Homological algebra notes

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Contents

1. Introduction

Homework:

- Must submit more than 50% correct solutions in order to take the final exam (oral).
- Posted on Monday after the lecture
- Submitted on following Monday in lecture
- Discuss following Tuesday, 16:45, room 430

Part I.

Homological algebra

2. Chain complexes

Homological algebra is about chain complexes.

Definition 1 (chain complex)**.** Let R be a ring.[1] A chain complex of R-modules, denoted $(C_\bullet, d)$, consists of the following data:

- For each $n \in \mathbb{Z}$, an R-module C_n, and
- R-linear maps $d_n \colon C_n \longrightarrow C_{n-1}$, called the *differentials*,

such that the differentials satisfy the relation

$$d_{n-1} \circ d_n = 0.$$

Example 2 (Syzygies)**.** Let $R = \mathbb{C}[x_1, \ldots, x_k]$. Following Hilbert, we consider a finitely generated[2] R-module M. The ring R is Noetherian, so by Hilbert's basis theorem, M is Noetherian, hence has finitely many generators $a_1, \ldots, a_k$.

Denote by R^k the free R-module with k generators, and consider

$$\phi \colon R^k \twoheadrightarrow M; \qquad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $\mathrm{Syz}^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^{k} \lambda_i e_i \overset{\phi}{\mapsto} \sum_{i=1}^{k} \lambda_i a_i = 0.$$

This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\mathrm{Syz}^1(M)$ is again a finitely generated R-module. Hence, pick finite set of generators b_i, $1 \leq i \leq \ell$, and look at

$$R^\ell \twoheadrightarrow \mathrm{Syz}^1(M); \qquad e_i \mapsto b_i.$$

In general, there is non-trivial kernel, denoted $\mathrm{Syz}^2(M)$. This consists of relation between relations.

We can keep going. This gives us a sequence $\mathrm{Syz}^\bullet(M)$ corresponding to relations be-

[1] Associative, with unit. Not necessarily commutative.
[2] i.e. expressible as an R-linear combination of finitely many generators.

tween relations between…between relations.

$$\begin{tikzcd}
 & & \mathrm{Syz}^2(M) \arrow[dr, hook] & & & & \\
\cdots & R^m \arrow[ur, two heads] \arrow[rr, "d"] & & R^l \arrow[rr, "d"] \arrow[dr, two heads] & & R^k \arrow[r] & M \\
\mathrm{Syz}^3 \arrow[ur, hook] & & & & \mathrm{Syz}^1(M) \arrow[ur, hook] & &
\end{tikzcd}$$

From this we build a chain complex $(C_\bullet, d)$, where C_n corresponds to the nth copy of R^k, known as the *free resolution* of M.

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \ldots, x_n]$-module admits a free resolution of length at most n.

Example 3 (simplicial complexes)**.** Let $N \geq 0$. A collection $K \subset \mathcal{P}(\{0, \ldots, n\})$ of subsets is called a *simplicial complex* if it is closed under the taking of subsets in the sense that if for all $\sigma \in K$,

$$\tau \subset \sigma \implies \tau \in K.$$

For every $n \geq 0$, we define the *n-simplices* of K to be the set

$$K_n = \{\sigma \in K \mid |\sigma| = n + 1\}.$$

Every $\sigma \in K_n$ can be expressed uniquely as $\sigma = \{x_0, \ldots, x_n\}$, with $x_0 < \cdots < x_n$. Define the *ith face* of σ to be

$$\partial_i \sigma = \{x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n\}.$$

Let R be a ring. Define

$$C_n(K, R) = \bigoplus_{\sigma \in K_n} R_{e_\sigma},$$

where the subscripts are simply labels. Further, define

$$d \colon C_n(K, R) \to C_{n-1}(K, R); \qquad e_\sigma \mapsto \sum_{i=0}^{n} (-1)^i e_{\partial_i \sigma}.$$

This is a complex because…

Example 4. Let k be a field, and A an associative k-algebra. We define a complex of vector spaces

$$\cdots \xrightarrow{d} A^{\otimes n} \xrightarrow{d} \cdots \xrightarrow{d} A^{\otimes 3} \xrightarrow{d} A \otimes A \xrightarrow{d} A\ ,$$

where d is defined by the formula

$$d(a_0 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n + (-1)^n a_n a_0 \otimes \cdots \otimes a_{n-1}.$$

This complex is known as the *cyclic bar complex.*

Definition 5 (cycle, boundary, homology)**.** Let $(C_\bullet, d)$ be a chain complex. We make the following definitions.

- The space $\ker d_m$ is known as the space of n-cycles, and denoted $Z_n(C_\bullet)$.
- The space $\operatorname{im} d_{m+1}$ is known as the space of n-boundaries, and denoted $B_n(C_\bullet)$.
- The space
$$\frac{Z_n}{B_n} = H_n(C_\bullet)$$
is known as the nth homology of C.

Definition 6 (exact)**.** We say that a chain complex $(C_\bullet, d)$ is exact at $n \in \mathbb{Z}$ if $H_n(C_\bullet = 0)$.

Definition 7 (exact sequence)**.** We say that a sequence

$$C_k \xrightarrow{d} C_{k-1} \xrightarrow{d} \cdots \xrightarrow{d} C_\ell$$

with $d^2 = 0$ is exact if it is exact at $k-1, k-2, \ldots, \ell+1$.

Definition 8 (short exact sequence)**.** A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0 \ .$$

Equivalently, such a sequence is exact if

- f is a monomorphism
- g is a epimorphism
- $H_B = 0$.

Example 9.

- The sequence
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$
is exact.
- The sequence
$$0 \longrightarrow \mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is not exact since the map $x \mapsto 2x$ is not injective.

- The sequence

$$\mathbb{Z}/4\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z}/4\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

is exact.

3. Basic algebra

Strictly speaking, this chapter shouldn't need to exist, but my algebra knowledge is pretty hopeless so I'll be putting stuff I should know already here.

3.1. Exactness

Proposition 10. Let R be a ring, and M a left R-module. Then both the hom functors

$$\mathrm{Hom}(M, -), \qquad \mathrm{Hom}(-, M)$$

are left exact.

Proof. Let

$$0 \longrightarrow A \xhookrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. First, we show that $\mathrm{Hom}(M, A)$ is exact, i.e. that

$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$

is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha \colon M \to A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and

$$f \circ \alpha = 0 \implies \alpha = 0$$

because f is a monomorphism.

Similarly, $\operatorname{im} f \subseteq \ker g$ because

$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \operatorname{im} f$. To this end, let $\gamma\colon M \to B$ such that $g_*\gamma = 0$.

$$\begin{tikzcd}
 & & M \arrow[dl, dashed, "\exists!\delta"'] \arrow[d, "\gamma"] \arrow[dr, "0"] & & \\
0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0
\end{tikzcd}$$

Then $\operatorname{im}\gamma \subset \ker g = \operatorname{im} f$, so γ factors through A; that is to say, there exists a $\delta\colon M \to A$ such that $f_*\delta = \gamma$. □

This theorem is a shadow of a much, much stronger result: Exactness of functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories is a consequence of preservation of certain (co)limits. More concretely, we have the following.

3.2. Tensor products

Definition 11 (tensor product)**.** Let R be a (not necessarily commutative) ring M a right R-module, and N a left R-module. The tensor product $M \otimes_R N$ satisfies the following universal property…

Even for modules over non-commutative rings, we have the following version of tensor-hom adjunction.

Proposition 12. Let R be a ring, M a right R-module, and N a left R-module. There is an adjunction
$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \operatorname{Hom}_{\mathbf{Ab}}(N, -).$$

Proof. We show that there is a natural bijection
$$\operatorname{Hom}_{\mathbf{Ab}}(M \otimes_R N, P) \simeq \operatorname{Hom}_{\mathbf{Ab}}(M, \operatorname{Hom}_{\mathbf{Mod}\text{-}R}(N, P)).$$

□

3.3. Projective and injective modules

Definition 13 (projective, injective module)**.** Let R be a ring. An R-module M is said to be projective if the functor $\operatorname{Hom}(P, -)$ is exact, and injective if the functor $\operatorname{Hom}(-, P)$ is exact.

We know (from Proposition 10) that the hom functor is left exact in both slots. Therefore, we have the following immediate classification of projective objects.

Proposition 14. An R-module P in is projective if and only if for every for every epimorphism $g\colon B \twoheadrightarrow C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the folowing diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\tilde{p}"'] \arrow[d, "p"] \\ B \arrow[r, two heads, "g"'] & C \end{tikzcd}$$

Similarly, a module Q is injective if for every monomorphism $A \hookrightarrow B$ and map $g\colon A \to Q$, there exists a homomorphism $\tilde{q}$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "f"] \arrow[d, "p"'] & B \arrow[dl, dashed, "\exists\tilde{p}"] \\ Q & \end{tikzcd}$$

Proof. In the projective case, this condition precisely encodes the condition that f_* is an epimorphism; in the projective case, □

Proposition 15. Let R be a ring. A module P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M.$$

Proof. First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & \ker\pi \arrow[r, hook] & F \arrow[r, two heads, "\pi"] & P \arrow[r] & 0 \end{tikzcd}$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker\pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{tikzcd} & P \oplus M \arrow[dl, dashed, "\exists(j,k)"'] \arrow[d, "(f,g)"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists j"'] \arrow[d, "f"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

exhibiting P as projective. $\square$

4. Abelian categories

4.1. Basics

4.1.1. Building blocks

Ab-enriched categories

In this chapter we recall some basic results.

Definition 16 (**Ab**-enriched category)**.** A category $\mathcal{C}$ is called **Ab**-*enriched* if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$.

Lemma 17. Let C be an Ab-enriched category with finite products. Then the product $X_1 \times \cdots \times X_n$ satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

Proof. First, we show that initial and terminal objects coincide. Let $\emptyset$ be initial and $*$ terminal in C.

Since $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on $\emptyset$ *and* the identity element of $\mathrm{Hom}_{\mathsf{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, $\mathrm{id}_* = 0$ in $\mathrm{Hom}_{\mathsf{C}}(*, *)$.

Since $\emptyset$ is initial and $*$ is final, $\mathrm{Hom}_{\mathsf{C}}(\emptyset, *)$ has exactly one element f. We are thus guaranteed that $\mathrm{Hom}_{\mathsf{C}}(*, \emptyset)$ has at least one element, g. Then $g \circ f = 0 = \mathrm{id}$ and $f \circ g = 0 = \mathrm{id}$, so $f\colon \emptyset \to *$ is an isomorphism. Thus, initial and terminal objects coincide, so C has a zero object 0.

We prove the lemma for binary products. The general case is preciesly the same.

For each $X, Y \in \mathcal{C}$, we can define $i_X\colon X \to X \times Y$ using the universal property for products as follows.

$$\begin{tikzcd}
& X \arrow[dl, "\mathrm{id}"'] \arrow[d, dashed, "i_X"] \arrow[dr, "0"] & \\
X & X \times Y \arrow[l] \arrow[r] & Y
\end{tikzcd}$$

Similarly, get $i_Y\colon Y \to X \times Y$.

The claim is that $X \times Y$, together with i_X and i_Y, satisfies the universal property for coproducts; that is, that we can find a unique map ϕ making the below diagram commute.

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & X \times Y \arrow[d, dashed, "\phi"] & Y \arrow[l, "i_Y"'] \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

In fact, $\phi = f \circ \pi_X + g \circ \pi_Y$ satisfies the universal property. It is clear that this makes the diagram commute since $\pi_X \circ i_X = \mathrm{id}$, $\pi_X \circ i_Y = 0$, and similarly for Y.

Note that, by the universal property for products, there is a unique downward map. But both $\mathrm{id}_{X\times Y}$ and $i_X \circ \pi_X + i_Y \circ \pi_Y$ make it commute, so they must be equal.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_Y"'] \arrow[d] \arrow[dr, "\pi_X"] & \\ X & X \times Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y \end{tikzcd}$$

Now we are ready to show that ϕ is unique. Suppose we have another candidate ψ which also makes the above diagram commute. Then

$$\psi = \psi \circ \mathrm{id}_{X\times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

□

Additive categories

Definition 18 (additive category)**.** A category C is additive if it has products and is Ab-enriched.

Definition 19 (additive functor)**.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between additive categories. We say that F is additive if for each $X, Y \in \mathrm{Obj}(\mathcal{C})$ the map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Pre-abelian categories

Definition 20 (kernel, cokernel)**.** Let $f\colon A \to B$ be a morphism in an abelian category.

- A pair (K, ι), where $\iota\colon K \to A$ is a morphism, is a kernel of f if for every object X and morphism $g\colon X \to A$ such that $f \circ g = 0$, there exists a unique morphism

$\alpha\colon X \to K$ such that $f \circ \alpha = 0$.

$$\begin{tikzcd}
& X \arrow[dl, dashed, "\exists!"'] \arrow[d, "g"] \arrow[dr, "0"] & \\
K \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

- A pair (C, π), where $\pi\colon B \to C$ is a morphism, is a cokernel of f if for every object Y and morphism $g\colon B \to Y$ such that $g \circ f = 0$, there exists a unique morphism $\beta\colon C \to Y$ such that $\beta \circ f = 0$.

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "0"'] & B \arrow[r, "\pi"] \arrow[d, "g"] & C \arrow[dl, dashed, "\exists!"] \\
& Y &
\end{tikzcd}$$

Definition 21 (pre-abelian category)**.** A pre-abelian category is an additive category such that every morphism has a kernel and a cokernel.

Lemma 22. Pre-abelian categories have all finite limits and colimits.

Proof. In a pre-abelian category, the equalizer of f and g is the kernel of $f - g$, and the coequalizer is the cokernel of $f - g$. Thus, pre-abelian categories have finite products and equalizers, and finite coproducts and coequalizers. □

Proposition 23. In a pre-abelian category, kernels are monic and cokernels are epic.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category, and let $\iota\colon K \to A$ be a kernel of f. Let $\alpha\colon X \to K$ be a morphism such that $\alpha \circ \iota = 0$.

$$\begin{tikzcd}
& X \arrow[dl, "\alpha"'] \arrow[d, "0"] \arrow[dr, "0"] & \\
K \arrow[r, "\iota"] & A \arrow[r, "f"] & B
\end{tikzcd}$$

The universal property for kernels tells us that α is the *unique* morphism such that $0 = \iota \circ \alpha$. But setting $\alpha = 0$ certainly satisfies this equality, so we must have that $\alpha = 0$.

The cokernel case is dual. □

Proposition 24. Let $\mathcal{C}$ be a pre-abelian category, and let $f\colon A \to B$ be a morphism.

1. The morphism f is a monomorphism if and only if $\ker f = 0$.
2. The morphism f is an epimorphism if and only if $\operatorname{coker} f = 0$.

Proof.

1. Suppose f is a monomorphism, and consider a morphism g making the following diagram commute.

$$\begin{tikzcd} & Z \arrow[d, "g"] \arrow[dr, "0"] & \\ \ker f \arrow[r, hook, "\iota"] & A \arrow[r, hook, "f"] & B \end{tikzcd}$$

Because f is a monomorphism, we have $g = 0$. By the universal property for kernels, there exists a unique map $Z \to \ker f$ making the above diagram commute.

$$\begin{tikzcd} & Z \arrow[d, "g"] \arrow[dr, "0"] \arrow[dl, dashed, "\exists!"'] & \\ \ker f \arrow[r, hook, "\iota"] & A \arrow[r, hook, "f"] & B \end{tikzcd}$$

But this remains true if we replace $\ker f$ by 0, so by definition we must have $\ker f = 0$.

Now suppose that $\ker f = 0$, and let $g\colon X \to A$ such that $f \circ g = 0$. The universal property guarantees us a morphism $\alpha\colon X \to K$ making the left-hand triangle below commute.

$$\begin{tikzcd} & Z \arrow[d, "g"] \arrow[dr, "0"] \arrow[dl, dashed, "\exists!"'] & \\ \ker f = 0 \arrow[r, hook, "\iota"] & A \arrow[r, "f"] & B \end{tikzcd}$$

However, the only way the left-hand triangle can commute is if $g = 0$. Thus, f is mono.

2. Dual.

□

4.1.2. Abelian categories

Definition 25 (abelian category)**.** A pre-abelian category is said to be abelian if every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Proposition 26. Let $\mathcal{A}$ be an abelian category, and let $f\colon A \to B$ be a morphism in $\mathcal{A}$. Then f is an isomorphism if and only if $\ker f = 0$ and $\operatorname{coker} f = 0$.

Proof. If f is an isomorphism, then it is mono and epi, and we are done by Proposition 24.

Conversely, if $\ker f = 0$ and $\operatorname{coker} f = 0$, then f is mono and epi. We need only to show that any morphism in an abelian category which is monic and epic is an isomorphism.

Let f be monic and epic. Then f is the kernel of its cokernel and the cokernel of its kernel, and both of these are zero.

$$\begin{tikzcd}
& & B \arrow[dl, dashed, "\exists! f_R^{-1}"'] \arrow[d, equal] \arrow[dr, "0"] & \\
\ker f = 0 \arrow[r] & A \arrow[r, hook, two heads, "f"] & B \arrow[r] & 0 = \operatorname{coker} f \\
& A \arrow[ul, "0"] \arrow[u, equal] \arrow[ur, dashed, "\exists! f_L^{-1}"'] & &
\end{tikzcd}$$

This gives us morphisms f_L^{-1} and f_R^{-1} such that

$$f \circ f_R^{-1} = \mathrm{id}_A, \qquad f_L^{-1} \circ f = \mathrm{id}_B.$$

The usual trick shows that $f_L^{-1} = f_R^{-1}$. □

As the following proposition shows, additive functors from additive categories preserve direct sums.

Lemma 27. Let A and B be objects in an additive category $\mathcal{A}$, and let X be an object in $\mathcal{A}$ equipped with morphisms as follows,

$$\begin{tikzcd}
A \arrow[dr, "i_A"] & & A \\
& X \arrow[ur, "\pi_A"] \arrow[dr, "\pi_A"'] & \\
B \arrow[ur, "i_B"'] & & B
\end{tikzcd}$$

such that the following equations hold.

$$\begin{gathered}
i_A \circ \pi_A + i_B \circ \pi_B = \mathrm{id}_X \\
\pi_A \circ i_B = 0 = \pi_B \circ i_A \\
\pi_A \circ i_A = \mathrm{id}_A \\
\pi_B \circ i_B = \mathrm{id}_B
\end{gathered}$$

Then the πs and is exhibit $X \cong A \oplus B$.

Proof. It suffices to show that X is a product of A and B. Let $f\colon Z \to A$ and $g\colon Z \to B$. We need to show that there exists a unique $\phi\colon Z \to X$ making the following diagram commute.

$$\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[d, dashed, "\phi"] \arrow[dr, "g"] & \\
A & X \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B
\end{tikzcd}$$

That is, we need

$$\pi_A \circ \phi = f, \qquad \pi_B \circ \phi = g.$$

Thus, we need

$$\begin{aligned} i_A \circ f + \pi_B \circ g &= i_A \circ \pi_A \circ \phi + i_B \circ \pi_B \circ \phi \\ &= \phi, \end{aligned}$$

so ϕ exists and is unique. □

Corollary 28. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(a \oplus b) \simeq F(a) \oplus F(b).$$

Once it is known that a category $\mathcal{A}$ is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in $\mathcal{A}$ is abelian, as we will see in
- For any small category I, the category $\mathbf{Fun}(I, \mathcal{A})$ of I-diagrams in $\mathcal{A}$ is abelian.

4.2. Embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

In abelian categories, one can defines kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero obect: f is a kernel for g if and only the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pullback.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \arrow[d, "g"] \\ 0 \arrow[r] & C \end{tikzcd}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category $\mathbf{Set}_*$ of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category $\mathcal{A}$, suppose one could find an embedding $\mathcal{H}\colon \mathcal{A} \to \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \to B \to C$ in $\mathcal{A}$, mapped it into $\mathbf{Set}_*$ finding a sequence $\mathcal{H}(A) \to \mathcal{H}(B) \to \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \to B \to C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in $\mathbf{Set}_*$ using $\mathcal{H}$ and check exactness there. Effectively, one could check exactness of a sequence in $\mathcal{A}$ by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking the that the image of the diagram under our functor $\mathcal{H}$ commutes.

In fact, for $\mathcal{A}$ small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to $\mathbf{Set}_*$.

Definition 29 (category of contravariant epimorphisms)**.** Let $\mathcal{A}$ be a small abelian category. Define a category $\mathcal{A}_{\twoheadleftarrow}$ with $\mathrm{Obj}(\mathcal{A}_{\twoheadleftarrow}) = \mathrm{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\mathrm{Hom}_{\mathcal{A}_{\twoheadleftarrow}}(X, Y) = \{f\colon Y \to X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity morphism is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A\colon \mathcal{A}_{\twoheadleftarrow} \to \mathbf{Set}_*; \qquad Z \mapsto \mathrm{Hom}_{\mathcal{A}}(Z, A)$$

and sends a morphism $f\colon Z_1 \to Z_2$ in $\mathcal{A}_{\twoheadleftarrow}$ (which is to say, an epimorphism $\tilde{f}\colon Z_2 \twoheadrightarrow Z_1$ in $\mathcal{A}$) to the map

$$\mathcal{H}_A(f)\colon\ \mathrm{Hom}_{\mathcal{A}}(Z_1, A) \to \mathrm{Hom}_{\mathcal{A}}(Z_2, A); \quad (\alpha\colon Z_1 \to A) \mapsto (\alpha \circ \tilde{f}\colon Z_2 \to A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 30 (member functor)**.** Let $\mathcal{A}$ be a small abelian category. We define a functor $\mathcal{M}\colon \mathcal{A} \to \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \operatorname{colim} \mathcal{H}_A.$$

On morphisms, functorality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(\mathcal{A})$ have a much simpler interpretation.

Proposition 31. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \operatorname{Hom}(X, A)/\sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{tikzcd}
Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\
X' \arrow[r, "g'"'] & A
\end{tikzcd}$$

Proof. The colimit can be computed using the following coequalizer.

$$\coprod_{\substack{f \in \operatorname{Morph}(\mathcal{A}_{\twoheadleftarrow}) \\ \tilde{f}\colon Y \to X}} \operatorname{Hom}_{\mathcal{A}}(X, A) \overset{\mathrm{id}}{\underset{-\circ \tilde{f}}{\rightrightarrows}} \coprod_{Z \in \operatorname{Obj}(\mathcal{A}_{\twoheadleftarrow})} \operatorname{Hom}(Z, A) \xrightarrow{\text{coeq}} \mathcal{M}(A)$$

On elements, we have the following.

$$\begin{tikzcd}
 & (g\colon X \to A) \arrow[d, equal, "!"] \\
(g\colon X \to A) \arrow[ur, mapsto, "\mathrm{id}"] \arrow[r, mapsto, "-\circ \tilde{f}"'] & (g \circ \tilde{f}\colon Y \to A)
\end{tikzcd}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \operatorname{Obj}(\mathcal{A}_{\twoheadleftarrow})} \operatorname{Hom}(Z, A)/\sim,$$

where $\sim$ is the equivalence relation generated by the relation

$$(g\colon X \to A) R (g'\colon X' \to A) \iff \exists f\colon X' \twoheadrightarrow X \text{ such that } g' = g \circ \tilde{f}.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g\colon X \to A) \sim (g'\colon X' \to A) \iff \exists f : Z \twoheadrightarrow X, f' : Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "f"] \arrow[d, two heads, "f'"'] & X \arrow[d, "g"] \\ X' \arrow[r, "g'"'] & A \end{tikzcd}$$

□

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphsims into A modulo the above relation, and for a morphism $f\colon A \to B$ in $\mathcal{A}$, $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f)\colon [g] \mapsto [g \circ f].$$

Lemma 32. Let $f : A \to B$ be a morphism in a small abelian category $\mathcal{A}$ such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms making the following diagram commute.

$$\begin{tikzcd} Z \arrow[r, two heads, "g"] \arrow[d, two heads] & A \arrow[d, "f"] \\ 0 \arrow[r] & B \end{tikzcd}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. □

Now we introduce some load-lightening notation: we write $\hat{(-)} = \mathcal{M}(-)$.

Lemma 33. Let $f : A \to B$ be a morphism in a small abelian category $\mathcal{A}$. Then $f = 0$ if and only if $\hat{f} : \hat{A} \to \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g, so $\hat{f} = 0$.

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\mathrm{id}_A]) = [0]$. But

$$\hat{f}([\mathrm{id}_A]) = [\mathrm{id}_A \circ f] = [f].$$

By Lemma 32, $[f] = [0]$ implies $f = 0$. $\square$

Corollary 34. A diagram commutes in $\mathcal{A}$ if and only if its image in $\mathbf{Set}_*$ under $\mathcal{M}$ commutes.

Proof. A diagram commutes in $\mathcal{A}$ if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. $\square$

Lemma 35. Let $f : A \to B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

4.3. Chain complexes

Definition 36 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex $C_\bullet$ in $\mathcal{A}$ is a collection of objects $C_n \in \mathcal{A}$, $n \in \mathbb{Z}$, together with morphisms $d_n : C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms $C_\bullet \to D_\bullet$ of chain complexes, i.e. for each $n \in \mathbb{Z}$ a map $f_n : C_n \to D_n$ such that all of the squares form commute.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes (i.e. the full subcategory consisting of chain complexes $C_\bullet$ such that $C_n = 0$ for small enough n), $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Proposition 37. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the $\mathbf{Ab}$-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd}
 & Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\
C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet
\end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned}(d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q,\end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd} & Q_\bullet \arrow[dl, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\ \ker(f_\bullet) \arrow[r, "\iota_\bullet"] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \end{tikzcd}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{4.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi\colon Q_n \to \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 4.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned}\iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n\end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

Definition 38 (image, coimage, exact sequence)**.** Given a morphism $f\colon A \to B$, the image of f is $\operatorname{im} f = \ker \operatorname{coker} f$ and the coimage of f is $\operatorname{coker} \ker f$.

Given such an f, we build the following commuting diagram.

$$\begin{tikzcd}
& \operatorname{im} f \arrow[d, "i"] \arrow[dr, "0"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "p"] & C \\
& \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a morphism $\phi\colon \operatorname{im} f \to \ker g$.

$$\begin{tikzcd}
& \operatorname{im} f \arrow[r, "\phi"] \arrow[dr, "i"] & \ker g \arrow[r] \arrow[d] & H_n \\
C_{n+1} \arrow[rr, "f"] & & C_n \arrow[r, "g"] \arrow[d, "p"] & C_{n-1} \\
& & \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

We say that the above is exact if ϕ is an isomorphism. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

Definition 39 (homology)**.** Let $(C_\bullet, d)$ be a complex. The nth homology of $C_\bullet$ is the cokernel of the map $\phi\colon \operatorname{im} d_{n+1} \to \ker d_n$ described in Definition 38, denoted $H_n(C_\bullet)$.

It is useful to have the following picture in mind.

Proposition 40. Homology extends to a family of additive functors

$$H_n\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d_n^C \to \ker d_n^D$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. □

We can also go the other way, by defining a map

$$\iota_n \colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\begin{tikzcd} {} \arrow[r, dotted] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

concentrated in degree n.

Example 41. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex which is exact except in degree zero. Then $H_0(C_\bullet) = \operatorname{coker} f$.

Similarly, if

$$0 \longrightarrow C_0 \xrightarrow{f} C_{-1} \longrightarrow \cdots$$

is a chain complex whose non-trivial homology is in degree 0, then $H_0(C_\bullet) = \ker f$.

Definition 42 (quasi-isomorphism)**.** Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Quasi-isomorphisms are a fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope quasi-isomorphisms themselves were invertible. Unfortunately, we are not so lucky. Consider the following

By Example 41, we

Definition 43 (resolution)**.** Let A be an object in an abelian category. A resolution of A consists of a chain complex $C_\bullet$ together with a quasi-isomorphism $C_\bullet \to \iota(A)$.

4.4. Diagram lemmas

4.4.1. The splitting lemma

Lemma 44 (Splitting lemma)**.** Consider the following solid exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[l, bend left, dashed, "\pi_A"] \arrow[r, "\pi_C"] & C \arrow[l, bend left, dashed, "i_C"] \arrow[r] & 0 \end{tikzcd}$$

The following are equivalent.

- There exists a morphism $\pi_A \colon B \to A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
- There exists a morphism $i_C \colon C \to B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
- B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

4.4.2. The snake lemma

Lemma 45. Let $f \colon B \twoheadrightarrow C$ be an epimorphism, and let $g \colon D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd}
S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "g"] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K also functions as the kernel of f'. Of course, in order for this statement to make sense we need a map $i \colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists! i"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v \colon Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.

$$\begin{tikzcd}
& Q \arrow[dl, dashed, "\exists! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{tikzcd}
Q \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left, "0"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By the universal property, there exists a unique map $Q \longrightarrow S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\begin{array}{c}\text{elements of pullback}\\ \text{which map to } 0\end{array}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{array}{c}\text{elements of } B\\ \text{which map to } C\end{array}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\begin{array}{c}\text{elements of } B\\ \text{which map to } 0\end{array}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 46 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed, bend left] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd} \tag{4.2}$$

The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\operatorname{coker} g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 4.2 the following diagram.

$$\begin{tikzcd}
 & & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}$$

Take a pullback and a pushout, and using Lemma 45 (and its dual), we find the following.

$$\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}$$

Consider the map

$$\delta_0 = \quad S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \ .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \ \operatorname{coker} f \to \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 47 (snake lemma II)**.** Given the following commutative diagram with exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

we get a exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

4.4.3. The long exact sequence on homology

Corollary 48. Given an exact sequence of complexes

$$\begin{tikzcd} 0 \arrow[r] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \arrow[r, "g_\bullet"] & C_\bullet \arrow[r] & 0 \end{tikzcd}$$

we get a long exact sequence on homology

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\
H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\
H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta"] \\
H_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. For each n, we have a diagram of the following form.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d^A_{n+1}"'] & B_{n+1} \arrow[r] \arrow[d, "d^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma (Theorem 46) gives us, for each n, two exact sequences as follows.

$$\begin{tikzcd} 0 \arrow[r] & \ker d^A_{n+1} \arrow[r] & \ker d^B_{n+1} \arrow[r] & \ker d^C_{n+1} \end{tikzcd}$$

and

$$\begin{tikzcd} \operatorname{coker} d^A_n \arrow[r] & \operatorname{coker} d^B_n \arrow[r] & \operatorname{coker} d^C_n \arrow[r] & 0 \end{tikzcd}$$

We can put these together in a diagram with exact rows as follows.

$$\begin{tikzcd}
 & \operatorname{coker} f_{n+1} & \operatorname{coker} g_{n+1} & \operatorname{coker} h_{n+1} & 0 \\
0 & \ker f_n & \ker g_n & \ker h_n &
\end{tikzcd}$$

Recall that for each n, we get a map

$$\phi\colon \operatorname{im} f_{n+1} \to \ker f_n$$

□

Lemma 49. Let $\mathcal{A}$ be an abelian category, and let f be a morphism of short exact sequences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{tikzcd}
0 \arrow[r] & A'_\bullet \arrow[r, "\alpha_\bullet"] \arrow[d, "f'_\bullet"] & A_\bullet \arrow[r, "\alpha'_\bullet"] \arrow[d, "f_\bullet"] & A''_\bullet \arrow[r] \arrow[d, "f''_\bullet"] & 0 \\
0 \arrow[r] & B'_\bullet \arrow[r, "\beta_\bullet"'] & B_\bullet \arrow[r, "\beta'_\bullet"'] & B''_\bullet \arrow[r] & 0
\end{tikzcd}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha')"] \arrow[d, "H_n(f)"'] & H_n(A'') \arrow[r, "\delta"] \arrow[d, "H_n(f'')"] & H_{n-1}(A') \arrow[r, "H_{n-1}(\alpha)"] \arrow[d, "H_n(f')"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_n(f)"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "H_n(\beta')"] & H_n(B'') \arrow[r, "\delta"] & H_{n-1}(B') \arrow[r, "H_{n-1}(\beta)"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, the connecting morphism δ is functorial.

4.4.4. The five lemma

Lemma 50 (four lemmas)**.** Let

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads] & C \arrow[r] \arrow[d] & D \arrow[r] \arrow[d, two heads] & E \arrow[d, hook] \\
B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then

Theorem 51 (five lemma)**.**

4.4.5. The nine lemma

Theorem 52 (nine lemma)**.**

5. Homological algebra in abelian categories

5.1. Exactness in abelian categories

Definition 53 (exact functor)**.** Let $F \colon \mathcal{A} \to \mathcal{B}$ be an additive functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

  is exact

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

  is exact

- We call F exact if it is both left exact and right exact.

Example 54. We showed in Lemma 10 that both hom functors $\mathrm{Hom}(A, -)$ and $\mathrm{Hom}(-, B)$ are left exact.

According to the Yoneda lemma, to check that two objects are isomorphic it suffices to check that their images under the Yoneda embedding are isomorphic. In the context of abelian categories, the following result shows that we can also check the exactness of a sequence by checking the exactness of the image of the sequence under the Yoneda embedding.

Lemma 55. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\mathrm{im}\, f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\mathrm{im}\, \iota = \ker g \subset \mathrm{im}\, f$. □

Proposition 56. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
    - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.
    - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.
3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

Any functor G which is a right adjoint preserves limits. By the above reasoning, G certainly preserves zero objects and kernels. Thus, G preserves the first two conditions, which means precisely that G is left exact.

Dually, any functor F which is a left adjoint preserves colimits, hence zero objects and cokernels. Thus, F preserves the last two conditions, which means that F is right exact. □

Note that the previous theorem makes no demand that F be additive, which may seem surprising; after all, by Definition 53, only exact funcotors are allowed the honor of being called additive. However, it turns out that a functor which preserves either finite limits or finite colimits is always additive. To see this, note that by applying any functor which is either left or right exact to a split exact sequence gives a split exact sequence.

This is often useful when trying to check the exactness of a functor which is a left or right adjoint, by the following proposition.

Lemma 57. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence, and let X be any object. Then the sequence

Corollary 58. $0 \longrightarrow \mathrm{Hom}(X, A) \xrightarrow{f_*} \mathrm{Hom}(X, B) \xrightarrow{g_*} \mathrm{Hom}(X, C)$

is an exact sequence of abelian groups.

Proof. The hom functor □

5.2. Chain Homotopies

Definition 59 (chain homotopy)**.** Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A chain homotopy

Lemma 60. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 61. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 62. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

Definition 63 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

Lemma 64. The homotopy category $\mathcal{K}(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

In fact, the homotopy category satisfies the following universal property.

Proposition 65. Let $F\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{C}$ be an additive functor such that $\mathcal{F}(f)$ is an isomorphism for all quasi-isomorphisms f. Then there exists a unique functor $\mathcal{K}(\mathcal{A}) \to \mathcal{C}$ making the following diagram commute.

Proposition 66. Let $\mathcal{F}\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{B}$ be an additive functor which takes quasi-isomorphisms to isomorphisms. Then for homotopic morphisms $f \sim g$ in $\mathbf{Ch}(\mathcal{A})$, we have $F(f) = F(g)$.

5.3. Projectives and injectives

Definition 67 (projective, injective)**.** An object P in an abelian category is said to be projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A, it is very easy to see that an object P in an abelian category $\mathcal{A}$ is projective if for every epimorphism $f\colon B \to C$ and every morphism $p\colon P \to C$, there exists a morphism $\tilde{p}\colon P \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[d, "p"] \arrow[dl, dashed, "\exists\tilde{p}"'] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g\colon A \to B$ and every morphism $q\colon A \to Q$ there exists a morphism $\tilde{q}\colon B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists\tilde{q}"] \\ Q & \end{tikzcd}$$

Definition 68 (enough projectives)**.** Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

Proposition 69. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, we get a morphism $P \to B$ making the triangle commute.

$$\begin{tikzcd} & & & & P \arrow[dl, "\exists g"'] \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma 44). ◻

Corollary 70. Let F be an additive functor, and let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be an exact sequence with P projective. Then F applied to the sequence gives an exact sequence.

Proof. Applying F to a split sequence gives a split sequence, and all split sequences are exact. ◻

Corollary 71. The category R-**Mod** has enough projectives.

Proof. By Proposition 15, free modules are projective, and every module is a quotient of a free module. ◻

Proposition 72. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Dually, any abelian category with enough injectives has injective resolutions.

Proof. We provide running commentary on the following diagram. ◻

Definition 73 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \to \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \to Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

Lemma 74. Let $f\colon P_\bullet \to \iota(M)$ be a projective resolution. Then $f\colon P_0 \to M$ is an epimorphism.

Proof. We know that $H_0(f)\colon H_0(P) \to H_0(\iota(M))$ is an isomorphism. But $H_0(\iota(m)) \simeq \iota_M$, and that ◻

In fact, we can say more.

Lemma 75. Let $f_\bullet \colon P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\begin{tikzcd} \cdots\cdots\cdots & P_2 \arrow[r] & P_1 \arrow[r] & P_0 \arrow[r, "f_0"] & A \arrow[r] & 0 \end{tikzcd}$$

is exact.

Theorem 76 (extended horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "\tilde{g}"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

Furthermore, given a morphism $M \to N$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd} & \vdots \arrow[d, dotted, no head] & & \vdots \arrow[d, dotted, no head] & \\ 0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"] & M \arrow[r, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd} 0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f\circ p'_0"'] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since by Lemma 74 p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P_0' \oplus P_0'' \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p_0' \circ \pi_{P_0'} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P_1'' to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r] \arrow[d] & \ker p \arrow[r] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] & P_0 \arrow[r] & P_0'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Proposition 77. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P_\bullet' \to M'$$

be projective resolutions. Then for every morphism $f\colon M \to M'$ there exists a lift

$\tilde{f} : P_\bullet \to P'_\bullet$ making the diagram

$$\begin{tikzcd} P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P'_\bullet \arrow[d] \\ \iota M \arrow[r, "\iota f"'] & \iota M' \end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Corollary 78. For any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}\mathcal{A})$.

5.4. Mapping cones

Definition 79 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the <u>k-shifted chain complex</u> $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 80 (mapping cone)**.** Let $f : C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\operatorname{cone}(f)$ as follows.

- For each n, define
$$\operatorname{cone}(f)_n = C_{n-1} \oplus D_n.$$
- Define $d_n^{\operatorname{cone}(f)}$ by
$$d_n^{\operatorname{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$
which is shorthand for
$$d_n^{\operatorname{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 81. The complex $\operatorname{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow C_\bullet \xrightarrow{\iota} \operatorname{cone}(f)_\bullet \xrightarrow{\pi} D_\bullet \longrightarrow 0 .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 82. Let $f_\bullet : C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\operatorname{cone}(f)$ is an exact complex.

Proof. By Corollary 48, we get a long exact sequence

$$\cdots \to H_n(X) \xrightarrow{\delta} H_n(Y) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(X) \to \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see (although I'm missing a sign somewhere); picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0, f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. □

Proposition 83. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$. Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g\colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots \\
\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r =$

$f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
 & & P_1 \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists!"] \arrow[dd, "r"] \arrow[ddr, "0"] & \\
 & K \arrow[dr, hook] & & \\
C_2 \arrow[ur, two heads] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\operatorname{cone}(f)$ is exact, which means that $\iota\colon P \to \operatorname{cone}(f)$ is homotopic to the zero morphism.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d_{i+1}^P"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d_i^P"] \arrow[dl, dashed, "\tilde{h}_i"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d_{i+1}^{\operatorname{cone}(f)}"'] & C_{i-1} \oplus P_i \arrow[r, "d_i^{\operatorname{cone}(f)}"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}$$

That is, there exist $\tilde{h}_i\colon P_i \to \operatorname{cone}(f)_{i+1}$ such that

$$d_{i+1}^{\operatorname{cone} f} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \tag{5.1}$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i\colon P_i \to C_i \qquad \text{and} \qquad \gamma_i\colon P_{i-1} \to P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet \colon P_\bullet \to C_\bullet$. Indeed, writing Equation 5.1 in components, we find the following.

$$\begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$
$$\begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that[1]

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

5.5. Localization at weak equivalences: a love story

We can embed any abelian category $\mathcal{A}$ into the corresponding category $\mathbf{Ch}(\mathcal{A})$ of chain complexes via inclusion into degree zero. This is obviously a lossless procedure, in the sense that we recover $\mathcal{A}$ by restricting to degree 0.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "(-)_0"] \end{tikzcd}$$

Equivalently, we recover our category $\mathcal{A}$ by taking zeroth homology.

$$\begin{tikzcd} \mathcal{A} \arrow[r, bend left, "\iota"] & \mathbf{Ch}(\mathcal{A}) \arrow[l, bend left, "H_0"] \end{tikzcd}$$

However, we notice that there are much better-behaved embeddings of $\mathcal{A}$ into $\mathbf{Ch}(\mathcal{A})$ such that we recover $\mathcal{A}$ when taking zeroth homology. For example, taking projective resolutions of objects and lifting morphisms using Proposition 77 will do the trick. Unfortunately, this is not in general uniquely defined: one can take many different pro-

[1]Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

jective resolutions, and lift morphisms in many different ways. There is no reason to expect these data to assemble themselves functorially.

However, if we were to modify $\mathbf{Ch}(\mathcal{A})$ in such a way that all quasi-isomorphisms became bona-fide isomorphisms, then we would have this functoriality, thanks to Proposition 66.

Thus we find ourselves in a common situation: we have a category $\mathbf{Ch}(\mathcal{A})$, and identified a collection of morphisms inside this category which we would like to view as weak equivalences, namely the quasi-isomorphisms. In an ideal world, we would simply promote quasi-isomorphisms to bona fide isomorphisms. That is, we would like to form the localization

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}(\mathcal{A}) \twoheadrightarrow \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. In fact, this is universal in the sense that every other additive functor sending quasi-isomorphisms to isomorphisms factors through it. In particular, the functor

$$\mathbf{Ch}(\mathcal{A}) \to \mathbf{Ch}(\mathcal{A})[\{\text{quasi-isomorphisms}^{-1}\}]$$

factors through it.

But now we are in a really wonderful position, as long as $\mathcal{A}$ has enough projectives: we can (by REF we have already proved) find a projective resolution functor

$$P \colon \mathcal{A} \to \mathcal{K}^{+}(\mathcal{A}).$$

In fact, this functor is an equivalence of categories.

5.6. Homological δ-functors and derived functors

We have noticed that we can

Definition 84 (derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the left derived functor to be the cohomological δ-functor $\{L_nF\}$ defined by

$$L_nF = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

  where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the right derived functor to be the cohomological δ-functor

$$R^nF(X) = H^n \circ F \circ Q$$

  where Q is an injective resolution functor.

It may seem that we still have something left to prove; after all, we have not shown that the result of our composition is independant of the projective/injective resolution functor we used. However,

Because $\{H_n\}$ is a homological δ-functor, it is trivial that L_nF is a homological delta-functor. Note that it really does extend F in the following sense.

Proposition 85. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 75, the following sequence is exact.

$$P_2 \longrightarrow P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_2) \longrightarrow F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But

$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$

Similarly, on morphisms $\square$

Definition 86 (homological δ-functor)**.** A homological δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n\colon \mathcal{A} \to \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n \colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

3. For every morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

and every n, the diagram

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

This seems like an inelegant definition, and indeed it is.

Definition 87 (cohomological δ-functors)**.** Dual to Definition 86.

Example 88. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors,* come from homology.

5.7. Double complexes

Definition 89 (double complex)**.** Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d_{i,j}\colon M_{i,j} \to M_{i-1,j}, \qquad \delta_{i,j}\colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $d^2 = 0$
- $\delta^2 = 0$
- $d\delta + \delta d = 0$

We say that $M_{\bullet,\bullet}$ is first-quadrant if $M_{i,j} = 0$ for $i, j < 0$.

That is, for M a double complex the following diagram anticommutes rather than commutes.

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} & M_{1,0} \arrow[l] & M_{2,0} \arrow[l] & M_{3,0} \arrow[l]
\end{tikzcd}$$

Given any complex of complexes

$$\Big(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \Big) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

one can construct a double complex by multiplying every other differential by -1.

Definition 90 (total complex)**.** Let $M_{\bullet,\bullet}$ be a first-quadrant[2] double complex. The total

[2]This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

Lemma 91. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d+\delta)\circ(d+\delta) = d^2 + d\circ\delta + \delta\circ d + \delta^2 = 0.$$

$\square$

Theorem 92. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex, without loss of generality with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$\begin{tikzcd}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} \arrow[d] & = & \mathrm{Tot}(M)_3 \arrow[d] \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} \arrow[d] & = & \mathrm{Tot}(M)_2 \arrow[d] \\
M_{1,0} \oplus M_{0,1} \arrow[d] & = & \mathrm{Tot}(M)_1 \arrow[d] \\
M_{0,0} \arrow[d] & = & \mathrm{Tot}(M)_0 \arrow[d] \\
0 & & 0
\end{tikzcd}$$

is exact.

The construction is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we construct the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \qquad d_{11}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}} n = m$.

We make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker\delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta d n_{1,2},$$

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned} d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d+\delta)n_{2,1} + (d+\delta)n_{1,2} + (d+\delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. □

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain
$$C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$
hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;
- As a morphism between double complexes, hence between totalizations
$$\mathrm{Tot}(\alpha)_\bullet \colon \mathrm{Tot}(C)_\bullet \to \mathrm{Tot}(D)_\bullet.$$

Lemma 93. These two points of view agree in the sense that
$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 94. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let
$$\begin{tikzcd} \cdots \arrow[r] \arrow[d] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet \end{tikzcd}$$
be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence
$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$
is exact. By Theorem 92, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 93 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 82) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 95. Let
$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$
be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence
$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now Theorem 94 gives us the result we want. □

5.8. Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let $\mathcal{A}$ be an abelian category and let

$$-\otimes -\colon \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{H},$$

be an additive functor, where $\mathcal{H}$ is some abelian category equipped with an exact functor to **Ab**. Further suppose that

Definition 96 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$(C \otimes D)_\bullet = \operatorname{Tot}(C_\bullet \otimes D_\bullet),$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 97 (internal hom)**.** Let $\mathcal{A}$ be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We have

5.9. The Künneth formula

This section takes place in R-**Mod**, where R is a PID. For example, everything we are saying holds in **Ab**.

Lemma 98. Let $C_\bullet$ be a chain complex of free R-modules with trivial differential, and let $C'_\bullet$ be an arbitrary chain complex. Then there is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \cong H_n(C_\bullet \otimes C'_\bullet).$$

Proof. We write

$$Z_p = Z_p(C_\bullet), \qquad B_p = B_p(C_\bullet), \qquad Z'_p = Z_p(C'_\bullet), \qquad B'_p = B_p(C'_\bullet),$$

The sequence

$$0 \longrightarrow Z'_q \hookrightarrow C'_q \twoheadrightarrow B'_{q-1} \longrightarrow 0$$

is exact by definition, and since $Z_p = C_p$ is by assumption free, the sequence

By definition, $Z_p = C_p$ is free. Thus the sequence

$$0 \longrightarrow Z_p \otimes Z'_q \hookrightarrow Z_p \otimes C'_q \twoheadrightarrow Z_p \otimes B'_{q-1} \longrightarrow 0$$

is exact.

The differential of the tensor product complex $C_\bullet \otimes C'_\bullet$ is

$$\begin{aligned} d_n^{\mathrm{tot}} &= \sum_{i=0}^{n} d_i^C \otimes \mathrm{id}_{C_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C'} \\ &= \sum_{i=0}^{n} (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C}. \end{aligned}$$

Since C_i is by assumption free, tensoring with it doesn't change homology, and we have

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

and

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

□

6. Basic applications

6.1. The Tor functor

Let R be a ring, and N a right R-module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by Proposition 56, is right exact. Thus, we may form the left derived functor.

Definition 99 (Tor functor)**.** The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 100. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R-module R/rR. We have the short exact sequence

$$0 \longrightarrow R \xhookrightarrow{r\cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r\cdot} R \longrightarrow 0$$

as a free resolution of R/rR. Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i\left(0 \longrightarrow R \otimes_R N \xrightarrow{r\cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & n = 0 \\ {}_rN, & n = 1 \\ 0, & \text{otherwise.} \end{cases}$$

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

Proposition 101. The functor Tor is balanced; that is,

$$L_i \mathrm{Hom}(M, -)(N) \simeq L_i \mathrm{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \to M$ and $P'_\bullet \to N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to acutally make it a double complex).

$$\begin{tikzcd}
M \otimes_R P'_1 \arrow[d] & P_0 \otimes_R P'_1 \arrow[l, dashed] \arrow[d] & P_1 \otimes_R P'_1 \arrow[l] \arrow[d] \\
M \otimes_R P'_0 \arrow[d, dashed] & P_0 \otimes_R P'_0 \arrow[l, dashed] \arrow[d, dashed] & P_1 \otimes_R P'_0 \arrow[l] \arrow[d, dashed] \\
M \otimes_R N & P_0 \otimes_R N \arrow[l, dashed] & P_1 \otimes_R N \arrow[l]
\end{tikzcd}$$

$$\cdots \longrightarrow P_\bullet \otimes P'_2 \longrightarrow P_\bullet \otimes P'_1 \longrightarrow P_\bullet \otimes P'_0 \longrightarrow 0$$

Since each P'_i is projective, $- \otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\mathrm{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

6.2. Ext and extensions

We know that $\mathrm{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 102 (Ext functor)**.** Let R be a ring, and $N \in R\text{-}\mathbf{Mod}$ an R-module. The right derived functors

$$R^i \mathrm{Hom}(N, -)\colon R\text{-}\mathbf{Mod} \to \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i \mathrm{Hom}(N, -) = \mathrm{Ext}^i_R(N, -).$$

An argument very similar to that given for Tor in Proposition 101 shows that Ext is balanced; that is, when computing

$$\mathrm{Ext}^i_R(M, N),$$

it doesn't matter whether we take an injective resolution of N or a projective resolution of M; we will get the same result either way. Rather than spelling this out in detail, we will wait for Chapter 7, where a spectral sequences argument gives the result almost immediately.

There turns out to be a deep connection between the Ext groups and extensions.

Definition 103 (extension)**.** Let A and B be R-modules. An extension of A by B is a short exact sequence

$$\begin{tikzcd} \xi\colon & 0 \arrow[r] & B \arrow[r, hook] & X \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}.$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{tikzcd} \xi\colon & 0 \arrow[r] & B \arrow[r, hook] \arrow[d, "\mathrm{id}_B"] & X \arrow[r, two heads] \arrow[d, "\varphi"] & A \arrow[r] \arrow[d, "\mathrm{id}_A"] & 0 \\ \xi'\colon & 0 \arrow[r] & B \arrow[r, hook] & X' \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}.$$

The five lemma (Theorem 51) implies that φ is an isomorphism.

Let ξ be an extension of A by B. Applying the Ext functor we find an associated long exact sequence on cohomology.

$$\begin{tikzcd} \mathrm{Ext}^1(A, B) \arrow[r] & \mathrm{Ext}^1(X, B) \arrow[r] & \cdots \\ \mathrm{Hom}(A, B) \arrow[r] & \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[ull, "\delta", rounded corners] \end{tikzcd}$$

Definition 104 (obstruction class)**.** The image of id_B under the connecting homomorphism $\delta(\mathrm{id}_B) \in \mathrm{Ext}^1(A, B)$ is known as the obstruction class of ξ. For any extension ξ, we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 105. An extension ξ is split if and only if the correspoding obstruction class is zero.

Proof. We are given an extension

$$\xi: \qquad \begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[r] & 0 \end{tikzcd}.$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$\mathrm{Hom}(X, B) \longrightarrow \mathrm{Hom}(B, B) \longrightarrow \mathrm{Ext}^1(A, B)$$

If $\delta\colon \mathrm{id} \to 0$, then there exists $a \in \mathrm{Hom}(X, B)$ such that $a \circ f = \mathrm{id}$. Thus, by the splitting lemma (Lemma 44), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a\colon A \to X$ such that $a \circ f = \mathrm{id}_B$.

$$\begin{tikzcd} 0 \arrow[r] & B \arrow[r, hook, "f"] & X \arrow[r, two heads, "g"] & A \arrow[l, bend left, "a"] \arrow[r] & 0 \end{tikzcd}.$$

Since $\mathrm{Ext}^1(-, B)$ is additive, the sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \mathrm{Ext}^1(X, B) \arrow[l, bend left, "\mathrm{Ext}^1(a,B)"] \arrow[r, two heads, "\mathrm{Ext}^1(f,B)"] & \mathrm{Ext}^1(B, B) \arrow[r] & 0 \end{tikzcd}$$

is split exact. Thus, $\mathrm{Ext}^1(g, B)$ is a monomorphism, so it has trivial kernel.

$$\begin{tikzcd} \mathrm{Ext}^1(A, B) \arrow[r, hook, "\mathrm{Ext}^1(g,B)"] & \cdots & \\ \mathrm{Hom}(A, B) \arrow[r] & \mathrm{Hom}(X, B) \arrow[r] & \mathrm{Hom}(B, B) \arrow[ull, "\delta"] \end{tikzcd}$$

Exactness then forces $\delta = 0$. $\square$

Proposition 105 explains the name *obstruction class:* $\Theta(\xi)$ is the obstruction to the splitness of ξ. The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\mathrm{Ext}^1(A, B)$ parametrizes extensions of A by B.

Theorem 106. There is a natural bijection

$$\left\{\begin{smallmatrix}\text{Extension classes of}\\ \text{extensions of } A \text{ by } B\end{smallmatrix}\right\} \xrightarrow{\cong} \mathrm{Ext}^1(A,B); \qquad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ, the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}(X,B) \arrow[r] & \mathrm{Hom}(B,B) \arrow[r, "\delta"] & \mathrm{Ext}^1(A,B) \\
\mathrm{Hom}(X,B) \arrow[r] \arrow[u, "\mathrm{id}"] & \mathrm{Hom}(B,B) \arrow[r, "\delta"'] \arrow[u, "\mathrm{id}"] & \mathrm{Ext}^1(A,B) \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ. Since R-**Mod** has enough projectives, we can find a projective P which surjects onto A. Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] & P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}$$

We construct from such an exact sequence a map which takes an element of $\mathrm{Ext}^1(A,B)$ and gives an extension of A by B. First applying $\mathrm{Ext}^*(-,B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\mathrm{Hom}(K,B) \xrightarrow{\delta} \mathrm{Ext}^1(A,B) \longrightarrow \mathrm{Ext}^1(P,B)$$

Since P is projective, the group $\mathrm{Ext}^1(P,B)$ vanishes, so $\delta\colon \mathrm{Hom}(K,B) \to \mathrm{Ext}^1(A,B)$ is surjective. Thus, given any $e \in \mathrm{Ext}^1(A,B)$, we can find a (not unique!) lift to $\phi \in \mathrm{Hom}(K,B)$. Take the pushout in the following diagram.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads] \arrow[d, "i_P"] & A \arrow[r] & 0 \\
 & B \arrow[r, hook, "i_B"] & B \amalg_K^\phi P & &
\end{tikzcd}$$

By the dual to Lemma 45, the cokernel of i_B is equal to the cokernel of ψ, so we get an induced short exact sequence as follows.

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\phi"'] & P \arrow[r, two heads] \arrow[d, "i_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "i_B"] & B \amalg_K^\phi P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}$$

It remains to show that Ψ as defined is independent of the choice of ϕ, and that it really is an inverse to Φ. To this end, pick a different $\tilde{\phi} \in \mathrm{Hom}(N, B)$ such that $\delta(\tilde{\phi}) = e$. Then there exists some $\rho \in \mathrm{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$.

This gives us a new morphism of exact sequences as follows

$$
\begin{tikzcd}
0 \arrow[r] & K \arrow[r, hook, "\psi"] \arrow[d, "\tilde{\phi}"'] & P \arrow[r, two heads] \arrow[d, "\tilde{i}_P"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, hook, "\tilde{i}_B"] & B \amalg_K^{\tilde{\phi}} P \arrow[r, two heads] & A \arrow[r] & 0
\end{tikzcd}
\tag{6.1}
$$

Consider the following outer square.

$$
\begin{tikzcd}
K \arrow[r, "\psi"] \arrow[d, "\phi"'] & P \arrow[d, "i_P"] \arrow[ddr, bend left, "\tilde{i}_P - \tilde{i}_B \circ \rho"] & \\
B \arrow[r, "i_B"] \arrow[drr, bend right, "\tilde{i}_B"'] & B \amalg_N^{\phi} P \arrow[dr, dashed, "\exists! \alpha"] & \\
 & & B \amalg_N^{\tilde{\phi}} P
\end{tikzcd}
$$

Using the commutativity of the left-hand square in Diagram 6.1, we find

$$
\begin{aligned}
(\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
&= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
&= \tilde{i}_B \circ \phi,
\end{aligned}
$$

so this outer square commutes, giving us a unique morphism α. $\square$

6.3. Quiver representations

Definition 107 (quiver). A quiver Q consists of the following data.

- A set Q_0 of *vertices*
- A set Q_1 of *arrows*
- A pair of maps

  $$s, t \colon Q_1 \to Q_0$$

  called the *source* and *target* maps respectively.

6.4. Group (co)homology

6.4.1. The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 108 (normalized chain complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2}d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1}d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} NA_{n-1} \xrightarrow{(-1)^{n-1}d_{n-1}} \cdots$$

Definition 109 (Moore complex)**.** Let $A\colon \Delta^{\mathrm{op}} \to \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n-chains A_n and differential

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned}
\partial^2 &= \left(\sum_{j=0}^{n-1} (-1)^j d_j\right) \circ \left(\sum_{i=0}^{n} (-1)^i d_i\right) \\
&= \sum\sum (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_j \circ d_i + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= \sum_{0\le j<i\le n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0\le i\le j\le n-1} (-1)^{i+j} d_j \circ d_i \\
&= 0.
\end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A, unless this is confusing. In this case, we will denote it by MA.

Definition 110 (alternating face maps chain modulo degeneracies)**.** Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial\colon A_n \to A_{n-1}$ takes degerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain modulo degeneracies.[1]

Lemma 111. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \xhookrightarrow{i} A \overset{p}{\twoheadrightarrow} A/D(A)$$

Proof. We need only show that the following diagram commutes.

$$\begin{tikzcd}
NA_n \arrow[r, hook] \arrow[d, "(-1)^n d_n"'] & A_n \arrow[r, two heads] \arrow[d, "\partial"] & A_n/DA_n \arrow[d, "{[\partial]}"] \\
NA_{n-1} \arrow[r, hook] & A_{n-1} \arrow[r, two heads] & A_{n-1}/DA_{n-1}
\end{tikzcd}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n, and the right-hand square commutes trivially. □

Theorem 112. The composite

$$p \circ i\colon NA \to A/D(A)$$

is an isomorphism of chain complexes.

Let $f\colon [m] \to [n]$ be a morphism in the simplex category Δ. By functoriality, f induces a map

$$N\Delta^n \to N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0\colon [n] \to [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]\colon \Delta \to \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

and on morphisms by

$$f\colon [m] \to [n] \mapsto$$

[1] The nLab is responsible for this terminology.

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N : \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \tag{6.2}$$

Theorem 113. The adjunction in Equation 6.2 is an equivalence of categories

Proof. later, if I have time. □

Also later if time, Dold-Puppe correspondence: this works not only for **Ab**, but for any abelian category.

6.4.2. The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta\colon \mathrm{id}_\mathcal{C} \to RL$ and counit $\varepsilon\colon LR \to \mathrm{id}_\mathcal{D}$.

Recall that this data gives a comonad in $\mathcal{D}$, i.e. a comonoid internal to the category $\mathrm{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$LRLR \xLeftarrow{L\eta R} LR \xRightarrow{\varepsilon} \mathrm{id}_\mathcal{D}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category $\mathcal{C}$ and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M}\colon \Delta \to \mathcal{C}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, dashed, "\exists!\tilde{M}"] \\ & \mathcal{C} \end{tikzcd}$$

In particular, with $\mathcal{C} = \mathrm{End}(\mathcal{D})^{\mathrm{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \to \mathrm{End}(\mathcal{D})^{\mathrm{op}}$.

$$\begin{tikzcd} \{*\} \arrow[r, "{[0]}"] \arrow[dr, "M"'] & \Delta_+ \arrow[d, "\exists!"] \\ & \mathrm{End}(\mathcal{D})^{\mathrm{op}} \end{tikzcd}$$

Equivalently, this gives a functor

$$B\colon \Delta_+^{\mathrm{op}} \longrightarrow \mathrm{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\mathrm{ev}_d\colon \mathrm{End}(\mathcal{D}) \longrightarrow \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in $\mathcal{D}$.

$$S_d = \Delta_+^{\mathrm{op}} \xrightarrow{B} \mathrm{End}(\mathcal{D}) \xrightarrow{\mathrm{ev}_d} \mathcal{D}$$

Definition 114 (bar construction)**.** The composition $\mathrm{ev}_d \circ B$ is called the bar construction.

In the case that the category $\mathcal{D}$ is abelian, we can form the Moore complex (Definition 109) of the bar construction.

Definition 115 (bar complex)**.** Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b\colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Set}$ the bar construction of b.

The bar complex Bar_b of b is the Moore complex of the bar construction of b.

Example 116. Consider the functor

$$U : \mathbb{Z}G\text{-}\mathbf{Mod} \longrightarrow \mathbf{Ab}$$

which forgets multiplication. This has left adjoint

$$\mathbb{Z}G \otimes_{\mathbb{Z}} -\colon \mathbf{Ab} \longrightarrow \mathbb{Z}G\text{-}\mathbf{Mod},$$

which assigns to an abelian group A the left $\mathbb{Z}G$-module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta\colon \mathrm{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_Z -)$ and counit $\varepsilon\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components

$$\eta_A\colon A \longrightarrow U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \qquad a \longmapsto 1 \otimes a$$

and

$$\varepsilon_M\colon \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \longrightarrow M; \qquad g \otimes m \longmapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z, we find the following comonad in $\mathbb{Z}G$-$\mathbf{Mod}$, where $\nabla =$

$\mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{tikzcd} Z \circ Z \\ Z \arrow[u, Rightarrow, "\nabla"'] \arrow[d, Rightarrow, "\varepsilon"] \\ \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \end{tikzcd}$$

This gives us a simplicial object in End($\mathbb{Z}G$-**Mod**).

$$\begin{tikzcd} \cdots\cdots\cdots \; Z \circ Z \circ Z \arrow[r, Rightarrow, shift left=6, "ZZ\varepsilon"] \arrow[r, Leftarrow, shift left=4] \arrow[r, Rightarrow, shift left=2] \arrow[r, Leftarrow] \arrow[r, Rightarrow, shift right=2] \arrow[r, Leftarrow, shift right=4] \arrow[r, Rightarrow, shift right=6, "\varepsilon ZZ"'] & Z \circ Z \arrow[r, Rightarrow, shift left=3, "Z\varepsilon"] \arrow[r, Leftarrow, "\nabla"] \arrow[r, Rightarrow, shift right=3, "\varepsilon Z"'] & Z \arrow[r, Rightarrow] & \mathrm{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \end{tikzcd}$$

Evaluating on a specific $\mathbb{Z}G$-module M then gives a simplicial object in $\mathbb{Z}G$-**Mod**.

We continue in the special case of $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$-module. In this case,

$$Z^{\circ n}(Z) = \overbrace{\mathbb{Z}G \otimes_Z \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G}^{n \text{ times}} \otimes_{\mathbb{Z}} \mathbb{Z}.$$

We are justified in suppressing the last copy of $\mathbb{Z}$.

The face maps d_i remove the ith copy of Z; on a generator, we have the action of $d_i \colon Z^{n+1}\mathbb{Z} \to Z^n\mathbb{Z}$ given by

$$d_i \colon x \otimes x_1 \otimes \cdots \otimes x_n \otimes a \mapsto \begin{cases} xx_1 \otimes \cdots \otimes x_n, & i = 0 \\ x \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, & 0 < i < n+1 \\ x \otimes \cdots \otimes x_{n-1}, & i = n+1. \end{cases}$$

It is traditional to use the notation

$$x \otimes x_1 \otimes \cdots \otimes x_n \otimes a = x[x_1|\cdots|x_n].$$

This notation is the origin of the name *bar complex.*

6.4.3. Group cohomology

One would like to be able to apply the techniques of homological algebra to the study of groups. Unfortunately, the category **Grp** is not very nicely behaved; in particular it is not abelian.

Out of any group G, one can form the group ring $\mathbb{Z}G$. Categorically speaking, the category of rings is arguably even worse then that of groups, so one might question the sanity of such a construction. However, one can study groups using homological techiques by studying modules over the group ring $\mathbb{Z}G$; as a category of modules, it is sure to be abelian.

Given a group G, a G-module is a functor $\mathbf{B}G \to \mathbf{Ab}$. More generally, the category of such G-modules is the category $\mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$. We will denote this category by G-**Mod**.

More explicitly, a G-module consists of an abelian group M, and a left G-action on M. However, since we are in **Ab**, we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$-module structure on M.

Because of the equivalence $G\text{-}\mathbf{Mod} \simeq \mathbb{Z}G\text{-}\mathbf{Mod}$, we know that G-**Mod** has enough projectives.

Note that there is a canonical functor $\mathrm{triv}\colon \mathbf{Ab} \to \mathbf{Fun}(\mathbf{B}G, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

Proposition 117. The functor triv has left adjoint

$$(-)_G\colon M \mapsto M_G = M/\langle g \cdot m - m \mid g \in G, m \in M\rangle$$

and right adjoint

$$(-)^G\colon M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \mathrm{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\mathrm{Hom}_{\mathbf{Ab}}(M_G, A) \equiv \mathrm{Hom}_{G\text{-}\mathbf{Mod}}(M, \mathrm{triv}\, A).$$

Starting on the left with a homomorphism $\alpha\colon M_G \to A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}\colon M \to A$ such that $\hat{\alpha}(g \cdot m - g) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}\colon M \to A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \tag{6.3}$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G-linear map $M \to \mathrm{triv}\, A$. In fact, G-linear maps $M \to \mathrm{triv}\, A$ are precisely those satisfying Equation 6.3, showing that this is really an isomorphism.

The other case is similar. □

Definition 118 (invariants, coinvariants)**.** For a G-module M, we call M^G the invariants of M and M_G the coinvariants.

Note that we actually have a fairly good handle on invariants and coinvariants.

Lemma 119. We have the formulae

$$(-)^G \simeq \mathrm{Hom}_{\mathbb{Z}G\text{-}\mathbf{Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula $\mathbb{Z}$ is taken to be a trivial right $\mathbb{Z}G$-module, and in the second a trivial left $\mathbb{Z}G$-module.

Proof. A $\mathbb{Z}G$-linear map $\mathbb{Z} \to M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$-linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. □

This tells us immediately that the invariants functor $(-)^G$ is left exact, and the coinvariant functor $(-)_G$ is right. We could have discovered this by explicit investigation, and formed the right and left derived hom functors. However, this would have required picking resolutions for each object under consideration, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of $\mathbb{Z}$ as a $\mathbb{Z}G$-module and get on with our lives.[2]

Example 120. Let $T \cong \mathbb{Z}$ be an infinite cyclic group with a single generator t. Then

$$\mathbb{Z}T \cong \mathbb{Z}[t, t^{-1}],$$

and we have an exact sequence

$$0 \longrightarrow \mathbb{Z}T \xrightarrow{t-1} \mathbb{Z}T \longrightarrow \mathbb{Z} \longrightarrow 0$$

giving a free resolution of $\mathbb{Z}$ as a $\mathbb{Z}T$-module.

In general, our life is not so easy. Fortunately, we still have, for any group G, a general construction of a free resolution of $\mathbb{Z}$ as a left $\mathbb{Z}G$-module, known as the *bar construction.*

[2]In the case of invariants (which correspond to Ext), we need to pick a resolution of $\mathbb{Z}$ as a *left* $\mathbb{Z}G$-module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of $\mathbb{Z}$ as a *right* $\mathbb{Z}G$-module.

Definition 121 (bar complex)**.** Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$-module

$$B_n = \bigoplus_{(g_i)\in(G\setminus\{e\})^n} \mathbb{Z}G[g_1|\cdots|g_n],$$

where $[g_1|\cdots|g_n]$ is simply a symbol denoting to the basis element corresponding to $(g_1,\ldots,g_n)$. The differential is defined by

$$d([g_1|\cdots|g_n]) = g_1[g_2|\cdots|g_n] + \sum_{i=1}^{n-1}(-1)^i[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] + (-1)^n[g_1|\cdots|g_{n-1}].$$

This is in fact a chain complex; the verification of this is identical to the verification of the simplicial identities. First, we note that we can write

$$d_n = d_0 + \sum_{i=1}^{n-1}(-1)^i d_i + (-1)^n d_n,$$

where the d_i satisfy the simplicial identities

$$d_i d_j = d_{j-1}d_i, \qquad i < j.$$

Thus, we have

$$\begin{aligned}
d_n \circ d_{n-1} &= \left(\sum_{j=0}^{n-1}(-1)^j d^j\right) \circ \left(\sum_{i=0}^{n}(-1)^i d^i\right)\\
&= \sum\sum(-1)^{i+j} d^j \circ d^i\\
&= \sum_{0\le j<i\le n}(-1)^{i+j} d^j \circ d^i + \sum_{0\le i\le j\le n-1}(-1)^{i+j} d^j \circ d^i\\
&= \sum_{0\le j<i\le n}(-1)^{i+j} d^{i-1} \circ d^j + \sum_{0\le i\le j\le n-1}(-1)^{i+j} d^j \circ d^i\\
&= 0.
\end{aligned}$$

Proposition 122. For any group G, the bar construction gives a free resolution of $\mathbb{Z}$ as a (left) $\mathbb{Z}G$-module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of $\mathbb{Z}$-modules, and providing a $\mathbb{Z}$-

linear homotopy between the identity and the zero map.

$$\begin{tikzcd}
B_2 \arrow[r] \arrow[d, "\mathrm{id}"] & B_1 \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_1"] & B_0 \arrow[r, "\epsilon"] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_0"] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] \arrow[dl, dashed, "h_{-1}"] & 0 \\
B_2 \arrow[r] & B_1 \arrow[r] & B_0 \arrow[r, "\epsilon"'] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

We define

$$h_{-1} \colon \mathbb{Z} \to B_0; \qquad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n \colon B_n \mapsto B_{n+1}; \qquad g[g_1|\cdots|g_n] \mapsto [g|g_1|\cdots|g_n].$$

We then have

$$\epsilon \circ h_{-1} \colon n \mapsto n[\cdot] \mapsto n,$$

and doing the butterfly

$$\begin{tikzcd}
B_n \arrow[r, "d_n"] & B_{n-1} \arrow[dl, dashed, "h_{n-1}"] \\
B_n &
\end{tikzcd}
\quad + \quad
\begin{tikzcd}
& B_n \arrow[dl, dashed, "h_n"] \\
B_{n+1} \arrow[r, "d_{n+1}"'] & B_n
\end{tikzcd}$$

gives us in one direction

$$g[g_1|\cdots|g_n] \longmapsto \begin{aligned} & gg_1[g_2|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i g[g_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &+ (-1)^n g[g_1|\cdots|g_{n-1}] \end{aligned}$$

$$\longmapsto \begin{aligned} & [gg_1|\cdots|g_n] \\ &+ \sum_{i=1}^{n-1}(-1)^i [gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ &+ (-1)^n [gg_1|\cdots|g_n] \end{aligned}$$

and in the other

$$\begin{tikzcd} & g[g_1|\cdots|g_n] \arrow[dl, mapsto] \\ {[g|g_1|\cdots|g_n]} \arrow[r, mapsto] & \begin{array}{l} g[g_1|\cdots|g_n] \\ -[gg_1|g_2|\cdots|g_n] \\ -\sum_{i=1}^{n-1}[gg_1|\cdots|g_ig_{i+1}|\cdots|g_n] \\ -(-1)^n[gg_1|\cdots|g_{n-1}] \end{array} \end{tikzcd}.$$

The sum of these is simply $g[g_1|\cdots|g_n]$, i.e. we have

$$h \circ d + d \circ h = \mathrm{id}.$$

This proves exactness as desired. □

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G, A) = H^i\left(0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G, A) = H_i\left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

The story so far is as follows.

1. We fixed a group G and considered the category G-**Mod** of functors $\mathbf{B}G \to \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \to G$-**Mod**, and that taking left- and right adjoints to this gave us interesting things.
   - The right adjoint $(-)^G$ applied to a G-module A gave us the subgroup of A stabilized by G.
   - The left adjoint $(-)_G$ applied to a G-module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
   - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
   - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived

functors $R^i(-)^G$.

4. We denote
$$L_i(-)_G = H_i(G,-), \qquad R^i(-)^G = H^i(G,-),$$
and call them *group homology* and *group cohomology* respectively.

5. We notice that, taking $\mathbb{Z}$ as a trivial $\mathbb{Z}G$-module, we have
$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \qquad A^G \equiv \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$
and thus that
$$H_i(G,A) = \mathrm{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \qquad H^i(G,A) = \mathrm{Ext}^i_{\mathbb{Z}G}(\mathbb{Z}, A).$$
This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of $\mathbb{Z}$ as a free $\mathbb{Z}G$-module, rather than having to take resolutions of A for all A. The bar complex gave us such a resolution.

Example 123. Let G be a group, and consider $\mathbb{Z}$ as a free $\mathbb{Z}G$-module. Let us compute $H_1(G,\mathbb{Z})$ and $H^1(G,\mathbb{Z})$.

To compute $H_1(G,\mathbb{Z})$, consider the sequence
$$\begin{tikzcd}
\cdots \arrow[r, "d_2\otimes \mathrm{id}_{\mathbb{Z}}"] & \bigoplus_{g\in G\smallsetminus\{1\}}[g]\mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r, "d_1\otimes \mathrm{id}_{\mathbb{Z}}"] \arrow[d, equal] & {[\cdot]}\mathbb{Z}G \otimes_{\mathbb{Z}G} \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r] & \bigoplus [g]\mathbb{Z} \arrow[r] & {[\cdot]}\mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$
The differential $d_1 \otimes \mathrm{id}_{\mathbb{Z}}$ acts on $[g]$ by sending it to
$$[\cdot]g - [\cdot] = [\cdot] - [\cdot],$$
i.e. everything is a cycle. The differential d_2 sends
$$[g|h] \mapsto [g]h - [gh] + [h] = [g] + [h] - [gh],$$
i.e. the terms $[gh]$ and $[g]+[h]$ are identitfied. Thus, $H_1(G,\mathbb{Z})$ is simply the abelianization of G.

To compute $H^1(G,\mathbb{Z})$, consider the sequence
$$0 \longrightarrow \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot], \mathbb{Z}) \xrightarrow{(d_1)^*} \mathrm{Hom}_{\mathbb{Z}G}\Big(\bigoplus_{g\in G\smallsetminus\{1\}} \mathbb{Z}G[g], \mathbb{Z}\Big) \xrightarrow{(d_2)^*} \cdots .$$
Notice immediately that since the hom functor preserves direct sums in the first slot,

we have

$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}G}\left(\bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z}\right) &\cong \bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\operatorname{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1|\cdots|g_n],\mathbb{Z})\\ &\cong \bigoplus_{(g_i)\in(G\smallsetminus\{1\})^n}\mathbb{Z}[g_1|\cdots|g_n].\end{aligned}$$

This notation deserves some explanation; recall that the symbols $[g_1|\cdots|g_n]$ are merely visual aids, reminding us where we are and how the differential acts. The differential

$$(d_n)^*\colon [g_1|\cdots|g_n].$$

In the low cases we have the sequence

7. Spectral sequences

7.1. Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section 5.7, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the building blocks of the homology of the total complex such a double complex.

They do this via a series of successive approximations. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
E_{0,3} \arrow[d, "d^v"'] & E_{1,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,2} \arrow[d, "d^v"'] & E_{1,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,1} \arrow[d, "d^v"'] & E_{1,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,0} & E_{1,0} \arrow[l, "d^h"'] & E_{2,0} \arrow[l, "d^h"'] & E_{3,0} \arrow[l, "d^h"'] & \cdots \arrow[l]
\end{tikzcd}$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$E^0_{\bullet,\bullet}$

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^0_{0,3} \arrow[d, "d^v"'] & E^0_{1,3} \arrow[d, "d^v"'] & E^0_{2,3} \arrow[d, "d^v"'] & E^0_{3,3} \arrow[d, "d^v"'] \\
E^0_{0,2} \arrow[d, "d^v"'] & E^0_{1,2} \arrow[d, "d^v"'] & E^0_{2,2} \arrow[d, "d^v"'] & E^0_{3,2} \arrow[d, "d^v"'] \\
E^0_{0,1} \arrow[d, "d^v"'] & E^0_{1,1} \arrow[d, "d^v"'] & E^0_{2,1} \arrow[d, "d^v"'] & E^0_{3,1} \arrow[d, "d^v"'] \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{tikzcd}
\tag{7.1}
$$

One then takes homology of the corresponding vertical complexes, replacing $E^0_{p,q}$ by the homology $H_q(E^0_{p,\bullet})$. To ease notation, one writes

$$H_q(E^0_{p,\bullet}) = E^1_{p,q}.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$
\begin{tikzcd}
E^1_{0,3} & E^1_{1,3} \arrow[l, "H_3(d^h)"'] & E^1_{2,3} \arrow[l, "H_3(d^h)"'] & E^1_{3,3} \arrow[l, "H_3(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "H_2(d^h)"'] & E^1_{2,2} \arrow[l, "H_2(d^h)"'] & E^1_{3,2} \arrow[l, "H_2(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "H_1(d^h)"'] & E^1_{2,1} \arrow[l, "H_1(d^h)"'] & E^1_{3,1} \arrow[l, "H_1(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "H_0(d^h)"'] & E^1_{2,0} \arrow[l, "H_0(d^h)"'] & E^1_{3,0} \arrow[l, "H_0(d^h)"'] & \cdots \arrow[l]
\end{tikzcd}
\tag{7.2}
$$

We now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{\bullet,q})$.

The claim is that the terms $E^r_{p,q}$ are pieces of a successive approximation of the homology of the total complex $\mathrm{Tot}(E)$. We will prove this claim much later. However, in the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Example 124. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes $E_{0,\bullet}$ and $E_{1,\bullet}$ and a morphism $f\colon E_{1,\bullet} \to E_{0,\bullet}$, padded by zeroes on either side.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
E_{0,3} \arrow[d, "d^0_3"'] & E_{1,3} \arrow[l, "f_3"'] \arrow[d, "-d^1_3"] \\
E_{0,2} \arrow[d, "d^0_2"'] & E_{1,2} \arrow[l, "f_2"'] \arrow[d, "-d^1_2"] \\
E_{0,1} \arrow[d, "d^0_1"'] & E^{1,1} \arrow[l, "f_1"'] \arrow[d, "-d^1_1"] \\
E_{0,0} \arrow[d, "d^0_0"'] & E_{1,0} \arrow[l, "f_0"'] \arrow[d, "-d^1_0"] \\
E_{-0,1} \arrow[d] & E_{-1,1} \arrow[l, "f_{-1}"'] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Note that we have multiplied the diferentials of $E_{1,\bullet}$ by -1 so that we have a double complex. This does not enter into the discussion.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$.

Let $T = \mathrm{Tot}(E)$; that is, let $T = \mathrm{Cone}(f)$.

Recall that $\mathrm{Cone}(f)$ fits into the following short exact seqence,

$$0 \longrightarrow E_{\bullet,1} \hookrightarrow E_{\bullet,0} \twoheadrightarrow \mathrm{Cone}(f)_\bullet \longrightarrow 0$$

and that this gives us the following long exact sequence on homology.

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{n+1}(\mathrm{Cone}(f)) \arrow[dlll, out=-10, in=170, "\delta"] \\ & H_n(E_{\bullet,1}) \arrow[r, "H_n(f)"] & H_n(E_{\bullet,0}) \arrow[r, "H_n(p)"] & H_n(\mathrm{Cone}(f)_\bullet) \arrow[dll, out=-10, in=170, "\delta"] \\ & H_{n-1}(E_{\bullet,1}) \arrow[r] & \cdots & \end{tikzcd}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \operatorname{coker}(H_n(f)) \hookrightarrow H_n(\mathrm{Cone}(f)) \twoheadrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation; that is, we have a short exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0 \ .$$

7.2. Homology spectral sequences

7.2.1. Notation and terminology

Definition 125 (homology spectral sequence)**.** Let $\mathcal{A}$ be an abelian category. A homology spectral sequence starting at E^a in $\mathcal{A}$ consists of the following data.

1. A family $E^r_{p,q}$ of objects of $\mathcal{A}$, defined for all integers p, q, and $r \geq a$.
2. Maps
$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r+1}$$
that are differentials in the sense that $d^r \circ d^r = 0$.
3. Isomorphisms
$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1})$$
between $E^{r+1}_{p,q}$ and the homology at the corresponding position of the E^r

We denote such a spectral sequence by E. A morphism $E' \to E$ is a family of maps

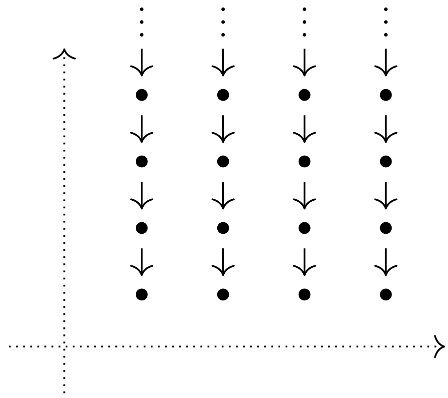
$$f^r_{p,q} \colon E'^r_{p,q} \to E^r_{p,q}$$

such that $d^r f^r = f^r d^r$ such that $f^{r+1}_{p,q}$ is induced on homology by $f^r_{p,q}$.

For each r, a spectral sequence E has a two-dimensional array of objects $E^r_{p,q}$. For a given r, one calls the objects $E^r_{p,q}$ the rth *page* of the spectral sequence E.

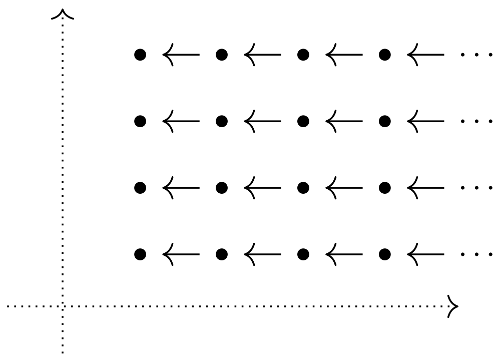
We have already seen an example of the first two pages of a spectral sequence (in Diagram 7.1 and Diagram 7.2). Schematically, the differentials on the zeroth page point down.

E^0 :



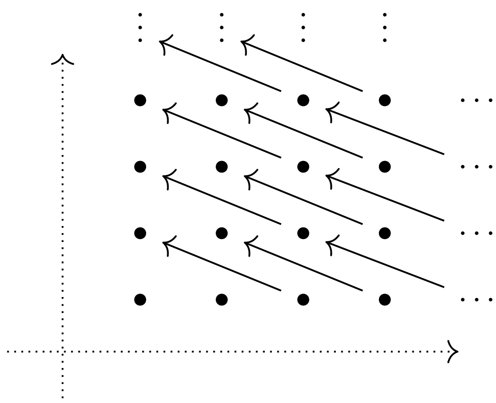
The differentials on the first page point to the left.

E^1 :



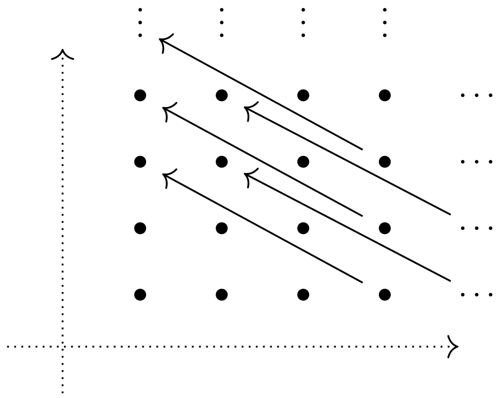
As r increases, the arrows rotate clockwise. The second page looks like this.

E^2 :



The third page looks like this.

E^3 :



7.2.2. Stabilization and convergence

We have so far placed no conditions on $E^r_{p,q}$. However, we will confine our study of certain classes of spectral sequences.

Definition 126 (bounded, first quadrant)**.** Let E be a spectral sequence starting at E^a.

- We say that E is bounded if each $n \in \mathbb{Z}$, there are only finitely many terms of total degree n on the ath page, i.e. terms of the form $E^a_{p,q}$ for $n = p + q$.
- We say that E is first-quadrant if $E^a_{p,q} = 0$ for $p < 0$ and $q < 0$.

Clearly, every first-quadrant spectral sequence is bounded.

Note that if a spectral sequence is bounded, then

We get $E^{r+1}_{p,q}$ by taking the homology of a complex on the rth page. If E is a first-quadrant spectral sequence, then for $r > \max(p, q + 1)$, the differential entering $E^r_{p,q}$ comes from the fourth quadrant (hence from a zero object), and the differential leaving it lands in the second quadrant (also a zero object). Thus, $E^{r+1}_{p,q} = E^r_{p,q}$.

Definition 127 (stabilize)**.** A spectral sequence starting at E^a is said to stabilize at position (p, q) if there exists some $r \geq a$ such that $E^{r'}_{p,q} = E^r_{p,q}$ for all $r' \geq r$. In this case, we write $E^\infty_{p,q}$ for the stable value at position (p, q).

By the argument above, each position in a first-quadrant spectral sequence eventually stabilizes. We will mostly be interested in first-quadrant spectral sequences. However, boundedness is (clearly!) sufficient to guarantee stabilization.

Suppose one has been given a bounded spectral sequence, which we now know eventually stabilizes at each position. We would like to get some useful information out of such a spectral sequence.

Definition 128 (filtration)**.** Let $C_\bullet$ be a chain complex. A filtration F of $C_\bullet$ is a chain of inclusions of subcomplexes $F_pC_\bullet$ of $C_\bullet$.

$$\cdots \subset F_{p-1}C_\bullet \subset F_pC_\bullet \subset F_{p+1}C_\bullet \subset \cdots \subset C_\bullet$$

Definition 129 (bounded convergence)**.** Let E be a bounded spectral sequence starting at E^a in an abelian category $\mathcal{A}$, and let $\{H_i\}_{i\in Z}$ be a collection of objects in $\mathcal{A}$, each equipped with a finite filtration

$$0 = F_sH_n \subset \cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset F_tH_n = H_n.$$

We say that E converges to $\{H_i\}_{i\in\mathbb{Z}}$ if we are given isomorphisms

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

We express this convergence by writing

$$E^a_{p,q} \Rightarrow H_{p+q}.$$

Example 130. Consider the spectral sequence E starting at E^0 from Example 124. This is certainly bounded, and stabilizes at each position (p,q) after the second page; that is, we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

The exact sequence

$$\begin{tikzcd} 0 \arrow[r] & E^2_{p-1,q+1} \arrow[r, hook] & H_{p+q}(T) \arrow[r, two heads] & E^2_{p,q} \arrow[r] & 0 \end{tikzcd}$$

gives us, in a wholly unsatisfying and trivial way, a filtration

$$\begin{tikzcd} F_{-1}H_n \arrow[d, equal] & F_0H_n \arrow[d, equal] & F_1H_n \arrow[d, equal] \\ 0 \arrow[r, hook] & E^2_{0,n} \arrow[r, hook] & H_n \end{tikzcd}$$

of H_n. The claim is that

$$E^0_{p,q} \Rightarrow H_{p+q}.$$

In order to check this, we have to check that

$$E^2_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q. The only non-trivial cases are when $p = 0$ or 1. When $p = 0$, we have

$$\begin{aligned} E^\infty_{0,n} &\overset{!}{\cong} F_0H_n/F_{-1}H_n \\ &\cong E^2_{0,n}/0 \\ &\cong E^2_{0,n}, \end{aligned}$$

and when $p = 1$ we have

$$\begin{aligned} E^\infty_{1,n} &\overset{!}{\cong} F_1H_{n+1}/F_0H_{n+1} \\ &\cong H_{n+1}/E^2_{0,n+1} \\ &\cong E^2_{1,n} \end{aligned}$$

as required.

7.2.3. The spectral sequence of a filtered complex

Theorem 131. A filtration F of a chain complex $C_\bullet$ naturally determines a spectral sequence starting with

$$E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}, \qquad E^1_{p,q} = H_{p+1}(E^0_{p,\bullet})$$

Proof. We construct explicitly the spectral sequence above .Let $C_\bullet$ be a chain complex, and F a filtration. We will use the following notation.

- We will denote by η_p the quotient

  $$\eta_p \colon F_pC \twoheadrightarrow F_pC/F_{p-1}C.$$

  coming from the short exact sequence

  $$0 \longrightarrow F_{p-1}C \hookrightarrow F_pC \twoheadrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

- We will denote by A^r_p the subobject

  $$A^r_p = \{c \in F_pC \colon dc \in F_{p-r}C\} \subset F_pC$$

  of those elements whose differentials survive r many levels of the grading.

- We will denote by Z^r_p the image

  $$Z^r_p = \eta_p(A^r_p).$$

- We will denote by B_p^r the image

$$B_p^r = \eta_p(d(a_{p+r-1}^{r-1})).$$

Examining the definitions, we see that we have the following inclusion.

$$\begin{tikzcd} dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ dA_{p+r}^{r} \arrow[r, equal] & \{dc \mid c \in F_{p+r},\ dc \in F_pC\} \end{tikzcd}$$

Both of these are subobjects of F_pC; by applying η_p and taking a sharp look at the definition of the Bs, we find an inclusion

$$B_p^r \subset B_p^{r+1}.$$

Working inductively, we are left with the following sequence of inclusions.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^r \subset \cdots$$

Defining $B_p^\infty = \bigcup_r B_p^r$, we can crown our sequence of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty.$$

Again examining definitions, we find the following inclusion.

$$\begin{tikzcd} A_p^r \arrow[r, equal] \arrow[d, hook] & \{c \in F_pC \mid dc \in F_{p-r}C\} \arrow[d, hook] \\ A_p^{r+1} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r-1}C\} \end{tikzcd}$$

As before, applying η_p and working inductively gives us a chain

$$\cdots \subset Z_p^r \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0$$

and defining $Z_p^\infty = \bigcap_r Z_p^r$ gives us

$$Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0,$$

But we should not rest on our laurels. Instead, note that for each r, r', we have an inclusion

$$\begin{tikzcd} dA^{r-1}_{p+r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ A^{r'}_p \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r'}C\} \end{tikzcd}$$

since $d^2 = 0$. Thus, we can graft our chains of inclusions as follows.

$$0 = B^0_p \subset B^1_p \subset \cdots \subset B^\infty_p \subset Z^\infty_p \subset \cdots \subset Z^1_p \subset Z^0_p = E^0_p.$$

We defined

$$Z^r_p = \eta_p(A^r_p).$$

Recall that η_p was defined to be the canoncal projection corresponding to a short exact sequence. Taking subobjects (where the left-hand square is a pullback), we find the following monomorphism of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1}C \cap A^p_r \arrow[r, hook] \arrow[d, hook] & A^p_r \arrow[r, two heads] \arrow[d, hook] & Z^p_r \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F_{p-1}C \arrow[r, hook] & F_pC \arrow[r, two heads] & F_pC/F_{p-1}C \arrow[r] & 0 \end{tikzcd}$$

Rewriting the definitions, we find that

$$F_{p-1}C \cap A^p_r = \{c \in F_{p-1}C \mid dc \in F_{p-r}C\} = A^{r-1}_{p-1},$$

so

$$Z^r_p \cong A^r_p / A^{r-1}_{p-1}.$$

We now define

$$E^r_p = \frac{Z^r_p}{B^r_p}.$$

We can write

$$\frac{Z^r_p}{B^r_p} \cong \frac{A^r_p + F_{p-1}C}{d(A^{r-1}_{p+r-1}) + F_{p-1}C} \cong \frac{A^r_p}{d(A^{r-1}_{p+r-1}) + A^{r-1}_{p-1}}.$$

I don't understand this step, and I don't have time to do the rest. □

Theorem 132.

- Let $C_\bullet$ be a chain complex, and F a bounded filtration on $C_\bullet$.

7.2.4. The spectral sequence of a double complex

7.3. Hyper-derived functors

Consider abelian categories and right exact functors as follows.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

One might wonder about the relationship between LF, LG, and $L(G \circ F)$.

The problem is that we don't know how to make sense of the composition $LG \circ LF$, since LF lands in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ rather than $\mathcal{A}$; derived functors go from an abelian category to a category of chain complexes.

7.3.1. Cartan-Eilenberg resolutions

Definition 133 (Cartan-Eilenberg resolution)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $A \in \mathbf{Ch}^{+}(\mathcal{A})$. A Cartan-Eilenberg resolution of A is an upper-half-plane double complex $(P_{\bullet,\bullet}, d^h, d^v)$, equipped with an augmentation map

$$\epsilon \colon P_{\bullet,0} \to A_{\bullet}$$

such that the following criteria are satsfied.

1. $B(P_{p,\bullet}, d^h) \xrightarrow{\simeq} B(A_p, d^A)$ is a projective resolution.
2. $H(P_{p,\bullet}, d^h) \xrightarrow{\simeq} H(A_p, d^A)$ is a projective resolution.
3. $A_p \cong 0 \implies P_{p,\bullet} = 0$.

Here, $B(P_{p,\bullet}, d^h)$ are the horizontal boundaries of the chain complex $(P_{p,\bullet}, d^h)$, and

$$H(P_{p,\bullet}, d^h) \cong \frac{Z(P_{p,\bullet}, d^h)}{B(P_{p,\bullet}, d^h)}$$

is defined similarly.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & P_{-r,1} \arrow[l] \arrow[d] & \cdots \arrow[l] & P_{0,1} \arrow[l] \arrow[d] & P_{1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[u, leftarrow] \arrow[d, dashed] & P_{-r,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] & P_{0,0} \arrow[l] \arrow[d, dashed] & P_{1,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
0 & A_{-r} \arrow[l] & \cdots \arrow[l] & A_0 \arrow[l] & A_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Proposition 134. We also have that the following are projective resolutions.

1. $Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)$
2. $P_{p,\bullet} \xrightarrow{\simeq} A_p$

Proof.

1. We have essentially by definition the following short exact sequence.

$$0 \longrightarrow B(P_{p,\bullet}, d^h) \hookrightarrow Z(P_{p,\bullet}, d^h) \twoheadrightarrow H(P_{p,\bullet}, d^h) \longrightarrow 0$$

   The outer terms are projective because they are the terms of projective resolutions, so $Z(P_{p,\bullet}, d^h)$ is also projective.

   LES on homology implies

$$Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)$$

   is a projective resolution.

2.

□

Proposition 135. Every chain complex $A_\bullet$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$.

Proof. We construct a Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ explicitly. Pick projective resolutions

$$P^B_{p,\bullet} \to B(A_p, d^h), \qquad B^H_{p,\bullet} \xrightarrow{\simeq} H(A_p, d^h).$$

By the horseshoe lemma, we can find a projective resolution $P^Z_{p,\bullet} \xrightarrow{\simeq} Z(A_p, d^h)$ fitting into the following exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & P^B_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^Z_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^H_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & B(A_p, d^h) \arrow[r, hook] & Z(A_p, d^h) \arrow[r, two heads] & H(A_p, d^h) \arrow[r] & 0
\end{tikzcd} \tag{7.3}$$

Playing the same game again, we find a projective resolution of A_p.

$$\begin{tikzcd}
0 \arrow[r] & P^Z_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^A_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & Z(A_p, d^h) \arrow[r, hook] & A(A_p, d^h) \arrow[r, two heads] & B(A_{p-1}, d^h) \arrow[r] & 0
\end{tikzcd} \tag{7.4}$$

We then define our Cartan-Eilenberg resolution to be the double complex whose pth column is $(P^A_{p,\bullet}, (-1)^p d^{P^A})$, and whose horizontal differentials are given by the composition

$$\begin{tikzcd}
P^A_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^Z_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^A_{p-1,\bullet} \arrow[d, "\simeq"] \\
A_p \arrow[r, "d^A"] & B_{p-1} \arrow[r, hook] & Z_{p-1} \arrow[r, hook] & A_{p-1}
\end{tikzcd} \tag{7.5}$$

In this case, all the criteria are satisfied by definition. □

Proposition 136. Let $A_\bullet$ and $A'_\bullet$ be bounded-below chain complexes, and let $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$ and $P'_{\bullet,\bullet} \xrightarrow{\simeq} A'_\bullet$ be Cartan-Eilenberg resolutions. Then any morphism of chain complexes $f: A_\bullet \to A'_\bullet$ can be lifted to a morphism $P^A_{\bullet,\bullet} \to P^B_{\bullet,\bullet}$ of double complex.

Proof. We get □

Proposition 137. Let $P^A_{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A_\bullet$. Then the canonical map

$$\mathrm{Tot}(P^A_{\bullet,\bullet}) \to A$$

is a quasi-isomorphism

Proof. Spectral sequence with vertical filtration. □

Part II.

Algebraic topology

8. Homology

8.1. Basic definitions and examples

We assume a basic knowledge of simplicial sets just to get the ball rolling.

Definition 138 (singular complex, singular homology)**.** Let X be a topological space. The singular chain complex $C_\bullet(X)$ is defined level-wise by

$$C_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where $\mathcal{F}$ denotes the free group functor and M is the Moore functor (Definition 109). That is, it has differentials

$$d_n : C_n \to C_{n-1}; \qquad (\alpha : \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(C_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f : X \to Y$ gives a chain map $C(f) : C(X) \to C(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 139. Denote by pt the one-point topological space. Then $C(\mathrm{pt})_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & C_3 \arrow[r, "d_3"'] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Example 140. Let X be a path-connected topological space. Then $H_0(X) = \mathbb{Z}$. To see this, consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

8.2. The Hurewicz homomorphism

Theorem 141 (Hurewicz)**.** For any path-connected topological space X, there is an isomorphism

$$h_X \colon H_1(X) \cong \pi_1(X)_{\mathrm{ab}},$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism between the functors

$$h \colon H_1 \Rightarrow (\pi_1)_{\mathrm{ab}}.$$

Example 142. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\mathrm{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \qquad n > 1.$$

Example 143. Since

$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

8.3. Homotopy equivalence

Proposition 144. Let $f, g \colon X \to Y$ be continuous maps between topological spaces, and let $H \colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 145. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 146. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

8.4. Relative homology; the long exact sequence of a pair of spaces

Definition 147 (relative homology)**.** Let X be a topological space, and let $X \subset A$. The relative chain complex of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Denote by **Pair** the category whose objects are pairs (X, A), where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \to (Y, B)$ are maps $f\colon X \to Y$ such that $F(A) \subset B$.

Lemma 148. For each n, relative homology provides a functor $H_n\colon \mathbf{Pair} \to \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{tikzcd} S(X)_\bullet \arrow[r, "f"] \arrow[d, two heads] & S(Y)_\bullet \arrow[d, two heads] \\ S(X)_\bullet/S(A)_\bullet \arrow[r, dashed] & S(Y)_\bullet/S(B)_\bullet \end{tikzcd}$$

The dashed arrow is well-defined because of the assumption that $f(A) \subset B$. Functoriality now follows from the functoriality of H_n. □

Proposition 149. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{tikzcd}
& & \cdots \arrow[r] & H_{j+1}(X,A) \arrow[dll, "\delta", rounded corners] \\
H_j(A) \arrow[r] & H_j(X) \arrow[r] & H_j(X,A) \arrow[dll, "\delta", rounded corners] & \\
H_{j-1}(A) \arrow[r] & \cdots & &
\end{tikzcd}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X,A) \longrightarrow 0$$

□

Example 150. Let $X = \mathbb{D}^n$, the n-disk, and $A = \mathbb{S}^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{tikzcd}
& & \cdots \arrow[r] & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta", rounded corners] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & H_j(\mathbb{D}^n) \arrow[r] & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta", rounded corners] & \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots & &
\end{tikzcd}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

8.5. Barycentric subdivision

Fact 151. Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X. Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha\colon \Delta^n \to X$$

whose images are completely contained in some open set in the open cover $\mathfrak{U}$. That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$

induces a cochain structure on $S^{\mathfrak{U}}_{\bullet}(X)$.

$$\begin{tikzcd}
\cdots \arrow[r] & S^{\mathfrak{U}}_2(X) \arrow[r] \arrow[d, hook] & S^{\mathfrak{U}}_1(X) \arrow[r] \arrow[d, hook] & S^{\mathfrak{U}}_0(X) \arrow[r] \arrow[d, hook] & 0 \\
\cdots \arrow[r] & S_2(X) \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & 0
\end{tikzcd}$$

In fact, this inclusion is homotopic to the identity, hence induces an isomorphism

$$H^{\mathfrak{U}}_n(X) := H_n(S^{\mathfrak{U}}(X)_{\bullet}) \equiv H_n(X).$$

This fact allows us almost immediately to read of two important theorems.

8.5.1. Excision

Theorem 152 (excision)**.** Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{tikzcd}
A \setminus W \arrow[r, hook, "i"] \arrow[d, hook] & A \arrow[d, hook] \\
X \setminus W \arrow[r, hook, "i"] & X
\end{tikzcd}$$

induce an isomorphism

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A).$$

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A.

8.5.2. The Mayer-Vietoris sequence

Theorem 153 (Mayer-Vietoris)**.** Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X, i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, out=-10, in=170, "\delta"'] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n(X) \arrow[dll, out=-10, in=170, "\delta"'] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}
$$

Proof. We can draw our inclusions as the following pushout.

$$
\begin{tikzcd}
 & X_1 \arrow[dr, hook, "\kappa_1"] & \\
X_1 \cap X_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\
 & X_2 \arrow[ur, hook, "\kappa_1"'] &
\end{tikzcd}
$$

We have, almost by definition, the following short exact sequence.

$$
\begin{tikzcd}
0 \arrow[r] & S_\bullet(X_1 \cap X_2) \arrow[r, hook, "(i_1,i_2)"] & S_\bullet(X_1) \oplus S_\bullet(X_2) \arrow[r, two heads, "\kappa_1-\kappa_2"] & S_\bullet^{\mathfrak{U}}(X) \arrow[r] & 0
\end{tikzcd}
$$

This gives the following long exact sequence on homology.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H^{\mathfrak{U}}_{n+1}(X) \arrow[dll, out=-10, in=170, "\delta"'] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H^{\mathfrak{U}}_n(X) \arrow[dll, out=-10, in=170, "\delta"'] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}
$$

We have seen that $H^{\mathfrak{U}}(n)(X) \cong H_n(X)$; the result follows. ◻

Example 154 (Homology groups of spheres)**.** We can decompose $\mathbb{S}^n$ as

$$\mathbb{S}^n = (\mathbb{S}^n \smallsetminus N) \cup (\mathbb{S}^n \smallsetminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$
\begin{tikzcd}
 & \mathbb{S}^n \smallsetminus N \arrow[dr, hook] & \\
(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & \mathbb{S}^n \smallsetminus S \arrow[ur, hook] &
\end{tikzcd}
$$

The Mayer-Vietoris sequence is as follows.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, rounded corners, "\delta" description] \\
H_j((S^n \smallsetminus N) \cap (S^n \smallsetminus S)) \arrow[r] & H_j(S^n \smallsetminus N) \oplus H_j(S^n \smallsetminus S) \arrow[r] & H_j^{\mathfrak{U}}(X) \arrow[dll, rounded corners, "\delta" description] \\
H_{j-1}((\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S)) \arrow[r] & \cdots &
\end{tikzcd}
$$

We know that

$$\mathbb{S}^n \smallsetminus N \cong \mathbb{S}^n \smallsetminus S \cong \mathbb{D}^n \simeq \mathrm{pt}$$

and that

$$(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, rounded corners, "\delta" description] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & 0 \arrow[r] & H_j(\mathbb{S}^n) \arrow[dll, rounded corners, "\delta" description] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}
$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \equiv H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X, $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\mathrm{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$j = 3$	$\mathbb{Z}$	0		
$j = 2$	$\mathbb{Z}$	0		
$j = 1$	$\mathbb{Z}$	$\mathbb{Z}$		
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \qquad i > 1$$

allows us to fill in the above table as follows.

$j = 3$	$\mathbb{Z}$	0	0	$\mathbb{Z}$
$j = 2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$j = 1$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

(8.1)

Example 155. Above, we used the Hurewicz homomorphism to see that

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed $\mathbb{S}^n$ as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$\begin{tikzcd}
 & X^+ \arrow[dr, hook] & \\
X^+ \cap X^- \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & X^- \arrow[ur, hook] &
\end{tikzcd}$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^n) \arrow[dll, "\delta"] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0,i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r] & \cdots \end{tikzcd}$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^-, with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+, and $i_1(p)$ is a point of X^-. Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[dll, "\delta"] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0,i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r, "H_0(\kappa_1)-H_0(\kappa_2)"] & H_0(\mathbb{S}^1) \end{tikzcd}$$

We can immediately replace things we know, finding the following.

$$(a, b) \longmapsto (a + b, a + b)$$

$$0 \longrightarrow H_1(\mathbb{S}^1) \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z}$$

$$(c, d) \longmapsto c - d$$

The kernel of f is the free group generated by (a, a). Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

8.5.3. The relative Mayer-Vietoris sequence

Theorem 156 (relative Mayer-Vietoris sequence)**.** Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X, A \cup B) \arrow[dll, "\delta", rounded corners] \\
H_n(X, A \cap B) \arrow[r] & H_n(X, A) \oplus H_n(X, B) \arrow[r] & H_n(X, A \cup B) \arrow[dll, "\delta", rounded corners] \\
H_{n-1}(X, A \cap B) \arrow[r] & \cdots &
\end{tikzcd}
$$

Proof. Consider the following chain complex of chain complexes.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem 52) implies that the last row is also exact.

Consider the following map of chain complexes; the first row is the last column of the above grid.

$$
\begin{tikzcd}
0 \arrow[r] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r, hook] \arrow[d, "\phi"'] & S_n(X) \arrow[r, two heads] \arrow[d, equal] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & S_n(A \cup B) \arrow[r] & S_n(X) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0
\end{tikzcd}
$$

This gives us, by Lemma 49, a morphism of long exact sequences on homology.

$$
\begin{tikzcd}
H_n(S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\phi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_\bullet(X)/S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\psi)"] & H_{n-1}(S_\bullet \mathfrak{U}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\phi)"] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}
$$

We have seen (in Fact 151) that $H_i(\phi)$ is an isomorphism for all i. Thus, the five lemma (Theorem 51) tells us that $H_n(\psi)$ is an isomorphism. □

8.6. Reduced homology

It would be hard to argue that Table 8.1 is not pretty, but it would be much prettier were it not for the $\mathbb{Z}$s in the first column. We have to carry these around because every non-empty space has at least one connected component.

The solution is to define a new homology $\tilde{H}_n(X)$ which agrees with $H_n(X)$ in positive degrees, and is missing a copy of $\mathbb{Z}$ in the zeroth degree. There are three equivalent ways of doing this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Denoting the unique map $X \to \mathrm{pt}$ by ϵ, one can define relative homology by
$$\tilde{H}_n(X) = \ker H_n(\epsilon).$$

2. **In between:** One can replace homology $H_n(X)$ by relative homology
$$\tilde{H}_n(X) = H_n(X, x),$$
where $x \in X$ is any point of x.

3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of $\mathbb{Z}$ in degree -1, so that
$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$
Then one can define
$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the following. In constructing the singular chain complex of our space X, we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1)-simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through $\mathcal{F}$ thus gives a copy of $\mathbb{Z}$ as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- (1 ⇔ 2): Since the composition

$$\mathrm{pt} \xhookrightarrow{x} X \xrightarrow{\epsilon} \mathrm{pt}$$

is a weak retract, the sequence

$$0 \longrightarrow S_n(\mathrm{pt}) \longrightarrow S_n(X) \longrightarrow S_n(X, \{x\}) \longrightarrow 0$$

Proposition 157. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Proof. Trivial from algebraic definition. □

Many of our results for regular homology hold also for reduced homology.

Proposition 158. There is a long exact sequence for a pair of spaces

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{j+1}(X, A) \arrow[dll, "\delta"] \\ \tilde{H}_j(A) \arrow[r] & \tilde{H}_j(X) \arrow[r] & \tilde{H}_j(X, A) \arrow[dll, "\delta"] \\ \tilde{H}_{j-1}(A) \arrow[r] & \cdots & \end{tikzcd}$$

Proposition 159. We have a reduced Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{n+1}(X) \arrow[dll, "\delta"] \\ \tilde{H}_n(X_1 \cap X_2) \arrow[r] & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \arrow[r] & \tilde{H}_n(X) \arrow[dll, "\delta"] \\ \tilde{H}_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots & \end{tikzcd}$$

Proposition 160. Let $\{(X_i, x_i)\}_{i \in I}$, be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$[1] we have

$$\tilde{H}_n\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in E} \tilde{H}_n(X_i).$$

[1]This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of Proposition 159 tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 161 (good pair)**.** A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X.
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U.

$$A \hookrightarrow U \overset{r}{\twoheadrightarrow} A$$

Proposition 162. Let (X, A) be a good pair. Let $\pi\colon X \to X/A$ be the canonical projection. Then

$$\tilde{H}(X, A) \cong \tilde{H}_n(X/A) \qquad \text{for all } n > 0.$$

Proof. □

Theorem 163 (suspension isomorphism)**.** Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

8.7. Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n. Let $f\colon \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d\,\mu_n, \qquad \text{for some } d \in \mathbb{Z}.$$

Definition 164 (mapping degree)**.** We call $d \in \mathbb{Z}$ as above the mapping degree of f, and denote it by $\deg(f)$.

Example 165. Consider the map

$$\omega\colon [0,1] \longrightarrow \mathbb{S}^1; \qquad t \mapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism (Theorem 141), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1; \qquad x \mapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism (Theorem 141) tells us that the following diagram commutes.

$$\begin{tikzcd} \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[r, "\pi_1(f_n)_{\mathrm{ab}}"] \arrow[d, "h_{\mathbb{S}^1}"'] & \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[d, "h_{\mathbb{S}^1}"] \\ H_1(\mathbb{S}^1) \arrow[r, "H_1(f_n)"'] & H_1(\mathbb{S}^1) \end{tikzcd}$$

8.8. CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 166 (cell)**.** Let X be a topological space. We say that X is an n-cell if X is homeomorphic to $\mathbb{R}^n$. We call the number n the dimension of X.

Definition 167 (cell decomposition)**.** A cell decomposition of a topological space X is a decomposition

$$X = \coprod_{i \in I} X_i, \qquad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topologial spaces.

Definition 168 (CW complex)**.** A Hausdorff topological space is known as a CW complex[2] if it satisfies the following conditions.

[2] Axiom (CW2) is called the *closure-finiteness* condition. This is the 'C' in CW complex. Axiom (CW3) says that X carries the *weak topology* and is responsible for the 'W'.

(CW1) For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial \mathbb{D}^n$ to the union of cells of dimension of at most $n-1$.

(CW2) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X.

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ.
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map.*

Example 169. Consider the unit interval $I = [0,1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n+1$ 0-cells and n 1-cells. which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma_k^0 = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \qquad \text{and} \qquad \sigma_\infty^0 = 0$$

and one-cells

$$\sigma_k^1 = \left(\frac{1}{k}, \frac{1}{k+1}\right), \qquad k \in \mathbb{N}^{\geq 1}.$$

At first glance, this looks like a CW decomposition; it is certainly satisfies Axiom (CW1) and Axiom (CW2). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right),$$

is the midpoint of the interval σ_k^1. We have $A \cap \sigma_k^0 = \emptyset$ for all k, and $A \cap \sigma_k^1 = \{a_k\}$ for all k. In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i. However, the set A is not closed in I, since it does not contain its limit point $\lim_{n\to\infty} a_n = 0$.

Definition 170 (skeleton, dimension)**.** Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n-skeleton of X. If X is equal to its n-skeleton but not equal to its $(n-1)$-skeleton, we say that X is n-dimensional.

Note 171. Axiom (CW3) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 172 (subcomplex, CW pair)**.** Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$

We call such a pair (X, Y) a CW pair.

Fact 173. Let X and Y be CW complexes such that X is locally compact. Then $X \times Y$ is a CW complex.

Lemma 174. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 175. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact[3] if and only if it is locally finite.[4]

Proof. It is clear that 1. $\Rightarrow$ 2., since X is a subset of itself. Similarly, it is clear that 2. $\Rightarrow$ 3., since □

Corollary 176. If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n.

$$\begin{tikzcd}
 & & & & K \arrow[d, "f"] \arrow[dlll, dashed, "\exists\tilde{f}"'] \\
X^{n-1} \arrow[r, hook] & X^n \arrow[r, hook] & X^{n+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 177. Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.

Lemma 178. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborbood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.
- Every CW complex is paracompact, locally path-connected, and locally contractible.

[3] I.e. every point of X has a compact neighborhood.
[4] I.e. if every point has a neighborhood which is contained in only finitely many cells.

- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 179. Let X be a CW complex. We have the following decompositions.

1.
$$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2.
$$X^n/X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n-cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n-cell is homeomorphic to the open n-ball.
2. For every n-cell σ, the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$-skeleton.

□

8.9. Cellular homology

Lemma 180. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 178, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 162, and the second from Lemma 179. □

Lemma 181. Consider the inclusion $i_n \colon X^n \hookrightarrow X$.

- The induced map
$$H_n(i_n)\colon H_n(X^n) \to H_n(X)$$
is surjective.
- On the $(n+1)$-skeleton we get an isomorphism
$$H_n(i_{n+1})\colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n). The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by Lemma 180,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n)\colon X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{tikzcd} & & H_{n+1}(X^{m+1}, X^m) \arrow[dll, "\delta"'] \\ H_n(X^m) \arrow[r] & H_n(X^{m+1}) \arrow[r] & H_n(X^{m+1}, X^m) \end{tikzcd}$$

But again by Lemma 180, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \to H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r] \arrow[drrr] & X^{n+1} \arrow[r] \arrow[drr] & X^{n+2} \arrow[r] \arrow[dr] & \cdots \\ & & & & X \end{tikzcd}$$

Taking nth singular homology, we find the following.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X^n) \arrow[r, two heads, "\alpha_1"] & H_n(X^{n+1}) \arrow[r, equal, "\alpha_2"] & H_n(X^{n+2}) \arrow[r, equal, "\alpha_3"] & \cdots \arrow[r] & H_n(X) \end{tikzcd}$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \qquad \sigma_i\colon \Delta^n \to X.$$

Since the standard n-simplex Δ^n is compact, Corollary 176 implies that each σ_i factors through some X^{n_i}. Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \qquad \tilde{\sigma}_i\colon \Delta^n \to X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma \right]$$

□

Corollary 182. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q-cells, then $H_q(X) \cong 0$.
3. In particular, for an n-dimensional CW-complex X (Definition 170), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 181.
2.

□

Definition 183 (cellular chain complex)**.** Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}).

Theorem 184 (comparison of cellular and singular homology)**.** Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n \colon H_n(C_\bullet(X), d) \cong H_n(X).$$

Example 185 (complex projective space)**.** Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \mathrm{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots | x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{tikzcd}
2n & 2n-1 & 2n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i,\ 0 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Example 186 (real projective space)**.** As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{tikzcd}
n & n-1 & n-2 & \cdots & 1 & 0 \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \cdots \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

8.10. Homology with coefficients

Definition 187 (homology with coefficients)**.** Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X; G) = S(X) \otimes_{\mathbb{Z}} G.$$

The nth singular homology of X with coefficients in G is the nth homology

$$H_n(X; G) = H_n(S(X; G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G.

Theorem 188 (topological universal coefficient thoerem for homology)**.** For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X; G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0 \ .$$

Furthermore, this sequence splits non-canonicaly, telling us that

$$H_n(X; G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. REF to be updated with UCT. □

8.11. The topological Künneth formula

Recall that the Künneth formula (Section 5.9) tells us that the following sequence is short exact.

TODO

It turns out that we can relate $H_p(X) \otimes H_q(Y)$ and $H_{p+q}(X \times Y)$. We first define a so-called *homology cross product*

$$\times \colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y),$$

and then show that it descends to homology.

Our goal is to find a way of making a p-simplex in X and a q-simplex in Y into a $(p+q)$-simplex in $X \times Y$. At least in the case that one of p or q is equal to zero, it is clear what to do, since the Cartesian product of a p-simplex with a zero-simplex is again a p-simplex. In more general cases, however, we have to be clever. The strategy is to write down a list of properties that we would like our homology cross product to have, and then show that there exists a unique map satisfying them.

Lemma 189. We can define a homomorphism

$$\times\colon S_p(X) \otimes S_q(Y) \to S_{p+q}(X \times Y), \qquad p, q \geq 0,$$

with the following properites.

1. For all points $x_0 \in X$ viewed as zero-chains in $S_0(X)$ and all $\beta\colon \Delta^q \to Y$, we have

   $$(x_0 \times \beta)(t_0, \dots, t_q) = (x_0, \beta(t_0, \dots, t_q));$$

   conversely, for $\alpha \in S_p(X)$ and $y_0 \in Y$, we have

   $$(\alpha \times y_0)(t_0, \dots, t_p) = (\alpha(t_0, \dots, t_q), y_0).$$

2. The map $\times$ is natural in the sense that the square

   $$\begin{tikzcd} S_p(X) \otimes S_q(Y) \arrow[r, "\times"] \arrow[d] & S_{p+q}(X \times Y) \arrow[d] \\ S_p(X') \otimes S_q(Y') \arrow[r, "\times"] & S_{p+q}(X' \times Y') \end{tikzcd}$$

   commutes.

3. The map $\times$ satisfies the Leibniz rule in the sense that

   $$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

Proof. As a warm up, let us work out some hypothetical consequences of our formulae, in the special case that $X = \Delta^p$ and $Y = \Delta^q$. In this case, we have

$$\mathrm{id}_{\Delta^p} \in S_p(\Delta^p), \qquad \mathrm{id}_{\Delta^q} \in S_q(\Delta^q),$$

so we can take the homology cross product $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} \in S_{p+q}(\Delta^p \times \Delta^q)$. By the Leibniz rule, we have

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \Delta^p \times \partial(\mathrm{id}_{\Delta^q}).$$

This is now a perfectly well-defined element of $S_{p+q-1}(\Delta^p \times \Delta^q)$, which we will give the nickname R.

A trivial computation shows that $\partial R = 0$, so there exists a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$. We choose some such c and define $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} = c$.

We now use the trick of expressing some $\alpha\colon \Delta^p \to X$ as $S_p(\alpha)(\mathrm{id}_{\Delta^p})$. Then

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}).$$

But then the naturality forces our hand; chasing $\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}$ around the naturality square

$$\begin{tikzcd} S_p(\Delta^p) \otimes S_q(\Delta^q) \arrow[r, "\times"] \arrow[d, "S_p(\alpha)\otimes S_q(\beta)"'] & S_{p+q}(\Delta^p \times \Delta^q) \arrow[d, "S_{p+q}(\alpha,\beta)"] \\ S_p(X) \otimes S_p(Y) \arrow[r, "\times"] & S_{p+q}(X \times Y) \end{tikzcd}$$

tells us that

$$\begin{aligned} S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) &= S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) \\ S_{p+q}(\alpha, \beta)(c) &= \alpha \times \beta. \end{aligned}$$

□

Theorem 190 (topological Künneth formula)**.** For any topological spaces X and Y, we have the following short exact sequence.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \longrightarrow 0$$

8.12. The Eilenberg-Steenrod axioms

Denote by T the functor

$$T\colon \mathbf{Pair} \to \mathbf{Pair}; \qquad (X, A) \mapsto (A, \emptyset); \qquad (f\colon (X, A) \to (Y, B)) \mapsto f|_A.$$

Definition 191 (homology theory)**.** Let A be an abelian group. A homology theory with coefficients in A is a sequence of functors

$$H_n\colon \mathbf{Pair} \to \mathbf{Ab}, \qquad n \geq 0$$

together with natural transformations

$$\delta\colon H_n \Rightarrow H_{n-1} \circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,

$$f \sim g \implies H_n(f) = H_n(g) \qquad \text{for all } f, g, n.$$

2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(i)\colon H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$$

for all n.

3. **Dimension:** We have
$$H_n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.

9. Cohomology

9.1. Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition 191) which govern cohomology theories.

Definition 192 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n : \mathbf{Pair}^{\mathrm{op}} \longrightarrow \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial : H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$H^n(i)\colon (X, A) \cong H^n(X \smallsetminus U, A \smallsetminus U)$$
for all n.
3. **Dimension:** We have
$$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$
4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H^n(X) = \prod_\alpha H^n(X_\alpha).$$
5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

9.2. Singular cohomology

Definition 193 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet, G)$$

We will denote the evaluation map $\mathrm{Hom}(A,G) \otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 194. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G) \otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet) \otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha \colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned} \langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\ &= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\ &= \langle \alpha, a \rangle. \end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned} \langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\ &= \langle \alpha, a \rangle + \langle \beta, da \rangle \\ &= \langle \alpha, a \rangle \end{aligned}$$

□

Via $\otimes$-hom adjunction, we get a map

$$\kappa \colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 195 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space. There is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X;G) \twoheadrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

Example 196. We have seen (in Example 185) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n,\ k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \leq k \leq n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \operatorname{Ext}^1(0, \mathbb{Z}) \oplus \operatorname{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \leq k \leq n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \operatorname{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \operatorname{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

9.3. The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \to \mathbb{Z},$$

allowing an n-cochain to eat an n-chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q-simplex, on which our cochain can act, and a singular $n - q$-simplex, which is along for the ride.

Definition 197 (front, rear face). Let $a\colon \Delta^n \to X$ be a singular n-simplex, and let $0 \leq q \leq n$.

- The $(n-q)$-dimensional front face of a is given by

  $$F^{n-q}(a) = \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X,$$

  where i is induced by the inclusion

  $$\{0, \ldots, n-q\} \hookrightarrow \{1, \ldots, n\}; \qquad k \mapsto k.$$

- The q-dimensional rear face of a is given by

  $$R^q(a) = \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X,$$

where r is the map

$$\{0, \dots q\} \to \{0, \dots, n\}; \qquad k \mapsto n - q + k.$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \to S_{n-q}(X); \qquad \alpha \otimes a \mapsto F^{n-q}(a)\langle a, R^q(a)\rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 198 (cap product)**.** The cap product is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \to S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b)\rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b)\rangle = 0$ by definition of relative cohomology.

Lemma 199. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial\alpha \otimes r).$$

Proposition 200. The cup product descends to a map on homology; that is, we get a map

$$\frown : H^q(X, A; R) \otimes H_n(X, A; R) \to H_{n-q}(X; R); \qquad [\alpha \otimes a \otimes r] \mapsto [F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r].$$