Homological algebra notes

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1 Introduction

Homework:

- Must submit more than 50% correct solutions in order to take the final exam (oral).
- Posted on Monday after the lecture
- Submitted on following Monday in lecture
- Discuss following Tuesday, 16:45, room 430

2 Chain complexes

Homological algebra is about chain complexes.

Definition 1 (chain complex). Let R be a ring.¹ A <u>chain complex</u> of R-modules, denoted (C_{\bullet}, d) , consists of the following data:

- For each $n \in \mathbb{Z}$, an R-module C_n , and
- *R*-linear maps $d_n: C_n \to C_{n-1}$, called the *differentials*,

such that the differentials satisfy the relation

$$d_{n-1} \circ d_n = 0.$$

Example 2 (Syzygies). Let $R = \mathbb{C}[x_1, \dots, x_k]$. Following Hilbert, we consider a finitely generated R-module M. The ring R is Noetherian, so by Hilbert's basis theorem, M is Noetherian, hence has finitely many generators a_1, \dots, a_k .

Denote by R^k the free R-module with k generators, and consider

$$\phi: \mathbb{R}^k \twoheadrightarrow M; \qquad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $Syz^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^k \lambda_i e_i \stackrel{\phi}{\mapsto} \sum_{i=1}^k \lambda_i a_i = 0.$$

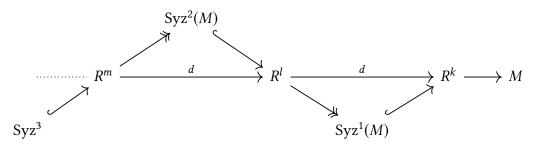
This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\operatorname{Syz}^1(M)$ is again a finitely generated R-module. Hence, pick finite set of generators b_i , $1 \le i \le \ell$, and look at

$$R^{\ell} Syz^1(M); \qquad e_i \mapsto b_i.$$

In general, there is non-trivial kernel, denoted $\operatorname{Syz}^2(M)$. This consists of relation between relations.

We can keep going. This gives us a sequence $\operatorname{Syz}^{\bullet}(M)$ corresponding to relations between relations between relations.



¹Associative, with unit. Not necessarily commutative.

²i.e. expressible as an *R*-linear combination of finitely many generators.

From this we build a chain complex (C_{\bullet}, d) , where C_n corresponds to the nth copy of \mathbb{R}^k , known as the *free resolution* of M.

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \ldots, x_n]$ -module admits a free resolution of length at most n.

Example 3 (simplicial complexes). Let $N \ge 0$. A collection $K \subset \mathcal{P}(\{0, ..., n\})$ of subsets is called a *simplicial complex* if it is closed under the taking of subsets in the sense that if for all $\sigma \in K$,

$$\tau \subset \sigma \implies \tau \in K$$
.

For every $n \ge 0$, we define the *n*-simplices of *K* to be the set

$$K_n = \{ \sigma \in K \mid |\sigma| = n+1 \}.$$

Every $\sigma \in K_n$ can be expressed uniquely as $\sigma = \{x_0, \dots, x_n\}$, with $x_0 < \dots < x_n$. Define the *ith face* of σ to be

$$\partial_i \sigma = \{x_0, \dots, x_{i-1}, x_{i+1}, \dots, x_n\}.$$

Let *R* be a ring. Define

$$C_n(K,R) = \bigoplus_{\sigma \in K_n} R_{e_\sigma},$$

where the subscripts are simply labels. Further, define

$$d: C_n(K,R) \to C_{n-1}(K,R); \qquad e_{\sigma} \mapsto \sum_{i=0}^n (-1)^i e_{\partial_i \sigma}.$$

This is a complex because...

Example 4. Let k be a field, and A an associative k-algebra. We define a complex of vector spaces

$$\cdots \xrightarrow{d} A^{\otimes n} \xrightarrow{d} \cdots \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} A \otimes A \xrightarrow{d} A$$

where *d* is defined by the formula

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

This complex is known as the *cyclic bar complex*.

Definition 5 (cycle, boundary, homology). Let (C_{\bullet}, d) be a chain complex. We make the following definitions.

- The space ker d_m is known as the space of n-cycles, and denoted $Z_n(C_{\bullet})$.
- The space im d_{m+1} is known as the space of n-boundaries, and denoted $B_n(C_{\bullet})$.
- The space

$$\frac{Z_n}{B_n} = H_n(C_{\bullet})$$

is known as the *n*th homology of *C*.

Definition 6 (exact). We say that a chain complex (C_{\bullet}, d) is exact at $n \in \mathbb{Z}$ if $H_n(C_{\bullet} = 0)$.

Definition 7 (exact sequence). We say that a sequence

$$C_k \xrightarrow{d} C_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} C_{\ell}$$

with $d^2 = 0$ is exact if it is exact at $k - 1, k - 2, ..., \ell + 1$.

Definition 8 (short exact sequence). A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0 .$$

Equivalently, such a sequence is exact if

- f is a monomorphism
- g is a epimorphism
- $H_B = 0$.

Example 9.

· The sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

• The sequence

$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is not exact since the map $x \mapsto 2x$ is not injective.

• The sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

3 Basic algebra

4 Abelian categories

4.1 Basics

In this chapter we provide results without proof.

A category is **Ab**-enriched if it is enriched over the symmetric monoidal category (**Ab**, \otimes_Z). In an **Ab**-enriched category with finite products, products agree with coproducts, and we call both direct sums. We call an **Ab**-enriched category with finite direct sums an *additive* category, and we call a functor between additive categories such that the maps

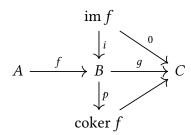
$$\operatorname{Hom}(A, B) \to \operatorname{Hom}(F(A), F(B))$$

are homomorphisms of abelian groups. A *pre-abelian* category is an additive category such that every morphism has a kernel and a cokernel, and an *abelian category* is a pre-abelian category such that every monic is the kernel of its cokernel, and every epic is the cokernel of its kernel.

This immediately implies the following results.

- · Kernels are monic and cokernels are epic
- Since the equalizer of f and g is the kernel of f g, abelian categories have all finite limits, and dually colimits.

Given a morphism $f: A \to B$, the image im $f = \ker \operatorname{coker} f$ and the coimage is coker $\ker f$. From this data, we build the following commuting diagram.

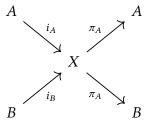


This defines a morphism $\phi \colon \operatorname{im} f \to \ker g$. We say that the above is *exact* if ϕ is an isomorphism.

Additive functors from additive categories preserve direct sums.

Lemma 10. Let A and B be objects in an additive category A, and let X be an object in A

equipped with morphisms



such that the following equations hold.

$$i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{id}_X$$

 $\pi_A \circ i_B = 0 = \pi_B \circ i_A$
 $\pi_A \circ i_A = \mathrm{id}_A$
 $\pi_B \circ i_B = \mathrm{id}_B$

Then the π s and is exhibit $X \simeq A \oplus B$.

Corollary 11. Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(a \oplus b) \simeq F(a) \oplus F(b)$$
.

4.2 Homological algebra in abelian categories

4.2.1 The snake lemma and its compatriots

Lemma 12. Consider the following exact sequence in an abelian category A.

Lemma 13. Let $f: B \rightarrow C$ be an epimorphism, and let $g: D \rightarrow C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\ker f' \xrightarrow{i} S \xrightarrow{f'} D$$

$$\parallel \qquad \qquad \qquad \qquad \qquad \downarrow^g$$

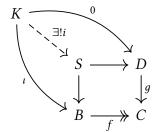
$$\ker f \xrightarrow{\iota} B \xrightarrow{f} C$$

Proof. Consider the following pullback square, where f is an epimorphism.

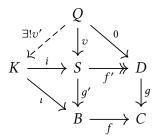
$$\begin{array}{ccc}
S & \xrightarrow{f'} & D \\
g' \downarrow & & \downarrow g \\
B & \xrightarrow{f} & C
\end{array}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K also functions as the kernel of f'. Of course, in order for this statement to make sense we need a map $i: K \to S$. This is given to us by the universal property of the pullback as follows.



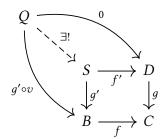
Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v: Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.



By definition $f' \circ v = 0$. Thus,

$$f \circ g' \circ v = g \circ f' \circ v$$
$$= g \circ 0$$
$$= 0.$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.



By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\ker f' \xrightarrow{i} S \xrightarrow{f'} D$$

$$\downarrow g' \qquad \downarrow g$$

$$\ker f \xrightarrow{\iota} B \xrightarrow{f} C$$

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{cases} \text{elements of pullback} \\ \text{which map to 0} \end{cases} \hookrightarrow \begin{cases} \text{elements of } B \\ \text{which map to } C \end{cases} \longrightarrow B$$

Theorem 14 (snake lemma). Consider the following commutative diagram with exact rows.

$$0 \longrightarrow A \xrightarrow{m} B \xrightarrow{e} C \longrightarrow 0$$

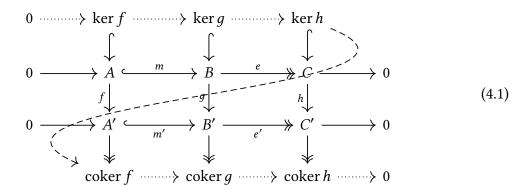
$$f \downarrow \qquad \qquad g \downarrow \qquad \qquad \downarrow h$$

$$0 \longrightarrow A' \xrightarrow{m'} B' \xrightarrow{e'} C' \longrightarrow 0$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

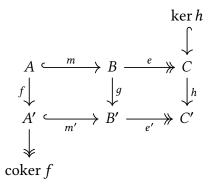
Proof. We provide running commentary on the diagram below.



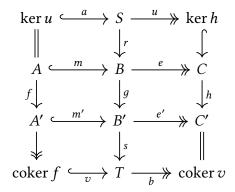
The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\operatorname{coker} g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual. I omit them.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at ker h and coker f.

Extract from the data of Diagram 4.1 the following diagram.



Take a pullback and a pushout, and using Lemma 13 (and its dual), we find the following.



Consider the map

$$\delta_0 = S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T$$
.

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1$$
: ker $h \to T$.

Composing this with b gives 0, hence we get a map

$$\delta$$
: coker $f \to \ker h$.

Theorem 15 (horseshoe lemma).

4.2.2 Projectives

Definition 16 (projective). An object P in an abelian category \mathcal{A} is said to be <u>projective</u> if for every exact sequence $B \to C \to 0$ and every morphism $p \colon P \to C$, there exists a morphism $\tilde{p} \colon P \to B$ such that the following diagram commutes.

Proposition 17. Let R be a ring. A module P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M$$
.

Proof. First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$0 \longrightarrow \ker \pi \hookrightarrow F \xrightarrow{\pi} P \longrightarrow 0$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker \pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$P \oplus M$$

$$\exists (j,k) \qquad \qquad \downarrow (f,g)$$

$$N \xrightarrow{\swarrow} M$$

In particular, this gives us the following solution to our lifting problem,



exhibiting *P* as projective.

4.2.3 Exactness in abelian categories

Lemma 18. Let A be an abelian category, and let

$$A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C$$

be objects and morphisms in A. If for all X the abelian groups and homomorphisms

$$\operatorname{Hom}_{\mathcal{A}}(X,A) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{A}}(X,B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{A}}(X,C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take X = A, giving the following sequence.

$$\operatorname{Hom}_{\mathcal{A}}(A,A) \xrightarrow{f_*} \operatorname{Hom}_{\mathcal{A}}(A,B) \xrightarrow{g_*} \operatorname{Hom}_{\mathcal{A}}(A,C)$$

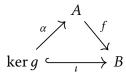
Exactness implies that

$$0 = (q_* \circ f_*)(id) = (q \circ f)(id) = q \circ f,$$

so im $f \subset \ker g$. Now take $X = \ker g$.

$$\operatorname{Hom}_{\mathcal{A}}(\ker g,A) \xrightarrow{\ f_{*} \ } \operatorname{Hom}_{\mathcal{A}}(\ker g,B) \xrightarrow{\ g_{*} \ } \operatorname{Hom}_{\mathcal{A}}(\ker g,C)$$

The canonical inclusion ι : $\ker g \to B$ is mapped to zero under g_* , hence is mapped to under f_* by some α : $\ker g \to A$. That is, we have the following commuting triangle.



Thus im $\iota = \ker g \subset \operatorname{im} f$.

Lemma 19. Let

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

$$0 \longrightarrow \operatorname{Hom}(X,A) \xrightarrow{f_*} \operatorname{Hom}(X,B) \xrightarrow{g_*} \operatorname{Hom}(X,C)$$

is an exact sequence of abelian groups.

Proof. To see that f_* is monic, suppose that $f_*\alpha = 0$ for some $\alpha: X \to A$. Then $f \circ \alpha = 0$, so $\alpha = 0$ by monomorphicity of f.

To see that im $f_* \subseteq \ker g_*$, let there be some β such that

$$g_*f_*\beta = \beta \circ g \circ f = \beta \circ 0 = 0.$$

Now let $\gamma: X \to B$ such that $g_*\gamma = g \circ \gamma = 0$. Then, because (A, f) functions as a kernel for g, we get a factorization

$$\gamma = f \circ \alpha = f_* \alpha$$

so ker $q_* \subseteq \text{im } f_*$.

Definition 20 (exact functor). Let $F: \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \stackrel{f}{\longrightarrow} B \stackrel{g}{\longrightarrow} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

• We call *F* left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

is exact

• We call *F* right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

is exact

• We call *F* exact if it is both left exact and right exact.

Example 21. By Lemma 19, the hom functor $\operatorname{Hom}(A, -) : A \to \operatorname{Ab}$ is left exact. It turns out that $\operatorname{Hom}(-, A)$ is also a left exact functor $A^{\operatorname{op}} \to \operatorname{Ab}$. It may seem that this should be right exact, but not so! There are two instances of contravariance which cancel each other: the contravariance of the hom functor itself, and the contravariance of...to be continued.

Theorem 22. Let

$$L: \mathcal{A} \longleftrightarrow \mathcal{B}: R$$

be an adjunction of additive functors between abelian categories. Then F