

1 Algebraic topology

1.1 The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 1 (normalized chain complex). Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{d_{n+2}} NA_{n+1} \xrightarrow{-d_{n+1}} NA_n \xrightarrow{d_n} NA_{n-1} \xrightarrow{-d_{n-1}} \cdots$$

Definition 2 (Moore complex). Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n -cycles A_n and differential

$$\partial = \sum_{i=0}^n (-1)^i d_i.$$

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This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
 \partial^2 &= \left(\sum_{j=0}^{n-1} (-1)^j d_j \right) \circ \left(\sum_{i=0}^n (-1)^i d_i \right) \\
 &= \sum_{j=0}^{n-1} \sum_{i=0}^n (-1)^{i+j} d_j \circ d_i \\
 &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_j \circ d_i + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\
 &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\
 &= 0.
 \end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A , unless this is confusing. In this case, we will denote it by MA .

We define a simplicial object

$$\mathbb{Z}[-]: \Delta \rightarrow \mathbf{Ab}$$

by the composition

$$\Delta \xrightarrow{\mathcal{Y}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{M} \mathbf{Ch}_+(\mathbf{Ab})$$

which takes each object $[n]$ to the corresponding Moore complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$M : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \quad (1.1)$$

Theorem 3. The adjunction in [Equation 1.1](#) is an equivalence of categories

Proof. later, if I have time. □

1.2 Basic definitions and examples

We assume a basic knowledge of simplicial sets just to get the ball rolling.

Definition 4 (singular complex, singular homology). Let X be a topological space. The singular chain complex $C_\bullet(X)$ is defined level-wise by

$$C_n(X) = N(\mathcal{F}(\mathbf{Sing}(X)))$$

where \mathcal{F} denotes the free group functor and M is the Moore functor ([Definition 2](#)). That

is, it has differentials

$$d_n: C_n \rightarrow C_{n-1}; \quad (\alpha: \Delta^n \rightarrow X) \mapsto \sum_{i=0}^n \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(C_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f: X \rightarrow Y$ gives a chain map $C(f): C(X) \rightarrow C(Y)$. This immediately implies that the n th homology of a space X is invariant under homeomorphism.

Example 5. Denote by pt the one-point topological space. Then $C(\text{pt})_\bullet$ is given level-wise as follows.

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\ \cdots & \xrightarrow{d_4} & C_3 & \xrightarrow{d_3} & C_2 & \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{d_0} & C_{-1} & \xrightarrow{d_{-1}} & \cdots \end{array}$$

Thus, the n th homology of the point is

$$H_n(\text{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Example 6. Let X be a path-connected topological space. Then $H_0(X) = \mathbb{Z}$. To see this, consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation “there is a path from x to y .” However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

1.3 The Hurewicz homomorphism

Theorem 7 (Hurewicz). For any path-connected topological space X , there is an isomorphism

$$h_X: H_1(X) \cong \pi_1(X)_{\text{ab}},$$

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where $(-)_\text{ab}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism between the functors

$$h: H_1 \Rightarrow (\pi_1)_\text{ab}.$$

Example 8. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_\text{ab} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \quad n > 1.$$

Example 9. Since

$$\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \cong \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

1.4 Homotopy equivalence

Proposition 10. Let $f, g: X \rightarrow Y$ be continuous maps between topological spaces, and let $H: X \times [0, 1] \rightarrow Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 11. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 12. Any contractible space is homotopy equivalent to the one point space pt . Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

1.5 Relative homology; the long exact sequence of a pair of spaces

Definition 13 (relative homology). Let X be a topological space, and let $X \subset A$. The relative chain complex of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Denote by **Pair** the category whose objects are pairs (X, A) , where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \rightarrow (Y, B)$ are maps $f: X \rightarrow Y$ such that $f(A) \subset B$.

Lemma 14. For each n , relative homology provides a functor $H_n: \mathbf{Pair} \rightarrow \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{array}{ccc} S(X)_\bullet & \xrightarrow{f} & S(Y)_\bullet \\ \downarrow & & \downarrow \\ S(X)_\bullet / S(A)_\bullet & \dashrightarrow & S(Y)_\bullet / S(B)_\bullet \end{array}$$

The dashed arrow is well-defined because of the assumption that $f(A) \subset B$. Functoriality now follows from the functoriality of H_n . \square

Proposition 15. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{j+1}(X, A) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_j(A) & \longrightarrow & H_j(X) \longrightarrow H_j(X, A) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_{j-1}(A) & \longrightarrow & \cdots \end{array}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \longrightarrow 0$$

\square

Example 16. Let $X = \mathbb{D}^n$, the n -disk, and $A = \mathbb{S}^{n-1}$ its boundary n -sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_j(\mathbb{S}^{n-1}) & \longrightarrow & H_j(\mathbb{D}^n) \longrightarrow H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_{j-1}(\mathbb{S}^{n-1}) & \longrightarrow & \cdots \end{array}$$

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We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

1.6 Barycentric subdivision

Fact 17. Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X . Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha: \Delta^n \rightarrow X$$

whose images are completely contained in some open set in the open cover \mathfrak{U} . That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$ induces a cochain structure on $S_{\bullet}^{\mathfrak{U}}(X)$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_2^{\mathfrak{U}}(X) & \longrightarrow & S_1^{\mathfrak{U}}(X) & \longrightarrow & S_0^{\mathfrak{U}}(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & S_2(X) & \longrightarrow & S_1(X) & \longrightarrow & S_0(X) \longrightarrow 0 \end{array}$$

In fact, this inclusion is homotopic to the identity, hence induces an isomorphism

$$H_n^{\mathfrak{U}}(X) := H_n(S_{\bullet}^{\mathfrak{U}}(X)) \cong H_n(X).$$

This fact allows us almost immediately to read off two important theorems.

1.6.1 Excision

Theorem 18 (excision). Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{array}{ccc} A \setminus W & \xhookrightarrow{i} & A \\ \downarrow & & \downarrow \\ X \setminus W & \xhookrightarrow{i} & X \end{array}$$

induce an isomorphism

$$H_n(i): H_n(X \setminus W, A \setminus W) \cong H_n(X, A).$$

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A .

1.6.2 The Mayer-Vietoris sequence

Theorem 19 (Mayer-Vietoris). Let X be a topological space, and let $\mathcal{U} = \{X_1, X_2\}$ be an open cover of X , i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & H_{n+1}(X) & \\ & \swarrow & & \delta & \searrow & & \\ H_n(X_1 \cap X_2) & \longrightarrow & H_n(X_1) \oplus H_n(X_2) & \longrightarrow & H_n(X) & & \\ & \swarrow & & \delta & \searrow & & \\ H_{n-1}(X_1 \cap X_2) & \longrightarrow & \cdots & & & & \end{array}$$

Proof. We can draw our inclusions as the following pushout.

$$\begin{array}{ccc} & X_1 & \\ i_1 \nearrow & & \searrow \kappa_1 \\ X_1 \cap X_2 & & X \\ i_2 \searrow & & \nearrow \kappa_2 \\ & X_2 & \end{array}$$

We have, almost by definition, the following short exact sequence.

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{\kappa_1 - \kappa_2} S_\bullet^{\mathcal{U}}(X) \longrightarrow 0$$

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This gives the following long exact sequence on homology.

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & H_{n+1}^{\text{ll}}(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 & H_n(X_1 \cap X_2) & \longrightarrow & H_n(X_1) \oplus H_n(X_2) & \longrightarrow & H_n^{\text{ll}}(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 & H_{n-1}(X_1 \cap X_2) & \longrightarrow & \cdots & & &
 \end{array}$$

We have seen that $H^{\text{ll}}(n)(X) \cong H_n(X)$; the result follows. \square

Example 20 (Homology groups of spheres). We can decompose \mathbb{S}^n as

$$\mathbb{S}^n = (\mathbb{S}^n \setminus N) \cup (\mathbb{S}^n \setminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$\begin{array}{ccccc}
 & & \mathbb{S}^n \setminus N & & \\
 & \swarrow & & \searrow & \\
 (\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) & & & & \mathbb{S}^n \\
 & \searrow & & \swarrow & \\
 & & \mathbb{S}^n \setminus S & &
 \end{array}$$

The Mayer-Vietoris sequence is as follows.

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & H_{j+1}(\mathbb{S}^n) & \\
 & \swarrow & & \delta & \searrow & & \\
 & H_j((\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S)) & \longrightarrow & H_j(\mathbb{S}^n \setminus N) \oplus H_j(\mathbb{S}^n \setminus S) & \longrightarrow & H_j^{\text{ll}}(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 & H_{j-1}((\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S)) & \longrightarrow & \cdots & & &
 \end{array}$$

We know that

$$\mathbb{S}^n \setminus N \cong \mathbb{S}^n \setminus S \cong \mathbb{D}^n \simeq \text{pt}$$

and that

$$(\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces

(for $j > 1$) to

$$\begin{array}{c}
 \cdots \rightarrow H_{j+1}(\mathbb{S}^n) \\
 \curvearrowleft \quad \delta \quad \curvearrowright \\
 H_j(\mathbb{S}^{n-1}) \longrightarrow 0 \longrightarrow H_j(\mathbb{S}^n) \\
 \curvearrowleft \quad \delta \quad \curvearrowright \\
 H_{j-1}(\mathbb{S}^{n-1}) \rightarrow \cdots
 \end{array}$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X , $H_1(X) \cong \pi_1(X)_{\text{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\text{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$j = 3$	\mathbb{Z}	0		
$j = 2$	\mathbb{Z}	0		
$j = 1$	\mathbb{Z}	\mathbb{Z}		
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \quad i > 1$$

allows us to fill in the above table as follows.

$j = 3$	\mathbb{Z}	0	0	\mathbb{Z}
$j = 2$	\mathbb{Z}	0	\mathbb{Z}	0
$j = 1$	\mathbb{Z}	\mathbb{Z}	0	0
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

(1.2)

We can immediately replace things we know, finding the following.

$$(a, b) \mapsto (a + b, a + b)$$

$$0 \longrightarrow H_1(\mathbb{S}^1) \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$(c, d) \mapsto c - d$$

The kernel of f is the free group generated by (a, a) . Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

1.6.3 The relative Mayer-Vietoris sequence

Theorem 22 (relative Mayer-Vietoris sequence). Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & H_{n+1}(X, A \cup B) & \\ & & & \delta & \searrow & & \\ H_n(X, A \cap B) & \longrightarrow & H_n(X, A) \oplus H_n(X, B) & \longrightarrow & H_n(X, A \cup B) & & \\ & & & \delta & \searrow & & \\ H_{n-1}(X, A \cap B) & \longrightarrow & \cdots & & & & \end{array}$$

Proof. Consider the following chain complex of chain complexes.

$$\begin{array}{ccccccc} & 0 & & 0 & & 0 & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 & \longrightarrow & S_n(A \cap B) & \longrightarrow & S_n(A) \oplus S_n(B) & \longrightarrow & S_n^{\mathfrak{U}}(A \cup B) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_n(X) & \longrightarrow & S_n(X) \oplus S_n(X) & \longrightarrow & S_n(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & S_n(X, A \cap B) & \longrightarrow & S_n(X, A) \oplus S_n(X, B) & \longrightarrow & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & 0 & & 0 & & 0 \end{array}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem ??) implies that the last row is also exact.

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Consider the following map of chain complexes; the first row is the last column of the above grid.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_n^{\mathbb{U}}(A \cup B) & \hookrightarrow & S_n(X) & \twoheadrightarrow & S_n(X)/S_n^{\mathbb{U}}(A \cup B) \longrightarrow 0 \\
 & & \phi \downarrow & & \parallel & & \downarrow \psi \\
 0 & \longrightarrow & S_n(A \cup B) & \longrightarrow & S_n(X) & \twoheadrightarrow & S_n(X, A \cup B) \longrightarrow 0
 \end{array}$$

This gives us, by Lemma ??, a morphism of long exact sequences on homology.

$$\begin{array}{ccccccccc}
 H_n(S_{\bullet}^{\mathbb{U}}(A \cup B)) & \rightarrow & H_n(X) & \rightarrow & H_n(S_{\bullet}(X)/S_{\bullet}^{\mathbb{U}}(A \cup B)) & \rightarrow & H_{n-1}(S_{\bullet}^{\mathbb{U}}(A \cup B)) & \rightarrow & H_{n-1}(X) \\
 \downarrow H_n(\phi) & & \parallel & & \downarrow H_n(\psi) & & \downarrow H_{n-1}(\phi) & & \parallel \\
 H_n(A \cup B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A \cup B) & \longrightarrow & H_{n-1}(A \cup B) & \longrightarrow & H_{n-1}(X)
 \end{array}$$

We have seen (in Fact 17) that $H_i(\phi)$ is an isomorphism for all i . Thus, the five lemma (Theorem ??) tells us that $H_n(\psi)$ is an isomorphism. \square

1.7 Reduced homology

It would be hard to argue that Table 1.2 is not pretty, but it would be much prettier were it not for the \mathbb{Z} s in the first column. We have to carry these around because every non-empty space has at least one connected component.

The solution is to define a new homology $\tilde{H}_n(X)$ which agrees with $H_n(X)$ in positive degrees, and is missing a copy of \mathbb{Z} in the zeroth degree. There are three equivalent ways of doing this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Denoting the unique map $X \rightarrow \text{pt}$ by ϵ , one can define relative homology by

$$\tilde{H}_n(X) = \ker H_n(\epsilon).$$

2. **In between:** One can replace homology $H_n(X)$ by relative homology

$$\tilde{H}_n(X) = H_n(X, x),$$

where $x \in X$ is any point of x .

3. **Algebraic:** One can augment the singular chain complex $C_{\bullet}(X)$ by adding a copy of \mathbb{Z} in degree -1 , so that

$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$

Then one can define

$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the following. In constructing the singular chain complex of our space X , we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor \mathbf{Sing} now has a component corresponding to (-1) -simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in \mathbf{Top} . Passing through \mathcal{F} thus gives a copy of \mathbb{Z} as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex. These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- $(1 \Leftrightarrow 2)$: Since the composition

$$\mathrm{pt} \xhookrightarrow{x} X \xrightarrow{\epsilon} \twoheadrightarrow \mathrm{pt}$$

is a weak retract, the sequence

$$0 \longrightarrow S_n(\mathrm{pt}) \longrightarrow S_n(X) \longrightarrow S_n(X, \{x\}) \longrightarrow 0$$

Proposition 23. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Proof. Trivial from algebraic definition. □

Many of our results for regular homology hold also for reduced homology.

Proposition 24. There is a long exact sequence for a pair of spaces

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & \tilde{H}_{j+1}(X, A) & & \\ & \searrow & & & \delta & \searrow & \\ & \tilde{H}_j(A) & \longrightarrow & \tilde{H}_j(X) & \longrightarrow & \tilde{H}_j(X, A) & \\ & \swarrow & & & \delta & \swarrow & \\ & \tilde{H}_{j-1}(A) & \longrightarrow & \cdots & & & \end{array}$$

Proposition 25. We have a reduced Mayer-Vietoris sequence.

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & \tilde{H}_{n+1}(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 & \tilde{H}_n(X_1 \cap X_2) & \longrightarrow & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) & \longrightarrow & \tilde{H}_n(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 & \tilde{H}_{n-1}(X_1 \cap X_2) & \longrightarrow & \cdots & & &
 \end{array}$$

Proposition 26. Let $\{(X_i, x_i)\}_{i \in I}$ be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$ ¹ we have

$$\tilde{H}_n \left(\bigvee_{i \in I} X_i \right) \cong \bigoplus_{i \in E} \tilde{H}_n(X_i).$$

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of [Proposition 25](#) tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 27 (good pair). A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X .
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U .

$$A \hookrightarrow U \xrightarrow{r} \twoheadrightarrow A$$

Proposition 28. Let (X, A) be a good pair. Let $\pi: X \rightarrow X/A$ be the canonical projection. Then

$$\tilde{H}(X, A) \cong \tilde{H}_n(X/A) \quad \text{for all } n > 0.$$

¹This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Proof.

□

Theorem 29 (suspension isomorphism). Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \quad \text{for all } n > 0.$$

1.8 Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n . Let $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d \mu_n, \quad \text{for some } d \in \mathbb{Z}.$$

Definition 30 (mapping degree). We call $d \in \mathbb{Z}$ as above the mapping degree of f , and denote it by $\deg(f)$.

Example 31. Consider the map

$$\omega: [0, 1] \rightarrow \mathbb{S}^1; \quad t \mapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism ([Theorem 7](#)), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1; \quad x \mapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism ([Theorem 7](#)) tells us that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1)_{\text{ab}} & \xrightarrow{\pi_1(f_n)_{\text{ab}}} & \pi_1(\mathbb{S}^1)_{\text{ab}} \\ h_{\mathbb{S}^1} \downarrow & & \downarrow h_{\mathbb{S}^1} \\ H_1(\mathbb{S}^1) & \xrightarrow{H_1(f_n)} & H_1(\mathbb{S}^1) \end{array}$$

1.9 CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 32 (cell). Let X be a topological space. We say that X is an n -cell if X is homeomorphic to \mathbb{R}^n . We call the number n the dimension of X .

Definition 33 (cell decomposition). A cell decomposition of a topological space X is a decomposition

$$X = \bigsqcup_{i \in I} X_i, \quad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topological spaces.

Definition 34 (CW complex). A Hausdorff topological space is known as a CW complex² if it satisfies the following conditions.

(CW1) For every n -cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma: \mathbb{D}^n \rightarrow X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n - 1$.

(CW2) For every n -cell σ , the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X .

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ .
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map*.

Example 35. Consider the unit interval $I = [0, 1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n + 1$ 0-cells and n 1-cells, which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma_k^0 = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \quad \text{and} \quad \sigma_\infty^0 = 0$$

and one-cells

$$\sigma_k^1 = \left(\frac{1}{k}, \frac{1}{k+1} \right), \quad k \in \mathbb{N}^{\geq 1}.$$

²**Axiom (CW2)** is called the *closure-finiteness* condition. This is the ‘C’ in CW complex. **Axiom (CW3)** says that X carries the *weak topology* and is responsible for the ‘W’.

At first glance, this looks like a CW decomposition; it certainly satisfies [Axiom \(CW1\)](#) and [Axiom \(CW2\)](#). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2} \left(\frac{1}{k} + \frac{1}{k+1} \right),$$

is the midpoint of the interval σ_k^1 . We have $A \cap \sigma_k^0 = \emptyset$ for all k , and $A \cap \sigma_k^1 = \{a_k\}$ for all k . In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i . However, the set A is not closed in I , since it does not contain its limit point $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 36 (skeleton, dimension). Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n -skeleton of X . If X is equal to its n -skeleton but not equal to its $(n-1)$ -skeleton, we say that X is n -dimensional.

Note 37. [Axiom \(CW3\)](#) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 38 (subcomplex, CW pair). Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$.

We call such a pair (X, Y) a CW pair.

Fact 39. Let X and Y be CW complexes such that X is locally compact. Then $X \times Y$ is a CW complex.

Lemma 40. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 41. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact³ if and only if it is locally finite.⁴

Proof. It is clear that 1. \Rightarrow 2., since X is a subset of itself. Similarly, it is clear that 2. \Rightarrow 3., since □

³I.e. every point of X has a compact neighborhood.

⁴I.e. if every point has a neighborhood which is contained in only finitely many cells.

Corollary 42. If $f: K \rightarrow X$ is a continuous map from a compact space K to a CW complex X , then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n .

$$\begin{array}{ccccccc} & & & & K & & \\ & & & \nearrow \exists \tilde{f} & \downarrow f & & \\ X^{n-1} & \hookrightarrow & X^n & \hookrightarrow & X^{n+1} & \hookrightarrow & \dots \hookrightarrow X \end{array}$$

Proposition 43. Let A be a subcomplex of a CW complex X . Then $X \times \{0\} \cup A \times [0, 1]$ is a strong deformation retract of $X \times [0, 1]$.

Lemma 44. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborhood U of A in X together with a strong deformation retract to A . In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U .
- Every CW complex is paracompact, locally path-connected, and locally contractible.
- Every CW complex is semi-locally 1-connected, hence possesses a universal covering space.

Lemma 45. Let X be a CW complex. We have the following decompositions.

1.

$$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2.

$$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n -cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n -cell is homeomorphic to the open n -ball.
2. For every n -cell σ , the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$ -skeleton.

□

1.10 Cellular homology

Lemma 46. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n / X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 44, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 28, and the second from Lemma 45. \square

Lemma 47. Consider the inclusion $i_n: X^n \hookrightarrow X$.

- The induced map

$$H_n(i_n): H_n(X^n) \rightarrow H_n(X)$$

is surjective.

- On the $(n+1)$ -skeleton we get an isomorphism

$$H_n(i_{n+1}): H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n) . The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by Lemma 46,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n): X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{array}{c} \xrightarrow{\quad H_{n+1}(X^{m+1}, X^m) \quad} \\ \delta \nearrow \\ H_n(X^m) \longrightarrow H_n(X^{m+1}) \longrightarrow H_n(X^{m+1}, X^m) \end{array}$$

But again by Lemma 46, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \rightarrow H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} \longrightarrow \cdots \\ & & & & \searrow & \searrow & \searrow \\ & & & & & & X \end{array}$$

1 Algebraic topology

Taking n th singular homology, we find the following.

$$\cdots \longrightarrow H_n(X^n) \xrightarrow{\alpha_1} H_n(X^{n+1}) \xrightarrow{\alpha_2} H_n(X^{n+2}) \xrightarrow{\alpha_3} \cdots \longrightarrow H_n(X)$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \quad \sigma_i: \Delta^n \rightarrow X.$$

Since the standard n -simplex Δ^n is compact, [Corollary 42](#) implies that each σ_i factors through some X^{n_i} . Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \quad \tilde{\sigma}_i: \Delta^n \rightarrow X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma \right]$$

□

Corollary 48. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q -cells, then $H_q(X) \cong 0$.

Proof.

1. This follows immediately from [Lemma 47](#).
- 2.

□