1 Basic applications

> You know how it is when someone asks you to ride in a terrific sports car, and then you wish you hadn't?
>
> John Adams

In this chapter, we will apply homological algebra to the study of various classes of objects. Our main goal will be to implement the following cockamamie scheme in two contexts: quiver representations and groups.

Cockamamie Scheme 1. Given a category of algebraic objects:

1. Find a way of turning the objects in question into rings.
2. Study modules over those rings using homological algebra.
3. ???
4. Profit.

1.1 Quiver representations

The first problem to which we will apply Cockamamie Scheme 1 is the study of *quiver representations*. Unfortunately, due to time constraints, we will only get as far as step 2.

Definition 2 (quiver). A quiver Q consists of the following data.

- A finite[1] set Q_0 of *vertices*
- A finite set Q_1 of *arrows*
- A pair of maps
$$s, t\colon Q_1 \to Q_0$$
called the *source* and *target* maps respectively.

Definition 3 (quiver representation). Let Q be a quiver, and let k be a field. A k-linear representation V of Q consists of the following data.

[1] There is no fundamental reason for this finiteness condition, except that it will make the following analysis more convenient.

- For every vertex $x \in Q_0$, a vector space V_x.
- For every arrow $\rho \in Q_1$, a k-linear map $V_{s(\rho)} \to V_{t(\rho)}$.

A morphism f between k-linear representations V and W consists of, for each vertex $x \in Q_0$ a linear map $f_x \colon V_x \to W_x$, such that for edge $s \in Q_1$, the square

$$\begin{tikzcd} V_{s(\rho)} \arrow[r] \arrow[d] & V_{t(\rho)} \arrow[d] \\ W_{s(\rho)} \arrow[r] & W_{t(\rho)} \end{tikzcd}$$

commutes.

This allows us to define a category $\mathrm{Rep}_k(Q)$ of k-linear representations of Q.

Following our grand plan, we first find a way of turning a quiver representation into a ring.

First, we form the free category over the quiver Q, which we denote by $\mathcal{F}(Q)$; that is, the objects of the category $\mathcal{F}(Q)$ are the vertices of Q, and the morphisms $x \to y$ consist of finite chains of arrows starting at y and ending at x.

$$\begin{tikzcd} & z_1 \arrow[dr, "\rho_1"] & & & & z_5 \arrow[dr, "\rho_5"] & \\ y \arrow[ur, "\rho_0"] & & z_2 \arrow[dr, "\rho_2"] & & z_4 \arrow[ur, "\rho_4"] & & x \\ & & & z_3 \arrow[ur, "\rho_3"] & & & \end{tikzcd}$$

We will denote this morphism by

$$\rho_5\rho_4\rho_3\rho_2\rho_1\rho_0.$$

We call such a finite non-empty chain a *non-trivial path.* Note that in addition to the non-trivial paths, we must formally adjoin identity arrows e_x with $s(e_x) = t(e_x) = x$; these are the *trivial paths.* Note that the notation introduced above is a good one, since in the category $\mathcal{F}(Q)$ composition is given by

$$(\rho_n \cdots \rho_1) \circ (\rho'_m \cdots \rho'_1) = \rho_n \cdots \rho_1 \rho'_m \cdots \rho'_1.$$

Definition 4 (path algebra)**.** Let k be a field and Q a quiver. The path algebra of Q over k is, as a vector space, the free vector space over the set $\mathrm{Mor}(\mathcal{F}(Q))$. The multiplication is given by composition

$$\rho \cdot \sigma = \begin{cases} \rho\sigma, & s(\rho) = t(\sigma) \\ 0, & s(\rho) \neq t(\sigma). \end{cases}$$

We will denote the path algebra of Q over k by kQ.

Another way of expressing the above composition rule is as follows: the multiplication of two paths ρ, σ in kQ is given by the composition $\rho \circ \sigma$ in the category $\mathcal{F}(Q)$ if the morphisms ρ and σ are composable (in the sense that ρ begins where σ ends) and zero otherwise.

From now on, we fix (arbitrarily) a bijection of Q with $\{1, 2, \cdots, n\}$. It is clear that

$$\mathrm{id}_{kQ} = e_1 + \cdots + e_n.$$

Example 5. Consider the following quiver.

$$\begin{tikzcd} \bullet_1 \arrow[loop right, "\rho"] \end{tikzcd}$$

Then paths are in correspondence with natural numbers, with the empty path being e_1. The path algebra is therefore

$$kQ \cong ke_1 \oplus k\rho \oplus k\rho\rho \oplus k\rho\rho\rho \oplus \cdots \cong k[\rho].$$

Example 6. Let Q be any quiver, and denote by $A = kQ$ its path algebra. For any vertex i and any path ρ, the multiplication law for the path algebra tells us that

$$\rho e_i = \begin{cases} e_i, & s(\rho) = i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the left ideal Ae_i consists of those paths which start at i. Similarly, the right ideal e_iA consists of those paths which terminate at i.

Since any path either starts at i or doesn't, we have

$$A \cong Ae_i \oplus A(1 - e_i).$$

This means (by Proposition **??**) that each Ae_i is a projective module. The same argument tells us that e_iA is projective for all i.

Proposition 7. Let Q be a quiver and k its path algebra. There is an equivalence of categories

$$\mathrm{Rep}_k(Q) \xrightarrow{\simeq} kQ\text{-}\mathbf{Mod}.$$

Proof. First, we construct out of any k-linear represenation V of Q a kQ-module. Our underlying space will be the vector space $\bigoplus_{i \in Q_0} V_i$, which we will also denote by V.

Now let $\rho \colon i \to j \in Q_0$, corresponding under the representation V to the map $f : V_i \to V_j$. We send this to the map $V \to V$ given by the matrix with f in the (i, j)th position and zeroes elsewhere. In particular, we send e_i to the matrix with a 1 in the ith place along the diagonal, and zeroes elsewhere.

It is easy to see that this assignment respects addition and composition, hence defines our functor on objects. On objects, we send $\alpha_i \colon V_i \to W_i$ to

$$\operatorname{diag}(0, \ldots, 0, \overset{i}{\alpha_x}, 0, \ldots, 0).$$

It is painfully clear that this is the right thing to do.

We now construct a functor in the opposite direction. □

Proposition 8. Let M be a left A-module. We have an exact sequence of left A-modules

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)} M \hookrightarrow \bigoplus_{i \in Q_0} Ae_i \otimes_k e_i M \twoheadrightarrow M \longrightarrow 0 \ ,$$

exhibiting a resolution of M by projective A-modules.

Corollary 9. Let M, N be left A-modules. Then

$$\operatorname{Ext}^i(M, N) = 0, \qquad i \geq 2.$$

1.2 Group (co)homology

Next, we will apply Cockamamie Scheme 1 to the study of groups. In order to do this, we need some theory.

1.2.1 The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A \colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 10 (normalized chain complex)**.** Let $A \colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2}d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1}d_{n+1}} NA_n \xrightarrow{(-1)^{n}d_{n}} NA_{n-1} \xrightarrow{(-1)^{n-1}d_{n-1}} \cdots$$

Definition 11 (Moore complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n-chains A_n and differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
\partial^2 &= \left(\sum_{j=0}^{n-1} (-1)^j d_j\right) \circ \left(\sum_{i=0}^{n} (-1)^i d_i\right) \\
&= \sum\sum (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_j \circ d_i + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= 0.
\end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A, unless this is confusing. In this case, we will denote it by MA.

Definition 12 (alternating face maps chain modulo degeneracies)**.** Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial\colon A_n \to A_{n-1}$ takes degerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain modulo degeneracies.[2]

Lemma 13. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \overset{i}{\hookrightarrow} A \overset{p}{\twoheadrightarrow} A/D(A)$$

[2] The nLab is responsible for this terminology.

Proof. We need only show that the following diagram commutes.

$$\begin{tikzcd}
NA_n \arrow[r, hook] \arrow[d, "(-1)^n d_n"'] & A_n \arrow[r, two heads] \arrow[d, "\partial"] & A_n/DA_n \arrow[d, "{[\partial]}"] \\
NA_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, two heads] & A_{n-1}/DA_{n-1}
\end{tikzcd}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n, and the right-hand square commutes trivially. □

Theorem 14. The composite

$$p \circ i \colon NA \to A/D(A)$$

is an isomorphism of chain complexes.

Let $f\colon [m] \to [n]$ be a morphism in the simplex category Δ. By functoriality, f induces a map

$$N\Delta^n \to N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0\colon [n] \to [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]\colon \Delta \to \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

and on morphisms by

$$f\colon [m] \to [n] \mapsto$$

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \tag{1.1}$$

Theorem 15 (Dold-Kan)**.** The adjunction in Equation 1.1 is an equivalence of categories

Proof. later, if I have time. □

Also later if time, Dold-Puppe correspondence: this works not only for **Ab**, but for any abelian category.

1.2.2 The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to RL$ and counit $\varepsilon\colon LR \to \mathrm{id}_{\mathcal{D}}$.

Recall that this data gives a comonad in $\mathcal{D}$, i.e. a comonoid internal to the category $\mathrm{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$LRLR \xLeftarrow{L\eta R} LR \xRightarrow{\varepsilon} \mathrm{id}_{\mathcal{D}}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category $\mathcal{C}$ and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M}\colon \Delta \to \mathcal{C}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, dashed, "\exists! \tilde{M}"] \\ & \mathcal{C} \end{tikzcd}$$

In particular, with $\mathcal{C} = \mathrm{End}(\mathcal{D})^{\mathrm{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \to \mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, "\exists!"] \\ & \mathrm{End}(\mathcal{D})^{\mathrm{op}} \end{tikzcd}$$

Equivalently, this gives a functor

$$B\colon \Delta_+^{\mathrm{op}} \to \mathrm{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\mathrm{ev}_d\colon\ \mathrm{End}(\mathcal{D}) \to \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in $\mathcal{D}$.

$$S_d = \ \Delta_+^{\mathrm{op}} \xrightarrow{B} \mathrm{End}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{D}$$

Definition 16 (bar construction)**.** The composition $\mathrm{ev}_d \circ B$ is called the bar construction.

Example 17. I believe this example comes from John Baez, although I can't find the exact source at the moment.

Consider the free-forgetful adjunction

$$F : \mathbf{Set} \longleftrightarrow \mathbf{Ab} : G.$$

The unit $\eta \colon \mathrm{id}_{\mathbf{Set}} \to U \circ F$ has components

$$\eta_S \colon S \to \mathbb{Z}[S]$$

which assigns any set S to the set underlying the free abelian group on it by mapping elements to their corresponding generators. The counit $\varepsilon \colon F \circ U \to \mathrm{id}_{\mathbf{Ab}} \to U \circ F$ has components

$$\epsilon_A \colon \mathbb{Z}[A] \to A$$

which takes formal linear combinations of group elements to actual linear combinations of group elements.

Denoting $F \circ U$ by Z, we find the following comonad in $\mathbf{Ab}$, where $\nabla = F\eta U$.

$$\begin{tikzcd}
Z \circ Z \\
Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\
\mathrm{id}_{\mathbf{Ab}}
\end{tikzcd}$$

This gives us a simplicial object in $\mathrm{End}(\mathbf{Ab})$.

Post-composing with the evaluation functor $\mathrm{ev}_{\mathbb{Z}}$ gives the following simplicial object in $\mathbf{Ab}$.

$$\begin{tikzcd}
\cdots\cdots\cdots & \mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]] \arrow[r, shift left=6, "(ZZ\varepsilon)_{\mathbb{Z}}"] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[r, shift right=6, "(\varepsilon ZZ)_{\mathbb{Z}}"'] & \mathbb{Z}[\mathbb{Z}[\mathbb{Z}]] \arrow[l, shift right=4] \arrow[l] \arrow[l, shift left=4] \arrow[r, shift left=4, "(Z\varepsilon)_{\mathbb{Z}}"] \arrow[r, shift right=4, "(\varepsilon Z)_{\mathbb{Z}}"'] & \mathbb{Z}[\mathbb{Z}] \arrow[l, "\nabla_{\mathbb{Z}}"'] \arrow[r, "\varepsilon_{\mathbb{Z}}"] & \mathbb{Z}
\end{tikzcd}$$

Let's examine some of these objects in detail.

- The first object is the friendly and reassuring $\mathbb{Z}$, whose elements are integers. An example of an element of $\mathbb{Z}$ is

  $$17.$$

- The next object, $\mathbb{Z}[\mathbb{Z}]$, consists of formal linear combinations of elements of $\mathbb{Z}$, i.e. formal sums of the form

  $$3 \times (5) + 2 \times (-1).$$

  The funny notation of enclosing the elements of (the set underlying) $\mathbb{Z}$ in parentheses will come in handy shortly.

- The next object, $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, consists of formal linear combinations of elements of $\mathbb{Z}[\mathbb{Z}]$, i.e. formal linear combinations of formal linear combinations of integers. An element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$ looks like this.

$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$

- I wonder if you can figure out for yourself what the elements of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]]$ are.

Next, we should understand the face maps in some low degrees.

Consider some element of $\mathbb{Z}[\mathbb{Z}]$, say

$$3 \times (5) + 2 \times (-1)$$

The map $\varepsilon_{\mathbb{Z}} = d_0$ collapses the formal multiplication into actual multiplication:

$$\varepsilon_{\mathbb{Z}} \colon 3 \times (5) + 2 \times (-1) \mapsto 15 - 2 = 13.$$

Consider now some element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, say

$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$

The map $\mathbb{Z}\varepsilon_{\mathbb{Z}} = d_0$ sends this to

$$3 \times (17) - 4 \times (3);$$

that is, it strips off the inner-most parentheses. The map $(\varepsilon Z)_{\mathbb{Z}} = d_1$ sends it to

$$6 \times (1) + 9 \times (5) - 4 \times (3);$$

that is, it strips off the outer-most parentheses.

Note that, as the simplicial identities promised us, $\epsilon_{\mathbb{Z}}$ sends both of these to

$$3 \times 17 - 4 \times 3 = 6 \times 1 + 9 \times 5 - 4 \times 3 = 39.$$

In the case that the category $\mathcal{D}$ is abelian, we can form from a simplicial object the chain complex of alternating face maps modulo degeneracies.

Definition 18 (bar complex)**.** Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b \colon \Delta^{\mathrm{op}} \to \mathbf{Set}$ the bar construction of b.

The bar complex Bar_b of b is the chain complex of face maps modulo degeneracies (Definition 12) of S_b.

We will see an example of the bar complex in the next chapter.

1.2.3 Invariants and coinvariants

Now we are ready to apply Cockamamie Scheme 1 to the study of groups. Here we have a somewhat popular method of turning a group G into a ring, namely by constructing the group ring $\mathbb{Z}G$. As is tradition, we will study modules over this ring.

In Section 1.1, we had a nice interpretation of modules over our rings, as corresponding to k-linear representations of a quiver. We will work backwards, finding another perspective on modules over the group ring $\mathbb{Z}G$ of a group G as *G-modules.*

Given a group G, a G-module is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-$\mathbf{Mod}$.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in $\mathbf{Ab}$, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the obvious equivalence G-$\mathbf{Mod} \simeq \mathbb{Z}G$-$\mathbf{Mod}$, we know that G-$\mathbf{Mod}$ has enough projectives.

Note that there is a canonical functor $\mathrm{triv}\colon \mathbf{Ab} \to \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 19. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \mathrm{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \mathrm{Hom}_{G\text{-}\mathbf{Mod}}(M, \mathrm{triv}\, A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \to A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \to A$ such that $\hat{\alpha}(g \cdot m - g) = 0$

for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \to A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{1.2}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \to \operatorname{triv} A$. In fact, G-linear maps $M \to \operatorname{triv} A$ are precisely those satisfying Equation 1.2, showing that this is really an isomorphism.

The other case is similar. □

Definition 20 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Clearly, the functors taking invariants and coinvariants are interesting things that deserve to be studied in their own right, and in keeping with Cockamamie Scheme 1 we could procede by deriving them. However, now something of a miracle occurs: these functors are really special cases of the hom and tensor product!

Lemma 21. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\text{-}\mathbf{Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula $\mathbb{Z}$ is taken to be a trivial right $\mathbb{Z}G$-module, and in the second a trivial left $\mathbb{Z}G$-module.

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

Lemma 21 means that we already have a good handle on the derived functors of $(-)_G$ and $(-)^G$: they are given by the Ext and Tor functors we have already seen. More specifically,

$$L_i(-)_G = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad R^i(-)^G = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$

This is a stroke of luck. Had we proceded naïvely by forming the right and left derived functors of invariants and coinvariants using Definition 20, we would have have been stuck picking resolutions for each object under consideration, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[3]

[3] In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \to G$-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
   - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.
   - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived functors $R^i(-)^G$.
4. We denote
$$L_i(-)_G = H_i(G,-), \qquad R^i(-)^G = H^i(G,-),$$
and call them *group homology* and *group cohomology* respectively.
5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have
$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$
and thus that
$$H_i(G,A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$
This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 22. Let $G = \mathbb{Z}$, and denote the generator by t. Suppose we want to compute $H_*(G;A)$ and $H^*(G;A)$ for some $\mathbb{Z}G$-module A.

Thanks to Lemma 21, instead of computing a projective resolution of some specific A, we may compute a projective resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

$$0 \longrightarrow \mathbb{Z}G \xhookrightarrow{t-1} \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

This gives us a free resolution $F_\bullet \xrightarrow{\simeq} \mathbb{Z}$.

By definition,

$$H_i(G;A) = H_i(F_\bullet \otimes_{\mathbb{Z}G} A) = H_i\left(0 \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}G} A \xrightarrow{f} \mathbb{Z}G \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right).$$

The map f takes $t \otimes a \mapsto a - t \otimes a$; the homology $H_0(G;A)$ is thus

$$\operatorname{coker} f = A/\langle t \cdot a - a \mid a \in A \rangle = A_G.$$

Similarly, we have

$$H_1(G;A) = \ker f = A^G.$$

For $n > 1$, we have $H_n(G;A) = 0$.

Now turning our attention to group cohomology, we find by definition

$$H^n(G;A) = H^n\left(0 \longrightarrow \operatorname{Hom}(\mathbb{Z}G, A) \xrightarrow{g} \operatorname{Hom}(\mathbb{Z}G, A) \longrightarrow 0 \right),$$

where the map g sends

$$g\colon\ (f : t \mapsto f(t)) \mapsto (gf : t \mapsto f(t-1))\,.$$

Note that under the correspondence $f \mapsto f(1)$, we have

$$\operatorname{Hom}(\mathbb{Z}G; A) \cong A.$$

Having made this identification, g simply sends $f \mapsto tf - f$. Thus, similarly to before, we have

$$H^0(G;A) = \ker(g) = A^G, \qquad H^1(G;A) = \operatorname{coker}(g) = A_G,$$

and $H^n(G;A) = 0$ for $n > 1$.

1.2.4 The bar construction for group rings

In the last section, we saw that we could compute group homology and cohomology of a group G by finding a projective resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module. This is nice, but computing projective resolutions is in general rather difficult.

It turns out that the bar construction (explored in Subsection 1.2.2) computes a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module for any group G.

Consider the functor

$$U : \mathbb{Z}G\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

which forgets multiplication. This has left adjoint

$$\mathbb{Z}G \otimes_{\mathbb{Z}} - \colon \mathbf{Ab} \to \mathbb{Z}G\text{-}\mathbf{Mod},$$

which assigns to an abelian group A the left $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta\colon \mathrm{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_Z -)$ and counit $\varepsilon\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components

$$\eta_A \colon A \to U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \qquad a \mapsto 1 \otimes a$$

and

$$\varepsilon_M \colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \to M; \qquad g \otimes m \mapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z, we find the following comonad in $\mathbb{Z}G$-**Mod**, where $\nabla = \mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{tikzcd}
Z \circ Z \\
Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\
\mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

This gives us a simplicial object in $\mathrm{End}(\mathbb{Z}G\text{-}\mathbf{Mod})$.

$$\begin{tikzcd}
\cdots\cdots\cdots\; Z \circ Z \circ Z
\arrow[r, Rightarrow, shift left=6, "ZZ\varepsilon"]
\arrow[r, Leftarrow, shift left=4]
\arrow[r, Rightarrow, shift left=2]
\arrow[r, Leftarrow]
\arrow[r, Rightarrow, shift right=2]
\arrow[r, Leftarrow, shift right=4]
\arrow[r, Rightarrow, shift right=6, "\varepsilon ZZ"']
& Z \circ Z
\arrow[r, Rightarrow, shift left=3, "Z\varepsilon"]
\arrow[r, Leftarrow, "\nabla" description]
\arrow[r, Rightarrow, shift right=3, "\varepsilon Z"']
& Z \arrow[r, Rightarrow] & \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}
\end{tikzcd}$$

Evaluating on a specific $\mathbb{Z}G$-module M then gives a simplicial object in $\mathbb{Z}G$-**Mod**. We continue in the special case $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$-module.

$$\begin{tikzcd}
\cdots\cdots\cdots\; \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}
\arrow[r, shift left=6, "d_0"]
\arrow[r, leftarrow, shift left=4]
\arrow[r, shift left=2]
\arrow[r, leftarrow]
\arrow[r, shift right=2]
\arrow[r, leftarrow, shift right=4]
\arrow[r, shift right=6, "d_2"']
& \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_Z \mathbb{Z}
\arrow[r, shift left=3, "d_0"]
\arrow[r, leftarrow, "\nabla" description]
\arrow[r, shift right=3, "d_1"']
& \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

We are justified in notationally suppressing the last copy of $\mathbb{Z}$.

The face map d_i removes the ith copy of Z; on a generator, we have the action of d_i given by

$$d_i\colon x\otimes x_1\otimes\cdots\otimes x_n \mapsto \begin{cases} xx_1\otimes\cdots\otimes x_n, & i=0\\ x\otimes x_1\otimes\cdots\otimes x_ix_{i+1}\otimes\cdots\otimes x_n, & 0<i<n+1\\ x\otimes\cdots\otimes x_{n-1}, & i=n+1.\end{cases}$$

and the degeneracy maps s_i send

$$x\otimes x_1\otimes\cdots\otimes x_{n-1}\mapsto x\otimes x_1\otimes\cdots\otimes x_{i-1}\otimes 1\otimes x_i\otimes\cdots\otimes x_{n-1}.$$

It is traditional to use the notation

$$x\otimes x_1\otimes\cdots\otimes x_n = x[x_1|\cdots|x_n].$$

This notation is the origin of the name *bar construction.*

We now form the bar complex; the objects B_n are given by

$$B_n = \mathbb{Z}G^{\otimes n}/\{\text{degenerate simplices}\};$$

that is, whenever we see something of the form

$$x[x_1|\cdots|1|\cdots|x_n],$$

we set it to zero.

The elements of B_n are simply formal $\mathbb{Z}$-linear combinations of the $x[x_1|\cdots|x_n]$. Therefore, we can also write

$$B_n = \bigoplus_{(g_i)\in(G\smallsetminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n], \qquad n\geq 0.$$

That is, we can write the bar complex as follows.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\smallsetminus\{1\})^2}\mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g\in G\smallsetminus\{1\}}\mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow \mathbb{Z}$$

To summarize, we have the following definition.

Definition 23 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module

$$B_n = \bigoplus_{(g_i)\in(G\smallsetminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$

where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to

$(g_1, \ldots, g_n)$. The differential is defined by

$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

Proposition 24. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
& B_n \arrow[dl, dashed, "h_n"] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}$$

gives us in one direction

$$
\begin{tikzcd}
g[g_1|\cdots|g_n] \arrow[r, mapsto] & \begin{array}{r} gg_1[g_2|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i g[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n g[g_1|\cdots|g_{n-1}] \end{array} \arrow[dl, mapsto] \\
\begin{array}{r} [gg_1|\cdots|g_n] \\ + \sum_{i=1}^{n-1}(-1)^i [gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ + (-1)^n [gg_1|\cdots|g_n] \end{array} &
\end{tikzcd}
$$

and in the other

$$
\begin{tikzcd}
& g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\
{[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{r} g[g_1|\cdots|g_n] \\ - [gg_1|g_2|\cdots|g_n] \\ - \sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ - (-1)^n[gg_1|\cdots|g_{n-1}] \end{array}
\end{tikzcd}
.
$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. $\square$

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G, A) = H^i\left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G, A) = H_i\left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

1.2.5 Computations in low degrees

First homology and abelianizations

Let G be a group, and consider $\mathbb{Z}$ as a free $\mathbb{Z}G$-module. Let us compute $H_1(G, \mathbb{Z})$.

To compute $H_1(G, \mathbb{Z})$, consider the first few terms in the bar complex for G.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\setminus\{1\})^2} \mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g\in G\setminus\{1\}} \mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow 0$$

Tensoring over $\mathbb{Z}G$ with $\mathbb{Z}$ is the same as trivializing the action, giving us the following complex.

$$\cdots \longrightarrow \bigoplus_{(g_1,g_2)\in(G\setminus\{1\})^2} \mathbb{Z}[g_1|g_2] \longrightarrow \bigoplus_{g\in G\setminus\{1\}} \mathbb{Z}[g] \longrightarrow \mathbb{Z}[\cdot] \longrightarrow 0$$

The differential $d_1 \otimes \mathrm{id}_{\mathbb{Z}}$ acts on a generator $[g]$ by sending it to

$$[\cdot]g - [\cdot] = [\cdot] - [\cdot] = 0,$$

implying that everything is a 1-cycle. The differential d_2 sends a generator $[g|h]$ to

$$[g]h - [gh] + [h] = [g] + [h] - [gh].$$

Thus, taking cycles modulo boundaries is the same as

$$\{[g] \in G\}/\langle [gh] - [g] - [h] \rangle \cong G_{\mathrm{ab}}.$$

Thus, $H_1(G, \mathbb{Z})$ is simply the abelianization of G.

First cohomology and semidirect products

To compute $H^1(G, A)$, consider the sequence

$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot], A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(\textstyle\bigoplus_{g\in G\setminus\{1\}} \mathbb{Z}G[g], A) \xrightarrow{(d_2)^*} \cdots .$$

Notice immediately that since the hom functor preserves direct sums in the first slot, we have

$$\mathrm{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i)\in(G\setminus\{1\})^n} \mathbb{Z}G[g_1|\cdots|g_n], A\right) \cong \bigoplus_{(g_i)\in(G\setminus\{1\})^n} \mathrm{Hom}_{\mathbb{Z}G}\left(\mathbb{Z}G[g_1|\cdots|g_n], A\right)$$

An element of $\mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1|\cdots|g_n])$ is determined completely by where it sends the generator of $\mathbb{Z}G$. We can therefore re-package the RHS as consisting of functions

$$\phi\colon (G \setminus \{1\})^n \to A,$$

or equivalently as functions

$$\phi\colon G^n \to A, \qquad \phi(g_1, \ldots, 1, \ldots g_n) = 0.$$

The differential $d^n = (d_n)_*$ sends $\phi \mapsto d^n\phi = \phi \circ d_n$, where

$$\begin{aligned}\phi \circ d_n\colon (g_1, \ldots, g_{n+1}) \mapsto & g_1\phi(g_2, \ldots, g_{n+1}) \\ & + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}) \\ & + (-1)^{n+1}\phi(g_1, \ldots, g_n).\end{aligned}$$

We will denote the nth term in Equation 1.2.5 by C^n; that is, we have

$$C^n \cong \{\phi\colon G^n \to A \mid \phi(g_1, \ldots, 1, \ldots, g_n) = 0\}.$$

Let us explicitly compute $H^1(G; A)$.

The set of 1-coboundaries corresponds to functions $\psi\colon G \to A$ with

$$d\psi\colon G^2 \to A; \qquad d\psi(g, h) = g\psi(h) - \psi(gh) + \psi(g) = 0,$$

i.e. functions

$$\phi\colon G \to A; \qquad \psi(g, h) = \psi(g) + g\psi(h).$$

We will call these *crossed homomorphisms.*

The set of 1-cocycles corresponds to the image $d^0(C^0)$, i.e. those functions

$$d\phi\colon G \to A, \qquad d\phi([g]) = g\phi([\cdot]) - \phi([\cdot]).$$

We will call these *principal crossed homomorphisms.*

Thus, in this case we have

$$H^1(G; A) = \frac{\{\text{crossed homomorphisms from } G \text{ to } A\}}{\{\text{principal crossed homomorphisms from } G \text{ to } A\}}.$$

Now suppose that A is a trivial G-module. In that case, crossed homomorphisms are simply homomorphisms $G \to A$, and principal crossed homomorphisms are zero. Thus,

$H^1(G;A)$ consists of homomorphisms $G \to A$, i.e.

$$H^1(G;A) = \mathrm{Hom}_{\mathbf{Ab}}(G_{\mathrm{ab}}, A).$$

Definition 25 (semidirect product)**.** Let G be a group, and A a G-module. The semidirect product of A and G is the group whose underlying set is $A \times G$, and whose group operation is given by

$$(a, g) \cdot (a', g') = (a + ga', gg').$$

We denote the semidirect product of A and G by $A \rtimes B$.

Note that we have a short exact sequence of groups[4]

$$0 \longrightarrow A \hookrightarrow A \rtimes G \twoheadrightarrow G \longrightarrow 1\ .$$

Let $\sigma\colon A \rtimes G \to A \rtimes G$ be a group homomorphism. We say that σ *stabilizes* A and G if the following diagram commutes.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, hook] \arrow[d, equal] & A \rtimes G \arrow[r, two heads] \arrow[d, "\sigma"] & G \arrow[r] \arrow[d, equal] & 1 \\
0 \arrow[r] & A \arrow[r, hook] & A \rtimes G \arrow[r, two heads] & G \arrow[r] & 1
\end{tikzcd}$$

Given any crossed homomorphism ϕ, we get a group homomorphism σ_ϕ by the formula

$$\sigma_\phi\colon (a, g) \mapsto (a + \phi(g), g).$$

This is an easy check; we have

$$\begin{aligned}
\sigma_\phi((a, g)(a', g')) &= \sigma_\phi(a + ga', gg') \\
&= (a + ga' + \phi(gg'), gg') \\
&= (a + \phi(g) + g(a' + \phi(g')), gg') \\
&= (a + \phi(g), g)(a' + \phi(g'), g') \\
&= \sigma_\phi(a, g)\sigma_\phi(a', g').
\end{aligned}$$

In fact, σ_ϕ trivially stabilizes A and G. This gives us a way of turning a cocycle in $Z^1(G, A)$ into an an automorphism of $A \rtimes G$ which stabilizes A and G.

Proposition 26. The map

$$\psi\colon Z^1(G;A) \to \mathrm{Aut}(A \rtimes G); \qquad \phi \mapsto \sigma_\phi$$

[4] We are cheating a bit. Since **Grp** is not an abelian category, we have no notion of an exact sequence of groups. We trust that the reader will not be confused.

is an isomorphism onto the subgroup of $\mathrm{Aut}(A \rtimes G)$

Proof. The fact that ψ is a homomorphism is trivial. We show that we have an inverse given by □

This is miserable.

Second cohomology and extensions

Definition 27 (extension of a group)**.** Let G be a group and A an abelian group. An extension of G by A is a short exact sequence of groups

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0 .$$

Two such extensions are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook, "i"] \arrow[d, equal] & E \arrow[r, two heads, "\pi"] \arrow[d] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r, hook, "i'"] & E \arrow[r, two heads, "\pi'"] & G \arrow[r] & 0 \end{tikzcd} .$$

1.2.6 Periodicity in group homology

In this section, we will prove a combinatorial version of the following result.

Fact 28. Let G be a group. Suppose G acts freely on a sphere of dimension $2k-1$. Then G has $2k$-periodic group homology.

Simplicial complexes

Definition 29 (simplicial complex)**.** Let I be a finite set. A simplicial complex K on I is a collection of subsets $K \subset \mathcal{P}(I)$ such that for $\sigma \in K$ and $\tau \subset \sigma$, $\tau \in K$.

Let K be a simplicial complex on a set I, and let K' be a simplicial complex on a set I'. A morphism $K \to K'$ is a function $f\colon I \to I'$ such that for every $\sigma \in K$, $f(\sigma) \in K'$.

We define a cosimplicial object in the category of simplicial complexes by

$$[n] \mapsto \mathcal{P}(\{0, \ldots, n\}).$$

Taking nerves gives us, for every simplicial complex K, a simplicial set $\tilde{K}$.

Note that for each $\sigma \in K_n$, there are $n!$ simplices in $\tilde{K}_n$, one for each permutation of the elements of σ. In **Set** there is not much we can do about this; however, applying the free abelian group functor $\mathcal{F}$ brings us to **Ab**, where we can mod out by this ambiguity.

We thus define a simplicial complex $\widetilde{C}(K)$ to be the alternating face maps complex of the simplicial object $\mathcal{F} \circ \tilde{K}$.

A more concrete description of what we have just done is as follows. We have

$$\widetilde{C}(K)_n = \bigoplus_{\{x_0,\ldots,x_n\} \in K_n} \mathbb{Z} e_{(x_0,\ldots,x_n)},$$

and the differential is

$$\sum_{i=0}^{n} (-1)^i d_i,$$

where the face maps d_i are the $\mathbb{Z}$-linear extensions of the maps

$$d_i \colon e_{(x_0,\ldots,x_n)} \mapsto e_{(x_0,\ldots,\widehat{x_i},\ldots,x_n)}.$$

Now that we are in **Ab**, we can get rid of the extraneous simplices by defining

$$C(K)_n = \widetilde{C}(K)_n / \langle e_{(x_1,\ldots,x_n)} - \operatorname{sign}(\sigma) e_{(x_{\sigma(1)},\ldots,x_{\sigma(n)})} \mid \sigma \in S_n \rangle.$$

It is not hard to see that the differential on $\widetilde{C}(K)$ descends to $C(K)$.

Example 30. Let $K = \mathcal{P}(\{0,1\})$. Then $e_{(0,1)} = -e_{(1,0)}$ in $C(K)_1$.

Let $f \colon K \to K$ be an automorphism of simplicial complexes K. This descends to an automorphism of chain complexes $C_\bullet(K) \to C_\bullet(K)$ defined on generators by

$$e_{(x_0,\ldots,x_n)} \mapsto e_{(f(x_0),\ldots,f(x_n))}.$$