1 Homological algebra in abelian categories

1.1 Exact functors

Definition 1 (exact functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

  is exact

- We call F exact if it is both left exact and right exact.

Example 2. We showed in Lemma **??** that both hom functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ are left exact.

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 3. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\operatorname{im} f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\operatorname{im} \iota = \ker g \subset \operatorname{im} f$. □

Proposition 4. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f : A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
    - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.
    - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.
3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

Any functor G which is a right adjoint preserves limits. By the above reasoning, G certainly preserves zero objects and kernels. Thus, G preserves the first two conditions, which means precisely that G is left exact.

Dually, any functor F which is a left adjoint preserves colimits, hence zero objects and cokernels. Thus, F preserves the last two conditions, which means that F is right exact. □

Example 5. Consider the category $\mathcal{D}$ with objects and morphisms as follows.

$$\begin{tikzcd} & \star \arrow[d] \\ * \arrow[r] & \bullet \end{tikzcd}$$

Let $\mathcal{A}$ be an abelian category. By Example **??**, the category of functors $\mathcal{D} \to \mathcal{A}$ is an abelian category.

Consider a functor $\mathcal{D} \to \mathcal{A}$ which yields the following diagram in $\mathcal{A}$.

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Taking the pullback of the above diagram yields the kernel of f.

A natural transformation ϕ between functors $\mathcal{D} \to \mathcal{A}$ consists of the data of a commuting square as follows.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_A"'] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

That is, we may view taking kernels as a functor from the category of morphisms in $\mathcal{A}$ to $\mathcal{A}$. Furthermore, because this functor is the result of a limiting procedure it is complete. Thus, given an exact sequence of morphisms in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d, "f"'] & B \arrow[r, two heads] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads] & C \arrow[r] & 0 \end{tikzcd}$$

one gets an exact sequence of kernels

$$\begin{tikzcd} 0 \arrow[r] & \ker f \arrow[r, hook] & \ker g \arrow[r] & \ker h \end{tikzcd} .$$

Conversely, taking cokernels is cocomplete, hence right exact, so one has also an exact sequence of cokernels

$$\begin{tikzcd} \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r, two heads] & \operatorname{coker} h \arrow[r] & 0 \end{tikzcd}$$

Note that the previous theorem makes no demand that F be additive, which may seem surprising; after all, by Definition 1, only exact functors are allowed the honor of being called additive. However, it turns out that a functor which preserves either finite limits or finite colimits is always additive. To see this, note that by applying any functor which is either left or right exact to a split exact sequence gives a split exact sequence.

This is often useful when trying to check the exactness of a functor which is a left or right adjoint, by the following proposition.

Lemma 6. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

be an exact sequence, and let X be any object. Then the sequence

Corollary 7. $\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(X, A) \arrow[r, "f_*"] & \mathrm{Hom}(X, B) \arrow[r, "g_*"] & \mathrm{Hom}(X, C) \end{tikzcd}$

is an exact sequence of abelian groups.

Proof. The hom functor ☐

1.2 Chain Homotopies

Definition 8 (chain homotopy)**.** Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A chain homotopy between $f_\bullet$ and $g_\bullet$ is a family of maps

$$h_n \colon C_n \to D_{n+1}$$

such that

$$d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n = f - g.$$

Lemma 9. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 10. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 11. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

Definition 12 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

Lemma 13. The homotopy category $\mathcal{K}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

In fact, the homotopy category satisfies the following universal property.

Proposition 14. Let $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{C}$ be an additive functor such that $\mathcal{F}(f)$ is an isomorphism for all quasi-isomorphisms f. Then there exists a unique functor $\mathcal{K}(\mathcal{A}) \to \mathcal{C}$ making the following diagram commute.

Proposition 15. Let $\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$ be an additive functor which takes quasi-isomorphisms to isomorphisms. Then for homotopic morphisms $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, we have $F(f) = F(g)$.

1.3 Projectives and injectives

One of the drawbacks of diagrammatic reasoning in abelian categories (which is to say, reasoning on elements in R-**Mod**) is that not every maneuver one can perform on elements has a diagrammatic analog. One of the most egregious examples of this is the property of lifting against epimorphisms. Given an epimorphism $f\colon A \twoheadrightarrow B$ and an element of B, one can always find an element of A which maps to it under f. Unfortunately, it is *not* in general the case that given a map $X \to B$, one can lift it to a map $X \to A$ making the triangle formed commute.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\nexists"'] \arrow[d] \\ A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

Nowever, certain objects do have this property. For example, if X is a free R-module, then we can find a lift by lifting generators; this is always possible. It will be interesting to find objects X which allow this. Such objects are called *projective.*

Definition 16 (projective, injective)**.** An object P in an abelian category $\mathcal{A}$ is <u>projective</u> if for every epimorphism $f\colon B \to C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\tilde{p}"'] \arrow[d, "p"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g\colon A \to B$ and every morphism $q\colon A \to Q$ there exists a morphism $\tilde{q}\colon B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists\tilde{q}"] \\ Q & \end{tikzcd}$$

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A, it is very easy to see that an object P in an abelian category is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

A recurring theme of homological algebra is that projective and injective objects are extremely nice to work with, and whenever possible we would like to be able to trade in an object for equivalent projective data. We will go into more detail about precisely what we mean by this later. This will be possible when our abelian category has *enough projectives.*

Definition 17 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Proposition 18. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, we get a morphism $P \longrightarrow B$ making the triangle commute.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] \arrow[dl, "\exists g"'] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma **??**). □

Corollary 19. Let F be an additive functor, and let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

be an exact sequence with P projective. Then F applied to the sequence gives an exact sequence.

Proof. Applying F to a split sequence gives a split sequence, and all split sequences are exact. □

Corollary 20. The category R-**Mod** has enough projectives.

Proof. By Proposition **??**, free modules are projective, and every module is a quotient of a free module. $\square$

Proposition 21. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Dually, any abelian category with enough injectives has injective resolutions.

Proof. Pick a projective which surjects onto A, $\square$

Definition 22 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \to \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \to Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

Lemma 23. Let $f\colon P_\bullet \to \iota(M)$ be a projective resolution. Then $f\colon P_0 \to M$ is an epimorphism.

Proof. We know that $H_0(f)\colon H_0(P) \to H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(m)) \simeq \iota_M$, and that $\square$

In fact, we can say more.

Lemma 24. Let $f_\bullet\colon P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots\cdots P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Theorem 25 (horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$0 \longrightarrow M' \xhookrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "f"'] & M \arrow[r, two heads, "\tilde{g}"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted, no head] & & \vdots \arrow[d, dotted, no head] & \\
0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[d] \arrow[r] & 0 \\
0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[d] \arrow[r] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "f"] & M \arrow[r, two heads, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f\circ p'_0"] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since by Lemma 23 p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P'_0 \oplus P''_0 \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P''_1 to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r, dashed] \arrow[d] & \ker p \arrow[r, dashed] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook] & M \arrow[r, two heads] & M'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. We get the dashed map $\ker p' \to \ker p''$

That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Theorem 26 (horseshoe lemma for morphisms)**.** Given a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, hook] \arrow[d, "f'"] & A \arrow[r, two heads] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, hook] & B \arrow[r, two heads] & B'' \arrow[r] & 0
\end{tikzcd}$$

Proposition 27. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P_\bullet' \to M'$$

be projective resolutions. Then for every morphism $f\colon M \to M'$ there exists a lift $\tilde{f}\colon P_\bullet \to P_\bullet'$ making the diagram

$$\begin{tikzcd}
P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P_\bullet' \arrow[d] \\
\iota M \arrow[r, "\iota f"'] & \iota M'
\end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Corollary 28. For any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}^+(\mathcal{A})$.

1.4 Mapping cones

Definition 29 (shift functor). For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the <u>k-shifted chain complex</u> $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 30 (mapping cone). Let $f: C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\mathrm{cone}(f)$ as follows.

- For each n, define
$$\mathrm{cone}(f)_n = C_{n-1} \oplus D_n.$$
- Define $d_n^{\mathrm{cone}(f)}$ by
$$d_n^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$
which is shorthand for
$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1}x_{n-1}).$$

Lemma 31. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow D_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} C[-1]_\bullet \longrightarrow 0 \ .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 32. Let $f_\bullet: C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. By Corollary **??**, we get a long exact sequence

$$\cdots \to H_n(C) \xrightarrow{\delta} H_n(D) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(C) \to \cdots \ .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see; picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
& & x \arrow[d, mapsto] \\
& (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0,f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. $\square$

Proposition 33. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$. Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g\colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots \\
\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
& & P_1 \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists!"] \arrow[dd, "r"] \arrow[ddr, "0"] & \\
& K \arrow[dr, hook] & & \\
C_2 \arrow[ur, two heads] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$
\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}
$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$
\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}
$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\mathrm{cone}(f)$ is exact, which means that $\iota\colon P \to \mathrm{cone}(f)$ is homotopic to the zero morphism.

$$
\begin{tikzcd}
P_{i+1} \arrow[r, "d_{i+1}^P"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d_i^P"] \arrow[dl, dashed, "\tilde{h}_i"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d_{i+1}^{\mathrm{cone}(f)}"'] & C_{i-1} \oplus P_i \arrow[r, "d_i^{\mathrm{cone}(f)}"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}
$$

That is, there exist $\tilde{h}_i\colon P_i \to \mathrm{cone}(f)_{i+1}$ such that

$$
d_{i+1}^{\mathrm{cone}\, f} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \tag{1.1}
$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$
\beta_i\colon P_i \to C_i \qquad \text{and} \qquad \gamma_i\colon P_{i-1} \to P_i,
$$

we find what looks tantalizingly like a chain map $\beta_\bullet\colon P_\bullet \to C_\bullet$. Indeed, writing Equation 1.1 in components, we find the following.

$$
\begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}
$$

$$
\begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}
$$

The first line tells us that β is a chain map. The second line tells us that[1]

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

1.5 Localization at weak equivalences: a love story

We can embed any abelian category $\mathcal{A}$ into the corresponding category $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ of chain complexes via inclusion into degree zero. This is obviously a lossless procedure, in the sense that we recover $\mathcal{A}$ by restricting to degree 0.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

Equivalently, we recover our category $\mathcal{A}$ by taking zeroth homology.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "H_0"] \end{tikzcd}$$

However, we notice that there are much better-behaved embeddings of $\mathcal{A}$ into $\mathbf{Ch}(\mathcal{A})$ such that we recover $\mathcal{A}$ when taking zeroth homology. For example, taking projective resolutions of objects and lifting morphisms using Proposition 27 will do the trick. Unfortunately, this is not in general uniquely defined: one can take many different projective resolutions, and lift morphisms in many different ways. There is no reason to expect these data to assemble themselves functorially.

However, if we were to modify $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ in such a way that all quasi-isomorphisms became bona-fide isomorphisms, then we would have this functoriality, thanks to Proposition 15.

Thus we find ourselves in a common situation: we have a category $\mathbf{Ch}_{\geq 0}(\mathcal{A})$, and identified a collection of morphisms inside this category which we would like to view as

[1] Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

weak equivalences, namely the quasi-isomorphisms. In an ideal world, we would simply promote quasi-isomorphisms to bona fide isomorphisms. That is, we would like to form the localization

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{Ch}_{\geq 0}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms.

But now we are in a really wonderful position, as long as $\mathcal{A}$ has enough projectives: we can by Corollary 28 find a projective resolution functor

Definition 34 (homological δ-functor)**.** A homological δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n \colon \mathcal{A} \to \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n \colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.

2. For every short exact sequence as above, there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

3. For every morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

   and every n, the diagram

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}$$

   commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

A morphism $S \to T$ of δ-functors consists of, for each n, a natural transformation $S_n \to T_n$ compatible with the δ.

This seems like an inelegant definition, and indeed it is.

Example 35. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors*, come from homology.

Definition 36 (cohomological δ-functors)**.** Dual to Definition 34.

Definition 37 (derived functor)**.** Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the left derived functor to be the homological δ-functor $\{L_nF\}$ defined by

$$L_nF = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the right derived functor to be the cohomological δ-functor

$$R^nF(X) = H^n \circ F \circ Q$$

where Q is an injective resolution functor.

We still need to show that derived functors are in fact homological delta-functors.

Theorem 38. For any right exact functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ has enough projectives, the left derived functor $\{L_nF\}$ is a homological δ-functor.

Proof. We check each condition separately.

1. L_nF is computed by taking the nth homology of a chain complex concentrated in positive degree.
2. Start with a short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

in $\mathcal{A}$.

We compute the image of this sequence under the derived functors L_nF by using our chosen projective resolution functor to find projective resolutions and lift maps, and take homology. In general, we cannot be guaranteed that we can lift our exact sequence up to an exact sequence of projective resolutions

$$0 \longrightarrow P^A_\bullet \hookrightarrow P^B_\bullet \twoheadrightarrow P^C_\bullet \longrightarrow 0 .$$

However, we have shown that, up to isomorphism, it doesn't matter which projective resolutions we choose, so we can choose those given by the horseshoe lemma (Theorem 25), giving us such an exact sequence. Applying F then gives us another exact sequence, and taking homology gives a long exact sequence.

3. We do the same thing, but now apply the horseshoe lemma on morphisms (Theorem 26).

$\square$

Proposition 39. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories.

We have that
$$L_0F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 24, the following sequence is exact.
$$P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$
Since F is right exact, the following sequence is exact.
$$F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$
This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But
$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$
$\square$

Given a right exact functor F, the above theorem shows that, at the lowest level, the left derived functor gives F itself. It turns out that $\{L_nF\}$ is the *universal* homological δ-functor with this property:

Proposition 40. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories where $\mathcal{A}$ has enough projectives, let $\{T_n\}$ be a homological δ-functor, and let $\phi_0\colon T_0 \Rightarrow F = L_0F$ be a natural transformation. Then there exists a unique lift of ϕ_0 to a morphism $\{\phi_i\colon T_i \to L_iF\}$.

Proof. First we construct $\phi_1\colon T_1 \Rightarrow L_1F$. For each object $A \in \mathcal{A}$, pick a projective which surjects onto it and take a kernel, giving the following short exact sequence.
$$0 \longrightarrow K \hookrightarrow P \twoheadrightarrow A \longrightarrow 0$$
Applying each of our homological δ-functors and using the fact that $L_iFP = 0$ for $i > 0$, we get the following data.
$$\begin{tikzcd} & T_1A \arrow[r, "\delta"] & T_0K \arrow[r] \arrow[d, "(\phi_0)_K"] & T_0P \arrow[r] \arrow[d, "(\phi_0)_P"] & T_0A \arrow[r] \arrow[d, "(\phi_0)_A"] & 0 \\ 0 \arrow[r] & L_1FA \arrow[r, hook, "\delta"] & FK \arrow[r] & FP \arrow[r] & FA \arrow[r] & 0 \end{tikzcd}$$
Since $L_1FA \to FK$ is a kernel of $T_0K \to T_0P$, we get a unique map $T_1A \to L_1FA$, which we declare to be $(\phi_1)_A$.

It remains to show that these are really the components of a natural transformation. Let $f\colon A \to A'$ be a morphism in $\mathcal{A}$. Consider any induced morphisms of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook] \arrow[d] & P \arrow[r, two heads] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K' \arrow[r, hook] & P' \arrow[r, two heads] & A' \arrow[r] & 0
\end{tikzcd}$$

Since $\{L_nF\}$ and $\{T_n\}$ are homological δ-functors, this gives us the following cube, where the components $(\phi_1)_A$ and $(\phi_1)_{A'}$ are dashed. All faces except the one on the left commute by definition.

$$\begin{tikzcd}
& T_1A' \arrow[rr] \arrow[dd, dashed, "(\phi_1)_{A'}"] & & T_0K' \arrow[dd] \\
T_1A \arrow[ur, "T_1f"] \arrow[rr] \arrow[dd, dashed, "(\phi_1)_A"'] & & T_0K \arrow[ur] \arrow[dd] & \\
& L_1FA' \arrow[rr, hook, "\delta"] & & FK' \\
L_1FA \arrow[ur, "L_1Ff"] \arrow[rr] & & FK \arrow[ur] &
\end{tikzcd}$$

In order to check that the $(\phi_1)_A$ form the components of a natural transformation, we need to check that the left face of the above cube commutes. Because $\delta\colon L_1FA' \to FK'$ is a monomorphism, it suffices to check that

$$\delta \circ (\phi_1)_{A'} \circ T_1F = \delta \circ L_1Ff \circ (\phi_1)_A.$$

But given the commutativity of the other faces of the cube, this is clear.

We now continue inductively, constructing ϕ_2, etc.

The last step is to show that the collection $\{\phi_n\}$ is really a morphism of homological δ-functors, i.e. that for any short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & A'' \arrow[r, hook] & A \arrow[r, two heads] & A' \arrow[r] & 0
\end{tikzcd}$$

the squares

$$\begin{tikzcd}
T_nA'' \arrow[r, "\delta"] \arrow[d, "(\phi_n)_{A''}"'] & T_{n-1}A' \arrow[d, "(\phi_{n-1})_{A'}"] \\
L_nFA'' \arrow[r, "\delta"'] & L_{n-1}FA'
\end{tikzcd}$$

commute □

1.6 Double complexes

Given any abelian category $\mathcal{A}$, we have considered the category of chain complexes in $\mathcal{A}$, and shown (in Theorem **??**) that this is an abelian category. It is therefore natural to consider chain complexes in the category of chain complexes.

$$\Big(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \Big) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

Such a beast is a lattice of commuting squares.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & M_{i-1,j} \arrow[r] \arrow[d] & M_{i,j} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M_{i-1,j-1} \arrow[r] \arrow[d] & M_{i,j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

However, it turns out to convenient to do something slightly different, namely work with complexes whose squares anticommute rather than commute. This is not a restriction, as one can make from any lattice of commuting squares a lattice of anticommuting squares by multiplying the differentials of every other column by -1. This is known as the *sign trick.*

Definition 41 (double complex)**.** Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d^h_{i,j} \colon M_{i,j} \to M_{i-1,j}, \qquad d^v_{i,j} \colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $(d^h)^2 = 0$
- $(d^v)^2 = 0$
- $d^h \circ d^v + d^v \circ d^h = 0$

We say that a double complex is $M_{\bullet,\bullet}$ is *first-quadrant* if $M_{i,j} = 0$ for $i, j < 0$.

Given any complex of complexes, one can construct a double complex by multiplying the differentials in every other row or column by -1.

Definition 42 (total complex)**.** Let $M_{\bullet,\bullet}$ be a first-quadrant[2] double complex. The total

[2] This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

Lemma 43. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Theorem 44. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & \iddots & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & \iddots & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \iddots \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$
\begin{array}{ccc}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} & = & \mathrm{Tot}(M)_3 \\
\downarrow & & \downarrow \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} & = & \mathrm{Tot}(M)_2 \\
\downarrow & & \downarrow \\
M_{1,0} \oplus M_{0,1} & = & \mathrm{Tot}(M)_1 \\
\downarrow & & \downarrow \\
M_{0,0} & = & \mathrm{Tot}(M)_0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

is exact.

The proof is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we give the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \qquad d_{11}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}}n = m$.

It turns out that we can make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker \delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta d n_{1,2},$$

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned} d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d+\delta)n_{2,1} + (d+\delta)n_{1,2} + (d+\delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. □

Example 45. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Form from this a double complex by multiplying the differential of $C_\bullet$ by -1 and padding by zeroes on either side.

$$\begin{tikzcd} & \vdots \arrow[d] & \vdots \arrow[d] & \\ 0 & D_3 \arrow[l] \arrow[d, "d_3^D"'] & C_3 \arrow[l, "f_3"'] \arrow[d, "-d_3^C"] & 0 \arrow[l] \\ 0 & D_2 \arrow[l] \arrow[d, "d_2^D"'] & C_2 \arrow[l, "f_2"'] \arrow[d, "-d_2^C"] & 0 \arrow[l] \\ 0 & D_1 \arrow[l] \arrow[d, "d_1^D"'] & C_1 \arrow[l, "f_1"'] \arrow[d, "-d_1^C"] & 0 \arrow[l] \\ 0 & D_0 \arrow[l] \arrow[d, "d_0^D"'] & C_0 \arrow[l, "f_0"'] \arrow[d, "-d_0^C"] & 0 \arrow[l] \\ 0 & D_{-1} \arrow[l] \arrow[d] & C_{-1} \arrow[l, "f_{-1}"'] \arrow[d] & 0 \arrow[l] \\ & \vdots & \vdots & \end{tikzcd}$$

The total complex of this is simply $\mathrm{cone}(f)$.

Now suppose that $f_\bullet$ is an isomorphism, so the rows are exact. Then by Theorem 44, $\mathrm{Tot}(f) = \mathrm{cone}(f)$ is exact. We have already seen this (in one direction) in Corollary 32.

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain
$$\cdots \longrightarrow C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$
hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;
- As a morphism between double complexes, hence between totalizations
$$\mathrm{Tot}(\alpha)_\bullet \colon \mathrm{Tot}(C)_\bullet \to \mathrm{Tot}(D)_\bullet.$$

Lemma 46. These two points of view agree in the sense that
$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 47. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let
$$\begin{tikzcd} \cdots \arrow[r] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet \end{tikzcd}$$
be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence
$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$
is exact. By Theorem 44, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 46 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 32) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 48. Let
$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$
be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence
$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write
$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Flipping along the main diagonal, we have the following resolution.

$$\begin{tikzcd}
M_{\bullet,2} \arrow[r] & M_{\bullet,1} \arrow[r] & M_{\bullet,0} \arrow[r] & H_0(M_{\bullet,\bullet})
\end{tikzcd}$$

Now Theorem 47 gives us the result we want. □

1.7 Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let $\mathcal{A}$ be an abelian category and let

$$- \otimes - \colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{H},$$

be an additive functor, where $\mathcal{H}$ is some abelian category equipped with an exact functor to **Ab**. Further suppose that

Definition 49 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$(C \otimes D)_\bullet = \mathrm{Tot}(C_\bullet \otimes D_\bullet),$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 50 (internal hom)**.** Let $\mathcal{A}$ be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We have

1.8 The Künneth formula

This section takes place in R-**Mod**, where R is a PID. For example, everything we are saying holds in **Ab**.

Lemma 51. Let $C_\bullet$ be a chain complex of free R-modules with trivial differential, and let $C'_\bullet$ be an arbitrary chain complex. Then there is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \cong H_n(C_\bullet \otimes C'_\bullet).$$

Proof. We write

$$Z_p = Z_p(C_\bullet), \qquad B_p = B_p(C_\bullet), \qquad Z'_p = Z_p(C'_\bullet), \qquad B'_p = B_p(C'_\bullet),$$

The sequence

$$0 \longrightarrow Z'_q \hookrightarrow C'_q \twoheadrightarrow B'_{q-1} \longrightarrow 0$$

is exact by definition, and since $Z_p = C_p$ is by assumption free, the sequence

By definition, $Z_p = C_p$ is free. Thus the sequence

$$0 \longrightarrow Z_p \otimes Z'_q \hookrightarrow Z_p \otimes C'_q \twoheadrightarrow Z_p \otimes B'_{q-1} \longrightarrow 0$$

is exact.

The differential of the tensor product complex $C_\bullet \otimes C'_\bullet$ is

$$\begin{aligned} d_n^{\text{tot}} &= \sum_{i=0}^{n} d_i^C \otimes \mathrm{id}_{C_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C'} \\ &= \sum_{i=0}^{n} (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C}. \end{aligned}$$

Since C_i is by assumption free, tensoring with it doesn't change homology, and we have

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

and

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

□