1 Homology

1.1 Basic definitions and examples

1.1.1 Ordinary homology

Singular homology is the study of topological spaces by associating to them chain complexes in the following way.

$$\mathbf{Top} \xrightarrow[\text{nerve}]{\text{singular}} \mathbf{Set}_\Delta \xrightarrow[\text{group}]{\text{free abelian}} \mathbf{Ab}_\Delta \xrightarrow[\text{correspondence}]{\text{Dold-Kan}} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Definition 1 (singular complex, singular homology)**.** Let X be a topological space. The singular chain complex $S_\bullet(X)$ is defined level-wise by

$$S_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where $\mathcal{F}$ denotes the free group functor and M is the Moore functor (Definition **??**). That is, it has differentials

$$d_n \colon S_n \to S_{n-1}; \qquad (\alpha \colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(S_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f \colon X \to Y$ gives a chain map $S(f) \colon S(X) \to S(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 2. Denote by pt the one-point topological space. Then $\mathbf{Sing}(\mathrm{pt})_n = \{*\}$, and the chain complex $S_\bullet(\mathrm{pt})$ is at each level simply $\mathbb{Z}$.

The differential d_n is given by

$$d_n = \sum_{i=0}^{n} (-1)^i \partial^i; \qquad * \mapsto \sum_{i=0}^{n} (-1)^i * .$$

If n is odd, then there are as many summands with even sign as there are with odd sign, and they cancel each other out. If n is even, then there is one more term with positive sign than with negative sign, and one ends up with the identity. Thus $S(\mathrm{pt})_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & S_3 \arrow[r, "d_3"'] & S_2 \arrow[r, "d_2"'] & S_1 \arrow[r, "d_1"'] & S_0 \arrow[r, "d_0"'] & S_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

1.1.2 Reduced homology

Recall that in addition to the *ordinary*, or *topologist's simplex category* Δ, there is the so-called *extended*, or *algebraist's simplex category* Δ_+, which includes the object $[-1] = \emptyset$. If we use this category to index our simplicial sets, our chain complexes have a term $\tilde{S}_{-1} = \mathbb{Z}$. It is traditional to denote the differential $d_0 \colon \tilde{S}_0 \to \tilde{S}_{-1}$ by ε, and call it the *augmentation map.*

$$\cdots \longrightarrow \tilde{S}_2 \xrightarrow{d_2} \tilde{S}_1 \xrightarrow{d_1} \tilde{S}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Definition 3 (reduced singular chain complex)**.** The chain complex $\tilde{S}$ is called the reduced singular chain complex of X.

We see immediately that

$$S_i(X) = \tilde{S}_i(X), \qquad i \geq 0,$$

implying

$$H_i(X) = \tilde{H}_i(X), \qquad i \geq 1.$$

The difference between ordinary and reduced homology is that, as one might suspect, ordinary homology is better-behaved from a topological point of view, and reduced homology is better behaved algebraically (for example, as a functor reduced homology turns wedge products into direct sums).

There is also another interpretation of reduced homology. Recall that we have used the name *augmentation map* before, when dealing with resolutions. There we reinterpreted the augmentation map ε as a chain map to a complex concentrated in degree zero. We can pull exactly the same trick here, viewing the augmentation map as a map of chain complexes

$$\begin{tikzcd}
\cdots \arrow[r] & S_2 \arrow[r, "d_2"] \arrow[d] & S_1 \arrow[r, "d_1"] \arrow[d] & S_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}.$$

Example 4. We can immediately read off that the reduced homology of the point is

$$\tilde{H}_n(\mathrm{pt}) = 0 \qquad \text{for all } n.$$

Example 5. Let X be a nonempty path-connected topological space. Then $H_0(X) \cong \mathbb{Z}$. Specifically,

$$\varepsilon\colon H_0(X) \to \mathbb{Z}$$

is an isomorphism.

To see this, let $x \in S_0(X)$ be a point. Since $\varepsilon(nx) = n$ for any $n \in \mathbb{Z}$, we know that ε is surjective. If we can show that ε is injective, then we are done.

At face value, ε does not look injective; after all, there are as many elements of $S_0(X)$

consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

Example 6. More generally, for any (not necessarily path connected) space X,

$$H_0(X) = \mathbb{Z}^{\pi_0(X)}.$$

Suppose that X can be written as a disjoint union

$$X = \coprod_i X_i.$$

Then, since a map

$$\sigma\colon \Delta^n \to X$$

cannot hit more that one of the X_i, we have

$$\mathbf{Top}\left(\Delta^n, \coprod_i X_i\right) \cong \coprod_i \mathbf{Top}(\Delta^n, X_i).$$

Thus, since the free abelian group functor $\mathcal{F}$ and the Moore functor M are left adjoint, they preserve colimits and we have

$$S_n\left(\coprod_i X_i\right) \cong \bigoplus_i S_n(X_i).$$

We will come back to reduced homology in Section 1.7, when we have the tools to understand it properly.

1.2 The Hurewicz homomorphism

Let X be a topological space, and let $x \in X$. Let γ be a loop in X which starts and ends at x, i.e.

$$\gamma\colon \Delta^1 \to X; \qquad \gamma(0) = x = \gamma(1).$$

We can view γ either as a singular 1-simplex in X, or as a representative of a homotopy class in $\pi_1(X, x)$. It turns out that these points of view are compatible.

Lemma 7. The assignment $\gamma \mapsto [\gamma]_{H_1}$ respects homotopy, and thus descends to a map of sets out of $\pi_1(X, x)$.

$$\begin{tikzcd} \left\{\begin{matrix}\text{Loops starting and}\\ \text{ending at } x\end{matrix}\right\} \arrow[r, "\text{homology class}"] \arrow[d, two heads, "\text{homotopy class}"'] & H_1(X) \\ \pi_1(X, x) \arrow[ur, dashed, "\exists ! \tilde{h}_X"'] & \end{tikzcd}$$

Proof. Let γ_1 and γ_2 be loops based at x.

First, suppose that $\gamma_1 \overset{H}{\sim} \gamma_2$, i.e.

In order to see that □

The map $\tilde{h}_X$ is very well-behaved.

Lemma 8. For a topological space X and point $x \in X$, the map

$$\tilde{h}_X\colon \pi_1(X, x) \to H_1(X)$$

is a group homomorphism.

Proof. We need to show that it sends the identity to the identity and respects composition. □

Because $H_1(X)$ is abelian, $\tilde{h}_X$ descends to a map out of $\pi_1(X, x)_{\mathrm{ab}}$.

Definition 9 (Hurewicz homomorphism)**.** Let X be a path-connected topological space, and let $x \in X$. The Hurewicz homomorphism is the map

$$h_X\colon \pi_1(X, x)_{\mathrm{ab}} \to H_1(X).$$

Theorem 10 (Hurewicz)**.** For any path-connected topological space X, the Hurewicz homomorphism is an isomorphism

$$h_X \colon \pi_1(X, x_0)_{\mathrm{ab}} \cong H_1(X),$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism.

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, bend left, "(\pi_1)_{\mathrm{ab}}"{name=U}] \arrow[r, bend right, "H_1"'{name=D}] & \mathbf{Ab} \arrow[Rightarrow, from=U, to=D, "h"] \end{tikzcd}$$

Proof. We construct an inverse explicitly.

For each point $x \in X$, pick a path

$$\gamma_x \colon \Delta^1 \longrightarrow X; \qquad \gamma_x(0) = x_0, \quad \gamma_x(1) = x$$

connecting x_0 to x. For $x = x_0$, choose γ_{x_0} to be the constant path.

For each generator $\alpha \colon \Delta^1 \longrightarrow X$ in $H_1(X)$, the concatenation

$$\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}$$

is a path starting and ending at x. We may thus define a map

$$\phi \colon S_1(X) \longrightarrow \pi_1(X, x)_{\mathrm{ab}}$$

on generators by

$$\alpha \longmapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}],$$

and addition via

$$\alpha + \beta \longmapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}][\gamma_{\beta(0)} * \beta * \overline{\gamma_{\beta(1)}}].$$

In order to check that defining this homomorphism on generators descends to a map on homology, we have to check that it sends boundaries $\partial\sigma$ to zero. □

Example 11. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\mathrm{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \qquad n > 1.$$

Example 12. Denote by Σ_g the two-dimensional surface of genus g. We know that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_{2g}, b_{2g} | \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

The abelianization of this is simply $\mathbb{Z}^{2g}$, so

$$H_1(\Sigma_g) = \mathbb{Z}^{2g}.$$

Example 13. Since

$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

1.3 The method of acyclic models

We need to break the flow here for a theorem which will show up several times, called the *method of acyclic models.*

Definition 14 (category with models)**.** A category with models is a pair $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is a category and $\mathcal{M} \subset \mathrm{Obj}(\mathcal{C})$ is a set of objects of $\mathcal{C}$.

Definition 15 (free, acyclic functor)**.** Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let $F\colon \mathcal{C} \to \mathbf{Ch}_{\geq 0}(R\text{-}\mathbf{Mod})$.

1. We say that F is acyclic on $\mathcal{M}$ if for each $M \in \mathcal{M}$, $F(M)$ is acyclic in positive degree, i.e. if $H_n(F(M)) = 0$ for $n > 0$.
2. Let J be a set, and $\mathcal{M}_J \subset \mathcal{M}$ a J-indexed set of objects in $\mathcal{M}$. Note that we allow the possibility that each $M \in \mathcal{M}$ can appear more than once in $\mathcal{M}_J$, or not at all.

   Let $F_*\colon \mathcal{C} \to R\text{-}\mathbf{Mod}$ be a functor. An $\mathcal{M}_J$-basis for F_* is, for each $j \in J$ an element $m_j \in F_*(M_j)$ (forming an indexed collection $\{m_j \in F_*(M_j)\}_{j \in J}$) such that for any $X \in \mathcal{C}$ the indexed collection

   $$\{F_*(f)(m_j)\}_{j \in J, f \in \mathrm{Hom}(M_j, X)}$$

   is a basis for $F_*(X)$ as a free R-module; that is, that we can write

   $$F_*(X) = \mathbb{Z}\{F_*(f)(m_j)\}_{j \in J, f \in \mathrm{Hom}(M_j, X)}.$$

   We say that F is free on $\mathcal{M}$ if for each $q \geq 0$ there exists some set J_q and indexed set $\mathcal{M}_{J_q}$ such that each F_q has an $\mathcal{M}_{J_q}$-basis.

Example 16. Consider the category **Top** with models $\{\Delta^n\}_{n=0,1,\ldots}$. The functor $S\colon \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ is both free and acyclic. Acyclicity is clear since Δ^n is contractible for all n.

To see freeness, note that the singleton

$$\{\mathrm{id}_{\Delta^n} \colon \Delta^n \to \Delta^n\}$$

forms a basis for F_n because the abelian group $S_n(X)$ is free with generating set

$$\{S(\alpha)(\mathrm{id}_{\Delta^n}) = \alpha\}_{\alpha \in \mathrm{Hom}(\Delta^n, X)}.$$

The freeness condition can be interpreted as telling us that in order to know how the functor F behaves on any object, it is enough to know how it behaves on the M_α.

Theorem 17 (acyclic model theorem)**.** Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let F, G be functors $\mathcal{C} \to \mathbf{Ch}_{\geq 0}(R\text{-}\mathbf{Mod})$ such that F is free on $\mathcal{M}$ and G is acyclic on $\mathcal{M}$.

1. Any natural transformation $\bar{\tau}_0 \colon H_0(F) \to H_0(G)$ is induced by a natural chain map $\tau \colon F \to G$.
2. Two natural chain maps τ, $\tau' \colon F \to G$ inducing the same natural transformation $H_0(G) \to H_0(G')$ are naturally chain homotopic.

Proof.

1. Let $\tau \colon F \Rightarrow G$ be a natural transformation. First, we collect some results about the natural transformation $\tau_q \colon F_q \Rightarrow G_q$. Note that, since

   $$F_q(X) \cong R\{F_q(f)(m_j)\}_{j \in J_q, f \in \mathrm{Hom}(M_j, X)}$$

   we can specify $\tau_q(X)$ (the component of τ_q at X) completely by specifying how it acts on those elements of $F(q)(X)$ of the form $F_q(f)(m_j)$, and that we are free in choosing this action.

   Consider the following naturality square for some map $f \colon M_j \to X$.

   $$\begin{tikzcd}
F_q(M_j) \arrow[rrr, "\tau_q(m_j)"] \arrow[ddd, "F_q(f)"'] & & & G_q(M_j) \arrow[ddd, "G_q(f)"] \\
& m_j \arrow[r, mapsto] \arrow[d, mapsto] & \tau_q(M_j)(m_j) \arrow[d, mapsto] & \\
& F_q(f)(m_j) \arrow[r, mapsto] & \star & \\
F_q(X) \arrow[rrr, "\tau_q(X)"'] & & & G_q(X)
\end{tikzcd}$$

   Naturality requires that

   $$\tau_q(X)(F(f)(m_j)) = G_q(f)(\tau_q(M_j)(m_j)).$$

Thus, in order to define $\tau_q(X)$ for all X, we need only specify how $\tau_q(M_j)$ behaves on $m_j \in F_q(M_j)$. Defining it in this way and extending it by naturality will ensure that the maps we construct form the components of a natural transformation.

For $q = 0$, this is easy, since we have a natural transformation $\bar{\tau}_0$. Since everything in $H_0(F)$ is a cycle, we can define $\tau_0(M_j)(m_j)$ to be any representative of the equivalence class

$$\bar{\tau}_0(M_j)[m_j] \in H_0(G(M_j)).$$

Now suppose we have defined natural transformations τ_{q-1}, for $q > 0$. For $j \in J_q$, we define $\tau_q(M_j(m_j))$ by

$$\partial \tau_q(M_j)(m_j) = \tau_{q-1}(M_j(\partial m_j)).$$

This is well-defined precisely because

$$\partial \tau_{q-1}(M_j)(\partial m_j) = \tau_{q-2}(\partial^2 m_j) = 0$$

since τ_{q-1} is by assumption a chain map and G is by assumption acyclic on $\mathcal{M}$.

2. One defines a chain homotopy inductively using the same trick.

□

1.4 Homotopy equivalence

We would like that two maps which are homotopic induce the same maps on homology. As alluded to in Section ??, we will see that a homotopy H between maps $f, g\colon X \to Y$ induces a chain homotopy h between $S_n(f)$ and $S_n(g)$.

1.4.1 Homotopy invariance: the high brow approach

Proposition 18. Let f, $g\colon X \to Y$ be continuous maps between topological spaces, and let $H\colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$.

Proof. Suppose we are given a homotopy $f \overset{H}{\sim} g$.

$$\begin{tikzcd}
X \arrow[d, "\mathrm{id}\times\{0\}"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, "H"] & X \\
X \arrow[u, "\mathrm{id}\times\{1\}"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Consider the functors

$$F = S_\bullet(-),\ G = S_\bullet(I \times -) \colon \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab}).$$

Take **Top** with models $\mathcal{M} = \{\Delta^q \mid q = 0, 1, \dots\}$, $J_n = \{*\}$, $\mathcal{M}_{J_n} = \{\Delta^n\}$ and $m_* = \mathrm{id}_{\Delta^n}$. For all n, we have that F is free because

$$S_n(X) \cong \mathbb{Z}\{S_n(\alpha)(\mathrm{id}_{\Delta^n}) \mid \alpha \in \mathrm{Hom}(\Delta^n, X)\},$$

and that G is acyclic because $\Delta^n \times I$ is contractible for all n.

Denote by ι_X^0 the map $X \to X \times I$, which sends $x \mapsto (x, 0)$. Define ι_X^1 analogously.

Consider the natural transformations

$$i^0, i^1 \colon F \Rightarrow G$$

with components

$$i_X^0 = S_\bullet(\iota^0), \qquad i_X^1 = S_\bullet(\iota^1).$$

These clearly agree on H_0. Thus, by Theorem 17, i^0 and i^1 are chain homotopic via some chain homotopy D.

If D is homotopy between i^0 and i^1, then $S(H) \circ D$ is a homotopy between $S(H) \circ i^0$ and $S(H) \circ i^1$. But $S(H) \circ i^0 = S(f)$, and $S(H) \circ i^1 = S(g)$. □

Corollary 19. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 20. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

1.4.2 Homotopy-invariance: the low-brow approach

Such a homotopy H is a map

$$H \colon X \times [0,1] \to Y.$$

It seems reasonable that we would get the chain homotopy in question by relating the singular homology of X and the singular homology of $X \times [0, 1]$.

Here is the game plan.

- We notice that there are n obvious ways of of mapping

$$p_i \colon \Delta^{n+1} \to \Delta^n \times [0,1],$$

and $\Delta^n \times [0,1]$ is the union of the images, which overlap only along their boundaries.

- Given an n-simplex $\alpha\colon \Delta^n \to X$, we can produce an $(n+1)$-simplex in $X \times [0,1]$ by pulling back:
$$P_i\colon \Delta^{n+1} \to \Delta^n \times [0,1] \to X \times [0,1].$$
- Given a homotopy
$$H\colon X \times [0,1] \to Y$$
between f and g, we can pull back by the P_i, giving us an $(n+1)$-simplex
$$P_i(\alpha)\colon \Delta^{n+1} \to \Delta^n \times [0,1] \to X \times [0,1] \to Y.$$
The association
$$\alpha \mapsto P_i\alpha$$
is a homomorphism.
- If we take a sum of the $P_i\alpha$, we get a cellular decomposition of the image of α. However, by taking an alternating sum
$$P = \sum_{i=0}^{n} P_i$$
we can get (upon passing to boundaries) the interior walls to cancel each other out. Thus, the composition
$$d \circ P$$
gives a cellular decomposition of the boundary of the image of the cylinder. The composition $P \circ d$ gives you a cellular decomposition of the sides of the cylinder with the opposite sign. Thus, the sum $d \circ P + P \circ d$ gives you the (difference of) the top and the bottom of the cylinder.
- This tells us that *homotopic maps between topological spaces induce the same map between homotopy groups.*

1.5 Relative homology

1.5.1 Relative homology

Denote by **Pair** the category whose objects are pairs (X, A), where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \to (Y, B)$ are maps $f\colon X \to Y$ such that $F(A) \subset B$. That is, the category **Pair** is the full subcategory on the category **Fun**$(I, \mathbf{Top})$ on monomorphisms; the objects are diagrams

$$A \hookrightarrow X$$

and the morphisms are commuting squares

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ B \arrow[r, hook] & Y \end{tikzcd}$$

Definition 21 (relative homology)**.** Let (X, A) be a pair of spaces. The inclusion $i\colon A \hookrightarrow X$ induces a map of chain complexes $A \hookrightarrow X$. The relative chain complex of (X, A) is the cokernel of $C_\bullet(i)$.

Since colimits are computed level-wise, the relative chain complex is given by the level-wise quotient

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Lemma 22. For each n, relative homology provides a functor $H_n\colon \mathbf{Pair} \to \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{tikzcd} S(X)_\bullet \arrow[r, "f"] \arrow[d, two heads] & S(Y)_\bullet \arrow[d, two heads] \\ S(X)_\bullet/S(A)_\bullet \arrow[r, dashed] & S(Y)_\bullet/S(B)_\bullet \end{tikzcd}$$

The dashed arrow is uniquely well-defined because of the assumption that $f(A) \subset B$, meaning that the relative homology construction is well-defined on the level of the relative chain complex. Functoriality of the relative homology now follows from the functoriality of H_n. □

1.5.2 The long exact sequence on a pair of spaces

Proposition 23. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{j+1}(X,A) \arrow[dll, "\delta"] \\ & H_j(A) \arrow[r] & H_j(X) \arrow[r] & H_j(X,A) \arrow[dll, "\delta"] \\ & H_{j-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \longrightarrow 0$$

□

Example 24. Let $X = \mathbb{D}^n$, the n-disk, and $A = \mathbb{S}^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\ & H_j(\mathbb{S}^{n-1}) \arrow[r] & H_j(\mathbb{D}^n) \arrow[r] & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\ & H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots & \end{tikzcd}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible. Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

Proposition 25. Suppose $i\colon A \hookrightarrow X$ is a weak retract, i.e. that there is an $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_A"'] & X \arrow[r, two heads, "r"] & A \end{tikzcd}$$

Then

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

Proof. Applying the functor H_n to the diagram above, we find that

$$H_n(r) \circ H_n(i) = \mathrm{id}_{H_n(A)},$$

implying that $H_n(i)$ is injective. Consider the long exact sequence on the pair (X, A).

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X,A) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, hook, "H_n(i)"] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, hook, "H_{n-1}(i)"] & \cdots &
\end{tikzcd}$$

The injectivity of $H_n(i)$ implies that the connecting homomorphisms are zero, meaning that the following sequence is short exact for all n.

$$\begin{tikzcd}
0 \arrow[r] & H_n(A) \arrow[r, hook, "H_n(i)"] & H_n(X) \arrow[r, two heads] & H_n(X,A) \arrow[r] & 0
\end{tikzcd}$$

As $H_n(r)$ is a left inverse to $H_n(i)$, the sequence above splits from the left, implying by the splitting lemma (Lemma **??**) the result. □

Proposition 26. Let $i\colon A \to X$ be a deformation retract,[1] i.e. that there is a homotopy

$$R\colon X \times [0,1] \to X$$

such that the following conditions are satisfied.

1. $R(x, 0) = x$ for all $x \in X$
2. $R(x, 1) \in A$ for all $x \in X$
3. $R(a, 1) = a$ for all $a \in A$

Then $H_n(i)\colon H_n(A) \to H_n(X)$ is an isomorphism.

Proof. Let

$$r = R(-, 1)\colon X \to A.$$

Then 3. implies that $r \circ i = \mathrm{id}_A$.

Furthermore, $R(x, -)$ provides a homotopy between $i \circ r$ and id_X. Thus, X and A are homotopy equivalent, and $H_n(i)$ is an isomorphism. □

Corollary 27. If $i\colon X \to A$ is a deformation retract, then $H_n(X, A) = 0$ for all n.

1.5.3 The braided monstrosity on a triple of spaces

Definition 28 (triple of spaces)**.** Let X be a topological space, and let $B \subset A \subset X$ be subspaces. We call (X, A, B) a <u>triple</u>.

[1] I believe we only need i to be a homotopy equivalence; a deformation retract is a homotopy equivalence in which one of the two homotopies is an identity.

A triple of spaces is in particular three pairs of spaces: (X, A), (X, B), and (A, B). All of these have associated long exact sequences. There is also a fourth long exact sequence.

Proposition 29. There is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(A,B) \arrow[dll, out=-10, in=170, "\delta"] \\
H_j(A,B) \arrow[r] & H_j(X,B) \arrow[r] & H_j(X,A) \arrow[dll, out=-10, in=170, "\delta"] \\
H_{j-1}(A,B) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. This comes from the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & S_n(A)/S_n(B) \arrow[r, hook] & S_n(X)/S_n(B) \arrow[r, two heads] & S_n(X)/S_n(A) \arrow[r] & 0 .
\end{tikzcd}$$

□

We can put all four of these together in the following handsome commutative diagram.

$$\begin{tikzcd}
\cdots \arrow[dr] & & & & & & \cdots \\
 & H_{n+1}(X,A) \arrow[rr, bend left, "\delta"] \arrow[dr, "\delta"] & & H_n(A,B) \arrow[rr, bend left, "\delta"] \arrow[dr] & & H_{n-1}(B) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr, "\delta"] & & H_n(A) \arrow[ur] \arrow[dr] & & H_n(X,B) \arrow[ur, "\delta"] \arrow[dr] & & \cdots \\
 & H_n(B) \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,A) \arrow[ur, "\delta"] \arrow[dr, "\delta"] & \\
\cdots \arrow[ur, "\delta"] & & & & & & \cdots
\end{tikzcd}$$

1.6 Barycentric subdivision

Fact 30. Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X. Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha \colon \Delta^n \to X$$

whose images are completely contained in some open set in the open cover $\mathfrak{U}$. That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$ induces a chain structure on $S_\bullet^{\mathfrak{U}}(X)$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_2^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_1^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_0^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & 0 \\
\cdots \arrow[r] & S_2(X) \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & 0
\end{tikzcd}$$

In fact, this inclusion is homotopic to the identity, hence induces an isomorphism

$$H_n^{\mathfrak{U}}(X) := H_n(S^{\mathfrak{U}}(X)_\bullet) \equiv H_n(X).$$

This fact allows us almost immediately to read of two important theorems.

1.6.1 Excision

Theorem 31 (excision)**.** Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{tikzcd}
A \setminus W \arrow[r, hook, "i"] \arrow[d, hook] & A \arrow[d, hook] \\
X \setminus W \arrow[r, hook, "i"] & X
\end{tikzcd}$$

induce an isomorphism

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A).$$

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A.

1.6.2 The Mayer-Vietoris sequence

Theorem 32 (Mayer-Vietoris)**.** Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X, i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, out=-5, in=175, "\delta" description] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n(X) \arrow[dll, out=-5, in=175, "\delta" description] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. We can draw our inclusions as the following pushout.

$$\begin{tikzcd}
 & X_1 \arrow[dr, hook, "\kappa_1"] & \\
X_1 \cap X_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\
 & X_2 \arrow[ur, hook, "\kappa_1"'] &
\end{tikzcd}$$

We have, almost by definition, the following short exact sequence.

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xhookrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{\kappa_1 - \kappa_2} S_\bullet^{\mathfrak{U}}(X) \longrightarrow 0$$

This gives the following long exact sequence on homology.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}^{\mathfrak{U}}(X) \arrow[dll, out=-5, in=175, "\delta" description] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n^{\mathfrak{U}}(X) \arrow[dll, out=-5, in=175, "\delta" description] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}$$

We have seen that $H_n^{\mathfrak{U}}(X) \cong H_n(X)$; the result follows. □

Example 33 (Homology groups of spheres)**.** We can decompose $\mathbb{S}^n$ as

$$\mathbb{S}^n = (\mathbb{S}^n \smallsetminus N) \cup (\mathbb{S}^n \smallsetminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$\begin{tikzcd}
 & \mathbb{S}^n \smallsetminus N \arrow[dr, hook] & \\
(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & \mathbb{S}^n \smallsetminus S \arrow[ur, hook] &
\end{tikzcd}$$

The Mayer-Vietoris sequence is as follows.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_j((S^n \smallsetminus N) \cap (S^n \smallsetminus S)) \arrow[r] & H_j(S^n \smallsetminus N) \oplus H_j(S^n \smallsetminus S) \arrow[r] & H_j^{\mathfrak{U}}(X) \arrow[dll, "\delta"] \\
H_{j-1}((\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S)) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that

$$\mathbb{S}^n \smallsetminus N \cong \mathbb{S}^n \smallsetminus S \cong \mathbb{D}^n \simeq \mathrm{pt}$$

and that

$$(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & 0 \arrow[r] & H_j(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X, $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\mathrm{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$$\begin{array}{c|cccc} j=3 & \mathbb{Z} & 0 & & \\ j=2 & \mathbb{Z} & 0 & & \\ j=1 & \mathbb{Z} & \mathbb{Z} & & \\ j=0 & \mathbb{Z}\oplus\mathbb{Z} & 0 & 0 & 0 \\ \hline H_i(\mathbb{S}^j) & i=0 & i=1 & i=2 & i=3 \end{array}$$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \qquad i > 1$$

allows us to fill in the above table as follows.

$$\begin{array}{c|cccc} j=3 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\ j=2 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\ j=1 & \mathbb{Z} & \mathbb{Z} & 0 & 0 \\ j=0 & \mathbb{Z}\oplus\mathbb{Z} & 0 & 0 & 0 \\ \hline H_i(\mathbb{S}^j) & i=0 & i=1 & i=2 & i=3 \end{array} \tag{1.1}$$

Example 34. Above, we used the Hurewicz homomorphism to see that

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed $\mathbb{S}^n$ as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$\begin{tikzcd} & X^+ \arrow[dr, hook] & \\ X^+ \cap X^- \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\ & X^- \arrow[ur, hook] & \end{tikzcd}$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^n) \arrow[dll, "\delta"', rounded corners, to path={}] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0,i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r] & \cdots \end{tikzcd}$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^-, with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+, and $i_1(p)$ is a point of X^-. Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[dlll, "\delta"'] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0,i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r, "H_0(\kappa_1)-H_0(\kappa_2)"] & H_0(\mathbb{S}^1) \end{tikzcd}$$

We can immediately replace things we know, finding the following.

$$\begin{tikzcd} & & (a,b) \arrow[r, mapsto] & (a+b, a+b) & \\ 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[r, hook] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\ & & & (c,d) \arrow[r, mapsto] & c - d \end{tikzcd}$$

The kernel of f is the free group generated by (a, a). Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

1.6.3 The relative Mayer-Vietoris sequence

Theorem 35 (relative Mayer-Vietoris sequence)**.** Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X, A \cup B) \arrow[dll, "\delta"] \\
H_n(X, A \cap B) \arrow[r] & H_n(X, A) \oplus H_n(X, B) \arrow[r] & H_n(X, A \cup B) \arrow[dll, "\delta"] \\
H_{n-1}(X, A \cap B) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Consider the following commuting diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem **??**) implies that the last row is also exact.

Consider the following map of short exact sequences; the first row is the last column of the above grid.

$$\begin{tikzcd}
0 \arrow[r] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r, hook] \arrow[d, "\phi"'] & S_n(X) \arrow[r, two heads] \arrow[d, equal] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & S_n(A \cup B) \arrow[r] & S_n(X) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0
\end{tikzcd}$$

This gives us, by Lemma **??**, a morphism of long exact sequences on homology.

$$\begin{tikzcd}
H_n(S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\phi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_\bullet(X)/S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\psi)"] & H_{n-1}(S_\bullet \mathfrak{U}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\phi)"] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

We have seen (in Fact 30) that $H_i(\phi)$ is an isomorphism for all i. Thus, the five lemma (Theorem **??**) tells us that $H_n(\psi)$ is an isomorphism. Thus, the bottom row of our grid can be written

$$0 \longrightarrow S_n(X, A \cap B) \hookrightarrow S_n(X, A) \oplus S_n(X, B) \twoheadrightarrow S_n(X, A \cup B) \longrightarrow 0 \ ,$$

and taking the long exact sequence on homology gives the result. ◻

1.7 Reduced homology

We have seen that reduced homology $\tilde{H}_n(X)$ agrees with $H_n(X)$ in positive degrees, and is missing a copy of $\mathbb{Z}$ in the zeroth degree. There are three equivalent ways of understanding this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Picking any point $x \in X$, one can define the reduced homology of X by
$$\tilde{H}_n(X) = H_n(X, x).$$

2. **In between:** One can define
$$\tilde{H}_n(X) = \ker(H_n(X) \to H_n(\mathrm{pt})).$$

3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of $\mathbb{Z}$ in degree -1, so that
$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$
Then one can define
$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the one we have taken so far. In constructing the singular chain complex of our space X, we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1)-simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through $\mathcal{F}$ thus gives a copy of $\mathbb{Z}$ as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- (1 ⇔ 2): Applying the functor $H_n \circ S$ to the diagram

$$\begin{tikzcd} \{x\} \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_{\{x\}}"'] & X \arrow[r, two heads, "\epsilon"] & \{x\} \end{tikzcd}$$

  one finds that $H_n(i)$ is an injection. Therefore, the connecting homomorphisms for the long exact sequence on the pair (X, x)

$$\cdots \longrightarrow H_{n+1}(X, x) \xrightarrow{\delta} H_n(\{x\}) \overset{i}{\hookrightarrow} H_n(X) \longrightarrow \cdots$$

  must be zero, so the sequence

$$0 \longrightarrow H_n(\{x\}) \overset{i}{\hookrightarrow} H_n(X) \twoheadrightarrow H_n(X, x) \longrightarrow 0$$

  is exact. However, we can say more: thanks again to to

Proposition 36. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

although this isomorphism is not canonical.

Proof. Trivial from algebraic definition. □

Proposition 37. Let $A \subset X$ be a closed subspace, and suppose that A is a deformation retract of an open neighborhood $A \subset U$. Then

$$H_n(X, A) \cong \tilde{H}_n(X/A).$$

Proof. Let $\pi\colon X \to X/A$ be the canonoical projection, and $b = \pi(A)$. □

1.7.1 Deja vu all over again

Many of our results for regular homology hold also for reduced homology.

Proposition 38. There is a long exact sequence for a pair of spaces

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{j+1}(X, A) \arrow[dll, "\delta"] \\ \tilde{H}_j(A) \arrow[r] & \tilde{H}_j(X) \arrow[r] & \tilde{H}_j(X, A) \arrow[dll, "\delta"] \\ \tilde{H}_{j-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

We're now ready to prove quite a nice theorem.

Proposition 39 (Brouwer)**.** For any $n \geq 1$, any continuous bijective map $f\colon \mathbb{D}^n \longrightarrow \mathbb{D}^n$ has at least one fixed point.

Proof. Suppose we have such a map. Then x and $f(x)$ are different for all x, and we can draw a ray starting at $f(x)$ and passing through $f(x)$. This ray intersects $\partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$ at exactly one place $r(x)$, giving us a map

$$r\colon \mathbb{D}^n \to \mathbb{S}^{n-1}; \qquad x \mapsto r(x).$$

This is clearly continuous, and it fixes $\partial\mathbb{D}^n \subset \mathbb{D}^n$. Thus, we have the following commutative diagram.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook, "\text{incl.}"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{S}^{n-1}}"'] & \mathbb{D}^n \arrow[r, two heads, "r"] & \mathbb{S}^{n-1} \end{tikzcd}$$

Applying $\tilde{H}_n$ gives the following commutative diagram.

$$\begin{tikzcd} \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[r, hook, "\tilde{H}_{n-1}(\text{incl.})"] \arrow[rr, bend right, "\mathrm{id}_{\tilde{H}_n(\mathbb{S}^{n-1})}"'] & \tilde{H}_{n-1}(\mathbb{D}^n) \arrow[r, two heads, "\tilde{H}_{n-1}(r)"] & \tilde{H}_{n-1}\mathbb{S}^{n-1}) \end{tikzcd}$$

But $\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ and $\tilde{H}_{n-1}(\mathbb{D}^n) \cong 0$ for all $n \geq 1$, so $\tilde{H}_{n-1}(r)$ cannot be surjective. $\square$

Proposition 40. We have a reduced Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{n+1}(X) \arrow[dll, "\delta"] \\ \tilde{H}_n(X_1 \cap X_2) \arrow[r] & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \arrow[r] & \tilde{H}_n(X) \arrow[dll, "\delta"] \\ \tilde{H}_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots & \end{tikzcd}$$

Proposition 41. Let $\{(X_i, x_i)\}_{i\in I}$, be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$[2] we have

$$\tilde{H}_n\left(\bigvee_{i\in E} X_i\right) \cong \bigoplus_{i\in E} \tilde{H}_n(X_i).$$

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of Proposition 40 tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 42 (good pair)**.** A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X.
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U.

$$A \hookrightarrow U \overset{r}{\twoheadrightarrow} A$$

Proposition 43. Let (X, A) be a good pair. Let $\pi : X \to X/A$ be the canonical projection. Then

$$H(X, A) \cong \tilde{H}_n(X/A) \qquad \text{for all } n \geq 0.$$

Proof. First, note that since $X \setminus A \cong (X/A) \setminus \{b\}$ and $U \setminus A \cong (U/A) \setminus \{b\}$, we have an isomorphism of pairs

$$(X \setminus A, U \setminus A) \cong ((X/A) \setminus \{b\}, (U/A) \setminus \{b\}). \tag{1.2}$$

[2] This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Thus, we have the following chain of isomorphisms.

$$\begin{aligned}
H_n(X, A) &\cong H_n(X, U) && \binom{A \text{ deformation}}{\text{retract of } U} \\
&\cong H_n(X \setminus A, U \setminus A) && \text{(excision)} \\
&\cong H_n((X/A) \setminus \{b\}, (U/A) \setminus \{b\}) && \text{(Equation 1.2)} \\
&\cong H_n(X/A, U/A) && \text{(excision)} \\
&\cong H_n(X/A, \{b\}) && \binom{\{b\} \text{ deformation}}{\text{retract of } U/A} \\
&\cong \tilde{H}_n(X/A)
\end{aligned}$$

□

Theorem 44 (suspension isomorphism)**.** Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

1.8 Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d\, \mu_n, \qquad \text{for some } d \in \mathbb{Z}.$$

Definition 45 (mapping degree)**.** We call $d \in \mathbb{Z}$ as above the mapping degree of f, and denote it by $\deg(f)$.

Example 46. Consider the map

$$\omega\colon [0, 1] \longrightarrow \mathbb{S}^1; \qquad t \longmapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism (Theorem 10), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1; \qquad x \longmapsto x^n.$$

We have

$$\begin{aligned}
H_1(f_n)(\omega) &= [f_n \circ \omega] \\
&= [e^{2\pi i n t}].
\end{aligned}$$

The naturality of the Hurewicz isomorphism (Theorem 10) tells us that the following diagram commutes.

$$\begin{tikzcd}
\pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[r, "\pi_1(f_n)_{\mathrm{ab}}"] \arrow[d, "h_{\mathbb{S}^1}"'] & \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[d, "h_{\mathbb{S}^1}"] \\
H_1(\mathbb{S}^1) \arrow[r, "H_1(f_n)"'] & H_1(\mathbb{S}^1)
\end{tikzcd}$$

1.9 CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 47 (cell)**.** Let X be a topological space. We say that X is an <u>n-cell</u> if X is homeomorphic to $\mathbb{R}^n$. We call the number n the <u>dimension</u> of X.

Definition 48 (cell decomposition)**.** A <u>cell decomposition</u> of a topological space X is a decomposition

$$X = \coprod_{i\in I} X_i, \qquad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topologial spaces.

Definition 49 (CW complex)**.** A Hausdorff topological space X, together with a cell decomposition, is known as a <u>CW complex</u>[3] if it satisfies the following conditions.

(CW1) For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma\colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n-1$.

(CW2) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X.

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ.
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map.*

Example 50. Consider the unit interval $I = [0, 1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n + 1$ 0-cells and n 1-cells. which looks like n intervals glued together at their endpoints.

[3] Axiom (CW2) is called the *closure-finiteness* condition. This is the ‘C’ in CW complex. Axiom (CW3) says that X carries the *weak topology* and is responsible for the ‘W’.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma_k^0 = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \qquad \text{and} \qquad \sigma_\infty^0 = 0$$

and one-cells

$$\sigma_k^1 = \left(\frac{1}{k}, \frac{1}{k+1}\right), \qquad k \in \mathbb{N}^{\geq 1}.$$

At first glance, this looks like a CW decomposition; it is certainly satisfies Axiom (CW1) and Axiom (CW2). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right)$$

is the midpoint of the interval σ_k^1. We have $A \cap \sigma_k^0 = \emptyset$ for all k, and $A \cap \sigma_k^1 = \{a_k\}$ for all k. In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i. However, the set A is not closed in I, since it does not contain its limit point $\lim_{n\to\infty} a_n = 0$.

Definition 51 (skeleton, dimension)**.** Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n-skeleton of X. If X is equal to its n-skeleton but not equal to its $(n-1)$-skeleton, we say that X is n-dimensional.

Note 52. Axiom (CW3) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 53 (subcomplex, CW pair)**.** Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$

We call such a pair (X, Y) a CW pair.

Fact 54. Let X and Y be CW complexes such that X is locally compact.[4] Then $X \times Y$ is a CW complex.

Lemma 55. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 56. Let X be a CW complex.

[4] I.e. if for every point x there is an open neighborhood U containing x and a compact set K containing U.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact if and only if it is locally finite.[5]

Proof. It is clear that 1. $\Rightarrow$ 2., since X is a subset of itself. Similarly, it is clear that 2. $\Rightarrow$ 3., since □

Corollary 57. If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n.

$$\begin{tikzcd}
 & & & & K \arrow[d, "f"] \arrow[dlll, dashed, "\exists \tilde{f}"'] \\
X^{n-1} \arrow[r, hook] & X^n \arrow[r, hook] & X^{n+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 58. Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.

Lemma 59. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborbood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.
- Every CW complex is paracompact, locally path-connected, and locally contractible.
- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 60. Let X be a CW complex. We have the following decompositions.

1. $$X^n \smallsetminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$
2. $$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

1. Since $X^n \smallsetminus X^{n-1}$ is simply the union of all n-cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n-cell is homeomorphic to the open n-ball.

[5] I.e. if every point has a neighborhood which is contained in only finitely many cells.

2. For every n-cell σ, the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$-skeleton.

$\square$

1.10 Cellular homology

Lemma 61. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 59, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 43, and the second from Lemma 60. $\square$

Lemma 62. Consider the inclusion $i_n\colon X^n \hookrightarrow X$.

- The induced map
$$H_n(i_n)\colon H_n(X^n) \to H_n(X)$$
is surjective.
- On the $(n+1)$-skeleton we get an isomorphism
$$H_n(i_{n+1})\colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n). The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by Lemma 61,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n)\colon X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{tikzcd} & & H_{n+1}(X^{m+1}, X^m) \arrow[dll, "\delta"'] \\ H_n(X^m) \arrow[r] & H_n(X^{m+1}) \arrow[r] & H_n(X^{m+1}, X^m) \end{tikzcd}$$

But again by Lemma 61, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \to H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r] \arrow[drrr] & X^{n+1} \arrow[r] \arrow[drr] & X^{n+2} \arrow[r] \arrow[dr] & \cdots \\ & & & & X \end{tikzcd}$$

Taking nth singular homology, we find the following.

$$\cdots \longrightarrow H_n(X^n) \overset{\alpha_1}{\twoheadrightarrow} H_n(X^{n+1}) \overset{\alpha_2}{=\!=} H_n(X^{n+2}) \overset{\alpha_3}{=\!=} \cdots \longrightarrow H_n(X)$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \qquad \sigma_i \colon \Delta^n \to X.$$

Since the standard n-simplex Δ^n is compact, Corollary 57 implies that each σ_i factors through some X^{n_i}. Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \qquad \tilde{\sigma}_i \colon \Delta^n \to X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma \right]$$

□

Corollary 63. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q-cells, then $H_q(X) \cong 0$.
3. In particular, for an n-dimensional CW-complex X (Definition 51), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 62.

2.

□

Definition 64 (cellular chain complex)**.** Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}).

Theorem 65 (comparison of cellular and singular homology)**.** Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n \colon H_n(C_\bullet(X), d) \cong H_n(X).$$

Example 66 (complex projective space)**.** Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \mathrm{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots | x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{tikzcd}
2n & 2n-1 & 2n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i,\ 0 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Example 67 (real projective space)**.** As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{tikzcd}
n & n-1 & n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

1.10.1 Euler characteristic

Here's one last application of CW stuff: euler characteristic.

Definition 68 (Euler characteristic)**.** Let X be a finite CW complex. The Euler characteristic of X is defined to be

$$\chi(X) = \sum_{n \geq 0} (-1)^n \ \mathrm{rk}(H_n(X;\mathbb{Z}))$$

where for an abelian group A, $\mathrm{rk}(A)$ denotes the number of free summands.

Note that the above sum must be finite since there is some simplex of highest degree, say k, so $H_{k'}(X;\mathbb{Z}) \cong 0$ for all $k' > k$.

Proposition 69. For X a finite CW complex, denote by $c_n(X)$ the number of n-cells. Then

$$\chi(X) = \sum_{n \geq 0} (-1)^n \, c_n(X).$$

Proof. We will use the following fact. For any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0\ ,$$

we have the equality

$$\mathrm{rk}(B) = \mathrm{rk}(A) + \mathrm{rk}(C).$$

Consider the short exact sequences

$$0 \longrightarrow Z_n \hookrightarrow C_n \twoheadrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \hookrightarrow Z_n \twoheadrightarrow H_n \longrightarrow 0\ .$$

Applying our rank formula to both and summing leads to the equation

$$\mathrm{rk}(C_n) = \mathrm{rk}(B_{n-1}) + \mathrm{rk}(B_n) + \mathrm{rk}(H_n).$$

Substituting $C_n(X)$ by the above and summing results in the B_i terms cancelling each other in the alternating sum, giving us the result we want. □

Proposition 70. For finite CW complexes X and Y, we have

$$\chi(X \times Y) \cong \chi(X) \times \chi(Y).$$

Proof. First, note that for $\mathbb{S}^n$ □

1.11 Homology with coefficients

Definition 71 (homology with coefficients)**.** Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X; G) = S(X) \otimes_{\mathbb{Z}} G.$$

The nth singular homology of X with coefficients in G is the nth homology

$$H_n(X; G) = H_n(S(X; G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G.

Theorem 72 (topological universal coefficient thoerem for homology)**.** For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X; G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0\ .$$

Furthermore, this sequence splits non-canonicaly, telling us that

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. Theorem 72. □

1.12 The topological Künneth formula

Let X and Y be topological spaces. Plugging $C = S(X)$ and $D = S(Y)$ into Theorem **??** tells us that the following sequence is split exact.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \hookrightarrow H_n(S(X)_\bullet \otimes S(Y)_\bullet) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X), H_q(Y)) \longrightarrow 0 \ .$$

It turns out that we can relate $H_n(S(X)_\bullet \otimes S(Y)_\bullet)$ and $H_n(X \times Y)$. In fact, they turn out to be isomorphic; this is known as the *Eilenberg-Zilber theorem.*

1.12.1 The Eilenberg-Zilber theorem: the high-brow approach

This turns out to be a direct consequence of the acyclic model theorem (Theorem 17).

Lemma 73. The functors

$$F = S(-) \otimes S(-) \colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

and

$$G = S(- \times -) \colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

are both free and acyclic (Definition 15) on

$$\mathcal{M} = \{(\Delta^p, \Delta^q)\}_{p,q=0,1,\dots}.$$

Proof. Clear, basically. □

Proposition 74. There is a chain homotopy equivalence between F and G as defined in Lemma 73.

Proof. Consider the following diagram.

$$\begin{tikzcd} S_0(X) \otimes S_0(Y) \arrow[rr, bend left, "f_0 \colon x\otimes y \mapsto (x,y)"] & & S_0(X \times Y) \arrow[ll, bend left, "g_0 \colon (x,y) \mapsto x \otimes y"] \end{tikzcd}$$

We can find lifts… □

1.12.2 The Eilenberg-Zilber theorem: the low-brow approach

We first define a so-called *homology cross product*

$$\times\colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y),$$

and then show that it descends to homology.

Our goal is to find a way of making a p-simplex in X and a q-simplex in Y into a $(p+q)$-simplex in $X \times Y$, called the *homology cross product.* At least in the case that one of p or q is equal to zero, it is clear what to do, since the Cartesian product of a p-simplex with a zero-simplex is in an obvious way a p-simplex. In more general cases, however, we have to be clever. The strategy is to write down a list of properties that we would like our homology cross product to have, and then show that there exists a unique map satisfying them.

Lemma 75. We can define a homomorphism

$$\times\colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y), \qquad p, q \geq 0,$$

with the following properites.

1. For all points $x_0 \in X$ viewed as zero-chains in $S_0(X)$ and all $\beta\colon \Delta^q \to Y$, we have

   $$(x_0 \times \beta)(t_0, \ldots, t_q) = (x_0, \beta(t_0, \ldots, t_q));$$

   conversely, for $\alpha \in S_p(X)$ and $y_0 \in Y$, we have

   $$(\alpha \times y_0)(t_0, \ldots, t_p) = (\alpha(t_0, \ldots, t_q), y_0).$$

2. The map $\times$ is natural in the sense that the square

   $$\begin{array}{ccc} S_p(X) \otimes S_q(Y) & \xrightarrow{\times} & S_{p+q}(X \times Y) \\ \downarrow & & \downarrow \\ S_p(X') \otimes S_q(Y') & \xrightarrow{\times} & S_{p+q}(X' \times Y') \end{array}$$

   commutes.

3. The map $\times$ satisfies the Leibniz rule in the sense that

   $$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

Proof. As a warm up, let us work out some hypothetical consequences of our formulae, in the special case that $X = \Delta^p$ and $Y = \Delta^q$. In this case, we have

$$\mathrm{id}_{\Delta^p} \in S_p(\Delta^p), \qquad \mathrm{id}_{\Delta^q} \in S_q(\Delta^q),$$

so we can take the homology cross product $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} \in S_{p+q}(\Delta^p \times \Delta^q)$. By the Leibniz rule, we have

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times \partial(\mathrm{id}_{\Delta^q}).$$

This is now a perfectly well-defined element of $S_{p+q-1}(\Delta^p \times \Delta^q)$, which we will give the nickname R.[6]

A trivial computation shows that $\partial R = 0$, so there exists a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$. We choose some such c and define $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} = c$.

We now use the trick of expressing some $\alpha\colon \Delta^p \to X$ as $S_p(\alpha)(\mathrm{id}_{\Delta^p})$. Then

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}).$$

But then the naturality forces our hand; chasing $\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}$ around the naturality square

$$\begin{tikzcd}
S_p(\Delta^p) \otimes S_q(\Delta^q) \arrow[r, "\times"] \arrow[d, "S_p(\alpha)\otimes S_q(\beta)"'] & S_{p+q}(\Delta^p \times \Delta^q) \arrow[d, "S_{p+q}(\alpha,\beta)"] \\
S_p(X) \otimes S_p(Y) \arrow[r, "\times"] & S_{p+q}(X \times Y)
\end{tikzcd}$$

tells us that

$$\begin{aligned}
S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) &= S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) \\
S_{p+q}(\alpha, \beta)(c) &= \alpha \times \beta.
\end{aligned}$$

Thus, we can define

$$\alpha \times \beta = S_{p+q}(\alpha, \beta)(c).$$

This satisfies all the desired properties by construction. □

Proposition 76. Any natural transformations f, g with components

$$f_{X,Y},\ g_{X,Y}\colon (S(X) \otimes S(Y))_\bullet \to S_\bullet(X \times Y)$$

which agree in degree zero and send $x_0 \otimes y_0 \mapsto (x_0, y_0)$ are chain homotopic.

Proof. First, suppose that $X = \Delta^p$ and $Y = \Delta^q$. Then $(S(\Delta^p) \otimes S(\Delta^q))_\bullet$ is free, hence certainly projective, and $S(\Delta^p \times \Delta^q)$ is acyclic, so the result follows from Lemma **??**: that

[6] I think there's an induction argument hidden here.

is, we get a chain homotopy H with components

$$H_n \colon (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_n \to S_{n+1}(\Delta^p \times \Delta^q)$$

such that

$$\partial H_n + H_{n-1}\partial = f_n - g_n.$$

□

1.12.3 The topological Künneth formula

Theorem 77 (topological Künneth formula)**.** For any topological spaces X and Y, we have the following short exact sequence, natural in X and Y.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \longrightarrow 0$$

This sequence splits, but not canonically, and the splitting is not natural.

Example 78. Consider the torus $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$. We have that the following sequence is exact.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(\mathbb{S}^1) \otimes H_q(\mathbb{S}^1) \longrightarrow H_n(\mathbb{S}^1 \times \mathbb{S}^1) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(\mathbb{S}^1), H_q(\mathbb{S}^1)) \longrightarrow 0$$

Because $H_i(\mathbb{S})$ is either 0 or $\mathbb{Z}$, hence certainly projective, we get that

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1) \cong \bigoplus_{p+q=n} H_p(\mathbb{S}) \otimes H_q(\mathbb{S}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

By induction, $H_n(\mathbb{S}^k) = \mathbb{Z}^{\binom{n}{k}}$.

Example 79. For a space of the form $X \times \mathbb{S}^n$, we get

$$H_n(X \times \mathbb{S}^n) \cong H_q(X) \oplus H_{q-n}(X).$$

1.13 The Eilenberg-Steenrod axioms

Denote by T the functor

$$T \colon \mathbf{Pair} \to \mathbf{Pair}; \qquad (X, A) \mapsto (A, \emptyset); \qquad (f \colon (X, A) \to (Y, B)) \mapsto f|_A.$$

Definition 80 (homology theory)**.** Let A be an abelian group. A homology theory with coefficients in A is a sequence of functors

$$H_n \colon \mathbf{Pair} \longrightarrow \mathbf{Ab}, \qquad n \geq 0$$

together with natural transformations

$$\delta \colon H_n \Rightarrow H_{n-1} \circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,
$$f \sim g \implies H_n(f) = H_n(g) \qquad \text{for all } f, g, n.$$
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i \colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$H_n(i) \colon H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$$
for all n.
3. **Dimension:** We have
$$H_n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$
4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$
5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.