1 Algebraic topology

1.1 Basic definitions and examples

We assume a basic knowledge of simplicial sets just to get the ball rolling.

Definition 1 (singular complex, homology)**.** Let X be a topological space. The singular chain complex $C_\bullet(X)$ is defined level-wise by

$$C_n(X) = \mathcal{F}(\mathbf{Sing}(X)_n), \qquad C_n = 0 \quad \text{for } n < 0,$$

where $\mathcal{F}$ denotes the free group functor, and with

$$d_n \colon C_n \to C_{n-1}; \qquad (\alpha \colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(C_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f \colon X \to Y$ gives a chain map $C(f) \colon C(X) \to C(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 2. Denote by pt the one-point topological space. Then $C(X)_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & C_3 \arrow[r, "d_3"'] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Example 3. Let X be a path-connected topological space. Then $H_0(X) = 0$. To see this, consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the equivalence relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

Theorem 4 (Hurewicz)**.** For any path-connected topological space X, there is an isomorphism

$$H_1(X) \cong \pi_1(X)_{\mathrm{ab}},$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization.

Example 5. We can now confidently say that

$$H_1(S^1) = \mathbb{Z}_{\mathrm{ab}} = \mathbb{Z}.$$

Proposition 6. Let $f, g \colon X \to Y$ be continuous maps between topological spaces, and let $H \colon X \times [0, 1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 7. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 8. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

Definition 9 (relative homology)**.** Let X be a topological space, and let $X \subset A$. The relative chain complex of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(C_\bullet(X, A)).$$

Note that essentially by definition, we have a short exact sequence of chain complexes

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \longrightarrow 0 \ .$$

Example 10. Let $X = D^n$, the n-disk, and $A = S^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on homology coming from the above short exact seqence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(D^n, S^{n-1}) \arrow[dll, "\delta"] \\
H_j(S^{n-1}) \arrow[r] & H_j(D^n) \arrow[r] & H_j(D^n, S^{n-1}) \arrow[dll, "\delta"] \\
H_{j-1}(S^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that $H_j(D^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(D^n, S^{n-1}) \cong H_{j-1}(S^{n-1})$$

for $j > 1$ and $n \geq 1$.