1 Spectral sequences

1.1 Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section ??, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the building blocks of the homology of the total complex such a double complex.

They do this via a series of successive approximations. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
E_{0,3} \arrow[d, "d^v"'] & E_{1,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,2} \arrow[d, "d^v"'] & E_{1,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,1} \arrow[d, "d^v"'] & E_{1,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,0} & E_{1,0} \arrow[l, "d^h"'] & E_{2,0} \arrow[l, "d^h"'] & E_{3,0} \arrow[l, "d^h"'] & \cdots \arrow[l]
\end{tikzcd}
$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$E^0_{\bullet,\bullet}$

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^0_{0,3} \arrow[d, "d^v"'] & E^0_{1,3} \arrow[d, "d^v"'] & E^0_{2,3} \arrow[d, "d^v"'] & E^0_{3,3} \arrow[d, "d^v"'] \\
E^0_{0,2} \arrow[d, "d^v"'] & E^0_{1,2} \arrow[d, "d^v"'] & E^0_{2,2} \arrow[d, "d^v"'] & E^0_{3,2} \arrow[d, "d^v"'] \\
E^0_{0,1} \arrow[d, "d^v"'] & E^0_{1,1} \arrow[d, "d^v"'] & E^0_{2,1} \arrow[d, "d^v"'] & E^0_{3,1} \arrow[d, "d^v"'] \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{tikzcd}
\tag{1.1}
$$

One then takes homology of the corresponding vertical complexes, replacing $E^0_{p,q}$ by the homology $H_q(E^0_{p,\bullet})$. To ease notation, one writes

$$H_q(E^0_{p,\bullet}) = E^1_{p,q}.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$
\begin{tikzcd}
E^1_{0,3} & E^1_{1,3} \arrow[l, "H_3(d^h)"'] & E^1_{2,3} \arrow[l, "H_3(d^h)"'] & E^1_{3,3} \arrow[l, "H_3(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "H_2(d^h)"'] & E^1_{2,2} \arrow[l, "H_2(d^h)"'] & E^1_{3,2} \arrow[l, "H_2(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "H_1(d^h)"'] & E^1_{2,1} \arrow[l, "H_1(d^h)"'] & E^1_{3,1} \arrow[l, "H_1(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "H_0(d^h)"'] & E^1_{2,0} \arrow[l, "H_0(d^h)"'] & E^1_{3,0} \arrow[l, "H_0(d^h)"'] & \cdots \arrow[l]
\end{tikzcd}
\tag{1.2}
$$

We now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{\bullet,q})$.

The claim is that the terms $E^r_{p,q}$ are pieces of a successive approximation of the homology of the total complex $\mathrm{Tot}(E)$. We will prove this claim much later. However, in the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Example 1. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes $E_{0,\bullet}$ and $E_{1,\bullet}$ and a morphism $f\colon E_{1,\bullet} \to E_{0,\bullet}$, padded by zeroes on either side.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
E_{0,3} \arrow[d, "d^0_3"'] & E_{1,3} \arrow[l, "f_3"'] \arrow[d, "-d^1_3"] \\
E_{0,2} \arrow[d, "d^0_2"'] & E_{1,2} \arrow[l, "f_2"'] \arrow[d, "-d^1_2"] \\
E_{0,1} \arrow[d, "d^0_1"'] & E^{1,1} \arrow[l, "f_1"'] \arrow[d, "-d^1_1"] \\
E_{0,0} \arrow[d, "d^0_0"'] & E_{1,0} \arrow[l, "f_0"'] \arrow[d, "-d^1_0"] \\
E_{-0,1} \arrow[d] & E_{-1,1} \arrow[l, "f_{-1}"'] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Note that we have multiplied the diferentials of $E_{1,\bullet}$ by -1 so that we have a double complex. This does not enter into the discussion.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$.

Let $T = \mathrm{Tot}(E)$; that is, let $T = \mathrm{Cone}(f)$.

Recall that $\mathrm{Cone}(f)$ fits into the following short exact seqence,

$$\begin{tikzcd}
0 \arrow[r] & E_{\bullet,1} \arrow[r, hook] & E_{\bullet,0} \arrow[r, two heads] & \mathrm{Cone}(f)_\bullet \arrow[r] & 0
\end{tikzcd}$$

and that this gives us the following long exact sequence on homology.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(\mathrm{Cone}(f)) \arrow[dll, "\delta"] \\
H_n(E_{\bullet,1}) \arrow[r, "H_n(f)"] & H_n(E_{\bullet,0}) \arrow[r, "H_n(p)"] & H_n(\mathrm{Cone}(f)_\bullet) \arrow[dll, "\delta"] \\
H_{n-1}(E_{\bullet,1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \mathrm{coker}(H_n(f)) \hookrightarrow H_n(\mathrm{Cone}(f)) \twoheadrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation; that is, we have a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0 \ .$$

1.2 Homology spectral sequences

1.2.1 Notation and terminology

Definition 2 (homology spectral sequence)**.** Let $\mathcal{A}$ be an abelian category. A homology spectral sequence starting at E^a in $\mathcal{A}$ consists of the following data.

1. A family $E^r_{p,q}$ of objects of $\mathcal{A}$, defined for all integers p, q, and $r \geq a$.
2. Maps
$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r+1}$$
that are differentials in the sense that $d^r \circ d^r = 0$.
3. Isomorphisms
$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1})$$
between $E^{r+1}_{p,q}$ and the homology at the corresponding position of the E^r

We denote such a spectral sequence by E. A morphism $E' \to E$ is a family of maps

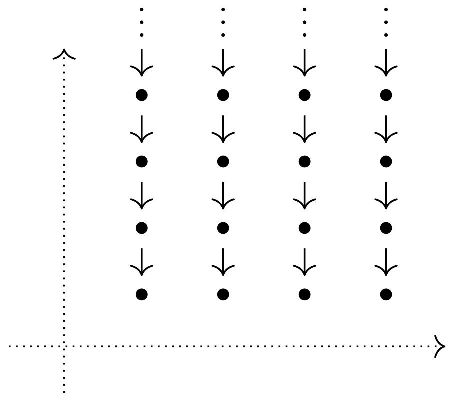
$$f^r_{p,q} \colon E'^r_{p,q} \to E^r_{p,q}$$

such that $d^r f^r = f^r d^r$ such that $f^{r+1}_{p,q}$ is induced on homology by $f^r_{p,q}$.

For each r, a spectral sequence E has a two-dimensional array of objects $E^r_{p,q}$. For a given r, one calls the objects $E^r_{p,q}$ the rth *page* of the spectral sequence E.

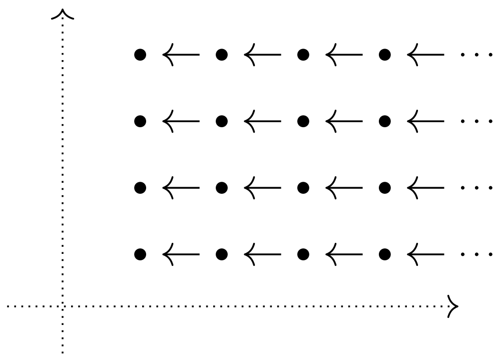
We have already seen an example of the first two pages of a spectral sequence (in Diagram 1.1 and Diagram 1.2). Schematically, the differentials on the zeroth page point down.

E^0 :



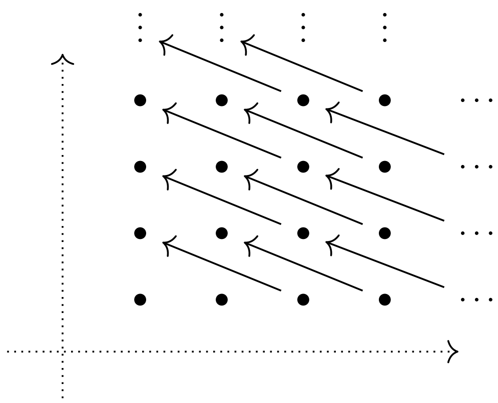
The differentials on the first page point to the left.

E^1 :



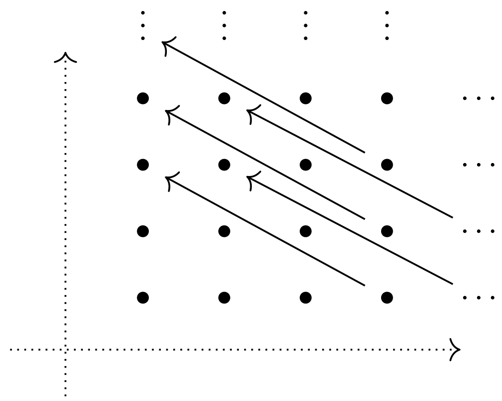
As r increases, the arrows rotate clockwise. The second page looks like this.

E^2 :



The third page looks like this.

E^3 :



1.2.2 Stabilization and convergence

We have so far placed no conditions on $E^r_{p,q}$. However, we will confine our study of certain classes of spectral sequences.

Definition 3 (bounded, first quadrant)**.** Let E be a spectral sequence starting at E^a.

- We say that E is bounded if each $n \in \mathbb{Z}$, there are only finitely many terms of total degree n on the ath page, i.e. terms of the form $E^a_{p,q}$ for $n = p + q$.
- We say that E is first-quadrant if $E^a_{p,q} = 0$ for $p < 0$ and $q < 0$.

Clearly, every first-quadrant spectral sequence is bounded.

Note that if a spectral sequence is bounded, then

We get $E^{r+1}_{p,q}$ by taking the homology of a complex on the rth page. If E is a first-quadrant spectral sequence, then for $r > \max(p, q + 1)$, the differential entering $E^r_{p,q}$ comes from the fourth quadrant (hence from a zero object), and the differential leaving it lands in the second quadrant (also a zero object). Thus, $E^{r+1}_{p,q} = E^r_{p,q}$.

Definition 4 (stabilize)**.** A spectral sequence starting at E^a is said to stabilize at position (p, q) if there exists some $r \geq a$ such that $E^{r'}_{p,q} = E^r_{p,q}$ for all $r' \geq r$. In this case, we write $E^\infty_{p,q}$ for the stable value at position (p, q).

By the argument above, each position in a first-quadrant spectral sequence eventually stabilizes. We will mostly be interested in first-quadrant spectral sequences. However, boundedness is (clearly!) sufficient to guarantee stabilization.

Suppose one has been given a bounded spectral sequence, which we now know eventually stabilizes at each position. We would like to get some useful information out of such a spectral sequence.

Definition 5 (filtration)**.** Let $C_\bullet$ be a chain complex. A filtration F of $C_\bullet$ is a chain of inclusions of subcomplexes $F_pC_\bullet$ of $C_\bullet$.

$$\cdots \subset F_{p-1}C_\bullet \subset F_pC_\bullet \subset F_{p+1}C_\bullet \subset \cdots \subset C_\bullet$$

Definition 6 (bounded convergence)**.** Let E be a bounded spectral sequence starting at E^a in an abelian category $\mathcal{A}$, and let $\{H_i\}_{i\in Z}$ be a collection of objects in $\mathcal{A}$, each equipped with a finite filtration

$$0 = F_sH_n \subset \cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset F_tH_n = H_n.$$

We say that E converges to $\{H_i\}_{i\in\mathbb{Z}}$ if we are given isomorphisms

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

We express this convergence by writing

$$E^a_{p,q} \Rightarrow H_{p+q}.$$

Example 7. Consider the spectral sequence E starting at E^0 from Example 1. This is certainly bounded, and stabilizes at each position (p,q) after the second page; that is, we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

The exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0$$

gives us, in a wholly unsatisfying and trivial way, a filtration

$$\begin{tikzcd}
F_{-1}H_n \arrow[d, equal] & F_0H_n \arrow[d, equal] & F_1H_n \arrow[d, equal] \\
0 \arrow[r, hook] & E^2_{0,n} \arrow[r, hook] & H_n
\end{tikzcd}$$

of H_n. The claim is that

$$E^0_{p,q} \Rightarrow H_{p+q}.$$

In order to check this, we have to check that

$$E^2_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q. The only non-trivial cases are when $p = 0$ or 1. When $p = 0$, we have

$$\begin{aligned} E^\infty_{0,n} &\overset{!}{\cong} F_0 H_n / F_{-1} H_n \\ &\cong E^2_{0,n}/0 \\ &\cong E^2_{0,n}, \end{aligned}$$

and when $p = 1$ we have

$$\begin{aligned} E^\infty_{1,n} &\overset{!}{\cong} F_1 H_{n+1} / F_0 H_{n+1} \\ &\cong H_{n+1}/E^2_{0,n+1} \\ &\cong E^2_{1,n} \end{aligned}$$

as required.

1.2.3 The spectral sequence of a filtered complex

Theorem 8. A filtration F of a chain complex $C_\bullet$ naturally determines a spectral sequence starting with

$$E^0_{p,q} = F_p C_{p+q} / F_{p-1} C_{p+q}, \qquad E^1_{p,q} = H_{p+1}(E^0_{p,\bullet})$$

Proof. We construct explicitly the spectral sequence above .Let $C_\bullet$ be a chain complex, and F a filtration. We will use the following notation.

- We will denote by η_p the quotient

  $$\eta_p \colon F_p C \twoheadrightarrow F_p C / F_{p-1} C.$$

  coming from the short exact sequence

  $$0 \longrightarrow F_{p-1} C \hookrightarrow F_p C \twoheadrightarrow F_p C / F_{p-1} C \longrightarrow 0$$

- We will denote by A^r_p the subobject

  $$A^r_p = \{c \in F_p C \colon dc \in F_{p-r} C\} \subset F_p C$$

  of those elements whose differentials survive r many levels of the grading.

- We will denote by Z^r_p the image

  $$Z^r_p = \eta_p(A^r_p).$$

- We will denote by B_p^r the image

$$B_p^r = \eta_p(d(a_{p+r-1}^{r-1})).$$

Examining the definitions, we see that we have the following inclusion.

$$\begin{tikzcd} dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ dA_{p+r}^{r} \arrow[r, equal] & \{dc \mid c \in F_{p+r},\ dc \in F_pC\} \end{tikzcd}$$

Both of these are subobjects of F_pC; by applying η_p and taking a sharp look at the definition of the Bs, we find an inclusion

$$B_p^r \subset B_p^{r+1}.$$

Working inductively, we are left with the following sequence of inclusions.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^r \subset \cdots$$

Defining $B_p^\infty = \bigcup_r B_p^r$, we can crown our sequence of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty.$$

Again examining definitions, we find the following inclusion.

$$\begin{tikzcd} A_p^r \arrow[r, equal] \arrow[d, hook] & \{c \in F_pC \mid dc \in F_{p-r}C\} \arrow[d, hook] \\ A_p^{r+1} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r-1}C\} \end{tikzcd}$$

As before, applying η_p and working inductively gives us a chain

$$\cdots \subset Z_p^r \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0$$

and defining $Z_p^\infty = \bigcap_r Z_p^r$ gives us

$$Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0,$$

But we should not rest on our laurels. Instead, note that for each r, r', we have an inclusion

$$\begin{tikzcd} dA^{r-1}_{p+r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ A^{r'}_p \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r'}C\} \end{tikzcd}$$

since $d^2 = 0$. Thus, we can graft our chains of inclusions as follows.

$$0 = B^0_p \subset B^1_p \subset \cdots \subset B^\infty_p \subset Z^\infty_p \subset \cdots \subset Z^1_p \subset Z^0_p = E^0_p.$$

We defined

$$Z^r_p = \eta_p(A^r_p).$$

Recall that η_p was defined to be the canoncal projection corresponding to a short exact sequence. Taking subobjects (where the left-hand square is a pullback), we find the following monomorphism of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1}C \cap A^p_r \arrow[r, hook] \arrow[d, hook] & A^p_r \arrow[r, two heads] \arrow[d, hook] & Z^p_r \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F_{p-1}C \arrow[r, hook] & F_pC \arrow[r, two heads] & F_pC/F_{p-1}C \arrow[r] & 0 \end{tikzcd}$$

Rewriting the definitions, we find that

$$F_{p-1}C \cap A^p_r = \{c \in F_{p-1}C \mid dc \in F_{p-r}C\} = A^{r-1}_{p-1},$$

so

$$Z^r_p \cong A^r_p / A^{r-1}_{p-1}.$$

We now define

$$E^r_p = \frac{Z^r_p}{B^r_p}.$$

We can write

$$\frac{Z^r_p}{B^r_p} \cong \frac{A^r_p + F_{p-1}C}{d(A^{r-1}_{p+r-1}) + F_{p-1}C} \cong \frac{A^r_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}}.$$

I don't understand this step, and I don't have time to do the rest. □

Theorem 9.

- Let $C_\bullet$ be a chain complex, and F a bounded filtration on $C_\bullet$.

1.2.4 The spectral sequence of a double complex

1.3 Hyper-derived functors

Consider abelian categories and right exact functors as follows.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

One might wonder about the relationship between LF, LG, and $L(G \circ F)$.

The problem is that we don't know how to make sense of the composition $LG \circ LF$, since LF lands in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ rather than $\mathcal{A}$; derived functors go from an abelian category to a category of chain complexes.

1.3.1 Cartan-Eilenberg resolutions

Definition 10 (Cartan-Eilenberg resolution)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $A \in \mathbf{Ch}^{+}(\mathcal{A})$. A Cartan-Eilenberg resolution of A is an upper-half-plane double complex $(P_{\bullet,\bullet}, d^h, d^v)$, equipped with an augmentation map

$$\epsilon \colon P_{\bullet,0} \to A_{\bullet}$$

such that the following criteria are satsfied.

1. $B(P_{p,\bullet}, d^h) \xrightarrow{\simeq} B(A_p, d^A)$ is a projective resolution.
2. $H(P_{p,\bullet}, d^h) \xrightarrow{\simeq} H(A_p, d^A)$ is a projective resolution.
3. $A_p \cong 0 \implies P_{p,\bullet} = 0$.

Here, $B(P_{p,\bullet}, d^h)$ are the horizontal boundaries of the chain complex $(P_{p,\bullet}, d^h)$, and

$$H(P_{p,\bullet}, d^h) \cong \frac{Z(P_{p,\bullet}, d^h)}{B(P_{p,\bullet}, d^h)}$$

is defined similarly.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & P_{-r,1} \arrow[l] \arrow[d] & \cdots \arrow[l] & P_{0,1} \arrow[l] \arrow[d] & P_{1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[u, leftarrow] \arrow[d, dashed] & P_{-r,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] & P_{0,0} \arrow[l] \arrow[d, dashed] & P_{1,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
0 & A_{-r} \arrow[l] & \cdots \arrow[l] & A_0 \arrow[l] & A_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

Proposition 11. We also have that the following are projective resolutions.

1. $Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)$
2. $P_{p,\bullet} \xrightarrow{\simeq} A_p$

Proof.

1. We have essentially by definition the following short exact sequence.

$$
0 \longrightarrow B(P_{p,\bullet}, d^h) \hookrightarrow Z(P_{p,\bullet}, d^h) \twoheadrightarrow H(P_{p,\bullet}, d^h) \longrightarrow 0
$$

   The outer terms are projective because they are the terms of projective resolutions, so $Z(P_{p,\bullet}, d^h)$ is also projective.

   LES on homology implies

$$
Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)
$$

   is a projective resolution.

2.

□

Proposition 12. Every chain complex $A_\bullet$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$.

Proof. We construct a Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ explicitly. Pick projective resolutions

$$
P^B_{p,\bullet} \to B(A_p, d^h), \qquad B^H_{p,\bullet} \xrightarrow{\simeq} H(A_p, d^h).
$$

By the horseshoe lemma, we can find a projective resolution $P^Z_{p,\bullet} \xrightarrow{\simeq} Z(A_p, d^h)$ fitting into the following exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & P^B_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^Z_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^H_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & B(A_p, d^h) \arrow[r, hook] & Z(A_p, d^h) \arrow[r, two heads] & H(A_p, d^h) \arrow[r] & 0 \end{tikzcd} \tag{1.3}$$

Playing the same game again, we find a projective resolution of A_p.

$$\begin{tikzcd} 0 \arrow[r] & P^Z_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^A_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & Z(A_p, d^h) \arrow[r, hook] & A(A_p, d^h) \arrow[r, two heads] & B(A_{p-1}, d^h) \arrow[r] & 0 \end{tikzcd} \tag{1.4}$$

We then define our Cartan-Eilenberg resolution to be the double complex whose pth column is $(P^A_{p,\bullet}, (-1)^p d^{P^A})$, and whose horizontal differentials are given by the composition

$$\begin{tikzcd} P^A_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^Z_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^A_{p-1,\bullet} \arrow[d, "\simeq"] \\ A_p \arrow[r, "d^A"] & B_{p-1} \arrow[r, hook] & Z_{p-1} \arrow[r, hook] & A_{p-1} \end{tikzcd} \tag{1.5}$$

In this case, all the criteria are satisfied by definition. □

Proposition 13. Let $A_\bullet$ and $A'_\bullet$ be bounded-below chain complexes, and let $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$ and $P'_{\bullet,\bullet} \xrightarrow{\simeq} A'_\bullet$ be Cartan-Eilenberg resolutions. Then any morphism of chain complexes $f\colon A_\bullet \to A'_\bullet$ can be lifted to a morphism $P^A_{\bullet,\bullet} \to P^B_{\bullet,\bullet}$ of double complex.

Proof. From the extended horseshoe lemma (Theorem **??**), we can f to a morphism of the chain complexes in Diagram 1.3 and Diagram 1.4. We also get a lift of Diagram 1.5, since this is made by pasting the commuting squares from the other short exact sequence. □

Proposition 14. Let $P^A_{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A_\bullet$. Then the canonical map

$$\mathrm{Tot}(P^A_{\bullet,\bullet}) \to A$$

is a quasi-isomorphism

Proof. Spectral sequence with vertical filtration. □