi'\colon & 0 \arrow[r] & B \arrow[r, hook] & X' \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}\ .$$

The five lemma (Theorem **??**) implies that φ is an isomorphism.

Let ξ be an extension of A by B. Applying the Ext functor we find an associated long exact sequence on cohomology.

$$\begin{tikzcd}
& & \mathrm{Ext}^1(A,B) \arrow[r] & \mathrm{Ext}^1(X,B) \arrow[r] & \cdots \\
\mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[u, "\delta"] & &
\end{tikzcd}$$

Definition 6 (obstruction class)**.** The image of id_B under the connecting homomorphism $\delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A, B)$ is known as the obstruction class of ξ. For any extension ξ, we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 7. An extension ξ is split if and only if the correspoding obstruction class is zero.

Proof. We are given an extension

$$\xi: \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xrightarrow{g} A \longrightarrow 0 \ .$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$\mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(B, B) \longrightarrow \mathrm{Ext}^1(A, B)$$

If $\delta\colon \mathrm{id} \to 0$, then there exists $a \in \mathrm{Hom}(X, B)$ such that $a \circ f = \mathrm{id}$. Thus, by the splitting lemma (Lemma **??**), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a\colon A \to X$ such that $a \circ f = \mathrm{id}_B$.

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] \arrow[l, bend left, "a"] & 0 \ . \end{tikzcd}$$

Since $\mathrm{Ext}^1(-, B)$ is additive, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(X, B) \arrow[r, two heads, "\mathrm{Ext}^1(f,B)"] \arrow[l, bend left, "\mathrm{Ext}^1(a,B)"] & \mathrm{Ext}^1(B, B) \arrow[r] & 0 \end{tikzcd}$$

is split exact. Thus, $\mathrm{Ext}^1(g, B)$ is a monomorphism, so it has trivial kernel.

$$\begin{tikzcd} & & \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \cdots \\ \mathrm{Hom}(A, B) \arrow[r] & \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[u, "\delta"] & \end{tikzcd}$$

Exactness then forces $\delta = 0$. ☐

Proposition 7 explains the name *obstruction class:* $\Theta(\xi)$ is the obstruction to the splitness of ξ. The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\mathrm{Ext}^1(A, B)$ parametrizes extensions of A by B.

Theorem 8. There is a natural bijection

$$\left\{\substack{\text{Extension classes of} \\ \text{extensions of } A \text{ by } B}\right\} \xrightarrow{\cong} \mathrm{Ext}^1(A, B); \qquad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A,B) \\
\mathrm{Hom}(X,B) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{Hom}(B,B) \arrow[r, "\delta"'] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1(A,B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ. Since R-**Mod** has enough projectives, we can find a projective P which surjects onto A. Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$0 \longrightarrow K \overset{\psi}{\hookrightarrow} P \overset{\gamma}{\twoheadrightarrow} A \longrightarrow 0$$

We construct from such an exact sequence a map which takes an element of $\mathrm{Ext}^1(A,B)$ and gives an extension of A by B. First applying $\mathrm{Ext}^*(-,B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\mathrm{Hom}(K,B) \xrightarrow{\delta} \mathrm{Ext}^1(A,B) \longrightarrow \mathrm{Ext}^1(P,B)$$

Since P is projective, the group $\mathrm{Ext}^1(P,B)$ vanishes, so $\delta\colon \mathrm{Hom}(K,B) \to \mathrm{Ext}^1(A,B)$ is surjective. Thus, given any $e \in \mathrm{Ext}^1(A,B)$, we can find a (not unique!) lift to $\phi \in \mathrm{Hom}(K,B)$. Take the pushout in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] & 0 \\
& B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P & &
\end{tikzcd}$$

By the dual to Lemma **??**, the cokernel of i_B is equal to the cokernel of ψ, so we get an induced short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

It remains to show that Ψ as defined is independent of the choice of ϕ, and that it really is an inverse to Φ. To this end, pick a different $\tilde{\phi} \in \mathrm{Hom}(N,B)$ such that $\delta(\tilde{\phi}) = e$.

This gives us a new morphism of exact sequences as follows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\tilde{\phi}"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "\tilde{i}_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\tilde{\phi}} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd} \tag{1.1}$$

Because of the way we chose $\tilde{\phi}$, there exists some $\rho \in \mathrm{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$. Consider the following outer square.

$$\begin{tikzcd}
K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[d, "i_P"] \arrow[ddr, bend left, "\tilde{i}_P - \tilde{i}_B \circ \rho"] & \\
B \arrow[r, hook, "i_B"] \arrow[drr, bend right, "\tilde{i}_B"'] & B \amalg_N^{\phi} P \arrow[dr, dashed, "\exists ! \alpha"] & \\
 & & B \amalg_N^{\tilde{\phi}} P
\end{tikzcd} \tag{1.2}$$

Using the commutativity of the left-hand square in Diagram 1.1, we find

$$\begin{aligned}
(\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
&= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
&= \tilde{i}_B \circ \phi,
\end{aligned}$$

so this outer square commutes, giving us the unique morphism α.

The claim that our two extensions are equivalent unwraps to the claim that the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook, "i_B"] \arrow[d, equal] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes. The left-hand square is the lower commuting triangle in Diagram 1.2.

Reminding ourselves of the proof of Lemma **??**, we recall that the morphism π is defined by a pushout: it is the unique map $P \amalg_K^{\phi} P \to A$ such that $\pi \circ i_B = 0$ and $\pi \circ i_P = \gamma$. If we can show that $\tilde{\pi} \circ \alpha$ also satisfies these equations, then we can content ourselves that the diagram commutes. Indeed, we have

$$\begin{aligned}
(\tilde{\pi} \circ \alpha) \circ i_B &= \tilde{\pi} \circ \tilde{i}_B \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}(\tilde{\pi} \circ \alpha) \circ i_P &= \tilde{\pi} \circ (\tilde{i}_P - \tilde{i}_B \circ \rho) \\ &= \gamma.\end{aligned}$$

Our last job, which was really the whole point of the endeavor, is to check that the maps Φ and Ψ are inverse to each other. To that end, fix some A and B, and pick $e \in \operatorname{Ext}^1(A, B)$. By construction of Ψ, we get a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \tag{1.3}$$

where ϕ is an element of $\operatorname{Hom}(K, A)$ which is mapped to by e under δ. The next step in computing $\Phi(\Psi(e))$ would now be to stick $\Psi(e)$ into the hom functor $\operatorname{Hom}(-, B)$, and look at the image of id_B under the connecting homomorphism. However, using the fact that Ext is a homological δ-functor, we can stick the whole morphism of exact sequences Equation 1.3 in and get a morphism of long exact sequences out. Picking out the square containing the relevant boundary map, we find the following.

$$\begin{tikzcd} \operatorname{Hom}(B, B) \arrow[r, "\delta"] \arrow[d, "\operatorname{Hom}(\phi, B)"'] & \operatorname{Ext}^1(A, B) \arrow[d, equal] \\ \operatorname{Hom}(K, B) \arrow[r, "\delta'"] & \operatorname{Ext}^1(A, B) \end{tikzcd}$$

Chasing the identity around in both directions tells us that

$$\delta(\mathrm{id}_B) = \delta'(\phi) = e.$$

Next, we start with an extension

$$\xi: \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] & 0 \end{tikzcd}.$$

Using the connecting homomorphism, we get $e = \delta(\mathrm{id}_B) \in \operatorname{Ext}^1(A, B)$. Now picking $\square$