Homological Algebra

With Applications to Algebraic Topology

(Under construction)

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$$
\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed, bend left] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd}
$$

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0. Introduction

This document grew out of notes I took during a course on homological algebra, taught by Prof. Tobias Dyckerhoff, and algebraic topology, taught by Prof. Birgit Richter, during Summer Semester 2019 at the University of Hamburg.

Part I, on homological algebra, is fairly close to the course taught by Prof. Dyckerhoff, but Part II departs somewhat from the approach taken by Prof. Richter, mainly because of the level of homological algebra at our disposal.

Part I.

Homological algebra

1. Abelian categories

1.1. Basics

Abelian categories provide a categorical axiomatization of the properties of the category R-**Mod** of modules over a (not necessarily commutative) ring R; in fact, one can show that any small abelian category is a full subcategory of R-**Mod** for some R. Homological is the study of chain complexes of R-modules, and hence of the study of chain complexes in an abelian category.

One of the benefits of working with abelian categories rather than working directly R-**Mod** is that many things which are not obviously abelian categories turn out to be abelian categories; for example, the category of chain complexes in an abelian category is itself an abelian category, so when one is studying abelian categories one is already learning something about chain complexes in an abelian category.

1.1.1. Building blocks

Additive categories

Definition 1 (**Ab**-enriched category)**.** A category $\mathcal{C}$ is called **Ab**-*enriched* if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$.

Definition 2 (additive category)**.** A category $\mathcal{C}$ is additive if it has finite products (including a terminal object) and is **Ab**-enriched.

Proposition 3. Let $\mathcal{C}$ be an additive category. Then any terminal object is also an initial object, and the product $A \times B$ satisfies the universal property for a coproduct $A \amalg B$.

Proof. First we show that any terminal object is also initial. Denote by $*$ a terminal object in $\mathcal{C}$. The unique morphism $\mathrm{id}_*\colon * \to *$ must be the additive identity $e_{*\to*}$ in the abelian group $\mathrm{Hom}(*, *)$. Let X be any other object in $\mathcal{C}$, and let $f\colon * \to X$. Then $e_{*\to*} \circ f = f$. However, since composition is bilinear, we also have $e_{*\to*} \circ f = e_{*\to X}$. Thus, there is a unique map $* \to X$ for all X, so $*$ is an initial object. Hence it is a zero object. From now on, we will write 0 for zero objects.

Now we show that binary products are also coproducts. By the universal property for products, we get a unique map $i_A \colon A \to A \times B$ making the following diagram commute.

$$\begin{tikzcd} & A \arrow[dl, "\mathrm{id}_A"'] \arrow[d, "\exists! i_A"] \arrow[dr, "0"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

Similarly, there exists a unique map $i_B \colon B \to A \times B$ making the diagram

$$\begin{tikzcd} & B \arrow[dl, "0"'] \arrow[d, "\exists! i_B"] \arrow[dr, "\mathrm{id}_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

commute.

Consider the following diagram.

$$\begin{tikzcd} & A \times B \arrow[dl, "\pi_A"'] \arrow[d, dashed, "\exists!"] \arrow[dr, "\pi_B"] & \\ A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B \end{tikzcd}$$

Obviously, the identity map $\mathrm{id}_{A \times B}$ makes this diagram commute. However

$$\pi_A \circ (i_A \circ \pi_A + i_B \circ \pi_B) = \pi_A$$

and

$$\pi_B \circ (i_A \circ \pi_A + i_B \circ \pi_B) = \pi_B,$$

so the map $i_A \circ \pi_A + i_B \circ \pi_B$ also does. Thus,

$$\pi_A \circ i_A + \pi_B \circ i_B = \mathrm{id}_{A \times B}.$$

We now claim that $A \times B$, together with the maps i_A and i_B, satisfy the universal property for coproducts. To see this, let X be an object and $f \colon A \to A \times B$ and $g \colon B \to A \times B$ be morphisms.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "f"'] & A \times B \arrow[d, dashed, "\phi"] & B \arrow[l, "i_B"'] \arrow[dl, "g"] \\ & X & \end{tikzcd}$$

We need to show that there exists a unique morphism ϕ making the diagram commute. Note that, by logic identical to that above, taking

$$\phi = f \circ \pi_A + g \circ \pi_B$$

makes the above diagram commute. Now suppose that there exists another $\tilde{\phi}$ making the diagram commute. Then

$$
\begin{aligned}
\tilde{\phi} &= \tilde{\phi} \circ \mathrm{id}_{A \times B} \\
&= \tilde{\phi} \circ (i_A \circ \pi_A + i_B \circ \pi_B) \\
&= f \circ \pi_A + g \circ \pi_B \\
&= \phi.
\end{aligned}
$$

□

Proposition 3 tells us that, in an additive category with products, we also have coproducts, and that these agree with the products. Dually, we could have defined an additive category to be a category with coproducts; the dual to Proposition 3 would then have shown us that these agreed with products. Rather than preferencing either the notation for products or the notation for coproducts, we will use a new symbol.

Definition 4 (direct sum)**.** Let $\mathcal{C}$ be a category with products $\times$ (including a terminal object $*$) and coproducts $\amalg$ (including an initial object $\emptyset$). If products and coproducts agree in the sense that the canonical map $A \amalg B \longrightarrow A \times B$ is an isomorphism, then we say that $\mathcal{C}$ has <u>direct sums</u>, and denote the (co)product of A and B by $A \oplus B$.

Lemma 5. Let A and B be objects in an additive category $\mathcal{A}$, and let X be an object in $\mathcal{A}$ equipped with morphisms as follows,

$$
\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_A"'] & \\
B \arrow[ur, "i_B"'] & & B
\end{tikzcd}
$$

such that the following equations hold.

$$
\begin{gathered}
i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{id}_X \\
\pi_A \circ i_B = 0 = \pi_B \circ i_A \\
\pi_A \circ i_A = \mathrm{id}_A \\
\pi_B \circ i_B = \mathrm{id}_B
\end{gathered}
$$

Then the πs and is exhibit $X \cong A \oplus B$.

Proof. It suffices to show that X is a product of A and B. Let $f \colon Z \longrightarrow A$ and $g \colon Z \longrightarrow B$. We need to show that there exists a unique $\phi \colon Z \longrightarrow X$ making the following diagram commute.

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dashed, "\phi"] \arrow[dr, "g"] & \\
A & X \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}$$

That is, we need

$$\pi_A \circ \phi = f, \qquad \pi_B \circ \phi = g.$$

Thus, we need

$$\begin{aligned} i_A \circ f + \pi_B \circ g &= i_A \circ \pi_A \circ \phi + i_B \circ \pi_B \circ \phi \\ &= \phi. \end{aligned}$$

It is easy to check that ϕ defined in this way *does* make the diagram commute, so ϕ exists and is unique. □

Definition 6 (additive functor)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between additive categories. We say that F is additive if for each $X, Y \in \mathrm{Obj}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Corollary 7. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(A \oplus B) \simeq F(A) \oplus F(B).$$

Conversely, if F preserves products,

Proof. Apply F to the data of Lemma 5. □

Pre-abelian categories

Definition 8 (kernel, cokernel)**.** Let $f\colon A \to B$ be a morphism in an abelian category.

- A pair (K, ι), where $\iota\colon K \to A$ is a morphism, is a kernel of f if $f \circ \iota = 0$ and for every object X and morphism $g\colon X \to A$ such that $f \circ g = 0$, there exists a unique morphism $\alpha\colon X \to K$ such that $f \circ \alpha = 0$.

$$\begin{tikzcd}
& X \arrow[dl, dashed, "\exists!"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
K \arrow[r, "\iota"] \arrow[rr, bend right, "0"'] & A \arrow[r, "f"] & B
\end{tikzcd}$$

- A pair (C, π), where $\pi\colon B \to C$ is a morphism, is a cokernel of f if $\pi \circ f = 0$ and for every object Y and morphism $g\colon B \to Y$ such that $g \circ f = 0$, there exists a unique morphism $\beta\colon C \to Y$ such that $\beta \circ f = 0$.

$$\begin{tikzcd}
A \arrow[r, "f"'] \arrow[rr, bend left, "0"] \arrow[dr, "0"'] & B \arrow[r, "\pi"'] \arrow[d, "g"] & C \arrow[dl, dashed, "\exists!"] \\
& Y &
\end{tikzcd}$$

Here are some equivalent ways of defining the kernel of a morphism $f\colon A \to B$.

- The equalizer of f with the zero map.

$$\begin{tikzcd}
\ker f = \mathrm{eq} \arrow[r] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "0"'] & B
\end{tikzcd}$$

- The pullback along the zero morphism

$$\begin{tikzcd}
\ker f \arrow[r] \arrow[d] & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}$$

Definition 9 (pre-abelian category)**.** A pre-abelian category is an additive category such that every morphism has a kernel and a cokernel.

Lemma 10. Pre-abelian categories have all finite limits and colimits.

Proof. In a pre-abelian category, the equalizer of f and g is the kernel of $f - g$, and the coequalizer is the cokernel of $f - g$. Thus, pre-abelian categories have finite products and equalizers, and finite coproducts and coequalizers. □

Proposition 11. In a pre-abelian category, kernels are monic and cokernels are epic.

Proof. The kernel of $f\colon A \to B$ is the pullback of the morphism $0 \to B$, which is monic. The pullback of a monic is a monic, which gives the result. □

Proposition 12. Let $\mathcal{C}$ be a pre-abelian category, and let $f\colon A \to B$ be a morphism.

1. The morphism f is a monomorphism if and only if $\ker f = 0$.
2. The morphism f is an epimorphism if and only if $\operatorname{coker} f = 0$.

Proof.

1. Suppose f is a monomorphism, and consider a morphism g making the following diagram commute.

$$\begin{tikzcd}
& Z \arrow[d, "g"] \arrow[dr, "0"] & \\
\ker f \arrow[r, hook, "\iota"] & A \arrow[r, hook, "f"] & B
\end{tikzcd}$$

   Because f is a monomorphism, we have $g = 0$. By the universal property for kernels, there exists a unique map $Z \to \ker f$ making the above diagram commute.

$$\begin{tikzcd}
& Z \arrow[d, "g"] \arrow[dr, "0"] \arrow[dl, dashed, "\exists!"'] & \\
\ker f \arrow[r, hook, "\iota"] & A \arrow[r, hook, "f"] & B
\end{tikzcd}$$

   But this remains true if we replace $\ker f$ by 0, so by definition we must have $\ker f = 0$.

   Now suppose that $\ker f = 0$, and let $g \colon X \to A$ such that $f \circ g = 0$. The universal property guarantees us a morphism $\alpha \colon X \to K$ making the left-hand triangle below commute.

$$\begin{tikzcd}
& Z \arrow[d, "g"] \arrow[dr, "0"] \arrow[dl, dashed, "\exists!"'] & \\
\ker f = 0 \arrow[r, hook, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

   However, the only way the left-hand triangle can commute is if $g = 0$. Thus, f is mono.

2. Dual.

□

1.1.2. Abelian categories

We have seen that in a pre-abelian category, kernels and cokernels more or less behave as they do in the category of modules over some ring R. However, the first isomorphism theorem tells us that every submodule is the kernel of some module homomorphism (namely the cokernel), and every quotient module of some module homomorphism (namely the kernel). We therefore add an extra axiom to enforce this behavior.

Definition 13 (abelian category)**.** A pre-abelian category is said to be abelian if every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

The above definition is equivalent to the following: an abelian category is:

- a category $\mathcal{A}$;
- enriched over $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$;
- with finite products (hence direct sums);
- such that every morphism has a kernel and a cokernel;
- such that every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Proposition 14. Let $\mathcal{A}$ be an abelian category, and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Then f is an isomorphism if and only if $\ker f = 0$ and $\operatorname{coker} f = 0$.

Proof. If f is an isomorphism, then it is mono and epi, and we are done by Proposition 12.

Conversely, if $\ker f = 0$ and $\operatorname{coker} f = 0$, then f is mono and epi. We need only to show that any morphism in an abelian category which is monic and epic is an isomorphism.

Let f be monic and epic. Then f is the kernel of its cokernel and the cokernel of its kernel, and both of these are zero.

$$\begin{tikzcd}
 & & B \arrow[dl, dashed, "\exists! f_R^{-1}"'] \arrow[d, equal] \arrow[dr, "0"] & \\
\ker f = 0 \arrow[r] & A \arrow[r, hook, two heads, "f"] \arrow[d, equal] & B \arrow[r] \arrow[dl, dashed, "\exists! f_L^{-1}"] & 0 = \operatorname{coker} f \\
 & A \arrow[ul, "0"] & &
\end{tikzcd}$$

The universal properties for kernels and cokernels give us respectively morphisms f_L^{-1} and f_R^{-1} such that

$$f \circ f_R^{-1} = \mathrm{id}_A, \qquad f_L^{-1} \circ f = \mathrm{id}_B.$$

The usual trick shows that $f_L^{-1} = f_R^{-1}$. □

As the following proposition shows, additive functors from additive categories preserve direct sums.

Example 15. Once it is known that a category $\mathcal{A}$ is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is abelian, as we will see in
- For any small category I, the category $\mathbf{Fun}(I, \mathcal{A})$ of I-diagrams in $\mathcal{A}$ is abelian. To see this, note the following.
  - Addition of natural transformations, defined pointwise, is an $\mathbf{Ab}$-enrichment.
  - Finite products exist because limits in functor categories are taken pointwise.

- Every morphism has a kernel and cokernel; and every monomorphism is the kernel of its cokernel; and every epimorphism is the cokernel of its kernel; because limits and colimits in functor categories are taken pointwise.

1.1.3. Abelian-ness is a property, not a structure

We have defined an abelian category to be an **Ab**-enriched category with products, kernels, and cokernels, such that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel. This makes it sound like an ordinary category could never aspire to be an abelian category. After all, an addition law on morphisms is extra structure, which one would have to provide to get an **Ab**-enriched category from an ordinary category.

Not so! It turns out that demanding that a category $\mathcal{C}$ admit finite direct sums is sufficient to uniquely determine the structure of a commutative monoid on each hom-set, as will follow from the Eckmann-Hilton argument. Furthermore, the axioms regarding the existence and properties of kernels and cokernels are enough to imply that each commutative monoid is an abelian group.

Proposition 16 (Eckmann-Hilton)**.** Let S be a set and let $\bullet$ and $\star$ be two unital binary operations on S, with units $0_\bullet$ and $0_\star$, satisfying the *interchange rule*

$$(a \bullet b) \star (c \bullet d) = (a \star c) \bullet (b \star d).$$

Then the operations $\bullet$ and $\star$ coincide, and they are (it is) associative and commutative.

Proof. We immediately invent new notation, writing

$$a \bullet b = (a\ b), \qquad \text{and} \qquad a \star b = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then the interchange rule can be written

$$\begin{pmatrix} (a\ b) \\ (c\ d) \end{pmatrix} = \left(\begin{pmatrix} a \\ c \end{pmatrix} \begin{pmatrix} b \\ d \end{pmatrix} \right).$$

We will therefore without fear of confusion omit parentheses, denoting this operation by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

First, we note that the units $0_\bullet$ and $0_\star$ are equal since

$$0_\bullet = (0_\bullet\ 0_\bullet) = \begin{pmatrix} 0_\bullet & 0_\star \\ 0_\star & 0_\bullet \end{pmatrix} = \begin{pmatrix} 0_\star \\ 0_\star \end{pmatrix} = 0_\star.$$

From now on, we will write $0_\bullet = 0_\star = 0$. To see that $\star$ and $\bullet$ are equal and commutative, note that

$$(a\ b) = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} = (b\ a).$$

$\square$

For any object A, denote the diagonal map $A \to A \oplus A$ by Δ, and the codiagonal map $A \oplus A \to A$ by ∇.

Definition 17. Let $\mathcal{A}$ be an abelian category, and let A, B in $\mathcal{A}$. Define the following operations on the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$.

1. Define $f \underset{L}{+} g$ to be the composition

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{(f,g)} B\ .$$

2. Define $f \underset{R}{+} g$ to be the composition

$$A \xrightarrow{(f,g)} B \oplus B \xrightarrow{\nabla} B\ .$$

Proposition 18. Let $\mathcal{C}$ be a category with finite direct sums[1] (Definition 4). Then $\mathcal{C}$ has a canonical enrichment in commutative monoids.

Proof. We show that $\underset{L}{+}$ and $\underset{R}{+}$ are unital with unit 0, and satisfy the exchange law. This suffices by Proposition 16. To see that $\underset{L}{+}$ is unital, note that the compositions

$$A \oplus A \xrightarrow{(f,0)} B \qquad \text{and} \qquad A \oplus A \xrightarrow{\pi_1} A \xrightarrow{f} B$$

agree since there is a unique horizontal morphism making the diagram

$$\begin{tikzcd}
A \arrow[d, "i_1"'] \arrow[rrd, bend left, "f"] & & \\
A \oplus A \arrow[r, "\pi_1"] & A \arrow[r, "f"] & B \\
A \arrow[u, "i_2"] \arrow[rru, bend right, "0"'] & &
\end{tikzcd}$$

[1] Including 0-ary direct sums, i.e. a zero object.

commute. Thus, $f \underset{L}{+} 0$ is given by

$$\begin{aligned}(f, 0) \circ \Delta &= f \circ \pi_1 \circ \Delta \\ &= f \circ \mathrm{id}_A \\ &= f.\end{aligned}$$

The case $0 \underset{L}{+} f = f$ is exactly analogous. The case of $\underset{R}{+}$ is dual.

Now we show that $\underset{L}{+}$ and $\underset{R}{+}$ obey the interchange law. Consider the composition

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} B \oplus B \xrightarrow{\nabla} B \ ,$$

where we have employed the matrix shorthand for the following map given to us by the universal properties for products and coproducts.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad = \quad \begin{tikzcd} A \arrow[r, "a"] \arrow[dr, "c" near start] & B \\ A \arrow[r, "d"'] \arrow[ur, "b"' near start] & B \end{tikzcd} \ .$$

Consider composition of the first two maps

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} B \oplus B \ .$$

Writing $B \oplus B = B'$, we can write this as

$$A \xrightarrow{\Delta} A \oplus A \xrightarrow{\left(\begin{pmatrix} a \\ c \end{pmatrix}, \begin{pmatrix} b \\ d \end{pmatrix}\right)} B' \ .$$

This composition is nothing else but

$$\begin{pmatrix} a \\ c \end{pmatrix} \underset{L}{+} \begin{pmatrix} b \\ d \end{pmatrix}.$$

Post-composing with ∇ then gives

$$(a \underset{R}{+} c) \underset{L}{+} (b \underset{R}{+} d),$$

and composing in the other direction gives

$$(a \underset{L}{+} b) \underset{R}{+} (c \underset{L}{+} d).$$

□

Already we have shown the following.

Corollary 19. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories. Then F is additive if and only if it preserves direct sums.

Proof. Suppose F preserves direct sums. Then □

1.2. Embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

In abelian categories, one can defines kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero obect: f is a kernel for g if and only the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pushout.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category $\mathbf{Set}_*$ of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category $\mathcal{A}$, suppose one could find an embedding $\mathcal{H}\colon \mathcal{A} \to \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \to B \to C$ in $\mathcal{A}$, mapped it into $\mathbf{Set}_*$ finding a sequence $\mathcal{H}(A) \to \mathcal{H}(B) \to \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \to B \to C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in $\mathbf{Set}_*$ using $\mathcal{H}$ and check exactness there. Effectively, one could check exactness of a sequence in $\mathcal{A}$ by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking the that the image of the diagram under our functor $\mathcal{H}$ commutes.

In fact, for $\mathcal{A}$ small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to $\mathbf{Set}_*$.

Definition 20 (category of contravariant epimorphisms)**.** Let $\mathcal{A}$ be a small abelian category. Define a category $\mathcal{A}_{\twoheadleftarrow}$ with $\mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow}) = \mathrm{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}_{\twoheadleftarrow}}(X, Y) = \{f\colon Y \to X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity morphism is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A\colon \mathcal{A}_{\twoheadleftarrow} \to \mathbf{Set}_*; \qquad Z \mapsto \mathrm{Hom}_{\mathcal{A}}(Z, A)$$

and sends a morphism $f\colon Z_1 \to Z_2$ in $\mathcal{A}_{\twoheadleftarrow}$ (which is to say, an epimorphism $\tilde{f}\colon Z_2 \twoheadrightarrow Z_1$ in $\mathcal{A}$) to the map

$$\mathcal{H}_A(f)\colon\ \mathrm{Hom}_{\mathcal{A}}(Z_1, A) \to \mathrm{Hom}_{\mathcal{A}}(Z_2, A); \quad (\alpha\colon Z_1 \to A) \mapsto (\alpha \circ \tilde{f}\colon Z_2 \to A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 21 (member functor)**.** Let $\mathcal{A}$ be a small abelian category. We define a functor $\mathcal{M}\colon \mathcal{A} \to \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \operatorname{colim} \mathcal{H}_A.$$

On morphisms, functorality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(\mathcal{A})$ have a much simpler interpretation.

Proposition 22. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \mathrm{Hom}(X, A)/\sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{tikzcd}
Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\
X' \arrow[r, "g'"'] & A
\end{tikzcd}$$

Proof. The colimit can be computed using the following coequalizer.

$$\begin{tikzcd}
\coprod\limits_{\substack{f \in \mathrm{Morph}(\mathcal{A}_{\twoheadleftarrow}) \\ \tilde{f}\colon Y \to X}} \mathrm{Hom}_{\mathcal{A}}(X, A) \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "- \circ \tilde{f}"'] & \coprod\limits_{Z \in \mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow})} \mathrm{Hom}(Z, A) \arrow[r, two heads, "\mathrm{coeq}"] & \mathcal{M}(A)
\end{tikzcd}$$

On elements, we have the following.

$$\begin{tikzcd}
 & (g\colon X \to A) \arrow[d, equal, "!"] \\
(g\colon X \to A) \arrow[ur, mapsto, "\mathrm{id}"] \arrow[r, mapsto, "- \circ \tilde{f}"'] & (g \circ \tilde{f}\colon Y \to A)
\end{tikzcd}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow})} \mathrm{Hom}(Z, A)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(g\colon X \to A) R (g'\colon X' \to A) \iff \exists f\colon X' \twoheadrightarrow X \text{ such that } g' = g \circ \tilde{f}.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g\colon X \to A) \sim (g'\colon X' \to A) \iff \exists f\colon Z \twoheadrightarrow X, f'\colon Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\ X' \arrow[r, "g'"'] & A \end{tikzcd}$$

$\square$

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphsims into A modulo the above relation, and for a morphism $f\colon A \to B$ in $\mathcal{A}$, $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f)\colon [g] \mapsto [g \circ f].$$

Lemma 23. Let $f\colon A \to B$ be a morphism in a small abelian category $\mathcal{A}$ such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms making the following diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "g"] \arrow[d, two heads] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. $\square$

Now we introduce some load-lightening notation: we write $\hat{(-)} = \mathcal{M}(-)$.

Lemma 24. Let $f\colon A \to B$ be a morphism in a small abelian category $\mathcal{A}$. Then $f = 0$ if and only if $\hat{f}\colon \hat{A} \to \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g, so $\hat{f} = 0$.

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\mathrm{id}_A]) = [0]$. But

$$\hat{f}([\mathrm{id}_A]) = [\mathrm{id}_A \circ f] = [f].$$

By Lemma 23, $[f] = [0]$ implies $f = 0$. $\square$

Corollary 25. A diagram commutes in $\mathcal{A}$ if and only if its image in $\mathbf{Set}_*$ under $\mathcal{M}$ commutes.

Proof. A diagram commutes in $\mathcal{A}$ if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. $\square$

Lemma 26. Let $f\colon A \to B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

2. Homological algebra in abelian categories

2.1. Chain complexes

Definition 27 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex $C_\bullet$ in $\mathcal{A}$ is a collection of objects $C_n \in \mathcal{A}$, $n \in \mathbb{Z}$, together with morphisms $d_n \colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms $C_\bullet \to D_\bullet$ of chain complexes, i.e. for each $n \in \mathbb{Z}$ a map $f_n \colon C_n \to D_n$ such that all of the squares form commute.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes (i.e. the full subcategory consisting of chain complexes $C_\bullet$ such that $C_n = 0$ for small enough n), $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Theorem 28. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the $\mathbf{Ab}$-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd}
 & Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\
C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet
\end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned}(d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q,\end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd} & & Q_\bullet \arrow[dll, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\ \ker(f_\bullet) \arrow[rr, "\iota_\bullet"] & & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \end{tikzcd}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{2.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi\colon Q_n \to \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 2.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned}\iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n\end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^{+}(\mathcal{A})$, etc., are also abelian.

2.2. Homology

We would like to replicate the notion of the homology of a chain complex $C_\bullet$ in the context of abelian categories.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n_1} \xrightarrow{d_{n-1}} \cdots$$

This will mean finding appropriate notions of every symbol appearing in the equation.

$$H_n(C_\bullet) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

We already have a notion of the kernel, which we can use to define a notion of the image in the following way. Given a map $f\colon A \to B$, we heuristically think of the cokernel of f as being $B/\operatorname{im} f$. Thus, taking the kernel of the cokernel should give us the image of f.

Definition 29 (image)**.** Given a morphism $f\colon A \to B$, the image of f is $\operatorname{im} f = \ker \operatorname{coker} f$.

Now we need to figure out what it means to quotient the image of one map by the kernel of the next. Again, the cokernel saves our necks; we just need a map from $\operatorname{im} d_{n+1}$ into $\ker d_n$. The homology will be the cokernel of this map.

Definition 30 (homology)**.** Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be morphisms such that $g \circ f = 0$. Using the universal properties of the kernel and cokernel, we can build the following commutative diagram.

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[d, hook, "i"] \arrow[dr, "0"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, two heads, "p"] & C \\
 & \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a monomorphism $\phi\colon \operatorname{im} f \to \ker g$.

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[r, hook, "\phi"] \arrow[dr, hook, "i"'] & \ker g \arrow[r, two heads] \arrow[d, hook] & H_n & \\
A \arrow[rr, "f"] & & B \arrow[rr, "g"] \arrow[d, two heads, "p"] & & C \\
 & & \operatorname{coker} f \arrow[urr] & &
\end{tikzcd}$$

The homology of the sequence $A \to B \to C$ at B is then $H_n = \operatorname{coker} \phi$.

Definition 31 (exact sequence)**.** Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

be objects and morphisms in an abelian category. We say that the above sequence is exact if ϕ_i is an isomorphism, i.e. if the homology is zero. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Proposition 32. Homology extends to a family of additive functors

$$H_n\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d^C_n \to \ker d^D_n$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. ◻

We can also go the other way, by defining a map

$$\iota_n\colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\begin{tikzcd} {} \arrow[r, dotted] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

concentrated in degree n.

Example 33. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then $H_0(C_\bullet) = \operatorname{coker} f$.

Similarly, if

$$0 \longrightarrow C_0 \xrightarrow{f} C_{-1} \longrightarrow \cdots$$

is a chain complex in $\mathbf{Ch}_{\leq 0}(\mathcal{A})$, then $H_0(C_\bullet) = \ker f$.

2.3. Diagram lemmas

2.3.1. The splitting lemma

Lemma 34 (Splitting lemma)**.** Consider the following solid short exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[r, "\pi_C"] \arrow[l, dashed, bend left, "\pi_A"] & C \arrow[r] \arrow[l, dashed, bend left, "i_C"] & 0 \end{tikzcd}$$

The following are equivalent.

1. The sequence *splits from the left,* i.e. there exists a morphism $\pi_A \colon B \to A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
2. The sequence *splits from the right,* i.e. there exists a morphism $i_C \colon C \to B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
3. B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

Proof. That $3 \Rightarrow 1$ and $3 \Rightarrow 2$ is obvious. We now show that $1 \Rightarrow 2$.

By exactness, $i_A \colon A \to B$ is a kernel of π_C. Consider the map $\phi = \mathrm{id}_B - i_C \circ \pi_C$. Since

$$\begin{aligned} \pi_C \circ \phi &= \pi_C - \pi_C \circ i_C \circ \pi_C \\ &= \pi_C - \mathrm{id} \circ \pi_C \\ &= 0, \end{aligned}$$

the universal property for the kernel guarantees us a map $\psi \colon B \to A$ such that $i_A \circ \psi = \phi$.

Now we use the same universal property in a different situation. Consider the following diagram.

$$\begin{tikzcd}
& A \arrow[dl, dashed] \arrow[d, "i_A"] \arrow[dr, "0"] & \\
A \arrow[r, "i_A"] & B \arrow[r, "\pi_C"] & C
\end{tikzcd}$$

The universal property guarantees us a unique dashed map making everything commute. But id_A and $i_A \circ \psi$ both do the job since

$$\begin{aligned} i_A \circ \psi \circ i_A &= i_A - i_C \circ \pi_C \circ i_A \\ &= i_A. \end{aligned}$$

□

2.3.2. The snake lemma

Lemma 35. Given a square

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
C \arrow[r, "k"] & D
\end{tikzcd}$$

in any abelian category, consider the sequence

$$A \xrightarrow{(f,-g)} B \oplus C \xrightarrow{(h,k)} D \ .$$

We have the following results.

1. The composition $(h, k) \circ (f, -g) = 0$ if and only if the square commutes.
2. The morphism $(f, -g) = \ker(h, k)$ if and only if the square is a pullback.
3. The morphism $(h, k) = \operatorname{coker}(f, -g)$ if and only if the square is a pushout.

Proof.

1. Obvious.
2.

□

Lemma 36. Let $f \colon A \to C$ be an epimorphism, and let $g \colon B \to C$ be any morphism. Then $(f, g) \colon A \oplus B \to C$ is an epimorphism.

Proof. Let $\alpha\colon C \to Z$ such that $\alpha \circ (f, g) = 0$. Consider the following diagram.

$$\begin{tikzcd}
A \arrow[r, hook, "i_A"] \arrow[dr, two heads, "f"'] \arrow[ddr, bend right, "\alpha\circ f"'] & A \oplus B \arrow[d, "{(f,g)}"] & B \arrow[l, hook', "i_C"'] \arrow[dl, "g"] \arrow[ddl, bend left, "\alpha\circ g"] \\
& C \arrow[d, "\alpha"] & \\
& Z &
\end{tikzcd}$$

We need to show that $\alpha = 0$. Certainly, we have that

$$\alpha \circ (f, g) \circ \iota_A = 0.$$

But this means that $\alpha \circ f = 0$; since f is an epimorphism this means that $\alpha = 0$. □

Lemma 37. Let $\mathcal{A}$ be an abelian category, and let the diagram

$$\begin{tikzcd}
A \times_C B \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "h"] \\
A \arrow[r, two heads, "k"] & C
\end{tikzcd}$$

be a pullback in $\mathcal{A}$. Then f is an epimorphism.

Proof. Since the above square is a pullback, the morphism $(f, -g) = \ker(h, k)$. But (h, k) is an epimorphism, so it is the cokernel of its kernel, implying that the square is also a pushout square. Since pushouts reflect epimorphisms, f is an epimorphism. □

Lemma 38. Let $f\colon B \twoheadrightarrow C$ be an epimorphism, and let $g\colon D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd}
S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

By Lemma 37 the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K is also a kernel of f'. Of course, in order for this statement to make sense we need a map $i\colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists ! i"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v\colon Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.

$$\begin{tikzcd}
& Q \arrow[dl, dashed, "\exists ! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{tikzcd}
Q \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists !"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{elements of pullback}\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{matrix}\text{elements of } B\\ \text{which map to } D\end{matrix}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\begin{matrix}\text{elements of } B\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 39 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
& \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd} \tag{2.2}$$

We will see why the dotted portions are exact later.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 2.2 the following diagram.

$$\begin{tikzcd}
 & & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}$$

Take a pullback and a pushout, and using Lemma 38 (and its dual), we find the following.

$$\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}$$

Consider the map

$$\delta_0 = \; S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \; .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \operatorname{coker} f \to \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 40 (snake lemma II)**.** Given the following commutative diagram with exact rows,

$$\begin{tikzcd}
 & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' &
\end{tikzcd}$$

we get a exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

2.3.3. The long exact sequence on homology

Corollary 41. Given an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta"] \\
H_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. For each n, we have a diagram of the following form.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d^A_{n+1}"'] & B_{n+1} \arrow[r] \arrow[d, "d^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma (Theorem 39) gives us, for each n, two exact sequences as follows.

$$0 \longrightarrow \ker d^A_{n+1} \longrightarrow \ker d^B_{n+1} \longrightarrow \ker d^C_{n+1}$$

and

$$\operatorname{coker} d^A_n \longrightarrow \operatorname{coker} d^B_n \longrightarrow \operatorname{coker} d^C_n \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{tikzcd}
 & \operatorname{coker} d^A_n \arrow[r] & \operatorname{coker} d^B_n \arrow[r] & \operatorname{coker} d^C_n \arrow[r] & 0 \\
0 \arrow[r] & \ker d^A_{n-1} \arrow[r] & \ker d^B_{n-1} \arrow[r] & \ker d^C_{n-1} &
\end{tikzcd}$$

Recall that for each n, we get a map

$$\phi^A_{n-1} \colon \operatorname{im} d^A_{n-1} \to \ker d^A_n,$$

and similarly for $B_\bullet$ and $C_\bullet$.

Adding these in gives the following.

$$\begin{tikzcd}
& \operatorname{coker} d^A_n \arrow[r] \arrow[d] & \operatorname{coker} d^B_n \arrow[r] \arrow[d] & \operatorname{coker} d^C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker d^A_{n-1} \arrow[r] & \ker d^B_{n-1} \arrow[r] & \ker d^C_{n-1} &
\end{tikzcd}$$

A second application of the snake lemma then gives a long exact sequence

$$\ker \phi_f \longrightarrow \ker \phi_g \longrightarrow \ker \phi_h \longrightarrow \operatorname{coker} \phi_f \longrightarrow \operatorname{coker} \phi_g \longrightarrow \operatorname{coker} \phi_h$$

□

Lemma 42. Let $\mathcal{A}$ be an abelian category, and let f be a morphism of short exact sequences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{tikzcd}
0 \arrow[r] & A'_\bullet \arrow[r, "\alpha_\bullet"] \arrow[d, "f'_\bullet"] & A_\bullet \arrow[r, "\alpha'_\bullet"] \arrow[d, "f_\bullet"] & A''_\bullet \arrow[r] \arrow[d, "f''_\bullet"] & 0 \\
0 \arrow[r] & B'_\bullet \arrow[r, "\beta_\bullet"'] & B_\bullet \arrow[r, "\beta'_\bullet"'] & B''_\bullet \arrow[r] & 0
\end{tikzcd}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha')"] \arrow[d, "H_n(f)"'] & H_n(A'') \arrow[r, "\delta"] \arrow[d, "H_n(f'')"] & H_{n-1}(A') \arrow[r, "H_{n-1}(\alpha)"] \arrow[d, "H_n(f')"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_n(f)"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "H_n(\beta')"] & H_n(B'') \arrow[r, "\delta"] & H_{n-1}(B') \arrow[r, "H_{n-1}(\beta)"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, the connecting morphism δ is functorial.

2.3.4. The five lemma

Lemma 43 (four lemmas)**.**

1. Let

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads, "b"'] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, two heads, "d"] & E \arrow[d, hook, "e"] \\
B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is an epimorphism.

2. Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "a"'] & B \arrow[r] \arrow[d, hook, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, hook, "d"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is a monomorphism.

Proof.

1. Expand the diagram with image-kernel factorization as follows.

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[d, two heads, "b"] & I_1 \arrow[r, hook] \arrow[d, "i"] & C \arrow[r, two heads] \arrow[d, "c"] & I_2 \arrow[r, hook] \arrow[d, "j"] & D \arrow[r, two heads] \arrow[d, two heads, "d"] & I_3 \arrow[r, hook] \arrow[d, "k"] & E \arrow[d, hook, "e"] \\
B' \arrow[r, two heads] & I_1' \arrow[r, hook] & C' \arrow[r, two heads] & I_2' \arrow[r, hook] & D' \arrow[r, two heads] & I_3' \arrow[r, hook] & E'
\end{tikzcd}$$

It is immediate that i is epi, and that k is mono. Now, consider the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_2 \arrow[r, hook] \arrow[d, "j"] & D \arrow[r, two heads] \arrow[d, two heads, "d"] & I_3 \arrow[r] \arrow[d, hook, "k"] & 0 \\
0 \arrow[r] & I_2' \arrow[r, hook] & D' \arrow[r, two heads] & I_3' \arrow[r] & 0
\end{tikzcd}$$

The snake lemma (Theorem 39) tells us that we have a short exact sequence

$$0 = \ker k \longrightarrow \operatorname{coker} j \longrightarrow \operatorname{coker} d = 0,$$

implying that $\operatorname{coker} j = 0$, i.e. that j is epi.

We have another morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_1 \arrow[r, hook] \arrow[d, two heads, "i"] & C \arrow[r, two heads] \arrow[d, "c"] & I_2 \arrow[r] \arrow[d, two heads, "j"] & 0 \\
0 \arrow[r] & I_1' \arrow[r, hook] & C' \arrow[r, two heads] & I_2' \arrow[r] & 0
\end{tikzcd}$$

Using the information that i and j are epi, the snake lemma tells us that j is epi as required.

2. Dual.

□

Theorem 44 (five lemma)**.** Let

$$\begin{tikzcd} A \arrow[r] \arrow[d, two heads] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "\cong"] & E \arrow[d, hook] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E' \end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms, epimorphisms, and isomorphisms as marked. Then c is an isomorphism.

Proof. The first part of Lemma 43 tells us that c is epi, and the second part tells us that c is mono. □

2.3.5. The nine lemma

Theorem 45 (nine lemma)**.** Given the following commuting diagram whose rows are exact sequences,

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & A_2 \arrow[r, hook] \arrow[d] & B_2 \arrow[r, two heads, "p"] \arrow[d] & C_2 \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & A_1 \arrow[r, hook] \arrow[d] & B_1 \arrow[r, two heads] \arrow[d] & C_1 \arrow[r] \arrow[d, "g"] & 0 \\ 0 \arrow[r] & A_0 \arrow[r, hook] \arrow[d] & B_0 \arrow[r, two heads] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

we have the following results. Note that we do not assume that the columns are chain complexes.

1. If the first and the second columns $A_\bullet$ and $B_\bullet$ are exact sequences, then third column $C_\bullet$ is also an exact sequence.
2. If $B_\bullet$ and $C_\bullet$ are exact sequence, then $A_\bullet$ is an exact sequence.
3. If $A_\bullet$ and $C_\bullet$ are exact and $B_\bullet$ is a chain complex, then $B_\bullet$ is exact.

Proof.

1. First, we show that $C_\bullet$ is a chain complex, i.e. that $g \circ f = 0$. Since p is an epimorphism, it suffices to show that $g \circ f \circ p = 0$. But this is zero by commutativity of the diagram and the fact that $B_\bullet$ is a chain complex.

Thus, the diagram above is a short exact sequence

$$0 \longrightarrow A_\bullet \hookrightarrow B_\bullet \twoheadrightarrow C_\bullet \longrightarrow 0$$

of chain complexes, and we get a long exact sequence on homology.

$$\begin{tikzcd}
 & & 0 \arrow[dll, "\delta"'] \\
H_2(A_\bullet) \arrow[r] & H_2(B_\bullet) \arrow[r] & H_2(C_\bullet) \arrow[dll, "\delta"'] \\
H_1(A_\bullet) \arrow[r] & H_1(B_\bullet) \arrow[r] & H_1(C_\bullet) \arrow[dll, "\delta"'] \\
H_0(A_\bullet) \arrow[r] & H_0(B_\bullet) \arrow[r] & H_0(C_\bullet) \arrow[dll, "\delta"'] \\
0 & &
\end{tikzcd}$$

If the first two columns are zeroes, then $H_n(C_\bullet) = 0$ by exactness; that is, $C_\bullet$ is exact.

2. Formally dual.
3. Identical.

□

2.4. Exact functors

Definition 46 (exact functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact.
- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

is exact.

- We call F exact if it is both left exact and right exact.

Example 47. Let

$$0 \longrightarrow A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \longrightarrow 0$$

be an exact sequence. First, we show that $\mathrm{Hom}(M, -)$ is left exact, i.e. that

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha\colon M \to A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and

$$f \circ \alpha = 0 \implies \alpha = 0$$

because f is a monomorphism.

Similarly, $\mathrm{im}\, f \subseteq \ker g$ because

$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \mathrm{im}\, f$. To this end, let $\gamma\colon M \to B$ such that $g_*\gamma = 0$.

$$\begin{tikzcd} & & M \arrow[dll, dashed, "\exists!\delta"'] \arrow[d, "\gamma"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0 \end{tikzcd}$$

Then $\mathrm{im}\, \gamma \subset \ker g = \mathrm{im}\, f$, so γ factors through A; that is to say, there exists a $\delta\colon M \to A$ such that $f_*\delta = \gamma$.

One can also check that the functor $\mathrm{Hom}(-, B)$ is left exact via the same sort of arguments. However, we will see in the next section that one can make a much more general statement by relating left and right exactness to preservation of limits. The next theorem gives us a taste of this.

Lemma 48. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories which is either left or right exact. Then F is additive, i.e. preserves finite (co)products.

Proof. First, assume that F is left exact. By Corollary 19, it suffices to show that F preserves direct sums. To this end, consider the image of a split exact sequence under a left-exact functor F.

$$0 \longrightarrow F(A) \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(\pi_B)} F(B) \longrightarrow 0$$

Because we still have the identities

$$F(\pi_A) \circ F(i_A) = \mathrm{id}_{F(A)}, \qquad F(\pi_B) \circ F(i_B) = \mathrm{id}_{F(B)},$$

the morphism $F(\pi_B)$ is still epic; hence the splitting lemma (Theorem 34) implies that the sequence is still split exact. Thus, F preserves direct sums.

The case in which F is right exact is dual. □

2.4.1. Exactness and preservation of (co)limits

The above definitions of left and right exactness are evocative, but one can cast them in a more useful way. One starting point is to notice that they are related to the preservation of limits.

Proposition 49. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
   - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.

- If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.

3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

□

As may now be clear, the converse is also true; any functor which is left exact preserves finite limits, and any functor which is right exact preserves finite colimits.

Proposition 50. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is left-exact, then F preserves finite limits.
- If F is right-exact, then F preserves finite colimits.

Proof. Suppose F is left exact. Then F preserves kernels and products, hence all finite limits. If F is right exact, then F preserves all cokernels and coproducts, hence all finite colimits. □

Our definition of homology now allows us to cleanly note the following.

Proposition 51. Exact functors preserve homology; that is, $H_n(F(C_\bullet)) = F(H_n(C_\bullet))$.

Proof. Exact functors preserve kernels and cokernels. □

Example 52. Consider the category $\mathcal{D}$ with objects and morphisms as follows.

$$
\begin{tikzcd}
 & \star \arrow[d] \\
* \arrow[r] & \bullet
\end{tikzcd}
$$

Let $\mathcal{A}$ be an abelian category. By Example 15, the category of functors $\mathcal{D} \to \mathcal{A}$ is an abelian category.

Consider a functor $\mathcal{D} \to \mathcal{A}$ which yields the following diagram in $\mathcal{A}$.

$$
\begin{tikzcd}
 & 0 \arrow[d] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

Taking the pullback of the above diagram yields the kernel of f.

A natural transformation ϕ between functors $\mathcal{D} \to \mathcal{A}$ consists of the data of a commuting square as follows.

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\phi_A"'] & B \arrow[d, "\phi_B"] \\
A' \arrow[r, "f'"] & B'
\end{tikzcd}
$$

That is, we may view taking kernels as a functor from the category of morphisms in $\mathcal{A}$ to $\mathcal{A}$. Furthermore, because this functor is the result of a limiting procedure it preserves limits. Thus, given an exact sequence of morphisms in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d, "f"'] & B \arrow[r, two heads] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads] & C \arrow[r] & 0 \end{tikzcd}$$

one gets an exact sequence of kernels

$$0 \longrightarrow \ker f \hookrightarrow \ker g \longrightarrow \ker h \ .$$

Conversely, taking cokernels is cocomplete, hence right exact, so one has also an exact sequence of cokernels

$$\operatorname{coker} f \longrightarrow \operatorname{coker} g \twoheadrightarrow \operatorname{coker} h \longrightarrow 0$$

Note that the data of a kernel

Example 53. Let J be a filtered category, and let $\mathcal{A}$ be an abelian category. Then the category $\mathcal{A}^J$ is abelian (see Example 15).

2.4.2. Exactness of the hom functor and tensor products

We have already seen that the hom functor $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is left exact in both slots. Now we understand precisely why this is true: the hom functor preserves limits in both slots![1]

Since any functor which is a right adjoint preserves limits, any right adjoint functor between abelian categories must be left exact. For example, there is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \mathrm{Hom}_{\mathbf{Ab}}(N,-)$$

which comes from replacing a map out of the tensor product by its corresponding bilinear map, and currying along the first entry. Thus, the map $- \otimes_R N$ is a left adjoint, hence preserves colimits, hence is right exact.

We can use this to check exactness (hence limit preservation) of other functors using the following beefed-up version of the Yoneda lemma.

Lemma 54. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

[1] In the first slot, one must take the limits in the category $\mathcal{A}^{\mathrm{op}}$.

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X,C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A,C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\mathrm{im}\, f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\mathrm{im}\, \iota = \ker g \subset \mathrm{im}\, f$. □

2.5. Quasi-isomorphism

One of the organizing principles of homological algebra is that the fundamental data contained in a chain complex is the level-wise homology of that chain complex, rather than the chain complex itself. We will see many examples (especially in Part II, on algebraic topology) in which one can perform a drastic simplification of a problem by replacing a chain complex by a better-behaved one, together with a homology-preserving map between them.

For this reason, one would like to understand morphisms of chain complexes $f_\bullet\colon C_\bullet \to D_\bullet$ which induce isomorphisms on homology. One even has a special name for such a morphism.

Definition 55 (quasi-isomorphism). Let $f_\bullet\colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Quasi-isomorphisms are a surprisingly fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope that quasi-isomorphisms themselves were invertible; that is, that one could find a *quasi-inverse.* Unfortunately, we are not so lucky. Consider the following morphism of chain complexes.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

This map induces an isomorphism on homology, but there is no non-trivial map in the other direction.

2.6. Chain Homotopies

Chain homotopies are, at this point, best thought of as a technical tool which allow us to better understand quasi-isomorphisms. The interested reader should look forward to Part II, where we will reach an understanding of chain homotopies as an algebraic analog of a homotopy between maps of topological spaces.

Definition 56 (chain homotopy)**.** Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A chain homotopy between $f_\bullet$ and $g_\bullet$ is a family of maps

$$h_n \colon C_n \to D_{n+1}$$

such that

$$d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n = f - g.$$

Chain homotopies will help us to understand the slippery notion of quasi-isomorphism because chain homotopic maps will turn out to induce the same maps on homology.

Proposition 57. Let $f, g \colon C_\bullet \to D_\bullet$ be chain homotopic maps via a homotopy h. Then for all n,

$$H_n(f) = H_n(g).$$

Proof. Since homology is additive, it suffices to show that any map which is homotopic to zero induces the zero map on homology, i.e. that if we can write

$$f = d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n,$$

then $H_n(f) = 0$ for all n. □

This is very useful, both theoretically and practically. We will often exploit this in order to prove that a complex is exact by showing that the identity morphism is homotopic to the zero morphism.

Lemma 58. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 59. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 60. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

2.7. Projectives and injectives

One of the drawbacks of diagrammatic reasoning in abelian categories (which is to say, reasoning on elements in R-**Mod**) is that not every maneuver one can perform on elements has a diagrammatic analog. One of the most egregious examples of this is the property of lifting against epimorphisms. Given an epimorphism $f \colon A \twoheadrightarrow B$ and an element of B, one can always find an element of A which maps to it under f. Unfortunately, it is *not* in general the case that given a map $X \to B$, one can lift it to a map $X \to A$

making the triangle formed commute.

$$\begin{tikzcd} & X \arrow[d] \arrow[dl, dashed, "\nexists"'] \\ A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

However, certain objects X do have this property. For example, if X is a free R-module, then we can find a lift by lifting generators; this is always possible. It will be interesting to find objects X which allow this. Such objects are called *projective*.

We will eventually be interested in projective objects because of the control they give us over quasi-isomorphisms.

2.7.1. Basic properties

Definition 61 (projective, injective). An object P in an abelian category $\mathcal{A}$ is projective if for every epimorphism $f\colon B \to C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[dl, dashed, "\exists \tilde{p}"'] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g\colon A \to B$ and every morphism $q\colon A \to Q$ there exists a morphism $\tilde{q}\colon B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists \tilde{q}"] \\ Q & \end{tikzcd}$$

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A, it is very easy to see simply by unwrapping the definitions that an object P in an abelian category is projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

2.7.2. Enough projectives

A recurring theme of homological algebra is that projective and injective objects are extremely nice to work with, and whenever possible we would like to be able to trade in an object or chain complex for an object or chain complex of projectives which is (quasi-)isomorphic. We will go into more detail about precisely what we mean by this later. This will be possible when our abelian category has *enough projectives*. Dually, we can replace objects or chain complexes by injectives when we have *enough injectives.* As the abstract theory of projectives is dual to that of injectives, we will for the remainder of this chapter confine ourselves to the study of projectives.

Definition 62 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

The category in which we have mainly been interested is R-**Mod**. We will now see that R-**Mod** has enough projectives; we need only prove the following lemma.

Lemma 63. Let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, we get a morphism $P \to B$ making the triangle commute.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] \arrow[dl, "\exists g"'] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma 34). □

Proposition 64. The category R-**Mod** has enough projectives.

We see this by proving the following characterization of projectives in R-**Mod**: P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M.$$

This will be sufficient because any module can be written as a quotient of a free module.

First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$0 \longrightarrow \ker \pi \hookrightarrow F \xrightarrow{\pi} P \longrightarrow 0$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker \pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{tikzcd} & P \oplus M \arrow[dl, dashed, "\exists(j,k)"'] \arrow[d, "(f,g)"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists j"'] \arrow[d, "f"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

exhibiting P as projective.

2.7.3. Here be dragons

There are a few subtleties to the theory of projectives and injectives of which one should be aware.

- We have said that the abstract theory of injectives is dual to that of projectives. This is true, but does not imply that projectives and injectives are somehow on equal footing. For example, in R-**Mod** we have given a complete classification of projective modules: they are direct summands of free modules. There is no such classification of injective modules. Despite this, R-**Mod** does turn out to have enough injectives.
- We have so far mostly avoided the use of element-theoretic arguments, which are possible thanks to the Freyd-Mitchell embedding theorem. This is partially due to an aesthetic distaste, and partially because they can be tricky.

2.7.4. Projective resolutions

We are now ready to enact a part of our goal of replacing an object or chain complex with a (quasi-)isomorphic chain complex of projectives: if we are working in an abelian category with enough projectives, we can replace an object A with a quasi-isomorphic chain complex of projectives.

More concretely, let $\mathcal{A}$ be an abelian category, and denote by ι the functor which takes an object of $\mathcal{A}$ and returns a chain complex with that object concentrated in degree zero. Clearly, the homology of $\iota(A)$ is

$$H_n(\iota(A)) = \begin{cases} A, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Our goal, namely replacing A by a chain complex which contains the same homological data, can certainly be achieved by finding a chain complex with the same homology, and a quasi-isomorphism from it to $\iota(A)$. Such an object is known as a *resolution.*

Definition 65 (resolution)**.** Let A be an object in an abelian category. A resolution of A consists of a chain complex $C_\bullet$ together with a quasi-isomorphism $C_\bullet \xrightarrow{\simeq} \iota(A)$.

What we want is a *projective resolution,* i.e. a resolution made out of projective objects.

Definition 66 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \xrightarrow{\simeq} \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \xrightarrow{\simeq} Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

We said that our success in our goal would be predicated on the existence of enough projectives. Indeed, this is the case. In order to see this, we change our view on projective resolutions. Rather than a quasi-isomorphism between chain complexes whose homology is concentrated in degree zero, we should look at a projective resolution as a single, exact chain complex.

Lemma 67. Let $f_\bullet \colon P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots\cdots\; P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Proposition 68. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Proof. Pick a projective which surjects onto A, and □

2.7.5. Lifting data to projective resolutions

We have seen that for every object in in abelian category $\mathcal{A}$ with enough projectives, we can find a *projective resolution,* i.e. a complex $P_\bullet$ of projectives and a quasi-isomorphism $P_\bullet \xrightarrow{\simeq} \iota(A)$. We would like to be able to extend this to a functor. This idea turns out not to work precisely as we would like, but it is nonetheless helpful to pursue it for a while.

The first thing we need is a way of lifting morphisms in $\mathcal{A}$ to morphisms between projective resolutions.

Lemma 69. Let $P_\bullet$ be a chain complex of projectives with $P_n = 0$ for $n \leq 0$, and let $D_\bullet$ be a chain complex with $D_n = 0$ for $n \leq 0$ such that $H_n(D) = 0$ for $n > 0$. Let $f\colon P_\bullet \to D_\bullet$ such that $f_0 = 0$.

1. For any $f_0\colon P_0 \to D_0$, there exists a lift to a chain map $f\colon P \to D$.
2. Any such lifts $f, f'\colon P \to D$ are chain homotopic.

Proof. 1. We construct such a lift inductively.

2. It suffices to show that any lift of the zero morphism is chain homotopic to zero.

□

Proposition 70. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P'_\bullet \to M'$$

be projective resolutions. Then for every morphism $f\colon M \to M'$ there exists a lift $\tilde{f}\colon P_\bullet \to P'_\bullet$ making the diagram

$$\begin{tikzcd} P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P'_\bullet \arrow[d] \\ \iota M \arrow[r, "\iota f"'] & \iota M' \end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Theorem 71 (horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, hook, "f"] & M \arrow[r, two heads, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd}
0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "f"'] & M \arrow[r, two heads, "\tilde{g}"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd}
 & \vdots \arrow[d, dotted, no head] & & \vdots \arrow[d, dotted, no head] & \\
0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook, "f"] & M \arrow[r, two heads, "g"] & M'' \arrow[r] & 0
\end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd}
0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f\circ p'_0"'] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0
\end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P'_0 \oplus P''_0 \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P_1'' to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$
\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r, dashed] \arrow[d] & \ker p \arrow[r, dashed] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook] & M \arrow[r, two heads] & M'' \arrow[r] & 0
\end{tikzcd}
$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. We get the dashed map $\ker p' \to \ker p''$

That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Theorem 72 (horseshoe lemma for morphisms)**.** Given a morphism of short exact sequences

$$
\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, hook] \arrow[d, "f'"] & A \arrow[r, two heads] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, hook] & B \arrow[r, two heads] & B'' \arrow[r] & 0
\end{tikzcd}
$$

Theorem 73. Let $\mathcal{A}$ be an abelian category with enough projectives, and let $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then there exists a

2.8. The homotopy category

In the previous section, we proved many theorems which were schematically of the following form:

Given some data in an abelian category, we can lift it to data between projective resolutions.

One might hope that this data would assemble itself functorially, but this is far too much to ask; we have no reason to believe, having picked lifts $\tilde{f}$ for our morphisms f, that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

In this, however, we are saved by the notion of homotopy! While $\widetilde{f \circ g}$ and $\tilde{f} \circ \tilde{g}$ may not be equal, they are certainly homotopic, thanks to Proposition 70. Thus, we will have a shot at functoriality, if we formally invert all homotopies.

And this we can do!

Definition 74 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

This is well-defined by Lemma 59 and Lemma 60.

We will in general not be interested in the full homotopy category, but instead in the bounded-below homotopy category $\mathcal{K}^{+}(\mathcal{A})$ of *bounded-below* chain complexes up to homotopy.

Lemma 75. The homotopy category $\mathcal{K}^{+}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{K}^{+}(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

The upshot of the above discussion is that for any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}^{+}(\mathcal{A})$ defined as follows.

1. Pick a projective resolution of every object in $\mathcal{A}$.
2. Pick a lift of every morphism of $\mathcal{A}$, and take equivalence classes.

This functor has

2.9. Mapping cones

Mapping cones are a technical tool, borrowed from algebraic topology. We will use them to great profit in the next section. For now,

Definition 76 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the k-shifted chain complex $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 77 (mapping cone)**.** Let $f\colon C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\operatorname{cone}(f)$ as follows.

- For each n, define
$$\operatorname{cone}(f)_n = C_{n-1} \oplus D_n.$$

- Define $d_n^{\mathrm{cone}(f)}$ by
$$d_n^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$
which is shorthand for
$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 78. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow D_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} C[-1]_\bullet \longrightarrow 0 \ .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 79. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. By Corollary 41, we get a long exact sequence

$$\cdots \to H_n(C) \xrightarrow{\delta} H_n(D) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(C) \to \cdots \ .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see; picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0, f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. □

Proposition 80. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$.

Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g\colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots \\
\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
 & & P_1 \arrow[ddll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists!"] \arrow[dd, "r"] \arrow[ddr, "0"] & \\
 & K \arrow[dr, hook] & & \\
C_2 \arrow[ur, two heads] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\operatorname{cone}(f)$ is exact, which means that $\iota\colon P \to$

cone(f) is homotopic to the zero morphism.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d^P_{i+1}"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d^P_i"] \arrow[dl, dashed, "\tilde{h}_i"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d^{\mathrm{cone}(f)}_{i+1}"'] & C_{i-1} \oplus P_i \arrow[r, "d^{\mathrm{cone}(f)}_i"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}$$

That is, there exist $\tilde{h}_i \colon P_i \to \mathrm{cone}(f)_{i+1}$ such that

$$d^{\mathrm{cone}\, f}_{i+1} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d^P_i = \iota. \tag{2.3}$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i \colon P_i \to C_i \qquad \text{and} \qquad \gamma_i \colon P_{i-1} \to P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet \colon P_\bullet \to C_\bullet$. Indeed, writing Equation 2.3 in components, we find the following.

$$\begin{pmatrix} -d^C_i & 0 \\ -f_i & d^P_{i+1} \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d^P_i \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$
$$\begin{pmatrix} -d^C_i \circ \beta_i + \beta_{i-1} \circ d^P_i \\ -f_i \circ \beta_i + d^P_{i+1} \circ \gamma_i + \gamma_{i-1} \circ d^P_i \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that

$$d^P_{i+1} \circ \gamma_i + \gamma_{i-1} \circ d^P_i = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

2.10. Derived functors

2.10.1. Derived categories

Hopefully, the reader will now agree that chain complexes are interesting enough to study, and that given a chain complex, the level-wise homology (and therefore the slippery notion of quasi-isomorphism) is an interesting invariant. Of course, one considers the category of chain complexes, but this category has the annoying feature that quasi-isomorphisms do not necessarily have quasi-inverses. That is, it is difficult to view quasi-isomorphism as a notion of equivalence because is it is not invertible.

The way out of this is to replace the category $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ by a category in which all quasi-isomorphisms have quasi-inverses. There is a fiddly and conceptually unclear way to do that, namely by forming the categorical localization.

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{Ch}_{\geq 0}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. By

2.10.2. Derived functors

Definition 81 (derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the <u>left derived functor</u> to be the homological δ-functor $\{L_nF\}$ defined by

$$L_nF = \ \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

  where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the <u>right derived functor</u> to be the cohomological δ-functor

$$R^nF(X) = H^n \circ F \circ Q$$

  where Q is an injective resolution functor.

Definition 82 (homological δ-functor)**.** A <u>homological δ-functor</u> between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor
$$T_n \colon \mathcal{A} \to \mathcal{B}.$$
2. For every short exact sequence
$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$
in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism
$$\delta_n \colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence
$$\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$
3. For every morphism of short exact sequences
$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$
and every n, the diagram
$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}$$
commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

A morphism $S \to T$ of δ-functors consists of, for each n, a natural transformation $S_n \to$

T_n compatible with the δ.

This seems like an inelegant definition, and indeed it is.

Example 83. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors,* come from homology.

Definition 84 (cohomological δ-functors)**.** Dual to Definition 82.

We still need to show that derived functors are in fact homological delta-functors.

Theorem 85. For any right exact functor $F \colon \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ has enough projectives, the left derived functor $\{L_n F\}$ is a homological δ-functor.

Proof. We check each condition separately.

1. $L_n F$ is computed by taking the nth homology of a chain complex concentrated in positive degree.
2. Start with a short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

   in $\mathcal{A}$.

   We compute the image of this sequence under the derived functors $L_n F$ by using our chosen projective resolution functor to find projective resolutions and lift maps, and take homology. In general, we cannot be guaranteed that we can lift our exact sequence up to an exact sequence of projective resolutions

$$0 \longrightarrow P^A_\bullet \hookrightarrow P^B_\bullet \twoheadrightarrow P^C_\bullet \longrightarrow 0\ .$$

   However, we have shown that, up to isomorphism, it doesn't matter which projective resolutions we choose, so we can choose those given by the horseshoe lemma (Theorem 71), giving us such an exact sequence. Applying F then gives us another exact sequence, and taking homology gives a long exact sequence.
3. We do the same thing, but now apply the horseshoe lemma on morphisms (Theorem 72).

□

Proposition 86. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0 F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 67, the following sequence is exact.

$$P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But

$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$

□

Given a right exact functor F, the above theorem shows that, at the lowest level, the left derived functor gives F itself. It turns out that $\{L_nF\}$ is the *universal* homological δ-functor with this property:

Proposition 87. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories where $\mathcal{A}$ has enough projectives, let $\{T_n\}$ be a homological δ-functor, and let $\phi_0\colon T_0 \Rightarrow F = L_0F$ be a natural transformation. Then there exists a unique lift of ϕ_0 to a morphism $\{\phi_i\colon T_i \to L_iF\}$.

Proof. First we construct $\phi_1\colon T_1 \Rightarrow L_1F$. For each object $A \in \mathcal{A}$, pick a projective which surjects onto it and take a kernel, giving the following short exact sequence.

$$0 \longrightarrow K \hookrightarrow P \twoheadrightarrow A \longrightarrow 0$$

Applying each of our homological δ-functors and using the fact that $L_iFP = 0$ for $i > 0$, we get the following data.

$$\begin{tikzcd}
 & T_1A \arrow[r, "\delta"] & T_0K \arrow[r] \arrow[d, "(\phi_0)_K"] & T_0P \arrow[r] \arrow[d, "(\phi_0)_P"] & T_0A \arrow[r] \arrow[d, "(\phi_0)_A"] & 0 \\
0 \arrow[r] & L_1FA \arrow[r, hook, "\delta"] & FK \arrow[r] & FP \arrow[r] & FA \arrow[r] & 0
\end{tikzcd}$$

Since $L_1FA \to FK$ is a kernel of $T_0K \to T_0P$, we get a unique map $T_1A \to L_1FA$, which we declare to be $(\phi_1)_A$.

It remains to show that these are really the components of a natural transformation. Let $f\colon A \to A'$ be a morphism in $\mathcal{A}$. Consider any induced morphisms of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook] \arrow[d] & P \arrow[r, two heads] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K' \arrow[r, hook] & P' \arrow[r, two heads] & A' \arrow[r] & 0
\end{tikzcd}$$

Since $\{L_nF\}$ and $\{T_n\}$ are homological δ-functors, this gives us the following cube, where the components $(\phi_1)_A$ and $(\phi_1)_{A'}$ are dashed. All faces except the one on the left commute by definition.

$$
\begin{tikzcd}
& T_1A' \arrow[rr] \arrow[dd, dashed, "(\phi_1)_{A'}"] & & T_0K' \arrow[dd] \\
T_1A \arrow[ur, "T_1f"] \arrow[rr] \arrow[dd, dashed, "(\phi_1)_A"'] & & T_0K \arrow[ur] \arrow[dd] & \\
& L_1FA' \arrow[rr, hook, "\delta"] & & FK' \\
L_1FA \arrow[ur, "L_1Ff"] \arrow[rr] & & FK \arrow[ur] &
\end{tikzcd}
$$

In order to check that the $(\phi_1)_A$ form the components of a natural transformation, we need to check that the left face of the above cube commutes. Because $\delta\colon L_1FA' \to FK'$ is a monomorphism, it suffices to check that

$$\delta \circ (\phi_1)_{A'} \circ T_1F = \delta \circ L_1Ff \circ (\phi_1)_A.$$

But given the commutativity of the other faces of the cube, this is clear.

We now continue inductively, constructing ϕ_2, etc.

The last step is to show that the collection $\{\phi_n\}$ is really a morphism of homological δ-functors, i.e. that for any short exact sequence

$$0 \longrightarrow A'' \hookrightarrow A \twoheadrightarrow A' \longrightarrow 0$$

the squares

$$
\begin{tikzcd}
T_nA'' \arrow[r, "\delta"] \arrow[d, "(\phi_n)_{A''}"'] & T_{n-1}A' \arrow[d, "(\phi_{n-1})_{A'}"] \\
L_nFA'' \arrow[r, "\delta"'] & L_{n-1}FA'
\end{tikzcd}
$$

commute □

2.11. Double complexes

Given any abelian category $\mathcal{A}$, we have considered the category of chain complexes in $\mathcal{A}$, and shown (in Theorem 28) that this is an abelian category. It is therefore natural to consider chain complexes in the category of chain complexes.

$$\left(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \right) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

Such a beast is a lattice of commuting squares.

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & M_{i-1,j} \arrow[r] \arrow[d] & M_{i,j} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M_{i-1,j-1} \arrow[r] \arrow[d] & M_{i,j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}
$$

However, when manipulating these complexes it turns out to convenient to do something slightly different, namely work with complexes whose squares anticommute rather than commute. This is not a fundamental difference, as one can make from any lattice of commuting squares a lattice of anticommuting squares by multiplying the differentials of every other column by -1. This is known as the *sign trick.*

Definition 88 (double complex). Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d^h_{i,j} \colon M_{i,j} \to M_{i-1,j}, \qquad d^v_{i,j} \colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $(d^h)^2 = 0$
- $(d^v)^2 = 0$
- $d^h \circ d^v + d^v \circ d^h = 0$

We say that a double complex is $M_{\bullet,\bullet}$ is *first-quadrant* if $M_{i,j} = 0$ for $i, j < 0$.

Given any complex of complexes, one can construct a double complex by multiplying the differentials in every other row or column by -1.

Definition 89 (total complex). Let $M_{\bullet,\bullet}$ be a first-quadrant[2] double complex. The total complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

[2]This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

Lemma 90. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Example 91. Let $C_\bullet \in \mathbf{Ch}(\mathbf{Mod}\text{-}R)$ and $D_\bullet \in \mathbf{Ch}(R\text{-}\mathbf{Mod})$ be chain complexes. We can form a double complex by sticking $C_\bullet$ into the first slot of the tensor product, $D_\bullet$ into the second, and using the sign trick to make the squares anticommute. The total complex of this double complex is known as the *tensor product* of $C_\bullet$ and $D_\bullet$.

In general, computing the homology of a double complex is very difficult; we will explore some techniques for this in Chapter 5. However, under some special conditions, we can say something.

Theorem 92. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & \iddots & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & \iddots & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \iddots \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$\begin{array}{ccc}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} & = & \mathrm{Tot}(M)_3 \\
\downarrow & & \downarrow \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} & = & \mathrm{Tot}(M)_2 \\
\downarrow & & \downarrow \\
M_{1,0} \oplus M_{0,1} & = & \mathrm{Tot}(M)_1 \\
\downarrow & & \downarrow \\
M_{0,0} & = & \mathrm{Tot}(M)_0 \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

is exact.

The proof is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we give the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \qquad d_{11}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}}n = m$.

It turns out that we can make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker\delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta d n_{1,2},$$

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned} d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d+\delta)n_{2,1} + (d+\delta)n_{1,2} + (d+\delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. □

Example 93. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Form from this a double complex by multiplying the differential of $C_\bullet$ by -1 and padding by zeroes on either side.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & D_3 \arrow[l] \arrow[d, "d_3^D"'] & C_3 \arrow[l, "f_3"'] \arrow[d, "-d_3^C"] & 0 \arrow[l] \\
0 & D_2 \arrow[l] \arrow[d, "d_2^D"'] & C_2 \arrow[l, "f_2"'] \arrow[d, "-d_2^C"] & 0 \arrow[l] \\
0 & D_1 \arrow[l] \arrow[d, "d_1^D"'] & C_1 \arrow[l, "f_1"'] \arrow[d, "-d_1^C"] & 0 \arrow[l] \\
0 & D_0 \arrow[l] \arrow[d, "d_0^D"'] & C_0 \arrow[l, "f_0"'] \arrow[d, "-d_0^C"] & 0 \arrow[l] \\
0 & D_{-1} \arrow[l] \arrow[d] & C_{-1} \arrow[l, "f_{-1}"'] \arrow[d] & 0 \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

The total complex of this is simply $\mathrm{cone}(f)$.

Now suppose that $f_\bullet$ is an isomorphism, so the rows are exact. Then by Theorem 92, $\mathrm{Tot}(f) = \mathrm{cone}(f)$ is exact. We have already seen this (in one direction) in Corollary 79.

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0.

We can view this in two ways.

- As a chain

$$\cdots \longrightarrow C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$

  hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;
- As a morphism between double complexes, hence between totalizations

$$\mathrm{Tot}(\alpha)_\bullet \colon \mathrm{Tot}(C)_\bullet \to \mathrm{Tot}(D)_\bullet.$$

Lemma 94. These two points of view agree in the sense that

$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 95. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{tikzcd}
\cdots \arrow[r] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet
\end{tikzcd}$$

be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$

is exact. By Theorem 92, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 94 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 79) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 96. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now Theorem 95 gives us the result we want. □

2.11.1. Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, in the category R-**Mod**, there *is* a canonical tensor product, and we would like to be able to use homological algebra to understand it.

One of the fundamental properties of the tensor product (indeed, the property that justifies giving a universal construction the name tensor product) is that there is an adjunction between the tensor product and the hom functor. In this section, we will define that the tensor product on R-**Mod** extends to a tensor product on **Ch**(R-**Mod**), and that this is left adjoint to an internal hom functor on **Ch**(R-**Mod**).

Definition 97 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$(C \otimes D)_\bullet = \operatorname{Tot}(C_\bullet \otimes D_\bullet),$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

The next step is to introduce the notion of the internal hom. The way to do this is the obvious one: we should have

$$\mathbf{Hom}(C_\bullet, D_\bullet) = \mathrm{Tot}(\mathrm{Hom}(C_\bullet, D_\bullet)).$$

Unfortunately, even if $C_\bullet$ and $D_\bullet$ are bounded below, the double complex above will be second-quadrant rather than first, so our definition Definition 89 of the total complex will not do. The reason for this mismatch is that, for double complexes which are not first quadrant, one must choose between the product and the coproduct, which no longer agree for infinite indexing sets.

In our case, the product will be the correct choice. For this reason, we make the following definition.

Definition 98 (internal hom)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. The internal hom functor on $\mathbf{Ch}(\mathcal{A})$ is the functor

$$\mathbf{Hom}_{\mathcal{A}} \colon \mathbf{Ch}(\mathcal{A})^{\mathrm{op}} \times \mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})$$

is defined by

$$(\mathbf{Hom}_{\mathcal{A}})_n = \prod_{p+q=n} \mathrm{Hom}(C_{-p}, D_q),$$

and

$$d = d^C + (-1)^p d^D,$$

where the factor of $(-1)^p$ implements the sign trick.

Theorem 99. There is an isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(C_\bullet \otimes_R D_\bullet, E_\bullet) \cong \mathrm{Hom}_{\mathcal{A}}(C_\bullet, \mathbf{Hom}(D_\bullet, E_\bullet)),$$

natural in everything.

Proof. This is clear level-wise, since

$$\begin{aligned}
\mathrm{Hom}((C_\bullet \otimes_R D_\bullet)_{-n}, E_m) &\cong \mathrm{Hom}\Big(\coprod_{p+q=-n} C_p \otimes_R D_q, E_m\Big) \\
&\cong \prod_{p+q=-n} \mathrm{Hom}(C_p \otimes_R D_q, E_m) \\
&\cong \prod_{p+q=-n} \mathrm{Hom}(C_p, \mathrm{Hom}(D_q, E_m))
\end{aligned}$$

□

2.11.2. The Künneth formula

This section takes place in **Ab**. The same result holds in R-**Mod**, for R a PID, but we will only need the **Ab** case.

Theorem 100 (Künneth)**.** Let $C_\bullet, D_\bullet \in \mathbf{Ch}(\mathbf{Ab})$, and suppose that $C_\bullet$ is free, that is, that C_n is free for all n. Then there is a short exact sequence

$$0 \to \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \hookrightarrow H_n(C_\bullet \otimes D_\bullet) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \to 0 \ .$$

Proof. Denote by $Z^C_\bullet$ the chain complex given level-wise by $Z^C_n = Z_n(C_\bullet)$, and with trivial differentials. Define $B^C_\bullet$ and $H^C_\bullet$ similarly. Then, essentially by definition, we have the following short exact sequences of chain complexes.

$$0 \longrightarrow Z^C_\bullet \hookrightarrow C_\bullet \twoheadrightarrow B^C[-1]_\bullet \longrightarrow 0 \tag{2.4}$$

$$0 \longrightarrow B^C_\bullet \hookrightarrow Z^C_\bullet \twoheadrightarrow H^C_\bullet \longrightarrow 0 \tag{2.5}$$

We happily note that Sequence 2.5, taken level-wise, gives a free resolution of $H^C_\bullet$.

Since Sequence 2.4 is level-wise split, applying $- \otimes D$ takes the rows to short exact sequences. Since a short exact sequence of chain complexes is simply a sequence of chain complexes whose rows are short exact, the following sequence of chain complexes is short exact.

$$0 \longrightarrow (Z^C \otimes D)_\bullet \hookrightarrow (C \otimes D)_\bullet \twoheadrightarrow (B^C[-1] \otimes D)_\bullet \longrightarrow 0 \ .$$

Thus, we get a long exact sequence on homology.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(B^C[-1] \otimes D) \arrow[dll, "\delta"] \\
H_n(Z^C \otimes D) \arrow[r] & H_n(C \otimes D) \arrow[r] & H_n(B^C[-1] \otimes D) \arrow[dll, "\delta"] \\
H_{n-1}(Z^C \otimes D) \arrow[r] & \cdots &
\end{tikzcd}$$

Consider several terms of the chain complex $(B^C[-1] \otimes D)_\bullet$.

$$\cdots \longrightarrow \bigoplus_{p+q=n+1} B^C_{p-1} \otimes D_q \longrightarrow \bigoplus_{p+q=n} B^C_{p-1} \otimes D_q \longrightarrow \cdots$$

Because of the triviality of the differential of $B^C[-1]$, the above morphisms are a direct sum of morphisms

$$B^C_{p-1} \otimes D_q \xrightarrow{\mathrm{id}_B \otimes d^D} B^C_{p-1} \otimes D_{q-1}\ .$$

Hence, because B^C_{p-1} is free, the homology of $(B^C[-1] \otimes D)_\bullet$ is

$$\begin{aligned} H_n(B[-1] \otimes D) &= \bigoplus_{p+q=n} B_{p-1} \otimes H_n(D) \\ &= \bigoplus_{p+q=n=1} B_p \otimes H_n(D) \\ &= (B^C \otimes H^D)_{n-1}, \end{aligned}$$

where we have defined $H^D_\bullet$ in analogy with $H^C_\bullet$.

Exactly the same reasoning as above shows that

$$H_n(Z \otimes D) = (Z \otimes H^D)_n.$$

Making these replacements in our long exact sequence gives us the following.

$$\begin{tikzcd}
 & \cdots \arrow[r] & (B^C \otimes H^D)_n \arrow[dll, "\delta"'] \\
(Z^C \otimes H^D)_n \arrow[r] & H_n(C \otimes D) \arrow[r] & (B^C \otimes H^D)_{n-1} \arrow[dll, "\delta"'] \\
(Z^C \otimes H^D)_{n-1} \arrow[r] & \cdots &
\end{tikzcd}$$

A quick computation shows that

$$\delta = \bigoplus_{p+q=n} (B^C_p \hookrightarrow Z^C_p) \otimes \mathrm{id}_{H^D_q}.$$

The kernel and cokernel of the connecting homomorphism are refreshingly familiar. The kernel corresponds precisely to our definition of Tor coming from the projective resolution from Sequence (2.5), and the cokernel is the definition of nth homology.

$$\begin{tikzcd}
\bigoplus_{p+q=n} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H^D_q) \arrow[d, equal] & & & \bigoplus_{p+q=n} H_p(C) \otimes H^D_q \arrow[d, equal] \\
\ker \delta \arrow[r, hook] & \bigoplus_{p+q=n} B^C_p \otimes H^D_q \arrow[r, "\delta"] & \bigoplus_{p+q=n} Z^C_p \otimes H^D_q \arrow[r, two heads] & \operatorname{coker} \delta
\end{tikzcd}\ .$$

We can decorate our long exact sequence with these to find the following solid short exact sequence.

$$\begin{tikzcd}
& 0 \arrow[dr] & & & & & \\
& & \bigoplus\limits_{p+q=n} H_p(C) \otimes H_q^D \arrow[dr, hook] & & & & \\
\cdots \arrow[r] & (Z^C \otimes H^D)_n \arrow[ur, dashed] \arrow[rr, dashed] & & H_n(C \otimes D) \arrow[rr, dashed] \arrow[dr, two heads] & & (B^C \otimes H^D)_{n-1} \arrow[r] & \cdots \\
& & & & \bigoplus\limits_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H_q^D) \arrow[ur, dashed] \arrow[dr] & & \\
& & & & & 0 &
\end{tikzcd}$$

Replacing H_n^C by $H_n(C)$ and H_n^D by $H_n(D)$, we find precisely the sequence we were looking for. □

Proposition 101. The short exact sequence in the Künneth formula splits, but not canonically.

Corollary 102 (universal coefficient theorem)**.** Let $C_\bullet$ be a chain complex of free abelian groups, and let A be an abelian group. Then there is a short exact sequence

$$0 \longrightarrow H_n(C) \otimes A \hookrightarrow H_n(C \otimes A) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(C), A) \longrightarrow 0\ .$$

3. Some important derived functors

3.1. The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $-\otimes_R N$ preserves colimits, hence by Proposition 49 is right exact. Thus, we may form the left derived functor.

Definition 103 (Tor functor)**.** The left derived functor of $-\otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 104. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free (hence projective) resolution of R/rR. Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i\left(0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & i = 0 \\ {}_rN, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for any abelian group D, $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, D) =_n D$, the n-torsion elements of D.

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

Proposition 105. The functor Tor is *balanced;* that is,

$$L_i \operatorname{Hom}(M, -)(N) \simeq L_i \operatorname{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
 & P_0 \otimes_R N & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\operatorname{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

Example 106. Let N be an R-module. The Tor functor controls flatness in the sense that the following are equivalent:

1. The module N is flat
2. For all R-modules M, we have $\operatorname{Tor}_1^R(M,N) = 0$.

To see that 1. ⇒ 2., suppose that N is flat and pick a projective resolution $P_\bullet \xrightarrow{\simeq} M$. Then

$$\begin{aligned}\operatorname{Tor}_1(M,N) &= H_1(P_\bullet \otimes_R N) \\ &= H_1(P_\bullet) \otimes_R N \\ &= 0.\end{aligned}$$

Conversely, suppose that for all M, $\operatorname{Tor}_1(M,N) = 0$. Then for any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0\ ,$$

we have a long exact sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow \operatorname{Tor}_1(A,N) \longrightarrow \cdots\ .$$

But $\operatorname{Tor}_1(A,N) = 0$, so the sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \twoheadrightarrow C \otimes_R N \longrightarrow 0\ .$$

is short exact. Thus, $- \otimes_R N$ is exact, which means by definition that N is flat.

3.2. The Ext functor

We know that $\operatorname{Hom}(N,-)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 107 (Ext functor)**.** Let R be a ring, and $N \in R\text{-}\mathbf{Mod}$ an R-module. The right derived functors

$$R^i \operatorname{Hom}(N,-)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i \operatorname{Hom}(N,-) = \operatorname{Ext}^i_R(N,-).$$

An argument very similar to that given for Tor in Proposition 105 shows that Ext is balanced; that is, when computing $\operatorname{Ext}^i_R(M,N)$, it doesn't matter whether we take an projective resolution of N or a injectiv resolution of M; we will get the same result either way. Rather than spelling this out in detail, we will wait for Chapter 5, where a spectral sequences argument gives the result almost immediately.

Example 108. Consider $R = \mathbb{Z}$, so R-**Mod** = **Ab**. Let us calculate

$$\mathrm{Ext}^*(\mathbb{Z}/(p), \mathbb{Z}).$$

We can take either an injective resolution of $\mathbb{Z}$ or a projective resolution of $\mathbb{Z}/(p)$. Each $\mathbb{Z}/(p)$ has a canonical projective resolution

$$0 \longrightarrow \mathbb{Z} \xhookrightarrow{\cdot p} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/(p) \longrightarrow 0 \ ,$$

which we might as well take advantage of. Doing so, we find

$$\mathrm{Ext}^* = H^*\left(0 \longrightarrow \overset{0}{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})} \cong \mathbb{Z} \xrightarrow{\cdot p} \overset{1}{\mathrm{Hom}(\mathbb{Z}, \mathbb{Z})} \cong \mathbb{Z} \longrightarrow 0 \right).$$

We can immediately calculate

$$\mathrm{Ext}^0(\mathbb{Z}/(p), \mathbb{Z}) \cong \ker \cdot p \cong 0, \qquad \mathrm{Ext}^1(\mathbb{Z}/(p), \mathbb{Z}) \cong \operatorname{coker} \cdot p \cong \mathbb{Z}/(p),$$

and $\mathrm{Ext}^i(\mathbb{Z}/(p), \mathbb{Z}) = 0$ for $i > 1$.

Note that balancedness immediately

3.2.1. Ext and extensions

There turns out to be a deep connection between the Ext groups and extensions.

Definition 109 (extension)**.** Let A and B be R-modules. An extension of A by B is a short exact sequence

$$\xi\colon \qquad 0 \longrightarrow B \hookrightarrow X \twoheadrightarrow A \longrightarrow 0 \ .$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd}
\xi\colon & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\mathrm{id}_B"] & X \arrow[r, two heads] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\
\xi'\colon & 0 \arrow[r] & B \arrow[r, hook] & X' \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd} .$$

The five lemma (Theorem 44) implies that φ is an isomorphism.

Let ξ be an extension of A by B. Applying the Ext functor we find an associated long exact sequence on cohomology.

$$\begin{tikzcd}
\mathrm{Ext}^1(A,B) \arrow[r] & \mathrm{Ext}^1(X,B) \arrow[r] & \cdots \\
\mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[ull, "\delta" description]
\end{tikzcd}$$

Definition 110 (obstruction class)**.** The image of id_B under the connecting homomorphism $\delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A,B)$ is known as the obstruction class of ξ. For any extension ξ, we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 111. An extension ξ is split if and only if the correspoding obstruction class is zero.

Proof. We are given an extension

$$\xi\colon \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xtwoheadrightarrow{g} A \longrightarrow 0 \;.$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$\mathrm{Hom}(X,B) \longrightarrow \mathrm{Hom}(B,B) \longrightarrow \mathrm{Ext}^1(A,B)$$

If $\delta(\mathrm{id}_B) = 0$, then there exists $a \in \mathrm{Hom}(X,B)$ such that $a \circ f = \mathrm{id}$. Thus, by the splitting lemma (Lemma 34), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a\colon A \to X$ such that $a \circ f = \mathrm{id}_B$.

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[l, bend left, "a"] \arrow[r] & 0 \;.
\end{tikzcd}$$

Since $\mathrm{Ext}^1(-,B)$ is additive, the sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ext}^1(A,B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(X,B) \arrow[l, bend left, "\mathrm{Ext}^1(a,B)"] \arrow[r, two heads, "\mathrm{Ext}^1(f,B)"] & \mathrm{Ext}^1(B,B) \arrow[r] & 0
\end{tikzcd}$$

is split exact. Thus, $\mathrm{Ext}^1(g,B)$ is a monomorphism, so it has trivial kernel.

$$\begin{tikzcd}
\mathrm{Ext}^1(A,B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \cdots \\
\mathrm{Hom}(A,B) \arrow[r] & \mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[ull, "\delta" description]
\end{tikzcd}$$

Exactness then forces $\delta = 0$. □

Proposition 111 explains the name *obstruction class:* $\Theta(\xi)$ is the obstruction to the splitness of ξ. The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\mathrm{Ext}^1(A, B)$ parametrizes extensions of A by B.

Theorem 112. There is a natural bijection

$$\left\{\begin{matrix}\text{Extension classes of}\\ \text{extensions of } A \text{ by } B\end{matrix}\right\} \xrightarrow{\cong} \mathrm{Ext}^1(A,B); \qquad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A,B) \\
\mathrm{Hom}(X,B) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{Hom}(B,B) \arrow[r, "\delta"'] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1(A,B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ. Since R-**Mod** has enough projectives, we can find a projective P which surjects onto A. Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$0 \longrightarrow K \xhookrightarrow{\psi} P \xrightarrow{\gamma} A \longrightarrow 0 \tag{3.1}$$

We construct from such an exact sequence a map which takes an element of $\mathrm{Ext}^1(A, B)$ and gives an extension of A by B. First applying $\mathrm{Ext}^*(-, B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\mathrm{Hom}(K,B) \xrightarrow{\delta} \mathrm{Ext}^1(A,B) \longrightarrow \mathrm{Ext}^1(P,B)$$

Since P is projective, the group $\mathrm{Ext}^1(P, B)$ vanishes, so $\delta\colon \mathrm{Hom}(K, B) \to \mathrm{Ext}^1(A, B)$ is surjective. Thus, given any $e \in \mathrm{Ext}^1(A, B)$, we can find a (not unique!) lift to $\phi \in \mathrm{Hom}(K, B)$. Take the pushout in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] & 0 \\
& B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P & &
\end{tikzcd}$$

By the dual to Lemma 38, the cokernel of i_B is equal to the cokernel of ψ, so we get an induced short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg^{\phi}_{K} P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0
\end{tikzcd}$$

It remains to show that Ψ as defined is independent of the choice of ϕ, and that it really is an inverse to Φ. To this end, pick a different $\tilde{\phi} \in \mathrm{Hom}(N, B)$ such that $\delta(\tilde{\phi}) = e$.

This gives us a new morphism of exact sequences as follows

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\tilde{\phi}"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "\tilde{i}_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg^{\tilde{\phi}}_{K} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd} \tag{3.2}$$

Because of the way we chose $\tilde{\phi}$, there exists some $\rho \in \mathrm{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$. Consider the following outer square.

$$\begin{tikzcd}
K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[d, "i_P"] \arrow[ddr, bend left, "\tilde{i}_P - \tilde{i}_B \circ \rho"] & \\
B \arrow[r, hook, "i_B"] \arrow[drr, bend right, "\tilde{i}_B"'] & B \amalg^{\phi}_{N} P \arrow[dr, dashed, "\exists!\alpha"] & \\
 & & B \amalg^{\tilde{\phi}}_{N} P
\end{tikzcd} \tag{3.3}$$

Using the commutativity of the left-hand square in Diagram 3.2, we find

$$\begin{aligned}
(\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
&= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
&= \tilde{i}_B \circ \phi,
\end{aligned}$$

so this outer square commutes, giving us the unique morphism α.

The claim that our two extensions are equivalent unwraps to the claim that the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, hook, "i_B"] \arrow[d, equal] & B \amalg^{\phi}_{K} P \arrow[r, two heads, "\pi"] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg^{\phi}_{K} P \arrow[r, two heads, "\tilde{\pi}"] & A \arrow[r] & 0
\end{tikzcd}$$

commutes. The left-hand square is the lower commuting triangle in Diagram 3.3.

Reminding ourselves of the proof of Lemma 38, we recall that the morphism π is defined by a pushout: it is the unique map $P \amalg_K^{\phi} P \to A$ such that $\pi \circ i_B = 0$ and $\pi \circ i_P = \gamma$. If we can show that $\tilde{\pi} \circ \alpha$ also satisfies these equations, then we can content ourselves that the diagram commutes. Indeed, we have

$$\begin{aligned} (\tilde{\pi} \circ \alpha) \circ i_B &= \tilde{\pi} \circ \tilde{i}_B \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} (\tilde{\pi} \circ \alpha) \circ i_P &= \tilde{\pi} \circ (\tilde{i}_P - \tilde{i}_B \circ \rho) \\ &= \gamma. \end{aligned}$$

Our last job, which was really the whole point of the endeavor, is to check that the maps Φ and Ψ are inverse to each other. To that end, fix some A and B, and pick $e \in \mathrm{Ext}^1(A, B)$. By construction of Ψ, we get a morphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads, "\gamma"] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^{\phi} P \arrow[r, two heads, "\pi"] & A \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

where ϕ is an element of $\mathrm{Hom}(K, A)$ which is mapped to by e under δ. The next step in computing $\Phi(\Psi(e))$ would now be to stick $\Psi(e)$ into the hom functor $\mathrm{Hom}(-, B)$, and look at the image of id_B under the connecting homomorphism. However, using the fact that Ext is a homological δ-functor, we can stick the whole morphism of exact sequences Equation 3.4 in and get a morphism of long exact sequences out. Picking out the square containing the relevant boundary map, we find the following.

$$\begin{tikzcd} \mathrm{Hom}(B, B) \arrow[r, "\delta"] \arrow[d, "\mathrm{Hom}(\phi, B)"'] & \mathrm{Ext}^1(A, B) \arrow[d, equal] \\ \mathrm{Hom}(K, B) \arrow[r, "\delta'"] & \mathrm{Ext}^1(A, B) \end{tikzcd}$$

Chasing the identity around in both directions tells us that

$$\delta(\mathrm{id}_B) = \delta'(\phi) = e.$$

Next, we start with an extension

$$\xi: \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] & 0 \end{tikzcd} .$$

Using the connecting homomorphism, we get a representative for the obstruction class $e = \delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A, B)$. Using our short exact sequence from Equation 3.1, we can find the fo ☐

Example 113. Let's do a sort of capstone computation, which pulls in just about everything we've done so far.

Consider the ring $R = \mathbb{Z}[x]$. We will compute the obstruction class of the extension

$$\xi\colon \qquad \begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\cdot x"] & \mathbb{Z}[x]/(x^2) \arrow[r, two heads] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd},$$

where we have written $\mathbb{Z} = \mathbb{Z}[x]/(x)$.

At a very high level of abstraction, we compute the obstruction class of ξ by taking the long exact sequence associated to it under the right derived functor of $\mathrm{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z})$, namely

$$\begin{tikzcd} & \mathrm{Ext}^1_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}) \arrow[r] & \cdots \\ \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]/(x^2), \mathbb{Z}) \arrow[r] & \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}) \arrow[ul, "\delta"'] \end{tikzcd}$$

and finding the image of the identity $\mathrm{id}_{\mathbb{Z}}$ under δ.

In order to find this long exact sequence, we need to lift ξ to a short exact sequence of projective resolutions. If we can find a projective resolution of $\mathbb{Z}$, the horseshoe lemma will do the rest of the work for us.

Our work on syzygies will help us here. We note that $\mathbb{Z}$ has a single generator, namely 1, so we begin our resolution

$$\begin{tikzcd} \mathbb{Z}[x] \arrow[r, "\pi"] & \mathbb{Z} \end{tikzcd}.$$

The kernel of this is the ideal (x).

$$\begin{tikzcd} (x) \arrow[dr, hook] & & \\ & \mathbb{Z}[x] \arrow[r, "\pi"] & \mathbb{Z} \end{tikzcd}.$$

This again has a single generator, namely x.

$$\begin{tikzcd} & (x) \arrow[dr, hook] & & \\ \mathbb{Z}[x] \arrow[ur, two heads, "\cdot x"] \arrow[rr, "\cdot x"] & & \mathbb{Z}[x] \arrow[r, "\pi"] & \mathbb{Z} \end{tikzcd}.$$

The composition gives us our next differential. Note that multiplication by x is injective, so our projective resolution terminates.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}[x] \arrow[r, "\cdot x"] & \mathbb{Z}[x] \arrow[r] & \cdots \end{tikzcd}.$$

Now we apply the horseshoe lemma. We first build our horseshoe.

$$\begin{tikzcd} & \mathbb{Z}[x] \arrow[d] & & \mathbb{Z}[x] \arrow[d] & \\ & \mathbb{Z}[x] \arrow[d] & & \mathbb{Z}[x] \arrow[d] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\cdot x"] & \mathbb{Z}[x]/(x^2) \arrow[r, two heads, "\pi"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The center column is then given by the direct sum of the outer columns, and the horizontal maps are the canonical injection and projection respectively.

$$\begin{tikzcd} & \mathbb{Z}[x] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(3)"] & \mathbb{Z}[x]^2 \arrow[r, two heads] \arrow[d, "\alpha"] \arrow[dr, phantom, "(4)"] & \mathbb{Z}[x] \arrow[d] & \\ & \mathbb{Z}[x] \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "(1)"] & \mathbb{Z}[x]^2 \arrow[r, two heads] \arrow[d, "\beta"] \arrow[dr, phantom, "(2)"] & \mathbb{Z}[x] \arrow[d] & \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, hook, "\cdot x"] & \mathbb{Z}[x]/(x^2) \arrow[r, two heads, "\pi"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

The horseshoe lemma guarantees the existence of maps α and β above such that all the squares commute and the middle column forms a projective resolution, but being assured of existence is of no help to us; we must find them explicitly. We are helped tremendously that what we have found are free resolutions, so we can write the maps as matrices. We first figure out what we can about the morphisms α and β from the commutativity of the squares labelled (1)-(4).

(1) Taking the low road maps

$$1 \mapsto 1 \mapsto x,$$

while taking the high road maps

$$1 \mapsto (1,0) \mapsto \beta(1,0).$$

Thus,

$$\beta\colon (1,0) \mapsto x$$

(2) The high road maps

$$(0,1) \mapsto 1 \mapsto 1,$$

and the low road maps

$$(0,1) \mapsto \beta(0,1) \mapsto \beta(0,1).$$

Thus,

$$\beta\colon (0,1) \mapsto 1.$$

Already, this is enough to tell us that

$$\beta(a,b) = ax + b.$$

We make the ansatz

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(3) Taking the high road maps

$$1 \mapsto (1,0) \mapsto (a,c),$$

while taking the low road maps

$$1 \mapsto x \mapsto (x,0),$$

so $a = x$ and $c = 0$.

(4) Taking the high road maps

$$(0,1) \mapsto 1 \mapsto x,$$

and taking the low road maps

$$(0,1) \mapsto (b,d) \mapsto d$$

so we must have $d = x$.

Thus, we have

$$\alpha = \begin{pmatrix} a & b \\ 0 & x \end{pmatrix},$$

where we have not yet fixed b. However, we have not yet used that the middle column must be a chain complex, i.e. that $\beta \circ \alpha$ must be equal to zero. Multiplying out, we have

$$\beta \circ \alpha = \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} x & b \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x + bx \end{pmatrix}.$$

Thus, we must take $b = -1$, fixing

$$\alpha = \begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z}[x])$, we find the following diagram.

At last, we can calculate $\Theta(\xi)$ as the image of $\mathrm{id}_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}[x])$ under the connecting homomorphism.

3.2.2. The Baer sum

By definition, $\mathrm{Ext}^1(A, B)$ has an abelian group structure. We have provided a bijection between equivalence classes of extensions of B by A and $\mathrm{Ext}^1(A, B)$, which allows us to transport our addition law from $\mathrm{Ext}^1(A, B)$ to the set of extension classes. However, it is not at all clear how to interpret the sum of two classes of extensions.

The group operation on the set of extension classes is known as the *Baer sum.* Let

$$\xi: \qquad 0 \longrightarrow B \overset{f}{\hookrightarrow} X \overset{g}{\twoheadrightarrow} A \longrightarrow 0$$

and

$$\xi': \qquad 0 \longrightarrow B \overset{f'}{\hookrightarrow} X' \overset{g'}{\twoheadrightarrow} A \longrightarrow 0$$

be extensions. We shall form a new extension from ξ and ξ' called the *Baer sum,* which will turn out to be in the class of $\Psi(\Phi(\xi + \xi'))$.

A word of apology: things are about to get very element-theoretic. Of course, this is not a problem thanks to the Freyd-Mitchell embedding theorem, but it's still a bit sad.

From the above extensions above, take the following pushout, and form $X'' = \ker \alpha$.

$$\begin{tikzcd}
B \arrow[rr, "f'"] \arrow[dr, "\exists! f''"] \arrow[dd, "f"'] & & X' \arrow[dd] \arrow[dddr, bend left, "-g'"] & \\
& X'' \arrow[dr, hook] & & \\
X \arrow[rr] \arrow[drrr, bend right, "g"'] & & X \amalg_B X' \arrow[dr, dashed, "\exists! \alpha"] & \\
& & & A
\end{tikzcd}$$

We have immediately the following morphism.

$$\begin{tikzcd}
0 & B \arrow[r, "f''"] & X'' & A & 0
\end{tikzcd}$$

We would like to fill in morphisms so that we have an exact sequence. The obvious choice of morphism $X'' \to A$, namely the composition $\alpha \circ \ker \alpha$, won't work because it's zero, so we have do define something else.

Warning: brutal element-wise computation ahead.

We have an explicit description of X'' as

$$X \amalg_B X' = X \oplus X' / \langle f(b) = f'(b) \rangle.$$

The kernel X'' is therefore the part of this which goes to zero under α, namely $(x, x') \in X \amalg_B X'$ such that

$$\alpha(x, x') = g(x) - g'(x') = 0,$$

i.e.

$$X'' = \{(x, x') \in X \amalg_B X' \mid g(x) = g'(x')\}.$$

The map f'' then simply sends $b \mapsto (f(b), 0)$, which by definition is equivalent to $(0, f'(b))$.

Note that f' is clearly injective.

Next, we define a map $g'' : X'' \to A$ by sending

$$(x, x') \mapsto g(x) = g'(x').$$

This is clearly surjective since g and g' are. The kernel of g'' consists precisely of those (x, x') such that $g(x) = 0$, which is to say, such that $x = f(b)$ by exactness of ξ. Thus, we have constructed a bona fide short exact sequence

$$\xi'' : \qquad 0 \longrightarrow B \xhookrightarrow{f''} X'' \xrightarrow{g''} A \longrightarrow 0 \,. \tag{3.5}$$

Definition 114 (Baer sum)**.** The extension ξ'' from Equation 3.5 is called the Baer sum of ξ and ξ'.

Lemma 115. The Baer sum is compatible with equivalences; that is, if $\xi \sim \zeta$, then $\xi + \xi' \sim \zeta + \xi'$.

Proof. Note that equivalence implies in particular □

This means that the Baer sum descends to an addition law on equivalence classes.

Lemma 116. Let

$$\eta : \qquad 0 \longrightarrow B \xhookrightarrow{i} B \oplus A \xrightarrow{p} A \longrightarrow 0$$

be a split extension. Then for any extension ξ, $\eta + \xi \sim \xi$.

Proof. □

Lemma 117. Let

$$\xi\colon \qquad 0 \longrightarrow B \xhookrightarrow{f} X \xrightarrow{g} A \longrightarrow 0\ .$$

be an extension. Then

$$-\xi\colon \qquad 0 \longrightarrow B \xhookrightarrow{-f} X \xrightarrow{g} A \longrightarrow 0$$

is the inverse under the Baer sum, i.e. $\xi + (-\xi) \sim \eta$.

3.2.3. Higher extensions

We can also interpret higher Ext groups in terms of extensions. Let

$$\xi\colon \qquad 0 \longrightarrow B \hookrightarrow X_0 \longrightarrow \cdots \longrightarrow X_{n-1} \twoheadrightarrow A \longrightarrow 0$$

be an extension. We define the obstruction class $\Theta(\xi)$ as follows: first, find a projective resolution $P_\bullet \to A$, and a lift of the identity id_A as follows.

$$\begin{tikzcd}
P_{n+1} \arrow[r] \arrow[d] & P_n \arrow[r] \arrow[d, "f"] & P_{n-1} \arrow[r] \arrow[d] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d] & A \arrow[d, equal] \\
0 \arrow[r] & B \arrow[r] & X_0 \arrow[r] & \cdots \arrow[r] & X_{n-1} \arrow[r] & A
\end{tikzcd}$$

Now apply $\mathrm{Hom}(-, B)$, and take nth homology. The morphism $f \in \mathrm{Hom}(P_n, B)$ descends to an element of $\mathrm{Ext}^n(A, B)$ (since we can write it as $f = f^*\mathrm{id}_B$, implying that it is a cycle). This is what we call $\Theta(\xi)$.

3.3. Inverse and direct limits

3.3.1. Direct limits

Definition 118 (filtered poset)**.** A poset $(I, \leq)$ is called filtered if every finite subset of I has an upper bound; that is, thinking about a poset as a category, if for every $i, j \in I$, there exists $k \in I$ and arrows $i \to k$, $j \to k$.

$$\begin{tikzcd}
i \arrow[dr] & \\
& k \\
j \arrow[ur] &
\end{tikzcd}$$

Example 119. Let X be a set, and consider the poset of finite subsets $\mathcal{P}_{\mathrm{fin}}(X)$. This is a filtered poset since every finite collection $X_i \in \mathcal{P}_{\text{fin}}(X)$ is contained in the set $\bigcup_i X_i$, which is again finite.

Let I be a filtered poset, and let $F\colon I \to R$-**Mod** be a functor. Using the description of a colimit as a coequalizer, we can calculate that

$$\operatorname{colim} F \cong \left(\coprod_{i \in I} F(i) \right) / E,$$

where E is the subgroup generated by elements of the form $x - x'$, where $x \in F(i)$ and $x' \in F(i')$, such that there exists $j \in I$, $y \in F(j)$, and maps $u\colon i \to j$ and $u'\colon i' \to j$ such that $F(u)(x) = y$ and $F(u')(x') = y$.

$$\begin{tikzcd} F(i) \arrow[dr, "F(u)"] & \\ & F(j) \\ F(i') \arrow[ur, "F(u')"'] & \end{tikzcd}, \qquad \begin{tikzcd} x \arrow[dr, mapsto] & \\ & y \\ x' \arrow[ur, mapsto] & \end{tikzcd}$$

That is, two elements of $\coprod_{i \in I} F(i)$ are identified if they 'eventually' become equal.

Lemma 120. Let I be a filtered poset, and let $A\colon I \to R$-**Mod** be a functor.

1. Every element $a \in \operatorname{colim}$ is the image of some $a_i \in A_i$ (for some $i \in I$) under the canonical map $A_i \to \operatorname{colim} A$.
2. For every i, the kernel of the canoncal map $A_i \to \operatorname{colim} A$ is the union of the kernels of the maps $A(\phi)\colon A_i \to A_j$ (for $\phi\colon i \to j$ a map in I).

Proof.

1. Denote the canonical map $A_i \to \operatorname{colim} F$ by λ_i. A priori, we are only guaranteed that $a \in \operatorname{colim} A$ can be written as an equivalence class

$$\left[\sum_{i \in J} \lambda_i(a_i) \right], \qquad a_i \in A_i,$$

for some finite subset $J \subset I$. Since $J = \{i_1, \ldots, i_n\}$ is finite, we can find an upper bound $j \geq i_\ell$ for all ℓ. In the quotient, we are justified in replacing $\lambda_i(a_i)$ by $\lambda_j(\hat{a}_i)$, where $\hat{a}_i$ is the image of a_i in A_j. Thus, we can write our element a as

$$a = \lambda_j \left(\sum_{i \in I} \hat{a}_i \right),$$

where the sum is over elements of A_j, and hence itself belongs to A_j.

2.

∎

Theorem 121. Let I be a filtered poset. The colimit functor

$$\operatorname{colim}\colon R\text{-}\mathbf{Mod}^I \longrightarrow R\text{-}\mathbf{Mod}$$

is exact.

Proof. By general abstract nonsense, we know that colimits commute with colimits, and thus that the colimit functor preserves colimits, hence is right exact. We need only show that colim preserves monics.

To this end, let $t \in \operatorname{Hom}_{R\text{-}\mathbf{Mod}^I}(A, B)$ be monic, and choose some $a \in \operatorname{colim} A$ such that $t(a) = 0$ in $\operatorname{colim} B$. By Lemma 120, there exists some $a_i \in A_i$ which maps to a under the canonical map $A_i \longrightarrow \operatorname{colim} A$. Since each $t_i\colon A_i \longrightarrow B_i$ is monic this maps to some $t_i(a_i) \neq 0$ in B_i. The element $t_i(a_i)$ vanishes in $\operatorname{colim} B$, so there exists some $j \in I$ and $\phi\colon i \longrightarrow j$ such that

$$0 = B(\phi)(t_i(a_i)) = t_j(A(\phi)(a_i)) \qquad \text{in } B_j.$$

However, t_j is monic, so $A(\phi)(a_i) = 0$ in A_j. Thus a maps to zero in $\operatorname{colim} A$. ∎

Corollary 122. The functor

$$\operatorname{colim}\colon \mathbf{Ch}(R\text{-}\mathbf{Mod})^I \longrightarrow \mathbf{Ch}(R\text{-}\mathbf{Mod})$$

is exact.

Proof. Colimits are computed level-wise. ∎

Since direct limits are exact, there is no need to take derived functors. However, the situation for limits is not so pretty.

3.3.2. Inverse limits

Let I be a small category, and let $\mathcal{A}$ be an abelian category such that for every set-indexed collection $\{A_s\}_{s\in S}$, the product $\prod_{s\in S} A_s$ exists. Then $\mathcal{A}^I$ is an abelian category (which we have seen in REF) and has enough injectives (which we will not show unless we have time).

The limit functor

$$\lim\colon \mathcal{A}^I \to \mathcal{A}$$

commutes with limits, hence is left exact. We can form the right derived functors

$$\lim{}^i := R^i \lim\colon A^I \to I.$$

For us it will be sufficient to consider the case in which $I = \mathbb{N}$ and $\mathcal{A} = \mathbf{Ab}$, so a functor $I \to \mathcal{A}$ is a sequence of abelian groups and morphisms as follows.

$$\cdots \to A_2 \to A_1 \to A_0$$

We will call functors $\mathbb{N} \to \mathbf{Ab}$ *towers*, and denote $A\colon \mathbb{N} \to \mathbf{Ab}$ by $\{A_i\}$.

Lemma 123. Let $\{A_i\}$ be a tower in **Ab**, and denote $A(j \to i) = \lambda_{j,i}$. We can express the limit $\lim_i A_i$ as a kernel

$$\ker \Delta \hookrightarrow \prod_{i=0} A_i \xrightarrow{\Delta} \prod_{i=0} A_i$$

where Δ is the map

$$(\ldots, a_n, \ldots) \mapsto (\ldots, a_n - \lambda_{n+1,n} a_{n+1}, \ldots).$$

Proof. We can express any limit as a kernel

$$\ker \xi \hookrightarrow \prod_{a\in\mathbb{N}} A_a \xrightarrow{\Delta} \prod_{(m,n)\in D} A_m$$

where ξ is defined on components by

$$\xi\colon \prod_{a\in\mathbb{N}} A_a \to A_{\mathrm{cod}(m,n)=n}; \qquad (\cdots, a_n, \cdots) \mapsto a_m - \lambda_{n,m} a_n$$

where $D = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ indexes morphisms in $\mathbb{N}$. An element

$$a = (\ldots, a_n, \ldots, a_1, a_0)$$

is in the kenrnel of $\hat{\xi}$ if and only if

$$a_m - \lambda_{n,m}(a_n) = 0$$

for all $(m, n) \in D$. But because $\lambda_{k,j} \circ \lambda_{j,i} = \lambda_{k,i}$, this is true if and only

$$a_m - \lambda_{m+1,m}(a_{m+1}) = 0$$

for all m. But we can view this as a function from $\prod_{a\in\mathbb{N}} A_a$ to itself, giving us the required formula. □

Let

$$0 \longrightarrow \{A_i\} \hookrightarrow \{B_i\} \twoheadrightarrow \{C_i\} \longrightarrow 0$$

be a short exact sequence of towers, and consider the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \prod_i A_i \arrow[r, hook] \arrow[d, "\Delta"] & \prod_i B_i \arrow[r, two heads] \arrow[d, "\Delta"] & \prod_i C_i \arrow[r] \arrow[d, "\Delta"] & 0 \\
0 \arrow[r] & \prod_i A_i \arrow[r, hook] & \prod_i B_i \arrow[r, two heads] & \prod_i C_i \arrow[r] & 0
\end{tikzcd}$$

The snake lemma gives us a long exact sequence

$$0 \to \ker \Delta_A \to \ker \Delta_B \to \ker \Delta_C \to \operatorname{coker} \Delta_A \to \operatorname{coker} \Delta_B \to \operatorname{coker} \Delta_C \to 0.$$

We have already seen that $\ker \Delta_A = \lim\{A_i\}$. This tells us that

$$\lim^i = \begin{cases} \lim, & i = 0 \\ \operatorname{coker} \Delta & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a homological δ-functor. In fact, it turns out to be universal, hence the derived functor of lim.

Lemma 124. Suppose all the maps $\lambda_{i,i+1} \colon A_{i+1} \to A_i$ are epimorphisms. Then $\lim^1 A_i = 0$.

We need to show that Δ is an epimorphism. To that end, let

$$(\cdots, a_i, \cdots) \in \prod_i A_i.$$

Let $b_0 = a_0$ Define b_{i+1} inductively as a lift of $b_i - a_i$. Then Δ maps $(\cdots, b_i, \cdots)$ to

$$(\cdots, b_i - (b_i - a_i), \cdots) = (\cdots, a_i, \cdots).$$

Definition 125 (Mittag-Leffler condition)**.** Let $\{A_i\}$ be a tower of abelian groups. We say that $\{A_i\}$ satisfies the Mittag-Leffler condition if for each $n \geq 0$ there exists a $N = N(n) > n$ such that the for all $m \geq N$, the image of $A_m \to A_n$ is equal to the image of $A_N \to A_n$.

We say that $\{A_i\}$ satisfies the trivial Mittag-Leffler condition if for all $n \geq 0$ there exists $N(n) > n$ such that $\operatorname{im}(A_{N(n)} \to A_n) = 0$.

Example 126. If all the maps $A_n \to A_{n-1}$ are epimorphisms, then $\{A_i\}$ satisfies the

Mittag-Leffler condition with $N(n) = n + 1$.

Proposition 127. Suppose $\{A_i\}$ is a tower of abelian groups satisfying the Mittag-Leffler condition. Then

$$\lim^1 A_i = 0.$$

Proof. First, suppose that A_i satisfies the trivial Mittag-Leffler condition. We need to show that $\lim^1 A_i = 0$, i.e. that Δ is epi.

To that end, choose some $a_i \in A_i$, $i \geq 0$. We need to find $b_i \in A_i$, $i \geq 0$ such that

$$\Delta\colon (\dots, b_1, b_0) \mapsto (\dots, a_1, a_0).$$

Define

$$b_n = \sum_{m \geq n} \lambda_{n,m} a_m.$$

Note that this is a finite sum because $\lambda_{n,m} = 0$ for large enough m. Then

$$\begin{aligned}
\Delta(\cdots, b_n, \cdots) &= \left(\cdots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \lambda_{n,n+1} \sum_{m \geq n+1} \lambda_{m,n+1} a_m\right) \\
&= \left(\cdots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \sum_{m \geq n+1} \lambda_{n,m} a_m\right) \\
&= (\cdots, a_n, \cdots).
\end{aligned}$$

Now suppose that A_i satisfies the ordinary Mittag-Leffler condition. Without loss of generality we may assume that $N(n)$ is nondecreasing. Define

$$B_i = \lambda_{i,N(i)}(A_{N(i)}).$$

Consider the following short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & B_{i+1} \arrow[r, hook] & A_{i+1} \arrow[r, two heads] \arrow[d, "\lambda_{i,i+1}"'] & A_{i+1}/B_{i+1} \arrow[r] & 0 \\
0 \arrow[r] & B_i \arrow[r, hook] & A_i \arrow[r, two heads] & A_i/B_i \arrow[r] & 0
\end{tikzcd}$$

Because N is nondecreasing, the λs give us a map $B_{i+1} \to B_i$ making the diagram commute. The universal property for cokernels gives us an induced map $A_{i+1}/B_{i+1} \to A_i/B_i$

making the diagram commute.

$$
\begin{tikzcd}
0 \arrow[r] & B_{i+1} \arrow[r, hook] \arrow[d] & A_{i+1} \arrow[r, two heads] \arrow[d, "\lambda_{i,i+1}"'] & A_{i+1}/B_{i+1} \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & B_i \arrow[r, hook] & A_i \arrow[r, two heads] & A_i/B_i \arrow[r] & 0
\end{tikzcd}
$$

This defines a tower $\{C_i\}$ with $C_i = A_i/B_i$. By definition, this satisfies the trivial Mittag-Leffler condition.

Note that by definition, the maps of the tower $\{B_i\}$ are onto, so

$$\lim{}^1 B_i = 0.$$

The long exact sequence associated to the short exact sequence

$$0 \longrightarrow B_i \hookrightarrow A_i \twoheadrightarrow A_i/B_i \longrightarrow 0$$

thus tells us that $\lim^1 A_i = 0$, as it is sandwiched between two things which are zero. □

Example 128. Consider the tower

$$\cdots \longrightarrow \mathbb{Z}/(p^3) \longrightarrow \mathbb{Z}/(p^2) \longrightarrow \mathbb{Z}/(p)$$

Theorem 129 (Milnor sequence)**.** Let

$$\cdots \longrightarrow C_1 \longrightarrow C_0$$

be a tower of cochain complexes of abelian groups such that the Mittag-Leffler condition is satisfied (i.e. satisfied level-wise). Let $C = \lim C_i$. Then for each q there is an exact sequence

$$0 \longrightarrow \lim{}^1 H^{q-1}(C_i) \hookrightarrow H^q(C) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0\ .$$

Proof. We first fix notation. Let

$$B_i \subset Z_i \subset C_i$$

be the subcomplexes of C_i on boundaries and cycles, with trivial differentials; then

$$H^\bullet(C_i) = \operatorname{coker}(B_i \to Z_i).$$

Let B denote the level-wise coboundaries of C, and Z the level-wise cocycles, again with trivial differentials.

- Consider the exact sequence

$$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \longrightarrow \{C_i[1]\} \ .$$

Applying the left exact functor lim, we find that

$$\lim Z_i \cong \ker(d\colon C \to C[1]),$$

i.e. that $\lim Z_i$ is the subcomplex of cycles in C; that is, that

$$Z = \lim Z_i.$$

- Consider the short exact sequence

$$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \twoheadrightarrow \{B_i[1]\} \longrightarrow 0 \ .$$

The long exact sequence corresponding to $\lim^i$ is

$$\begin{tikzcd} 0 \arrow[r] & Z \arrow[r] & C \arrow[r] & \lim B_i[1] \arrow[dll, "\delta"'] & \\ & \lim^1 Z_i \arrow[r] & \lim^1 C^i \arrow[r] & \lim^1 B^i[1] \arrow[r] & 0 \end{tikzcd}$$

But since C_i satisfies the Mittag-Leffler condition, $\lim^1 C_i = 0$, implying that

$$\lim^1 B^i = 0.$$

Furthermore, $B[1] \cong \operatorname{coker}(Z \hookrightarrow C)$, so image-kernel factorization gives us

$$0 \longrightarrow B[1] \hookrightarrow \lim B_i[1] \twoheadrightarrow \lim^1(Z_i) \longrightarrow 0$$

- The long exact sequence on homology corresponding to

$$0 \longrightarrow \{B^i\} \hookrightarrow \{Z^i\} \twoheadrightarrow \{H^i\} \longrightarrow 0$$

is

$$\begin{tikzcd} 0 \arrow[r] & \lim B_i \arrow[r] & Z \arrow[r] & \lim H_i \arrow[dll, "\delta"'] & \\ & 0 \arrow[r] & \lim^1 Z^i \arrow[r] & \lim^1 H^i \arrow[r] & 0 \end{tikzcd}$$

.

This tells us both that

$$0 \longrightarrow \lim B_i \hookrightarrow Z \twoheadrightarrow \lim H_i \longrightarrow 0$$

is short exact, *and* that
$$\lim^1 Z^i \cong \lim^1 H_i.$$

The exact sequences we have discovered give us the chain of inclusions

$$0 \subset B \subset \lim B_i \subset Z \subset C.$$

The third isomorphism theorem or one of the axioms for triangulated categories or whatever gives that the sequence

$$0 \longrightarrow \tfrac{\lim B_i}{B} \hookrightarrow \tfrac{Z}{B} \twoheadrightarrow \tfrac{Z}{\lim B_i} \longrightarrow 0$$

is short exact. However,

$$\frac{\lim B_i}{B} \cong \lim^1 Z_i \cong \lim^1 H_i \qquad \frac{Z}{B} \cong H, \qquad \frac{Z}{\lim B_i} \cong \lim H_i.$$

As every chain complex we are working with has trivial differentials, this amounts to the statement needed. □

4. Basic applications

> You know how it is when someone asks you to ride in a terrific sports car, and then you wish you hadn't?
>
> John Adams

4.1. Syzygy theorem

Let $R = \mathbb{C}[x_1, \ldots, x_k]$. Following Hilbert, we consider a finitely generated[1] R-module M.

By Hilbert's basis theorem, M is Noetherian, hence has finitely many generators $a_1, \ldots,$ a_k.

Denote by R^k the free R-module with k generators $e_1, \ldots, e_k$, and consider

$$\phi \colon R^k \twoheadrightarrow M; \qquad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $\mathrm{Syz}^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^{k} \lambda_i e_i \stackrel{\phi}{\mapsto} \sum_{i=1}^{k} \lambda_i a_i = 0.$$

This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\mathrm{Syz}^1(M)$ is again a finitely generated R-module. Hence, pick finite set of generators b_i, $1 \le i \le \ell$, and look at

$$R^\ell \twoheadrightarrow \mathrm{Syz}^1(M); \qquad e_i \mapsto b_i.$$

In general, there is non-trivial kernel, denoted $\mathrm{Syz}^2(M)$. This consists of relation between relations.

[1] i.e. expressible as an R-linear combination of finitely many generators.

We can keep going. This gives us a sequence $\mathrm{Syz}^\bullet(M)$ corresponding to relations between relations between…between relations.

$$\begin{tikzcd}
& & \mathrm{Syz}^2(M) \arrow[dr, hook] & & & & \\
\cdots & R^m \arrow[ur, two heads] \arrow[rr, "d"] & & R^l \arrow[rr, "d"] \arrow[dr, two heads] & & R^k \arrow[r] & M \\
\mathrm{Syz}^3 \arrow[ur, hook] & & & & \mathrm{Syz}^1(M) \arrow[ur, hook] & &
\end{tikzcd}$$

From this we build a chain complex $(C_\bullet, d)$, where C_n corresponds to the nth copy of R^k, known as the *free resolution* of M.

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-module admits a free resolution of length at most n.

Definition 130 (Koszul complex)**.** Let R be a commutative ring, and let $x \in R$ be an element of R such that left multiplication by x is injective on first homology. The Koszul complex of x is the complex

$$K(x) = \ \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees 0 and 1. For elements $x_1, \ldots, x_n$, we define the Koszul complex of $x_1, \ldots, x_n$ to be

$$K(x_1, \ldots, x_n) = K(x_1) \otimes_R \cdots \otimes_R K(x_n).$$

Lemma 131. The Koszul complex $K(x)$ provides a free resolution of $R/(x)$.

Proof. There is an easy spectral sequence argument. □

4.2. Quiver representations

In the remainder of this chapter, we will apply homological algebra to the study of various classes of objects. Our main goal will be to implement the following cockamamie scheme in two contexts: quiver representations and groups.

Cockamamie Scheme 132. Given a category of algebraic objects:

1. Find a way of turning the objects in question into rings.
2. Study modules over those rings using homological algebra.
3. ???
4. Profit.

The first problem to which we will apply Cockamamie Scheme 132 is the study of *quiver representations.* Unfortunately, due to time constraints, we will only get as far as step 2.

Definition 133 (quiver)**.** A quiver Q consists of the following data.

- A finite[2] set Q_0 of *vertices*
- A finite set Q_1 of *arrows*
- A pair of maps
$$s, t \colon Q_1 \to Q_0$$
called the *source* and *target* maps respectively.

Definition 134 (quiver representation)**.** Let Q be a quiver, and let k be a field. A k-linear representation V of Q consists of the following data.

- For every vertex $x \in Q_0$, a vector space V_x.
- For every arrow $\rho \in Q_1$, a k-linear map $V_{s(\rho)} \to V_{t(\rho)}$.

A morphism f between k-linear representations V and W consists of, for each vertex $x \in Q_0$ a linear map $f_x \colon V_x \to W_x$, such that for edge $s \in Q_1$, the square

$$\begin{tikzcd}
V_{s(\rho)} \arrow[r] \arrow[d] & V_{t(\rho)} \arrow[d] \\
W_{s(\rho)} \arrow[r] & W_{t(\rho)}
\end{tikzcd}$$

commutes.

This allows us to define a category $\mathrm{Rep}_k(Q)$ of k-linear representations of Q.

Following our grand plan, we first find a way of turning a quiver representation into a ring.

First, we form the free category over the quiver Q, which we denote by $\mathcal{F}(Q)$; that is, the objects of the category $\mathcal{F}(Q)$ are the vertices of Q, and the morphisms $x \to y$ consist of finite chains of arrows starting at y and ending at x.

$$\begin{tikzcd}
 & z_1 \arrow[dr, "\rho_1"] & & & & z_5 \arrow[dr, "\rho_5"] & \\
y \arrow[ur, "\rho_0"] & & z_2 \arrow[dr, "\rho_2"] & & z_4 \arrow[ur, "\rho_4"] & & x \\
 & & & z_3 \arrow[ur, "\rho_3"] & & &
\end{tikzcd}$$

We will denote this morphism by

$$\rho_5 \rho_4 \rho_3 \rho_2 \rho_1 \rho_0.$$

[2]There is no fundamental reason for this finiteness condition, except that it will make the following analysis more convenient.

We call such a finite non-empty chain a *non-trivial path.* Note that in addition to the non-trivial paths, we must formally adjoin identity arrows e_x with $s(e_x) = t(e_x) = x$; these are the *trivial paths.* Note that the notation introduced above is a good one, since in the category $\mathcal{F}(Q)$ composition is given by

$$(\rho_n \cdots \rho_1) \circ (\rho'_m \cdots \rho'_1) = \rho_n \cdots \rho_1 \rho'_m \cdots \rho'_1.$$

Definition 135 (path algebra)**.** Let k be a field and Q a quiver. The path algebra of Q over k is, as a vector space, the free vector space over the set $\mathrm{Mor}(\mathcal{F}(Q))$. The multiplication is given by composition

$$\rho \cdot \sigma = \begin{cases} \rho\sigma, & s(\rho) = t(\sigma) \\ 0, & s(\rho) \neq t(\sigma). \end{cases}$$

We will denote the path algebra of Q over k by kQ.

Another way of expressing the above composition rule is as follows: the multiplication of two paths ρ, σ in kQ is given by the composition $\rho \circ \sigma$ in the category $\mathcal{F}(Q)$ if the morphisms ρ and σ are composable (in the sense that ρ begins where σ ends) and zero otherwise.

From now on, we fix (arbitrarily) a bijection of Q with $\{1, 2, \cdots, n\}$. It is clear that

$$\mathrm{id}_{kQ} = e_1 + \cdots + e_n.$$

Example 136. Consider the following quiver.

$$\begin{tikzcd} \bullet_1 \arrow[loop right, "\rho"] \end{tikzcd}$$

Then paths are in correspondence with natural numbers, with the empty path being e_1. The path algebra is therefore

$$kQ \cong ke_1 \oplus k\rho \oplus k\rho\rho \oplus k\rho\rho\rho \oplus \cdots \cong k[\rho].$$

Example 137. Let Q be any quiver, and denote by $A = kQ$ its path algebra. For any vertex i and any path ρ, the multiplication law for the path algebra tells us that

$$\rho e_i = \begin{cases} e_i, & s(\rho) = i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the left ideal Ae_i consists of those paths which start at i. Similarly, the right ideal e_iA consists of those paths which terminate at i.

Since any path either starts at i or doesn't, we have

$$A \cong Ae_i \oplus A(1 - e_i).$$

This means (by Example 64) that each Ae_i is a projective module. The same argument tells us that e_iA is projective for all i.

Proposition 138. Let Q be a quiver and k its path algebra. There is an equivalence of categories

$$\mathrm{Rep}_k(Q) \xrightarrow{\simeq} kQ\text{-}\mathbf{Mod}.$$

Proof. First, we construct out of any k-linear represenation V of Q a kQ-module. Our underlying space will be the vector space $\bigoplus_{i \in Q_0} V_i$, which we will also denote by V.

Now let $\rho\colon i \to j \in Q_0$, corresponding under the representation V to the map $f\colon V_i \to V_j$. We send this to the map $V \to V$ given by the matrix with f in the (i, j)th position and zeroes elsewhere. In particular, we send e_i to the matrix with a 1 in the ith place along the diagonal, and zeroes elsewhere.

It is easy to see that this assignment respects addition and composition, hence defines our functor on objects. On objects, we send $\alpha_i\colon V_i \to W_i$ to

$$\mathrm{diag}(0, \ldots, 0, \overset{i}{\alpha_x}, 0, \ldots, 0).$$

It is painfully clear that this is the right thing to do.

We now construct a functor in the opposite direction. ☐

Proposition 139. Let M be a left A-module. We have an exact sequence of left A-modules

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)} M \hookrightarrow \bigoplus_{i \in Q_0} Ae_i \otimes_k e_i M \twoheadrightarrow M \longrightarrow 0\ ,$$

exhibiting a resolution of M by projective A-modules.

Corollary 140. Let M, N be left A-modules. Then

$$\mathrm{Ext}^i(M, N) = 0, \qquad i \geq 2.$$

4.3. Group (co)homology

Next, we will apply Cockamamie Scheme 132 to the study of groups. In order to do this, we need some theory.

4.3.1. The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 141 (normalized chain complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2}d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1}d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} NA_{n-1} \xrightarrow{(-1)^{n-1}d_{n-1}} \cdots$$

Definition 142 (Moore complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n-chains A_n and differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
\partial^2 &= \left(\sum_{j=0}^{n-1} (-1)^j d_j\right) \circ \left(\sum_{i=0}^{n} (-1)^i d_i\right) \\
&= \sum \sum (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_j \circ d_i + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\
&= 0.
\end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A, unless this is confusing. In this case, we will denote it by MA.

Definition 143 (alternating face maps chain modulo degeneracies)**.** Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial\colon A_n \to A_{n-1}$ takes degerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain modulo degeneracies.[3]

Lemma 144. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \xhookrightarrow{i} A \overset{p}{\twoheadrightarrow} A/D(A)$$

Proof. We need only show that the following diagram commutes.

$$\begin{tikzcd}
NA_n \arrow[r, hook] \arrow[d, "(-1)^n d_n"'] & A_n \arrow[r, two heads] \arrow[d, "\partial"] & A_n/DA_n \arrow[d, "{[\partial]}"] \\
NA_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, two heads] & A_{n-1}/DA_{n-1}
\end{tikzcd}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n, and the right-hand square commutes trivially. □

Theorem 145. The composite

$$p \circ i\colon NA \to A/D(A)$$

is an isomorphism of chain complexes.

Let $f\colon [m] \to [n]$ be a morphism in the simplex category Δ. By functoriality, f induces a map

$$N\Delta^n \to N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0\colon [n] \to [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]\colon \Delta \to \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

[3] The nLab is responsible for this terminology.

and on morphisms by

$$f\colon [m] \to [n] \mapsto$$

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \tag{4.1}$$

Theorem 146 (Dold-Kan)**.** The adjunction in Equation 4.1 is an equivalence of categories

Proof. later, if I have time. □

Also later if time, Dold-Puppe correspondence: this works not only for **Ab**, but for any abelian category.

4.3.2. The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta\colon \mathrm{id}_\mathcal{C} \to RL$ and counit $\varepsilon\colon LR \to \mathrm{id}_\mathcal{D}$.

Recall that this data gives a comonad in $\mathcal{D}$, i.e. a comonoid internal to the category $\mathrm{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$LRLR \xLeftarrow{L\eta R} LR \xRightarrow{\varepsilon} \mathrm{id}_\mathcal{D}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category $\mathcal{C}$ and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M}\colon \Delta \to \mathcal{C}$.

$$\begin{tikzcd}
\{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, dashed, "\exists!\tilde{M}"] \\
& \mathcal{C}
\end{tikzcd}$$

In particular, with $\mathcal{C} = \mathrm{End}(\mathcal{D})^{\mathrm{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \to$

$\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, "\exists!"] \\ & \mathrm{End}(\mathcal{D})^{\mathrm{op}} \end{tikzcd}$$

Equivalently, this gives a functor

$$B\colon \Delta_+^{\mathrm{op}} \to \mathrm{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\mathrm{ev}_d\colon \mathrm{End}(\mathcal{D}) \to \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in $\mathcal{D}$.

$$S_d = \Delta_+^{\mathrm{op}} \xrightarrow{B} \mathrm{End}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{D}$$

Definition 147 (bar construction)**.** The composition $\mathrm{ev}_d \circ B$ is called the bar construction.

Example 148. I believe this example comes from John Baez, although I can't find the exact source at the moment.

Consider the free-forgetful adjunction

$$F : \mathbf{Set} \longleftrightarrow \mathbf{Ab} : G.$$

The unit $\eta\colon \mathrm{id}_{\mathbf{Set}} \to U \circ F$ has components

$$\eta_S\colon S \to \mathbb{Z}[S]$$

which assigns any set S to the set underlying the free abelian group on it by mapping elements to their corresponding generators. The counit $\varepsilon\colon F \circ U \to \mathrm{id}_{\mathbf{Ab}} \to U \circ F$ has components

$$\epsilon_A\colon \mathbb{Z}[A] \to A$$

which takes formal linear combinations of group elements to actual linear combinations of group elements.

Denoting $F \circ U$ by Z, we find the following comonad in **Ab**, where $\nabla = F\eta U$.

$$\begin{tikzcd} Z \circ Z \\ Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\ \mathrm{id}_{\mathbf{Ab}} \end{tikzcd}$$

This gives us a simplicial object in End(**Ab**).

Post-composing with the evaluation functor $\mathrm{ev}_{\mathbb{Z}}$ gives the following simplicial object in **Ab**.

$$\begin{tikzcd} \cdots\cdots\cdots \; \mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]] \arrow[r, shift left=6, "(ZZ\varepsilon)_{\mathbb{Z}}"] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6, "(\varepsilon ZZ)_{\mathbb{Z}}"'] & \mathbb{Z}[\mathbb{Z}[\mathbb{Z}]] \arrow[l, shift left=4] \arrow[l] \arrow[l, shift right=4] \arrow[r, shift left=4, "(Z\varepsilon)_{\mathbb{Z}}"] \arrow[r, shift right=4, "(\varepsilon Z)_{\mathbb{Z}}"'] & \mathbb{Z}[\mathbb{Z}] \arrow[l, "\nabla_{\mathbb{Z}}"'] \arrow[r, "\varepsilon_{\mathbb{Z}}"] & \mathbb{Z} \end{tikzcd}$$

Let's examine some of these objects in detail.

- The first object is the friendly and reassuring $\mathbb{Z}$, whose elements are integers. An example of an element of $\mathbb{Z}$ is
$$17.$$
- The next object, $\mathbb{Z}[\mathbb{Z}]$, consists of formal linear combinations of elements of $\mathbb{Z}$, i.e. formal sums of the form
$$3 \times (5) + 2 \times (-1).$$
The funny notation of enclosing the elements of (the set underlying) $\mathbb{Z}$ in parentheses will come in handy shortly.
- The next object, $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, consists of formal linear combinations of elements of $\mathbb{Z}[\mathbb{Z}]$, i.e. formal linear combinations of formal linear combinations of integers. An element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$ looks like this.
$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$
- I wonder if you can figure out for yourself what the elements of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]]$ are.

Next, we should understand the face maps in some low degrees.

Consider some element of $\mathbb{Z}[\mathbb{Z}]$, say

$$3 \times (5) + 2 \times (-1)$$

The map $\varepsilon_{\mathbb{Z}} = d_0$ collapses the formal multiplication into actual multiplication:

$$\varepsilon_{\mathbb{Z}}\colon 3 \times (5) + 2 \times (-1) \mapsto 15 - 2 = 13.$$

Consider now some element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, say

$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$

The map $\mathbb{Z}\varepsilon_{\mathbb{Z}} = d_0$ sends this to

$$3 \times (17) - 4 \times (3);$$

that is, it strips off the inner-most parentheses. The map $(\varepsilon Z)_{\mathbb{Z}} = d_1$ sends it to

$$6 \times (1) + 9 \times (5) - 4 \times (3);$$

that is, it strips off the outer-most parentheses.

Note that, as the simplicial identities promised us, $\epsilon_{\mathbb{Z}}$ sends both of these to

$$3 \times 17 - 4 \times 3 = 6 \times 1 + 9 \times 5 - 4 \times 3 = 39.$$

In the case that the category $\mathcal{D}$ is abelian, we can form from a simplicial object the chain complex of alternating face maps modulo degeneracies.

Definition 149 (bar complex)**.** Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b\colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ the bar construction of b.

The bar complex Bar_b of b is the chain complex of face maps modulo degeneracies (Definition 143) of S_b.

We will see an example of the bar complex in the next chapter.

4.3.3. Invariants and coinvariants

Now we are ready to apply Cockamamie Scheme 132 to the study of groups. Here a somewhat popular method exists for turning a group G into a ring, namely by constructing the group ring $\mathbb{Z}G$. As is tradition, we will study modules over this ring.

In Section 4.2, we had a nice interpretation of modules over our rings, as corresponding to k-linear representations of a quiver. We will work backwards, finding another perspective on modules over the group ring $\mathbb{Z}G$ of a group G as *G-modules.*

Given a group G, a G-module is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-$\mathbf{Mod}$.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in $\mathbf{Ab}$, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the obvious equivalence G-$\mathbf{Mod} \simeq \mathbb{Z}G$-$\mathbf{Mod}$, we know that G-$\mathbf{Mod}$ has enough projectives.

Note that there is a canonical functor $\operatorname{triv}\colon \mathbf{Ab} \to \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 150. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \operatorname{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\operatorname{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \operatorname{Hom}_{G\text{-}\mathbf{Mod}}(M, \operatorname{triv} A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \to A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \to A$ such that $\hat{\alpha}(g{\cdot}m{-}g) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \to A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{4.2}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \to \operatorname{triv} A$. In fact, G-linear maps $M \to \operatorname{triv} A$ are precisely those satisfying Equation 4.2, showing that this is really an isomorphism.

The other case is similar. $\square$

Definition 151 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Clearly, the functors taking invariants and coinvariants are interesting things that deserve to be studied in their own right, and in keeping with Cockamamie Scheme 132 we could procede by deriving them. However, now something of a miracle occurs: these functors are really special cases of the hom and tensor product!

Lemma 152. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\text{-}\mathbf{Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula $\mathbb{Z}$ is taken to be a trivial left $\mathbb{Z}G$-module, and in the second a trivial right $\mathbb{Z}G$-module.

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

Lemma 152 means that we already have a good handle on the derived functors of $(-)_G$ and $(-)^G$: they are given by the Ext and Tor functors we have already seen. More specifically,

$$L_i(-)_G = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad R^i(-)^G = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$

This is a stroke of luck. Had we proceded naïvely by forming the right and left derived functors of invariants and coinvariants using Definition 151, we would have have been stuck picking resolutions for each $\mathbb{Z}G$-module whose value under the derived functors we wanted to compute, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[4]

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \to G$-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
    - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.

[4] In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

  - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.

3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived functors $R^i(-)^G$.
4. We denote
$$L_i(-)_G = H_i(G, -), \qquad R^i(-)^G = H^i(G, -),$$
and call them *group homology* and *group cohomology* respectively.
5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have
$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$
and thus that
$$H_i(G, A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad H^i(G, A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$
This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 153. Let $G = \mathbb{Z}$, and denote the generator by t. Suppose we want to compute $H_*(G; A)$ and $H^*(G; A)$ for some $\mathbb{Z}G$-module A.

Thanks to Lemma 152, instead of computing a projective resolution of some specific A, we may compute a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

$$0 \longrightarrow \mathbb{Z}G \xhookrightarrow{t-1} \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

This gives us a free resolution $F_\bullet \xrightarrow{\simeq} \mathbb{Z}$.

By definition,

$$H_i(G; A) = H_i(F_\bullet \otimes_{\mathbb{Z}G} A) = H_i\left(0 \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}G} A \xrightarrow{f} \mathbb{Z}G \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right).$$

The map f takes $t \otimes a \mapsto a - t \otimes a$; the homology $H_0(G; A)$ is thus

$$\operatorname{coker} f = A / \langle t \cdot a - a \mid a \in A \rangle = A_G.$$

Similarly, we have
$$H_1(G;A) = \ker f = A^G.$$
For $n > 1$, we have $H_n(G;A) = 0$.

Now turning our attention to group cohomology, we find by definition
$$H^n(G;A) = H^n\left(0 \longrightarrow \mathrm{Hom}(\mathbb{Z}G, A) \xrightarrow{g} \mathrm{Hom}(\mathbb{Z}G, A) \longrightarrow 0 \right),$$
where the map g sends
$$g\colon\ (f\colon t \mapsto f(t)) \mapsto (gf\colon t \mapsto f(t-1))\,.$$
Note that under the correspondence $f \mapsto f(1)$, we have
$$\mathrm{Hom}(\mathbb{Z}G;A) \cong A.$$
Having made this identification, g simply sends $f \mapsto tf - f$. Thus, similarly to before, we have
$$H^0(G;A) = \ker(g) = A^G, \qquad H^1(G;A) = \mathrm{coker}(g) = A_G,$$
and $H^n(G;A) = 0$ for $n > 1$.

4.3.4. The bar construction for group rings

In the last section, we saw that we could compute group homology and cohomology of a group G by finding a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. This is nice, but computing projective resolutions is in general rather difficult.

It turns out that the bar construction (explored in Subsection 4.3.2) computes a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module for any group G.

Consider the functor
$$U\colon \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$$
which forgets multiplication. This has left adjoint
$$\mathbb{Z}G \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \to \mathbb{Z}G\text{-}\mathbf{Mod},$$
which assigns to an abelian group A the left $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta\colon \mathrm{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_Z -)$ and counit $\varepsilon\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components
$$\eta_A\colon A \to U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \qquad a \mapsto 1 \otimes a$$
and
$$\varepsilon_M\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \to M; \qquad g \otimes m \mapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z, we find the following comonad in $\mathbb{Z}G$-**Mod**, where $\nabla = \mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{tikzcd}
Z \circ Z \\
Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\
\mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

This gives us a simplicial object in $\mathrm{End}(\mathbb{Z}G\text{-}\mathbf{Mod})$.

$$\begin{tikzcd}[column sep=large]
\cdots\cdots\cdots \; Z \circ Z \circ Z
\arrow[r, Rightarrow, shift left=6, "ZZ\varepsilon"]
\arrow[r, Leftarrow, shift left=4]
\arrow[r, Rightarrow, shift left=2]
\arrow[r, Leftarrow]
\arrow[r, Rightarrow, shift right=2]
\arrow[r, Leftarrow, shift right=4]
\arrow[r, Rightarrow, shift right=6, "\varepsilon ZZ"']
& Z \circ Z
\arrow[r, Rightarrow, shift left=3, "Z\varepsilon"]
\arrow[r, Leftarrow, "\nabla" description]
\arrow[r, Rightarrow, shift right=3, "\varepsilon Z"']
& Z \arrow[r, Rightarrow] & \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

Evaluating on a specific $\mathbb{Z}G$-module M then gives a simplicial object in $\mathbb{Z}G$-**Mod**. We continue in the special case $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$-module.

$$\begin{tikzcd}[column sep=large]
\cdots\cdots\cdots \; \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}
\arrow[r, shift left=6, "d_0"]
\arrow[r, leftarrow, shift left=4]
\arrow[r, shift left=2]
\arrow[r, leftarrow]
\arrow[r, shift right=2]
\arrow[r, leftarrow, shift right=4]
\arrow[r, shift right=6, "d_2"']
& \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}
\arrow[r, shift left=3, "d_0"]
\arrow[r, leftarrow, "\nabla" description]
\arrow[r, shift right=3, "d_1"']
& \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Because we are tensoring over $\mathbb{Z}$, we are justified in notationally suppressing the last copy of $\mathbb{Z}$ in our tensor products.

The face map d_i removes the ith copy of Z; on a generator, we have the action of d_i given by

$$d_i \colon x \otimes x_1 \otimes \cdots \otimes x_n \mapsto \begin{cases} xx_1 \otimes \cdots \otimes x_n, & i = 0 \\ x \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, & 0 < i < n+1 \\ x \otimes \cdots \otimes x_{n-1}, & i = n+1. \end{cases}$$

and the degeneracy maps s_i send

$$x \otimes x_1 \otimes \cdots \otimes x_{n-1} \mapsto x \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n-1}.$$

It is traditional to use the notation

$$x \otimes x_1 \otimes \cdots \otimes x_n = x[x_1 | \cdots | x_n].$$

This notation is the origin of the name *bar construction.*

We now form the bar complex; the objects B_n are given by

$$B_n = \mathbb{Z}G^{\otimes n}/\{\text{degenerate simplices}\};$$

that is, whenever we see something of the form

$$x[x_1|\cdots|1|\cdots|x_n],$$

we set it to zero.

The elements of B_n are simply formal $\mathbb{Z}$-linear combinations of the $x[x_1|\cdots|x_n]$. Therefore, we can also write

$$B_n = \bigoplus_{(g_i)\in(G\smallsetminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n], \qquad n \geq 0.$$

That is, we can write the bar complex as follows.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\smallsetminus\{1\})^2} \mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow \mathbb{Z}$$

To summarize, we have the following definition.

Definition 154 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module

$$B_n = \bigoplus_{(g_i)\in(G\smallsetminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$

where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to $(g_1,\ldots,g_n)$. The differential is defined by

$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

Proposition 155. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
+
\begin{tikzcd}
 & B_n \arrow[dl, dashed, "h_n"'] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}$$

gives us in one direction

$$g[g_1|\cdots|g_n] \longmapsto \begin{aligned} & gg_1[g_2|\cdots|g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i g[g_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ & + (-1)^n g[g_1|\cdots|g_{n-1}] \end{aligned}$$

$$\longmapsto \begin{aligned} & [gg_1|\cdots|g_n] \\ & + \sum_{i=1}^{n-1} (-1)^i [gg_1|\cdots|g_i g_{i+1}|\cdots|g_n] \\ & + (-1)^n [gg_1|\cdots|g_n] \end{aligned}$$

and in the other

$$
\begin{tikzcd}
 & g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\
{[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{l} g[g_1|\cdots|g_n] \\ -\,[gg_1|g_2|\cdots|g_n] \\ -\displaystyle\sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ -\,(-1)^n[gg_1|\cdots|g_{n-1}] \end{array}
\end{tikzcd}
.
$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. □

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G,A) = H^i\left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0,A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1,A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G,A) = H_i\left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

4.3.5. Computations in low degrees

It turns out that one can find compelling interpretations of low homology classes. In the course of this section we will find the following interpretations.

- Taking $\mathbb{Z}$ as a trivial G-module, the first homology $H_1(G,\mathbb{Z})$ simply coincides with G_{ab}.
- For any G-module A, the first homology $H^1(G;A)$ corresponds to *crossed homomorphisms* $G \to A$ modulo *principal crossed homomorphisms* $G \to A$. In particular, if A is a trivial G-module, then $H^1(G,A)$ corresponds to $\mathrm{Hom}_{\mathbf{Ab}}(G_{\mathrm{ab}},A)$.
- Second cohomology $H^2(A;G)$ controls extensions of G by A, i.e. short exact sequences of groups

$$0 \longrightarrow A \hookrightarrow E \twoheadrightarrow G \longrightarrow 0\ .$$

First homology and abelianizations

Let G be a group, and consider $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. Let us compute $H_1(G,\mathbb{Z})$.

To compute $H_1(G,\mathbb{Z})$, consider the first few terms in the bar complex for G.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\smallsetminus\{1\})^2} \mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow 0$$

Tensoring over $\mathbb{Z}G$ with $\mathbb{Z}$ is the same as trivializing the action, giving us the following complex.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\smallsetminus\{1\})^2} \mathbb{Z}[g_1|g_2] \longrightarrow \bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}[g] \longrightarrow \mathbb{Z}[\cdot] \longrightarrow 0$$

The differential $d_1 \otimes \mathrm{id}_{\mathbb{Z}}$ acts on a generator $[g]$ by sending it to

$$[\cdot]g - [\cdot] = [\cdot] - [\cdot] = 0,$$

implying that everything is a 1-cycle. The differential d_2 sends a generator $[g|h]$ to

$$[g]h - [gh] + [h] = [g] + [h] - [gh].$$

Thus, taking cycles modulo boundaries is the same as

$$\{[g] \in G\}/\langle [gh] - [g] - [h] \rangle \cong G_{\mathrm{ab}}.$$

Thus, $H_1(G,\mathbb{Z})$ is simply the abelianization of G.

First cohomology and abelianizations

To compute $H^1(G,A)$, consider the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot],A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(\textstyle\bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}G[g],A) \xrightarrow{(d_2)^*} \cdots .$$

Notice immediately that since the hom functor preserves direct sums in the first slot, we have

$$\mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n} \mathbb{Z}G[g_1|\cdots|g_n], A \right) \cong \bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n} \mathrm{Hom}_{\mathbb{Z}G}\left(\mathbb{Z}G[g_1|\cdots|g_n],A\right)$$

An element of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1|\cdots|g_n])$ is determined completely by where it sends the generator of $\mathbb{Z}G$. We can therefore re-package the RHS as consisting of functions

$$\phi\colon (G \setminus \{1\})^n \to A,$$

or equivalently as functions

$$\phi\colon G^n \to A, \qquad \phi(\ldots, 1, \ldots) = 0.$$

The differential $d^n = (d_n)_*$ sends $\phi \mapsto d^n\phi = \phi \circ d_n$, where

$$\begin{aligned}\phi \circ d_n\colon (g_1, \ldots, g_{n+1}) \mapsto & g_1\phi(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1}\phi(g_1, \ldots, g_n).\end{aligned}$$

We will denote the nth term in Equation 4.3.5 by C^n; that is, we have

$$C^n \cong \{\phi\colon G^n \to A \mid \phi(g_1, \ldots, 1, \ldots, g_n) = 0\}.$$

Let us explicitly compute $H^1(G; A)$.

The set of 1-coboundaries corresponds to functions $\psi\colon G \to A$ with

$$d\psi\colon G^2 \to A; \qquad d\psi(g, h) = g\psi(h) - \psi(gh) + \psi(g) = 0,$$

i.e. functions

$$\phi\colon G \to A; \qquad \psi(g, h) = \psi(g) + g\psi(h).$$

We will call these *crossed homomorphisms.*

The set of 1-cocycles corresponds to the image $d^0(C^0)$, i.e. those functions

$$d\phi\colon G \to A, \qquad d\phi([g]) = g\phi([\cdot]) - \phi([\cdot]).$$

We will call these *principal crossed homomorphisms.*

Thus, in this case we have

$$H^1(G; A) = \frac{\{\text{crossed homomorphisms from } G \text{ to } A\}}{\{\text{principal crossed homomorphisms from } G \text{ to } A\}}.$$

Now suppose that A is a trivial G-module. In that case, crossed homomorphisms are simply homomorphisms $G \to A$, and principal crossed homomorphisms are zero. Thus,

$H^1(G;A)$ consists of homomorphisms $G \to A$, i.e.

$$H^1(G;A) = \mathrm{Hom}_{\mathbf{Ab}}(G_{\mathrm{ab}}, A).$$

First cohomology and semidirect products

Definition 156 (semidirect product)**.** Let G be a group, and A a G-module. The semidirect product of A and G is the group whose underlying set is $A \times G$, and whose group operation is given by

$$(a, g) \cdot (a', g') = (a + ga', gg').$$

We denote the semidirect product of A and G by $A \rtimes B$.

Note that we have a short exact sequence of groups[5]

$$0 \longrightarrow A \hookrightarrow A \rtimes G \twoheadrightarrow G \longrightarrow 1 \ .$$

Let $\sigma\colon A \rtimes G \to A \rtimes G$ be a group homomorphism. We say that σ *stabilizes* A and G if the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & A \rtimes G \arrow[r, two heads] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, equal] & 1 \\ 0 \arrow[r] & A \arrow[r, hook] & A \rtimes G \arrow[r, two heads] & G \arrow[r] & 1 \end{tikzcd}$$

Given any crossed homomorphism ϕ, we get a group homomorphism σ_ϕ by the formula

$$\sigma_\phi\colon (a, g) \mapsto (a + \phi(g), g).$$

This is an easy check; we have

$$\begin{aligned} \sigma_\phi((a,g)(a',g')) &= \sigma_\phi(a + ga', gg') \\ &= (a + ga' + \phi(gg'), gg') \\ &= (a + \phi(g) + g(a' + \phi(g')), gg') \\ &= (a + \phi(g), g)(a' + \phi(g'), g') \\ &= \sigma_\phi(a,g)\sigma_\phi(a',g'). \end{aligned}$$

In fact, σ_ϕ trivially stabilizes A and G. This gives us a way of turning a cocycle in $Z^1(G, A)$ into an an automorphism of $A \rtimes G$ which stabilizes A and G.

[5] We are cheating a bit. Since **Grp** is not an abelian category, we have no notion of an exact sequence of groups. We trust that the reader will not be confused.

Proposition 157. The map

$$\psi\colon Z^1(G;A) \longrightarrow \operatorname{Aut}(A \rtimes G); \qquad \phi \mapsto \sigma_\phi$$

is an isomorphism onto the subgroup of $\operatorname{Aut}(A \rtimes G)$

Proof. The fact that ψ is a homomorphism is trivial. We show that we have an inverse given by □

This is miserable.

Second cohomology and extensions

Definition 158 (extension of a group)**.** Let G be a group and A an abelian group. An extension of G by A is a short exact sequence of groups

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] & E \arrow[r, two heads, "\pi"] & G \arrow[r] & 0 \end{tikzcd} .$$

Two such extensions are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] \arrow[d, equal] & E \arrow[r, two heads, "\pi"] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i'"] & E \arrow[r, two heads, "\pi'"] & G \arrow[r] & 0 \end{tikzcd} .$$

We are no longer justified in using the snake lemma to show that σ is an isomorphism, but the logic in Example 52 *does* carry over to our current situation.

Definition 159 (split extension)**.** An extension

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] & E \arrow[r, two heads, "\pi"] & G \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd}$$

is said to be *split* if there exists a group homomorphism $s\colon G \to E$ such that $\pi \circ s = \mathrm{id}_G$.

Proposition 160. An extension of E of G by A is equivalent to the semidirect product $A \rtimes G$ if and only if it is split.

Proof. The semidirect product $A \rtimes G$ admits the extension $g \mapsto (0, g)$. Given a split extension

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] & E \arrow[r, two heads, "\pi"] & G \arrow[r] & 0 \end{tikzcd} ,$$

we can define a map

$$E \to A \rtimes G; \qquad e \mapsto (e(s\pi(e)^{-1}), \pi(e)).$$

□

As the last proposition shows, in general we cannot find a section of an extension. However, since π is surjective, given any extension,

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0 \ ,$$

we can find at least a *set-theoretic extension* of G by A, i.e. a set-function $G \to E$ such that $\pi \circ s = e_G$.

Note that E can act on A via conjugation; if $a \in A$ and $e \in E$, then $eae^{-1} \in A$, because

$$\begin{aligned}\pi(eae^{-1}) &= \pi(e)\pi(a)\pi(e^{-1}) \\ &= \pi(e)\pi(e)^{-1} \\ &= e_G.\end{aligned}$$

In fact, since A is abelian, it acts trivially on itself via conjugation, so it descends to an action of $G \cong E/A$ on A.

This gives a section of E relative to G. Given such a splitting, we can define for each pair $g, h \in G$ an element $f(g,h) \in A$ in the fiber over the identity $e_G \in G$ which corresponds to how far…

$$f(g,h) = s(g)s(h)s(gh)^{-1}.$$

The function $f\colon G \times G \to G$ is called the *factor set* of G. If s were a group homomorphism, the factor set would vanish; in general, the factor set measures how far apart $s(g)s(h)$ and $s(gh)$ are.

Lemma 161. Given an extension

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] & E \arrow[r, two heads, "\pi"] & G \arrow[l, bend left, "s"] \arrow[r] & 0 \end{tikzcd} \ ,$$

we can completely recover the group law on E from the injection i and the factor set f.

Proof. As a set, E is in bijection with $A \times G$ via

$$e \mapsto (e(s\pi(e)^{-1}), \pi(e)).$$

It is easy to check that this has inverse

$$(a, g) \mapsto i(a)s(g).$$

Let $e, e' \in E$ corresponding to (a, g) and (a', g') respectively. Then

$$\begin{aligned} ee' &= i(a)s(g)i(a')s(g') \\ &= i(a)s(g)i(a')s(g)^{-1}s(g)s(h) \\ &= i(a)i(g \cdot b)s(g)s(h) \\ &= i(a + g \cdot b)i(f(g,h))s(gh) \\ &= i(a + g \cdot b + f(g,h))s(gh) \end{aligned}$$

so under the bijection $E \cong A \times G$, ee' corresponds to $(a + g \cdot b + f(g,h), gh)$. □

Associativity now immediately implies that

$$((a,g)(b,h))(c,k) \overset{!}{=} (a,g)((b,h)(c,k));$$

expanding, we find that

$$(a + g \cdot b + f(g,h) + (gh) \cdot c + f(gh,k), ghk)$$

must be the same as

$$(a + g \cdot b + (gh) \cdot c + gf(h,k) + f(g,hk), ghk).$$

That means that

$$g \cdot f(h,k) - f(gh,k) + f(g,hk) - f(g,h) \overset{!}{=} 0.$$

Thus, the factor set f must be a 2-cocycle in $C^\bullet(G; A)$.

Our construction depends on the choice of section s. supposed we had picked a different section s'. Then the 'difference' $s'(g)s(g)^{-1}$ can be written $i(\phi(a))$ for some $a \in A$. We thus obtain a map $\phi\colon G \to A$.

To warm up, we do a computation.

$$s'(g)s'(h) = \cdots$$

In the end, we find

$$f_{s'}(g,h) - f_s(g,h) = g \cdot \phi(h) - \phi(gh) + \phi(g).$$

Theorem 162. Let G be a group, and A a G-module. Then there is a natural bijection

$$\left\{\begin{matrix}\text{Extension classes of } G \text{ by } A \\ \text{such that the action of } G \text{ on } A \\ \text{agrees with the module structure}\end{matrix}\right\} \xleftrightarrow{\cong} H^2(G;A).$$

4.3.6. Periodicity in group homology

In this section, we will prove a combinatorial version of the following result.

Fact 163. Let G be a group. Suppose G acts freely on a sphere of dimension $2k-1$. Then G has $2k$-periodic group homology.

Simplicial complexes

Definition 164 (simplicial complex)**.** Let I be a finite set. A simplicial complex K on I is a collection of subsets $K \subset \mathcal{P}(I)$ such that for $\sigma \in K$ and $\tau \subset \sigma$, $\tau \in K$.

Let K be a simplicial complex on a set I, and let K' be a simplicial complex on a set I'. A morphism $K \to K'$ is a function $f: I \to I'$ such that for every $\sigma \in K$, $f(\sigma) \in K'$.

We define a cosimplicial object in the category of simplicial complexes by

$$[n] \mapsto \mathcal{P}(\{0, \dots, n\}).$$

Taking nerves gives us, for every simplicial complex K, a simplicial set $\tilde{K}$.

Note that for each $\sigma \in K_n$, there are $n!$ simplices in $\tilde{K}_n$, one for each permutation of the elements of σ. In **Set** there is not much we can do about this; however, applying the free abelian group functor $\mathcal{F}$ brings us to **Ab**, where we can mod out by this ambiguity.

We thus define a simplicial complex $\widetilde{C}(K)$ to be the alternating face maps complex of the simplicial object $\mathcal{F} \circ \tilde{K}$.

A more concrete description of what we have just done is as follows. We have

$$\widetilde{C}(K)_n = \bigoplus_{\{x_0,\dots,x_n\} \in K_n} \mathbb{Z} e_{(x_0,\dots,x_n)},$$

and the differential is

$$\sum_{i=0}^{n} (-1)^i d_i,$$

where the face maps d_i are the $\mathbb{Z}$-linear extensions of the maps

$$d_i\colon e_{(x_0,\dots,x_n)} \mapsto e_{(x_0,\dots,\widehat{x_i},\dots,x_n)}.$$

Now that we are in **Ab**, we can get rid of the extraneous simplices by defining

$$C(K)_n = \widetilde{C}(K)_n / \langle e_{(x_1,\dots,x_n)} - \operatorname{sign}(\sigma) e_{(x_{\sigma(1)},\dots,x_{\sigma(n)})} \mid \sigma \in S_n \rangle.$$

It is not hard to see that the differential on $\widetilde{C}(K)$ descends to $C(K)$.

Example 165. Let $K = \mathcal{P}(\{0,1\})$. Then $e_{(0,1)} = -e_{(1,0)}$ in $C(K)_1$.

Let $f\colon K \to K$ be an automorphism of simplicial complexes K. This descends to an automorphism of chain complexes $C_\bullet(K) \to C_\bullet(K)$ defined on generators by

$$e_{(x_0,\dots,x_n)} \mapsto e_{(f(x_0),\dots,f(x_n))}.$$

Now suppose G acts trivially on

5. Spectral sequences

5.1. Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section 2.11, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the building blocks of the homology of the total complex of such a double complex.

They do this via a series of successive approximations. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
E_{0,3} \arrow[d, "d^v"'] & E_{1,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,2} \arrow[d, "d^v"'] & E_{1,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,1} \arrow[d, "d^v"'] & E_{1,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,0} & E_{1,0} \arrow[l, "d^h"'] & E_{2,0} \arrow[l, "d^h"'] & E_{3,0} \arrow[l, "d^h"'] & \cdots \arrow[l]
\end{tikzcd}$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$E^0_{\bullet,\bullet}$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^0_{0,3} \arrow[d, "d^v"'] & E^0_{1,3} \arrow[d, "d^v"'] & E^0_{2,3} \arrow[d, "d^v"'] & E^0_{3,3} \arrow[d, "d^v"'] \\
E^0_{0,2} \arrow[d, "d^v"'] & E^0_{1,2} \arrow[d, "d^v"'] & E^0_{2,2} \arrow[d, "d^v"'] & E^0_{3,2} \arrow[d, "d^v"'] \\
E^0_{0,1} \arrow[d, "d^v"'] & E^0_{1,1} \arrow[d, "d^v"'] & E^0_{2,1} \arrow[d, "d^v"'] & E^0_{3,1} \arrow[d, "d^v"'] \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{tikzcd} \tag{5.1}$$

One then takes homology of the corresponding vertical complexes, replacing $E^0_{p,q}$ by the homology $H_q(E^0_{p,\bullet})$. To ease notation, one writes

$$H_q(E^0_{p,\bullet}) = E^1_{p,q}.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$\begin{tikzcd}
E^1_{0,3} & E^1_{1,3} \arrow[l, "H_3(d^h)"'] & E^1_{2,3} \arrow[l, "H_3(d^h)"'] & E^1_{3,3} \arrow[l, "H_3(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "H_2(d^h)"'] & E^1_{2,2} \arrow[l, "H_2(d^h)"'] & E^1_{3,2} \arrow[l, "H_2(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "H_1(d^h)"'] & E^1_{2,1} \arrow[l, "H_1(d^h)"'] & E^1_{3,1} \arrow[l, "H_1(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "H_0(d^h)"'] & E^1_{2,0} \arrow[l, "H_0(d^h)"'] & E^1_{3,0} \arrow[l, "H_0(d^h)"'] & \cdots \arrow[l]
\end{tikzcd} \tag{5.2}$$

We now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{\bullet,q})$.

The claim is that the terms $E^r_{p,q}$ are pieces of a successive approximation of the homology of the total complex $\mathrm{Tot}(E)$. We will prove this claim much later. However, in the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Example 166. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes $E_{0,\bullet}$ and $E_{1,\bullet}$ and a morphism $f\colon E_{1,\bullet} \to E_{0,\bullet}$, padded by zeroes on either side.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
E_{0,3} \arrow[d, "d^0_3"'] & E_{1,3} \arrow[l, "f_3"'] \arrow[d, "-d^1_3"] \\
E_{0,2} \arrow[d, "d^0_2"'] & E_{1,2} \arrow[l, "f_2"'] \arrow[d, "-d^1_2"] \\
E_{0,1} \arrow[d, "d^0_1"'] & E^{1,1} \arrow[l, "f_1"'] \arrow[d, "-d^1_1"] \\
E_{0,0} \arrow[d, "d^0_0"'] & E_{1,0} \arrow[l, "f_0"'] \arrow[d, "-d^1_0"] \\
E_{-0,1} \arrow[d] & E_{-1,1} \arrow[l, "f_{-1}"'] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Note that we have multiplied the diferentials of $E_{1,\bullet}$ by -1 so that we have a double complex. This does not enter into the discussion.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$. Further, let $T = \mathrm{Tot}(E)$; that is, let $T = \mathrm{Cone}(f)$. Recall that $\mathrm{Cone}(f)$ fits into the following short exact seqence,

$$0 \longrightarrow E_{\bullet,1} \hookrightarrow \mathrm{cone}(f)_\bullet \twoheadrightarrow E_{\bullet,0}[-1] \longrightarrow 0$$

and that this gives us the following long exact sequence on homology.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_n(E_{\bullet,0}) \arrow[dll, "\delta" description] \\
H_n(E_{\bullet,1}) \arrow[r] & H_n(\mathrm{Cone}(f)_\bullet) \arrow[r] & H_{n-1}(E_{\bullet,0}) \arrow[dll, "\delta" description] \\
H_{n-1}(E_{\bullet,1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \operatorname{coker}(H_n(f)) \hookrightarrow H_n(\mathrm{Cone}(f)) \twoheadrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation; that is, we have a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0 \ .$$

5.2. Homology spectral sequences

5.2.1. Notation and terminology

Definition 167 (homology spectral sequence)**.** Let $\mathcal{A}$ be an abelian category. A homology spectral sequence starting at E^a in $\mathcal{A}$ consists of the following data.

1. A family $E^r_{p,q}$ of objects of $\mathcal{A}$, defined for all integers p, q, and $r \geq a$.
2. Maps
$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r+1}$$
that are differentials in the sense that $d^r \circ d^r = 0$.
3. Isomorphisms
$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1})$$
between $E^{r+1}_{p,q}$ and the homology at the corresponding position of the E^r

We denote such a spectral sequence by E. A morphism $E' \to E$ is a family of maps

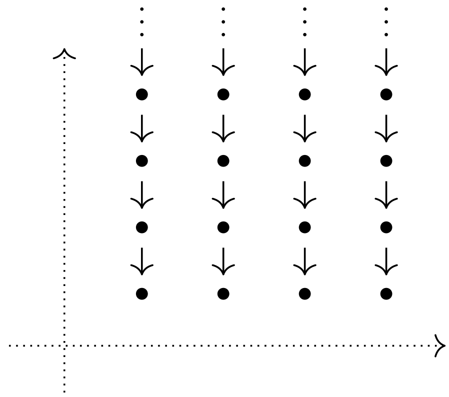
$$f^r_{p,q} : E'^r_{p,q} \to E^r_{p,q}$$

such that $d^r f^r = f^r d^r$ such that $f^{r+1}_{p,q}$ is induced on homology by $f^r_{p,q}$.

For each r, a spectral sequence E has a two-dimensional array of objects $E^r_{p,q}$. For a given r, one calls the objects $E^r_{p,q}$ the rth *page* of the spectral sequence E.

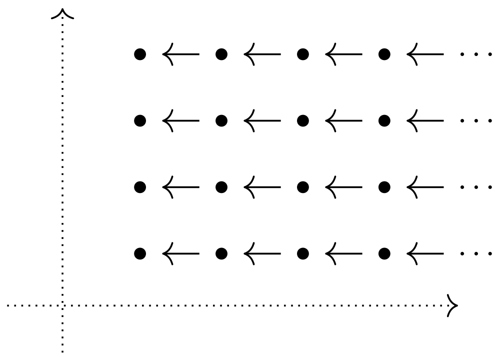
We have already seen an example of the first two pages of a spectral sequence (in Diagram 5.1 and Diagram 5.2). Schematically, the differentials on the zeroth page point down.

E^0 :



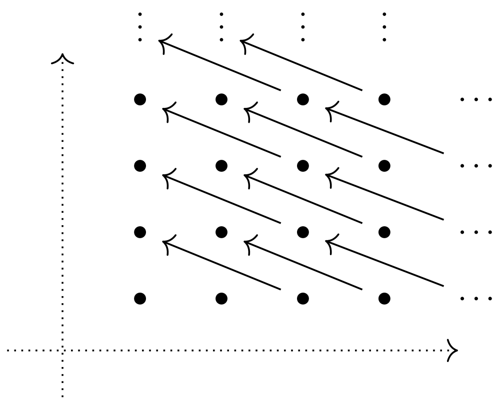
The differentials on the first page point to the left.

E^1 :



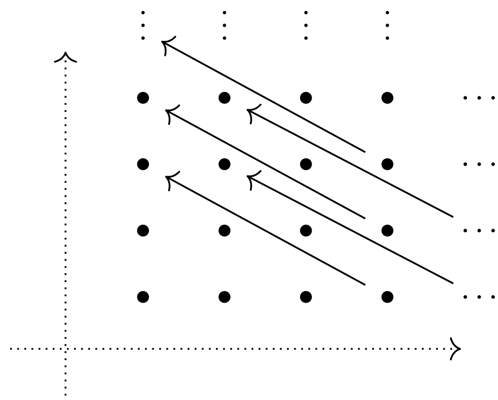
As r increases, the arrows rotate clockwise. The second page looks like this.

E^2 :



The third page looks like this.

$E^3:$



5.2.2. Stabilization and convergence

We have so far placed no conditions on $E^r_{p,q}$. However, we will confine our study of certain classes of spectral sequences.

Definition 168 (bounded, first quadrant)**.** Let E be a spectral sequence starting at E^a.

- We say that E is bounded if each $n \in \mathbb{Z}$, there are only finitely many terms of total degree n on the ath page, i.e. terms of the form $E^a_{p,q}$ for $n = p + q$.
- We say that E is first-quadrant if $E^a_{p,q} = 0$ for $p < 0$ and $q < 0$.

Clearly, every first-quadrant spectral sequence is bounded.

Note that if a spectral sequence is bounded, then

We get $E^{r+1}_{p,q}$ by taking the homology of a complex on the rth page. If E is a first-quadrant spectral sequence, then for $r > \max(p, q + 1)$, the differential entering $E^r_{p,q}$ comes from the fourth quadrant (hence from a zero object), and the differential leaving it lands in the second quadrant (also a zero object). Thus, $E^{r+1}_{p,q} = E^r_{p,q}$.

Definition 169 (stabilize)**.** A spectral sequence starting at E^a is said to stabilize at position (p, q) if there exists some $r \geq a$ such that $E^{r'}_{p,q} = E^r_{p,q}$ for all $r' \geq r$. In this case, we write $E^\infty_{p,q}$ for the stable value at position (p, q).

By the argument above, each position in a first-quadrant spectral sequence eventually stabilizes. We will mostly be interested in first-quadrant spectral sequences. However, boundedness is (clearly!) sufficient to guarantee stabilization.

Suppose one has been given a bounded spectral sequence, which we now know eventually stabilizes at each position. We would like to get some useful information out of such a spectral sequence.

Definition 170 (filtration)**.** Let $C_\bullet$ be a chain complex. A filtration F of $C_\bullet$ is a chain of inclusions of subcomplexes $F_pC_\bullet$ of $C_\bullet$.

$$\cdots \subset F_{p-1}C_\bullet \subset F_pC_\bullet \subset F_{p+1}C_\bullet \subset \cdots \subset C_\bullet$$

Definition 171 (bounded convergence)**.** Let E be a bounded spectral sequence starting at E^a in an abelian category $\mathcal{A}$, and let $\{H_i\}_{i\in\mathbb{Z}}$ be a collection of objects in $\mathcal{A}$, each equipped with a finite filtration

$$0 = F_sH_n \subset \cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset F_tH_n = H_n.$$

We say that E converges to $\{H_i\}_{i\in\mathbb{Z}}$ if there exist isomorphisms

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

We express this convergence by writing

$$E^a_{p,q} \Rightarrow H_{p+q}.$$

Example 172. Consider the spectral sequence E starting at E^0 from Example 166. This is certainly bounded, and stabilizes at each position (p,q) after the second page; that is, we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

The exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0$$

gives us, in a wholly unsatisfying and trivial way, a filtration

$$\begin{tikzcd}
F_{-1}H_n \arrow[d, equal] & F_0H_n \arrow[d, equal] & F_1H_n \arrow[d, equal] \\
0 \arrow[r, hook] & E^2_{0,n} \arrow[r, hook] & H_n
\end{tikzcd}$$

of H_n. The claim is that

$$E^0_{p,q} \Rightarrow H_{p+q}.$$

In order to check this, we have to check that

$$E^2_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q. The only non-trivial cases are when $p = 0$ or 1. When $p = 0$, we have

$$\begin{aligned} E^{\infty}_{0,n} &\overset{!}{\cong} F_0 H_n / F_{-1} H_n \\ &\cong E^2_{0,n}/0 \\ &\cong E^2_{0,n}, \end{aligned}$$

and when $p = 1$ we have

$$\begin{aligned} E^{\infty}_{1,n} &\overset{!}{\cong} F_1 H_{n+1} / F_0 H_{n+1} \\ &\cong H_{n+1}/E^2_{0,n+1} \\ &\cong E^2_{1,n} \end{aligned}$$

as required.

5.2.3. Exact couples

We now take a detour to the much simpler, more elegant world of exact couples.

Definition 173 (exact couple)**.** Let $\mathcal{A}$ be an abelian category. An exact couple in $\mathcal{A}$ consists of objects and morphisms

$$\begin{tikzcd} & E \arrow[dr, "\alpha"] & \\ A \arrow[ur, "\gamma"] & & E \arrow[ll, "\beta"] \end{tikzcd}$$

such that $\operatorname{im}\alpha = \ker\beta$

5.2.4. The spectral sequence of a filtered complex

Theorem 174. A filtration F of a chain complex $C_\bullet$ naturally determines a spectral sequence starting with

$$E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}, \qquad E^1_{p,q} = H_{p+q}(E^0_{p,\bullet})$$

Proof. We construct explicitly the spectral sequence above. Let $C_\bullet$ be a chain complex, and F a filtration. We will use the following notation.

- We will denote by η_p the quotient

  $$\eta_p\colon F_pC \twoheadrightarrow F_pC/F_{p-1}C.$$

  coming from the short exact sequence

  $$0 \longrightarrow F_{p-1}C \hookrightarrow F_pC \twoheadrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

- We will denote by A_p^r the subobject

  $$A_p^r = \{c \in F_pC\colon dc \in F_{p-r}C\} \subset F_pC$$

  of those elements whose differentials survive r many levels of the grading.

- We will denote by Z_p^r the image

  $$Z_p^r = \eta_p(A_p^r).$$

- We will denote by B_p^r the image

  $$B_p^r = \eta_p(d(a_{p+r-1}^{r-1})).$$

Examining the definitions, we see that we have the following inclusion.

$$\begin{tikzcd}
dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\
dA_{p+r}^{r} \arrow[r, equal] & \{dc \mid c \in F_{p+r},\ dc \in F_pC\}
\end{tikzcd}$$

Both of these are subobjects of F_pC; by applying η_p and taking a sharp look at the definition of the Bs, we find an inclusion

$$B_p^r \subset B_p^{r+1}.$$

Working inductively, we are left with the following sequence of inclusions.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^r \subset \cdots$$

Defining $B_p^\infty = \bigcup_r B_p^r$, we can crown our sequence of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty.$$

Again examining definitions, we find the following inclusion.

$$\begin{tikzcd} A_p^r \arrow[r, equal] \arrow[d, hook] & \{c \in F_pC \mid dc \in F_{p-r}C\} \arrow[d, hook] \\ A_p^{r+1} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r-1}C\} \end{tikzcd}$$

As before, applying η_p and working inductively gives us a chain

$$\cdots \subset Z_p^r \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0$$

and defining $Z_p^\infty = \bigcap_r Z_p^r$ gives us

$$Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0,$$

But we should not rest on our laurels. Instead, note that for each r, r', we have an inclusion

$$\begin{tikzcd} dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ A_p^{r'} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r'}C\} \end{tikzcd}$$

since $d^2 = 0$. Thus, we can graft our chains of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty \subset Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0.$$

We defined

$$Z_p^r = \eta_p(A_p^r).$$

Recall that η_p was defined to be the canoncal projection corresponding to a short exact sequence. Taking subobjects (where the left-hand square is a pullback), we find the following monomorphism of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1}C \cap A_r^p \arrow[r, hook] \arrow[d, hook] & A_r^p \arrow[r, two heads] \arrow[d, hook] & Z_r^p \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F_{p-1}C \arrow[r, hook] & F_pC \arrow[r, two heads] & F_pC/F_{p-1}C \arrow[r] & 0 \end{tikzcd}$$

Rewriting the definitions, we find that

$$F_{p-1}C \cap A_r^p = \{c \in F_{p-1}C \mid dc \in F_{p-r}C\} = A_{p-1}^{r-1},$$

so

$$Z_p^r \cong A_p^r / A_{p-1}^{r-1}.$$

We now define
$$E_p^r = \frac{Z_p^r}{B_p^r}.$$
We can write
$$\frac{Z_p^r}{B_p^r} \cong \frac{A_p^r + F_{p-1}C}{d(A_{p+r-1}^{r-1}) + F_{p-1}C} \cong \frac{A_p^r}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}}.$$
I don't understand this step, and I don't have time to do the rest. □

Theorem 175. Let $C_\bullet$ be a chain complex, and F a bounded filtration on $C_\bullet$. Then the spectral sequence constructed in Theorem 174 converges to the homology of the homology of $C_\bullet$; that is

$$E_{p,q}^1 = H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet) \Longrightarrow H_{j+q}(\mathrm{Tot}(C_\bullet)).$$

Example 176. Let E be the spectral sequence associated to a filtered complex $C_\bullet$, and suppose that
$$E_{p,q}^r = \begin{cases} A_p, & q = 0 \\ 0, & q \neq 0 \end{cases}, \qquad r > 0.$$

Then certainly the spectral sequence collapses.

The claim is that $A_p = H_p(C_\bullet)$; that is, the spectral sequence converges to A_p. To see this, we need to find filtrations

$$0 = F_{s-1}A_p \subset F_sA_p \subset F_{s+1}A_p \subset \cdots \subset F_tA_p \subset F_{t+1}A_p = A_p$$

such that, for each p and q, we have

$$E_{p,q}^\infty = F_pA_{p+q}/F_{p-1}A_{p+q}.$$

This requires that

$$\begin{aligned} 0 &= F_sA_p/F_{s-1}A_p \\ &\vdots \\ 0 &= F_{p-1}A_p/F_{p-2}A_p \\ A_p &= F_pA_p/F_{p-1}A_p \\ 0 &= F_{p+1}A_p/F_pA_p \\ &\vdots \\ 0 &= F_{t+1}A_p/F_tA_p. \end{aligned}$$

Starting at $F_{s-1}A_p = 0$, these equations tell us that

$$0 = F_{s-1}A_p = F_sA_p = F_{s+1}A_p = \cdots = F_{p-1}A_p.$$

We then have that

$$A_p = F_pA_p/0 = F_pA_p.$$

Again, our equations tell us that

$$F_pA_p = F_{p+1}A_p = \cdots = F_tA_p.$$

This tells us that $H_p(C_\bullet) = A_p$.

Note that we could have taken our filtration to be the one-step filtration

$$0 = F_{p-1}A_p \subset F_pA_p = A_p.$$

Example 177. Suppose instead of collapsing to a single row, our spectral sequence stabilizes to two rows:

$$E^\infty_{p,q} = \begin{cases} A_p, & q = 0 \\ B_p, & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Identical logic to that above tells us that

$$\begin{aligned} F_{p-1}H_p/F_{p-2}H_p &= 0 \\ F_pH_p/F_{p-1}H_p &= A_p \\ F_{p+1}H_p/F_pH_p &= B_{p-1} \\ F_{p+2}H_p/F_{p+1}H_p &= 0, \end{aligned}$$

i.e. that we have a two-step filtration

$$0 = F_{p-1}H_p \subset F_pH_p \subset F_{p+1}H_p = H_p,$$

with $F_pH_p = A_p$ and $F_{p+1}H_p/A_p = B_{p-1}$. This says precisely that we have a short exact sequence

$$0 \longrightarrow A_p \hookrightarrow H_p \twoheadrightarrow B_{p-1} \longrightarrow 0 .$$

That is, we know our homology up to an extension problem. This is exactly what happened in Example 166.

5.3. The spectral sequence of a double complex

A special case of the spectral sequence of a filtered complex is the spectral sequence of a double complex. Given a double complex $C_{\bullet,\bullet}$, we define a new double complex

$$\left({}^I\tau_{\leq n}C\right)_{p,q} = \begin{cases} C_{p,q} & p \leq n \\ 0, & p > n \end{cases}.$$

That is, the $\left({}^I\tau_{\leq n}C\right)$ is the double complex obtained by setting all the entries to the right of the column $C_{p,\bullet}$ to zero.

This leads to a filtered complex

$$0 \subset \cdots \subset \mathrm{Tot}({}^I\tau_{\leq 0}C)_{p,q} \subset \mathrm{Tot}({}^I\tau_{\leq 1}C)_{p,q} \subset \cdots \subset \mathrm{Tot}(C)_{p,q};$$

that is, with

$$F_pC = \mathrm{Tot}({}^I\tau_{\leq p}C).$$

This gives rise to a spectral sequence via Theorem 174.

Specifically, we have

$$\begin{aligned} {}^IE^0_{p,q} &= \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}} \\ &= \frac{\mathrm{Tot}({}^I\tau_{\leq p}C)_{p+q}}{\mathrm{Tot}({}^I\tau_{\leq p-1}C)_{p+q}} \\ &= \frac{\bigoplus_{i+j=p+q} C_{i,j}}{\bigoplus_{\substack{i+j=p+q \\ i\leq p-1}} C_{i,j}} \\ &= \frac{C_{0,p+q} \oplus \cdots \oplus C_{p,q}}{C_{0,p+q} \oplus \cdots \oplus C_{p-1,q+1}} \\ &= C_{p,q}. \end{aligned}$$

The differentials $C_{p,q} \longrightarrow C_{p,q-1}$ are simply the vertical differentials $d^v_{p,q}$.

Of course, we can also transpose our double complex before performing the above procedure; this gives a different spectral sequence ${}^{II}E^0_{p,q}$.

By Theorem 174, both of these converge to the homology of the total complex $H_{p+q}(\mathrm{Tot}(C))$.

Definition 178 (spectral sequences associated to a double complex)**.**

- The vertically filtered spectral sequence associated to a double complex is the spectral sequence ${}^IE^0_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(C))$.
- The horizontally filtered spectral sequence associated to a double complex is the

spectral sequence ${}^{II}E^0_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(C))$.

The vertically filtered spectral sequence associated to a double complex is the one given in Section 5.1.

Example 179 (balancing Tor again)**.** Let M and N be right and left R-modules respectively, and let $P_\bullet \xrightarrow{\simeq} M$ and $Q_\bullet \xrightarrow{\simeq} N$ be projective resolutions. We can form the double complex

$$M_{p,q} = P_p \otimes_R Q_q.$$

The totalization of this complex is then simply what we have been calling $P_\bullet \otimes_R Q_\bullet$.

Let us work out explicitly the vertical filtration. The zeroth page is as follows.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
P_0 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_1 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_2 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] \\
P_0 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_1 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_2 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] \\
P_0 \otimes_R Q_0 & P_1 \otimes_R Q_0 & P_2 \otimes_R Q_0
\end{tikzcd}$$

Since each P_i is projective, taking homology ignores it, and we get that

$$\begin{aligned} H_q(P_p \otimes_R Q_\bullet) &= P_p \otimes_R H_q(Q_\bullet) \\ &= \begin{cases} P_p \otimes_R N, & q = 0 \\ 0, & q \neq 0 \end{cases}. \end{aligned}$$

Therefore, the first page of our spectral sequence consists of a single line at the first page.

$$
\begin{array}{cccc}
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
P_0 \otimes_R N & \longleftarrow P_1 \otimes_R N & \longleftarrow P_2 \otimes_R N & \longleftarrow \cdots
\end{array}
$$

We now have

$$
\begin{aligned}
{}^{I}E^2_{p,q} &= H_p(P_\bullet \otimes_R N) \\
&= \operatorname{Tor}_p(M, N).
\end{aligned}
$$

At this point the spectral sequence collapses, telling us that

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(P_\bullet \otimes_R N) = \operatorname{Tor}_p(M, N).$$

The other filtration gives

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(M \otimes_R Q_\bullet).$$

Thus,

$$H_p(M \otimes_R Q_\bullet) = \operatorname{Tor}_p(M, N),$$

which says that Tor is balanced.

Example 180 (universal coefficient theorem)**.** Let $X_\bullet$ be a chain complex of free abelian groups, and let A be an abelian group. Then there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes A \hookrightarrow H_n(X \otimes A) \twoheadrightarrow \operatorname{Tor}_1(H_{n-1}(X), A) \longrightarrow 0 \ .$$

5.4. Hyper-derived functors

Consider abelian categories and right exact functors as follows.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

One might wonder about the relationship between LF, LG, and $L(G \circ F)$.

The problem is that we don't know how to make sense of the composition $LG \circ LF$, since LF lands in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ rather than $\mathcal{A}$; derived functors go from an abelian category to a category of chain complexes. The solution is to define a *projective resolution of a chain complex.*

5.4.1. Cartan-Eilenberg resolutions

Definition 181 (Cartan-Eilenberg resolution)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $A \in \mathbf{Ch}^{+}(\mathcal{A})$. A Cartan-Eilenberg resolution of A is an upper-half-plane double complex $(P_{\bullet,\bullet}, d^h, d^v)$, equipped with an augmentation map

$$\epsilon \colon P_{\bullet,0} \to A_{\bullet}$$

such that the following criteria are satsfied.

1. $B(P_{p,\bullet}, d^h) \xrightarrow{\simeq} B(A_p, d^A)$ is a projective resolution.
2. $H(P_{p,\bullet}, d^h) \xrightarrow{\simeq} H(A_p, d^A)$ is a projective resolution.
3. $A_p \cong 0 \implies P_{p,\bullet} = 0$.

Here, $B(P_{p,\bullet}, d^h)$ are the horizontal boundaries of the chain complex $(P_{p,\bullet}, d^h)$, and

$$H(P_{p,\bullet}, d^h) \cong \frac{Z(P_{p,\bullet}, d^h)}{B(P_{p,\bullet}, d^h)}$$

is defined similarly.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & P_{-r,1} \arrow[l] \arrow[d] & \cdots \arrow[l] & P_{0,1} \arrow[l] \arrow[d] & P_{1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[u, leftarrow] \arrow[d, dashed] & P_{-r,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] & P_{0,0} \arrow[l] \arrow[d, dashed] & P_{1,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
0 & A_{-r} \arrow[l] & \cdots \arrow[l] & A_0 \arrow[l] & A_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Proposition 182. The following are also projective resolutions.

1. $Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)$
2. $P_{p,\bullet} \xrightarrow{\simeq} A_p$

Proof.

1. We have essentially by definition the following short exact sequence.

$$0 \longrightarrow B(P_{p,\bullet}, d^h) \hookrightarrow Z(P_{p,\bullet}, d^h) \twoheadrightarrow H(P_{p,\bullet}, d^h) \longrightarrow 0$$

The outer terms are projective because they are the terms of projective resolutions, so $Z(P_{p,\bullet}, d^h)$ is also projective. Thus, $Z(P_{p,\bullet}, d^h)$ is at the very least a complex of projectives.

Further, we have the first few rows of the long exact sequence on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & Z(P_{p,1}, d^h) \arrow[r] & 0 \arrow[dll, "\delta"] \\
0 \arrow[r] & Z(P_{p,0}, d^h) \arrow[r] & 0 \arrow[dll, "\delta"] \\
B(A_p, d^A) \arrow[r] & Z(A_p, d^A) \arrow[r] & H(A_p, d^A) \arrow[r] & 0
\end{tikzcd}$$

This tells us that the homology of $Z(P_{p,\bullet})$ is

$$H_q(Z(P_{p,\bullet}, d^h)) = \begin{cases} Z(A_p, d^A), & q = 0 \\ 0, & q > 0 \end{cases},$$

i.e. that $Z(P_{p,\bullet}, d^h)$ is a projective resolution of $Z(A_p, d^A)$.

2. Exactly the same, but with the short exact sequence

$$0 \longrightarrow Z(P_{p,\bullet}, d^h) \hookrightarrow H(P_{p,\bullet}, d^h) \twoheadrightarrow B(P_{p-1,\bullet}, d^h) \longrightarrow 0$$

□

Proposition 183. Every chain complex $A_\bullet$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$.

Proof. We construct a Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ explicitly. Pick projective resolutions

$$P^B_{p,\bullet} \to B(A_p, d^h), \qquad B^H_{p,\bullet} \xrightarrow{\simeq} H(A_p, d^h).$$

By the horseshoe lemma, we can find a projective resolution $P^Z_{p,\bullet} \xrightarrow{\simeq} Z(A_p, d^h)$ fitting into the following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & P^B_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^Z_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^H_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & B(A_p, d^h) \arrow[r, hook] & Z(A_p, d^h) \arrow[r, two heads] & H(A_p, d^h) \arrow[r] & 0 \end{tikzcd} \tag{5.3}$$

Playing the same game again, we find a projective resolution of A_p.

$$\begin{tikzcd} 0 \arrow[r] & P^Z_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^A_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & Z(A_p, d^h) \arrow[r, hook] & A_p \arrow[r, two heads] & B(A_{p-1}, d^h) \arrow[r] & 0 \end{tikzcd} \tag{5.4}$$

We then define our Cartan-Eilenberg resolution to be the double complex whose pth column is $(P^A_{p,\bullet}, (-1)^p d^{P^A})$, and whose horizontal differentials are given by the composition

$$\begin{tikzcd} P^A_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^Z_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^A_{p-1,\bullet} \arrow[d, "\simeq"] \\ A_p \arrow[r, "d^A"] & B_{p-1} \arrow[r, hook] & Z_{p-1} \arrow[r, hook] & A_{p-1} \end{tikzcd} \tag{5.5}$$

In this case, all the criteria are satisfied by definition. □

Proposition 184. Let $A_\bullet$ and $A'_\bullet$ be bounded-below chain complexes, and let $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$ and $P'_{\bullet,\bullet} \xrightarrow{\simeq} A'_\bullet$ be Cartan-Eilenberg resolutions. Then any morphism of chain complexes $f\colon A_\bullet \to A'_\bullet$ can be lifted to a morphism $\tilde{f}\colon P^A_{\bullet,\bullet} \to P^B_{\bullet,\bullet}$ of double complex. Furthermore, this lift is unique up to a *homotopy of double complexes,* i.e. for any other lift $\tilde{\tilde{f}}$, there exist maps

$$s^h_{p,q}\colon P_{p,q} \to P'_{p+1,q}, \qquad s^v_{p,q}\colon P_{p,q} \to E_{p,q+1}$$

such that

Proof. We have the following lifts.

$$\begin{tikzcd}
& & Q^B_{p,\bullet} \arrow[dd] & & Q^Z_{p,\bullet} \arrow[dd] & & Q^H_{p,\bullet} \arrow[dd] & & \\
& P^B_{p,\bullet} \arrow[ur] \arrow[dd] & & P^Z_{p,\bullet} \arrow[dd] & & P^H_{p,\bullet} \arrow[ur] \arrow[dd] & & & \\
& 0 \arrow[r] & B_p(B) \arrow[rr] & & Z_p(B) \arrow[rr] & & H_p(B) \arrow[rr] & & 0 \\
0 \arrow[r] & B_p(A) \arrow[ur] \arrow[rr] & & Z_p(A) \arrow[ur] \arrow[rr] & & H_p(A) \arrow[ur] \arrow[rr] & & 0 &
\end{tikzcd}$$

The horseshoe lemma for morphisms (mutatis mutandis) allows us to complete this diagram in a compatible way as follows.

$$\begin{tikzcd}
& 0 \arrow[r] & Q^B_{p,\bullet} \arrow[dd] \arrow[rr] & & Q^Z_{p,\bullet} \arrow[dd] \arrow[rr] & & Q^H_{p,\bullet} \arrow[dd] \arrow[rr] & & 0 \\
0 \arrow[r] & P^B_{p,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & P^Z_{p,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & P^H_{p,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & 0 & \\
& 0 \arrow[r] & B_p(B) \arrow[rr] & & Z_p(B) \arrow[rr] & & H_p(B) \arrow[rr] & & 0 \\
0 \arrow[r] & B_p(A) \arrow[ur] \arrow[rr] & & Z_p(A) \arrow[ur] \arrow[rr] & & H_p(A) \arrow[ur] \arrow[rr] & & 0 &
\end{tikzcd}$$

Using the lifts we constructed above and identical logic, we get the following.

$$\begin{tikzcd}
& 0 \arrow[r] & Q^Z_{p,\bullet} \arrow[dd] \arrow[rr] & & Q_{p,\bullet} \arrow[dd] \arrow[rr] & & Q^B_{p-1,\bullet} \arrow[dd] \arrow[rr] & & 0 \\
0 \arrow[r] & P^Z_{p,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & P_{p,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & P^B_{p-1,\bullet} \arrow[ur] \arrow[dd] \arrow[rr] & & 0 & \\
& 0 \arrow[r] & Z_p(B) \arrow[rr] & & B_p(A) \arrow[rr] & & B_{p-1}(A) \arrow[rr] & & 0 \\
0 \arrow[r] & Z_p(A) \arrow[ur] \arrow[rr] & & A_p \arrow[ur] \arrow[rr] & & B_{p-1}(A) \arrow[ur] \arrow[rr] & & 0 &
\end{tikzcd}$$

Thus we have, for each $p \in \mathbb{Z}$, a morphism of complexes $P_{p,\bullet} \to Q_{p,\bullet}$. It remains only to show that each of the squares

$$\begin{tikzcd} P_{p,\bullet} \arrow[r] \arrow[d] & P_{p-1,\bullet} \arrow[d] \\ Q_{p,\bullet} \arrow[r] & Q_{p-1,\bullet} \end{tikzcd}$$

commutes. But we get these squares by pasting together squares stolen from the above diagrams as follows.

$$\begin{tikzcd} P_{p,\bullet} \arrow[r, "d^h"] \arrow[d] & P^Z_{p-1,\bullet} \arrow[r, hook] \arrow[d] & P^B_{p-1,\bullet} \arrow[r, hook] \arrow[d] & P_{p-1,\bullet} \arrow[d] \\ Q_{p,\bullet} \arrow[r, "d^h"] & Q^Z_{p-1,\bullet} \arrow[r, hook] & Q^B_{p-1,\bullet} \arrow[r, hook] & Q_{p-1,\bullet} \end{tikzcd}$$

□

Proposition 185. Let f, g be chain homotopic maps via the homotopy f.

Proposition 186. Let $P^A_{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A_\bullet$. Then the canonical map

$$\mathrm{Tot}(P^A_{\bullet,\bullet}) \to A$$

is a quasi-isomorphism

Proof. Spectral sequence with vertical filtration. □

5.4.2. Hyper-derived functors

Definition 187 (left hyper-derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor, and assume $\mathcal{A}$ has enough projectives. The left hyper-derived functor $\mathbb{L}_i F$ of F is the functor

$$\mathbb{L}_i F\colon \mathbf{Ch}_+(\mathcal{A}) \to \mathcal{B}; \qquad \mathbb{L}_i F(A_\bullet) = H_i(\mathrm{Tot}(F(P^A_{\bullet,\bullet}))),$$

where $P^A_{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of $A_\bullet$.

Hyper-derived functors are functorial by

Proposition 188. Let $A_\bullet$ be a bounded below chain complex in an abelian category $\mathcal{A}$. There are two spectral sequences

$${}^{I}E^2_{p,q} = H_p(L_q F(A_\bullet)) \Longrightarrow \mathbb{L}_{p+q}(A_\bullet),$$

$${}^{II}E^2_{p,q} = L_p F(H_q(A_\bullet)) \Longrightarrow \mathbb{L}_{p+q}(A_\bullet).$$

Proof. We use the spectral sequence for the double complex $F(P^A_{\bullet,\bullet})$. The horizontal filtration has no surprises. Using the vertical filtration, one must use that

$$H_p(F(P^A_{\bullet,\bullet})) = F(H_p(P^A_{\bullet,\bullet})),$$

which is true because the horizontal cycles and boundaries are □

This allows us to define the famous *Grothendieck spectral sequence,* which allows us to compute the composition of two derived functors.

Proposition 189. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories, and assume that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Let F and G as below be right exact, and assume that G sends projective objects to F-acyclic objects.[1] Then for all $A \in \mathcal{A}$, there exists a convergent spectral sequence

$$E^2_{p,q} = L_pF(L_qG(A)) \Longrightarrow L_{p+q}(F \circ G)(A).$$

Proof. Let $P_\bullet \xrightarrow{\simeq} A$ be a projective resolution. From Proposition 188, we find two spectral sequences. Applying the first to $G(P_\bullet)$, we find

$$H_p(L_q(F(G(P_\bullet)))) \Longrightarrow \mathbb{L}_{p+q}.$$

However, by assumption $G(P_\bullet)$ is a chain complex of F-acyclic objects, so

$$L_qF(G(P_\bullet)) = \begin{cases} F(G(P_\bullet)), & q = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Taking pth homology thus gives us, with $q = 0$,

$$\begin{aligned} H_p(L_0F(G(P_\bullet))) &= H_p(F \circ G(P_\bullet)) \\ &= L_p(F \circ G)(A). \end{aligned}$$

Thus, the spectral sequence collapses at the second page, and we have

$$\mathbb{L}_pF(G(P_\bullet)) \cong L_p(F \circ G)(A)$$

The second spectral sequence guarantees us that

$$L_pF(H_q(G(P_\bullet))) \Longrightarrow \mathbb{L}_{p+q}(G(P_\bullet)).$$

However, by definition

$$H_q(G(P_\bullet)) = L_qG(A),$$

[1] An object $B \in \mathcal{B}$ is called *F-acyclic* if for all $q > 0$, $L_qF(B) = 0$. For example, projective objects are F-acyclic for all F.

giving us a spectral sequence

$$E^2_{p,q} = (L_pF \circ L_qG)(A) \Longrightarrow L_{p+q}(F \circ G)(A)$$

as required. □

Example 190 (Lyndon-Hochschild-Serre spectral sequence)**.** Let G be a group, and N a normal subgroup. This gives us the following commuting[2] diagram.

$$\begin{tikzcd} \mathbb{Z}G\text{-}\mathbf{Mod} \arrow[rr, "(-)_N"] \arrow[dr, "(-)_G"'] & & \mathbb{Z}G/N\text{-}\mathbf{Mod} \arrow[dl, "(-)_{G/N}"] \\ & \mathbf{Ab} & \end{tikzcd}$$

If we can show that $(-)_N$ takes projectives to $(-)_{G/N}$-acyclics, then we are entitled to apply Proposition 189. Note that $(-)_N$ takes free $\mathbb{Z}G$-modules to free $\mathbb{Z}G/N$ modules, simply by cutting away the basis elements on which N acts trivially. Since any projective is a direct summand of some free module and $(-)_N$ is additive, it thus takes projectives to projectives.

Proposition 189 now tells us that there exists a convergent spectral sequence

$$E^2_{p,q} = (L_p(-)_{G/N} \circ L_q(-)_N)(A) \Longrightarrow L_{p+q}(-)_G(A).$$

That is, a spectral sequence

$$E^2_{p,q} = H_p(G/N, H_q(N, A)) \Longrightarrow H_{p+q}(G, A).$$

This is known as the *Lyndon-Hochschild-Serre spectral sequence.*

[2]On the nose!

6. Triangulated categories

Definition 191 (triangulated category)**.** A triangulated category consists of the following data.

1. An additive category $\mathcal{T}$.
2. An equivalence
$$\mathcal{T} \to \mathcal{T}; \qquad X \mapsto X[1],$$
called the translation functor.
3. A collection of distinguished triangles
$$\left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1] \right\}.$$

These must satisfy the following four axioms.

(TR1)
a) Every morphism f can be extended to a distinguished triangle as above.
b) The collection of distinguisned triangles is closed under isomorphism.
c) Given $X \in \mathcal{T}$, the diagram
$$X \xrightarrow{\mathrm{id}} X \longrightarrow 0 \longrightarrow X[1]$$
is also distinguished.

(TR2) A diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
if and only if the 'rotated' diagram
$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$
is distinguished.

(TR3) Given a solid commmutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

with distinguished rows, there exists a dashed morphism making everything commute.

(TR4) Suppose we are given the following distinguished triangles.

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

Then there is a distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

making the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"'] \arrow[d, "u"'] & Z \arrow[d, "v"] & & \\
& Y/X \arrow[r, "\phi"] & Z/X \arrow[r, "d''"] \arrow[d, "\psi"'] & X[1] \arrow[d, "f[1]"] & \\
& & Z/Y \arrow[r, "d'"] \arrow[rr, bend right, "\theta"'] & Y[1] \arrow[r, "u[1]"] & Y/X[1]
\end{tikzcd}$$

Part II.

Algebraic topology

7. Homology

7.1. Basic definitions and examples

7.1.1. Ordinary homology

Singular homology is the study of topological spaces by associating to them chain complexes in the following way.

$$\mathbf{Top} \xrightarrow[\text{nerve}]{\text{singular}} \mathbf{Set}_\Delta \xrightarrow[\text{group}]{\text{free abelian}} \mathbf{Ab}_\Delta \xrightarrow[\text{correspondence}]{\text{Dold-Kan}} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Definition 192 (singular complex, singular homology)**.** Let X be a topological space. The singular chain complex $S_\bullet(X)$ is defined level-wise by

$$S_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where $\mathcal{F}$ denotes the free group functor and M is the Moore functor (Definition 142). That is, it has differentials

$$d_n \colon S_n \to S_{n-1}; \qquad (\alpha \colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(S_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f \colon X \to Y$ gives a chain map $S(f) \colon S(X) \to S(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 193. Denote by pt the one-point topological space. Then $\mathbf{Sing}(\mathrm{pt})_n = \{*\}$, and the chain complex $S_\bullet(\mathrm{pt})$ is at each level simply $\mathbb{Z}$.

The differential d_n is given by

$$d_n = \sum_{i=0}^{n} (-1)^i \partial^i; \qquad * \mapsto \sum_{i=0}^{n} (-1)^i * .$$

If n is odd, then $n+1$ is even, and there are as many summands with a positive sign as there are with negative sign, and they cancel each other out. If n is even, then there is one more term with positive sign than with negative sign. Thus $S(\mathrm{pt})_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & S_3 \arrow[r, "d_3"'] & S_2 \arrow[r, "d_2"'] & S_1 \arrow[r, "d_1"'] & S_0 \arrow[r, "d_0"'] & S_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

7.1.2. Reduced homology

Recall that in addition to the *ordinary*, or *topologist's simplex category* Δ, there is the so-called *extended*, or *algebraist's simplex category* Δ_+, which includes the object $[-1] = \emptyset$. If we use this category to index our simplicial sets, our chain complexes have a term $\tilde{S}_{-1} = \mathbb{Z}$. It is traditional to denote the differential $d_0 \colon \tilde{S}_0 \to \tilde{S}_{-1}$ by ε, and call it the *augmentation map.*

That is, the chain complex $\tilde{S}_\bullet$ agrees with the chan complex $S_\bullet$ in all degrees except -1, where $\tilde{S}_{-1} = \mathbb{Z}$ and $S_{-1} = 0$.

$$\cdots \longrightarrow \tilde{S}_2 \xrightarrow{d_2} \tilde{S}_1 \xrightarrow{d_1} \tilde{S}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Definition 194 (reduced singular chain complex)**.** The chain complex $\tilde{S}$ is called the reduced singular chain complex of X.

We see immediately that

$$S_i(X) = \tilde{S}_i(X), \qquad i \geq 0,$$

implying

$$H_i(X) = \tilde{H}_i(X), \qquad i \geq 1.$$

There is also another interpretation of reduced homology. Recall that we have used the name *augmentation map* before, when dealing with resolutions. There we reinterpreted the augmentation map ε as a chain map to a complex concentrated in degree zero. We can pull exactly the same trick here, viewing the augmentation map as a map of chain complexes

$$\begin{tikzcd} \cdots \arrow[r] & S_2 \arrow[r, "d_2"] \arrow[d] & S_1 \arrow[r, "d_1"] \arrow[d] & S_0 \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}.$$

Example 195. We can immediately read off that the reduced homology of the point is

$$\tilde{H}_n(\mathrm{pt}) = 0 \qquad \text{for all } n.$$

One of the benefits of reduced homology is that it can be used to prove facts about ordinary homology.

Example 196. Let X be a nonempty path-connected topological space. Then $H_0(X) \cong \mathbb{Z}$. Specifically,

$$\varepsilon\colon H_0(X) \to \mathbb{Z}$$

is an isomorphism.

To see this, let $x \in S_0(X)$ be a point. Since $\varepsilon(nx) = n$ for any $n \in \mathbb{Z}$, we know that ε is surjective. If we can show that ε is injective, then we are done.

At face value, ε does not look injective; after all, there are as many elements of $S_0(X)$

consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

Example 197. More generally, for any (not necessarily path connected) space X,

$$H_0(X) = \mathbb{Z}^{\pi_0(X)}.$$

Suppose that X can be written as a disjoint union

$$X = \coprod_i X_i.$$

Then, since a map

$$\sigma\colon \Delta^n \to X$$

cannot hit more that one of the X_i, we have

$$\mathbf{Top}\left(\Delta^n, \coprod_i X_i\right) \cong \coprod_i \mathbf{Top}(\Delta^n, X_i).$$

Thus, since the free abelian group functor $\mathcal{F}$ and the Moore functor M are both left adjoints, they preserve colimits and we have

$$S_n\left(\coprod_i X_i\right) \cong \bigoplus_i S_n(X_i).$$

We will come back to reduced homology in Section 7.7, when we have the tools to understand it properly.

7.2. The Hurewicz homomorphism

It turns out that, for path connected spaces X, knowledge of $\pi_1(X)$ allows us to compute $H_1(X)$.

Let X be a path-connected topological space, and let $x \in X$. Let γ be a loop in X which starts and ends at x, i.e.

$$\gamma\colon \Delta^1 \to X; \qquad \gamma(0) = x = \gamma(1).$$

We can view γ either as a singular 1-simplex in X, or as a representative of a homotopy class in $\pi_1(X, x)$. It turns out that these points of view are compatible.

Lemma 198. The assignment $\gamma \mapsto [\gamma]_{H_1}$ respects homotopy, and thus descends to a map of sets out of $\pi_1(X, x)$.

$$\begin{tikzcd} \left\{\begin{array}{c}\text{Loops starting and}\\ \text{ending at } x\end{array}\right\} \arrow[r, "\text{homology class}"] \arrow[d, two heads, "\text{homotopy class}"'] & H_1(X) \\ \pi_1(X, x) \arrow[ur, dashed, "\exists! \tilde{h}_X"'] & \end{tikzcd}$$

Proof. First, let κ_x be the constant 1-simplex at x, and let σ_x be the constant 2-simplex. Then

$$\partial\sigma_x = \kappa_x - \kappa_x + \kappa_x = \kappa_x,$$

so κ_x is a boundary.

Let γ_1 and γ_2 be loops based at x.

Next, suppose that $\gamma_1 \overset{H}{\sim} \gamma_2$, i.e. let

$$H\colon \Delta^1 \times [0, 1] \to X$$

be a homotopy such that

$$H(t,0) = \gamma_1 \quad H(t,1) = \gamma_2, \quad H(0,s) = H(1,s) = x.$$

Since H is constant along $H(0,s)$, it descends to a map out of

$$\Delta^1 \times [0,1]/\{(0,s) \mid s \in [0,1]\} \cong \Delta^2.$$

The boundary of this 2-simplex is

$$\gamma_1 - \gamma_2 + \kappa_x.$$

Thus, γ_1 is homologous to γ_2. □

The map $\pi_1(X,x) \to H_1(x)$ we have constructed turns out, somewhat astonishingly, to be a group homomorphism.

Lemma 199. For a path-connected topological space X and point $x \in X$, the map

$$\tilde{h}_X \colon \pi_1(X,x) \to H_1(X)$$

is a group homomorphism.

Proof. We need to show that it sends the identity to the identity and respects the group law. The identity element of $\pi_1(X,x)$ is represented by κ_x, which we have already seen is homologous to zero. To see that it respects the group law, let γ_1 and γ_2 be loops. Then there is a 2-simplex with boundary

$$(\gamma_2, \gamma_1 * \gamma_2, \gamma_1),$$

showing that $\gamma_1 + \gamma_2$ is homologous to $\gamma_1 * \gamma_2$. □

Because $H_1(X)$ is abelian, $\tilde{h}_X$ descends to a map out of $\pi_1(X,x)_{\mathrm{ab}}$.

Definition 200 (Hurewicz homomorphism)**.** Let X be a path-connected topological space, and let $x \in X$. The Hurewicz homomorphism is the map

$$h_X \colon \pi_1(X,x)_{\mathrm{ab}} \to H_1(X).$$

Theorem 201 (Hurewicz)**.** For any path-connected topological space X, the Hurewicz homomorphism is an isomorphism

$$h_X \colon \pi_1(X,x_0)_{\mathrm{ab}} \cong H_1(X),$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism.

$$\begin{tikzcd} \mathbf{Top}_* \arrow[r, bend left, "(\pi_1)_{\mathrm{ab}}"] \arrow[r, bend right, "H_1"'] \arrow[r, phantom, "\Downarrow h"] & \mathbf{Ab} \end{tikzcd}$$

Proof. We construct an inverse explicitly.

For each point $x \in X$, pick a path

$$\gamma_x \colon \Delta^1 \to X; \qquad \gamma_x(0) = x_0, \quad \gamma_x(1) = x$$

connecting x_0 to x. For $x = x_0$, choose γ_{x_0} to be the constant path.

For each generator $\alpha \colon \Delta^1 \to X$ in $S_1(X)$, the concatenation

$$\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}$$

is a path starting and ending at x. We may thus define a map

$$\phi \colon S_1(X) \to \pi_1(X, x)_{\mathrm{ab}}$$

on generators by

$$\alpha \mapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}],$$

and addition via

$$\alpha + \beta \mapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}][\gamma_{\beta(0)} * \beta * \overline{\gamma_{\beta(1)}}].$$

In order to check that defining this homomorphism on generators descends to a map on homology, we have to check that it sends boundaries $\partial\sigma$ to zero. □

Example 202. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\mathrm{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \qquad n > 1.$$

Example 203. Denote by Σ_g the two-dimensional surface of genus g. We know that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_{2g}, b_{2g} | \prod_{i=1}^{g} [a_i, b_i] \rangle.$$

The abelianization of this is simply $\mathbb{Z}^{2g}$, so

$$H_1(\Sigma_g) = \mathbb{Z}^{2g}.$$

Example 204. Since
$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$
we have that
$$H_1(X \times Y) \cong H_1(X) \times H_1(Y).$$
For nice[1] X and Y, e.g. $X = Y = \mathbb{S}^1$, we have that
$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$
we have that
$$H_1(X \vee Y) \cong H_1(X) \times H_1(Y).$$

7.3. The method of acyclic models

We need to break the flow here for a theorem which will show up several times, called the *method of acyclic models.*

Definition 205 (category with models)**.** A category with models is a pair $(\mathcal{C}, \mathcal{M})$, where $\mathcal{C}$ is a category and $\mathcal{M} \subset \mathrm{Obj}(\mathcal{C})$ is a set of objects of $\mathcal{C}$.

Definition 206 (free, acyclic functor)**.** Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let $F\colon \mathcal{C} \to \mathbf{Ch}_{\geq 0}(R\text{-}\mathbf{Mod})$.

1. We say that F is acyclic on $\mathcal{M}$ if for each $M \in \mathcal{M}$, $F(M)$ is acyclic in positive degree, i.e. if $H_n(F(M)) = 0$ for $n > 0$.
2. Let J be a set, and $\mathcal{M}_J \subset \mathcal{M}$ a J-indexed set of objects in $\mathcal{M}$. Note that we allow the possibility that each $M \in \mathcal{M}$ can appear more than once in $\mathcal{M}_J$, or not at all.

   Let $F_*\colon \mathcal{C} \to R\text{-}\mathbf{Mod}$ be a functor. An $\mathcal{M}_J$-basis for F_* is, for each $j \in J$ an element $m_j \in F_*(M_j)$ (forming an indexed collection $\{m_j \in F_*(M_j)\}_{j \in J}$) such that for any $X \in \mathcal{C}$ the indexed collection
   $$\{F_*(f)(m_j)\}_{j \in J, f \in \mathrm{Hom}(M_j, X)}$$
   is a basis for $F_*(X)$ as a free R-module; that is, that we can write
   $$F_*(X) = \mathbb{Z}\{F_*(f)(m_j)\}_{j \in J, f \in \mathrm{Hom}(M_j, X)}.$$
   We say that F is free on $\mathcal{M}$ if for each $q \geq 0$ there exists some set J_q and indexed set $\mathcal{M}_{J_q}$ such that each F_q has an $\mathcal{M}_{J_q}$-basis.

[1] The more technical condition comes from Van Kampen's theorem.

Example 207. Consider the category **Top** with models $\{\Delta^n\}_{n=0,1,\dots}$. The functor $S\colon \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ is both free and acyclic. Acyclicity is clear since Δ^n is contractible for all n.

To see freeness, note that the singleton

$$\{\mathrm{id}_{\Delta^n}\colon \Delta^n \to \Delta^n\}$$

forms a basis for F_n because the abelian group $S_n(X)$ is free with generating set

$$\{S(\alpha)(\mathrm{id}_{\Delta^n}) = \alpha\}_{\alpha\in\mathrm{Hom}(\Delta^n,X)}.$$

The freeness condition can be interpreted as telling us that in order to know how the functor F behaves on any object, it is enough to know how it behaves on the M_α. In fact, this is sufficiently precise that it can be made into a theorem.

Proposition 208. Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let $F, G\colon \mathcal{C} \to \mathbf{Ab}$ be functors, with F $\mathcal{M}$-free. Then specifying a natural transformation $\eta\colon F \Rightarrow G$ is equivalent to specifying only the values

$$\eta_{M_i}(m_i),$$

Theorem 209 (acyclic model theorem)**.** Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let F, G be functors $\mathcal{C} \to \mathbf{Ch}_{\geq 0}(R\text{-}\mathbf{Mod})$ such that F is free on $\mathcal{M}$ and G is acyclic on $\mathcal{M}$.

1. Any natural transformation $\bar{\tau}_0\colon H_0(F) \to H_0(G)$ can be extended to a natural chain map $\tau\colon F \to G$.
2. This extendsion is unique up to homotopy in the sense that two natural chain maps τ, $\tau'\colon F \to G$ inducing the same natural transformation $H_0(G) \to H_0(G')$ are naturally chain homotopic.

Proof.

1. Let $\tau\colon F \Rightarrow G$ be a natural transformation. First, we collect some results about the natural transformation $\tau_q\colon F_q \Rightarrow G_q$. Note that, since

   $$F_q(X) \cong R\{F_q(f)(m_j)\}_{j\in J_q, f\in\mathrm{Hom}(M_j,X)}$$

   we can specify $\tau_q(X)$ (the component of τ_q at X) completely by specifying how it acts on those elements of $F(q)(X)$ of the form $F_q(f)(m_j)$, and that we are free in choosing this action.

Consider the following naturality square for some map $f\colon M_j \to X$.

$$
\begin{tikzcd}
F_q(M_j) \arrow[rrr, "\tau_q(m_j)"] \arrow[ddd, "F_q(f)"'] & & & G_q(M_j) \arrow[ddd, "G_q(f)"] \\
& m_j \arrow[r, mapsto] \arrow[d, mapsto] & \tau_q(M_j)(m_j) \arrow[d, mapsto] & \\
& F_q(f)(m_j) \arrow[r, mapsto] & \star & \\
F_q(X) \arrow[rrr, "\tau_q(X)"'] & & & G_q(X)
\end{tikzcd}
$$

Naturality requires that

$$\tau_q(X)(F(f)(m_j)) = G_q(f)(\tau_q(M_j)(m_j)).$$

Thus, in order to define $\tau_q(X)$ for all X, we need only specify how $\tau_q(M_j)$ behaves on $m_j \in F_q(M_j)$. Defining it in this way and extending it by naturality will ensure that the maps we construct form the components of a natural transformation.

For $q = 0$, this is easy, since we have a natural transformation $\bar{\tau}_0$. Since everything in $H_0(F)$ is a cycle, we can define $\tau_0(M_j)(m_j)$ to be any representative of the equivalence class

$$\bar{\tau}_0(M_j)[m_j] \in H_0(G(M_j)).$$

Now suppose we have defined natural transformations τ_{q-1}, for $q > 0$. For $j \in J_q$, we define $\tau_q(M_j(m_j))$ by

$$\partial\tau_q(M_j)(m_j) = \tau_{q-1}(M_j(\partial m_j)).$$

This is well-defined precisely because

$$\partial\tau_{q-1}(M_j)(\partial m_j) = \tau_{q-2}(\partial^2 m_j) = 0$$

since τ_{q-1} is by assumption a chain map and G is by assumption acyclic on $\mathcal{M}$.

2. One defines a chain homotopy $D\colon \tau \to \tau'$ inductively using the same trick: one defines

$$\partial D_q(M_j)(m_j) = \tau_q(M_j)(m_j) - \tau'_q(M_j)(M_j) - D_{q-1}(M_j)\partial m_j.$$

$\square$

7.4. Homotopy equivalence

In this section, we will see that homotopic maps induce the same map on homology. As alluded to in Section 2.6, we will do this by showing that a homotopy H between maps $f, g\colon X \to Y$ induces a chain homotopy h between $S_n(f)$ and $S_n(g)$.

7.4.1. Homotopy invariance: with acyclic models

Proposition 210. Let $f, g\colon X \to Y$ be continuous maps between topological spaces, and let $H\colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$.

Proof. Suppose we are given a homotopy $f \overset{H}{\sim} g$.

$$\begin{tikzcd} X \arrow[d, "\mathrm{id}\times\{0\}"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, "H"] & Y \\ X \arrow[u, "\mathrm{id}\times\{1\}"] \arrow[ur, "g"'] & \end{tikzcd}$$

Consider the functors

$$F = S_\bullet(-),\ G = S_\bullet(I \times -)\colon \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab}).$$

Take **Top** with models $\mathcal{M} = \{\Delta^q \mid q = 0, 1, \ldots\}$, $J_n = \{*\}$, $\mathcal{M}_{J_n} = \{\Delta^n\}$ and $m_* = \mathrm{id}_{\Delta^n}$. For all n, we have that F is free because

$$S_n(X) \cong \mathbb{Z}\{S_n(\alpha)(\mathrm{id}_{\Delta^n}) \mid \alpha \in \mathrm{Hom}(\Delta^n, X)\},$$

and that G is acyclic because $\Delta^n \times I$ is contractible for all n.

Denote by ι_X^0 the map $X \to X \times I$, which sends $x \mapsto (x, 0)$. Define ι_X^1 analogously.

Consider the natural transformations

$$i^0, i^1 : F \Rightarrow G$$

with components

$$i_X^0 = S_\bullet(\iota^0), \qquad i_X^1 = S_\bullet(\iota^1).$$

These clearly agree on H_0. Thus, by Theorem 209, i^0 and i^1 are chain homotopic via some chain homotopy D.

If D is homotopy between i^0 and i^1, then $S(H) \circ D$ is a homotopy between $S(H) \circ i^0$ and

$S(H) \circ i^1$. But $S(H) \circ i^0 = S(f)$, and $S(H) \circ i^1 = S(g)$. □

Corollary 211. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 212. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

7.4.2. Homotopy-invariance: without acyclic models

Such a homotopy H is a map

$$H\colon X \times [0,1] \to Y.$$

It seems reasonable that we would get the chain homotopy in question by relating the singular homology of X and the singular homology of $X \times [0,1]$.

Here is the game plan.

- We notice that there are n obvious ways of of mapping

  $$p_i\colon \Delta^{n+1} \to \Delta^n \times [0,1],$$

  and $\Delta^n \times [0,1]$ is the union of the images, which overlap only along their boundaries.
- Given an n-simplex $\alpha\colon \Delta^n \to X$, we can produce an $(n+1)$-simplex in $X \times [0,1]$ by pulling back:

  $$P_i\colon \Delta^{n+1} \to \Delta^n \times [0,1] \to X \times [0,1].$$
- Given a homotopy

  $$H\colon X \times [0,1] \to Y$$

  between f and g, we can pull back by the P_i, giving us an $(n+1)$-simplex

  $$P_i(\alpha)\colon \Delta^{n+1} \to \Delta^n \times [0,1] \to X \times [0,1] \to Y.$$

  The association

  $$\alpha \mapsto P_i\alpha$$

  is a homomorphism.
- If we take a sum of the $P_i\alpha$, we get a cellular decomposition of the image of α.

However, by taking an alternating sum

$$P = \sum_{i=0}^{n} P_i$$

we can get (upon passing to boundaries) the interior walls to cancel each other out. Thus, the composition

$$d \circ P$$

gives a cellular decomposition of the boundary of the image of the cylinder. The composition $P \circ d$ gives you a cellular decomposition of the sides of the cylinder with the opposite sign. Thus, the sum $d \circ P + P \circ d$ gives you the (difference of) the top and the bottom of the cylinder.

- This tells us that *homotopic maps between topological spaces induce the same map between homotopy groups.*

Definition 213 (prism maps)**.** The ith prism map is the map

$$p_i \colon \Delta^{n+1} \to \Delta^n \times I; \qquad (t_0, \ldots, t_{n+1}) \mapsto \left((t_0, \ldots, t_i + t_{i+1}, \ldots, t_n), \sum_{j=i+1}^{n} t_j \right)$$

The ith prism map sends the vertices $\{0, \ldots, n+1\} \in \Delta^{n+1}$ to the vertices of the prism as follows.

$$\begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & i & i+1 & \cdots & n+1 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ (0,0) & (1,0) & (2,0) & \cdots & (i,0) & (i,1) & \cdots & (n,1) \end{array}$$

We will denote the vertex $(j, 0)$ by a_j and $(j, 1)$ by b_j. We can thus summarize the situation by saying that

$$p_i \colon (0, \ldots, n+1) \to (a_1, \ldots, a_i, b_i, \ldots, b_n),$$

where the ith element of the last tuple corresponds to the location of the ith vertex in the prism $\Delta^n \times I$.

Definition 214 (prism operator)**.** We define maps

$$P_i \colon S_n(X) \to S_{n+1}(X \times I)$$

which send $\alpha \colon \Delta^n \to X$ to the composition

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times I \xrightarrow{\alpha \times \mathrm{id}_I} X \times I \ .$$

The prism operator is the alternating sum

$$\sum_{i=0}^{n} P_i \colon S_n(X) \to S_{n+1}(X \times I).$$

Given an n-simplex σ, the prism operator P_i creates an $n+1$-simplex $(\alpha \times \mathrm{id}) \circ p_i$. To keep track of the indices of the $n+1$-simplex we are using, we notate this by

$$(\sigma \times \mathrm{id})|_{[a_0,\ldots,a_i,b_i,\ldots,b_n]}\,.$$

Proposition 215. Denote by j_i, for $i = 0, 1$, the map

$$j_i \colon X \to X \times I; \qquad x \mapsto (x, i).$$

The prism operator is a chain homotopy between $S_\bullet(j_0)$ and $S_\bullet(j_1)$.

Proof. We simply do the horrible computations. □

7.5. Relative homology

7.5.1. Relative homology

Denote by **Pair** the category whose objects are pairs (X, A), where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \to (Y, B)$ are maps $f \colon X \to Y$ such that $F(A) \subset B$. That is, the category **Pair** is the full subcategory on the category **Fun**$(I, \mathbf{Top})$ on monomorphisms; the objects (X, A) are diagrams

$$A \hookrightarrow X$$

and morphisms $(A, X) \to (B, Y)$ are commuting squares

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & X \arrow[d] \\ B \arrow[r, hook] & Y \end{tikzcd}$$

Definition 216 (relative homology)**.** Let (X, A) be a pair of spaces. The inclusion $i \colon A \hookrightarrow X$ induces a map of chain complexes $A \hookrightarrow X$. The relative chain complex of (X, A) is the cokernel of $C_\bullet(i)$.

Since colimits are computed level-wise, the relative chain complex is given by the level-wise quotient

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Lemma 217. For each n, relative homology provides a functor $H_n \colon \mathbf{Pair} \to \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{tikzcd}
S(X)_\bullet \arrow[r, "f"] \arrow[d, two heads] & S(Y)_\bullet \arrow[d, two heads] \\
S(X)_\bullet/S(A)_\bullet \arrow[r, dashed] & S(Y)_\bullet/S(B)_\bullet
\end{tikzcd}$$

The dashed arrow is uniquely well-defined because of the assumption that $f(A) \subset B$, meaning that the relative homology construction is well-defined on the level of the relative chain complex. Functoriality of the relative homology now follows from the functoriality of H_n. □

7.5.2. The long exact sequence on a pair of spaces

Proposition 218. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{j+1}(X, A) \arrow[dll, "\delta"] \\
H_j(A) \arrow[r] & H_j(X) \arrow[r] & H_j(X, A) \arrow[dll, "\delta"] \\
H_{j-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r, hook] & C_\bullet(X) \arrow[r, two heads] & C_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

□

Proposition 219. The connecting homomorphism

$$\delta \colon H_n(X, A) \to H_{n-1}(A)$$

maps a class represented by a relative cycle $[\sigma]$ to its boundary class $[\partial\sigma]$.

Proof. We simply trace σ through the zig-zag defining the connecting homomorphism.

$$\begin{tikzcd}
0 \arrow[r] & S_n(A) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & S_n(X,A) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_{n-1}(A) \arrow[r] & S_{n-1}(X) \arrow[r] & S_{n-1}(X,A) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
 & \sigma \arrow[r, mapsto] \arrow[d, mapsto] & \sigma \\
\partial\sigma \arrow[r, mapsto] & \partial\sigma &
\end{tikzcd}$$

□

Example 220. Let $X = \mathbb{D}^n$, the n-disk, and $A = \mathbb{S}^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & H_j(\mathbb{D}^n) \arrow[r] & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible. Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

Proposition 221. Suppose $i\colon A \hookrightarrow X$ is a weak retract, i.e. that there is an $r\colon X \to A$ such that $r \circ i = \mathrm{id}_A$.

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_A"'] & X \arrow[r, two heads, "r"] & A
\end{tikzcd}$$

Then

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

Proof. Applying the functor H_n to the diagram above, we find that

$$H_n(r) \circ H_n(i) = \mathrm{id}_{H_n(A)},$$

implying that $H_n(i)$ is injective. Consider the long exact sequence on the pair (X, A).

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X,A) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, hook, "H_n(i)"] & H_n(X) \arrow[r] & H_n(X,A) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, hook, "H_{n-1}(i)"] & \cdots &
\end{tikzcd}$$

The injectivity of $H_n(i)$ implies that the connecting homomorphisms are zero, meaning that the following sequence is short exact for all n.

$$0 \longrightarrow H_n(A) \xhookrightarrow{H_n(i)} H_n(X) \twoheadrightarrow H_n(X,A) \longrightarrow 0$$

As $H_n(r)$ is a left inverse to $H_n(i)$, the sequence above splits from the left, implying by the splitting lemma (Lemma 34) the result. □

Proposition 222. Let $i\colon A \to X$ be a deformation retract,[2] i.e. that there is a homotopy

$$R\colon X \times [0,1] \to X$$

such that the following conditions are satisfied.

1. $R(x, 0) = x$ for all $x \in X$
2. $R(x, 1) \in A$ for all $x \in X$
3. $R(a, 1) = a$ for all $a \in A$

Then $H_n(i)\colon H_n(A) \to H_n(X)$ is an isomorphism.

Proof. Let

$$r = R(-, 1)\colon X \to A.$$

Then 3. implies that $r \circ i = \mathrm{id}_A$.

Furthermore, $R(x, -)$ provides a homotopy between $i \circ r$ and id_X. Thus, X and A are homotopy equivalent, and $H_n(i)$ is an isomorphism. □

Corollary 223. If $i\colon X \to A$ is a deformation retract, then $H_n(X, A) = 0$ for all n.

These results may not seem like much, but we are now in a position to show a truly fundamental result: invariance of dimension.

[2] I believe we only need i to be a homotopy equivalence; a deformation retract is a homotopy equivalence in which one of the two homotopies is an identity.

Example 224. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be open subsets, and let $f\colon A \to B$ be a homeomorphism. Then $m = n$.

To see this, let $a \in A$. Then

$$f|_{A \smallsetminus \{a\}} = \bar{f}\colon A \smallsetminus \{a\} \to B \smallsetminus \{f(a)\}$$

is a homeomorphism, so there is an isomorphism of pairs

$$(A, A \smallsetminus \{a\}) \cong (B, B \smallsetminus \{f(a)\}).$$

Thus

$$H_q(A, A \smallsetminus \{a\}) \cong H_q(B, B \smallsetminus \{f(a)\}).$$

By excision, we have

$$H_q(\mathbb{R}^m, \mathbb{R}^m \smallsetminus \{a\}) \cong H_q(\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{f(a)\}).$$

But we have a homotopy equivalence of pairs

$$(\mathbb{R}^m, \mathbb{R}^m \smallsetminus \{a\}) \cong (\mathbb{D}^m, \mathbb{S}^{m-1}), \qquad (\mathbb{R}^n, \mathbb{R}^n \smallsetminus \{f(a)\}) \cong (\mathbb{D}^m, \mathbb{S}^{n-1}),$$

so we have

$$H_q(A, A \smallsetminus \{a\}) \cong \begin{cases} \mathbb{Z}, & q = 0,\ m \\ 0, & \text{otherwise} \end{cases}$$

and

$$H_q(B, B \smallsetminus \{f(a)\}) \cong \begin{cases} \mathbb{Z}, & q = 0,\ n \\ 0, & \text{otherwise.} \end{cases}$$

For these homologies to agree for all q we must have $m = n$.

7.5.3. The braided monstrosity on a triple of spaces

Definition 225 (triple of spaces)**.** Let X be a topological space, and let $B \subset A \subset X$ be subspaces. We call (X, A, B) a triple.

A triple of spaces is in particular three pairs of spaces: (X, A), (X, B), and (A, B). All of these have associated long exact sequences. There is also a fourth long exact sequence.

Proposition 226. There is a long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(A,B) \arrow[dll, out=-10, in=170, "\delta" description] \\
H_j(A,B) \arrow[r] & H_j(X,B) \arrow[r] & H_j(X,A) \arrow[dll, out=-10, in=170, "\delta" description] \\
H_{j-1}(A,B) \arrow[r] & \cdots &
\end{tikzcd}
$$

Proof. This comes from the short exact sequence

$$
\begin{tikzcd}
0 \arrow[r] & S_n(A)/S_n(B) \arrow[r, hook] & S_n(X)/S_n(B) \arrow[r, two heads] & S_n(X)/S_n(A) \arrow[r] & 0 .
\end{tikzcd}
$$

□

We can put all four of these together in the following handsome commutative diagram.

$$
\begin{tikzcd}
\cdots \arrow[dr] & & & & & & \cdots \\
 & H_{n+1}(X,A) \arrow[rr, bend left, "\delta"] \arrow[dr, "\delta" description] & & H_n(A,B) \arrow[rr, bend left, "\delta"] \arrow[dr] & & H_{n-1}(B) \arrow[ur] \arrow[dr] & \\
\cdots \arrow[ur] \arrow[dr, "\delta" description] & & H_n(A) \arrow[ur] \arrow[dr] & & H_n(X,B) \arrow[ur, "\delta" description] \arrow[dr] & & \cdots \\
 & H_n(B) \arrow[ur] \arrow[rr, bend right] & & H_n(X) \arrow[ur] \arrow[rr, bend right] & & H_n(X,A) \arrow[ur, "\delta" description] \arrow[dr, "\delta" description] & \\
\cdots \arrow[ur, "\delta" description] & & & & & & \cdots
\end{tikzcd}
$$

7.6. Barycentric subdivision

Definition 227 (cone maps)**.** Let $v \in \Delta^p$ be a point, and let $\alpha \colon \Delta^n \to \Delta^p$ be a singular n-simplex. The cone of α with respect to v is

$$K_v(\alpha) \colon \Delta^{n+1} \to \Delta^p$$

defined by sending

$$(t_0, \ldots, t_n) \mapsto \begin{cases} (1 - t_{n+1})\, \alpha\left(\frac{t_0}{1-t_{n+1}}, \ldots, \frac{t_n}{1-t_{n+1}}\right) + t_{n+1} v, & t_{n+1} < 1 \\ v, & t_{n+1} = 1. \end{cases}$$

This is well-defined and continuous.

We extend K_v linearly to $S_\bullet(\Delta^p)$.

Lemma 228. The map $K_v \colon S_n(\Delta^p) \to S_{n+1}(\Delta^p)$ satisfies the following identities.

1. For a zero-simplex $c \in S_0(\Delta^p)$, we have

$$\partial K_v(c) = \epsilon(c)\kappa_v - c,$$

   where $\kappa_v(e_0) = v$, and $\epsilon \colon S_0(\Delta^p) \to \mathbb{Z}$ is the augmentation map.

2. For $n > 0$, we have the equality

$$\partial \circ K_v - K_v \circ \partial = (-1)^{n+1}\mathrm{id}$$

Proof.

1. It suffices to show the claim on generators, i.e. singular 0-simplices $\Delta^0 \to \Delta^p$. Let $\alpha \colon \Delta^0 \to \Delta^p$. Then $\epsilon(\alpha) = 1$, and

$$\begin{aligned}
\partial K_v(\alpha)(e_0) &= \partial_0 K_v(\alpha)(e_0) - \partial_1 K_v(\alpha)(e_0) \\
&= \kappa_v(\alpha)(d_0 e_0) - \kappa_v(\alpha)(d_1 e_0) \\
&= \kappa_v(\alpha)(e_1) - \kappa_v(\alpha)(e_0) \\
&= v - \alpha(e_0)
\end{aligned}$$

2. An easy calculation shows that

$$\partial_i K_v(\alpha) = k_v(\partial_i \alpha), \qquad i < n+1,$$

   and

$$\partial_{n+1} K_v(\alpha) = \alpha.$$

   Thus,

$$\partial K_v - K_v \partial = (-1)^{n+1}\mathrm{id}$$

   as required.

□

We first define *Barycentric subdivision* on models $\{\Delta^n\}$: it is a chain map

$$B \colon S_\bullet(\Delta^p) \to S_\bullet(\Delta^p)$$

defined on an 0-simplex $\alpha \colon \Delta^0 \to \Delta^p$ by

$$B(\alpha) = \alpha,$$

and for $\sigma\colon \Delta^n \longrightarrow \Delta^p$ inductively by

$$B(\sigma) = (-1)^n (K_v)(B(\partial\sigma)).$$

This is a bona-fide chain map because...

Since the functor $S_\bullet$ is $\{\Delta^n\}$-free, we can extend B to a natural chain map $S_\bullet \longrightarrow S_\bullet$ via

$$B_X(\alpha) = S_n(\alpha)(B_{\Delta^n}(\mathrm{id}_{\Delta^n})), \qquad \alpha\colon \Delta^n \longrightarrow X.$$

Theorem 229. The chain map $B\colon S_\bullet(X)$ is chain homotopic to the identity.

Proof. For a zero-simplex $\alpha\colon \Delta^0 \longrightarrow X$

$$\begin{aligned} B_X(\alpha) &= S_0(\alpha)(B_{\Delta^0}(\mathrm{id}_{\Delta^0})) \\ &= S_0(\alpha)(\mathrm{id}_{\Delta^0}) \\ &= \alpha, \end{aligned}$$

so B and $\mathrm{id}_{S_\bullet}$ agree on $H_0(S_\bullet)$. The claim follows from Theorem 209. □

Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X. Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha\colon \Delta^n \longrightarrow X$$

whose images are completely contained in some open set in the open cover $\mathfrak{U}$. That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$ induces a chain structure on $S_\bullet^{\mathfrak{U}}(X)$.

$$\begin{tikzcd} \cdots \arrow[r] & S_2^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_1^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_0^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & 0 \\ \cdots \arrow[r] & S_2(X) \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & 0 \end{tikzcd}$$

Fact 230. For any open cover $\mathfrak{U}$ of a topological space X, for every simplex $\alpha\colon \Delta^n \longrightarrow X$, there exists some $n \geq 0$ such that $B^n\alpha \in S^{\mathfrak{U}}(X)$.

Sketch of proof. This is really just some messing with Lebesgue numbers. One first shows that for any *affine simplex*[3] $\alpha\colon \Delta^n \to \Delta^p$,

$$\max_{\sigma} \operatorname{diam}(\sigma) \leq \left(\frac{n}{n+1}\right)^k \operatorname{diam}(\alpha),$$

where σ runs over all simplices in $B^k\alpha$. Thus, for any n,

$$\lim_{k\to\infty} \max \operatorname{diam}(B^k\alpha) = 0.$$

Lebesgue's number lemma tells us that for any compact metric space X and any open cover $\mathfrak{U} = \{U_i\}$ of X, there is some $\delta > 0$ such that any subset $S \subset X$ with $\operatorname{diam}(S) < \delta$ is completely contained in some U_i.

Now let X be any topological space, and let $\alpha\colon \Delta^n \to X$. Let $\mathfrak{U}$ be an open cover of X. Then the preimages of the open cover under α form an open cover of Δ^n. Since Δ^n is a compact metric space, Lebesgue's number lemma tells us that there exists $k \geq 0$ such that every simplex in $B^k\mathrm{id}_{\Delta^m}$ is completely contained in the preimage of at least one of the $U_i \in \mathfrak{U}$. Mapping everything forward with $S_\bullet(\alpha)$ gives us the result we want. □

We have proved the following.

Corollary 231. Every chain in $S_n(X)$ is homologous to a chain in $S^{\mathfrak{U}}(X)$

Proposition 232. There is a natural homotopy equivalence $S_\bullet^{\mathfrak{U}} \cong S_\bullet$.

Corollary 233.

$$H_n(X) \cong H_n^{\mathfrak{U}}(X).$$

This fact allows us almost immediately to read of two important theorems.

7.6.1. Excision

Theorem 234 (excision)**.** Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{tikzcd} A \smallsetminus W \arrow[r, hook, "i"] \arrow[d, hook] & A \arrow[d, hook] \\ X \smallsetminus W \arrow[r, hook, "i"] & X \end{tikzcd}$$

induce an isomorphism

$$H_n(i)\colon H_n(X \smallsetminus W, A \smallsetminus W) \cong H_n(X, A).$$

[3] I.e. a singular simplex which comes from an affine map $\mathbb{R}^{n+1} \to \mathbb{R}^{p+1}$

Proof. Consider the open cover $\mathfrak{U} = (\mathring{A}, X \setminus \bar{W})$ of X.

First, we show that $H_n(i)$ is surjective. Let $c \in S_n(X, A)$ be a relative cycle. By Corollary 231, we can replace (not uniquely, but who cares) c by a homologous sum $c' + c''$ with $c' \in S_n(\mathring{A})$ and $c'' \in S_n(X \setminus \bar{W})$.

NOTE: THIS IS WRONG. NEED TO FIX.

But $c' \sim 0$ in $H_n(X, A)$, so we have found $c \sim c''$, with $c'' \in S_n(X \setminus \bar{W})$.

Now we compute a boundary:

$$\partial c \sim \partial c''.$$

By definition,

$$\partial c'' \in S_{n-1}(X \setminus \bar{W}) \subset S_{n-1}(X \setminus W).$$

However, $c'' \in S_{n-1}(A)$, so

$$c'' \in S_{n-1}(X \setminus W) \cap S_{n-1}(A) = S_{n-1}(A \setminus W).$$

Therefore, c is homologous to a relative cycle in $S_n(X \setminus W, A \setminus W)$.

Now we show that $H_n(i)$ is injective. Suppose that $c \in S_n(X\ W)$ such that $\partial c \in S_n(A \setminus W)$. Further suppose that $H_n(i)[c] = 0$, i.e. that we can write

$$i(c) = \partial b + a'$$

with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$. Then $i(c)$ is homologous to □

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A.

7.6.2. The Mayer-Vietoris sequence

Theorem 235 (Mayer-Vietoris)**.** Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X, i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, "\delta"] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n(X) \arrow[dll, "\delta"] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. We can draw our inclusions as the following pushout.

$$\begin{tikzcd} & X_1 \arrow[dr, hook, "\kappa_1"] & \\ X_1 \cap X_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\ & X_2 \arrow[ur, hook, "\kappa_1"'] & \end{tikzcd}$$

We have, almost by definition, the following short exact sequence.

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xhookrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{\kappa_1 - \kappa_2} S_\bullet^{\mathfrak{U}}(X) \longrightarrow 0$$

This gives the following long exact sequence on homology.

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}^{\mathfrak{U}}(X) \arrow[dll, "\delta"] \\ H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n^{\mathfrak{U}}(X) \arrow[dll, "\delta"] \\ H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots & \end{tikzcd}$$

We have seen that $H_n^{\mathfrak{U}}(X) \cong H_n(X)$; the result follows. □

Example 236 (Homology groups of spheres)**.** We can decompose $\mathbb{S}^n$ as

$$\mathbb{S}^n = (\mathbb{S}^n \setminus N) \cup (\mathbb{S}^n \setminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$\begin{tikzcd} & \mathbb{S}^n \setminus N \arrow[dr, hook] & \\ (\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\ & \mathbb{S}^n \setminus S \arrow[ur, hook] & \end{tikzcd}$$

The Mayer-Vietoris sequence is as follows.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta" description] \\
H_j((S^n \setminus N) \cap (S^n \setminus S)) \arrow[r] & H_j(S^n \setminus N) \oplus H_j(S^n \setminus S) \arrow[r] & H_j^{\mathfrak{U}}(X) \arrow[dll, "\delta" description] \\
H_{j-1}((\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S)) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that

$$\mathbb{S}^n \setminus N \cong \mathbb{S}^n \setminus S \cong \mathbb{D}^n \simeq \mathrm{pt}$$

and that

$$(\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta" description] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & 0 \arrow[r] & H_j(\mathbb{S}^n) \arrow[dll, "\delta" description] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X, $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\mathrm{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$$
\begin{array}{c|cccc}
j=3 & \mathbb{Z} & 0 & & \\
j=2 & \mathbb{Z} & 0 & & \\
j=1 & \mathbb{Z} & \mathbb{Z} & & \\
j=0 & \mathbb{Z}\oplus\mathbb{Z} & 0 & 0 & 0 \\
\hline
H_i(\mathbb{S}^j) & i=0 & i=1 & i=2 & i=3
\end{array}
$$

The relation

$$
H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \qquad i > 1
$$

allows us to fill in the above table as follows.

$$
\begin{array}{c|cccc}
j=3 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
j=2 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
j=1 & \mathbb{Z} & \mathbb{Z} & 0 & 0 \\
j=0 & \mathbb{Z}\oplus\mathbb{Z} & 0 & 0 & 0 \\
\hline
H_i(\mathbb{S}^j) & i=0 & i=1 & i=2 & i=3
\end{array}
\tag{7.1}
$$

Example 237. Above, we used the Hurewicz homomorphism to see that

$$
H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j=1 \\ 0, & \text{otherwise} \end{cases}.
$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed $\mathbb{S}^n$ as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$
\begin{tikzcd}
 & X^+ \arrow[dr, hook] & \\
X^+ \cap X^- \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & X^- \arrow[ur, hook] &
\end{tikzcd}
$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$
\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^n) \arrow[dll, "\delta", rounded corners] \\
H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0, i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r] & \cdots
\end{tikzcd}
$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^-, with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+, and $i_1(p)$ is a point of X^-. Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[dll, "\delta"] \\
H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0,i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r, "H_0(\kappa_1)-H_0(\kappa_2)"] & H_0(\mathbb{S}^1) &
\end{tikzcd}$$

We can immediately replace things we know, finding the following.

$$\begin{tikzcd}
 & & (a,b) \arrow[r, mapsto] & (a+b, a+b) & \\
0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[r, hook] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & & & (c,d) \arrow[r, mapsto] & c - d
\end{tikzcd}$$

The kernel of f is the free group generated by (a, a). Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

7.6.3. The relative Mayer-Vietoris sequence

Theorem 238 (relative Mayer-Vietoris sequence)**.** Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X, A \cup B) \arrow[dll, "\delta"] \\
H_n(X, A \cap B) \arrow[r] & H_n(X, A) \oplus H_n(X, B) \arrow[r] & H_n(X, A \cup B) \arrow[dll, "\delta"] \\
H_{n-1}(X, A \cap B) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Consider the following commuting diagram.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem 45) implies that the last row is also exact.

Consider the following map of short exact sequences; the first row is the last column of the above grid.

$$\begin{tikzcd}
0 \arrow[r] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r, hook] \arrow[d, "\phi"'] & S_n(X) \arrow[r, two heads] \arrow[d, equal] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & S_n(A \cup B) \arrow[r] & S_n(X) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0
\end{tikzcd}$$

This gives us, by Lemma 42, a morphism of long exact sequences on homology.

$$\begin{tikzcd}
H_n(S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\phi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_\bullet(X)/S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\psi)"] & H_{n-1}(S_\bullet\mathfrak{U}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\phi)"] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

We have seen (in Fact 230) that $H_i(\phi)$ is an isomorphism for all i. Thus, the five lemma (Theorem 44) tells us that $H_n(\psi)$ is an isomorphism. Thus, the bottom row of our grid can be written

$$\begin{tikzcd}
0 \arrow[r] & S_n(X, A \cap B) \arrow[r, hook] & S_n(X, A) \oplus S_n(X, B) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0 ,
\end{tikzcd}$$

and taking the long exact sequence on homology gives the result. □

7.7. Reduced homology

We have seen that reduced homology $\tilde{H}_n(X)$ agrees with $H_n(X)$ in positive degrees, and is missing a copy of $\mathbb{Z}$ in the zeroth degree. There are three equivalent ways of understanding this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Picking any point $x \in X$, one can define the reduced homology of X by
$$\tilde{H}_n(X) = H_n(X, x).$$
2. **In between:** One can define
$$\tilde{H}_n(X) = \ker(H_n(X) \to H_n(\mathrm{pt})).$$
3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of $\mathbb{Z}$ in degree -1, so that
$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$
Then one can define
$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the one we have taken so far. In constructing the singular chain complex of our space X, we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1)-simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through $\mathcal{F}$ thus gives a copy of $\mathbb{Z}$ as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- $(1 \Leftrightarrow 2)$: Since $H_n(P)$ is free for all n, the sequence

$$0 \longrightarrow \ker H_n(\epsilon) \hookrightarrow H_n(X) \twoheadrightarrow H_n(\{x\}) \longrightarrow 0$$

splits, telling us that

$$H_n(X) \cong \ker H_n(\epsilon) \oplus H_n(\{X\}).$$

Applying the functor $H_n \circ S$ to the diagram

$$\begin{tikzcd} \{x\} \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_{\{x\}}"'] & X \arrow[r, two heads, "\epsilon"] & \{x\} \end{tikzcd}$$

one finds that $H_n(i)$ is an injection. Therefore, the connecting homomorphisms for the long exact sequence on the pair (X, x)

$$\cdots \longrightarrow H_{n+1}(X, x) \xrightarrow{\delta} H_n(\{x\}) \xhookrightarrow{i} H_n(X) \longrightarrow \cdots$$

must be zero, so the sequence

$$0 \longrightarrow H_n(\{x\}) \xhookrightarrow{i} H_n(X) \twoheadrightarrow H_n(X, x) \longrightarrow 0$$

is exact. Thus, $H_n(X, x) \cong H_n(X) \oplus H_n(X, x)$. But

$$H_n(X) \oplus H_n(X, x) \cong H_n(X) \oplus \ker H_n(\epsilon),$$

so $H_n(X, x) \cong \ker H_n(\epsilon)$.

- $(2 \Leftrightarrow 3)$: Consider the chain map $S_\bullet(X) \to S_\bullet(P)$.

$$\begin{tikzcd} \cdots \arrow[r] & S_2(X) \arrow[r, "d_2"] \arrow[d] & S_1(X) \arrow[r, "d_1"] \arrow[d] & S_0(X) \arrow[r] \arrow[d, "\varepsilon"] & 0 \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}.$$

Considering this as a chain complex

$$\cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \twoheadrightarrow \mathbb{Z} \longrightarrow 0\ ,$$

the homology $H_0(S_\bullet(X))$ is given by $\ker \epsilon / \operatorname{im} d_1$. But d_1 is epi, so $H_0(\tilde{C}_\bullet) \cong \ker \epsilon$.

Proposition 239. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

although this isomorphism is not canonical.

Proof. Trivial from algebraic definition. □

7.7.1. Deja vu all over again

Many of our results for regular homology hold also for reduced homology.

Proposition 240. There is a long exact sequence for a pair of spaces

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{j+1}(X,A) \arrow[dll, "\delta"] \\ \tilde{H}_j(A) \arrow[r] & \tilde{H}_j(X) \arrow[r] & \tilde{H}_j(X,A) \arrow[dll, "\delta"] \\ \tilde{H}_{j-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

We're now ready to prove quite a nice theorem.

Proposition 241 (Brouwer)**.** For any $n \geq 1$, any continuous bijective map $f\colon \mathbb{D}^n \to \mathbb{D}^n$ has at least one fixed point.

Proof. Suppose we have such a map. Then x and $f(x)$ are different for all x, and we can draw a ray starting at $f(x)$ and passing through $f(x)$. This ray intersects $\partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$ at exactly one place $r(x)$, giving us a map

$$r\colon \mathbb{D}^n \to \mathbb{S}^{n-1}; \qquad x \mapsto r(x).$$

This is clearly continuous, and it fixes $\partial\mathbb{D}^n \subset \mathbb{D}^n$. Thus, we have the following commutative diagram.

$$\begin{tikzcd} \mathbb{S}^{n-1} \arrow[r, hook, "\text{incl.}"] \arrow[rr, bend right, "\mathrm{id}_{\mathbb{S}^{n-1}}"'] & \mathbb{D}^n \arrow[r, two heads, "r"] & \mathbb{S}^{n-1} \end{tikzcd}$$

Applying $\tilde{H}_n$ gives the following commutative diagram.

$$\begin{tikzcd} \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \arrow[r, hook, "\tilde{H}_{n-1}(\text{incl.})"] \arrow[rr, bend right, "\mathrm{id}_{\tilde{H}_n(\mathbb{S}^{n-1})}"'] & \tilde{H}_{n-1}(\mathbb{D}^n) \arrow[r, two heads, "\tilde{H}_{n-1}(r)"] & \tilde{H}_{n-1}\mathbb{S}^{n-1}) \end{tikzcd}$$

But $\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ and $\tilde{H}_{n-1}(\mathbb{D}^n) \cong 0$ for all $n \geq 1$, so $\tilde{H}_{n-1}(r)$ cannot be surjective. $\square$

Proposition 242. We have a reduced Mayer-Vietoris sequence.

$$\cdots \longrightarrow \tilde{H}_{n+1}(X) \xrightarrow{\delta} \tilde{H}_n(X_1 \cap X_2) \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \xrightarrow{\delta} \tilde{H}_{n-1}(X_1 \cap X_2) \longrightarrow \cdots$$

Proposition 243. Let $\{(X_i, x_i)\}_{i\in I}$, be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$[4] we have

$$\tilde{H}_n\left(\bigvee_{i\in E} X_i\right) \cong \bigoplus_{i\in E} \tilde{H}_n(X_i).$$

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of Proposition 242 tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 244 (good pair)**.** A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X.
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U.

Proposition 245. Let (X, A) be a good pair. Let $\pi \colon X \to X/A$ be the canonical projection. Then

$$H_n(X, A) \cong \tilde{H}_n(X/A) \qquad \text{for all } n \geq 0.$$

[4] This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Proof. First, note that since $X \setminus A \cong (X/A) \setminus \{b\}$ and $U \setminus A \cong (U/A) \setminus \{b\}$, we have an isomorphism of pairs

$$(X \setminus A, U \setminus A) \cong ((X/A) \setminus \{b\}, (U/A) \setminus \{b\}). \tag{7.2}$$

Thus, we have the following chain of isomorphisms.

$$\begin{aligned}
H_n(X, A) &\cong H_n(X, U) && \left(\substack{A \text{ deformation} \\ \text{retract of } U}\right) \\
&\cong H_n(X \setminus A, U \setminus A) && \text{(excision)} \\
&\cong H_n((X/A) \setminus \{b\}, (U/A) \setminus \{b\}) && \text{(Equation 7.2)} \\
&\cong H_n(X/A, U/A) && \text{(excision)} \\
&\cong H_n(X/A, \{b\}) && \left(\substack{\{b\} \text{ deformation} \\ \text{retract of } U/A}\right) \\
&\cong \tilde{H}_n(X/A)
\end{aligned}$$

□

Theorem 246 (suspension isomorphism)**.** Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

This is natural in pairs (A, X).

Proof. Consider the following horizontal pairs. These are good pairs.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, hook] & X \arrow[d, hook] \\
CA \arrow[r, hook] \arrow[d, hook] & CX \arrow[d, hook] \\
\Sigma A \arrow[r, hook] & \Sigma X
\end{tikzcd}$$

Drawing diagrams should convince the ambivalent reader that

$$CA \cup X \hookrightarrow CX \qquad \text{and} \qquad CA \hookrightarrow X \cup CA$$

are also good pairs.

Consider the long exact sequence on the triple $CA \subset X \cup CA \subset CX$ (coming from

Proposition 226).

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(CX, CA \cup X) \arrow[dll, "\delta"] \\
H_n(X \cup CA, CA) \arrow[r] & H_n(CX, CA) \arrow[r] & H_n(CX, CA \cup X) \arrow[dll, "\delta"] \\
H_{n-1}(X \cup CA, CA) \arrow[r] & \cdots &
\end{tikzcd} .$$

By Proposition 245,

$$H_n(X \cup CA, CA) \cong \tilde{H}_n((X \cup CA)/CA), \qquad \tilde{H}_n(CX, (CA \cup X)) \cong \tilde{H}_n(CX/(CA \cup X)).$$

But (as another sketch should convince you), $(X \cup CA)/CA \cong X/A$, and $CX/(CA \cup X) \cong \Sigma X/\Sigma A$. Reading Proposition 245 backwards then tells us that

$$\tilde{H}_n((X \cup CA)/CA) \cong H_n(X, A), \qquad \tilde{H}_n(CX/(CA \cup X)) \cong H_n(\Sigma X, \Sigma A).$$

Note finally that $H_n(CX, CA) \cong 0$ because CX is contractible. Making these replacements in the long exact sequence on homology, the result follows. Naturality follows from the naturality of the connecting homomorphism. □

7.8. Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d\,\mu_n, \qquad \text{for some } d \in \mathbb{Z}.$$

Example 247. Consider the map

$$\omega\colon [0,1] \to \mathbb{S}^1; \qquad t \mapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz isomorphism (Theorem 201), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

In fact, there is a way of constructing inductively a generator. Recall the Mayer-Vietoris sequence we used to construct the homotopy groups of spheres in Example 236. This provided an isomoprhism

$$\delta\colon H_n(\mathbb{S}^n) \cong H_{n-1}(\mathbb{S}^n \smallsetminus \{N, S\}).$$

However, $\mathbb{S}^n \smallsetminus \{N, S\} \simeq \mathbb{S}^{n-1}$. Thus, we have a chain of isomorphisms

$$H_1(\mathbb{S}^1) \overset{D}{\cong} H_2(\mathbb{S}^2) \overset{D}{\cong} \cdots .$$

We can use these isomorphisms to transport the generator $[\omega] \in H_1(\mathbb{S}^1)$ up the line.

Definition 248 (mapping degree)**.** We call $d \in \mathbb{Z}$ as above the mapping degree of f, and denote it by $\deg(f)$.

Example 249. Consider the map

$$f_n\colon \mathbb{S}^1 \to \mathbb{S}^1; \qquad x \mapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism (Theorem 201) tells us that the following diagram commutes.

$$\begin{tikzcd} \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[r, "\pi_1(f_n)_{\mathrm{ab}}"] \arrow[d, "h_{\mathbb{S}^1}"'] & \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[d, "h_{\mathbb{S}^1}"] \\ H_1(\mathbb{S}^1) \arrow[r, "H_1(f_n)"'] & H_1(\mathbb{S}^1) \end{tikzcd}$$

Thus, $H_1(f_n)[\omega] = n[\omega]$.

Proposition 250. The following are immediate consequences of the definition of degree.

1. If f is homotopic to g, then $\deg f = \deg g$.
2. The degree of $\mathrm{id}_{\mathbb{S}^n}$ is 1.
3. Degree is multiplicative, i.e. $\deg(g \circ f) = \deg g \deg f$.
4. If f is not surjective, then $\deg f = 0$.

Proof. The only non-trivial one is 4., which is true because any non-surjective map to $\mathbb{S}^n$ is homotopic to a constant map. □

Lemma 251. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$. Then $\deg f = \deg \Sigma f$.

Proof. The suspension isomorphism of Theorem 246 is natural, so the following square commutes.

$$\begin{tikzcd} H_n(\Sigma\mathbb{S}^{n-1}) \arrow[r, "H_n(\Sigma f)"] \arrow[d, equal] & H_n(\Sigma\mathbb{S}^{n-1}) \arrow[d, equal] \\ H_{n-1}(S_{n-1}) \arrow[r, "H_n(f)"] & H_{n-1}(S_{n-1}) \end{tikzcd}$$

□

7.9. CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 252 (cell)**.** Let X be a topological space. We say that X is an n-cell if X is homeomorphic to $\mathbb{R}^n$. We call the number n the dimension of X.

Definition 253 (cell decomposition)**.** A cell decomposition of a topological space X is a decomposition

$$X = \coprod_{i\in I} X_i, \qquad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topologial spaces.

Definition 254 (CW complex)**.** A Hausdorff topological space X, together with a cell decomposition, is known as a CW complex[5] if it satisfies the following conditions.

(CW1) For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n-1$.

(CW2) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X.

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$; that is, it is a colimit over its cells.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ.
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map.*

[5] Axiom (CW2) is called the *closure-finiteness* condition. This is the 'C' in CW complex. Axiom (CW3) says that X carries the *weak topology* and is responsible for the 'W'.

Example 255. Consider the unit interval $I = [0, 1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n + 1$ 0-cells and n 1-cells. which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma_k^0 = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \qquad \text{and} \qquad \sigma_\infty^0 = 0$$

and one-cells

$$\sigma_k^1 = \left(\frac{1}{k}, \frac{1}{k+1}\right), \qquad k \in \mathbb{N}^{\geq 1}.$$

At first glance, this looks like a CW decomposition; it is certainly satisfies Axiom (CW1) and Axiom (CW2). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right)$$

is the midpoint of the interval σ_k^1. We have $A \cap \sigma_k^0 = \emptyset$ for all k, and $A \cap \sigma_k^1 = \{a_k\}$ for all k. In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i. However, the set A is not closed in I, since it does not contain its limit point $\lim_{n\to\infty} a_n = 0$.

Definition 256 (skeleton, dimension)**.** Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the <u>n-skeleton</u> of X. If X is equal to its n-skeleton but not equal to its $(n-1)$-skeleton, we say that X is <u>n-dimensional</u>.

Note 257. Axiom (CW3) implies that X carries the direct limit topology, i.e. that

$$X \cong \operatorname{colim} X^n.$$

Definition 258 (subcomplex, CW pair)**.** Let X be a CW complex. A subspace $Y \subset X$ is a <u>subcomplex</u> if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$

We call such a pair (X, Y) a <u>CW pair</u>.

Fact 259. Let X and Y be CW complexes such that X is locally compact.[6] Then $X \times Y$ is a CW complex.

[6] I.e. if for every point x there is an open neighborhood U containing x and a compact set K containing U.

Lemma 260. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 261. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact if and only if it is locally finite.[7]

Proof. It is clear that 1. $\Rightarrow$ 2., since X is a subset of itself. Similarly, it is clear that 2. $\Rightarrow$ 3., since □

Corollary 262. If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n.

$$\begin{tikzcd}
 & & & & K \arrow[d, "f"] \arrow[dlll, dashed, "\exists \tilde{f}"'] \\
X^{n-1} \arrow[r, hook] & X^n \arrow[r, hook] & X^{n+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 263. Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.

Lemma 264. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborbood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.
- Every CW complex is paracompact, locally path-connected, and locally contractible.
- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 265. Let X be a CW complex. We have the following decompositions.

1. $$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2. $$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

[7] I.e. if every point has a neighborhood which is contained in only finitely many cells.

Proof.

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n-cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n-cell is homeomorphic to the open n-ball.
2. For every n-cell σ, the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$-skeleton.

□

7.10. Cellular homology

Lemma 266. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 264, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 245, and the second from Lemma 265. □

Lemma 267. Consider the inclusion $i_n \colon X^n \hookrightarrow X$.

- The induced map
$$H_n(i_n) \colon H_n(X^n) \to H_n(X)$$
is surjective.
- On the $(n+1)$-skeleton we get an isomorphism
$$H_n(i_{n+1}) \colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n). The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by Lemma 266,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n) \colon X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{tikzcd} & & H_{n+1}(X^{m+1}, X^m) \arrow[dll, "\delta"] \\ H_n(X^m) \arrow[r] & H_n(X^{m+1}) \arrow[r] & H_n(X^{m+1}, X^m) \end{tikzcd}$$

But again by Lemma 266, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \to H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r] \arrow[drrr] & X^{n+1} \arrow[r] \arrow[drr] & X^{n+2} \arrow[r] \arrow[dr] & \cdots \\ & & & & X \end{tikzcd}$$

Taking nth singular homology, we find the following.

$$\cdots \longrightarrow H_n(X^n) \overset{\alpha_1}{\twoheadrightarrow} H_n(X^{n+1}) \overset{\alpha_2}{=\!=} H_n(X^{n+2}) \overset{\alpha_3}{=\!=} \cdots \longrightarrow H_n(X)$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \qquad \sigma_i \colon \Delta^n \to X.$$

Since the standard n-simplex Δ^n is compact, Corollary 262 implies that each σ_i factors through some X^{n_i}. Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \qquad \tilde{\sigma}_i \colon \Delta^n \to X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$\begin{aligned} [\alpha] &= \left[\sum_i \alpha^i i_N \circ \tilde{\sigma}_i \right] \\ &= H_n(i_n) \left[\sum_i \alpha^i \tilde{\sigma}_i \right]. \end{aligned}$$

□

Corollary 268. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q-cells, then $H_q(X) \cong 0$.
3. In particular, for an n-dimensional CW-complex X (Definition 256), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 267.
2.

□

Definition 269 (cellular chain complex)**.** Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}).

Theorem 270 (comparison of cellular and singular homology)**.** Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n \colon H_n(C_\bullet(X), d) \cong H_n(X).$$

Proof. Consider the following diagram.

$$\begin{tikzcd}
C_{n+1}(X) \arrow[r, equal] \arrow[dd, "d_{n+1}"'] & H_{n+1}(X^{n+1}, X^n) \arrow[dr, "\delta"] & \\
& & H_n(X) \arrow[dl, hook, "\varrho"] \\
C_n(X) \arrow[r, equal] \arrow[dd, "d_n"'] & H_n(X^n, X^{n-1}) \arrow[dr, "\delta"] & \\
& & H_{n-1}(X) \arrow[dl, hook, "\varrho"] \\
C_{n-1}(X) \arrow[r, equal] & H_{n-1}(X^{n-1}, X^{n-2}) &
\end{tikzcd}$$

1. Note that ρ is injective because $H_k(X^{k-1}) \cong 0$ for all k. In fact, $H_n(X) = \ker d_n$. To see this, let $c \in C_n(X)$ be a cycle. Then

$$dc = \varrho\delta c = 0 \implies \delta c = 0,$$

since ϱ is injective.

Computation…

For n even, we find

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & k = 0 \\ \mathbb{Z}/(2), & 1 \leq k \leq n-1,\ k \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

For n odd, we find

$$H_k(\mathbb{R}P^n) \cong \begin{cases} \mathbb{Z}, & k = 0, n \\ \mathbb{Z}/(2), & 1 \leq k \leq n-2,\ k \text{ odd} \\ 0, & \text{otherwise.} \end{cases}$$

□

Example 271 (complex projective space)**.** Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \text{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots : x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{tikzcd}
2n & 2n-1 & 2n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i,\ 0 \le i \le n \\ 0, & \text{otherwise.} \end{cases}$$

Example 272 (real projective space)**.** As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{tikzcd}
n & n-1 & n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

7.10.1. Euler characteristic

Here's one last application of CW stuff: euler characteristic.

Definition 273 (Euler characteristic)**.** Let X be a finite CW complex. The Euler characteristic of X is defined to be

$$\chi(X) = \sum_{n\geq 0}(-1)^n \ \mathrm{rk}(H_n(X;\mathbb{Z}))$$

where for an abelian group A, $\mathrm{rk}(A)$ denotes the number of free summands.

Note that the above sum must be finite since there is some simplex of highest degree, say k, so $H_{k'}(X;\mathbb{Z}) \cong 0$ for all $k' > k$.

Proposition 274. For X a finite CW complex, denote by $c_n(X)$ the number of n-cells. Then

$$\chi(X) = \sum_{n\geq 0}(-1)^n \, c_n(X).$$

Proof. We will use the following fact. For any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0 \ ,$$

we have the equality

$$\mathrm{rk}(B) = \mathrm{rk}(A) + \mathrm{rk}(C).$$

Consider the short exact sequences

$$0 \longrightarrow Z_n \hookrightarrow C_n \twoheadrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \hookrightarrow Z_n \twoheadrightarrow H_n \longrightarrow 0 \ .$$

Applying our rank formula to both and summing leads to the equation

$$\mathrm{rk}(C_n) = \mathrm{rk}(B_{n-1}) + \mathrm{rk}(B_n) + \mathrm{rk}(H_n).$$

Substituting $C_n(X)$ by the above and summing results in the B_i terms cancelling each other in the alternating sum, giving us the result we want. □

Proposition 275. For finite CW complexes X and Y, we have

$$\chi(X \times Y) \cong \chi(X) \times \chi(Y).$$

Proof. First, note that for $\mathbb{S}^n$ □

7.11. Homology with coefficients

Definition 276 (homology with coefficients)**.** Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X;G) = S(X) \otimes_{\mathbb{Z}} G.$$

The nth singular homology of X with coefficients in G is the nth homology

$$H_n(X;G) = H_n(S(X;G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G.

Theorem 277 (topological universal coefficient thoerem for homology)**.** For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X;G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0 \ .$$

Furthermore, this sequence splits non-canonicaly, telling us that

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. Theorem 277. □

7.12. The topological Künneth formula

Let X and Y be topological spaces. Plugging $C = S(X)$ and $D = S(Y)$ into Theorem 100 tells us that the following sequence is split exact.

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \hookrightarrow H_n(S(X)_\bullet \otimes S(Y)_\bullet) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(X), H_q(Y)) \to 0 \ .$$

It turns out that we can relate $H_n(S(X)_\bullet \otimes S(Y)_\bullet)$ and $H_n(X \times Y)$. In fact, they turn out to be isomorphic; this is known as the *Eilenberg-Zilber theorem.*

7.12.1. The Eilenberg-Zilber theorem: with acyclic models

This turns out to be a direct consequence of the acyclic model theorem (Theorem 209).

Lemma 278. The functors

$$F = S(-) \otimes S(-) \colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

and

$$G = S(- \times -) \colon \mathbf{Top} \times \mathbf{Top} \to \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

are both free and acyclic (Definition 206) on

$$\mathcal{M} = \{(\Delta^p, \Delta^q)\}_{p,q=0,1,\dots}.$$

Proof. Clear, basically. □

Proposition 279. There is a chain homotopy equivalence between F and G as defined in Lemma 278.

Proof. Consider the following diagram.

$$\begin{tikzcd} S_0(X) \otimes S_0(Y) \arrow[r, bend left, "f_0 \colon x\otimes y \mapsto (x,y)"] & S_0(X \times Y) \arrow[l, bend left, "g_0 \colon (x,y) \mapsto x \otimes y"] \end{tikzcd}$$

We can find lifts… □

7.12.2. The Eilenberg-Zilber theorem: without acyclic models

We first define a so-called *homology cross product*

$$\times \colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y),$$

and then show that it descends to homology.

Our goal is to find a way of making a p-simplex in X and a q-simplex in Y into a $(p+q)$-simplex in $X \times Y$, called the *homology cross product.* At least in the case that one of p or q is equal to zero, it is clear what to do, since the Cartesian product of a p-simplex with a zero-simplex is in an obvious way a p-simplex. In more general cases, however, we have to be clever. The strategy is to write down a list of properties that we would like our homology cross product to have, and then show that there exists a unique map satisfying them.

Lemma 280. We can define a homomorphism

$$\times \colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y), \qquad p, q \geq 0,$$

with the following properites.

1. For all points $x_0 \in X$ viewed as zero-chains in $S_0(X)$ and all $\beta\colon \Delta^q \to Y$, we have

$$(x_0 \times \beta)(t_0, \dots, t_q) = (x_0, \beta(t_0, \dots, t_q));$$

   conversely, for $\alpha \in S_p(X)$ and $y_0 \in Y$, we have

$$(\alpha \times y_0)(t_0, \dots, t_p) = (\alpha(t_0, \dots, t_q), y_0).$$

2. The map $\times$ is natural in the sense that the square

$$\begin{tikzcd} S_p(X) \otimes S_q(Y) \arrow[r, "\times"] \arrow[d] & S_{p+q}(X \times Y) \arrow[d] \\ S_p(X') \otimes S_q(Y') \arrow[r, "\times"] & S_{p+q}(X' \times Y') \end{tikzcd}$$

   commutes.

3. The map $\times$ satisfies the Leibniz rule in the sense that

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

Proof. As a warm up, let us work out some hypothetical consequences of our formulae, in the special case that $X = \Delta^p$ and $Y = \Delta^q$. In this case, we have

$$\mathrm{id}_{\Delta^p} \in S_p(\Delta^p), \qquad \mathrm{id}_{\Delta^q} \in S_q(\Delta^q),$$

so we can take the homology cross product $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} \in S_{p+q}(\Delta^p \times \Delta^q)$. By the Leibniz rule, we have

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \mathrm{id}_{\Delta^p} \times \partial(\mathrm{id}_{\Delta^q}).$$

This is now a perfectly well-defined element of $S_{p+q-1}(\Delta^p \times \Delta^q)$, which we will give the nickname R.[8]

A trivial computation shows that $\partial R = 0$, so there exists a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$. We choose some such c and define $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} = c$.

We now use the trick of expressing some $\alpha\colon \Delta^p \to X$ as $S_p(\alpha)(\mathrm{id}_{\Delta^p})$. Then

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}).$$

[8] I think there's an induction argument hidden here.

But then the naturality forces our hand; chasing $\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}$ around the naturality square

$$\begin{tikzcd}
S_p(\Delta^p) \otimes S_q(\Delta^q) \arrow[r, "\times"] \arrow[d, "S_p(\alpha)\otimes S_q(\beta)"'] & S_{p+q}(\Delta^p \times \Delta^q) \arrow[d, "S_{p+q}(\alpha,\beta)"] \\
S_p(X) \otimes S_p(Y) \arrow[r, "\times"] & S_{p+q}(X \times Y)
\end{tikzcd}$$

tells us that

$$\begin{aligned}
S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) &= S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) \\
S_{p+q}(\alpha, \beta)(c) &= \alpha \times \beta.
\end{aligned}$$

Thus, we can define

$$\alpha \times \beta = S_{p+q}(\alpha, \beta)(c).$$

This satisfies all the desired properties by construction. □

Proposition 281. Any natural transformations f, g with components

$$f_{X,Y},\ g_{X,Y} \colon (S(X) \otimes S(Y))_\bullet \to S_\bullet(X \times Y)$$

which agree in degree zero and send $x_0 \otimes y_0 \mapsto (x_0, y_0)$ are chain homotopic.

Proof. First, suppose that $X = \Delta^p$ and $Y = \Delta^q$. Then $(S(\Delta^p) \otimes S(\Delta^q))_\bullet$ is free, hence certainly projective, and $S(\Delta^p \times \Delta^q)$ is acyclic, so the result follows from Lemma 69: that is, we get a chain homotopy H with components

$$H_n \colon (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_n \to S_{n+1}(\Delta^p \times \Delta^q)$$

such that

$$\partial H_n + H_{n-1}\partial = f_n - g_n.$$

□

7.12.3. The topological Künneth formula

Theorem 282 (topological Künneth formula)**.** For any topological spaces X and Y, we have the following short exact sequence, natural in X and Y.

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

This sequence splits, but not canonically, and the splitting is not natural.

Example 283. Consider the torus $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$. We have that the following sequence is exact.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(\mathbb{S}^1) \otimes H_q(\mathbb{S}^1) \longrightarrow H_n(\mathbb{S}^1 \times \mathbb{S}^1) \longrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(\mathbb{S}^1), H_q(\mathbb{S}^1)) \longrightarrow 0$$

Because $H_i(\mathbb{S})$ is either 0 or $\mathbb{Z}$, hence certainly projective, we get that

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1) \cong \bigoplus_{p+q=n} H_p(\mathbb{S}) \otimes H_q(\mathbb{S}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

By induction, $H_n(\mathbb{S}^k) = \mathbb{Z}^{\binom{n}{k}}$.

Example 284. For a space of the form $X \times \mathbb{S}^n$, we get

$$H_n(X \times \mathbb{S}^n) \cong H_q(X) \oplus H_{q-n}(X).$$

Example 285 (products of Moore spaces)**.** Let A be an abelian group. For $n \geq 1$, the nth *Moore space* of A is the space $M(A, n)$ such that

$$H_r(M(A, n)) \cong \begin{cases} \mathbb{Z}, & r = 0 \\ A, & r = n \\ 0, & \text{otherwise} \end{cases}.$$

For example, $M(\mathbb{Z}, n) \cong \mathbb{S}^n$.

The topological Künneth formula allows us to compute products of Moore spaces. More specifically, it tells us that that we can express $H_r(M(A, n) \times M(B, m))$ as a direct sum

$$\left(\bigoplus_{p+q=r} H_p(M(A, n)) \otimes H_q(M(B, m))\right) \oplus \left(\bigoplus_{p+q=r-1} \mathrm{Tor}_1(H_p(M(A, n)), M(B, m))\right).$$

First, assume that $m \neq n$. Then

$$H_0(M(A, m) \times M(B, n)) \cong H_0(M(A, n)) \otimes H_0(M(B, m)) \cong \mathbb{Z}.$$

For $1 \geq r < \min(m, n)$, we have no non-trivial terms. Taking $m < n$ without loss of generality, the only non-trivial

Therefore, for $m \neq n$, we have that…

Thus,

$$H_r(M(A,m)\times M(B,n)) \cong \begin{cases} \mathbb{Z}, & r=0 \\ A, & r=m \\ B, & r=n \\ A\otimes B, & r=n+m \\ \operatorname{Tor}_1(A,B), & r=n+m+1 \\ 0, & \text{otherwise.} \end{cases}$$

For example,

7.13. The Eilenberg-Steenrod axioms

Denote by T the functor

$$T\colon \mathbf{Pair} \to \mathbf{Pair}; \qquad (X,A)\mapsto (A,\emptyset); \qquad (f\colon (X,A)\to (Y,B))\mapsto f|_A.$$

Definition 286 (homology theory)**.** Let A be an abelian group. A homology theory with coefficients in A is a sequence of functors

$$H_n\colon \mathbf{Pair}\to \mathbf{Ab}, \qquad n\geq 0$$

together with natural transformations

$$\delta\colon H_n \Rightarrow H_{n-1}\circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,

$$f\sim g \implies H_n(f)=H_n(g) \qquad \text{for all } f,g,n.$$

2. **Excision:** If (X,A) is a pair with $U\subset X$ such that $\bar{U}\subset \mathring{A}$, then the inclusion $i\colon (X\smallsetminus U, A\smallsetminus U)\hookrightarrow (X,A)$ induces an isomorphism

$$H_n(i)\colon H_n(X\smallsetminus U, A\smallsetminus U)\cong H_n(X,A)$$

for all n.

3. **Dimension:** We have

$$H_n(\mathrm{pt}) = \begin{cases} A, & n=0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$
5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.

8. Cohomology

8.1. Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition 286) which govern cohomology theories.

Definition 287 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n\colon \mathbf{Pair}^{\mathrm{op}} \to \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial\colon H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism

   $$H^n(i)\colon H^n(X, A) \cong H^n(X \smallsetminus U, A \smallsetminus U)$$

   for all n.
3. **Dimension:** We have

   $$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then

   $$H^n(X) = \prod_\alpha H^n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

8.2. Singular cohomology

Definition 288 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet(X), G).$$

We also define relative homology:

$$S^\bullet(X,A;G) = \mathrm{Hom}(S_\bullet(X,A), G).$$

We will denote the evaluation map $\mathrm{Hom}(A,G) \otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 289. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G) \otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet) \otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha \colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned}
\langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\
&= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\
&= \langle \alpha, a \rangle.
\end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned}
\langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\
&= \langle \alpha, a \rangle + \langle \beta, da \rangle \\
&= \langle \alpha, a \rangle
\end{aligned}$$

□

Via ⊗-hom adjunction, we get a map

$$\kappa \colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 290 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space, and G an abelian group. There is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X; G) \twoheadrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

Proof. We now ◻

Example 291. We have seen (in Example 271) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n,\ k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \leq k \leq n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(0, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \leq k \leq n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

8.3. The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \to \mathbb{Z},$$

allowing an n-cochain to eat an n-chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q-simplex, on which our cochain can act, and a singular $n - q$-simplex, which is along for the ride.

Definition 292 (front, rear face)**.** Let $a \colon \Delta^n \to X$ be a singular n-simplex, and let $0 \leq q \leq n$.

- The <u>$(n-q)$-dimensional front face</u> of a is given by

  $$F^{n-q}(a) = \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X,$$

  where i is induced by the inclusion

  $$\{0, \ldots, n-q\} \hookrightarrow \{1, \ldots, n\}; \qquad k \mapsto k.$$

- The q-dimensional rear face of a is given by

$$R^q(a) = \quad \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X \ ,$$

where r is the map

$$\{0, \ldots q\} \to \{0, \ldots, n\}; \qquad k \mapsto n - q + k.$$

Note that for $a \in S_n(X)$, we can write

$$F^{n-q}(a) = (\partial_{n-q+1} \circ \partial_{n-q+2} \circ \cdots \circ \partial_n)(a)$$

and

$$R^q(a) = (\overbrace{\partial_0 \circ \partial_0 \circ \cdots \circ \partial_0}^{q \text{ times}})(a).$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \to S_{n-q}(X); \qquad \alpha \otimes a \mapsto F^{n-q}(a)\langle a, R^q(a)\rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 293 (cap product)**.** The cap product is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \to S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b)\rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b)\rangle = 0$ by definition of relative cohomology.

Lemma 294. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial\alpha \otimes r).$$

Proof. We do three calculations.

$$\begin{aligned}\partial(\alpha \frown a) &= \partial(F^{n-q}(\alpha) \otimes \langle \alpha, R^q(a) \rangle) \\ &= \partial(F^{n-q}(a)) \otimes \langle \alpha, R^q(a) \rangle \\ &= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q+1} \circ \cdots \circ \partial_n)(a) \otimes \langle \alpha, R^q(a) \rangle\end{aligned}$$

$$\begin{aligned}(\delta\alpha) \frown a &= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a) \rangle \\ &= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a) \rangle \\ &= F^{n-q}(a) \otimes \langle \alpha, \partial R^q(a) \rangle \\ &= \sum_{i=0}^{q} (-1)^i F^{n-q}(a) \otimes \langle \alpha, \partial_i \circ \partial_0^q a \rangle\end{aligned}$$

$$\alpha \frown (\partial a) =$$

∎

Proposition 295. The cap product descends to a map on homology

$$\frown : H^q(X, A; R) \otimes H_n(X, A; R) \to H^{n-q}(X, A; R)$$

via the formula

$$[\alpha] \cap [a] := [\alpha \cap a].$$

Proof. Let us first collect some general results. Consider the formula

$$\partial(\alpha \frown a) = (\delta\alpha) \frown a + (-1)^q \alpha \frown \partial a.$$

1. Let α be a cocycle and a be a cycle. Then each term on the right-hand side is zero, so
$$\text{cocycle} \frown \text{cycle} = \text{cycle}.$$
2. Let α be arbitrary, so $\delta\alpha$ is a coboundary, and let a be a cycle. Then the above formula, read backwards, tells us that
$$\text{coboundary} \frown \text{cycle} = \text{boundary}.$$

3. Let a be arbitrary, so ∂a is a cycle, and let α be a coboundary. Then we get

$$\text{cocycle} \frown \text{boundary} = \text{boundary}.$$

Point 1. tells us that the formula is well-defined to begin with: $\alpha \cap a$ is really a cycle. Now suppose we had picked different representatives $\alpha + \delta\beta$ and $a + \partial b$. Then

$$\begin{aligned}
[\alpha + \delta\beta] \frown [a + \partial b] &= [(\alpha + \delta\beta) \frown (a + \partial b)] \\
&= [\alpha \frown a] + [\delta\beta \frown a] + [\alpha \frown \partial b] + [\delta\beta \frown \partial b] \\
&= [\alpha \frown a] \\
&= [\alpha] \frown [a],
\end{aligned}$$

where the the second term is zero because of 2. and the third term is zero because of 3. □

Proposition 296. The cap product is natural in the sense that for any $f : X \to X$ inducing a map of pairs $(X, A) \to (X, B)$, the diagram

$$\begin{tikzcd}
H^q(X,A;R) \otimes H^n(X,A;R) \arrow[rrr, "\frown"] \arrow[ddd] & & & H_{n-q}(X;R) \arrow[ddd] \\
& (f^*\beta, a) \arrow[r, mapsto] \arrow[d, mapsto] & F(a) \otimes \langle f^*\beta, R(a)\rangle \arrow[d, mapsto] & \\
& (\beta, f_*a) \arrow[r, mapsto] & \star & \\
H^q(X,B;R) \otimes H^n(X,B;R) \arrow[rrr, "\frown"] & & & H_{n-q}(X;R)
\end{tikzcd}$$

commutes, i.e.

$$f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) = F(f_*a) \otimes \langle \beta, R(f_*a)\rangle.$$

Proof.

$$\begin{aligned}
f_*(F(a) \otimes \langle f^*\beta, R(a)\rangle) &= f_*(F(a) \otimes \langle f^*\beta, R(\alpha)\rangle) \\
&= f_*(F(a) \otimes \langle \beta, f_*(R(a))\rangle) \\
&= f_*(F(a) \otimes \langle \beta, R(f_*a)\rangle) \\
&= F(f_*a) \otimes \langle \beta, R(f_*a)\rangle.
\end{aligned}$$

□

We can write this in a prettier way as

$$f_*(f^*\beta \frown a) = \beta \frown f_*(a).$$

Example 297. Suppose X is simply connected, and let

8.4. The cup product

This section takes place over a commutative ring with unit R. We will notationally suppress R.

Recall that we have constructed an Eilenberg-Zilber map

$$S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y).$$

in Subsection 7.12.1. There exist generalizations, which we will not construct explicitly.

Definition 298 (Eilenberg-Zilber map)**.** An Eilenberg-Zilber map is a natural chain map

$$S_\bullet(X \times Y, X \times B \cup A \times Y) \to S_\bullet(X, A) \otimes S_\bullet(Y, B).$$

Definition 299 (cohomology cross product)**.** Let EZ be any Eilenberg-Zilber map. For any $\alpha \in S^p(X, A)$ and $\beta \in S^q(Y, B)$, a cohomology cross product of α and β is the $(p+q)$-cochain $\alpha \times \beta$ defined by making the below diagram commute.

$$\begin{tikzcd}
S_n(X \times Y; X \times B \cup A \times Y) \arrow[d, "\mathrm{EZ}"'] \arrow[ddrr, "\alpha\times\beta"] & & \\
\bigoplus_{p+q=n} S_p(X) \otimes S_q(Y) \arrow[d, "\mathrm{proj}"'] & & \\
S_p(X, A) \otimes S_q(Y, B) \arrow[r, "\alpha\otimes\beta"'] & R \otimes R \arrow[r, "\mu"'] & R
\end{tikzcd}$$

Fact 300. Any homology cross product extends to a natural chain map

$$S^\bullet(X, A) \otimes S^\bullet(Y, B) \to S^\bullet(X \times Y, X \times B \cup A \times Y).$$

Note that since the Eilenberg-Zilber map is natural by assumption, projection is natural for any product, and evaluation is natural by definition of the tensor product, the homology cross product is natural in that the diagram

$$\begin{tikzcd}
S^p(X) \otimes S^q(Y) \arrow[r] \arrow[d] & S^{p+q}(X \times Y) \arrow[d] \\
S^p(X') \otimes S^q(Y') \arrow[r] & S^{p+q}(X' \times Y')
\end{tikzcd}$$

commutes; that is

$$(f,g)^*(\alpha \times \beta) = (f^*\alpha) \times (g^*\beta).$$

Fact 301. The homology cross product has the following properties.

1. The Leibniz rule holds, i.e.

$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|} \times (\delta\beta).$$

2. For the Kroenecker pairing we have for homology classes a, b, and cohomology classes α, β of corresponding degree

$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle.$$

3. For $1 \in R$ and thus $1 \in S^0(X, A)$, we have

$$1 \times \beta = p_2^*(\beta), \qquad \alpha \times 1 = p_1^*(\alpha),$$

where p_i denotes the projection onto the ith factor of $X \times Y$.

4. The cohomology cross product is associative.
5. The cohomology cross product satisfies a graded version of commutativity: the twist map $\tau\colon X \times Y \to Y \times X$ yields on homology

$$\alpha \times \beta = (-1)^{|\alpha||\beta|}\tau^*(\beta \times \alpha).$$

Definition 302 (cup product)**.** Denote by $\Delta\colon X \to X \times X$ the diagonal map. The cup product is the composition

$$H^p(X, A) \otimes H^q(X, B) \to H^{p+q}(X \times X, X \times B \cup A \times X) \to H^{p+q}(X, A \cup B),$$

where the first map is the cross product, and the second map is the pullback along the diagonal. That is,

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

Proposition 303. The cup product satisifes the following identities.

1. $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$
2. $\alpha \cup \beta = (-1)^{|\alpha||\beta|}$.
3. For $\partial\colon H^*(A) \to H^{*+1}(X, A)$, $\alpha \in H^*(A)$, $\beta \in H^*(X)$, we have

$$\partial(\alpha \cup i^*\beta) = (\partial\alpha) \cup \beta \in H^*(X, A).$$

4. For $f\colon X \to Y$, we have
$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

5. We can express the cohomology cross product via the cup product using the formula
$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta).$$

Proof.

1. Consequence of associativity of the cross product.
2.
3.
4.
5.

□

Definition 304 (diagonal approximation)**.** A diagonal approximation is a natural chain map
$$D\colon S_\bullet(X) \to S_\bullet(X) \otimes S_\bullet(X)$$
with $D(x) = x \otimes x$ for $x \in S_0(X)$.

Theorem 209 tells us immediately that any two diagonal approximations are naturally homotopic.

Example 305 (Alexander-Whitney map)**.** Consider the map
$$\mathrm{AW}\colon S_n(X) \to (S_\bullet(X) \times S_\bullet(X))_n, \qquad a \mapsto \sum_{p+q=n} F^p(A) \otimes R^q(a).$$

This is a chain map because

8.5. Manifold orientations

8.5.1. Basic definitions and results

Let M be an n-dimensional topological manifold. Without loss of generality, we may choose an atlas for M such that each each point $x \in M$ has a chart (U_x, ϕ_x) such that $U_x \cong \mathbb{R}^n$ and $\phi_x(x) = 0 \in \mathbb{R}^n$.

Let $x \in M$. Then

$$\begin{aligned} H_n(M, M \setminus \{x\}) &\cong H_n(M \setminus (M \setminus U_x), M \setminus \{x\} \setminus (M \setminus U_x)) && \text{(excision)} \\ &\cong H_n(U_x, U_x \setminus \{x\}) \\ &\cong H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) && \text{(homotopy)} \\ &\cong H_{n-1}(\mathbb{S}^{n-1}) && \text{(Example 220)} \\ &\cong \mathbb{Z}. \end{aligned}$$

We call a choice of generator of $H_n(X, X \setminus \{x\})$ a *local orientation* on M at x. For any other point $y \in U_x$, a local orientation at x gives a local orientation at y via the composition

$$H_n(X, X \setminus \{x\}) \cong H_n(X, X \setminus U_x) \cong H_n(X, X \setminus \{y\}).$$

We will use the following notation: for a pair X, A, we will write

$$H_n(X, A \setminus X) = H_n(X \mid A).$$

Furthermore, if $A = \{x\}$, we will write $H_n(X \mid x)$ instead of $H_n(X \mid \{x\})$.

We think of $H_n(X \mid A)$ as the *local homology* of A in X, since it depends (by excision) only on the closure of A in X.

Definition 306 (orientation cover)**.** Let M be an n-dimensional manifold. The orientation cover of M is, as a set,

$$\tilde{M} = \{\mu_x \mid x \in M,\ \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

There is a clear surjection from $\tilde{M}$ to M as follows.

$$\begin{array}{ccc} \tilde{M} & & \mu_x \\ \downarrow & & \downarrow \\ M & & x \end{array}$$

The set $\tilde{M}$ turns out to be a topological manifold, and the surjection $\tilde{M} \to M$ turns out to be a double cover. We will show this later.

Definition 307 (orientation)**.** Let M be an n-dimensional manifold. An orientation on M is a set-section σ of the bundle $\tilde{M} \to M$ satisfying the following condition: for any $x \in M$ and any $y \in U_x$, the induced local orientation at y is equal to $\sigma(y)$.

For a triple $B \subset A \subset X$ denote the canonical map on pairs $(X, X \setminus A) \to (X, X \setminus B)$ by $\rho_{B,A}$.

Lemma 308. Let M be a connected, orientable topological manifold of dimension m, and let $Ks\{M\}$ be a compact subset.

1. For all $q > m$, $H_q(M \mid K) = 0$.
2. If $a \in H_m(M \mid k)$, then $a = 0$ if and only if $(\rho_{x,K})_* a = 0$ for all $x \in X$.

Sketch of proof. The proof uses manifold induction, which consists of the following steps.

1. Prove the result the special case that our manifold M is simply $\mathbb{R}^n$ and K is a convex compact subset.
2. Using Mayer-Vietoris, show that the result holds when K is a union of two compact, convex subsets.
3. The case of $M = \mathbb{R}^n$ K a union of n compact, convex subsets follows by induction.
4. For K an arbitrary compact subset of $\mathbb{R}^m$, we cover K by finitely many closed balls B_i, then show that every cycle on $H_q(\mathbb{R}^m \mid K)$ comes from a cycle on $H_q(\mathbb{R}^m \mid \bigcup_i B_i)$. Since $\bigcup_i B_i$ is a finite union of compact, convex subsets, there are no nontrivial cycles, implying the claim.
5. For an arbitrary compact subset contained in a single chart, the claim reduces to the $\mathbb{R}^m$ case above.
6. For an arbitrary compact subset contained in two charts, the result follows from the Mayer-Vietoris sequence. The general claim follows by induction.

□

Proof.

1. First take $M = \mathbb{R}^m$ and $K \subset M$ compact and convex, so that any $x \in K$ is a deformation retract of K. Then, since K is closed and bounded, we can find an open ball $\mathring{\mathbb{D}}^m$ with $K \subset \mathring{\mathbb{D}}^m$. Then certainly

$$\overline{M \setminus \mathring{\mathbb{D}}^m} = M \setminus \mathbb{D}^m \subset M \setminus K = (M \overset{\circ}{\setminus} K),$$

so by excision and deformation retractness we have

$$\begin{aligned} H_q(M, M \setminus K) &\cong H_q(M \setminus (M \setminus \mathring{\mathbb{D}}^m), (M \setminus K) \setminus (M \setminus \mathring{\mathbb{D}}^m)) \\ &= H_q(\mathbb{D}^m, \mathbb{D}^m \setminus K) \\ &\cong H_q(\mathbb{D}^m, \mathbb{D}^m \setminus \{x\}) \\ &\cong H_q(\mathbb{D}^m, \mathbb{S}^{m-1}) \\ &\cong H_{q-1}(\mathbb{S}^{m-1}). \end{aligned}$$

For $q > m$, this is 0, and for $q = m$ we note that similar reasoning provides an isomorphism

$$H_q(M, M \setminus K) \cong H_q(M, M \setminus \{x\}) \text{ for all } x \in X.$$

2. Now let $M = \mathbb{R}^n$ and $K = K_1 \cup K_2$, where both K_i are compact and convex. Note that $K_1 \cap K_2$ is also compact and convex. Consider the following relative Mayer-Vietoris sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{q+1}(M \mid K_1 \cap K_2) \arrow[dll, "\delta"] \\
H_q(M \mid K_1 \cup K_2) \arrow[r] & H_q(M \mid K_1) \oplus H_q(M \mid K_2) \arrow[r] & H_q(M \mid K_1 \cap K_2) \arrow[dll, "\delta"] \\
H_{q-1}(M \mid K_1 \cup K_2) \arrow[r] & \cdots &
\end{tikzcd}$$

For $q > n$, Part 1. implies that

$$H_{q+1}(M \mid K_1 \cap K_2) = H_q(M \mid K_1) = H_q(M \mid K_2) = 0,$$

so by exactness $H_q(M \mid K_1 \cup K_2) = 0$.

3. For $M = \mathbb{R}^n$, $K = \bigcup_i K_i$ for K_i compact and contractible, we get the result by induction.

4. Let $M = \mathbb{R}^n$, and let K be an arbitrary compact subset.

   Choose some $a \in H_q(\mathbb{R}^n \mid K)$, and choose $\psi \in S_q(\mathbb{R}^m)$ representing a. Then because ψ is a relative cycle, $\partial\psi \in S_{q-1}(\mathbb{R}^n \setminus K)$.

   This means that we can express $\partial\psi$ as

$$\sum_{j=1}^{\ell} \lambda_j \tau_j, \qquad \tau_j \colon \Delta^{q-1} \to \mathbb{R}^m \setminus K.$$

   Since Δ^{q-1} is compact, the images $\tau(\Delta^{q-1})$ are compact, and their union

$$T = \bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1})$$

   is compact.

   There exists an open set $U \subset \mathbb{R}^m$ with $K \subset U$ and $T \cap U = \emptyset$. Thus, we know that

$$\partial\psi \in \mathbb{R}^m \setminus U \implies \psi \in Z_q(\mathbb{R}^m \mid U).$$

Denote the representative of ψ in $H_q(\mathbb{R}^m \mid U)$ by $[\psi] = a'$.

Due to general topology stuff, we can always find finitely many closed balls B_i satisfying the following conditions.

a) $B_i \subset U$ for all i

b) $B_i \cap K \neq \emptyset$

c) $K \subset \bigcup_i B_i$

Then $K \subset \bigcup_i B_i \subset U$, and we have restriction maps as follows.

$$\begin{tikzcd}
S_q(\mathbb{R}^m \mid U) \arrow[r, "(\rho_{\bigcup_i U_i, U})_*"] & S_q(\mathbb{R}^m \mid \bigcup_i B_i) \arrow[r, "(\rho_{K, \bigcup_i U_i})_*"] & S_q(\mathbb{R}^m \mid K) \\
a' \arrow[r, mapsto] & (\rho_{\bigcup_i B_i, U})_*(a') = a'' \arrow[r, mapsto] & a
\end{tikzcd}$$

Suppose $q > m$. We have already showed that $H_q(\mathbb{R}^m \mid \bigcup_i B_i) = 0$, so $a'' = 0$. Hence $a = 0$.

Now suppose $q = m$. We will provide running commentary on the following commutative diagram.

$$\begin{tikzcd}
H_m(X \mid U) \arrow[rr] & & H_m(X \mid \bigcup_i B_i) \arrow[rr] \arrow[d] \arrow[ddl] \arrow[ddr] & & H_m(X \mid K) \\
& & H_m(X \mid B_i) \arrow[dl] \arrow[dr] & & \\
& H_m(X \mid y) & & H_m(X \mid x) &
\end{tikzcd}$$

We have shown that given $a \in H_m(X \mid K)$, one can find $a' \in H_m(X \mid U)$ and $a'' \in H_m(X \mid \bigcup_i B_i)$ which map to each other as follows. By assumption, a'' maps to zero in $H_m(X \mid x)$.

$$\begin{tikzcd}
a' \arrow[r, mapsto] & a'' \arrow[r, mapsto] \arrow[dr, mapsto] & a \\
& & 0
\end{tikzcd}$$

Pick some B_i containing x. Since $(\rho_{x,B_i})_*$ is an isomorphism, we have that $(\rho_{B_i, \bigcup_i B_i})_*(a'') =$

0.

$$\begin{tikzcd}
a' \arrow[r, mapsto] & a'' \arrow[r, mapsto] \arrow[d, mapsto] \arrow[ddr, mapsto] & a \\
& 0 \arrow[dr, mapsto] & \\
& & 0
\end{tikzcd}$$

For any $y \in B_i$, we know that $(\rho_{y,B_i})_*$ is an isomorphism.

$$\begin{tikzcd}
& a' \arrow[r, mapsto] & a'' \arrow[r, mapsto] \arrow[d, mapsto] \arrow[ddl, mapsto] \arrow[ddr, mapsto] & a \\
& & 0 \arrow[dl, mapsto] \arrow[dr, mapsto] & \\
& 0 & & 0
\end{tikzcd}$$

Thus $(\rho_{y,\bigcup_i B_i})_*(a'') = 0$ for all $y \in \bigcup_i B_i$, so the result is implied by work above.

5. Mayer-Vietoris and induction.

□

Proposition 309. Let $K \subset M$ be compact, and assume that M is connected and oriented via an orientation $x \mapsto o_x \in H_m(X \mid x)$. Then there exists a unique orientation class $o_K \in H_m(M \mid K)$ such that $(\rho_{x,K})_* o_K = o_x$ for all $x \in K$.

Proof. First we show uniqueness, as we will need it later. Suppose that there exist two orientation classes o_K, o_K' such that

$$(\rho_{x,K})_* o_K = (\rho_{x,K})_* o_K' = o_x.$$

Taking a difference, we find

$$(\rho_{x,K})_*(o_K - o_K') = 0,$$

so by Lemma 308, we get uniqueness.

As a first step towards proving existence, suppose that K is contained in the domain $U_\alpha \cong \mathbb{R}^n$ of a single chart. Pick some $x \in K$, and consider the following morphisms in **Pair**

$$\begin{tikzcd}
M \setminus U_\alpha \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \setminus K \arrow[r, hook] \arrow[d, hook] & M \arrow[d, equal] \\
M \setminus \{x\} \arrow[r, hook] & M
\end{tikzcd}$$

which correspond to the commuting diagram.

$$\begin{tikzcd} H_m(M \mid U_\alpha) \arrow[r, equal] \arrow[d, "\phi"'] & H_m(M \mid x) \\ H_m(M \mid K) \arrow[ur] & \end{tikzcd}$$

Pulling $o_x \in H_m(M \mid x)$ back to $H_m(M \mid U_\alpha)$, then mapping it to $H_m(M \mid K)$ with ϕ gives an orientation class $o_K \in H_m(M \mid K)$. The commutativity of the diagram ensures that $(\rho_{x,K})_* o_K = o_x$.

Now suppose that K is not contained in a single chart. We can find an open cover of K by charts, and since K is compact, there exists a finite subcover.

Suppose K is contained in two charts U_1 and U_2; the general case follows by induction. Then we can express K as a union of compacta $K = K_1 \cup K_2$, where $K_i \subset U_i$. Consider the relative Mayer-Vietoris sequence

$$0 \longrightarrow H_m(M \mid K_1 \cup K_2) \overset{i}{\hookrightarrow} H_m(M \mid K_1) \oplus H_m(M \mid K_2) \xrightarrow{\kappa} H_m(M \mid K_1 \cap K_2) \longrightarrow \cdots$$

By our above work, we get unique orientation classes $o_{K_i} \in H_m(M \mid K_i)$ such that $(\rho_{x,K_i})_* = o_x$ for all $x \in K_i$, $i = 1, 2$. Applying κ, we find

$$\kappa(o_{K_1}, o_{K_2}) = \rho_{K_1 \cap K_2, K_1} o_{K_1} - \rho_{K_1 \cap K_2, K_2} o_{K_2}.$$

But by uniqueness, these must be equal, so there exists a unique o_K in $H_m(M \mid K)$ as in the theorem. □

In order to give $\tilde{M}$ a manifold structure, we define a topology on $\tilde{M}$ as follows. Let $B \subset \mathbb{R}^n \subset M$ be a ball of finite radius, and choose a generator $\mu_B \in H_n(M \mid B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \to M$ is two-sheeted covering. This immediately imples that $\tilde{M}$ is a manifold. In fact, $\tilde{M}$ is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} \mid \mu_b) \cong H_n(U(\mu_B) \mid \mu_b) \cong H_n(B \mid x),$$

where

Definition 310 (orientation class)**.** Let M be a connected, orientable, compact manifold of dimension m. An <u>orientation class</u> on M is an element $o_m \in H_m(M \colon \mathbb{Z})$ such that $o_x = (\rho_{x,M})_* o_m$ is an orientation.

Theorem 311. Let M be a compact, connected manifold of dimension m. The following are equivalent.

1. M is ($\mathbb{Z}$-)orientable.
2. There exists an orientation class $o_M \in H_m(M; \mathbb{Z})$.
3. $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof.

(1 ⇒ 2) Consequence of Proposition 309.

(2 ⇒ 3) Let $a \in H_m(M)$, and let $x \in M$. Then $(\rho_{x,M})_* a = k_x o_x$ for some $k_x \in \mathbb{Z}$. This defines a function $M \longrightarrow \mathbb{Z}$, $x \mapsto k_x$.

We aim to show that this function is constant. Choose a chart U_x around x, and let $y \in U_x$. Then the outside of the following diagram commutes by functoriality of H_m, and the lower triangle commutes by definition of an orientation class, so the upper triangle commutes.

$$
\begin{tikzcd}
& H_n(M) \arrow[dl] \arrow[dr] & \\
H_m(M \mid x) \arrow[rr, "\phi"] \arrow[dr, equal] & & H_m(M \mid y) \arrow[dl, equal] \\
& H_m(X \mid U_x) &
\end{tikzcd}
$$

This means in particular that

$$\phi(k_x o_x) = k_x \phi(o_x) = k_x o_y$$

must be equal to $k_y o_y$, so $k_x = k_y$. Thus, the function $x \mapsto k_x$ is locally constant, hence constant. Call this constant k.

Thus, $ka - o_M$ is zero when restricted to any $x \in M$, implying by Proposition 309 that $a = k o_M$. Thus, o_M generates $H_m(M)$.

We know that o_M is non-trivial because $(\rho_{x,M})_* o_M$ generates $H_m(M \mid x)$, and that it has infinite order because if $\ell o_M = 0$ for some $\ell \neq 0 \in \mathbb{Z}$ $H_m(M)$, then $\ell (\rho_{x,M})_* o_M = \ell o_x = 0$. This shows that $H_m(M)$ has a single generator of infinite order, i.e. is isomorphic to $\mathbb{Z}$.

(2 ⇒ 3) Pick a generator $o_M \in \mathbb{Z}$, and define an orientation by

$$o_x = (\rho_{x,M})_* o_M.$$

□

From now, we will sometimes denote an orientation class on M by $[M]$.

8.5.2. The orientation cover

Let M be an n-dimensional manifold, not necessarily orientable.

In order to give $\tilde{M}$ a manifold structure, we define a topology on $\tilde{M}$ as follows. Let $D\colon \mathbb{D}^m \hookrightarrow M$ be a compact subset of M homeomorphic to the closed ball, and pick $\mu_D \in H_m(M \mid \mathring{\mathbb{D}})$. We define a subset $U_B \subset \tilde{M}$ by

Let $\mathbb{D}^n \subset \mathbb{R}^n \subset M$ be a closed ball of finite radius and, and choose a generator $\mu_B \in H_n(M \mid B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \to M$ is two-sheeted covering. This immediately imples that $\tilde{M}$ is a manifold. In fact, $\tilde{M}$ is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} \mid \mu_b) \cong H_n(U(\mu_B) \mid \mu_b) \cong H_n(B \mid x),$$

where

8.5.3. Degree

Definition 312 (degree)**.** Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $g\colon M \to N$ be continuous. The degree of f is the integer d such that

$$f(o_M) = do_N.$$

Proposition 313. Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $f, g\colon M \to N$ be continuous.

1. The degree is multiplicative, i.e.

$$\deg(f \circ g) = \deg(f) \circ \deg(g)$$

2. If $\bar{M}$ is the same manifold as M but with the opposite orientation, then

$$\deg(f\colon M \to N) = -\deg(f\colon \bar{M} \to N).$$

3. If the degree of f is not trivial, then f is surjective.

Proof. All obvious. □

8.6. Cohomology with compact support

For this section, let R be a commutative ring with unit 1_R. For most of this section, we will work with cohomology over R, but notationally suppress this.

Definition 314 (singular cochains with compact support)**.** Let X be a topological space, and let R be a commutative ring with unit 1_R. The singular n-cochains with compact support are the set

$$S_c^n(X;R) = \left\{\phi\colon S_n(X) \to R \mid {}_{\phi(\sigma)=0 \text{ for all } \sigma\colon \Delta^n \to X \text{ with } \sigma(\Delta^n)\cap K_\phi = \emptyset}^{\text{there exists } K_\phi \subset X \text{ compact such that}}\right\}.$$

That is, singular cochains with compact support are those ϕ which ignore simplices which do not hit some compact subset K_ϕ.

Clearly $S_c^n(X) \subset S^n(X)$, and since the boundary of any simplex with compact support clearly has compact support, there is an induced chain complex structure

$$S_c^*(X).$$

However, there is a more categorical description of this chain complex coming from relative cohomology.

Let X be a topological space. Denote by $\mathcal{K}$ the set of compact subsets of X. This is clearly a poset under

$$K \le K' \iff K \subseteq K',$$

and it is filtered because for a finite set I, $\bigcup_i K_i$ is an upper bound for $\{K_i\}_{i\in I}$.

Fixing some X, consider the functor

$$\mathcal{K}^{\mathrm{op}} \to \mathbf{Pair}; \qquad K \mapsto (X, X \setminus K).$$

We get a functor $\mathcal{K} \to \mathbf{Coch}_{\ge 0}(R\text{-}\mathbf{Mod})$ via the composition

$$\mathcal{K} \longrightarrow \mathbf{Pair}^{\mathrm{op}} \xrightarrow{S^\bullet} \mathbf{Coch}_{\ge 0}(R\text{-}\mathbf{Mod}) \ .$$

For obvious reasons, we call this functor $S^\bullet(X \mid -)$

Proposition 315. The colimit

$$\operatorname*{colim}_K S^\bullet(X \mid K)$$

agrees with cohomology with compact support $S_c^\bullet(X)$.

Proof. Let $f \in S^n(X \mid K)$. Then by definition, f is a map $S_n(X \mid K) \to R$, i.e. a map $S_n(X) \to R$ such that for any $\sigma\colon \Delta^n \to X$ such that $\sigma(\Delta^n) \cap K = \emptyset$, $f(\sigma) = 0$. Thus,

$f \in S_c^n(X)$.

This inclusion gives a chain map $S^\bullet(X \mid K) \hookrightarrow S_c^\bullet(X)$, and these together induce an inclusion[1] $\operatorname{colim}_K S^\bullet(X \mid K) \to S_c^\bullet(X)$. In fact, this is a level-wise epimorphsim since any $g \in S_c^\bullet(X)$ is of the above form. □

Corollary 316. We have an isomorphism

$$H_c^n(X) \cong \operatorname*{colim}_K H^n(X \mid K).$$

Proof. Filtered colimits are exact by Corollary 122. □

8.7. Poincaré duality

This section takes place over a ring R with unit 1_R. We will usually notationally suppress R.

Let M be a connected, m-dimensional manifold with R-orientation $x \mapsto o_x$, and let $K \subset M$ be a compact subset. By Proposition 309, there exists an orientation class $o_K \in H_m(M \mid K)$ compatible with the orientation o_x.

Consider the cap product

$$H^q(M \mid K) \otimes H_m(M \mid K) \xrightarrow{\frown} H_{m-q}(M) .$$

Sticking $o_K \in H_m(M \mid K)$ into the second slot of the cap product and setting $p = n - q$, we find a map

$$H^{m-p}(M \mid K) \xrightarrow{(-)\frown o_K} H_p(M)$$

Proposition 317. The maps

$$(-) \frown o_K \colon H^{m-p}(M \mid K) \to H_p(M)$$

descend to a map

$$H_c^{m-p}(M) \to H_p(M).$$

Proof. By Proposition 296, the identity $\mathrm{id}_M \colon M \to M$ makes the following diagram commute.

$$\begin{tikzcd}
H^{m-p}(M \mid L) \otimes H_m(M \mid L) \arrow[r, "(-)\frown(-)"] \arrow[d] & H_p(M) \arrow[d, "(\mathrm{id}_M)_*"] \\
H^{m-p}(M \mid K) \otimes H_m(M \mid K) \arrow[r, "(-)\frown(-)"'] & H_p(M)
\end{tikzcd}$$

[1] This is a monomorphism because the components are monomorphisms, and by Corollary 122 filtered colimits are exact, hence preserve monomorphisms.

Since
$$(\mathrm{id}_M)_* = \mathrm{id}_{H_p(M)},$$
this in particular means that the triangles

$$\begin{tikzcd}
H^{m-p}(M \mid L) \arrow[d] \arrow[dr, "(-)\frown o_L"] & \\
 & H_p(M) \\
H^{m-p}(M \mid K) \arrow[ur, "(-)\frown o_K"'] &
\end{tikzcd}$$

commute. But this says that the maps $(-) \frown o_K$ form a cocone under $H^{m-p}(M \mid -)$. □

Definition 318 (Poincaré duality)**.** The map defined in Proposition 317 is called the Poincaré Duality map, and denoted

$$\mathsf{PD}_M \colon H_c^{m-q}(M) \to H_m(M).$$

Theorem 319 (Poincaré duality)**.** Let M be a connected manifold of dimension m with an R-orientation $x \mapsto o_x$. Then

$$\mathsf{PD}_M \colon H_c^{m-p}(M; R) \to H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Proof. Manifold induction! □

8.8. Alexander-Lefschetz duality

8.9. Application of duality

8.10. Duality and cup products

Definition 320 (cup pairing)**.** Let M be a connected closed m-manifold with an R-orientation for some commutative ring R. The cup pairing is the composition

$$H^k(M; R) \otimes_R H^{m-k}(M; R) \xrightarrow{\smile} H^m(M; R) \xrightarrow{\mathsf{PD}_M} H_0(M; R) \; .$$

8.11. The Milnor sequence

In Subsection 3.3.2, we showed that for any tower of cochain complexes of abelian groups $\{C_i\}$ satisfying the Mittag-Leffler condition[2] and any $q \in \mathbb{Z}$, we have a short exact sequence

$$0 \longrightarrow \lim^1 H^{q-1}(C_i) \hookrightarrow H^q(\lim_i C_i) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0 \ .$$

Theorem 321 (Milnor sequence for CW complexes)**.** Let X be a CW complex with filtration

$$X_0 \subset X_1 \subset \cdots, \qquad \text{so} \quad X \cong \bigcup_{i\geq 0} X_i.$$

Then for any abelian group G there is a short exact sequence

$$0 \longrightarrow \lim^1 H^{m-1}(X_n;G) \hookrightarrow H^m(X;G) \twoheadrightarrow \lim H^m(X_m;G) \longrightarrow 0 \ .$$

Proof. The tower of chain complexes $C_n^\bullet = \mathrm{Hom}(S_\bullet(X_n), G)$ satisfies the Mittag-Leffler condition (Definition 125) because the maps $C_n^\bullet \to C_{n-1}^\bullet$ dual to the inclusions $S_\bullet(X_{n-1}) \hookrightarrow S_\bullet(X_n)$ are epimorphisms (because the exact sequence on the pair splits).

Plugging this into in Theorem 129. we find that the sequence

$$0 \longrightarrow \lim^1 H^{q-1}(X_i) \hookrightarrow H^q(\lim \mathrm{Hom}(S_\bullet(X_i), G)) \twoheadrightarrow \lim H_q(X_i) \longrightarrow 0 \ .$$

It remains only to show that

$$H^q(\lim \mathrm{Hom}(S_\bullet(X_i), G)) \cong H^q(X;G),$$

i.e. that

$$\lim \mathrm{Hom}(S_\bullet(X_i), G) \cong \mathrm{Hom}(S_\bullet(X);G).$$

But this is true because Hom turns limits into colimits in the first slot, and $S_\bullet$ commutes with colimits because reasons. □

Example 322. Consider the tower (in **Top**!)

$$\mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \cdots \ .$$

Denote

$$\mathbb{C}P^\infty := \lim_i \mathbb{C}P^i.$$

The above tower is the CW filtration of $\mathbb{C}P^\infty$, including only even levels. Because the

[2] i.e. satisfying it level-wise

sequence

$$0 \longrightarrow S_\bullet(\mathbb{C}P^{n-1}) \hookrightarrow S_\bullet(\mathbb{C}P^n) \twoheadrightarrow S_\bullet(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \longrightarrow 0$$

splits and Hom and H_q are additive, we get that the maps

$$H_q(\mathbb{C}P^n) \to H_q(\mathbb{C}P^{n-1})$$

are epimorphisms. Thus, the tower $H^{q-1}(\mathbb{C}P^n)$ also satisfies the Mittag-Leffler condition, and we get that

$$H^q(\mathbb{C}P^\infty) \cong \lim_i H^q(\mathbb{C}P^i) \cong \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

8.12. Lens spaces