1 Basic applications

In this chapter, we will apply homological algebra to the study of various classes of objects as follows. Our main goal will be to implement the following.

Cockamamie Scheme 1.

1. Find a way of turning the objects in question into rings.
2. Study modules over those rings using homological algebra.
3. ???
4. Profit.

1.1 Quiver representations

The first problem to which we will apply Cockamamie Scheme 1 is the study of *quiver representations.* Unfortunately, due to time constraints, we will only get as far as step 2.

Definition 2 (quiver)**.** A quiver Q consists of the following data.

- A finite[1] set Q_0 of *vertices*
- A finite set Q_1 of *arrows*
- A pair of maps
$$s, t\colon Q_1 \to Q_0$$
called the *source* and *target* maps respectively.

Definition 3 (quiver representation)**.** Let Q be a quiver, and let k be a field. A k-linear representation V of Q consists of the following data.

- For every vertex $x \in Q_0$, a vector space V_x.
- For every arrow $\rho \in Q_1$, a k-linear map $V_{s(\rho)} \to V_{t(\rho)}$.

A morphism f between k-linear representations V and W consists of, for each vertex

[1]There is no fundamental reason for this finiteness condition, except that it will make the following analysis more convenient.

$x \in Q_0$ a linear map $f_x \colon V_x \to W_x$, such that for edge $s \in Q_1$, the square

$$\begin{tikzcd}
V_{s(\rho)} \arrow[r] \arrow[d] & V_{t(\rho)} \arrow[d] \\
W_{s(\rho)} \arrow[r] & W_{t(\rho)}
\end{tikzcd}$$

commutes.

This allows us to define a category $\mathrm{Rep}_k(Q)$ of k-linear representations of Q, whose objects.

Following our grand plan, we first find a way of turning a quiver representation into a ring.

First, we form the free category over the quiver Q, which we denote by $\mathcal{F}(Q)$; that is, the objects of the category $\mathcal{F}(Q)$ are the vertices of Q, and the morphisms $x \to y$ consist of finite chains of arrows starting at y and ending at x.

$$\begin{tikzcd}
 & z_1 \arrow[dr, "\rho_1"] & & & & z_5 \arrow[dr, "\rho_5"] & \\
y \arrow[ur, "\rho_0"] & & z_2 \arrow[dr, "\rho_2"] & & z_4 \arrow[ur, "\rho_4"] & & x \\
 & & & z_3 \arrow[ur, "\rho_3"] & & &
\end{tikzcd}$$

We will denote this morphism by

$$\rho_5\rho_4\rho_3\rho_2\rho_1\rho_0.$$

We call such finite non-empty chains *non-trivial paths.* Note that we must formally adjoin identity arrows e_x with $s(e_x) = t(e_x) = x$; these are the *trivial paths.* Note that the notation introduced above is a good one, since in the category $\mathcal{F}(Q)$ composition is given by

$$(\rho_n \cdots \rho_1) \circ (\rho'_m \cdots \rho'_1) = \rho_n \cdots \rho_1 \rho'_m \cdots \rho'_1.$$

Definition 4 (path algebra)**.** Let k be a field and Q a quiver. The path algebra of Q over k is, as a vector space, the free vector space over the set $\mathrm{Mor}(\mathcal{F}(Q))$. The multiplication is given by composition

$$x \cdot y = \begin{cases} \rho\sigma, & s(\rho) = t(\sigma) \\ 0, & s(\rho) \neq t(\sigma). \end{cases}$$

We will denote the path algebra of Q over k by kQ.

That is, the multiplication in kQ is given by the composition in the category $\mathcal{F}(Q)$ if the morphisms in question are composable (in the sense that the second ends where the first begins) and zero otherwise.

From now on, we fix (arbitrarily) a bijection of Q with $\{1, 2, \cdots, n\}$. It is clear that

$$\mathrm{id}_{kQ} = e_1 + \cdots + e_n.$$

Example 5. Let Q be any quiver, and denote by $A = kQ$ its path algebra. For any vertex i and any path ρ, the multiplication law for the path algebra tells us that

$$\rho e_i = \begin{cases} e_i, & s(\rho) = i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the left ideal Ae_i consists of those paths which start at i. Similarly, the right ideal e_iA consists of those paths which terminate at i.

Since any path either starts at i or doesn't, we have

$$A \cong Ae_i \oplus A(1 - e_i).$$

This means (by Proposition **??**) that each Ae_i is a projective module. The same argument tells us that e_iA is projective for all i.

Proposition 6. Let Q be a quiver and k its path algebra. There is an equivalence of categories

$$\mathrm{Rep}_k(Q) \xrightarrow{\simeq} kQ\text{-}\mathbf{Mod}.$$

Proof. First, we construct out of any k-linear represenation V of Q a kQ-module. Our underlying space will be the vector space $\bigoplus_{i \in Q_0} V_i$. □

1.2 Group (co)homology

1.2.1 The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A \colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 7 (normalized chain complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2}d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1}d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} NA_{n-1} \xrightarrow{(-1)^{n-1}d_{n-1}} \cdots$$

Definition 8 (Moore complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n-chains A_n and differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
\partial^2 &= \left(\sum_{j=0}^{n-1}(-1)^j d_j\right) \circ \left(\sum_{i=0}^{n}(-1)^i d_i\right) \\
&= \sum\sum(-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_j \circ d_i + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= 0.
\end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A, unless this is confusing. In this case, we will denote it by MA.

Definition 9 (alternating face maps chain modulo degeneracies)**.** Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial\colon A_n \to A_{n-1}$ takes degerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain modulo degeneracies.[2]

Lemma 10. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \overset{i}{\hookrightarrow} A \overset{p}{\twoheadrightarrow} A/D(A)$$

[2]The nLab is responsible for this terminology.

Proof. We need only show that the following diagram commutes.

$$\begin{tikzcd} NA_n \arrow[r, hook] \arrow[d, "(-1)^n d_n"'] & A_n \arrow[r, two heads] \arrow[d, "\partial"] & A_n/DA_n \arrow[d, "{[\partial]}"] \\ NA_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, two heads] & A_{n-1}/DA_{n-1} \end{tikzcd}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n, and the right-hand square commutes trivially. ☐

Theorem 11. The composite

$$p \circ i \colon NA \to A/D(A)$$

is an isomorphism of chain complexes.

Let $f\colon [m] \to [n]$ be a morphism in the simplex category Δ. By functoriality, f induces a map

$$N\Delta^n \to N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0\colon [n] \to [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]\colon \Delta \to \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

and on morphisms by

$$f\colon [m] \to [n] \mapsto$$

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \tag{1.1}$$

Theorem 12. The adjunction in Equation 1.1 is an equivalence of categories

Proof. later, if I have time. ☐

Also later if time, Dold-Puppe correspondence: this works not only for **Ab**, but for any abelian category.

1.2.2 The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta \colon \mathrm{id}_{\mathcal{C}} \to RL$ and counit $\varepsilon \colon LR \to \mathrm{id}_{\mathcal{D}}$.

Recall that this data gives a comonad in $\mathcal{D}$, i.e. a comonoid internal to the category $\mathrm{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$LRLR \xLeftarrow{L\eta R} LR \xRightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category $\mathcal{C}$ and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M} \colon \Delta \to \mathcal{C}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, dashed, "\exists! \tilde{M}"] \\ & \mathcal{C} \end{tikzcd}$$

In particular, with $\mathcal{C} = \mathrm{End}(\mathcal{D})^{\mathrm{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \to \mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, "\exists!"] \\ & \mathrm{End}(\mathcal{D})^{\mathrm{op}} \end{tikzcd}$$

Equivalently, this gives a functor

$$B \colon \Delta_+^{\mathrm{op}} \to \mathrm{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\mathrm{ev}_d \colon \mathrm{End}(\mathcal{D}) \to \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in $\mathcal{D}$.

$$S_d = \Delta_+^{\mathrm{op}} \xrightarrow{B} \mathrm{End}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{D}$$

Definition 13 (bar construction)**.** The composition $\mathrm{ev}_d \circ B$ is called the bar construction.

In the case that the category $\mathcal{D}$ is abelian, we can form the Moore complex (Definition 8) of the bar construction.

Definition 14 (bar complex)**.** Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ the bar construction of b.

The bar complex Bar_b of b is the Moore complex of the bar construction of b.

Example 15. Consider the functor

$$U \colon \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

which forgets multiplication. This has left adjoint

$$\mathbb{Z}G \otimes_{\mathbb{Z}} - \colon \mathbf{Ab} \to \mathbb{Z}G\text{-}\mathbf{Mod},$$

which assigns to an abelian group A the left $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta \colon \mathrm{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_Z -)$ and counit $\varepsilon \colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components

$$\eta_A \colon A \to U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \qquad a \mapsto 1 \otimes a$$

and

$$\varepsilon_M \colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \to M; \qquad g \otimes m \mapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z, we find the following comonad in $\mathbb{Z}G$-$\mathbf{Mod}$, where $\nabla = \mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{tikzcd}
Z \circ Z \\
Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\
\mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

This gives us a simplicial object in $\mathrm{End}(\mathbb{Z}G\text{-}\mathbf{Mod})$.

$$\begin{tikzcd}
\cdots\cdots\cdots\; Z \circ Z \circ Z \arrow[r, Rightarrow, shift left=9, "ZZ\varepsilon"] \arrow[r, Rightarrow, shift left=3] \arrow[r, Rightarrow, shift right=3] \arrow[r, Rightarrow, shift right=9, "\varepsilon ZZ"'] & Z \circ Z \arrow[l, Rightarrow, shift right=6] \arrow[l, Rightarrow] \arrow[l, Rightarrow, shift left=6] \arrow[r, Rightarrow, shift left=4, "Z\varepsilon"] \arrow[r, Rightarrow, shift right=4, "\varepsilon Z"'] & Z \arrow[l, Rightarrow, "\nabla"'] \arrow[r, Rightarrow] & \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

Evaluating on a specific $\mathbb{Z}G$-module M then gives a simplicial object in $\mathbb{Z}G$-$\mathbf{Mod}$.

We continue in the special case of $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$-module. In this case,

$$Z^{\circ n}(Z) = \overbrace{\mathbb{Z}G \otimes_Z \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G}^{n \text{ times}} \otimes_{\mathbb{Z}} \mathbb{Z}.$$

We are justified in suppressing the last copy of $\mathbb{Z}$.

The face maps d_i remove the ith copy of Z; on a generator, we have the action of $d_i \colon Z^{n+1}\mathbb{Z} \to Z^n\mathbb{Z}$ given by

$$d_i \colon x \otimes x_1 \otimes \cdots \otimes x_n \otimes a \mapsto \begin{cases} xx_1 \otimes \cdots \otimes x_n, & i = 0 \\ x \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, & 0 < i < n+1 \\ x \otimes \cdots \otimes x_{n-1}, & i = n+1. \end{cases}$$

It is traditional to use the notation

$$x \otimes x_1 \otimes \cdots \otimes x_n \otimes a = x[x_1|\cdots|x_n].$$

This notation is the origin of the name *bar complex.*

For intuition's sake, it is worth going through a concrete example. A zero-simplex in our complex is simply an element of $\mathbb{Z}G$; this is a finite formal sum of elements of G multiplied by integers.

1.2.3 Group cohomology

One would like to be able to apply the techniques of homological algebra to the study of groups. Unfortunately, the category **Grp** is not very nicely behaved; in particular it is not abelian.

Out of any group G, one can form the group ring $\mathbb{Z}G$. Categorically speaking, the category of rings is arguably even worse then that of groups, so one might question the sanity of such a construction. However, one can study groups using homological techiques by studying modules over the group ring $\mathbb{Z}G$; as a category of modules, it is sure to be abelian.

Given a group G, a G-module is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-$\mathbf{Mod}$.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in $\mathbf{Ab}$, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the equivalence G-**Mod** $\simeq \mathbb{Z}G$-**Mod**, we know that G-**Mod** has enough projectives.

Note that there is a canonical functor $\mathrm{triv}\colon \mathbf{Ab} \longrightarrow \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 16. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \mathrm{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \mathrm{Hom}_{G\text{-}\mathbf{Mod}}(M, \mathrm{triv}\, A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \longrightarrow A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \longrightarrow A$ such that $\hat{\alpha}(g \cdot m - g) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \longrightarrow A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{1.2}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \longrightarrow \mathrm{triv}\, A$. In fact, G-linear maps $M \longrightarrow \mathrm{triv}\, A$ are precisely those satisfying Equation 1.2, showing that this is really an isomorphism.

The other case is similar. □

Definition 17 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Note that we actually have a fairly good handle on invariants and coinvariants.

Lemma 18. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\text{-}\mathbf{Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula $\mathbb{Z}$ is taken to be a trivial right $\mathbb{Z}G$-module, and in the second a trivial left $\mathbb{Z}G$-module.

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element
$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$
By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

This tells us immediately that the invariants functor $(-)^G$ is left exact, and the coinvariant functor $(-)_G$ is right. We could have discovered this by explicit investigation, and formed the right and left derived hom functors. However, this would have required picking resolutions for each object under consideration, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[3]

Example 19. Let $T \cong \mathbb{Z}$ be an infinite cyclic group with a single generator t. Then
$$\mathbb{Z}T \cong \mathbb{Z}[t, t^{-1}],$$
and we have an exact sequence
$$0 \longrightarrow \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \longrightarrow \mathbb{Z} \longrightarrow 0$$
giving a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}T$-module.

In general, our life is not so easy. Fortunately, we still have, for any group G, a general construction of a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}G$-module, known as the *bar construction.*

Definition 20 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module
$$B_n = \bigoplus_{(g_i) \in (G \smallsetminus \{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$
where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to $(g_1, \ldots, g_n)$. The differential is defined by
$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

This is in fact a chain complex; the verification of this is identical to the verification of

[3] In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

the simplicial identities. First, we note that we can write

$$d_n = d_0 + \sum_{i=1}^{n-1} (-1)^i d_i + (-1)^n d_n,$$

where the d_i satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i, \qquad i < j.$$

Thus, we have

$$\begin{aligned}
d_n \circ d_{n-1} &= \left(\sum_{j=0}^{n-1} (-1)^j d^j\right) \circ \left(\sum_{i=0}^{n} (-1)^i d^i\right) \\
&= \sum \sum (-1)^{i+j} d^j \circ d^i \\
&= \sum_{0 \le j < i \le n} (-1)^{i+j} d^j \circ d^i + \sum_{0 \le i \le j \le n-1} (-1)^{i+j} d^j \circ d^i \\
&= \sum_{0 \le j < i \le n} (-1)^{i+j} d^{i-1} \circ d^j + \sum_{0 \le i \le j \le n-1} (-1)^{i+j} d^j \circ d^i \\
&= 0.
\end{aligned}$$

Proposition 21. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
+
\begin{tikzcd}
& B_n \arrow[dl, dashed, "h_n"'] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}$$

gives us in one direction

$$\begin{tikzcd}
g[g_1|\cdots|g_n] \arrow[r, mapsto] & \begin{array}{c} gg_1[g_2|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i g[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n g[g_1|\cdots|g_{n-1}] \end{array} \arrow[dl, mapsto] \\
\begin{array}{c} [gg_1|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i [gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n [gg_1|\cdots|g_n] \end{array} &
\end{tikzcd}$$

and in the other

$$\begin{tikzcd}
& g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\
{[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{c} g[g_1|\cdots|g_n] \\ - [gg_1|g_2|\cdots|g_n] \\ - \sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ - (-1)^n [gg_1|\cdots|g_{n-1}] \end{array}
\end{tikzcd} .$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. □

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G,A) = H^i\left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0,A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1,A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G,A) = H_i\left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \to G$-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
   - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.
   - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived functors $R^i(-)^G$.
4. We denote

   $$L_i(-)_G = H_i(G,-), \qquad R^i(-)^G = H^i(G,-),$$

   and call them *group homology* and *group cohomology* respectively.
5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have

   $$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z},A),$$

   and thus that

   $$H_i(G,A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z},A), \qquad H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z},A).$$

   This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 22. Let G be a group, and consider $\mathbb{Z}$ as a free $\mathbb{Z}G$-module. Let us compute

$H_1(G,\mathbb{Z})$ and $H^1(G,\mathbb{Z})$.

To compute $H_1(G,\mathbb{Z})$, consider the sequence

$$\begin{tikzcd}
\cdots \arrow[r, "d_2\otimes \mathrm{id}_{\mathbb{Z}}"] & \bigoplus_{g\in G\setminus\{1\}}[g]\mathbb{Z}G\otimes_{\mathbb{Z}G}\mathbb{Z} \arrow[r, "d_1\otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, equal] & {[\cdot]}\mathbb{Z}G\otimes_{\mathbb{Z}G}\mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \bigoplus[g]\mathbb{Z} \arrow[r] & {[\cdot]}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

The differential $d_1\otimes \mathrm{id}_{\mathbb{Z}}$ acts on $[g]$ by sending it to

$$[\cdot]g - [\cdot] = [\cdot] - [\cdot],$$

i.e. everything is a cycle. The differential d_2 sends

$$[g|h] \mapsto [g]h - [gh] + [h] = [g] + [h] - [gh],$$

i.e. the terms $[gh]$ and $[g]+[h]$ are identitfied. Thus, $H_1(G,\mathbb{Z})$ is simply the abelianization of G.

To compute $H^1(G,\mathbb{Z})$, consider the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot],\mathbb{Z}) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(\textstyle\bigoplus_{g\in G\setminus\{1\}}\mathbb{Z}G[g],\mathbb{Z}) \xrightarrow{(d_2)^*} \cdots .$$

Notice immediately that since the hom functor preserves direct sums in the first slot, we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i)\in(G\setminus\{1\})^n}\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z}\right) &\cong \bigoplus_{(g_i)\in(G\setminus\{1\})^n}\mathrm{Hom}_{\mathbb{Z}G}\left(\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z}\right) \\
&\cong \bigoplus_{(g_i)\in(G\setminus\{1\})^n}\mathbb{Z}[g_1|\cdots|g_n].
\end{aligned}$$

This notation deserves some explanation; recall that the symbols $[g_1|\cdots|g_n]$ are merely visual aids, reminding us where we are and how the differential acts. The differential

$$(d_n)^*\colon [g_1|\cdots|g_n].$$

In the low cases we have the sequence