

1 Homological algebra in abelian categories

1.1 Exact functors

Definition 1 (exact functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

is exact

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

is exact

- We call F exact if it is both left exact and right exact.

Example 2. We showed in Lemma ?? that both hom functors $\text{Hom}(A, -)$ and $\text{Hom}(-, B)$ are left exact.

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 3. Let \mathcal{A} be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

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be objects and morphisms in \mathcal{A} . If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \rightarrow B \rightarrow C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\mathrm{im} f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota: \ker g \rightarrow B$ is mapped to zero under g_* , hence is mapped to under f_* by some $\alpha: \ker g \rightarrow A$. That is, we have the following commuting triangle.

$$\begin{array}{ccc} & A & \\ \alpha \nearrow & & \searrow f \\ \ker g & \xrightarrow{\iota} & B \end{array}$$

Thus $\mathrm{im} \iota = \ker g \subset \mathrm{im} f$. □

Proposition 4. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f: A \rightarrow B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{array}{ccc} K & \longrightarrow & 0 \\ \iota \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since

initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \rightarrow A$ is a kernel of f .
2. Exactness at B means that $\text{im } f = \ker g$.
 - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g .
 - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f .
3. Exactness at C means that g is epi, i.e. $C \rightarrow 0$ is a cokernel of g .

Any functor G which is a right adjoint preserves limits. By the above reasoning, G certainly preserves zero objects and kernels. Thus, G preserves the first two conditions, which means precisely that G is left exact.

Dually, any functor F which is a left adjoint preserves colimits, hence zero objects and cokernels. Thus, F preserves the last two conditions, which means that F is right exact. \square

Note that the previous theorem makes no demand that F be additive, which may seem surprising; after all, by [Definition 1](#), only exact functors are allowed the honor of being called additive. However, it turns out that a functor which preserves either finite limits or finite colimits is always additive. To see this, note that by applying any functor which is either left or right exact to a split exact sequence gives a split exact sequence.

This is often useful when trying to check the exactness of a functor which is a left or right adjoint, by the following proposition.

Lemma 5. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

Corollary 6. $0 \longrightarrow \text{Hom}(X, A) \xrightarrow{f_*} \text{Hom}(X, B) \xrightarrow{g_*} \text{Hom}(X, C)$

is an exact sequence of abelian groups.

Proof. The hom functor \square

1.2 Chain Homotopies

Definition 7 (chain homotopy). Let $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ be morphisms of chain complexes. A chain homotopy

Lemma 8. Let $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ be a homotopic chain complexes via a homotopy h , and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 9. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \stackrel{h}{\sim} g$, then $g \stackrel{-h}{\sim} f$
- Transitivity: If $f \stackrel{h}{\sim} f'$ and $f' \stackrel{h'}{\sim} f''$, then $f \stackrel{h+h'}{\sim} f''$.

□

Lemma 10. Homotopy respects composition. That is, let $f \stackrel{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \stackrel{hr}{\sim} g \circ r$
- $r \circ f \stackrel{rh}{\sim} r \circ g$.

Proof. Obvious.

□

Definition 11 (homotopy category). Let \mathcal{A} be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in \mathcal{A} up to homotopy.

Lemma 12. The homotopy category $\mathcal{K}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$ is an additive functor.

Proof. Trivial.

□

In fact, the homotopy category satisfies the following universal property.

Proposition 13. Let $F: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{C}$ be an additive functor such that $\mathcal{F}(f)$ is an isomorphism for all quasi-isomorphisms f . Then there exists a unique functor $\mathcal{K}(\mathcal{A}) \rightarrow \mathcal{C}$ making the following diagram commute.

Proposition 14. Let $\mathcal{F}: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{B}$ be an additive functor which takes quasi-isomorphisms to isomorphisms. Then for homotopic morphisms $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, we have $\mathcal{F}(f) = \mathcal{F}(g)$.

1.3 Projectives and injectives

Definition 15 (projective, injective). An object P in an abelian category is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A , it is very easy to see that an object P in an abelian category \mathcal{A} is projective if for every epimorphism $f: B \rightarrow C$ and every morphism $p: P \rightarrow C$, there exists a morphism $\tilde{p}: P \rightarrow B$ such that the following diagram commutes.

$$\begin{array}{ccc} & P & \\ \exists \tilde{p} \swarrow & \downarrow p & \\ B & \xrightarrow{f} & C \end{array}$$

Dually, Q is injective if for every monomorphism $g: A \rightarrow B$ and every morphism $q: A \rightarrow Q$ there exists a morphism $\tilde{q}: B \rightarrow Q$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ q \downarrow & \nwarrow \exists \tilde{q} & \\ Q & & \end{array}$$

Definition 16 (enough projectives). Let \mathcal{A} be an abelian category. We say that \mathcal{A} has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Proposition 17. Let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \downarrow \mathrm{id} & & \\ 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{f} & P \longrightarrow 0 \end{array}$$

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By definition, we get a morphism $P \rightarrow B$ making the triangle commute.

$$\begin{array}{ccccccc}
 & & & & P & & \\
 & & & \swarrow \exists g & \downarrow \text{id} & & \\
 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{f} & P \longrightarrow 0
 \end{array}$$

But this says precisely that $f \circ g = \text{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma ??). \square

Corollary 18. Let F be an additive functor, and let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be an exact sequence with P projective. Then F applied to the sequence gives an exact sequence.

Proof. Applying F to a split sequence gives a split sequence, and all split sequences are exact. \square

Corollary 19. The category $R\text{-Mod}$ has enough projectives.

Proof. By Proposition ??, free modules are projective, and every module is a quotient of a free module. \square

Proposition 20. Let \mathcal{A} be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Dually, any abelian category with enough injectives has injective resolutions.

Proof. We provide running commentary on the following diagram. \square

Definition 21 (projective, injective resolution). Let \mathcal{A} be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \rightarrow \iota(A)$, where P_\bullet is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \rightarrow Q_\bullet$ where Q_\bullet is a complex of injectives.

Lemma 22. Let $f: P_\bullet \rightarrow \iota(M)$ be a projective resolution. Then $f: P_0 \rightarrow M$ is an epimorphism.

Proof. We know that $H_0(f): H_0(P) \rightarrow H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(M)) \simeq \iota_M$, and that \square

In fact, we can say more.

Lemma 23. Let $f_\bullet: P_\bullet \rightarrow \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \xrightarrow{f_0} A \rightarrow 0$$

is exact.

Theorem 24 (extended horseshoe lemma). Let \mathcal{A} be an abelian category with enough projectives, and let $P'_\bullet \rightarrow M'$ and $P''_\bullet \rightarrow M''$ be projective resolutions. Then given an exact sequence

$$0 \rightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \rightarrow 0$$

there is a projective resolution $P_\bullet \rightarrow M$ and maps \tilde{f} and \tilde{g} such that the following diagram has exact rows and commutes.

$$\begin{array}{ccccccc} 0 & \rightarrow & P'_\bullet & \xrightarrow{\tilde{f}} & P_\bullet & \xrightarrow{\tilde{g}} & P''_\bullet \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & M' & \xrightarrow{f} & M & \xrightarrow{\tilde{g}} & M'' \rightarrow 0 \end{array}$$

Furthermore, given a morphism $M \rightarrow N$

Proof. We construct P_\bullet inductively. We have specified data of the following form.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ 0 & \rightarrow & P'_1 & & P''_1 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & P'_0 & & P''_0 & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \rightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \rightarrow 0 \end{array}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{array}{ccccccc} 0 & \rightarrow & P'_0 & \xrightarrow{\iota} & P'_0 \oplus P''_0 & \xrightarrow{\pi} & P''_0 \rightarrow 0 \\ & & \downarrow p' & \searrow f \circ p'_0 & \downarrow p & \swarrow \exists q & \downarrow p'' \\ 0 & \rightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \rightarrow 0 \end{array}$$

We get the (dashed) map $P'_0 \rightarrow M$ by composition, and the (dashed) map $P''_0 \rightarrow M$ by projectivity of P''_0 (since by Lemma 22 p'' is an epimorphism). From these the universal

property for coproducts gives us the (dotted) map $p: P'_0 \oplus P''_0 \rightarrow M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of P_\bullet . Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P''_1 to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{array}{ccccccc} 0 & \longrightarrow & P'_1 & & P''_1 & \longrightarrow & 0 \\ & & \downarrow p'_1 & & \downarrow p''_1 & & \\ 0 & \longrightarrow & \ker p' & \longrightarrow & \ker p & \longrightarrow & \ker p'' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & P'_0 & \longrightarrow & P_0 & \longrightarrow & P''_0 \longrightarrow 0 \end{array}$$

The maps p'_1 and p''_1 come from the exactness of P'_\bullet and P''_\bullet . In fact, they are epimorphisms, because they are really cokernel maps in disguise. That means that we are in the same situation as before, and are justified in saying “we proceed inductively”. \square

Proposition 25. Let \mathcal{A} be an abelian category, let M and M' be objects of \mathcal{A} , and

$$P_\bullet \rightarrow M \quad \text{and} \quad P'_\bullet \rightarrow M'$$

be projective resolutions. Then for every morphism $f: M \rightarrow M'$ there exists a lift

$\tilde{f}: P_\bullet \rightarrow P'_\bullet$ making the diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\tilde{f}_\bullet} & P'_\bullet \\ \downarrow & & \downarrow \\ \iota M & \xrightarrow{\iota f} & \iota M' \end{array}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. \square

Corollary 26. For any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \rightarrow \mathcal{K}(\mathbf{Ch}_{\geq 0}\mathcal{A})$.

1.4 Mapping cones

Definition 27 (shift functor). For any chain complex C_\bullet and any $k \in \mathbb{Z}$, define the k -shifted chain complex $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \quad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 28 (mapping cone). Let $f: C_\bullet \rightarrow D_\bullet$ be a chain map. Define a new complex $\text{cone}(f)$ as follows.

- For each n , define

$$\text{cone}(f)_n = C_{n-1} \oplus D_n.$$

- Define $d_n^{\text{cone}(f)}$ by

$$d_n^{\text{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$

which is shorthand for

$$d_n^{\text{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 29. The complex $\text{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow C_\bullet \xrightarrow{\iota} \text{cone}(f)_\bullet \xrightarrow{\pi} D_\bullet \longrightarrow 0.$$

Proof. We need to specify ι and π . The morphism ι is the usual injection; the morphism π is given by minus the usual projection. \square

Corollary 30. Let $f_\bullet: C_\bullet \rightarrow D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\text{cone}(f)$ is an exact complex.

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Proof. By Corollary ??, we get a long exact sequence

$$\cdots \rightarrow H_n(X) \xrightarrow{\delta} H_n(Y) \rightarrow H_n(\text{cone}(f)) \rightarrow H_{n-1}(X) \rightarrow \cdots$$

We still need to check that $\delta = H_n(f)$. This is not hard to see (although I'm missing a sign somewhere); picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{array}{ccccc} & & & x & \\ & & & \downarrow & \\ & & (x, 0) & \xrightarrow{\quad} & -x \\ & & \downarrow & & \\ f(x) & \xrightarrow{\quad} & (0, f(x)) & & \\ \downarrow & & & & \\ [f(x)] & & & & \end{array}$$

If $\text{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n , then the maps to and from $H_n(\text{cone}(f))$ must be the zero maps, implying $H_n(\text{cone}(f)) = 0$ by exactness. \square

Proposition 31. Let \mathcal{A} be an abelian category, and let C_\bullet be a chain complex in \mathcal{A} . Let P_\bullet be a bounded below chain complex of projectives. Then any quasi-isomorphism $g: P_\bullet \rightarrow C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{array}{ccccccc} \cdots & P_1 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & 0 & \cdots \\ & \downarrow & \swarrow h_1 & \downarrow & \swarrow h_0 & \downarrow & \\ \cdots & C_1 & \xrightarrow{\quad} & C_0 & \xrightarrow{\quad} & C_{-1} & \cdots \end{array}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r =$

$f_1 - h_0 \circ d_1^P$ is not the map f_1 .

$$\begin{array}{ccccc}
 & & P_1 & & \\
 & \swarrow \exists h_1 & \downarrow r & \searrow 0 & \\
 & K & & & \\
 C_2 & \xrightarrow{d_2^C} & C_1 & \xrightarrow{d_1^C} & C_0
 \end{array}$$

Because

$$\begin{aligned}
 d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
 &= d_1^C \circ f_1 - d_1^C \circ f_1 \\
 &= 0,
 \end{aligned}$$

the morphism r factors through K . This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1 .

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
 d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
 &= f_1
 \end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f .

Now let $f: C_\bullet \rightarrow P_\bullet$ be a quasi-isomorphism, where P_\bullet is a bounded-below projective complex. Since f is a quasi-isomorphism, $\text{cone}(f)$ is exact, which means that $\iota: P \rightarrow \text{cone}(f)$ is homotopic to the zero morphism.

$$\begin{array}{ccccc}
 P_{i+1} & \xrightarrow{d_{i+1}^P} & P_i & \xrightarrow{d_i^P} & P_{i-1} \\
 \downarrow \iota_{i+1} & \searrow \tilde{h}_i & \downarrow \iota_i & \searrow \tilde{h}_{i-1} & \downarrow \iota_{i-1} \\
 C_i \oplus P_{i+1} & \xrightarrow{d_{i+1}^{\text{cone}(f)}} & C_{i-1} \oplus P_i & \xrightarrow{d_i^{\text{cone}(f)}} & C_{i-2} \oplus P_{i-1}
 \end{array}$$

That is, there exist $\tilde{h}_i: P_i \rightarrow \text{cone}(f)_{i+1}$ such that

$$d_{i+1}^{\text{cone}(f)} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \quad (1.1)$$

Writing \tilde{h}_i in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i: P_i \rightarrow C_i \quad \text{and} \quad \gamma_i: P_{i-1} \rightarrow P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet: P_\bullet \rightarrow C_\bullet$. Indeed, writing [Equation 1.1](#) in components, we find the following.

$$\begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \\ \end{pmatrix} = \begin{pmatrix} 0 \\ \text{id}_{P_i} \end{pmatrix}$$

$$\begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \text{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that¹

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \text{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\text{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\text{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which implies that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

1.5 Localization at weak equivalences: a love story

We can embed any abelian category \mathcal{A} into the corresponding category $\text{Ch}(\mathcal{A})$ of chain complexes via inclusion into degree zero. This is obviously a lossless procedure, in the sense that we recover \mathcal{A} by restricting to degree 0.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota} & \text{Ch}(\mathcal{A}) \\ & \xleftarrow{(-)_0} & \end{array}$$

Equivalently, we recover our category \mathcal{A} by taking zeroth homology.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\iota} & \text{Ch}(\mathcal{A}) \\ & \xleftarrow{H_0} & \end{array}$$

However, we notice that there are much better-behaved embeddings of \mathcal{A} into $\text{Ch}(\mathcal{A})$ such that we recover \mathcal{A} when taking zeroth homology. For example, taking projective resolutions of objects and lifting morphisms using [Proposition 25](#) will do the trick. Unfortunately, this is not in general uniquely defined: one can take many different pro-

¹Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

jective resolutions, and lift morphisms in many different ways. There is no reason to expect these data to assemble themselves functorially.

However, if we were to modify $\mathbf{Ch}(\mathcal{A})$ in such a way that all quasi-isomorphisms became bona-fide isomorphisms, then we would have this functoriality, thanks to [Proposition 14](#).

Thus we find ourselves in a common situation: we have a category $\mathbf{Ch}(\mathcal{A})$, and identified a collection of morphisms inside this category which we would like to view as weak equivalences, namely the quasi-isomorphisms. In an ideal world, we would simply promote quasi-isomorphisms to bona fide isomorphisms. That is, we would like to form the localization

$$\mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}(\mathcal{A}) \twoheadrightarrow \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. In fact, this is universal in the sense that every other additive functor sending quasi-isomorphisms to isomorphisms factors through it. In particular, the functor

$$\mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}]$$

factors through it.

But now we are in a really wonderful position, as long as \mathcal{A} has enough projectives: we can (by REF we have already proved) find a projective resolution functor

$$P: \mathcal{A} \rightarrow \mathcal{K}^+(\mathcal{A}).$$

In fact, this functor is an equivalence of categories.

1.6 Homological δ -functors and derived functors

We have noticed that we can

Definition 32 (derived functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and \mathcal{A} has enough projectives, we declare the left derived functor to be the cohomological δ -functor $\{L_n F\}$ defined by

$$L_n F = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

where $P: \mathcal{A} \rightarrow \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and \mathcal{A} has enough injectives, we define the right derived functor to be the cohomological δ -functor

$$R^n F(X) = H^n \circ F \circ Q$$

where Q is an injective resolution functor.

It may seem that we still have something left to prove; after all, we have not shown that the result of our composition is independant of the projective/injective resolution functor we used. However,

Because $\{H_n\}$ is a homological δ -functor, it is trivial that $L_n F$ is a homological delta-functor. Note that it really does extend F in the following sense.

Proposition 33. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0 F \simeq F.$$

Proof. Let $P_\bullet \rightarrow X$ be a projective resolution. By [Lemma 23](#), the following sequence is exact.

$$P_2 \longrightarrow P_1 \xrightarrow{f} P_0 \longrightarrow\!\!\!\twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_2) \longrightarrow F(P_1) \xrightarrow{F(f)} F(P_0) \longrightarrow\!\!\!\twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \text{coker}(F(f))$. But

$$\text{coker}(H(f)) \simeq H_0(P_\bullet) = L_0 F(X).$$

Similarly, on morphisms □

Definition 34 (homological δ -functor). A homological δ -functor between abelian categories \mathcal{A} and \mathcal{B} consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n: \mathcal{A} \rightarrow \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{A} and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n: T_n(C) \rightarrow T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & T_{n+1}(C) \\
 & & & & & \searrow \delta & \downarrow \\
 & & & & T_n(A) & \xrightarrow{T_n(f)} & T_n(B) \xrightarrow{T_n(g)} T_n(C) \\
 & & & & & \searrow \delta & \downarrow \\
 & & & & T_{n-1}(A) & \xrightarrow{T_{n-1}(f)} & T_{n-1}(B) \xrightarrow{T_{n-1}(g)} T_{n-1}(C) \\
 & & & & & \searrow \delta & \downarrow \\
 & & & & T_{n-2}(A) & \longrightarrow & \cdots
 \end{array}$$

3. For every morphism of short exact sequences

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & f \downarrow & & g \downarrow & & h \downarrow & & \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
 \end{array}$$

and every n , the diagram

$$\begin{array}{ccc}
 T_n(C) & \xrightarrow{\delta_n} & T_{n-1}(A) \\
 T_n(h) \downarrow & & \downarrow T_n(f) \\
 T_n(C') & \xrightarrow{\delta_n} & T_{n-1}(A')
 \end{array}$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

This seems like an inelegant definition, and indeed it is.

Definition 35 (cohomological δ -functors). Dual to [Definition 34](#).

Example 36. The prototypical example of a homological δ -functor is homology: the collection $H_n: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$, together with the collection of connecting homomorphisms,

is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors*, come from homology.

1.7 Double complexes

Definition 37 (double complex). Let \mathcal{A} be an abelian category. A double complex in \mathcal{A} consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d_{i,j}: M_{i,j} \rightarrow M_{i-1,j}, \quad \delta_{i,j}: M_{i,j} \rightarrow M_{i,j-1}$$

subject to the following conditions.

- $d^2 = 0$
- $\delta^2 = 0$
- $d\delta + \delta d = 0$

We say that $M_{\bullet,\bullet}$ is first-quadrant if $M_{i,j} = 0$ for $i, j < 0$.

That is, for M a double complex the following diagram anticommutes rather than commutes.

$$\begin{array}{ccccccc} M_{0,3} & \longleftarrow & M_{1,3} & \longleftarrow & M_{2,3} & \longleftarrow & M_{3,3} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_{0,2} & \longleftarrow & M_{1,2} & \longleftarrow & M_{2,2} & \longleftarrow & M_{3,2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_{0,1} & \longleftarrow & M_{1,1} & \longleftarrow & M_{2,1} & \longleftarrow & M_{3,1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M_{0,0} & \longleftarrow & M_{1,0} & \longleftarrow & M_{2,0} & \longleftarrow & M_{3,0} \end{array}$$

Given any complex of complexes

$$\left(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \right) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

one can construct a double complex by multiplying every other differential by -1 .

Definition 38 (total complex). Let $M_{\bullet,\bullet}$ be a first-quadrant² double complex. The total

²This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

complex $\text{Tot}(M)_\bullet$ is defined level-wise by

$$\text{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\text{Tot}} = d + \delta$.

Lemma 39. The total complex really is a complex, i.e.

$$d_{\text{Tot}} \circ d_{\text{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\text{Tot}} \circ d_{\text{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Theorem 40. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\text{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex, without loss of generality with exact rows.

$$\begin{array}{ccccccc}
 & & M_{3,0} & & & & \\
 & & \downarrow d_{3,0} & & & & \\
 & & M_{2,0} & \xleftarrow{\delta_{2,1}} & M_{2,1} & & \\
 & & \downarrow d_{2,0} & & \downarrow d_{2,1} & & \\
 & & M_{1,0} & \xleftarrow{\delta_{1,1}} & M_{1,1} & \xleftarrow{\delta_{1,2}} & M_{1,2} \\
 & & \downarrow d_{1,0} & & \downarrow d_{1,1} & & \downarrow d_{1,2} \\
 & & M_{0,0} & \xleftarrow{\delta_{0,1}} & M_{0,1} & \xleftarrow{\delta_{0,2}} & M_{0,2} & \xleftarrow{\delta_{0,3}} & M_{0,3}
 \end{array}$$

We want to show that the total complex

$$\begin{array}{ccc}
 M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} & = & \text{Tot}(M)_3 \\
 \downarrow & & \downarrow \\
 M_{2,0} \oplus M_{1,1} \oplus M_{0,2} & = & \text{Tot}(M)_2 \\
 \downarrow & & \downarrow \\
 M_{1,0} \oplus M_{0,1} & = & \text{Tot}(M)_1 \\
 \downarrow & & \downarrow \\
 M_{0,0} & = & \text{Tot}(M)_0 \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

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is exact.

The construction is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangeable, we construct the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \text{Tot}(M)_2$ which maps to 0 under d_2^{Tot} ; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \quad d_{1,1}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \text{Tot}(M)_3$$

such that $d_3^{\text{Tot}}n = m$.

We make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always find $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta dn_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker \delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta dn_{1,2},$$

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{0,3}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned} d^{\text{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d + \delta)n_{2,1} + (d + \delta)n_{1,2} + (d + \delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. \square

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain

$$C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_{\bullet}$$

hence (by inserting appropriate minus signs) a double complex $C_{\bullet,\bullet}^D$;

- As a morphism between double complexes, hence between totalizations

$$\mathrm{Tot}(\alpha)_{\bullet}: \mathrm{Tot}(C)_{\bullet} \rightarrow \mathrm{Tot}(D)_{\bullet}.$$

Lemma 41. These two points of view agree in the sense that

$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. \square

Theorem 42. Let $A_{\bullet} \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{2,\bullet} & \longrightarrow & M_{1,\bullet} & \longrightarrow & M_{0,\bullet} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \alpha_{\bullet} \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A_{\bullet} \end{array}$$

be a resolution of A_{\bullet} . Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A .

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_{\bullet} \longrightarrow 0$$

is exact. By [Theorem 40](#), the total complex $\mathrm{Tot}(M^A)$ is exact. But by [Lemma 41](#) the total complex is equivalently the cone of α_{\bullet} . However, we have seen (in [Corollary 30](#)) that exactness of $\mathrm{cone}(\alpha_{\bullet})$ means that α_{\bullet} is a quasi-isomorphism. \square

Corollary 43. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

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Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \text{coker } d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{array}{ccccccc}
 M_{0,3} & \longleftarrow & M_{1,3} & \longleftarrow & M_{2,3} & \longleftarrow & M_{3,3} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,2} & \longleftarrow & M_{1,2} & \longleftarrow & M_{2,2} & \longleftarrow & M_{3,2} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,1} & \longleftarrow & M_{1,1} & \longleftarrow & M_{2,1} & \longleftarrow & M_{3,1} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,0} & \longleftarrow & M_{1,0} & \longleftarrow & M_{2,0} & \longleftarrow & M_{3,0} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_0(M_{0,\bullet}) & \longleftarrow & H_0(M_{1,\bullet}) & \longleftarrow & H_0(M_{2,\bullet}) & \longleftarrow & H_0(M_{3,\bullet}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now [Theorem 42](#) gives us the result we want. □

1.8 Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let \mathcal{A} be an abelian category and let

$$- \otimes -: \mathcal{A}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{H},$$

be an additive functor, where \mathcal{H} is some abelian category equipped with an exact functor to \mathbf{Ab} . Further suppose that

Definition 44 (tensor product of chain complexes). Let C_\bullet, D_\bullet be chain complexes in \mathcal{A} . We define the tensor product of C_\bullet and D_\bullet by

$$(C \otimes D)_\bullet = \text{Tot}(C_\bullet \otimes D_\bullet),$$

where $\text{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 45 (internal hom). Let \mathcal{A} be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let C_\bullet, D_\bullet be chain complexes in \mathcal{A} . We have

1.9 The Künneth formula

This section takes place in $R\text{-Mod}$, where R is a PID. For example, everything we are saying holds in **Ab**.

Lemma 46. Let C_\bullet be a chain complex of free R -modules with trivial differential, and let C'_\bullet be an arbitrary chain complex. Then there is an isomorphism

$$\lambda: \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \cong H_n(C_\bullet \otimes C'_\bullet).$$

Proof. We write

$$Z_p = Z_p(C_\bullet), \quad B_p = B_p(C_\bullet), \quad Z'_p = Z_p(C'_\bullet), \quad B'_p = B_p(C'_\bullet),$$

The sequence

$$0 \longrightarrow Z'_q \hookrightarrow C'_q \twoheadrightarrow B'_{q-1} \longrightarrow 0$$

is exact by definition, and since $Z_p = C_p$ is by assumption free, the sequence

By definition, $Z_p = C_p$ is free. Thus the sequence

$$0 \longrightarrow Z_p \otimes Z'_q \hookrightarrow Z_p \otimes C'_q \twoheadrightarrow Z_p \otimes B'_{q-1} \longrightarrow 0$$

is exact.

The differential of the tensor product complex $C_\bullet \otimes C'_\bullet$ is

$$\begin{aligned} d_n^{\text{tot}} &= \sum_{i=0}^n d_i^C \otimes \text{id}_{C'_{n-i}} + (-1)^i \text{id}_{C_i} \otimes d_{n-i}^{C'} \\ &= \sum_{i=0}^n (-1)^i \text{id}_{C_i} \otimes d_{n-i}^C. \end{aligned}$$

Since C_i is by assumption free, tensoring with it doesn't change homology, and we have

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

and

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

□