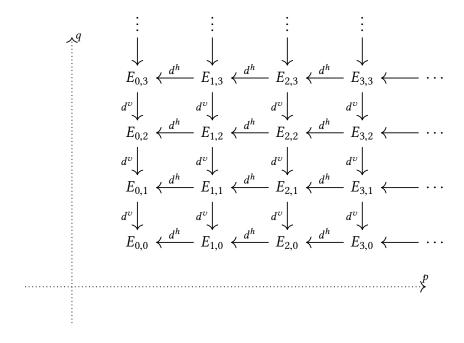
1 Spectral sequences

1.1 Motivation

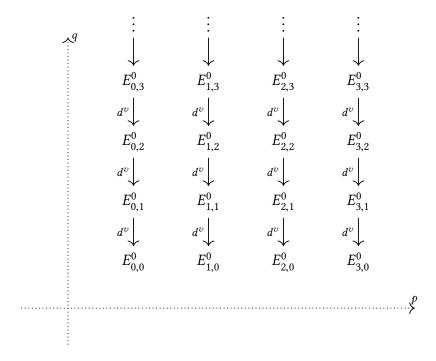
Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section ??, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the homology of the total complex such a double complex.

They do this via a series of successive 'approximations'. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.



One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

 $E^0_{\bullet,\bullet}$



One then takes homology of the corresponding vertical complexes, replacing $E_{p,q}^0$ by the homology $H_q(E_{p,\bullet}^0)$. To ease notation, one writes

$$H_q(E_{p,\bullet}^0) = E_{p,q}^1$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$E_{0,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{1,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{2,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{3,3}^{1} \longleftarrow \cdots$$

$$E_{0,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{1,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{2,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{3,2}^{1} \longleftarrow \cdots$$

$$E_{0,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{1,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{2,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{3,1}^{1} \longleftarrow \cdots$$

$$E_{0,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{1,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{2,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{3,0}^{1} \longleftarrow \cdots$$

We now write $E_{p,q}^2$ for the horizontal homology $H_p(E_{\bullet,q}^1)$.

The claim is that the terms $E_{p,q}^r$ are pieces of a successive approximation of the homology of the total complex Tot(E). In the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Proposition 1. Let *E* be a double complex where all but two adjacent columns are zero; that is, let *E* be a diagram consisting of two chain complexes and a morphism between them, padded by zeroes on either side.

$$\begin{array}{c}
\vdots \\
\downarrow \\
D_3 \leftarrow f_3 \qquad \downarrow \\
C_3
\end{array}$$

$$\begin{array}{c}
d_3^D \downarrow \\
D_2 \leftarrow f_2 \qquad \downarrow \\
C_2
\end{array}$$

$$\begin{array}{c}
d_2^D \downarrow \\
D_1 \leftarrow f_1 \qquad \downarrow \\
C_1
\end{array}$$

$$\begin{array}{c}
d_1^D \downarrow \\
D_0 \leftarrow C_0
\end{array}$$

$$\begin{array}{c}
f_0 \\
C_0
\end{array}$$

$$\begin{array}{c}
d_0^D \downarrow \\
D_{-1} \leftarrow C_{-1}
\end{array}$$

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}$$

$$\vdots \qquad \vdots$$

Note that we have multiplied the differentials of C by -1 so that we have a double complex.

Fix some $n \in \mathbb{Z}$, and let q = n - p.

Let T = Tot(E); that is, let T = Cone(f). Then we have a short exact sequence

$$0 \longrightarrow E_{p-1,q+1}^2 \hookrightarrow H_{p+q}(T) \longrightarrow E_{p,q}^2 \longrightarrow 0$$

Proof. Recall that Cone(f) fits into the following short exact sequence,

$$0 \longrightarrow C_{\bullet} \hookrightarrow D_{\bullet} \longrightarrow Cone(f)_{\bullet} \longrightarrow 0$$

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and that this gives us the following long exact sequence on homology.

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \operatorname{coker}(H_n(f)) \hookrightarrow H_n(\operatorname{Cone}(f)) \longrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation.

Of course, in a general situation we will not be so lucky. Let us examine the terms

$$E_{p,q}^2 = H_p(E_{\bullet,q}^1).$$

First, some notation. We will define in the obvious way $\mathbb{Z}_{p,q}^r$ and $\mathbb{B}_{p,q}^r$ so that

$$E_{p,q}^r = Z_{p,q}^r / B_{p,q}^r;$$

that is, we have

$$Z_{p,q}^r \subset E_{p,q}^{r-1}, \qquad B_{p,q}^r \subset E_{p,q}^{r-1}.$$

Proposition 2. There is an isomorphism

$$E_{p,q}^2 \cong U/V,$$

where

$$U = \{(a, b) \subset E_{p-1, q+1} \times E_{p, q} \mid d^{\upsilon}a + d^{h}b = d^{\upsilon}b = 0\}$$

and

$$V = \begin{pmatrix} (a,0) \colon d^v a = 0 \\ (d^h x, d^v x) \colon x \in E_{p,q+1} \\ (0, d^h c) \colon c \in E_{p+1,q}, d^v c = 0 \end{pmatrix} \subseteq U.$$

Proof. First we construct a homomorphism

$$\phi\colon U\to E_{p,q}^2.$$

Define

$$\phi^0 \colon U \to E^0_{p,q}; \qquad (a,b) \mapsto b.$$

Since by assumption $d^{v}b = 0$ we have that

$$d^{\upsilon}\phi^0(a,b) = d^{\upsilon}b = 0,$$

which is to say that im $\phi^0 \subset \ker d^v$, so ϕ^0 descends to $\ker d^v$, and therefore to a map

$$\phi^1 \colon U \to E^1_{p,q}; \qquad (a,b) \mapsto b + B^1_{p,q}.$$

Since

$$H_q(d^h)(b+B_{p,q}^1)=d^v(-a)+B_{p-1,q}^1=0+B_{p-1,q}^1,$$

we have that im $\phi^1 \subset \ker H_q(d^h)$, so the map ϕ^1 descends to $\ker H_q(d^h)$, and therefore to a map

$$\phi \colon U \to E_{p,q}^2; \qquad (a,b) \mapsto (b + B_{p,q}^1) + B_{p,q}^2.$$

Next, we show that ϕ is surjective. Pick some $\tilde{y} \in E_{p,q}^2$.

$$\tilde{y} = \underbrace{(y) + B_{p,q}^{1} + B_{p,q}^{2}}_{\in Z_{p,q}^{2}} + B_{p,q}^{2} \in E_{p,q}^{2}.$$

The fact that $y \in Z_{p,q}^1$ means that $d^v y = 0$. Thus, if we can find some $x \in E_{p+1,q-1}$ such that

$$d^{\upsilon}x + d^hy = 0,$$

then (x, y) will lie in U and $\phi(x, y) = \tilde{y}$, so we will be done.

The fact that $y+B_{p,q}^1\in Z_{p,q}^2$ means that $H_q(d^h)(y+B_{p,1}^1)=0+B_{p-1,q}^1$, i.e. that $d^hy\in B_{p-1,q}^1$. Thus, there exists $-x\in E_{p-1,q+1}$ such that

$$d^{\upsilon}(-x) = d^h y.$$

Thus, we have found our $(x, y) \in U$, so ϕ is surjective.

We will be done if we can show that $\ker \phi = V$.

The following calculations show that $V \subset \ker \phi$.

- $\phi(a,0) = (0 + B_{p,q}^1) + B_{p,q}^2$.
- $\phi(d^hx, d^vx) = (d^vx + B^1_{p,q}) + B^2_{p,q} = (0 + B^1_{p,q}) + B^2_{p,q}$
- $\phi(0,d^hc)=(d^hc+B^1_{p,q})+B^2_{p,q}=H_q(d^h)(c+B^1_{p+1,q})+B^2_{p,q}=\left[0\right]+B^2_{p,q}$

Now, let $(x, y) \in \ker \phi$. How in the hell...?

Definition 3 (filtration). Let C_{\bullet} be a chain complex. A filtration of C_{\bullet} is a chain of

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monomorphisms of subobjects $F_pC_{\bullet} \subset$ of C_{\bullet} .

$$\cdots \hookrightarrow F_{p-1}C_{\bullet} \hookrightarrow F_pC_{\bullet} \hookrightarrow F_{p+1}C_{\bullet} \hookrightarrow \cdots$$