1 Some important derived functors

1.1 The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by Proposition **??** is right exact. Thus, we may form the left derived functor.

Definition 1 (Tor functor)**.** The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 2. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free (hence projective) resolution of R/rR. Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i\left(0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & i = 0 \\ {}_rN, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for any abelian group D, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, D) =_n D$, the n-torsion elements of D.

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

Proposition 3. The functor Tor is *balanced;* that is,

$$L_i \operatorname{Hom}(M, -)(N) \simeq L_i \operatorname{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
 & P_0 \otimes_R N & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

Example 4. Let N be an R-module. The Tor functor controls flatness in the sense that the following are equivalent:

1. The module N is flat
2. For all R-modules M, we have $\mathrm{Tor}_1^R(M,N) = 0$.

To see that 1. $\Rightarrow$ 2., suppose that N is flat and pick a projective resolution $P_\bullet \xrightarrow{\simeq} M$. Then

$$\begin{aligned}\mathrm{Tor}_1(M,N) &= H_1(P_\bullet \otimes_R N)\\ &= H_1(P_\bullet) \otimes_R N\\ &= 0.\end{aligned}$$

Conversely, suppose that for all M, $\mathrm{Tor}_1(M,N) = 0$. Then for any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0\ ,$$

we have a long exact sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow \mathrm{Tor}_1(A,N) \longrightarrow \cdots\ .$$

But $\mathrm{Tor}_1(A,N) = 0$, so the sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \twoheadrightarrow C \otimes_R N \longrightarrow 0\ .$$

is short exact. Thus, $- \otimes_R N$ is exact, which means by definition that N is flat.

1.2 The Ext functor

We know that $\mathrm{Hom}(N,-)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 5 (Ext functor)**.** Let R be a ring, and $N \in R\text{-}\mathbf{Mod}$ an R-module. The right derived functors

$$R^i\,\mathrm{Hom}(N,-)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i\,\mathrm{Hom}(N,-) = \mathrm{Ext}_R^i(N,-).$$

An argument very similar to that given for Tor in Proposition 3 shows that Ext is balanced; that is, when computing $\mathrm{Ext}_R^i(M,N)$, it doesn't matter whether we take an projective resolution of N or a injectiv resolution of M; we will get the same result either way. Rather than spelling this out in detail, we will wait for Chapter **??**, where a spectral sequences argument gives the result almost immediately.

Example 6. Consider $R = \mathbb{Z}$, so R-**Mod** = **Ab**. Let us calculate

$$\operatorname{Ext}^*(\mathbb{Z}/(p), \mathbb{Z}).$$

We can take either an injective resolution of $\mathbb{Z}$ or a projective resolution of $\mathbb{Z}/(p)$. Each $\mathbb{Z}/(p)$ has a canonical projective resolution

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{\cdot p} \mathbb{Z} \longrightarrow\!\!\!\!\rightarrow \mathbb{Z}/(p) \longrightarrow 0\ ,$$

which we might as well take advantage of. Doing so, we find

$$\operatorname{Ext}^* = H^*\left(0 \longrightarrow \overset{0}{\operatorname{Hom}(\mathbb{Z}, \mathbb{Z})} \cong \mathbb{Z} \xrightarrow{\cdot p} \overset{1}{\operatorname{Hom}(\mathbb{Z}, \mathbb{Z})} \cong \mathbb{Z} \longrightarrow 0 \right).$$

We can immediately calculate

$$\operatorname{Ext}^0(\mathbb{Z}/(p), \mathbb{Z}) \cong \ker \cdot p \cong 0, \qquad \operatorname{Ext}^1(\mathbb{Z}/(p), \mathbb{Z}) \cong \operatorname{coker} \cdot p \cong \mathbb{Z}/(p),$$

and $\operatorname{Ext}^i(\mathbb{Z}/(p), \mathbb{Z}) = 0$ for $i > 1$.

Note that balancedness immediately

1.2.1 Ext and extensions

There turns out to be a deep connection between the Ext groups and extensions.

Definition 7 (extension)**.** Let A and B be R-modules. An extension of A by B is a short exact sequence

$$\xi: \qquad 0 \longrightarrow B \hookrightarrow X \longrightarrow\!\!\!\!\rightarrow A \longrightarrow 0\ .$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd}
\xi: & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\mathrm{id}_B"] & X \arrow[r, two heads] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
\xi': & 0 \arrow[r] & B \arrow[r, hook] & X' \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}.$$

The five lemma (Theorem **??**) implies that φ is an isomorphism.

Let ξ be an extension of A by B. Applying the Ext functor we find an associated long exact sequence on cohomology.

$$
\begin{tikzcd}
\mathrm{Ext}^1(A,B) \arrow[r] & \mathrm{Ext}^1(X,B) \arrow[r] & \cdots \\
\mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[ull, "\delta"]
\end{tikzcd}
$$

Definition 8 (obstruction class)**.** The image of id_B under the connecting homomorphism $\delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A,B)$ is known as the obstruction class of ξ. For any extension ξ, we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 9. An extension ξ is split if and only if the correspoding obstruction class is zero.

Proof. We are given an extension

$$
\xi: \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xrightarrow{g} A \longrightarrow 0\ .
$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$
\mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(B,B) \longrightarrow \mathrm{Ext}^1(A,B)
$$

If $\delta(\mathrm{id}_B) = 0$, then there exists $a \in \mathrm{Hom}(X,B)$ such that $a \circ f = \mathrm{id}$. Thus, by the splitting lemma (Lemma **??**), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a\colon A \to X$ such that $a \circ f = \mathrm{id}_B$.

$$
\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[l, bend left, "a"] \arrow[r] & 0\ .
\end{tikzcd}
$$

Since $\mathrm{Ext}^1(-,B)$ is additive, the sequence

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(A,B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(X,B) \arrow[l, bend left, "\mathrm{Ext}^1(a,B)"] \arrow[r, two heads, "\mathrm{Ext}^1(f,B)"] & \mathrm{Ext}^1(B,B) \arrow[r] & 0
\end{tikzcd}
$$

is split exact. Thus, $\mathrm{Ext}^1(g,B)$ is a monomorphism, so it has trivial kernel.

$$
\begin{tikzcd}
\mathrm{Ext}^1(A,B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \cdots \\
\mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[ull, "\delta"]
\end{tikzcd}
$$

Exactness then forces $\delta = 0$. □

Proposition 9 explains the name *obstruction class:* $\Theta(\xi)$ is the obstruction to the splitness of ξ. The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\mathrm{Ext}^1(A, B)$ parametrizes extensions of A by B.

Theorem 10. There is a natural bijection

$$\left\{\begin{smallmatrix}\text{Extension classes of}\\ \text{extensions of } A \text{ by } B\end{smallmatrix}\right\} \xrightarrow{\cong} \mathrm{Ext}^1(A, B); \qquad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A, B) \\
\mathrm{Hom}(X, B) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{Hom}(B, B) \arrow[r, "\delta"'] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1(A, B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ. Since R-**Mod** has enough projectives, we can find a projective P which surjects onto A. Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] & P \arrow[r, two heads, "\gamma"] & A \arrow[r] & 0
\end{tikzcd} \tag{1.1}$$

We construct from such an exact sequence a map which takes an element of $\mathrm{Ext}^1(A, B)$ and gives an extension of A by B. First applying $\mathrm{Ext}^*(-, B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\mathrm{Hom}(K, B) \xrightarrow{\delta} \mathrm{Ext}^1(A, B) \longrightarrow \mathrm{Ext}^1(P, B)$$

Since P is projective, the group $\mathrm{Ext}^1(P, B)$ vanishes, so $\delta\colon \mathrm{Hom}(K, B) \to \mathrm{Ext}^1(A, B)$ is surjective. Thus, given any $e \in \mathrm{Ext}^1(A, B)$, we can find a (not unique!) lift to $\phi \in \mathrm{Hom}(K, B)$. Take the pushout in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] & 0 \\
& B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P & &
\end{tikzcd}$$

By the dual to Lemma ??, the cokernel of i_B is equal to the cokernel of ψ, so we get an induced short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

It remains to show that Ψ as defined is independent of the choice of ϕ, and that it really is an inverse to Φ. To this end, pick a different $\tilde{\phi} \in \mathrm{Hom}(N, B)$ such that $\delta(\tilde{\phi}) = e$.

This gives us a new morphism of exact sequences as follows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\tilde{\phi}"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "\tilde{i}_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\tilde{\phi}} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd} \tag{1.2}$$

Because of the way we chose $\tilde{\phi}$, there exists some $\rho \in \mathrm{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$. Consider the following outer square.

$$\begin{tikzcd}
K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[d, "i_P"] \arrow[ddr, bend left, "\tilde{i}_P - \tilde{i}_B \circ \rho"] & \\
B \arrow[r, hook, "i_B"] \arrow[drr, bend right, "\tilde{i}_B"'] & B \amalg_N^{\phi} P \arrow[dr, dashed, "\exists! \alpha"] & \\
& & B \amalg_N^{\tilde{\phi}} P
\end{tikzcd} \tag{1.3}$$

Using the commutativity of the left-hand square in Diagram 1.2, we find

$$\begin{aligned}
(\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
&= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
&= \tilde{i}_B \circ \phi,
\end{aligned}$$

so this outer square commutes, giving us the unique morphism α.

The claim that our two extensions are equivalent unwraps to the claim that the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook, "i_B"] \arrow[d, equal] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes. The left-hand square is the lower commuting triangle in Diagram 1.3.

Reminding ourselves of the proof of Lemma **??**, we recall that the morphism π is defined by a pushout: it is the unique map $P \amalg_K^\phi P \to A$ such that $\pi \circ i_B = 0$ and $\pi \circ i_P = \gamma$. If we can show that $\tilde{\pi} \circ \alpha$ also satisfies these equations, then we can content ourselves that the diagram commutes. Indeed, we have

$$\begin{aligned}(\tilde{\pi} \circ \alpha) \circ i_B &= \tilde{\pi} \circ \tilde{i}_B \\ &= 0\end{aligned}$$

and

$$\begin{aligned}(\tilde{\pi} \circ \alpha) \circ i_P &= \tilde{\pi} \circ (\tilde{i}_P - \tilde{i}_B \circ \rho) \\ &= \gamma .\end{aligned}$$

Our last job, which was really the whole point of the endeavor, is to check that the maps Φ and Ψ are inverse to each other. To that end, fix some A and B, and pick $e \in \mathrm{Ext}^1(A, B)$. By construction of Ψ, we get a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^\phi P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \tag{1.4}$$

where ϕ is an element of $\mathrm{Hom}(K, A)$ which is mapped to by e under δ. The next step in computing $\Phi(\Psi(e))$ would now be to stick $\Psi(e)$ into the hom functor $\mathrm{Hom}(-, B)$, and look at the image of id_B under the connecting homomorphism. However, using the fact that Ext is a homological δ-functor, we can stick the whole morphism of exact sequences Equation 1.4 in and get a morphism of long exact sequences out. Picking out the square containing the relevant boundary map, we find the following.

$$\begin{tikzcd} \mathrm{Hom}(B, B) \arrow[r, "\delta"] \arrow[d, "\mathrm{Hom}(\phi, B)"'] & \mathrm{Ext}^1(A, B) \arrow[d, equal] \\ \mathrm{Hom}(K, B) \arrow[r, "\delta'"] & \mathrm{Ext}^1(A, B) \end{tikzcd}$$

Chasing the identity around in both directions tells us that

$$\delta(\mathrm{id}_B) = \delta'(\phi) = e.$$

Next, we start with an extension

$$\xi : \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xrightarrow{g} A \longrightarrow 0 .$$

Using the connecting homomorphism, we get a representative for the obstruction class $e = \delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A, B)$. Using our short exact sequence from Equation 1.1, we can find the fo □

Example 11. Let's do a sort of capstone computation, which pulls in just about everything we've done so far.

Consider the ring $R = \mathbb{Z}[x]$. We will compute the obstruction class of the extension

$$\xi: \qquad 0 \longrightarrow \mathbb{Z} \xhookrightarrow{\cdot x} \mathbb{Z}[x]/(x^2) \twoheadrightarrow \mathbb{Z} \longrightarrow 0\ ,$$

where we have written $\mathbb{Z} = \mathbb{Z}[x]/(x)$.

At a very high level of abstraction, we compute the obstruction class of ξ by taking the long exact sequence associated to it under the right derived functor of $\mathrm{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z})$, namely

$$\begin{tikzcd}
\mathrm{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) \arrow[r] & \cdots & \\
\mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]/(x^2),\mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z},\mathbb{Z}) \arrow[ull, "\delta"]
\end{tikzcd}$$

and finding the image of the identity $\mathrm{id}_{\mathbb{Z}}$ under δ.

In order to find this long exact sequence, we need to lift ξ to a short exact sequence of projective resolutions. If we can find a projective resolution of $\mathbb{Z}$, the horseshoe lemma will do the rest of the work for us.

Our work on syzygies will help us here. We note that $\mathbb{Z}$ has a single generator, namely 1, so we begin our resolution

$$\mathbb{Z}[x] \xrightarrow{\pi} \mathbb{Z}\ .$$

The kernel of this is the ideal (x).

$$\begin{tikzcd}
(x) \arrow[dr, hook] & & \\
& \mathbb{Z}[x] \arrow[r, "\pi"] & \mathbb{Z}
\end{tikzcd}\ .$$

This again has a single generator, namely x.

$$\begin{tikzcd}
& (x) \arrow[dr, hook] & & \\
\mathbb{Z}[x] \arrow[ur, two heads, "\cdot x"] \arrow[rr, "\cdot x"] & & \mathbb{Z}[x] \arrow[r, "\pi"] & \mathbb{Z}
\end{tikzcd}\ .$$

The composition gives us our next differential. Note that multiplication by x is injective, so our projective resolution terminates.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}[x] \arrow[r, "\cdot x"] & \mathbb{Z}[x] \arrow[r] & \cdots
\end{tikzcd}
.
$$

Now we apply the horseshoe lemma. We first build our horseshoe.

$$
\begin{tikzcd}
 & \mathbb{Z}[x] \arrow[d] & & \mathbb{Z}[x] \arrow[d] & \\
 & \mathbb{Z}[x] \arrow[d] & & \mathbb{Z}[x] \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\cdot x"] & \mathbb{Z}[x]/(x^2) \arrow[r, two heads, "\pi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

The center column is then given by the direct sum of the outer columns, and the horizontal maps are the canonical injection and projection respectively.

$$
\begin{tikzcd}
 & \mathbb{Z}[x] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(3)"] & \mathbb{Z}[x]^2 \arrow[r, two heads] \arrow[d, "\alpha"] \arrow[dr, phantom, "(4)"] & \mathbb{Z}[x] \arrow[d] & \\
 & \mathbb{Z}[x] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(1)"] & \mathbb{Z}[x]^2 \arrow[r, two heads] \arrow[d, "\beta"] \arrow[dr, phantom, "(2)"] & \mathbb{Z}[x] \arrow[d] & \\
0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\cdot x"] & \mathbb{Z}[x]/(x^2) \arrow[r, two heads, "\pi"] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}
$$

The horseshoe lemma guarantees the existence of maps α and β above such that all the squares commute and the middle column forms a projective resolution, but being assured of existence is of no help to us; we must find them explicitly. We are helped tremendously that what we have found are free resolutions, so we can write the maps as matrices. We first figure out what we can about the morphisms α and β from the commutativity of the squares labelled (1)-(4).

(1) Taking the low road maps
$$1 \mapsto 1 \mapsto x,$$
while taking the high road maps
$$1 \mapsto (1,0) \mapsto \beta(1,0).$$
Thus,
$$\beta\colon (1,0) \mapsto x$$

(2) The high road maps
$$(0,1) \mapsto 1 \mapsto 1,$$

and the low road maps

$$(0,1) \mapsto \beta(0,1) \mapsto \beta(0,1).$$

Thus,

$$\beta\colon (0,1) \mapsto 1.$$

Already, this is enough to tell us that

$$\beta(a,b) = ax + b.$$

We make the ansatz

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(3) Taking the high road maps

$$1 \mapsto (1,0) \mapsto (a,c),$$

while taking the low road maps

$$1 \mapsto x \mapsto (x,0),$$

so $a = x$ and $c = 0$.

(4) Taking the high road maps

$$(0,1) \mapsto 1 \mapsto x,$$

and taking the low road maps

$$(0,1) \mapsto (b,d) \mapsto d$$

so we must have $d = x$.

Thus, we have

$$\alpha = \begin{pmatrix} a & b \\ 0 & x \end{pmatrix},$$

where we have not yet fixed b. However, we have not yet used that the middle column must be a chain complex, i.e. that $\beta \circ \alpha$ must be equal to zero. Multiplying out, we have

$$\beta \circ \alpha = \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} x & b \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x + bx \end{pmatrix}.$$

Thus, we must take $b = -1$, fixing

$$\alpha = \begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z}[x])$, we find the following diagram.

At last, we can calculate $\Theta(\xi)$ as the image of $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}[x])$ under the connecting homomorphism.

1.2.2 The Baer sum

By definition, $\mathrm{Ext}^1(A, B)$ has an abelian group structure. We have provided a bijection between equivalence classes of extensions of B by A and $\mathrm{Ext}^1(A, B)$, which allows us to transport our addition law from $\mathrm{Ext}^1(A, B)$ to the set of extension classes. However, it is not at all clear how to interpret the sum of two classes of extensions.

The group operation on the set of extension classes is known as the *Baer sum.* Let

$$\xi : \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] & 0 \end{tikzcd}$$

and

$$\xi' : \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f'"] & X' \arrow[r, two heads, "g'"] & A \arrow[r] & 0 \end{tikzcd}$$

be extensions. We shall form a new extension from ξ and ξ' called the *Baer sum,* which will turn out to be in the class of $\Psi(\Phi(\xi + \xi'))$.

A word of apology: things are about to get very element-theoretic. Of course, this is not a problem thanks to the Freyd-Mitchell embedding theorem, but it's still a bit sad.

From the above extensions above, take the following pushout, and form $X'' = \ker \alpha$.

$$\begin{tikzcd} B \arrow[rr, "f'"] \arrow[dr, "\exists! f''"] \arrow[dd, "f"'] & & X' \arrow[dd] \arrow[dddr, bend left, "-g'"] & \\ & X'' \arrow[dr, hook] & & \\ X \arrow[rr] \arrow[drrr, bend right, "g"'] & & X \amalg_B X' \arrow[dr, dashed, "\exists! \alpha"] & \\ & & & A \end{tikzcd}$$

We have immediately the following morphism.

$$\begin{tikzcd} 0 & B \arrow[r, "f''"] & X'' & A & 0 \end{tikzcd}$$

We would like to fill in morphisms so that we have an exact sequence. The obvious choice of morphism $X'' \to A$, namely the composition $\alpha \circ \ker \alpha$, won't work because it's zero, so we have do define something else.

Warning: brutal element-wise computation ahead.

We have an explicit description of X'' as

$$X \amalg_B X' = X \oplus X' / \langle f(b) = f'(b) \rangle.$$

The kernel X'' is therefore the part of this which goes to zero under α, namely $(x, x') \in X \amalg_B X'$ such that

$$\alpha(x, x') = g(x) - g'(x') = 0,$$

i.e.

$$X'' = \{(x, x') \in X \amalg_B X' \mid g(x) = g'(x')\}.$$

The map f'' then simply sends $b \mapsto (f(b), 0)$, which by definition is equivalent to $(0, f'(b))$.

Note that f' is clearly injective.

Next, we define a map $g'' \colon X'' \to A$ by sending

$$(x, x') \mapsto g(x) = g'(x').$$

This is clearly surjective since g and g' are. The kernel of g'' consists precisely of those (x, x') such that $g(x) = 0$, which is to say, such that $x = f(b)$ by exactness of ξ. Thus, we have constructed a bona fide short exact sequence

$$\xi'' : \qquad 0 \longrightarrow B \xhookrightarrow{f''} X'' \xrightarrow{g''} A \longrightarrow 0 \ . \tag{1.5}$$

Definition 12 (Baer sum)**.** The extension ξ'' from Equation 1.5 is called the Baer sum of ξ and ξ'.

Lemma 13. The Baer sum is compatible with equivalences; that is, if $\xi \sim \zeta$, then $\xi + \xi' \sim \zeta + \xi'$.

Proof. Note that equivalence implies in particular □

This means that the Baer sum descends to an addition law on equivalence classes.

Lemma 14. Let

$$\eta : \qquad 0 \longrightarrow B \xhookrightarrow{i} B \oplus A \xrightarrow{p} A \longrightarrow 0$$

be a split extension. Then for any extension ξ, $\eta + \xi \sim \xi$.

Proof. □

Lemma 15. Let

$$\xi: \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xrightarrow{g} A \longrightarrow 0 \ .$$

be an extension. Then

$$-\xi: \qquad 0 \longrightarrow B \xhookrightarrow{-f} X \xrightarrow{g} A \longrightarrow 0$$

is the inverse under the Baer sum, i.e. $\xi + (-\xi) \sim \eta$.

1.2.3 Higher extensions

We can also interpret higher Ext groups in terms of extensions. Let

$$\xi: \qquad 0 \longrightarrow B \hookrightarrow X_0 \longrightarrow \cdots \longrightarrow X_{n-1} \twoheadrightarrow A \longrightarrow 0$$

be an extension. We define the obstruction class $\Theta(\xi)$ as follows: first, find a projective resolution $P_\bullet \to A$, and a lift of the identity id_A as follows.

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "f"] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[d, equal] \\
0 \arrow[r] & B \arrow[r] & X_0 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] & A
\end{tikzcd}$$

Now apply $\mathrm{Hom}(-, B)$, and take nth homology. The morphism $f \in \mathrm{Hom}(P_n, B)$ descends to an element of $\mathrm{Ext}^n(A, B)$ (since we can write it as $f = f^* \mathrm{id}_B$, implying that it is a cycle). This is what we call $\Theta(\xi)$.

1.3 Inverse and direct limits

1.3.1 Direct limits

Definition 16 (filtered poset)**.** A poset $(I, \leq)$ is called filtered if every finite subset of I has an upper bound; that is, thinking about a poset as a category, if for every $i, j \in I$, there exists $k \in I$ and arrows $i \to k$, $j \to k$.

$$\begin{tikzcd}
i \arrow[dr] & \\
& k \\
j \arrow[ur] &
\end{tikzcd}$$

Example 17. Let X be a set, and consider the poset of finite subsets $\mathcal{P}_{\text{fin}}(X)$. This is a filtered poset since every finite collection $X_i \in \mathcal{P}_{\text{fin}}(X)$ is contained in the set $\bigcup_i X_i$, which is again finite.

Let I be a filtered poset, and let $F\colon I \longrightarrow R$-**Mod** be a functor. Using the description of a colimit as a coequalizer, we can calculate that

$$\operatorname{colim} F \cong \left(\coprod_{i \in I} F(i) \right) /E,$$

where E is the subgroup generated by elements of the form $x - x'$, where $x \in F(i)$ and $x' \in F(i')$, such that there exists $j \in I$, $y \in F(j)$, and maps $u\colon i \longrightarrow j$ and $u'\colon i' \longrightarrow j$ such that $F(u)(x) = y$ and $F(u')(x') = y$.

$$\begin{tikzcd}
F(i) \arrow[dr, "F(u)"] & \\
& F(j) \\
F(i') \arrow[ur, "F(u')"'] &
\end{tikzcd}, \qquad
\begin{tikzcd}
x \arrow[dr, mapsto] & \\
& y \\
x' \arrow[ur, mapsto] &
\end{tikzcd}$$

That is, two elements of $\coprod_{i \in I} F(i)$ are identified if they 'eventually' become equal.

Lemma 18. Let I be a filtered poset, and let $A\colon I \longrightarrow R$-**Mod** be a functor.

1. Every element $a \in$ colim is the image of some $a_i \in A_i$ (for some $i \in I$) under the canonical map $A_i \longrightarrow \operatorname{colim} A$.
2. For every i, the kernel of the canoncal map $A_i \longrightarrow \operatorname{colim} A$ is the union of the kernels of the maps $A(\phi)\colon A_i \longrightarrow A_j$ (for $\phi\colon i \longrightarrow j$ a map in I).

Proof.

1. Denote the canonical map $A_i \longrightarrow \operatorname{colim} F$ by λ_i. A priori, we are only guaranteed that $a \in \operatorname{colim} A$ can be written as an equivalence class

$$\left[\sum_{i \in J} \lambda_i(a_i) \right], \qquad a_i \in A_i,$$

for some finite subset $J \subset I$. Since $J = \{i_1, \ldots, i_n\}$ is finite, we can find an upper bound $j \geq i_\ell$ for all ℓ. In the quotient, we are justified in replacing $\lambda_i(a_i)$ by $\lambda_j(\hat{a}_i)$, where $\hat{a}_i$ is the image of a_i in A_j. Thus, we can write our element a as

$$a = \lambda_j \left(\sum_{i \in I} \hat{a}_i \right),$$

where the sum is over elements of A_j, and hence itself belongs to A_j.

2.

$\square$

Theorem 19. Let I be a filtered poset. The colimit functor

$$\operatorname{colim}\colon R\text{-}\mathbf{Mod}^I \to R\text{-}\mathbf{Mod}$$

is exact.

Proof. By general abstract nonsense, we know that colimits commute with colimits, and thus that the colimit functor preserves colimits, hence is right exact. We need only show that colim preserves monics.

To this end, let $t \in \operatorname{Hom}_{R\text{-}\mathbf{Mod}^I}(A, B)$ be monic, and choose some $a \in \operatorname{colim} A$ such that $t(a) = 0$ in $\operatorname{colim} B$. By Lemma 18, there exists some $a_i \in A_i$ which maps to a under the canonical map $A_i \to \operatorname{colim} A$. Since each $t_i\colon A_i \to B_i$ is monic this maps to some $t_i(a_i) \neq 0$ in B_i. The element $t_i(a_i)$ vanishes in $\operatorname{colim} B$, so there exists some $j \in I$ and $\phi\colon i \to j$ such that

$$0 = B(\phi)(t_i(a_i)) = t_j(A(\phi)(a_i)) \qquad \text{in } B_j.$$

However, t_j is monic, so $A(\phi)(a_i) = 0$ in A_j. Thus a maps to zero in $\operatorname{colim} A$. $\square$

Corollary 20. The functor

$$\operatorname{colim}\colon \mathbf{Ch}(R\text{-}\mathbf{Mod})^I \to \mathbf{Ch}(R\text{-}\mathbf{Mod})$$

is exact.

Proof. Colimits are computed level-wise. $\square$

Since direct limits are exact, there is no need to take derived functors. However, the situation for limits is not so pretty.

1.3.2 Inverse limits

Let I be a small category, and let $\mathcal{A}$ be an abelian category such that for every set-indexed collection $\{A_s\}_{s\in S}$, the product $\prod_{s\in S} A_s$ exists. Then $\mathcal{A}^I$ is an abelian category (which we have seen in REF) and has enough injectives (which we will not show unless we have time).

The limit functor

$$\lim\colon \mathcal{A}^I \to \mathcal{A}$$

commutes with limits, hence is left exact. We can form the right derived functors

$$\lim^i := R^i \lim\colon A^I \to I.$$

For us it will be sufficient to consider the case in which $I = \mathbb{N}$ and $\mathcal{A} = \mathbf{Ab}$, so a functor $I \to \mathcal{A}$ is a sequence of abelian groups and morphisms as follows.

$$\cdots \to A_2 \to A_1 \to A_0$$

We will call functors $\mathbb{N} \to \mathbf{Ab}$ *towers,* and denote $A\colon \mathbb{N} \to \mathbf{Ab}$ by $\{A_i\}$.

Lemma 21. Let $\{A_i\}$ be a tower in $\mathbf{Ab}$, and denote $A(j \to i) = \lambda_{j,i}$. We can express the limit $\lim_i A_i$ as a kernel

$$\ker \Delta \hookrightarrow \prod_{i=0} A_i \xrightarrow{\Delta} \prod_{i=0} A_i$$

where Δ is the map

$$(\ldots, a_n, \ldots) \mapsto (\ldots, a_n - \lambda_{n+1,n} a_{n+1}, \ldots).$$

Proof. We can express any limit as a kernel

$$\ker \xi \hookrightarrow \prod_{a \in \mathbb{N}} A_a \xrightarrow{\Delta} \prod_{(m,n) \in D} A_m$$

where ξ is defined on components by

$$\xi\colon \prod_{a \in \mathbb{N}} A_a \to A_{\mathrm{cod}(m,n)=n}; \qquad (\cdots, a_n, \cdots) \mapsto a_m - \lambda_{n,m} a_n$$

where $D = \{(n,m) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ indexes morphisms in $\mathbb{N}$. An element

$$a = (\ldots, a_n, \ldots, a_1, a_0)$$

is in the kenrnel of $\hat{\xi}$ if and only if

$$a_m - \lambda_{n,m}(a_n) = 0$$

for all $(m,n) \in D$. But because $\lambda_{k,j} \circ \lambda_{j,i} = \lambda_{k,i}$, this is true if and only

$$a_m - \lambda_{m+1,m}(a_{m+1}) = 0$$

for all m. But we can view this as a function from $\prod_{a \in \mathbb{N}} A_a$ to itself, giving us the required formula. □

Let

$$0 \longrightarrow \{A_i\} \hookrightarrow \{B_i\} \twoheadrightarrow \{C_i\} \longrightarrow 0$$

be a short exact sequence of towers, and consider the following morphism of short exact sequences.

$$
\begin{tikzcd}
0 \arrow[r] & \prod_i A_i \arrow[r, hook] \arrow[d, "\Delta"] & \prod_i B_i \arrow[r, two heads] \arrow[d, "\Delta"] & \prod_i C_i \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \prod_i A_i \arrow[r, hook] & \prod_i B_i \arrow[r, two heads] & \prod_i C_i \arrow[r] & 0
\end{tikzcd}
$$

The snake lemma gives us a long exact sequence

$$0 \to \ker \Delta_A \to \ker \Delta_B \to \ker \Delta_C \to \operatorname{coker} \Delta_A \to \operatorname{coker} \Delta_B \to \operatorname{coker} \Delta_C \to 0.$$

We have already seen that $\ker \Delta_A = \lim\{A_i\}$. This tells us that

$$\lim^i = \begin{cases} \lim, & i = 0 \\ \operatorname{coker} \Delta & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a homological δ-functor. In fact, it turns out to be universal, hence the derived functor of lim.

Lemma 22. Suppose all the maps $\lambda_{i,i+1} \colon A_{i+1} \to A_i$ are epimorphisms. Then $\lim^1 A_i = 0$.

We need to show that Δ is an epimorphism. To that end, let

$$(\cdots, a_i, \cdots) \in \prod_i A_i.$$

Let $b_0 = a_0$ Define b_{i+1} inductively as a lift of $b_i - a_i$. Then Δ maps $(\cdots, b_i, \cdots)$ to

$$(\cdots, b_i - (b_i - a_i), \cdots) = (\cdots, a_i, \cdots).$$

Definition 23 (Mittag-Leffler condition)**.** Let $\{A_i\}$ be a tower of abelian groups. We say that $\{A_i\}$ satisfies the Mittag-Leffler condition if for each $n \geq 0$ there exists a $N = N(n) > n$ such that the for all $m \geq N$, the image of $A_m \to A_n$ is equal to the image of $A_N \to A_n$.

We say that $\{A_i\}$ satisfies the trivial Mittag-Leffler condition if for all $n \geq 0$ there exists $N(n) > n$ such that $\operatorname{im}(A_{N(n)} \to A_n) = 0$.

Example 24. If all the maps $A_n \to A_{n-1}$ are epimorphisms, then $\{A_i\}$ satisfies the Mittag-Leffler condition with $N(n) = n + 1$.

Proposition 25. Suppose $\{A_i\}$ is a tower of abelian groups satisfying the Mittag-Leffler condition. Then

$$\lim^1 A_i = 0.$$

Proof. First, suppose that A_i satisfies the trivial Mittag-Leffler condition. We need to show that $\lim^1 A_i = 0$, i.e. that Δ is epi.

To that end, choose some $a_i \in A_i$, $i \geq 0$. We need to find $b_i \in A_i$, $i \geq 0$ such that

$$\Delta\colon (\ldots, b_1, b_0) \mapsto (\ldots, a_1, a_0).$$

Define

$$b_n = \sum_{m \geq n} \lambda_{n,m} a_m.$$

Note that this is a finite sum because $\lambda_{n,m} = 0$ for large enough m. Then

$$\begin{aligned} \Delta(\cdots, b_n, \cdots) &= \left(\cdots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \lambda_{n,n+1} \sum_{m \geq n+1} \lambda_{m,n+1} a_m \right) \\ &= \left(\cdots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \sum_{m \geq n+1} \lambda_{n,m} a_m \right) \\ &= (\cdots, a_n, \cdots). \end{aligned}$$

Now suppose that A_i satisfies the ordinary Mittag-Leffler condition. Without loss of generality we may assume that $N(n)$ is nondecreasing. Define

$$B_i = \lambda_{i,N(i)}(A_{N(i)}).$$

Consider the following short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & B_{i+1} \arrow[r, hook] & A_{i+1} \arrow[r, two heads] \arrow[d, "\lambda_{i,i+1}"'] & A_{i+1}/B_{i+1} \arrow[r] & 0 \\ 0 \arrow[r] & B_i \arrow[r, hook] & A_i \arrow[r, two heads] & A_i/B_i \arrow[r] & 0 \end{tikzcd}$$

Because N is nondecreasing, the λs give us a map $B_{i+1} \to B_i$ making the diagram commute. The universal property for cokernels gives us an induced map $A_{i+1}/B_{i+1} \to A_i/B_i$ making the diagram commute.

$$\begin{tikzcd} 0 \arrow[r] & B_{i+1} \arrow[r, hook] \arrow[d] & A_{i+1} \arrow[r, two heads] \arrow[d, "\lambda_{i,i+1}"'] & A_{i+1}/B_{i+1} \arrow[r] \arrow[d, dashed] & 0 \\ 0 \arrow[r] & B_i \arrow[r, hook] & A_i \arrow[r, two heads] & A_i/B_i \arrow[r] & 0 \end{tikzcd}$$

This defines a tower $\{C_i\}$ with $C_i = A_i/B_i$. By definition, this satisfies the trivial Mittag-Leffler condition.

Note that by definition, the maps of the tower $\{B_i\}$ are onto, so

$$\lim^1 B_i = 0.$$

The long exact sequence associated to the short exact sequence

$$0 \longrightarrow B_i \hookrightarrow A_i \twoheadrightarrow A_i/B_i \longrightarrow 0$$

thus tells us that $\lim^1 A_i = 0$, as it is sandwiched between two things which are zero. □

Example 26. Consider the tower

$$\cdots \longrightarrow \mathbb{Z}/(p^3) \longrightarrow \mathbb{Z}/(p^2) \longrightarrow \mathbb{Z}/(p)$$

Theorem 27 (Milnor sequence)**.** Let

$$\cdots \longrightarrow C_1 \longrightarrow C_0$$

be a tower of cochain complexes of abelian groups such that the Mittag-Leffler condition is satisfied (i.e. satisfied level-wise). Let $C = \lim C_i$. Then for each q there is an exact sequence

$$0 \longrightarrow \lim^1 H^{q-1}(C_i) \hookrightarrow H^q(C) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0 \ .$$

Proof. We first fix notation. Let

$$B_i \subset Z_i \subset C_i$$

be the subcomplexes of C_i on boundaries and cycles, with trivial differentials; then

$$H^\bullet(C_i) = \operatorname{coker}(B_i \to Z_i).$$

Let B denote the level-wise coboundaries of C, and Z the level-wise cocycles, again with trivial differentials.

- Consider the exact sequence

  $$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \longrightarrow \{C_i[1]\} \ .$$

  Applying the left exact functor lim, we find that

  $$\lim Z_i \cong \ker(d\colon C \to C[1]),$$

  i.e. that $\lim Z_i$ is the subcomplex of cycles in C; that is, that

  $$Z = \lim Z_i.$$

- Consider the short exact sequence

  $$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \twoheadrightarrow \{B_i[1]\} \longrightarrow 0\ .$$

  The long exact sequence corresponding to $\lim^i$ is

  $$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r] & C \arrow[r] & \lim B_i[1] \arrow[dll, "\delta"] \\ & \lim^1 Z_i \arrow[r] & \lim^1 C^i \arrow[r] & \lim^1 B^i[1] \arrow[r] & 0 \end{tikzcd}$$

  But since C_i satisfies the Mittag-Leffler condition, $\lim^1 C_i = 0$, implying that

  $$\lim^1 B^i = 0.$$

  Furthermore, $B[1] \cong \operatorname{coker}(Z \hookrightarrow C)$, so image-kernel factorization gives us

  $$0 \longrightarrow B[1] \hookrightarrow \lim B_i[1] \twoheadrightarrow \lim^1(Z_i) \longrightarrow 0$$

- The long exact sequence on homology corresponding to

  $$0 \longrightarrow \{B^i\} \hookrightarrow \{Z^i\} \twoheadrightarrow \{H^i\} \longrightarrow 0$$

  is

  $$\begin{tikzcd} 0 \arrow[r] & \lim B_i \arrow[r] & Z \arrow[r] & \lim H_i \arrow[dll, "\delta"] \\ & 0 \arrow[r] & \lim^1 Z^i \arrow[r] & \lim^1 H^i \arrow[r] & 0 \end{tikzcd}$$

  .

  This tells us both that

  $$0 \longrightarrow \lim B_i \hookrightarrow Z \twoheadrightarrow \lim H_i \longrightarrow 0$$

  is short exact, *and* that

  $$\lim^1 Z^i \cong \lim^1 H_i.$$

The exact sequences we have discovered give us the chain of inclusions

$$0 \subset B \subset \lim B_i \subset Z \subset C.$$

The third isomorphism theorem or one of the axioms for triangulated categories or whatever gives that the sequence

$$0 \longrightarrow \frac{\lim B_i}{B} \hookrightarrow \frac{Z}{B} \twoheadrightarrow \frac{Z}{\lim B_i} \longrightarrow 0$$

is short exact. However,

$$\frac{\lim B_i}{B} \cong \lim^1 Z_i \cong \lim^1 H_i \qquad \frac{Z}{B} \cong H, \qquad \frac{Z}{\lim B_i} \cong \lim H_i$$

□