1 Spectral sequences

1.1 Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section **??**, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the homology of the total complex such a double complex.

They do this via a series of successive 'approximations'. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
E_{0,3} \arrow[d, "d^v"'] & E_{1,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,2} \arrow[d, "d^v"'] & E_{1,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,1} \arrow[d, "d^v"'] & E_{1,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,0} & E_{1,0} \arrow[l, "d^h"'] & E_{2,0} \arrow[l, "d^h"'] & E_{3,0} \arrow[l, "d^h"'] & \cdots \arrow[l]
\end{tikzcd}$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$E^0_{\bullet,\bullet}$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^0_{0,3} \arrow[d, "d^v"'] & E^0_{1,3} \arrow[d, "d^v"'] & E^0_{2,3} \arrow[d, "d^v"'] & E^0_{3,3} \arrow[d, "d^v"'] \\
E^0_{0,2} \arrow[d, "d^v"'] & E^0_{1,2} \arrow[d, "d^v"'] & E^0_{2,2} \arrow[d, "d^v"'] & E^0_{3,2} \arrow[d, "d^v"'] \\
E^0_{0,1} \arrow[d, "d^v"'] & E^0_{1,1} \arrow[d, "d^v"'] & E^0_{2,1} \arrow[d, "d^v"'] & E^0_{3,1} \arrow[d, "d^v"'] \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{tikzcd}$$

One then takes homology of the corresponding vertical complexes, replacing $E^0_{p,q}$ by the homology $H_q(E^0_{p,\bullet})$. To ease notation, one writes

$$H_q(E^0_{p,\bullet}) = E^1_{p,q}.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$\begin{tikzcd}
E^1_{0,3} & E^1_{1,3} \arrow[l, "H_3(d^h)"'] & E^1_{2,3} \arrow[l, "H_3(d^h)"'] & E^1_{3,3} \arrow[l, "H_3(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "H_2(d^h)"'] & E^1_{2,2} \arrow[l, "H_2(d^h)"'] & E^1_{3,2} \arrow[l, "H_2(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "H_1(d^h)"'] & E^1_{2,1} \arrow[l, "H_1(d^h)"'] & E^1_{3,1} \arrow[l, "H_1(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "H_0(d^h)"'] & E^1_{2,0} \arrow[l, "H_0(d^h)"'] & E^1_{3,0} \arrow[l, "H_0(d^h)"'] & \cdots \arrow[l]
\end{tikzcd}$$

We now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{\bullet,q})$.

The claim is that the terms $E^r_{p,q}$ are pieces of a successive approximation of the homology of the total complex $\mathrm{Tot}(E)$. In the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Proposition 1. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes and a morphism between them, padded by zeroes on either side.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
D_3 \arrow[d, "d_3^D"'] & C_3 \arrow[l, "f_3"'] \arrow[d, "-d_3^C"] \\
D_2 \arrow[d, "d_2^D"'] & C_2 \arrow[l, "f_2"'] \arrow[d, "-d_2^C"] \\
D_1 \arrow[d, "d_1^D"'] & C_1 \arrow[l, "f_1"'] \arrow[d, "-d_1^C"] \\
D_0 \arrow[d, "d_0^D"'] & C_0 \arrow[l, "f_0"'] \arrow[d, "-d_0^C"] \\
D_{-1} \arrow[d] & C_{-1} \arrow[l, "f_{-1}"'] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Note that we have multiplied the diferentials of C by -1 so that we have a double complex.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$.

Let $T = \mathrm{Tot}(E)$; that is, let $T = \mathrm{Cone}(f)$. Then we have a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0$$

Proof. Recall that $\mathrm{Cone}(f)$ fits into the following short exact seqence,

$$0 \longrightarrow C_\bullet \hookrightarrow D_\bullet \twoheadrightarrow \mathrm{Cone}(f)_\bullet \longrightarrow 0$$

and that this gives us the following long exact sequence on homology.

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{n+1}(\mathrm{Cone}(f)) \arrow[dll, "\delta"] \\ H_n(C_\bullet) \arrow[r, "H_n(f)"] & H_n(D_\bullet) \arrow[r, "H_n(p)"] & H_n(\mathrm{Cone}(f)_\bullet) \arrow[dll, "\delta"] \\ H_{n-1}(C_\bullet) \arrow[r] & \cdots \end{tikzcd}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \mathrm{coker}(H_n(f)) \hookrightarrow H_n(\mathrm{Cone}(f)) \twoheadrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation. □

Of course, in a general situation we will not be so lucky. Let us examine the terms

$$E^2_{p,q} = H_p(E^1_{\bullet,q}).$$

First, some notation. We will define in the obvious way $Z^r_{p,q}$ and $B^r_{p,q}$ so that

$$E^r_{p,q} = Z^r_{p,q}/B^r_{p,q};$$

that is, we have

$$Z^r_{p,q} \subset E^{r-1}_{p,q}, \qquad B^r_{p,q} \subset E^{r-1}_{p,q}.$$

Proposition 2. There is an isomorphism

$$E^2_{p,q} \cong U/V,$$

where

$$U = \{(a,b) \subset E_{p-1,q+1} \times E_{p,q} \mid d^v a + d^h b = d^v b = 0\}$$

and

$$V = \left\langle \begin{matrix} (a,0)\colon d^v a = 0 \\ (d^h x, d^v x)\colon x \in E_{p,q+1} \\ (0, d^h c)\colon c \in E_{p+1,q}, d^v c = 0 \end{matrix} \right\rangle.$$

Proof. First we construct a homomorphism

$$\phi\colon U \to E^2_{p,q}.$$

Define

$$\phi^0\colon U \to E^0_{p,q}; \qquad (a,b) \mapsto b.$$

Since by assumption $d^v b = 0$ we know that $\operatorname{im}\phi^0 \subset \ker d^v$, so ϕ^0 descends to $\ker d^v$, and therefore to a map

$$\phi^1 : U \longrightarrow E^1_{p,q}; \qquad (a, b) \mapsto b + B^1_{p,q}.$$

Since

$$H_q(d^h)(b + B^1_{p,q}) = d^v(-a) + B^1_{p-1,q} = 0 + B^1_{p-1,q},$$

we have that $\operatorname{im}\phi^1 \subset \ker H_q(d^h)$, so the map ϕ^1 descends to $\ker H_q(d^h)$, and therefore to a map

$$\phi\colon U \longrightarrow E^2_{p,q}; \qquad (a, b) \mapsto (b + B^1_{p,q}) + B^2_{p,q}.$$

Next, we show that ϕ is surjective. Let

$$\underbrace{(\overbrace{\ x\ }^{\in Z^1_{p,q}} + B^1_{p,q})}_{\in Z^2_{p,q}} + B^2_{p,q} \in E^2_{p,q}.$$

- The fact that $x \in Z^1_{p,q}$ means that $d^v x = 0$.
- The fact that $x + B^1_{p,q} \in Z^2_{p,q}$ means that $H_q(d^h)x = [0]$, i.e. that $d^h x \in B^1_{p-1,q}$.

□