Homological algebra notes

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Contents

1 Introduction

Homework:

- Must submit more than 50% correct solutions in order to take the final exam (oral).
- Posted on Monday after the lecture
- Submitted on following Monday in lecture
- Discuss following Tuesday, 16:45, room 430

2 Chain complexes

Homological algebra is about chain complexes.

Definition 1 (chain complex)**.** Let R be a ring.[1] A chain complex of R-modules, denoted $(C_\bullet, d)$, consists of the following data:

- For each $n \in \mathbb{Z}$, an R-module C_n, and
- R-linear maps $d_n\colon C_n \longrightarrow C_{n-1}$, called the *differentials*,

such that the differentials satisfy the relation

$$d_{n-1} \circ d_n = 0.$$

Example 2 (Syzygies)**.** Let $R = \mathbb{C}[x_1, \ldots, x_k]$. Following Hilbert, we consider a finitely generated[2] R-module M. The ring R is Noetherian, so by Hilbert's basis theorem, M is Noetherian, hence has finitely many generators $a_1, \ldots, a_k$.

Denote by R^k the free R-module with k generators, and consider

$$\phi\colon R^k \twoheadrightarrow M; \qquad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $\mathrm{Syz}^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^{k} \lambda_i e_i \overset{\phi}{\mapsto} \sum_{i=1}^{k} \lambda_i a_i = 0.$$

This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\mathrm{Syz}^1(M)$ is again a finitely generated R-module. Hence, pick finite set of generators b_i, $1 \le i \le \ell$, and look at

$$R^\ell \twoheadrightarrow \mathrm{Syz}^1(M); \qquad e_i \mapsto b_i.$$

[1] Associative, with unit. Not necessarily commutative.
[2] i.e. expressible as an R-linear combination of finitely many generators.

In general, there is non-trivial kernel, denoted $\mathrm{Syz}^2(M)$. This consists of relation between relations.

We can keep going. This gives us a sequence $\mathrm{Syz}^\bullet(M)$ corresponding to relations between relations between…between relations.

$$\begin{tikzcd}
 & & \mathrm{Syz}^2(M) \arrow[dr, hook] & & & & \\
{} \arrow[r, dash, dotted] & R^m \arrow[ur, two heads] \arrow[rr, "d"] & & R^l \arrow[rr, "d"] \arrow[dr, two heads] & & R^k \arrow[r] & M \\
\mathrm{Syz}^3 \arrow[ur, hook] & & & & \mathrm{Syz}^1(M) \arrow[ur, hook] & &
\end{tikzcd}$$

From this we build a chain complex $(C_\bullet, d)$, where C_n corresponds to the nth copy of R^k, known as the *free resolution* of M.

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \dots, x_n]$-module admits a free resolution of length at most n.

Example 3 (simplicial complexes)**.** Let $N \geq 0$. A collection $K \subset \mathcal{P}(\{0, \dots, n\})$ of subsets is called a *simplicial complex* if it is closed under the taking of subsets in the sense that if for all $\sigma \in K$,

$$\tau \subset \sigma \implies \tau \in K.$$

For every $n \geq 0$, we define the *n-simplices* of K to be the set

$$K_n = \{\sigma \in K \mid |\sigma| = n + 1\}.$$

Every $\sigma \in K_n$ can be expressed uniquely as $\sigma = \{x_0, \dots, x_n\}$, with $x_0 < \cdots < x_n$. Define the *ith face* of σ to be

$$\partial_i \sigma = \{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}.$$

Let R be a ring. Define

$$C_n(K, R) = \bigoplus_{\sigma \in K_n} R_{e_\sigma},$$

where the subscripts are simply labels. Further, define

$$d \colon C_n(K, R) \to C_{n-1}(K, R); \qquad e_\sigma \mapsto \sum_{i=0}^{n} (-1)^i e_{\partial_i \sigma}.$$

This is a complex because...

Example 4. Let k be a field, and A an associative k-algebra. We define a complex of vector spaces

$$\cdots \xrightarrow{d} A^{\otimes n} \xrightarrow{d} \cdots \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} A \otimes A \xrightarrow{d} A ,$$

where d is defined by the formula

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

This complex is known as the *cyclic bar complex.*

Definition 5 (cycle, boundary, homology)**.** Let $(C_\bullet, d)$ be a chain complex. We make the following definitions.

- The space $\ker d_m$ is known as the space of n-cycles, and denoted $Z_n(C_\bullet)$.
- The space $\operatorname{im} d_{m+1}$ is known as the space of n-boundaries, and denoted $B_n(C_\bullet)$.
- The space
$$\frac{Z_n}{B_n} = H_n(C_\bullet)$$
is known as the nth homology of C.

Definition 6 (exact)**.** We say that a chain complex $(C_\bullet, d)$ is exact at $n \in \mathbb{Z}$ if $H_n(C_\bullet = 0)$.

Definition 7 (exact sequence)**.** We say that a sequence

$$C_k \xrightarrow{d} C_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} C_\ell$$

with $d^2 = 0$ is exact if it is exact at $k-1, k-2, \ldots, \ell+1$.

Definition 8 (short exact sequence)**.** A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 .$$

Equivalently, such a sequence is exact if

- f is a monomorphism

- g is a epimorphism
- $H_B = 0$.

Example 9.

- The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is exact.
- The sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is not exact since the map $x \mapsto 2x$ is not injective.
- The sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is exact.

3 Basic algebra

Strictly speaking, this chapter shouldn't need to exist, but my algebra knowledge is pretty hopeless so I'll be putting stuff I should know already here.

3.1 Exactness

Proposition 10. Let R be a ring, and M a left R-module. Then both the hom functors

$$\mathrm{Hom}(M, -), \qquad \mathrm{Hom}(-, M)$$

are left exact.

Proof. Let

$$0 \longrightarrow A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \longrightarrow 0$$

be an exact sequence. First, we show that $\mathrm{Hom}(M, A)$ is exact, i.e. that

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha \colon M \to A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and

$$f \circ \alpha = 0 \implies \alpha = 0$$

because f is a monomorphism.

Similarly, $\mathrm{im}\, f \subseteq \ker g$ because

$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \mathrm{im}\, f$. To this end, let $\gamma \colon M \to B$ such that

$g_* \gamma = 0$.

$$\begin{tikzcd} & & M \arrow[dl, dashed, "\exists!\delta"'] \arrow[d, "\gamma"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Then $\operatorname{im}\gamma \subset \ker g = \operatorname{im} f$, so γ factors through A; that is to say, there exists a $\delta\colon M \to A$ such that $f_*\delta = \gamma$. □

3.2 Tensor products

Definition 11 (tensor product)**.** Let R be a (not necessarily commutative) ring M a right R-module, and N a left R-module. The tensor product $M \otimes_R N$ satisfies the following universal property…

Even for modules over non-commutative rings, we have the following version of tensor-hom adjunction.

Proposition 12. Let R be a ring, M a right R-module, and N a left R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \mathrm{Hom}_{\mathbf{Ab}}(N, -).$$

Proof. We show that there is a natural bijection

$$\mathrm{Hom}_{\mathbf{Ab}}(M \otimes_R N, P) \simeq \mathrm{Hom}_{\mathbf{Ab}}(M, \mathrm{Hom}_{\mathbf{Mod}\text{-}R}(N, P)).$$

□

3.3 Projective and injective modules

Definition 13 (projective, injective module)**.** Let R be a ring. An R-module M is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if the functor $\mathrm{Hom}(-, P)$ is exact.

We know (from Proposition 10) that the hom functor is left exact in both slots. Therefore, we have the following immediate classification of projective objects.

Proposition 14. An R-module P in is projective if and only if for every for every epimorphism $g\colon B \twoheadrightarrow C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the folowing diagram commutes.

$$\begin{tikzcd}
& P \arrow[d, "p"] \arrow[dl, dashed, "\exists\tilde{p}"'] \\
B \arrow[r, two heads, "g"'] & C
\end{tikzcd}$$

Similarly, a module Q is injective if for every monomorphism $A \hookrightarrow B$ and map $g\colon A \to Q$, there exists a homomorphism $\tilde{q}$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, hook, "f"] \arrow[d, "p"'] & B \arrow[dl, dashed, "\exists\tilde{p}"] \\
Q &
\end{tikzcd}$$

Proof. In the projective case, this condition precisely encodes the condition that f_* is an epimorphism; in the projective case, □

Proposition 15. Let R be a ring. A module P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M.$$

Proof. First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \ker\pi \arrow[r, hook] & F \arrow[r, two heads, "\pi"] & P \arrow[r] & 0
\end{tikzcd}$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker\pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{tikzcd}
& P \oplus M \arrow[d, "{(f,g)}"] \arrow[dl, dashed, "{\exists(j,k)}"'] \\
N \arrow[r, two heads] & M
\end{tikzcd}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists j"'] \arrow[d, "f"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

exhibiting P as projective. $\square$

4 Abelian categories

4.1 Basics

In this chapter we provide results without proof.

A category is **Ab**-*enriched* if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$. In an **Ab**-enriched category with finite products, products agree with coproducts, and we call both direct sums. We call an **Ab**-enriched category with finite direct sums an *additive* category, and we call a functor between additive categories such that the maps

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

are homomorphisms of abelian groups an *additive functor.* A *pre-abelian* category is an additive category such that every morphism has a kernel and a cokernel, and an *abelian category* is a pre-abelian category such that every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

This immediately implies the following results.

- Kernels are monic and cokernels are epic
- Since the equalizer of f and g is the kernel of $f - g$, abelian categories have all finite limits, and dually colimits.

Additive functors from additive categories preserve direct sums.

Lemma 16. Let A and B be objects in an additive category $\mathcal{A}$, and let X be an object in $\mathcal{A}$ equipped with morphisms

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_A"'] & \\
B \arrow[ur, "i_B"'] & & B
\end{tikzcd}$$

such that the following equations hold.

$$i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{id}_X$$
$$\pi_A \circ i_B = 0 = \pi_B \circ i_A$$
$$\pi_A \circ i_A = \mathrm{id}_A$$
$$\pi_B \circ i_B = \mathrm{id}_B$$

Then the πs and is exhibit $X \simeq A \oplus B$.

Corollary 17. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(a \oplus b) \simeq F(a) \oplus F(b).$$

The celebrated *Freyd-Mitchell embedding theorem* says that every small abelian category can be embedded into a category of R-modules via an exact functor; in particular, this implies that to prove a theorem in an abelian category, one can use the embedding to translate the theorem to a corresponding theorem about R-modules. Due to exactness, if the theorem holds for R-modules, it will hold in the abelian category as well.

4.2 Chain complexes

Definition 18 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex in $\mathcal{A}$ is…The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms of chain complexes, i.e.…

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes, $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Proposition 19. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the $\mathbf{Ab}$-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd} & Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\ C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet \end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned} (d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q, \end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd} & Q_\bullet \arrow[dl, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\ \ker(f_\bullet) \arrow[r, "\iota_\bullet"] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \end{tikzcd}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{4.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi\colon Q_n \to \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 4.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned} \iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n \end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

Definition 20 (exact sequence)**.** Given a morphism $f\colon A \to B$, the image $\operatorname{im} f = \ker \operatorname{coker} f$ and the coimage is $\operatorname{coker} \ker f$. From this data, we build the following commuting diagram.

$$\begin{tikzcd} & \operatorname{im} f \arrow[d, "i"] \arrow[dr, "0"] & \\ A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\ & \operatorname{coker} f \arrow[ur] & \end{tikzcd}$$

This defines a morphism $\phi\colon \operatorname{im} f \to \ker g$. We say that the above is exact if ϕ is an isomorphism. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

Definition 21 (homology)**.** Let $(C_\bullet, d)$ be a complex. The nth homology of $C_\bullet$ is the cokernel of the map $\phi\colon \operatorname{im} d_{n+1} \to \ker d_n$ described in Definition 20, denoted $H_n(C_\bullet)$.

Proposition 22. Homology extends to a family of additive functors

$$H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d_{n+1}^C"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d_n^C"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d_{n+1}^D"'] & D_n \arrow[r, "d_n^D"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d_n^C \to \ker d_n^D$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. □

We can also go the other way, by defining a map

$$\iota_n \colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\begin{tikzcd}
{} \arrow[r, dotted] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r, dotted] & {}
\end{tikzcd}$$

concentrated in degree n.

Definition 23 (quasi-isomorphism)**.** Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a <u>quasi-isomorphism</u> if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Definition 24 (resolution)**.** Let A be an object in an abelian category. A <u>resolution</u> of A consists of a chain complex $\bullet$ together with a quasi-isomorphism $C_\bullet \to \iota(A)$.

4.3 The snake lemma and its compatriots

Lemma 25 (Splitting lemma)**.** Consider the following solid exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[r, "\pi_C"] \arrow[l, dashed, bend left, "\pi_A"] & C \arrow[r] \arrow[l, dashed, bend left, "i_C"] & 0
\end{tikzcd}$$

The following are equivalent.

- There exists a morphism $\pi_A \colon B \to A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
- There exists a morphism $i_C \colon C \to B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
- B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

Lemma 26. Let $f \colon B \twoheadrightarrow C$ be an epimorphism, and let $g \colon D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd}
S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K also functions as the kernel of f'. Of course, in order for this statement to make sense we need a map $i \colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$\begin{tikzcd}
K \arrow[dr, dashed, "\exists! i"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v \colon Q \to S$ be a map such that $f' \circ v = 0$.

We need to find a unique factorization of v through K.

$$\begin{tikzcd}
 & Q \arrow[dl, dashed, "\exists! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
 & B \arrow[r, "f"'] & C
\end{tikzcd}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
 & S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
 & B \arrow[r, "f"'] & C
\end{tikzcd}$$

By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\substack{\text{elements of pullback}\\ \text{which map to } 0}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\substack{\text{elements of } B\\ \text{which map to } C}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\substack{\text{elements of } B\\ \text{which map to } 0}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 27 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd} \tag{4.2}$$

The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\operatorname{coker} g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 4.2 the following diagram.

$$\begin{tikzcd}
& & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}$$

Take a pullback and a pushout, and using Lemma 26 (and its dual), we find the following.

$$\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}$$

Consider the map

$$\delta_0 = \quad S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \ .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \operatorname{coker} f \to \ker h.$$

□

Corollary 28. Given an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "\delta" description] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "\delta" description] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, rounded corners, to path={ -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-1ex]\tikztostart.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}, "\delta" description] \\
H_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. First note that by the additivity of the functors H_n, we get for each n a (not exact!) complex

$$0 \longrightarrow H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \longrightarrow 0$$

□

4.4 Projectives and injectives

Definition 29 (projective)**.** An object P in an abelian category $\mathcal{A}$ is said to be <u>projective</u> if for every epimorphism $f: B \to C$ and every morphism $p: P \to C$, there exists a morphism $\tilde{p}: P \to B$ such that the following diagram commutes.

$$\begin{tikzcd}
 & P \arrow[d, "p"] \arrow[dl, dashed, "\exists\tilde{p}"'] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Dually, Q is said to be <u>injective</u> if for every monomorphism $g: A \to B$ and every morphism $q: A \to Q$ there exists a morphism $\tilde{q}: B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd}
A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists\tilde{q}"] \\
Q &
\end{tikzcd}$$

Definition 30 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that

$\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Corollary 31. The category R-**Mod** has enough projectives.

Proof. By Proposition 15, free modules are projective, and every module is a quotient of a free module. ◻

Definition 32 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \longrightarrow \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \longrightarrow Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

Lemma 33. Let $f : P_\bullet \longrightarrow \iota(M)$ be a projective resolution. Then $f : P_0 \longrightarrow \iota(M)$ is an epimorphism.

Proof. We know that $H_0(f) : H_0(P) \longrightarrow H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(m)) \simeq \iota_M$, and that ◻

Theorem 34 (extended horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \longrightarrow M'$ and $P''_\bullet \longrightarrow M''$ be projective resolutions. Then given an exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

there is a projective resolution $P_\bullet \longrightarrow M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "\tilde{g}"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

Furthermore, given a morphism $M \longrightarrow N$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd}
 & {} \arrow[d, dotted, no head] & & {} \arrow[d, dotted, no head] & \\
0 \arrow[r] & P_1' \arrow[d] & & P_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[d] & & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We define $P_0 = P_0' \oplus P_0''$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd}
0 \arrow[r] & P_0' \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f \circ p_0'"'] & P_0' \oplus P_0'' \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P_0'' \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

We get the (dashed) map $P_0' \to M$ by composition, and the (dashed) map $P_0'' \to M$ by projectivity of P_0'' (since by Lemma 33 p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P_0' \oplus P_0'' \to M$.

At this point, the innocent reader may believe that we are in the clear, but not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p_0' \circ \pi_{P_0'} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P_1'' to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r] \arrow[d] & \ker p \arrow[r] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0 \arrow[r] & P_0'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms; to see this, hote that □

4.5 Exactness in abelian categories

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 35. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\operatorname{im} f \subset \ker g$. Now take $X = \ker g$.

$$\operatorname{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\operatorname{im} \iota = \ker g \subset \operatorname{im} f$. $\square$

Lemma 36. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

$$0 \longrightarrow \operatorname{Hom}(X, A) \xrightarrow{f_*} \operatorname{Hom}(X, B) \xrightarrow{g_*} \operatorname{Hom}(X, C)$$

is an exact sequence of abelian groups.

Proof. To see that f_* is monic, suppose that $f_*\alpha = 0$ for some $\alpha\colon X \to A$. Then $f \circ \alpha = 0$, so $\alpha = 0$ by monomorphicity of f.

To see that $\operatorname{im} f_* \subseteq \ker g_*$, let there be some β such that

$$g_* f_* \beta = \beta \circ g \circ f = \beta \circ 0 = 0.$$

Now let $\gamma\colon X \to B$ such that $g_*\gamma = g \circ \gamma = 0$. Then, because (A, f) functions as a kernel for g, we get a factorization

$$\gamma = f \circ \alpha = f_*\alpha$$

so $\ker g_* \subseteq \operatorname{im} f_*$. $\square$

Definition 37 (exact functor). Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

  is exact

- We call F exact if it is both left exact and right exact.

Example 38. By Lemma 36, the hom functor $\mathrm{Hom}(A, -): \mathcal{A} \to \mathbf{Ab}$ is left exact. It turns out that $\mathrm{Hom}(-, A)$ is also a left exact functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$. It may seem that this should be right exact, but not so! I

This is analogous (and closely related!) to the behavior of the hom functor with respect to limits. It is often stated that the hom functor $\mathrm{Hom}_{\mathcal{C}}(-, A)$ turns colimits into limits; this is true in the sense that it turns limits in $\mathcal{C}$ into colimits in **Set**. Howevever, $\mathrm{Hom}_{\mathcal{C}}(-, A)$ is a functor whose domain is not $\mathcal{C}$ but $\mathcal{C}^{\mathrm{op}}$, so it would be more accurate to say that the hom functor preserves limits in both slots.

4.6 Mapping cones

Definition 39 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the k-shifted chain complex $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 40 (mapping cone)**.** Let $f: C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\mathrm{cone}(f)$ as follows.

- For each n, define

$$\mathrm{cone}(f)_n = C_{n-1} \oplus D_n.$$

- Define $d_n^{\mathrm{cone}(f)}$ by

$$d_n^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$

which is shorthand for

$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 41. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow C_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} D_\bullet \longrightarrow 0 \ .$$

Proof. We need to specify ι and π. Note that we can always □

Corollary 42. Let $f_\bullet : C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. By Corollary 28, we get a long exact sequence

$$\cdots \to H_n(X) \xrightarrow{\delta} H_n(Y) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see (although I'm missing a sign somewhere); picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & x \\
-f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0,-f(x)) & \\
{[-f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. □

4.7 The Tor functor