Homological algebra notes

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Contents

1 Introduction

Homework:

- Must submit more than 50% correct solutions in order to take the final exam (oral).
- Posted on Monday after the lecture
- Submitted on following Monday in lecture
- Discuss following Tuesday, 16:45, room 430

2 Chain complexes

Homological algebra is about chain complexes.

Definition 1 (chain complex)**.** Let R be a ring.[1] A chain complex of R-modules, denoted $(C_\bullet, d)$, consists of the following data:

- For each $n \in \mathbb{Z}$, an R-module C_n, and
- R-linear maps $d_n \colon C_n \longrightarrow C_{n-1}$, called the *differentials*,

such that the differentials satisfy the relation

$$d_{n-1} \circ d_n = 0.$$

Example 2 (Syzygies)**.** Let $R = \mathbb{C}[x_1, \ldots, x_k]$. Following Hilbert, we consider a finitely generated[2] R-module M. The ring R is Noetherian, so by Hilbert's basis theorem, M is Noetherian, hence has finitely many generators $a_1, \ldots, a_k$.

Denote by R^k the free R-module with k generators, and consider

$$\phi \colon R^k \twoheadrightarrow M; \qquad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $\mathrm{Syz}^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^{k} \lambda_i e_i \overset{\phi}{\mapsto} \sum_{i=1}^{k} \lambda_i a_i = 0.$$

This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\mathrm{Syz}^1(M)$ is again a finitely generated R-module. Hence, pick finite set of generators b_i, $1 \leq i \leq \ell$, and look at

$$R^\ell \twoheadrightarrow \mathrm{Syz}^1(M); \qquad e_i \mapsto b_i.$$

[1] Associative, with unit. Not necessarily commutative.
[2] i.e. expressible as an R-linear combination of finitely many generators.

In general, there is non-trivial kernel, denoted $\mathrm{Syz}^2(M)$. This consists of relation between relations.

We can keep going. This gives us a sequence $\mathrm{Syz}^\bullet(M)$ corresponding to relations between relations between. . . between relations.

$$\begin{tikzcd}
 & & \mathrm{Syz}^2(M) \arrow[dr, hook] & & & & \\
{} \arrow[r, dotted, no head] & R^m \arrow[ur, two heads] \arrow[rr, "d"] & & R^l \arrow[rr, "d"] \arrow[dr, two heads] & & R^k \arrow[r] & M \\
\mathrm{Syz}^3 \arrow[ur, hook] & & & & \mathrm{Syz}^1(M) \arrow[ur, hook] & &
\end{tikzcd}$$

From this we build a chain complex $(C_\bullet, d)$, where C_n corresponds to the nth copy of R^k, known as the *free resolution* of M.

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-module admits a free resolution of length at most n.

Example 3 (simplicial complexes)**.** Let $N \geq 0$. A collection $K \subset \mathcal{P}(\{0, \ldots, n\})$ of subsets is called a *simplicial complex* if it is closed under the taking of subsets in the sense that if for all $\sigma \in K$,

$$\tau \subset \sigma \implies \tau \in K.$$

For every $n \geq 0$, we define the *n-simplices* of K to be the set

$$K_n = \{\sigma \in K \mid |\sigma| = n + 1\}.$$

Every $\sigma \in K_n$ can be expressed uniquely as $\sigma = \{x_0, \ldots, x_n\}$, with $x_0 < \cdots < x_n$. Define the *ith face* of σ to be

$$\partial_i\sigma = \{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}.$$

Let R be a ring. Define

$$C_n(K, R) = \bigoplus_{\sigma \in K_n} R_{e_\sigma},$$

where the subscripts are simply labels. Further, define

$$d\colon C_n(K, R) \to C_{n-1}(K, R); \qquad e_\sigma \mapsto \sum_{i=0}^{n} (-1)^i e_{\partial_i \sigma}.$$

This is a complex because…

Example 4. Let k be a field, and A an associative k-algebra. We define a complex of vector spaces

$$\cdots \xrightarrow{d} A^{\otimes n} \xrightarrow{d} \cdots \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} A \otimes A \xrightarrow{d} A\ ,$$

where d is defined by the formula

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

This complex is known as the *cyclic bar complex.*

Definition 5 (cycle, boundary, homology)**.** Let $(C_\bullet, d)$ be a chain complex. We make the following definitions.

- The space $\ker d_m$ is known as the space of n-cycles, and denoted $Z_n(C_\bullet)$.
- The space $\operatorname{im} d_{m+1}$ is known as the space of n-boundaries, and denoted $B_n(C_\bullet)$.
- The space
  $$\frac{Z_n}{B_n} = H_n(C_\bullet)$$
  is known as the nth homology of C.

Definition 6 (exact)**.** We say that a chain complex $(C_\bullet, d)$ is exact at $n \in \mathbb{Z}$ if $H_n(C_\bullet = 0)$.

Definition 7 (exact sequence)**.** We say that a sequence

$$C_k \xrightarrow{d} C_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} C_\ell$$

with $d^2 = 0$ is exact if it is exact at $k-1, k-2, \ldots, \ell+1$.

Definition 8 (short exact sequence)**.** A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0\ .$$

Equivalently, such a sequence is exact if

- f is a monomorphism

- g is a epimorphism
- $H_B = 0$.

Example 9.

- The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is exact.
- The sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is not exact since the map $x \mapsto 2x$ is not injective.
- The sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

  is exact.

3 Basic algebra

Strictly speaking, this chapter shouldn't need to exist, but my algebra knowledge is pretty hopeless so I'll be putting stuff I should know already here.

3.1 Exactness

Proposition 10. Let R be a ring, and M a left R-module. Then both the hom functors

$$\mathrm{Hom}(M, -), \qquad \mathrm{Hom}(-, M)$$

are left exact.

Proof. Let

$$0 \longrightarrow A \xhookrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. First, we show that $\mathrm{Hom}(M, A)$ is exact, i.e. that

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha\colon M \to A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and

$$f \circ \alpha = 0 \implies \alpha = 0$$

because f is a monomorphism.

Similarly, $\operatorname{im} f \subseteq \ker g$ because

$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \operatorname{im} f$. To this end, let $\gamma\colon M \to B$ such that

$g_*\gamma = 0$.

$$\begin{tikzcd}
 & & M \arrow[dl, dashed, "\exists!\delta"'] \arrow[d, "\gamma"'] \arrow[dr, "0"] & & \\
0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Then $\operatorname{im}\gamma \subset \ker g = \operatorname{im} f$, so γ factors through A; that is to say, there exists a $\delta\colon M \to A$ such that $f_*\delta = \gamma$. □

This theorem is a shadow of a much, much stronger result: Exactness of functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is a consequence of preservation of certain (co)limits. More concretely, we have the following.

3.2 Tensor products

Definition 11 (tensor product)**.** Let R be a (not necessarily commutative) ring M a right R-module, and N a left R-module. The tensor product $M \otimes_R N$ satisfies the following universal property…

Even for modules over non-commutative rings, we have the following version of tensor-hom adjunction.

Proposition 12. Let R be a ring, M a right R-module, and N a left R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \operatorname{Hom}_{\mathbf{Ab}}(N, -).$$

Proof. We show that there is a natural bijection

$$\operatorname{Hom}_{\mathbf{Ab}}(M \otimes_R N, P) \simeq \operatorname{Hom}_{\mathbf{Ab}}(M, \operatorname{Hom}_{\mathbf{Mod}\text{-}R}(N, P)).$$

□

3.3 Projective and injective modules

Definition 13 (projective, injective module)**.** Let R be a ring. An R-module M is said to be projective if the functor $\operatorname{Hom}(P, -)$ is exact, and injective if the functor $\operatorname{Hom}(-, P)$ is exact.

We know (from Proposition 10) that the hom functor is left exact in both slots. Therefore, we have the following immediate classification of projective objects.

Proposition 14. An R-module P in is projective if and only if for every for every epimorphism $g\colon B \twoheadrightarrow C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the folowing diagram commutes.

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[dl, dashed, "\exists\tilde{p}"'] \\ B \arrow[r, two heads, "g"'] & C \end{tikzcd}$$

Similarly, a module Q is injective if for every monomorphism $A \hookrightarrow B$ and map $g\colon A \to Q$, there exists a homomorphism $\tilde{q}$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "p"'] & B \arrow[dl, dashed, "\exists\tilde{p}"] \\ Q & \end{tikzcd}$$

Proof. In the projective case, this condition precisely encodes the condition that f_* is an epimorphism; in the projective case, □

Proposition 15. Let R be a ring. A module P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M.$$

Proof. First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$0 \longrightarrow \ker\pi \hookrightarrow F \xrightarrow{\pi} P \longrightarrow 0$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker\pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{tikzcd} & P \oplus M \arrow[dl, dashed, "\exists(j,k)"'] \arrow[d, "(f,g)"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists j"'] \arrow[d, "f"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

exhibiting P as projective. □

4 Abelian categories

4.1 Basics

In this chapter we recall some basic results.

A category is **Ab**-*enriched* if it is enriched over the symmetric monoidal category ($\mathbf{Ab}, \otimes_{\mathbb{Z}}$). In an **Ab**-enriched category with finite products, products agree with coproducts, and we call both direct sums. We call an **Ab**-enriched category with finite direct sums an *additive* category, and we call a functor between additive categories such that the maps

$$\mathrm{Hom}(A, B) \to \mathrm{Hom}(F(A), F(B))$$

are homomorphisms of abelian groups an *additive functor.* A *pre-abelian* category is an additive category such that every morphism has a kernel and a cokernel, and an *abelian category* is a pre-abelian category such that every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

This immediately implies the following results.

- Kernels are monic and cokernels are epic
- Since the equalizer of f and g is the kernel of $f - g$, abelian categories have all finite limits, and dually colimits.

As the following proposition shows, additive functors from additive categories preserve direct sums.

Lemma 16. Let A and B be objects in an additive category $\mathcal{A}$, and let X be an object in $\mathcal{A}$ equipped with morphisms as follows,

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_A"] & \\
B \arrow[ur, "i_B"] & & B
\end{tikzcd}$$

such that the following equations hold.

$$\begin{aligned} i_A \circ \pi_A + i_B \circ \pi_B &= \mathrm{id}_X \\ \pi_A \circ i_B = 0 &= \pi_B \circ i_A \\ \pi_A \circ i_A &= \mathrm{id}_A \\ \pi_B \circ i_B &= \mathrm{id}_B \end{aligned}$$

Then the πs and is exhibit $X \simeq A \oplus B$.

Corollary 17. Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(a \oplus b) \simeq F(a) \oplus F(b).$$

Once it is known that a category $\mathcal{A}$ is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is abelian, as we will see in
- For any small category I, the category $\mathbf{Fun}(I, \mathcal{A})$ of I-diagrams in $\mathcal{A}$ is abelian.

4.2 Embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

In abelian categories, one can defines kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero obect: f is a kernel for g if and only the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pullback.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category $\mathbf{Set}_*$ of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category $\mathcal{A}$, suppose one could find an embedding $\mathcal{H}\colon \mathcal{A} \to \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \to B \to C$ in $\mathcal{A}$, mapped it into $\mathbf{Set}_*$ finding a sequence $\mathcal{H}(A) \to \mathcal{H}(B) \to \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \to B \to C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in $\mathbf{Set}_*$ using $\mathcal{H}$ and check exactness there. Effectively, one could check exactness of a sequence in $\mathcal{A}$ by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking the that the image of the diagram under our functor $\mathcal{H}$ commutes.

In fact, for $\mathcal{A}$ small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to $\mathbf{Set}_*$.

Definition 18 (category of contravariant epimorphisms)**.** Let $\mathcal{A}$ be a small abelian category. Define a category $\mathcal{A}_{\twoheadleftarrow}$ with $\mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow}) = \mathrm{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}_{\twoheadleftarrow}}(X,Y) = \{f\colon Y \to X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity functor is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A\colon \mathcal{A}_{\twoheadleftarrow} \to \mathbf{Set}_*; \qquad Z \mapsto \mathrm{Hom}_{\mathcal{A}}(Z,A)$$

and sends a morphism $f\colon Z_1 \to Z_2$ in $\mathcal{A}_{\twoheadleftarrow}$ (which is to say, an epimorphism $\tilde{f}\colon Z_2 \twoheadrightarrow Z_1$ in $\mathcal{A}$) to the map

$$\mathcal{H}_A(f)\colon\ \mathrm{Hom}_{\mathcal{A}}(Z_1,A) \to \mathrm{Hom}_{\mathcal{A}}(Z_2,A); \quad (\alpha\colon Z_1 \to A) \mapsto (\alpha \circ \tilde{f}\colon Z_2 \to A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 19 (member functor)**.** Let $\mathcal{A}$ be a small abelian category. We define a functor $\mathcal{M}\colon \mathcal{A} \to \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \operatorname{colim} \mathcal{H}_A.$$

On morphisms, functorality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(\mathcal{A})$ have a much simpler interpretation.

Proposition 20. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \mathrm{Hom}(X,A)/\sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\ X' \arrow[r, "g'"'] & A \end{tikzcd}$$

Proof. The colimit can be computed using the following coequalizer.

$$\coprod_{\substack{f \in \mathrm{Morph}(\mathcal{A}_{\twoheadleftarrow}) \\ \tilde{f}\colon Y \to X}} \mathrm{Hom}_{\mathcal{A}}(X, A) \overset{\mathrm{id}}{\underset{- \circ \tilde{f}}{\rightrightarrows}} \coprod_{Z \in \mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow})} \mathrm{Hom}(Z, A) \xrightarrow{\mathrm{coeq}} \mathcal{M}(A)$$

On elements, we have the following.

$$\begin{tikzcd}
 & (g\colon X \to A) \arrow[d, equal, "!"] \\
(g\colon X \to A) \arrow[ur, mapsto, "\mathrm{id}"] \arrow[r, mapsto, "- \circ \tilde{f}"'] & (g \circ \tilde{f}\colon Y \to A)
\end{tikzcd}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow})} \mathrm{Hom}(Z, A)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(g\colon X \to A) R (g'\colon X' \to A) \iff \exists f\colon X' \twoheadrightarrow X \text{ such that } g' = g \circ \tilde{f}.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g\colon X \to A) \sim (g'\colon X' \to A) \iff \exists f\colon Z \twoheadrightarrow X, f'\colon Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{tikzcd}
Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\
X' \arrow[r, "g'"'] & A
\end{tikzcd}$$

□

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphsims into A modulo the above relation, and for a morphism $f\colon A \to B$ in $\mathcal{A}$, $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f)\colon [g] \mapsto [g \circ f].$$

Lemma 21. Let $f\colon A \to B$ be a morphism in a small abelian category $\mathcal{A}$ such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms making the following diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "g"] \arrow[d, two heads] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. □

Now we introduce some load-lightening notation: we write $\hat{(-)} = \mathcal{M}(-)$.

Lemma 22. Let $f\colon A \to B$ be a morphism in a small abelian category $\mathcal{A}$. Then $f = 0$ if and only if $\hat{f}\colon \hat{A} \to \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g, so $\hat{f} = 0$.

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\mathrm{id}_A]) = [0]$. But

$$\hat{f}([\mathrm{id}_A]) = [\mathrm{id}_A \circ f] = [f].$$

By Lemma 21, $[f] = [0]$ implies $f = 0$. □

Corollary 23. A diagram commutes in $\mathcal{A}$ if and only if its image in $\mathbf{Set}_*$ under $\mathcal{M}$ commutes.

Proof. A diagram commutes in $\mathcal{A}$ if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. □

Lemma 24. Let $f\colon A \to B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

4.3 Chain complexes

Definition 25 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex in $\mathcal{A}$ is…The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms of chain complexes, i.e.…

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes, $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Proposition 26. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the **Ab**-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd} & Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\ C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet \end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned} (d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q, \end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd} & Q_\bullet \arrow[dl, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\ \ker(f_\bullet) \arrow[r, "\iota_\bullet"] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \end{tikzcd}$$

That is, we must have
$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{4.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map
$$\psi \colon Q_n \to \ker f_{n-1}$$
such that
$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 4.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned} \iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n \end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

Definition 27 (exact sequence)**.** Given a morphism $f \colon A \to B$, the image $\operatorname{im} f = \ker \operatorname{coker} f$ and the coimage is $\operatorname{coker} \ker f$. From this data, we build the following commuting diagram.

$$\begin{tikzcd}
& \operatorname{im} f \arrow[d, "i"] \arrow[dr, "0"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\
& \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a morphism $\phi\colon \operatorname{im} f \to \ker g$. We say that the above is exact if ϕ is an isomorphism. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

Definition 28 (homology)**.** Let $(C_\bullet, d)$ be a complex. The nth homology of $C_\bullet$ is the cokernel of the map $\phi\colon \operatorname{im} d_{n+1} \to \ker d_n$ described in Definition 27, denoted $H_n(C_\bullet)$.

It is useful to have the following picture in mind.

$$\begin{tikzcd}
& \operatorname{im} f \arrow[r, "\phi"] \arrow[dr, "i"] & \ker g \arrow[r] \arrow[d] & H_n & \\
C_{n+1} \arrow[rr, "f"] & & C_n \arrow[rr, "g"] \arrow[d, "p"] & & C_{n-1} \\
& & \operatorname{coker} f \arrow[urr] & &
\end{tikzcd}$$

Proposition 29. Homology extends to a family of additive functors

$$H_n\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d^C_n \to \ker d^D_n$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. □

We can also go the other way, by defining a map

$$\iota_n \colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\cdots\to 0 \longrightarrow A \longrightarrow 0 \cdots\to$$

concentrated in degree n.

Example 30. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex which is exact except in degree zero. Then $H_0(C_\bullet) = \operatorname{coker} f$.

Similarly, if

$$0 \longrightarrow C_{-1} \xrightarrow{f} C_{-2} \longrightarrow \cdots$$

is a chain complex whose non-trivial homology is in degree 0, then $H_0(C_\bullet) = \ker f$.

Definition 31 (quasi-isomorphism)**.** Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Quasi-isomorphisms are a fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope quasi-isomorphisms themselves were invertible. Unfortunately, we are not so lucky. Consider the following

By Example 30, we

Definition 32 (resolution)**.** Let A be an object in an abelian category. A resolution of A consists of a chain complex $C_\bullet$ together with a quasi-isomorphism $C_\bullet \to \iota(A)$.

4.4 The snake lemma and its compatriots

Lemma 33 (Splitting lemma)**.** Consider the following solid exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[r, "\pi_C"] \arrow[l, dashed, bend left, "\pi_A"] & C \arrow[r] \arrow[l, dashed, bend left, "i_C"] & 0 \end{tikzcd}$$

The following are equivalent.

- There exists a morphism $\pi_A \colon B \to A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
- There exists a morphism $i_C \colon C \to B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
- B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

Lemma 34. Let $f \colon B \twoheadrightarrow C$ be an epimorphism, and let $g \colon D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{tikzcd} \ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\ \ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd} S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K also functions as the kernel of f'. Of course, in order for this statement to make sense we need a map $i \colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$
\begin{tikzcd}
K \arrow[dr, dashed, "\exists! i"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}
$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v\colon Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.

$$
\begin{tikzcd}
& Q \arrow[dl, dashed, "\exists! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}
$$

By definition $f' \circ v = 0$. Thus,

$$
\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}
$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$
\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}
$$

By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{elements of pullback}\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{matrix}\text{elements of } B\\ \text{which map to } C\end{matrix}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\begin{matrix}\text{elements of } B\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 35 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed, rounded corners] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd} \tag{4.2}$$

The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\operatorname{coker} g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 4.2 the following diagram.

$$\begin{tikzcd}
 & & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}$$

Take a pullback and a pushout, and using Lemma 34 (and its dual), we find the following.

$$\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}$$

Consider the map

$$\delta_0 = \quad S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \; .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \operatorname{coker} f \to \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 36 (snake lemma II)**.** Given the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

we get a exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

Corollary 37. Given an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta"] \\
H_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. For each n, we have a diagram of the following form.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d^A_{n+1}"'] & B_{n+1} \arrow[r] \arrow[d, "d^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma (Theorem 35) gives us, for each n, two exact sequences as follows.

$$0 \longrightarrow \ker d^A_{n+1} \longrightarrow \ker d^B_{n+1} \longrightarrow \ker d^C_{n+1}$$

and

$$\operatorname{coker} d_n^A \longrightarrow \operatorname{coker} d_n^B \longrightarrow \operatorname{coker} d_n^C \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{matrix} & \operatorname{coker} f_{n+1} & \operatorname{coker} g_{n+1} & \operatorname{coker} h_{n+1} & 0 \\ 0 & \ker f_n & \ker g_n & \ker h_n & \end{matrix}$$

Recall that for each n, we get a map

$$\phi\colon \operatorname{im} f_{n+1} \to \ker f_n$$

$\square$

Lemma 38. Let $\mathcal{A}$ be an abelian category, and let f be a morphism of short exact sequences as below.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d, "f'"] & A \arrow[r] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & B' \arrow[r] & B \arrow[r] & B'' \arrow[r] & 0 \end{tikzcd}$$

4.5 Exactness in abelian categories

Definition 39 (exact functor). Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F <u>left exact</u> if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact

- We call F <u>right exact</u> if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

is exact

- We call F exact if it is both left exact and right exact.

Example 40. We showed in Lemma 10 that both hom functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ are left exact.

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 41. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\operatorname{im} f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\operatorname{im}\iota = \ker g \subset \operatorname{im} f$. □

Proposition 42. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
   - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.
   - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.
3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

Any functor G which is a right adjoint preserves limits. By the above reasoning, G certainly preserves zero objects and kernels. Thus, G preserves the first two conditions, which means precisely that G is left exact.

Dually, any functor F which is a left adjoint preserves colimits, hence zero objects and cokernels. Thus, F preserves the last two conditions, which means that F is right exact. □

Note that the previous theorem makes no demand that F be additive, which may seem surprising; after all, by Definition 39, only exact funcotors are allowed the honor of being called additive. However, it turns out that a functor which preserves either finite limits or finite colimits is always additive. To see this, note that by applying any functor which is either left or right exact to a split exact sequence gives a split exact sequence.

This is often useful when trying to check the exactness of a functor which is a left or right adjoint, by the following proposition.

Lemma 43. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

Corollary 44. $0 \longrightarrow \mathrm{Hom}(X, A) \xrightarrow{f_*} \mathrm{Hom}(X, B) \xrightarrow{g_*} \mathrm{Hom}(X, C)$

is an exact sequence of abelian groups.

Proof. The hom functor □

4.6 Chain Homotopies

Definition 45 (chain homotopy)**.** Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A chain homotopy

Lemma 46. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 47. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 48. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

Definition 49 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

Lemma 50. The homotopy category $\mathcal{K}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

In fact, the homotopy category satisfies the following universal property.

Proposition 51. Let $F \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{C}$ be an additive functor such that $\mathcal{F}(f)$ is an isomorphism for all quasi-isomorphisms f. Then there exists a unique functor $\mathcal{K}(\mathcal{A}) \to \mathcal{C}$ making the following diagram commute.

Proposition 52. Let $\mathcal{F} \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$ be an additive functor which takes quasi-isomorphisms to isomorphisms. Then for homotopic morphisms $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, we have $F(f) = F(g)$.

4.7 Projectives and injectives

Definition 53 (projective, injective)**.** An object P in an abelian category is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

Since the hom functor $\operatorname{Hom}(A, -)$ is left exact for every A, it is very easy to see that an object P in an abelian category $\mathcal{A}$ is projective if for every epimorphism $f\colon B \to C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\tilde{p}"'] \arrow[d, "p"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g\colon A \to B$ and every morphism $q\colon A \to Q$ there exists a morphism $\tilde{q}\colon B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists\tilde{q}"] \\ Q & \end{tikzcd}$$

Definition 54 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Proposition 55. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, we get a morphism $P \to B$ making the triangle commute.

$$\begin{tikzcd} & & & P \arrow[dl, "\exists g"'] \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right.

The result follows from the splitting lemma (Lemma 33). □

Corollary 56. Let F be an additive functor, and let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be an exact sequence with P projective. Then F applied to the sequence gives an exact sequence.

Proof. Applying F to a split sequence gives a split sequence, and all split sequences are exact. □

Corollary 57. The category R-**Mod** has enough projectives.

Proof. By Proposition 15, free modules are projective, and every module is a quotient of a free module. □

Proposition 58. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Dually, any abelian category with enough injectives has injective resolutions.

Proof. We provide running commentary on the following diagram. □

Definition 59 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \to \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \to Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

Lemma 60. Let $f : P_\bullet \to \iota(M)$ be a projective resolution. Then $f : P_0 \to M$ is an epimorphism.

Proof. We know that $H_0(f) : H_0(P) \to H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(m)) \simeq \iota_M$, and that □

In fact, we can say more.

Lemma 61. Let $f_\bullet : P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots\cdots\; P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Theorem 62 (extended horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "\tilde{g}"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

Furthermore, given a morphism $M \to N$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd} & \vdots \arrow[d, dotted, no head] & & \vdots \arrow[d, dotted, no head] & \\ 0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd} 0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f \circ p'_0"'] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since by Lemma 60 p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P'_0 \oplus P''_0 \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p_0' \circ \pi_{P_0'} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P_1'' to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r] \arrow[d] & \ker p \arrow[r] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0 \arrow[r] & P_0'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Proposition 63. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P_\bullet' \to M'$$

be projective resolutions. Then for every morphism $f : M \to M'$ there exists a lift $\tilde{f} : P_\bullet \to P'_\bullet$ making the diagram

$$\begin{tikzcd} P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P'_\bullet \arrow[d] \\ \iota M \arrow[r, "\iota f"'] & \iota M' \end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Corollary 64. For any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}\mathcal{A})$.

4.8 Mapping cones

Definition 65 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the <u>k-shifted chain complex</u> $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 66 (mapping cone)**.** Let $f : C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\mathrm{cone}(f)$ as follows.

- For each n, define
$$\mathrm{cone}(f)_n = C_{n-1} \oplus D_n.$$
- Define $d_n^{\mathrm{cone}(f)}$ by
$$d_n^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$
  which is shorthand for
$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 67. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow C_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} D_\bullet \longrightarrow 0 \ .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 68. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is an exact complex.

Proof. By Corollary 37, we get a long exact sequence

$$\cdots \to H_n(X) \xrightarrow{\delta} H_n(Y) \to H_n(\operatorname{cone}(f)) \to H_{n-1}(X) \to \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see (although I'm missing a sign somewhere); picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0, f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\operatorname{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\operatorname{cone}(f))$ must be the zero maps, implying $H_n(\operatorname{cone}(f)) = 0$ by exactness. □

Proposition 69. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$. Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g \colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots\cdots \\
\cdots\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots\cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C

and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
 & & P_1 \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists!"] \arrow[dd, "r"] \arrow[ddr, "0"] & \\
 & K \arrow[dr, hook] & & \\
C_2 \arrow[ur, two heads] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\operatorname{cone}(f)$ is exact, which means that $\iota\colon P \to \operatorname{cone}(f)$ is homotopic to the zero morphism.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d_{i+1}^P"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d_i^P"] \arrow[dl, dashed, "\tilde{h}_i"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d_{i+1}^{\operatorname{cone}(f)}"'] & C_{i-1} \oplus P_i \arrow[r, "d_i^{\operatorname{cone}(f)}"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}$$

That is, there exist $\tilde{h}_i\colon P_i \to \operatorname{cone}(f)_{i+1}$ such that

$$d_{i+1}^{\operatorname{cone} f} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \tag{4.3}$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i \colon P_i \longrightarrow C_i \qquad \text{and} \qquad \gamma_i \colon P_{i-1} \longrightarrow P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet \colon P_\bullet \to C_\bullet$. Indeed, writing Equation 4.3 in components, we find the following.

$$\begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$
$$\begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that[1]

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

4.9 Localization at weak equivalences: a love story

We can embed any abelian category $\mathcal{A}$ into the corresponding category $\mathbf{Ch}(\mathcal{A})$ of chain complexes via inclusion into degree zero. This is obviously a lossless procedure, in the sense that we recover $\mathcal{A}$ by restricting to degree 0.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

[1] Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

Equivalently, we recover our category A by taking zeroth homology.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "H_0"] \end{tikzcd}$$

However, we notice that there are much better-behaved embeddings of $\mathcal{A}$ into $\mathbf{Ch}(\mathcal{A})$ such that we recover $\mathcal{A}$ when taking zeroth homology. For example, taking projective resolutions of objects and lifting morphisms using Proposition 63 will do the trick, although this is not in general well-defined; one can take many different projective resolutions, and lift morphisms in many different ways. There is no reason to expect these to assemble themselves functorially.

However, if we were to modify $\mathbf{Ch}(\mathcal{A})$ in such a way that all quasi-isomorphisms became bona-fide isomorphisms, then we would have this functoriality, thanks to Proposition 52.

Thus we find ourselves in a common situation: we have a category $\mathbf{Ch}(\mathcal{A})$, and identified a collection of morphisms inside this category which we would like to view as weak equivalences, namely the quasi-isomorphisms. In an ideal world, we would simply promote quasi-isomorphisms to bona fide isomorphisms. That is, we would like to form the localization

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}(\mathcal{A}) \twoheadrightarrow \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. In fact, this is universal in the sense that every other additive functor sending quasi-isomorphisms to isomorphisms factors through it. In particular, the functor

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}^{-1}\}]$$

factors through it.

But now we are in a really wonderful position, as long as $\mathcal{A}$ has enough projectives: we can (by REF we have already proved) find a projective resolution functor

$$P\colon \mathcal{A} \to \mathcal{K}^{+}(\mathcal{A}).$$

In fact, this functor is an equivalence of categories.

4.10 Homological δ-functors and derived functors

We have noticed that we can

Definition 70 (derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the left derived functor to be the cohomological δ-functor $\{L_n F\}$ defined by

  $$L_n F = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

  where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the right derived functor to be the cohomological δ-functor

  $$R^n F(X) = H^n \circ F \circ Q$$

  where Q is an injective resolution functor.

It may seem that we still have something left to prove; after all, we have not shown that the result of our composition is independant of the projective/injective resolution functor we used. However,

Because $\{H_n\}$ is a homological δ-functor, it is trivial that $L_n F$ is a homological delta-functor. Note that it really does extend F in the following sense.

Proposition 71. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0 F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 61, the following sequence is exact.

$$P_2 \longrightarrow P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_2) \longrightarrow F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But

$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0 F(X).$$

Similarly, on morphisms □

Definition 72 (homological δ-functor)**.** A homological δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

   $$T_n\colon \mathcal{A} \to \mathcal{B}.$$

2. For every short exact sequence

   $$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

   in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism

   $$\delta_n\colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.

2. For every short exact sequence as above, there is a long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}
$$

3. For every morphism of short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}
$$

and every n, the diagram

$$
\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}
$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

This seems like an inelegant definition, and indeed it is.

Definition 73 (cohomological δ-functors)**.** Dual to Definition 72.

Example 74. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors,* come from homology.

4.11 Double complexes

Definition 75 (double complex)**.** Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d_{i,j} \colon M_{i,j} \to M_{i-1,j}, \qquad \delta_{i,j} \colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $d^2 = 0$
- $\delta^2 = 0$
- $d\delta + \delta d = 0$

We say that $M_{\bullet,\bullet}$ is first-quadrant if $M_{i,j} = 0$ for $i, j < 0$.

That is, for M a double complex the following diagram anticommutes rather than commutes.

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} & M_{1,0} \arrow[l] & M_{2,0} \arrow[l] & M_{3,0} \arrow[l]
\end{tikzcd}$$

Given any complex of complexes

$$\Big(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \Big) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

one can construct a double complex by multiplying every other differential by -1.

Definition 76 (total complex)**.** Let $M_{\bullet,\bullet}$ be a first-quadrant[2] double complex.

[2] This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

The total complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

Lemma 77. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Theorem 78. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex, without loss of generality with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$\begin{array}{ccc}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} & = & \mathrm{Tot}(M)_3 \\
\downarrow & & \downarrow \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} & = & \mathrm{Tot}(M)_2 \\
\downarrow & & \downarrow \\
M_{1,0} \oplus M_{0,1} & = & \mathrm{Tot}(M)_1 \\
\downarrow & & \downarrow \\
M_{0,0} & = & \mathrm{Tot}(M)_0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

is exact.

The construction is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we construct the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \qquad d_{11}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}}n = m$.

We make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker\delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} =$

$m_{1,1} - dn_{2,1}$. Thus
$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$
But
$$d\delta n_{1,2} = -\delta d n_{1,2},$$
so
$$\delta(m_{0,2} - dn_{1,2}) = 0.$$
Now we repeat this process, finding $n_{1,2}$ such that
$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$
and $n_{0,3}$ such that
$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$
Then
$$\begin{aligned} d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d+\delta)n_{2,1} + (d+\delta)n_{1,2} + (d+\delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$
as required. □

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain
$$C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$
hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;
- As a morphism between double complexes, hence between totalizations
$$\mathrm{Tot}(\alpha)_\bullet \colon \mathrm{Tot}(C)_\bullet \to \mathrm{Tot}(D)_\bullet.$$

Lemma 79. These two points of view agree in the sense that
$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 80. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[d] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet
\end{tikzcd}$$

be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$

is exact. By Theorem 78, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 79 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 68) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 81. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now Theorem 80 gives us the result we want. □

4.12 Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let $\mathcal{A}$ be an abelian category and let

$$- \otimes - \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{H},$$

be an additive functor, where $\mathcal{H}$ is some abelian category equipped with an exact functor to **Ab**. Further suppose that

Definition 82 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$(C \otimes D)_\bullet = \mathrm{Tot}(C_\bullet \otimes D_\bullet),$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 83 (internal hom)**.** Let $\mathcal{A}$ be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We have

4.13 The Künneth formula

This section takes place in R-**Mod**, where R is a PID.

Lemma 84. Let $C_\bullet$ be a chain complex with trivial differential, and let $C'_\bullet$ be an arbitrary chain complex. Then there is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \equiv H_n(C_\bullet \otimes C'_\bullet).$$

5 Tor and Ext

5.1 The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by Proposition 42, is right exact. Thus, we may form the left derived functor.

Definition 85 (Tor functor)**.** The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 86. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free resolution of R/rR. Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i\left(\ 0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0\ \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & n = 0 \\ {}_rN, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

There is a worrying asymmetry to our definition of Tor. The tensor product is 'morally' symmetric; that is, it should not matter whether we derive the first factor or the second. It turns out that Tor respects this.

Proposition 87. The functor Tor is balanced; that is,

$$L_i \operatorname{Hom}(M, -)(N) \simeq L_i \operatorname{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 \arrow[d, dashed] & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
M \otimes_R N & P_0 \otimes_R N \arrow[l, dashed] & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

5.2 Ext and extensions

5.3 Group (co)homology

Given a group G, a G-module is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-$\mathbf{Mod}$.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in $\mathbf{Ab}$, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the equivalence $G\text{-}\mathbf{Mod} \simeq \mathbb{Z}G\text{-}\mathbf{Mod}$, we know that G-$\mathbf{Mod}$ has enough projectives.

Note that there is a canonical functor $\mathrm{triv}\colon \mathbf{Ab} \to \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 88. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g\cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \mathrm{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \mathrm{Hom}_{G\text{-}\mathbf{Mod}}(M, \operatorname{triv} A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \to A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \to A$ such that $\hat{\alpha}(g \cdot m - g) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \to A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{5.1}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \to \operatorname{triv} A$. In fact, G-linear maps $M \to \operatorname{triv} A$ are precisely those satisfying Equation 5.1, showing that this is really an isomorphism.

The other case is similar. □

Definition 89 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Note that we actually have a fairly good handle on invariants and coinvariants.

Lemma 90. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\text{-}\mathbf{Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -.$$

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

This tells us immediately that the invariants functor $(-)^G$ is left exact, and the coinvariant functor $(-)_G$ is right. We could have discovered this by explicit investigation, and formed the right and left derived hom functors. However, this would have required picking resolutions for each object under consideration, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[1]

Example 91. Let $T \cong \mathbb{Z}$ be an infinite cyclic group with a single generator t. Then

$$\mathbb{Z}T \cong \mathbb{Z}[t, t^{-1}],$$

and we have an exact sequence

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \longrightarrow \mathbb{Z} \longrightarrow 0$$

giving a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}T$-module.

In general, our life is not so easy. Fortunately, we still have, for any group G, a general construction of a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}G$-module, known as the *bar construction.*

Definition 92 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module

$$B_n = \bigoplus_{(g_i) \in (G \smallsetminus \{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$

where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to $(g_1, \ldots, g_n)$. The differential is defined by

$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

This is in fact a chain complex; the verification of this is identical to the verification of the simplicial identities. First, we note that we can write

$$d_n = d_0 + \sum_{i=1}^{n-1}(-1)^i d_i + (-1)^n d_n,$$

where the d_i satisfy the simplicial identities

$$d_i d_j = d_{j-1} d_i, \qquad i < j.$$

[1] In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

Thus, we have

$$\begin{aligned}
d_n \circ d_{n-1} &= \left(\sum_{j=0}^{n-1}(-1)^j d^j\right) \circ \left(\sum_{i=0}^{n}(-1)^i d^i\right) \\
&= \sum\sum(-1)^{i+j} d^j \circ d^i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d^j \circ d^i + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d^j \circ d^i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d^{i-1} \circ d^j + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d^j \circ d^i \\
&= 0.
\end{aligned}$$

Proposition 93. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$
\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
& B_n \arrow[dl, dashed, "h_n"'] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}
$$

gives us in one direction

$$
\begin{tikzcd}
g[g_1|\cdots|g_n] \arrow[r, mapsto] & \begin{array}{l} gg_1[g_2|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i g[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n g[g_1|\cdots|g_{n-1}] \end{array} \arrow[dl, mapsto] \\
\begin{array}{l} [gg_1|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i [gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n [gg_1|\cdots|g_n] \end{array} &
\end{tikzcd}
$$

and in the other

$$
\begin{tikzcd}
& g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\
{[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{l} g[g_1|\cdots|g_n] \\ -[gg_1|g_2|\cdots|g_n] \\ - \sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ -(-1)^n[gg_1|\cdots|g_{n-1}] \end{array}
\end{tikzcd}
.
$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. $\square$

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G, A) = H^i \left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G, A) = H_i \left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to$ **Ab**.
2. We noticed that picking out the constant functor gave us a canonical functor **Ab** $\to$ G-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
   - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.
   - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived functors $R^i(-)^G$.
4. We denote

$$L_i(-)_G = H_i(G, -), \qquad R^i(-)^G = H^i(G, -),$$

   and call them *group homology* and *group cohomology* respectively.
5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have

$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$

   and thus that

$$H_i(G, A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad H^i(G, A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$

This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 94. Let G be a group, and consider $\mathbb{Z}$ as a free $\mathbb{Z}G$-module. Let us compute $H_1(G, \mathbb{Z})$ and $H^1(G, \mathbb{Z})$.

To compute $H_1(G, \mathbb{Z})$, consider the sequence

$$\begin{tikzcd} \cdots \arrow[r, "d_2 \otimes \mathrm{id}_{\mathbb{Z}}"] & \bigoplus_{g \in G \smallsetminus \{1\}} [g] \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r, "d_1 \otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, equal] & {[\cdot]} \mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\ \cdots \arrow[r] & \bigoplus [g] \mathbb{Z} \arrow[r] & {[\cdot]} \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The differential $d_1 \otimes \mathrm{id}_{\mathbb{Z}}$ acts on $[g]$ by sending it to

$$[\cdot]g - [\cdot] = [\cdot] - [\cdot],$$

i.e. everything is a cycle. The differential d_2 sends

$$[g|h] \mapsto [g]h - [gh] + [h] = [g] + [h] - [gh],$$

i.e. the terms $[gh]$ and $[g] + [h]$ are identitfied. Thus, $H_1(G, \mathbb{Z})$ is simply the abelianization of G.

To compute $H^1(G, \mathbb{Z})$, consider the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot], \mathbb{Z}) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{g \in G \smallsetminus \{1\}} \mathbb{Z}G[g], \mathbb{Z}\right) \xrightarrow{(d_2)^*} \cdots .$$

Notice immediately that since the hom functor preserves direct sums in the first slot, we have

$$\begin{aligned} \mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i) \in (G \smallsetminus \{1\})^n} \mathbb{Z}G[g_1|\cdots|g_n], \mathbb{Z} \right) &\cong \bigoplus_{(g_i) \in (G \smallsetminus \{1\})^n} \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1|\cdots|g_n], \mathbb{Z}) \\ &\cong \bigoplus_{(g_i) \in (G \smallsetminus \{1\})^n} \mathbb{Z}[g_1|\cdots|g_n]. \end{aligned}$$

This notation deserves some explanation; recall that the symbols $[g_1|\cdots|g_n]$ are merely visual aids, reminding us where we are and how the differential acts. The differential

$$(d_n)^* \colon [g_1|\cdots|g_n].$$

In the low cases we have the sequence

6 Algebraic topology

6.1 Basic definitions and examples

We assume a basic knowledge of simplicial sets just to get the ball rolling.

Definition 95 (singular complex, homology)**.** Let X be a topological space. The singular chain complex $C_\bullet(X)$ is defined level-wise by

$$C_n(X) = \mathcal{F}(\mathbf{Sing}(X)_n), \qquad C_n = 0 \quad \text{for } n < 0,$$

where $\mathcal{F}$ denotes the free group functor, and with

$$d_n \colon C_n \to C_{n-1}; \qquad (\alpha \colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(C_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f : X \to Y$ gives a chain map $C(f) \colon C(X) \to C(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 96. Denote by pt the one-point topological space. Then $C(X)_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & C_3 \arrow[r, "d_3"'] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Example 97. Let X be a path-connected topological space. Then $H_0(X) = 0$. To see this, consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the equivalence relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

Theorem 98 (Hurewicz)**.** For any path-connected topological space X, there is an isomorphism

$$H_1(X) \cong \pi_1(X)_{\mathrm{ab}},$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization.

Example 99. We can now confidently say that

$$H_1(S^1) = \mathbb{Z}_{\mathrm{ab}} = \mathbb{Z}.$$

Proposition 100. Let $f, g \colon X \to Y$ be continuous maps between topological spaces, and let $H \colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 101. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 102. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Definition 103 (relative homology)**.** Let X be a topological space, and let $X \subset A$. The <u>relative chain complex</u> of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(C_\bullet(X, A)).$$

Note that essentially by definition, we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \longrightarrow 0 \ .$$

Example 104. Let $X = D^n$, the n-disk, and $A = S^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on homology coming from the above short exact seqence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(D^n, S^{n-1}) \arrow[dll, "\delta"] \\
H_j(S^{n-1}) \arrow[r] & H_j(D^n) \arrow[r] & H_j(D^n, S^{n-1}) \arrow[dll, "\delta"] \\
H_{j-1}(S^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that $H_j(D^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(D^n, S^{n-1}) \cong H_{j-1}(S^{n-1})$$

for $j > 1$ and $n \geq 1$.