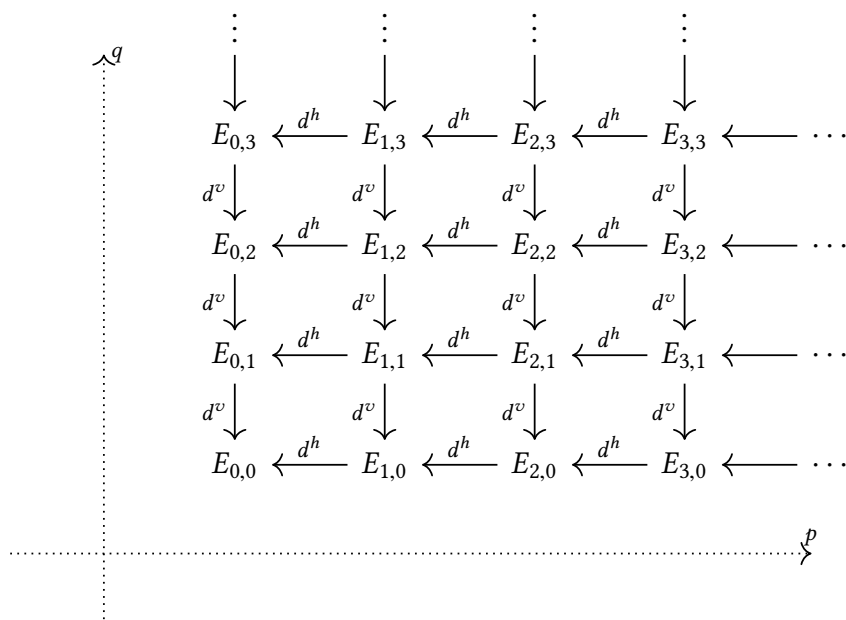


1 Spectral sequences

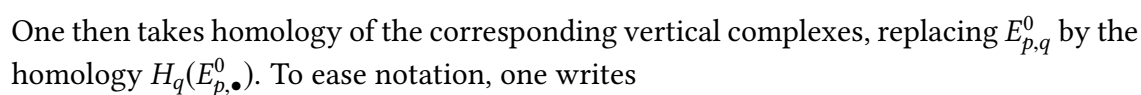
1.1 Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section ??, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the homology of the total complex of such a double complex.

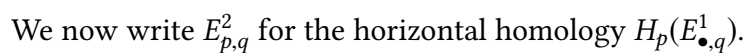
They do this via a series of successive ‘approximations’. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.



One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$$E^0_{\bullet,\bullet}$$


The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.



The claim is that the terms $E_{p,q}^r$ are pieces of a successive approximation of the homology of the total complex $\text{Tot}(E)$. In the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Proposition 1. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes and a morphism between them, padded by zeroes on either side.

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 D_3 & \xleftarrow{f_3} & C_3 \\
 d_3^D \downarrow & & \downarrow -d_3^C \\
 D_2 & \xleftarrow{f_2} & C_2 \\
 d_2^D \downarrow & & \downarrow -d_2^C \\
 D_1 & \xleftarrow{f_1} & C_1 \\
 d_1^D \downarrow & & \downarrow -d_1^C \\
 D_0 & \xleftarrow{f_0} & C_0 \\
 d_0^D \downarrow & & \downarrow -d_0^C \\
 D_{-1} & \xleftarrow{f_{-1}} & C_{-1} \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

Note that we have multiplied the differentials of C by -1 so that we have a double complex.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$.

Let $T = \text{Tot}(E)$; that is, let $T = \text{Cone}(f)$. Then we have a short exact sequence

$$0 \longrightarrow E_{p-1,q+1}^2 \hookrightarrow H_{p+q}(T) \twoheadrightarrow E_{p,q}^2 \longrightarrow 0$$

Proof. Recall that $\text{Cone}(f)$ fits into the following short exact sequence,

$$0 \longrightarrow C_{\bullet} \hookrightarrow D_{\bullet} \twoheadrightarrow \text{Cone}(f)_{\bullet} \longrightarrow 0$$

Since by assumption $d^v b = 0$ we know that $\text{im } \phi^0 \subset \ker d^v$, so ϕ^0 descends to $\ker d^v$, and therefore to a map

$$\phi^1: U \rightarrow E_{p,q}^1; \quad (a, b) \mapsto b + B_{p,q}^1.$$

Since

$$H_q(d^h)(b + B_{p,q}^1) = d^v(-a) + B_{p-1,q}^1 = 0 + B_{p-1,q}^1,$$

we have that $\text{im } \phi^1 \subset \ker H_q(d^h)$, so the map ϕ^1 descends to $\ker H_q(d^h)$, and therefore to a map

$$\phi: U \rightarrow E_{p,q}^2; \quad (a, b) \mapsto (b + B_{p,q}^1) + B_{p,q}^2.$$

Next, we show that ϕ is surjective. Let

$$\underbrace{\left(\overbrace{x}^{\in Z_{p,q}^1} + B_{p,q}^1 \right) + B_{p,q}^2}_{\in Z_{p,q}^2} \in E_{p,q}^2.$$

- The fact that $x \in Z_{p,q}^1$ means that $d^v x = 0$.
- The fact that $x + B_{p,q}^1 \in Z_{p,q}^2$ means that $H_q(d^h)x = [0]$, i.e. that $d^h x \in B_{p-1,q}^1$.

□