1 Cohomology

1.1 Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition **??**) which govern cohomology theories.

Definition 1 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n : \mathbf{Pair}^{\mathrm{op}} \to \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial : H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i : (X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$H^n(i) : H^n(X, A) \cong H^n(X \smallsetminus U, A \smallsetminus U)$$
for all n.
3. **Dimension:** We have
$$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$
4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H^n(X) = \prod_\alpha H^n(X_\alpha).$$
5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

1.2 Singular cohomology

Definition 2 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet(X), G).$$

We also define relative homology:

$$S^\bullet(X, A; G) = \mathrm{Hom}(S_\bullet(X, A), G).$$

We will denote the evaluation map $\mathrm{Hom}(A, G) \otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 3. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G) \otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet) \otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha \colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned} \langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\ &= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\ &= \langle \alpha, a \rangle. \end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned} \langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\ &= \langle \alpha, a \rangle + \langle \beta, da \rangle \\ &= \langle \alpha, a \rangle \end{aligned}$$

□

Via $\otimes$-hom adjunction, we get a map

$$\kappa \colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 4 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space, and G an abelian group. There is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X; G) \twoheadrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

Proof. We now □

Example 5. We have seen (in Example ??) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n,\ k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \leq k \leq n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(0, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \leq k \leq n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

1.3 The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \to \mathbb{Z},$$

allowing an n-cochain to eat an n-chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q-simplex, on which our cochain can act, and a singular $n - q$-simplex, which is along for the ride.

Definition 6 (front, rear face). Let $a\colon \Delta^n \to X$ be a singular n-simplex, and let $0 \leq q \leq n$.

- The <u>$(n-q)$-dimensional front face</u> of a is given by

  $$F^{n-q}(a) = \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X,$$

  where i is induced by the inclusion

  $$\{0, \ldots, n-q\} \hookrightarrow \{1, \ldots, n\}; \qquad k \mapsto k.$$

- The <u>q-dimensional rear face</u> of a is given by

  $$R^q(a) = \quad \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X \ ,$$

  where r is the map

  $$\{0, \dots q\} \to \{0, \dots, n\}; \qquad k \mapsto n - q + k.$$

Note that for $a \in S_n(X)$, we can write

$$F^{n-q}(a) = (\partial_{n-q+1} \circ \partial_{n-q+2} \circ \cdots \circ \partial_n)(a)$$

and

$$R^q(a) = (\overbrace{\partial_0 \circ \partial_0 \circ \cdots \circ \partial_0}^{q \text{ times}})(a).$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \to S_{n-q}(X); \qquad \alpha \otimes a \mapsto F^{n-q}(a)\langle a, R^q(a)\rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 7 (cap product)**.** The <u>cap product</u> is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \to S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b)\rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b)\rangle = 0$ by definition of relative cohomology.

Lemma 8. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial\alpha \otimes r).$$

Proof. We do three calculations.

$$\begin{aligned}
\partial(\alpha \frown a) &= \partial(F^{n-q}(\alpha) \otimes \langle \alpha, R^q(a)\rangle) \\
&= \partial(F^{n-q}(a)) \otimes \langle \alpha, R^q(a)\rangle \\
&= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q+1} \circ \cdots \circ \partial_n)(a) \otimes \langle \alpha, R^q(a)\rangle
\end{aligned}$$

$$\begin{aligned}
(\delta\alpha) \frown a &= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a)\rangle \\
&= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a)\rangle \\
&= F^{n-q}(a) \otimes \langle \alpha, \partial R^q(a)\rangle \\
&= \sum_{i=0}^{q} (-1)^i F^{n-q}(a) \otimes \langle \alpha, \partial_i \circ \partial_0^q a\rangle
\end{aligned}$$

$$\alpha \frown (\partial a) =$$

□

Proposition 9. The cap product descends to a map on homology

$$\frown : H^q(X, A; R) \otimes H_n(X, A; R) \to H^{n-q}(X, A; R)$$

via the formula

$$[\alpha] \cap [a] := [\alpha \cap a].$$

Proof. Let us first collect some general results. Consider the formula

$$\partial(\alpha \frown a) = (\delta\alpha) \frown a + (-1)^q \alpha \frown \partial a.$$

1. Let α be a cocycle and a be a cycle. Then each term on the right-hand side is zero, so
$$\text{cocycle} \frown \text{cycle} = \text{cycle}.$$

2. Let α be arbitrary, so $\delta\alpha$ is a coboundary, and let a be a cycle. Then the above formula, read backwards, tells us that
$$\text{coboundary} \frown \text{cycle} = \text{boundary}.$$

3. Let a be arbitrary, so ∂a is a cycle, and let α be a coboundary. Then we get
$$\text{cocycle} \frown \text{boundary} = \text{boundary}.$$

Point 1. tells us that the formula is well-defined to begin with: $\alpha \cap a$ is really a cycle. Now suppose we had picked different representatives $\alpha + \delta\beta$ and $a + \partial b$. Then

$$\begin{aligned}
[\alpha + \delta\beta] \frown [a + \partial b] &= [(\alpha + \delta\beta) \frown (a + \partial b)] \\
&= [\alpha \frown a] + [\delta\beta \frown a] + [\alpha \frown \partial b] + [\delta\beta \frown \partial b] \\
&= [\alpha \frown a] \\
&= [\alpha] \frown [a],
\end{aligned}$$

where the the second term is zero because of 2. and the third term is zero because of 3. ∎

Proposition 10. The cap product is natural in the sense that for any $f : X \to X$ inducing a map of pairs $(X, A) \to (X, B)$, the diagram

$$\begin{tikzcd}
H^q(X, A; R) \otimes H^n(X, A; R) \arrow[rrr, "\frown"] \arrow[ddd] & & & H_{n-q}(X; R) \arrow[ddd] \\
& (f^*\beta, a) \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \otimes \langle f^*\beta, R(a) \rangle \arrow[d, mapsto] & \\
& (\beta, f_* a) \arrow[r, mapsto] & \star & \\
H^q(X, B; R) \otimes H^n(X, B; R) \arrow[rrr, "\frown"] & & & H_{n-q}(X; R)
\end{tikzcd}$$

commutes, i.e.

$$f_*(F(a) \otimes \langle f^*\beta, R(a) \rangle) = F(f_* a) \otimes \langle \beta, R(f_* a) \rangle.$$

Proof.

$$\begin{aligned}
f_*(F(a) \otimes \langle f^*\beta, R(a) \rangle) &= f_*(F(a) \otimes \langle f^*\beta, R(\alpha) \rangle) \\
&= f_*(F(a) \otimes \langle \beta, f_*(R(a)) \rangle) \\
&= f_*(F(a) \otimes \langle \beta, R(f_* a) \rangle) \\
&= F(f_* a) \otimes \langle \beta, R(f_* a) \rangle.
\end{aligned}$$

∎

We can write this in a prettier way as

$$f_*(f^*\beta \frown a) = \beta \frown f_*(a).$$

Example 11. Suppose X is simply connected, and let

1.4 The cup product

This section takes place over a commutative ring with unit R. We will notationally suppress R.

Recall that we have constructed an Eilenberg-Zilber map

$$S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y).$$

in Subsection **??**. There exist generalizations, which we will not construct explicitly.

Definition 12 (Eilenberg-Zilber map)**.** An Eilenberg-Zilber map is a natural chain map

$$S_\bullet(X \times Y, X \times B \cup A \times Y) \to S_\bullet(X, A) \otimes S_\bullet(Y, B).$$

Definition 13 (cohomology cross product)**.** Let EZ be any Eilenberg-Zilber map. For any $\alpha \in S^p(X, A)$ and $\beta \in S^q(Y, B)$, a cohomology cross product of α and β is the $(p+q)$-cochain $\alpha \times \beta$ defined by making the below diagram commute.

$$\begin{tikzcd}
S_n(X \times Y; X \times B \cup A \times Y) \arrow[d, "\mathrm{EZ}"'] \arrow[ddrr, "\alpha\times\beta"] & & \\
\bigoplus_{p+q=n} S_p(X) \otimes S_q(Y) \arrow[d, "\mathrm{proj}"'] & & \\
S_p(X, A) \otimes S_q(Y, B) \arrow[r, "\alpha\otimes\beta"'] & R \otimes R \arrow[r, "\mu"'] & R
\end{tikzcd}$$

Fact 14. Any homology cross product extends to a natural chain map

$$S^\bullet(X, A) \otimes S^\bullet(Y, B) \to S^\bullet(X \times Y, X \times B \cup A \times Y).$$

Note that since the Eilenberg-Zilber map is natural by assumption, projection is natural for any product, and evaluation is natural by definition of the tensor product, the homology cross product is natural in that the diagram

$$\begin{tikzcd}
S^p(X) \otimes S^q(Y) \arrow[r] \arrow[d] & S^{p+q}(X \times Y) \arrow[d] \\
S^p(X') \otimes S^q(Y') \arrow[r] & S^{p+q}(X' \times Y')
\end{tikzcd}$$

commutes; that is

$$(f, g)^*(\alpha \times \beta) = (f^*\alpha) \times (g^*\beta).$$

Fact 15. The homology cross product has the following properties.

1. The Leibniz rule holds, i.e.

$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|} \times (\delta\beta).$$

2. For the Kroenecker pairing we have for homology classes a, b, and cohomology classes α, β of corresponding degree

$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle.$$

3. For $1 \in R$ and thus $1 \in S^0(X, A)$, we have

$$1 \times \beta = p_2^*(\beta), \qquad \alpha \times 1 = p_1^*(\alpha),$$

where p_i denotes the projection onto the ith factor of $X \times Y$.

4. The cohomology cross product is associative.
5. The cohomology cross product satisfies a graded version of commutativity: the twist map $\tau\colon X \times Y \to Y \times X$ yields on homology

$$\alpha \times \beta = (-1)^{|\alpha||\beta|} \tau^*(\beta \times \alpha).$$

Definition 16 (cup product)**.** Denote by $\Delta\colon X \to X \times X$ the diagonal map. The cup product is the composition

$$H^p(X, A) \otimes H^q(X, B) \to H^{p+q}(X \times X, X \times B \cup A \times X) \to H^{p+q}(X, A \cup B),$$

where the first map is the cross product, and the second map is the pullback along the diagonal. That is,

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

Proposition 17. The cup product satisifes the following identities.

1. $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$
2. $\alpha \cup \beta = (-1)^{|\alpha||\beta|}$.
3. For $\partial\colon H^*(A) \to H^{*+1}(X, A)$, $\alpha \in H^*(A)$, $\beta \in H^*(X)$, we have

$$\partial(\alpha \cup i^*\beta) = (\partial\alpha) \cup \beta \in H^*(X, A).$$

4. For $f\colon X \to Y$, we have

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

5. We can express the cohomology cross product via the cup product using the for-

mula

$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta).$$

Proof.

1. Consequence of associativity of the cross product.
2.
3.
4.
5.

□

Definition 18 (diagonal approximation)**.** A diagonal approximation is a natural chain map

$$D\colon S_\bullet(X) \to S_\bullet(X) \otimes S_\bullet(X)$$

with $D(x) = x \otimes x$ for $x \in S_0(X)$.

Theorem **??** tells us immediately that any two diagonal approximations are naturally homotopic.

Example 19 (Alexander-Whitney map)**.** Consider the map

$$\mathrm{AW}\colon S_n(X) \to (S_\bullet(X) \times S_\bullet(X))_n, \qquad a \mapsto \sum_{p+q=n} F^p(A) \otimes R^q(a).$$

This is a chain map because

1.5 Manifold orientations

1.5.1 Basic definitions and results

Let M be an n-dimensional topological manifold. Without loss of generality, we may choose an atlas for M such that each each point $x \in M$ has a chart (U_x, ϕ_x) such that $U_x \cong \mathbb{R}^n$ and $\phi_x(x) = 0 \in \mathbb{R}^n$.

Let $x \in M$. Then

$$\begin{aligned} H_n(M, M \setminus \{x\}) &\cong H_n(M \setminus (M \setminus U_x), M \setminus \{x\} \setminus (M \setminus U_x)) && \text{(excision)} \\ &\cong H_n(U_x, U_x \setminus \{x\}) \\ &\cong H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) && \text{(homotopy)} \\ &\cong H_{n-1}(\mathbb{S}^{n-1}) && \text{(Example ??)} \\ &\cong \mathbb{Z}. \end{aligned}$$

We call a choice of generator of $H_n(X, X \smallsetminus \{x\})$ a *local orientation* on M at x. For any other point $y \in U_x$, a local orientation at x gives a local orientation at y via the composition

$$H_n(X, X \smallsetminus \{x\}) \cong H_n(X, X \smallsetminus U_x) \cong H_n(X, X \smallsetminus \{y\}).$$

We will use the following notation: for a pair X, A, we will write

$$H_n(X, A \smallsetminus X) = H_n(X \mid A).$$

Furthermore, if $A = \{x\}$, we will write $H_n(X \mid x)$ instead of $H_n(X \mid \{x\})$.

We think of $H_n(X \mid A)$ as the *local homology* of A in X, since it depends (by excision) only on the closure of A in X.

Definition 20 (orientation cover)**.** Let M be an n-dimensional manifold. The orientation cover of M is, as a set,

$$\tilde{M} = \{\mu_x \mid x \in M,\ \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

There is a clear surjection from $\tilde{M}$ to M as follows.

$$\begin{tikzcd} \tilde{M} \arrow[d] & \mu_x \arrow[d, mapsto] \\ M & x \end{tikzcd}$$

The set $\tilde{M}$ turns out to be a topological manifold, and the surjection $\tilde{M} \to M$ turns out to be a double cover. We will show this later.

Definition 21 (orientation)**.** Let M be an n-dimensional manifold. An orientation on M is a set-section σ of the bundle $\tilde{M} \to M$ satisfying the following condition: for any $x \in M$ and any $y \in U_x$, the induced local orientation at y is equal to $\sigma(y)$.

For a triple $B \subset A \subset X$ denote the canonical map on pairs $(X, X \smallsetminus A) \to (X, X \smallsetminus B)$ by $\rho_{B,A}$.

Lemma 22. Let M be a connected, orientable topological manifold of dimension m, and let $Ks\{M\}$ be a compact subset.

1. For all $q > m$, $H_q(M \mid K) = 0$.
2. If $a \in H_m(M \mid k)$, then $a = 0$ if and only if $(\rho_{x,K})_* a = 0$ for all $x \in X$.

Sketch of proof. The proof uses manifold induction, which consists of the following steps.

1. Prove the result the special case that our manifold M is simply $\mathbb{R}^n$ and K is a convex compact subset.

2. Using Mayer-Vietoris, show that the result holds when K is a union of two compact, convex subsets.
3. The case of $M = \mathbb{R}^n$ K a union of n compact, convex subsets follows by induction.
4. For K an arbitrary compact subset of $\mathbb{R}^m$, we cover K by finitely many closed balls B_i, then show that every cycle on $H_q(\mathbb{R}^m \mid K)$ comes from a cycle on $H_q(\mathbb{R}^m \mid \bigcup_i B_i)$. Since $\bigcup_i B_i$ is a finite union of compact, convex subsets, there are no non-trivial cycles, implying the claim.
5. For an arbitrary compact subset contained in a single chart, the claim reduces to the $\mathbb{R}^m$ case above.
6. For an arbitrary compact subset contained in two charts, the result follows from the Mayer-Vietoris sequence. The general claim follows by induction.

□

Proof.

1. First take $M = \mathbb{R}^m$ and $K \subset M$ compact and convex, so that any $x \in K$ is a deformation retract of K. Then, since K is closed and bounded, we can find an open ball $\mathring{\mathbb{D}}^m$ with $K \subset \mathring{\mathbb{D}}^m$. Then certainly

$$\overline{M \setminus \mathring{\mathbb{D}}^m} = M \setminus \mathbb{D}^m \subset M \setminus K = (M \overset{\circ}{\setminus} K),$$

so by excision and deformation retractness we have

$$\begin{aligned} H_q(M, M \setminus K) &\cong H_q(M \setminus (M \setminus \mathring{\mathbb{D}}^m), (M \setminus K) \setminus (M \setminus \mathring{\mathbb{D}}^m)) \\ &= H_q(\mathbb{D}^m, \mathbb{D}^m \setminus K) \\ &\cong H_q(\mathbb{D}^m, \mathbb{D}^m \setminus \{x\}) \\ &\cong H_q(\mathbb{D}^m, \mathbb{S}^{m-1}) \\ &\cong H_{q-1}(\mathbb{S}^{m-1}). \end{aligned}$$

For $q > m$, this is 0, and for $q = m$ we note that similar reasoning provides an isomorphism

$$H_q(M, M \setminus K) \cong H_q(M, M \setminus \{x\}) \text{ for all } x \in X.$$

2. Now let $M = \mathbb{R}^n$ and $K = K_1 \cup K_2$, where both K_i are compact and convex. Note that $K_1 \cap K_2$ is also compact and convex. Consider the following relative Mayer-

Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & H_{q+1}(M \mid K_1 \cap K_2) \arrow[dll, "\delta"] \\ H_q(M \mid K_1 \cup K_2) \arrow[r] & H_q(M \mid K_1) \oplus H_q(M \mid K_2) \arrow[r] & H_q(M \mid K_1 \cap K_2) \arrow[dll, "\delta"] \\ H_{q-1}(M \mid K_1 \cup K_2) \arrow[r] & \cdots & \end{tikzcd}$$

For $q > n$, Part 1. implies that

$$H_{q+1}(M \mid K_1 \cap K_2) = H_q(M \mid K_1) = H_q(M \mid K_2) = 0,$$

so by exactness $H_q(M \mid K_1 \cup K_2) = 0$.

3. For $M = \mathbb{R}^n$, $K = \bigcup_i K_i$ for K_i compact and contractible, we get the result by induction.

4. Let $M = \mathbb{R}^n$, and let K be an arbitrary compact subset.

   Choose some $a \in H_q(\mathbb{R}^n \mid K)$, and choose $\psi \in S_q(\mathbb{R}^m)$ representing a. Then because ψ is a relative cycle, $\partial\psi \in S_{q-1}(\mathbb{R}^n \setminus K)$.

   This means that we can express $\partial\psi$ as

$$\sum_{j=1}^{\ell} \lambda_j \tau_j, \qquad \tau_j \colon \Delta^{q-1} \to \mathbb{R}^m \setminus K.$$

   Since Δ^{q-1} is compact, the images $\tau(\Delta^{q-1})$ are compact, and their union

$$T = \bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1})$$

   is compact.

   There exists an open set $U \subset \mathbb{R}^m$ with $K \subset U$ and $T \cap U = \emptyset$. Thus, we know that

$$\partial\psi \in \mathbb{R}^m \setminus U \implies \psi \in Z_q(\mathbb{R}^m \mid U).$$

   Denote the representative of ψ in $H_q(\mathbb{R}^m \mid U)$ by $[\psi] = a'$.

   Due to general topology stuff, we can always find finitely many closed balls B_i satisfying the following conditions.

   a) $B_i \subset U$ for all i

   b) $B_i \cap K \neq \emptyset$

c) $K \subset \bigcup_i B_i$

Then $K \subset \bigcup_i B_i \subset U$, and we have restriction maps as follows.

$$\begin{tikzcd}
S_q(\mathbb{R}^m \mid U) \arrow[r, "(\rho_{\bigcup_i U_i, U})_*"] & S_q(\mathbb{R}^m \mid \bigcup_i B_i) \arrow[r, "(\rho_{K, \bigcup_i U_i})_*"] & S_q(\mathbb{R}^m \mid K) \\
a' \arrow[r, mapsto] & (\rho_{\bigcup_i B_i, U})_*(a') = a'' \arrow[r, mapsto] & a
\end{tikzcd}$$

Suppose $q > m$. We have already showed that $H_q(\mathbb{R}^m \mid \bigcup_i B_i) = 0$, so $a'' = 0$. Hence $a = 0$.

Now suppose $q = m$. We will provide running commentary on the following commutative diagram.

$$\begin{tikzcd}
H_m(X \mid U) \arrow[rr] & & H_m(X \mid \bigcup_i B_i) \arrow[rr] \arrow[d] \arrow[dl] \arrow[dr] & & H_m(X \mid K) \\
& H_m(X \mid y) & H_m(X \mid B_i) \arrow[l] \arrow[r] & H_m(X \mid x) &
\end{tikzcd}$$

We have shown that given $a \in H_m(X \mid K)$, one can find $a' \in H_m(X \mid U)$ and $a'' \in H_m(X \mid \bigcup_i B_i)$ which map to each other as follows. By assumption, a'' maps to zero in $H_m(X \mid x)$.

$$\begin{tikzcd}
a' \arrow[r, mapsto] & a'' \arrow[r, mapsto] \arrow[dr, mapsto] & a \\
& & 0
\end{tikzcd}$$

Pick some B_i containing x. Since $(\rho_{x,B_i})_*$ is an isomorphism, we have that $(\rho_{B_i,\bigcup_i B_i})_*(a'') = 0$.

$$\begin{tikzcd}
a' \arrow[r, mapsto] & a'' \arrow[r, mapsto] \arrow[d, mapsto] \arrow[dr, mapsto] & a \\
& 0 \arrow[r, mapsto] & 0
\end{tikzcd}$$

For any $y \in B_i$, we know that $(\rho_{y,B_i})_*$ is an isomorphism.

$$\begin{tikzcd}
a' \arrow[rr, mapsto] & & a'' \arrow[rr, mapsto] \arrow[d, mapsto] \arrow[ddl, mapsto] \arrow[ddr, mapsto] & & a \\
& & 0 \arrow[dl, mapsto] \arrow[dr, mapsto] & & \\
& 0 & & 0 &
\end{tikzcd}$$

Thus $(\rho_{y,\bigcup_i B_i})_*(a'') = 0$ for all $y \in \bigcup_i B_i$, so the result is implied by work above.

5. Mayer-Vietoris and induction.

□

Proposition 23. Let $K \subset M$ be compact, and assume that M is connected and oriented via an orientation $x \mapsto o_x \in H_m(X \mid x)$. Then there exists a unique orientation class $o_K \in H_m(M \mid K)$ such that $(\rho_{x,K})_* o_K = o_x$ for all $x \in K$.

Proof. First we show uniqueness, as we will need it later. Suppose that there exist two orientation classes o_K, o'_K such that

$$(\rho_{x,K})_* o_K = (\rho_{x,K})_* o'_K = o_x.$$

Taking a difference, we find

$$(\rho_{x,K})_*(o_K - o'_K) = 0,$$

so by Lemma 22, we get uniqueness.

As a first step towards proving existence, suppose that K is contained in the domain $U_\alpha \cong \mathbb{R}^n$ of a single chart. Pick some $x \in K$, and consider the following morphisms in **Pair**

$$\begin{tikzcd}
M \setminus U_\alpha \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \setminus K \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \setminus \{x\} \arrow[r, hook] & M
\end{tikzcd}$$

which correspond to the commuting diagram.

$$\begin{tikzcd}
H_m(M \mid U_\alpha) \arrow[r, equal] \arrow[d, "\phi"'] & H_m(M \mid x) \\
H_m(M \mid K) \arrow[ur] &
\end{tikzcd}$$

Pulling $o_x \in H_m(M \mid x)$ back to $H_m(M \mid U_\alpha)$, then mapping it to $H_m(M|K)$ with ϕ gives an orientation class $o_K \in H_m(M \mid K)$. The commutativity of the diagram ensures that $(\rho_{x,K})_* o_K = o_x$.

Now suppose that K is not contained in a single chart. We can find an open cover of K by charts, and since K is compact, there exists a finite subcover.

Suppose K is contained in two charts U_1 and U_2; the general case follows by induction. Then we can express K as a union of compacta $K = K_1 \cup K_2$, where $K_i \subset U_i$. Consider the relative Mayer-Vietoris sequence

$$0 \longrightarrow H_m(M \mid K_1 \cup K_2) \xhookrightarrow{i} H_m(M \mid K_1) \oplus H_m(M \mid K_2) \xrightarrow{\kappa} H_m(M \mid K_1 \cap K_2) \longrightarrow \cdots$$

By our above work, we get unique orientation classes $o_{K_i} \in H_m(M|K_i)$ such that $(\rho_{x,K_i})_* = o_x$ for all $x \in K_i$, $i = 1, 2$. Applying κ, we find

$$\kappa(o_{K_1}, o_{K_2}) = \rho_{K_1 \cap K_2, K_1} o_{K_1} - \rho_{K_1 \cap K_2, K_2} o_{K_2}.$$

But by uniqueness, these must be equal, so there exists a unique o_K in $H_m(M \mid K)$ as in the theorem. □

In order to give $\tilde{M}$ a manifold structure, we define a topology on $\tilde{M}$ as follows. Let $B \subset \mathbb{R}^n \subset M$ be a ball of finite radius, and choose a generator $\mu_B \in H_n(M \mid B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \to M$ is two-sheeted covering. This immediately imples that $\tilde{M}$ is a manifold. In fact, $\tilde{M}$ is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} \mid \mu_b) \cong H_n(U(\mu_B) \mid \mu_b) \cong H_n(B \mid x),$$

where

Definition 24 (orientation class). Let M be a connected, orientable, compact manifold of dimension m. An orientation class on M is an element $o_m \in H_m(M \colon \mathbb{Z})$ such that $o_x = (\rho_{x,M})_* o_m$ is an orientation.

Theorem 25. Let M be a compact, connected manifold of dimension m. The following are equivalent.

1. M is ($\mathbb{Z}$-)orientable.
2. There exists an orientation class $o_M \in H_m(M; \mathbb{Z})$.
3. $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof.

(1 ⇒ 2) Consequence of Proposition 23.

(2 ⇒ 3) Let $a \in H_m(M)$, and let $x \in M$. Then $(\rho_{x,M})_* a = k_x o_x$ for some $k_x \in \mathbb{Z}$. This defines a function $M \longrightarrow \mathbb{Z}$, $x \mapsto k_x$.

We aim to show that this function is constant. Choose a chart U_x around x, and let $y \in U_x$. Then the outside of the following diagram commutes by functoriality of H_m, and the lower triangle commutes by definition of an orientation class, so the upper triangle commutes.

$$\begin{tikzcd} & H_n(M) \arrow[dl] \arrow[dr] & \\ H_m(M \mid x) \arrow[rr, "\phi"] \arrow[dr, equal] & & H_m(M \mid y) \arrow[dl, equal] \\ & H_m(X \mid U_x) & \end{tikzcd}$$

This means in particular that

$$\phi(k_x o_x) = k_x \phi(o_x) = k_x o_y$$

must be equal to $k_y o_y$, so $k_x = k_y$. Thus, the function $x \mapsto k_x$ is locally constant, hence constant. Call this constant k.

Thus, $ka - o_M$ is zero when restricted to any $x \in M$, implying by Proposition 23 that $a = k o_M$. o_M generates $H_m(M)$.

We know that o_M is non-trivial because $(\rho_{x,M})_* o_M$ generates $H_m(M|x)$, and that it has infinite order because if $\ell o_M = 0$ for some $\ell \neq 0 \in \mathbb{Z}\ H_m(M)$, then $\ell(\rho_{x,M})_* o_M = \ell o_x = 0$. This shows that $H_m(M)$ has a single generator of infinite order, i.e. is isomorphic to $\mathbb{Z}$.

(2 ⇒ 3) Pick a generator $o_M \in \mathbb{Z}$, and define an orientation by

$$o_x = (\rho_{x,M})_* o_M.$$

□

From now, we will sometimes denote an orientation class on M by $[M]$.

1.5.2 The orientation cover

Let M be an n-dimensional manifold, not necessarily orientable.

In order to give $\tilde{M}$ a manifold structure, we define a topology on $\tilde{M}$ as follows. Let $D\colon \mathbb{D}^m \hookrightarrow M$ be a compact subset of M homeomorphic to the closed ball, and pick

$\mu_D \in H_m(M \mid \mathring{\mathbb{D}})$. We define a subset $U_B \subset \tilde{M}$ by

Let $\mathbb{D}^n \subset \mathbb{R}^n \subset M$ be a closed ball of finite radius and, and choose a generator $\mu_B \in H_n(M \mid B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \to M$ is two-sheeted covering. This immediately imples that $\tilde{M}$ is a manifold. In fact, $\tilde{M}$ is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} \mid \mu_b) \cong H_n(U(\mu_B) \mid \mu_b) \cong H_n(B \mid x),$$

where

1.5.3 Degree

Definition 26 (degree)**.** Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $g\colon M \to N$ be continuous. The degree of f is the integer d such that

$$f(o_M) = do_N.$$

Proposition 27. Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $f, g\colon M \to N$ be continuous.

1. The degree is multiplicative, i.e.

$$\deg(f \circ g) = \deg(f) \circ \deg(g)$$

2. If $\bar{M}$ is the same manifold as M but with the opposite orientation, then

$$\deg(f\colon M \to N) = -\deg(f\colon \bar{M} \to N).$$

3. If the degree of f is not trivial, then f is surjective.

Proof. All obvious. □

1.6 Cohomology with compact support

For this section, let R be a commutative ring with unit 1_R. For most of this section, we will work with cohomology over R, but notationally suppress this.

Definition 28 (singular cochains with compact support)**.** Let X be a topological space, and let R be a commutative ring with unit 1_R. The singular n-cochains with compact support are the set

$$S^n_c(X;R) = \left\{\phi\colon S_n(X) \to R \mid {}_{\phi(\sigma)=0 \text{ for all } \sigma\colon \Delta^n \to X \text{ with } \sigma(\Delta^n)\cap K_\phi = \emptyset}^{\text{there exists } K_\phi \subset X \text{ compact such that}}\right\}.$$

That is, singular cochains with compact support are those ϕ which ignore simplices which do not hit some compact subset K_ϕ.

Clearly $S^n_c(X) \subset S^n(X)$, and since the boundary of any simplex with compact support clearly has compact support, there is an induced chain complex structure

$$S^*_c(X).$$

However, there is a more categorical description of this chain complex coming from relative cohomology.

Let X be a topological space. Denote by $\mathcal{K}$ the set of compact subsets of X. This is clearly a poset under

$$K \le K' \iff K \subseteq K',$$

and it is filtered because for a finite set I, $\bigcup_i K_i$ is an upper bound for $\{K_i\}_{i\in I}$.

Fixing some X, consider the functor

$$\mathcal{K}^{\mathrm{op}} \to \mathbf{Pair}; \qquad K \mapsto (X, X \smallsetminus K).$$

We get a functor $\mathcal{K} \to \mathbf{Coch}_{\ge 0}(R\text{-}\mathbf{Mod})$ via the composition

$$\mathcal{K} \longrightarrow \mathbf{Pair}^{\mathrm{op}} \xrightarrow{S^\bullet} \mathbf{Coch}_{\ge 0}(R\text{-}\mathbf{Mod}) \; .$$

For obvious reasons, we call this functor $S^\bullet(X \mid -)$

Proposition 29. The colimit

$$\operatorname*{colim}_K S^\bullet(X \mid K)$$

agrees with cohomology with compact support $S^\bullet_c(X)$.

Proof. Let $f \in S^n(X \mid K)$. Then by definition, f is a map $S_n(X \mid K) \to R$, i.e. a map $S_n(X) \to R$ such that for any $\sigma\colon \Delta^n \to X$ such that $\sigma(\Delta^n) \cap K = \emptyset$, $f(\sigma) = 0$. Thus, $f \in S^n_c(X)$.

This inclusion gives a chain map $S^\bullet(X \mid K) \hookrightarrow S^\bullet_c(X)$, and these together induce an inclusion[1] $\operatorname{colim}_K S^\bullet(X \mid K) \to S^\bullet_c(X)$. In fact, this is a level-wise epimorphsim since any $g \in S^\bullet_c(X)$ is of the above form. $\square$

[1] This is a monomorphism because the components are monomorphisms, and by Corollary ?? filtered colimits are exact, hence preserve monomorphisms.

Corollary 30. We have an isomorphism

$$H^n_c(X) \cong \operatorname*{colim}_K H^n(X \mid K).$$

Proof. Filtered colimits are exact by Corollary ??. □

1.7 Poincaré duality

This section takes place over a ring R with unit 1_R. We will usually notationally suppress R.

Let M be a connected, m-dimensional manifold with R-orientation $x \mapsto o_x$, and let $K \subset M$ be a compact subset. By Proposition 23, there exists an orientation class $o_K \in H_m(M \mid K)$ compatible with the orientation o_x.

Consider the cap product

$$H^q(M \mid K) \otimes H_m(M \mid K) \xrightarrow{\frown} H_{m-q}(M) \ .$$

Sticking $o_K \in H_m(M \mid K)$ into the second slot of the cap product and setting $p = n - q$, we find a map

$$H^{m-p}(M \mid K) \xrightarrow{(-)\frown o_K} H_p(M)$$

Proposition 31. The maps

$$(-) \frown o_K \colon H^{m-p}(M|K) \to H_p(M)$$

descend to a map

$$H^{m-p}_c(M) \to H_p(M).$$

Proof. By Proposition 10, the identity $\mathrm{id}_M \colon M \to M$ makes the following diagram commute.

$$\begin{tikzcd}
H^{m-p}(M \mid L) \otimes H_m(M \mid L) \arrow[r, "(-)\frown(-)"] \arrow[d] & H_p(M) \arrow[d, "(\mathrm{id}_M)_*"] \\
H^{m-p}(M \mid K) \otimes H_m(M \mid K) \arrow[r, "(-)\frown(-)"'] & H_p(M)
\end{tikzcd}$$

Since

$$(\mathrm{id}_M)_* = \mathrm{id}_{H_p(M)},$$

this in particular means that the triangles

$$\begin{tikzcd} H^{m-p}(M \mid L) \arrow[d] \arrow[dr, "(-)\frown o_L"] & \\ & H_p(M) \\ H^{m-p}(M \mid K) \arrow[ur, "(-)\frown o_K"'] & \end{tikzcd}$$

commute. But this says that the maps $(-) \frown o_K$ form a cocone under $H^{m-p}(M \mid -)$. □

Definition 32 (Poincaré duality)**.** The map defined in Proposition 31 is called the Poincaré Duality map, and denoted

$$\mathsf{PD}_M \colon H_c^{m-q}(M) \to H_m(M).$$

Theorem 33 (Poincaré duality)**.** Let M be a connected manifold of dimension m with an R-orientation $x \mapsto o_x$. Then

$$\mathsf{PD}_M \colon H_c^{m-p}(M; R) \to H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Proof. Manifold induction! □

1.8 Alexander-Lefschetz duality

1.9 Application of duality

1.10 Duality and cup products

Definition 34 (cup pairing)**.** Let M be a connected closed m-manifold with an R-orientation for some commutative ring R. The cup pairing is the composition

$$H^k(M; R) \otimes_R H^{m-k}(M; R) \xrightarrow{\smile} H^m(M; R) \xrightarrow{\mathsf{PD}_M} H_0(M; R) \ .$$

1.11 The Milnor sequence

In Subsection **??**, we showed that for any tower of cochain complexes of abelian groups $\{C_i\}$ satisfying the Mittag-Leffler condition[2] and any $q \in \mathbb{Z}$, we have a short exact se-

[2] i.e. satisfying it level-wise

quence

$$0 \longrightarrow \lim^1 H^{q-1}(C_i) \hookrightarrow H^q(\lim_i C_i) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0 \ .$$

Theorem 35 (Milnor sequence for CW complexes)**.** Let X be a CW complex with filtration

$$X_0 \subset X_1 \subset \cdots, \qquad \text{so} \quad X \cong \bigcup_{i \geq 0} X_i.$$

Then for any abelian group G there is a short exact sequence

$$0 \longrightarrow \lim^1 H^{m-1}(X_n; G) \hookrightarrow H^m(X; G) \twoheadrightarrow \lim H^m(X_m; G) \longrightarrow 0 \ .$$

Proof. The tower of chain complexes $C_n^\bullet = \mathrm{Hom}(S_\bullet(X_n), G)$ satisfies the Mittag-Leffler condition (Definition **??**) because the maps $C_n^\bullet \to C_{n-1}^\bullet$ dual to the inclusions $S_\bullet(X_{n-1}) \hookrightarrow S_\bullet(X_n)$ are epimorphisms (because the exact sequence on the pair splits).

Plugging this into in Theorem **??**. we find that the sequence

$$0 \longrightarrow \lim^1 H^{q-1}(X_i) \hookrightarrow H^q(\lim \mathrm{Hom}(S_\bullet(X_i), G)) \twoheadrightarrow \lim H_q(X_i) \longrightarrow 0 \ .$$

It remains only to show that

$$H^q(\lim \mathrm{Hom}(S_\bullet(X_i), G)) \cong H^q(X; G),$$

i.e. that

$$\lim \mathrm{Hom}(S_\bullet(X_i), G) \cong \mathrm{Hom}(S_\bullet(X); G).$$

But this is true because Hom turns limits into colimits in the first slot, and $S_\bullet$ commutes with colimits because reasons. □

Example 36. Consider the tower (in **Top**!)

$$\mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \cdots .$$

Denote

$$\mathbb{C}P^\infty := \lim_i \mathbb{C}P^i.$$

The above tower is the CW filtration of $\mathbb{C}P^\infty$, including only even levels. Because the sequence

$$0 \longrightarrow S_\bullet(\mathbb{C}P^{n-1}) \hookrightarrow S_\bullet(\mathbb{C}P^n) \twoheadrightarrow S_\bullet(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \longrightarrow 0$$

splits and Hom and H_q are additive, we get that the maps

$$H_q(\mathbb{C}P^n) \to H_q(\mathbb{C}P^{n-1})$$

are epimorphisms. Thus, the tower $H^{q-1}(\mathbb{C}P^n)$ also satisfies the Mittag-Leffler condition, and we get that

$$H^q(\mathbb{C}P^\infty) \cong \lim_i H^q(\mathbb{C}P^i) \cong \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

1.12 Lens spaces