1 Spectral sequences

1.1 Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in Section **??**, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the building blocks of the homology of the total complex such a double complex.

They do this via a series of successive approximations. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
E_{0,3} \arrow[d, "d^v"'] & E_{1,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,3} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,2} \arrow[d, "d^v"'] & E_{1,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,2} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,1} \arrow[d, "d^v"'] & E_{1,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{2,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & E_{3,1} \arrow[l, "d^h"'] \arrow[d, "d^v"'] & \cdots \arrow[l] \\
E_{0,0} & E_{1,0} \arrow[l, "d^h"'] & E_{2,0} \arrow[l, "d^h"'] & E_{3,0} \arrow[l, "d^h"'] & \cdots \arrow[l]
\end{tikzcd}
$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

$E^0_{\bullet,\bullet}$

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
E^0_{0,3} \arrow[d, "d^v"'] & E^0_{1,3} \arrow[d, "d^v"'] & E^0_{2,3} \arrow[d, "d^v"'] & E^0_{3,3} \arrow[d, "d^v"'] \\
E^0_{0,2} \arrow[d, "d^v"'] & E^0_{1,2} \arrow[d, "d^v"'] & E^0_{2,2} \arrow[d, "d^v"'] & E^0_{3,2} \arrow[d, "d^v"'] \\
E^0_{0,1} \arrow[d, "d^v"'] & E^0_{1,1} \arrow[d, "d^v"'] & E^0_{2,1} \arrow[d, "d^v"'] & E^0_{3,1} \arrow[d, "d^v"'] \\
E^0_{0,0} & E^0_{1,0} & E^0_{2,0} & E^0_{3,0}
\end{tikzcd} \tag{1.1}$$

One then takes homology of the corresponding vertical complexes, replacing $E^0_{p,q}$ by the homology $H_q(E^0_{p,\bullet})$. To ease notation, one writes

$$H_q(E^0_{p,\bullet}) = E^1_{p,q}.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$\begin{tikzcd}
E^1_{0,3} & E^1_{1,3} \arrow[l, "H_3(d^h)"'] & E^1_{2,3} \arrow[l, "H_3(d^h)"'] & E^1_{3,3} \arrow[l, "H_3(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,2} & E^1_{1,2} \arrow[l, "H_2(d^h)"'] & E^1_{2,2} \arrow[l, "H_2(d^h)"'] & E^1_{3,2} \arrow[l, "H_2(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,1} & E^1_{1,1} \arrow[l, "H_1(d^h)"'] & E^1_{2,1} \arrow[l, "H_1(d^h)"'] & E^1_{3,1} \arrow[l, "H_1(d^h)"'] & \cdots \arrow[l] \\
E^1_{0,0} & E^1_{1,0} \arrow[l, "H_0(d^h)"'] & E^1_{2,0} \arrow[l, "H_0(d^h)"'] & E^1_{3,0} \arrow[l, "H_0(d^h)"'] & \cdots \arrow[l]
\end{tikzcd} \tag{1.2}$$

We now write $E^2_{p,q}$ for the horizontal homology $H_p(E^1_{\bullet,q})$.

The claim is that the terms $E^r_{p,q}$ are pieces of a successive approximation of the homology of the total complex $\mathrm{Tot}(E)$. We will prove this claim much later. However, in the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Example 1. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes $E_{0,\bullet}$ and $E_{1,\bullet}$ and a morphism $f\colon E_{1,\bullet} \to E_{0,\bullet}$, padded by zeroes on either side.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
E_{0,3} \arrow[d, "d^0_3"'] & E_{1,3} \arrow[l, "f_3"'] \arrow[d, "-d^1_3"] \\
E_{0,2} \arrow[d, "d^0_2"'] & E_{1,2} \arrow[l, "f_2"'] \arrow[d, "-d^1_2"] \\
E_{0,1} \arrow[d, "d^0_1"'] & E^{1,1} \arrow[l, "f_1"'] \arrow[d, "-d^1_1"] \\
E_{0,0} \arrow[d, "d^0_0"'] & E_{1,0} \arrow[l, "f_0"'] \arrow[d, "-d^1_0"] \\
E_{-0,1} \arrow[d] & E_{-1,1} \arrow[l, "f_{-1}"'] \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

Note that we have multiplied the diferentials of $E_{1,\bullet}$ by -1 so that we have a double complex. This does not enter into the discussion.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$.

Let $T = \mathrm{Tot}(E)$; that is, let $T = \mathrm{Cone}(f)$.

Recall that $\mathrm{Cone}(f)$ fits into the following short exact seqence,

$$0 \longrightarrow E_{\bullet,1} \hookrightarrow \mathrm{cone}(f)_\bullet \twoheadrightarrow E_{\bullet,0}[-1] \longrightarrow 0$$

and that this gives us the following long exact sequence on homology.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_n(E_{\bullet,0}) \arrow[dll, "\delta"] \\
H_n(E_{\bullet,1}) \arrow[r] & H_n(\mathrm{Cone}(f)_\bullet) \arrow[r] & H_{n-1}(E_{\bullet,0}) \arrow[dll, "\delta"] \\
H_{n-1}(E_{\bullet,1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Through image-kernel factorization, we get the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{coker}(H_n(f)) \arrow[r, hook] & H_n(\mathrm{Cone}(f)) \arrow[r, two heads] & \ker(H_{n-1}(f)) \arrow[r] & 0
\end{tikzcd}$$

This is precisely the second page of the above computation; that is, we have a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & E^2_{p-1,q+1} \arrow[r, hook] & H_{p+q}(T) \arrow[r, two heads] & E^2_{p,q} \arrow[r] & 0
\end{tikzcd} .$$

1.2 Homology spectral sequences

1.2.1 Notation and terminology

Definition 2 (homology spectral sequence)**.** Let $\mathcal{A}$ be an abelian category. A homology spectral sequence starting at E^a in $\mathcal{A}$ consists of the following data.

1. A family $E^r_{p,q}$ of objects of $\mathcal{A}$, defined for all integers p, q, and $r \geq a$.
2. Maps
$$d^r_{p,q} \colon E^r_{p,q} \to E^r_{p-r,q+r+1}$$
that are differentials in the sense that $d^r \circ d^r = 0$.
3. Isomorphisms
$$E^{r+1}_{p,q} \cong \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1})$$
between $E^{r+1}_{p,q}$ and the homology at the corresponding position of the E^r

We denote such a spectral sequence by E. A morphism $E' \to E$ is a family of maps

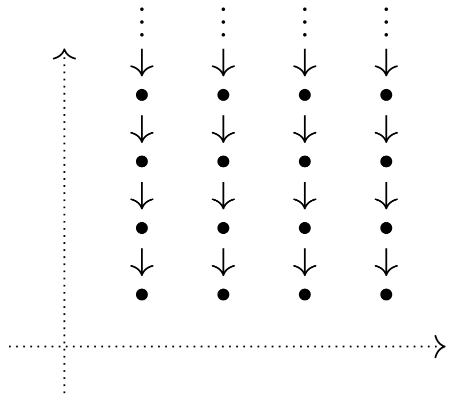
$$f^r_{p,q} \colon E'^r_{p,q} \to E^r_{p,q}$$

such that $d^r f^r = f^r d^r$ such that $f^{r+1}_{p,q}$ is induced on homology by $f^r_{p,q}$.

For each r, a spectral sequence E has a two-dimensional array of objects $E^r_{p,q}$. For a given r, one calls the objects $E^r_{p,q}$ the rth *page* of the spectral sequence E.

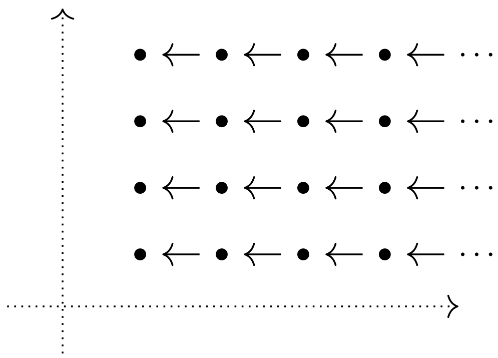
We have already seen an example of the first two pages of a spectral sequence (in Diagram 1.1 and Diagram 1.2). Schematically, the differentials on the zeroth page point down.

E^0 :



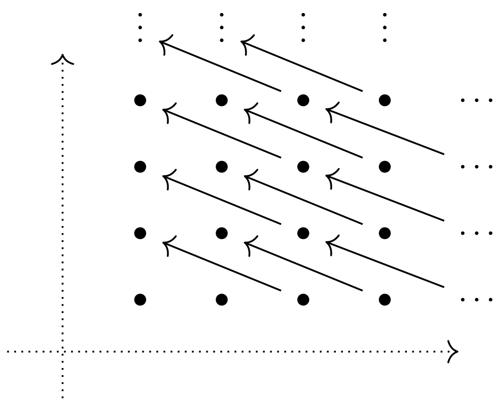
The differentials on the first page point to the left.

E^1 :



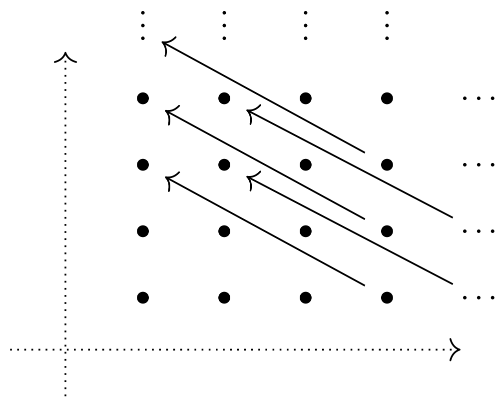
As r increases, the arrows rotate clockwise. The second page looks like this.

E^2 :



The third page looks like this.

E^3 :



1.2.2 Stabilization and convergence

We have so far placed no conditions on $E^r_{p,q}$. However, we will confine our study of certain classes of spectral sequences.

Definition 3 (bounded, first quadrant)**.** Let E be a spectral sequence starting at E^a.

- We say that E is bounded if each $n \in \mathbb{Z}$, there are only finitely many terms of total degree n on the ath page, i.e. terms of the form $E^a_{p,q}$ for $n = p + q$.
- We say that E is first-quadrant if $E^a_{p,q} = 0$ for $p < 0$ and $q < 0$.

Clearly, every first-quadrant spectral sequence is bounded.

Note that if a spectral sequence is bounded, then

We get $E^{r+1}_{p,q}$ by taking the homology of a complex on the rth page. If E is a first-quadrant spectral sequence, then for $r > \max(p, q + 1)$, the differential entering $E^r_{p,q}$ comes from the fourth quadrant (hence from a zero object), and the differential leaving it lands in the second quadrant (also a zero object). Thus, $E^{r+1}_{p,q} = E^r_{p,q}$.

Definition 4 (stabilize)**.** A spectral sequence starting at E^a is said to stabilize at position (p, q) if there exists some $r \geq a$ such that $E^{r'}_{p,q} = E^r_{p,q}$ for all $r' \geq r$. In this case, we write $E^\infty_{p,q}$ for the stable value at position (p, q).

By the argument above, each position in a first-quadrant spectral sequence eventually stabilizes. We will mostly be interested in first-quadrant spectral sequences. However, boundedness is (clearly!) sufficient to guarantee stabilization.

Suppose one has been given a bounded spectral sequence, which we now know eventually stabilizes at each position. We would like to get some useful information out of such a spectral sequence.

Definition 5 (filtration)**.** Let $C_\bullet$ be a chain complex. A filtration F of $C_\bullet$ is a chain of inclusions of subcomplexes $F_pC_\bullet$ of $C_\bullet$.

$$\cdots \subset F_{p-1}C_\bullet \subset F_pC_\bullet \subset F_{p+1}C_\bullet \subset \cdots \subset C_\bullet$$

Definition 6 (bounded convergence)**.** Let E be a bounded spectral sequence starting at E^a in an abelian category $\mathcal{A}$, and let $\{H_i\}_{i\in\mathbb{Z}}$ be a collection of objects in $\mathcal{A}$, each equipped with a finite filtration

$$0 = F_sH_n \subset \cdots \subset F_{p-1}H_n \subset F_pH_n \subset \cdots \subset F_tH_n = H_n.$$

We say that E converges to $\{H_i\}_{i\in\mathbb{Z}}$ if there exist isomorphisms

$$E^\infty_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}.$$

We express this convergence by writing

$$E^a_{p,q} \Rightarrow H_{p+q}.$$

Example 7. Consider the spectral sequence E starting at E^0 from Example 1. This is certainly bounded, and stabilizes at each position (p, q) after the second page; that is, we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

The exact sequence

$$0 \longrightarrow E^2_{p-1,q+1} \hookrightarrow H_{p+q}(T) \twoheadrightarrow E^2_{p,q} \longrightarrow 0$$

gives us, in a wholly unsatisfying and trivial way, a filtration

$$\begin{tikzcd} F_{-1}H_n \arrow[d, equal] & F_0H_n \arrow[d, equal] & F_1H_n \arrow[d, equal] \\ 0 \arrow[r, hook] & E^2_{0,n} \arrow[r, hook] & H_n \end{tikzcd}$$

of H_n. The claim is that

$$E^0_{p,q} \Rightarrow H_{p+q}.$$

In order to check this, we have to check that

$$E^2_{p,q} \cong F_pH_{p+q}/F_{p-1}H_{p+q}$$

for all p, q. The only non-trivial cases are when $p = 0$ or 1. When $p = 0$, we have

$$\begin{aligned} E^\infty_{0,n} &\overset{!}{\cong} F_0H_n/F_{-1}H_n \\ &\cong E^2_{0,n}/0 \\ &\cong E^2_{0,n}, \end{aligned}$$

and when $p = 1$ we have

$$\begin{aligned} E^\infty_{1,n} &\overset{!}{\cong} F_1H_{n+1}/F_0H_{n+1} \\ &\cong H_{n+1}/E^2_{0,n+1} \\ &\cong E^2_{1,n} \end{aligned}$$

as required.

1.2.3 Exact couples

We now take a detour to the much simpler, more elegant world of exact couples.

Definition 8 (exact couple)**.** Let $\mathcal{A}$ be an abelian category. An exact couple in $\mathcal{A}$ consists of objects and morphisms

$$\begin{tikzcd} & E \arrow[dr, "\alpha"] & \\ A \arrow[ur, "\gamma"] & & E \arrow[ll, "\beta"] \end{tikzcd}$$

such that $\operatorname{im}\alpha = \ker\beta$

1.2.4 The spectral sequence of a filtered complex

Theorem 9. A filtration F of a chain complex $C_\bullet$ naturally determines a spectral sequence starting with

$$E^0_{p,q} = F_pC_{p+q}/F_{p-1}C_{p+q}, \qquad E^1_{p,q} = H_{p+q}(E^0_{p,\bullet})$$

Proof. We construct explicitly the spectral sequence above .Let $C_\bullet$ be a chain complex, and F a filtration. We will use the following notation.

- We will denote by η_p the quotient

$$\eta_p \colon F_pC \twoheadrightarrow F_pC/F_{p-1}C.$$

coming from the short exact sequence

$$0 \longrightarrow F_{p-1}C \hookrightarrow F_pC \twoheadrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

- We will denote by A_p^r the subobject

$$A_p^r = \{c \in F_pC : dc \in F_{p-r}C\} \subset F_pC$$

  of those elements whose differentials survive r many levels of the grading.

- We will denote by Z_p^r the image

$$Z_p^r = \eta_p(A_p^r).$$

- We will denote by B_p^r the image

$$B_p^r = \eta_p(d(a_{p+r-1}^{r-1})).$$

Examining the definitions, we see that we have the following inclusion.

$$\begin{tikzcd} dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ dA_{p+r}^{r} \arrow[r, equal] & \{dc \mid c \in F_{p+r},\ dc \in F_pC\} \end{tikzcd}$$

Both of these are subobjects of F_pC; by applying η_p and taking a sharp look at the definition of the Bs, we find an inclusion

$$B_p^r \subset B_p^{r+1}.$$

Working inductively, we are left with the following sequence of inclusions.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^r \subset \cdots$$

Defining $B_p^\infty = \bigcup_r B_p^r$, we can crown our sequence of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty.$$

Again examining definitions, we find the following inclusion.

$$\begin{tikzcd} A_p^r \arrow[r, equal] \arrow[d, hook] & \{c \in F_pC \mid dc \in F_{p-r}C\} \arrow[d, hook] \\ A_p^{r+1} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r-1}C\} \end{tikzcd}$$

As before, applying η_p and working inductively gives us a chain

$$\cdots \subset Z_p^r \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0$$

and defining $Z_p^\infty = \bigcap_r Z_p^r$ gives us

$$Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0,$$

But we should not rest on our laurels. Instead, note that for each r, r', we have an inclusion

$$\begin{tikzcd} dA_{p+r-1}^{r-1} \arrow[r, equal] \arrow[d, hook] & \{dc \mid c \in F_{p+r-1},\ dc \in F_pC\} \arrow[d, hook] \\ A_p^{r'} \arrow[r, equal] & \{c \in F_pC \mid dc \in F_{p-r'}C\} \end{tikzcd}$$

since $d^2 = 0$. Thus, we can graft our chains of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty \subset Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0.$$

We defined

$$Z_p^r = \eta_p(A_p^r).$$

Recall that η_p was defined to be the canoncal projection corresponding to a short exact sequence. Taking subobjects (where the left-hand square is a pullback), we find the following monomorphism of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & F_{p-1}C \cap A_r^p \arrow[r, hook] \arrow[d, hook] & A_r^p \arrow[r, two heads] \arrow[d, hook] & Z_r^p \arrow[r] \arrow[d, hook] & 0 \\ 0 \arrow[r] & F_{p-1}C \arrow[r, hook] & F_pC \arrow[r, two heads] & F_pC/F_{p-1}C \arrow[r] & 0 \end{tikzcd}$$

Rewriting the definitions, we find that

$$F_{p-1}C \cap A_r^p = \{c \in F_{p-1}C \mid dc \in F_{p-r}C\} = A_{p-1}^{r-1},$$

so

$$Z_p^r \cong A_p^r / A_{p-1}^{r-1}.$$

We now define

$$E_p^r = \frac{Z_p^r}{B_p^r}.$$

We can write

$$\frac{Z_p^r}{B_p^r} \cong \frac{A_p^r + F_{p-1}C}{d(A_{p+r-1}^{r-1}) + F_{p-1}C} \cong \frac{A_p^r}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}}.$$

I don't understand this step, and I don't have time to do the rest. □

Theorem 10. Let $C_\bullet$ be a chain complex, and F a bounded filtration on $C_\bullet$. Then the spectral sequence constructed in Theorem 9 converges to the homology of the homology of $C_\bullet$; that is

$$E_{p,q}^1 = H_{p+q}(F_pC_\bullet/F_{p-1}C_\bullet).$$

Example 11. Let E be the spectral sequence associated to a filtered complex $C_\bullet$, and suppose that

$$E_{p,q}^r = \begin{cases} A_p, & q = 0 \\ 0, & q \neq 0 \end{cases}, \qquad r > 0.$$

Then certainly the spectral sequence collapses.

The claim is that $A_p = H_p(C_\bullet)$; that is, the spectral sequence converges to A_p. To see this, we need to find filtrations

$$0 = F_{s-1}A_p \subset F_sA_p \subset F_{s+1}A_p \subset \cdots \subset F_tA_p \subset F_{t+1}A_p = A_p$$

such that, for each p and q, we have

$$E_{p,q}^\infty = F_pA_{p+q}/F_{p-1}A_{p+q}.$$

This requires that

$$\begin{aligned} 0 &= F_sA_p/F_{s-1}A_p \\ &\vdots \\ 0 &= F_{p-1}A_p/F_{p-2}A_p \\ A_p &= F_pA_p/F_{p-1}A_p \\ 0 &= F_{p+1}A_p/F_pA_p \\ &\vdots \\ 0 &= F_{t+1}A_p/F_tA_p. \end{aligned}$$

Starting at $F_{s-1}A_p = 0$, these equations tell us that

$$0 = F_{s-1}A_p = F_sA_p = F_{s+1}A_p = \cdots = F_{p-1}A_p.$$

We then have that

$$A_p = F_p A_p / 0 = F_p A_p.$$

Again, our equations tell us that

$$F_p A_p = F_{p+1} A_p = \cdots = F_t A_p.$$

This tells us that $H_p(C_\bullet) = A_p$.

Note that we could have taken our filtration to be the one-step filtration

$$0 = F_{p-1} A_p \subset F_p A_p = A_p.$$

Example 12. Suppose instead of collapsing to a single row, our spectral sequence stabilizes to two rows:

$$E^\infty_{p,q} = \begin{cases} A_p, & q = 0 \\ B_p, & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Identical logic to that above tells us that

$$F_{p-1}H_p / F_{p-2}H_p = 0 F_p H_p / F_{p-1} H_p \quad = A_p F_{p+1} H_p / F_p H_p = B_{p-1} F_{p+2} H_p / F_{p+1} H_p \quad = 0,$$

i.e. that we have a two-step filtration

$$0 = F_{p-2} H_p \subset F_{p-1} H_p \subset F_p H_p = H_p,$$

with

1.3 The spectral sequence of a double complex

A special case of the spectral sequence of a filtered complex is the spectral sequence of a double complex. Given a double complex $C_{\bullet,\bullet}$, we define a new double complex

$$\left({}^I\tau_{\leq n} C\right)_{p,q} = \begin{cases} C_{p,q} & p \leq n \\ 0, & p > n \end{cases}.$$

That is, the $\left({}^I\tau_{\leq n} C\right)$ is the double complex obtained by setting all the entries to the right of the column $C_{p,\bullet}$ to zero.

This leads to a filtered complex

$$0 \subset \cdots \subset \mathrm{Tot}({}^I\tau_{\leq 0} C)_{p,q} \subset \mathrm{Tot}({}^I\tau_{\leq 1} C)_{p,q} \subset \cdots \subset \mathrm{Tot}(C)_{p,q};$$

that is, with

$$F_pC = \mathrm{Tot}({}^I\tau_{\le p}C).$$

This gives rise to a spectral sequence via Theorem 9.

Specifically, we have

$$\begin{aligned}
{}^IE^0_{p,q} &= \frac{F_pC_{p+q}}{F_{p-1}C_{p+q}}\\
&= \frac{\mathrm{Tot}({}^I\tau_{\le p}C)_{p+q}}{\mathrm{Tot}({}^I\tau_{\le p-1}C)_{p+q}}\\
&= \frac{\bigoplus_{i+j=p+q} C_{i,j}}{\bigoplus_{\substack{i+j=p+q\\ i\le p-1}} C_{i,j}}\\
&= \frac{C_{0,p+q}\oplus\cdots\oplus C_{p,q}}{C_{0,p+q}\oplus\cdots\oplus C_{p-1,q+1}}\\
&= C_{p,q}.
\end{aligned}$$

The differentials $C_{p,q} \to C_{p,q-1}$ are simply the vertical differentials $d^v_{p,q}$.

Of course, we can also transpose our double complex before performing the above procedure; this gives a different spectral sequence ${}^{II}E^0_{p,q}$.

By Theorem 9, both of these converge to the homology of the total complex $H_{p+q}(\mathrm{Tot}(C))$.

Definition 13 (spectral sequences associated to a double complex).

- The vertically filtered spectral sequence associated to a double complex is the spectral sequence ${}^IE^0_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(C))$.
- The horizontally filtered spectral sequence associated to a double complex is the spectral sequence ${}^{II}E^0_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(C))$.

Example 14 (balancing Tor again). Let M and N be right and left R-modules respectively, and let $P_\bullet \xrightarrow{\simeq} M$ and $Q_\bullet \xrightarrow{\simeq} N$ be projective resolutions. We can form the double complex

$$M_{p,q} = P_p \otimes_R Q_q.$$

The totalization of this complex is then simply what we have been calling $P_\bullet \otimes_R Q_\bullet$.

Let us work out explicitly the vertical filtration. The zeroth page is as follows.

$$
\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
P_0 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_1 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_2 \otimes_R Q_2 \arrow[d, "\mathrm{id}\otimes d^Q"'] \\
P_0 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_1 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] & P_2 \otimes_R Q_1 \arrow[d, "\mathrm{id}\otimes d^Q"'] \\
P_0 \otimes_R Q_0 & P_1 \otimes_R Q_0 & P_2 \otimes_R Q_0
\end{tikzcd}
$$

Since each P_i is projective, taking homology ignores it, and we get that

$$
\begin{aligned}
H_q(P_p \otimes_R Q_\bullet) &= P_p \otimes_R H_q(Q_\bullet) \\
&= \begin{cases} P_p \otimes_R N, & q = 0 \\ 0, & q \neq 0 \end{cases}.
\end{aligned}
$$

Therefore, the first page of our spectral sequence consists of a single line at the first page.

$$
\begin{tikzcd}
0 & 0 & 0 & \cdots \\
0 & 0 & 0 & \cdots \\
P_0 \otimes_R N & P_1 \otimes_R N \arrow[l] & P_2 \otimes_R N \arrow[l] & \cdots \arrow[l]
\end{tikzcd}
$$

We now have

$$
\begin{aligned}
{}^{I}E^2_{p,q} &= H_p(P_\bullet \otimes_R N) \\
&= \mathrm{Tor}_p(M, N).
\end{aligned}
$$

At this point the spectral sequence collapses, telling us that

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(P_\bullet \otimes_R N) = \mathrm{Tor}_p(M, N).$$

The other filtration gives

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(M \otimes_R Q_\bullet).$$

Thus,

$$H_p(M \otimes_R Q_\bullet) = \mathrm{Tor}_p(M, N),$$

which says that Tor is balanced.

1.4 Hyper-derived functors

Consider abelian categories and right exact functors as follows.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

One might wonder about the relationship between LF, LG, and $L(G \circ F)$.

The problem is that we don't know how to make sense of the composition $LG \circ LF$, since LF lands in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ rather than $\mathcal{A}$; derived functors go from an abelian category to a category of chain complexes.

1.4.1 Cartan-Eilenberg resolutions

Definition 15 (Cartan-Eilenberg resolution)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $A \in \mathbf{Ch}^{+}(\mathcal{A})$. A Cartan-Eilenberg resolution of A is an upper-half-plane double complex $(P_{\bullet,\bullet}, d^h, d^v)$, equipped with an augmentation map

$$\epsilon \colon P_{\bullet,0} \to A_\bullet$$

such that the following criteria are satsfied.

1. $B(P_{p,\bullet}, d^h) \xrightarrow{\simeq} B(A_p, d^A)$ is a projective resolution.
2. $H(P_{p,\bullet}, d^h) \xrightarrow{\simeq} H(A_p, d^A)$ is a projective resolution.
3. $A_p \cong 0 \implies P_{p,\bullet} = 0$.

Here, $B(P_{p,\bullet}, d^h)$ are the horizontal boundaries of the chain complex $(P_{p,\bullet}, d^h)$, and

$$H(P_{p,\bullet}, d^h) \cong \frac{Z(P_{p,\bullet}, d^h)}{B(P_{p,\bullet}, d^h)}$$

is defined similarly.

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & P_{-r,1} \arrow[l] \arrow[d] & \cdots \arrow[l] & P_{0,1} \arrow[l] \arrow[d] & P_{1,1} \arrow[l] \arrow[d] & \cdots \arrow[l] \\
0 \arrow[d, dashed] & P_{-r,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] & P_{0,0} \arrow[l] \arrow[d, dashed] & P_{1,0} \arrow[l] \arrow[d, dashed] & \cdots \arrow[l] \\
0 & A_{-r} \arrow[l] & \cdots \arrow[l] & A_0 \arrow[l] & A_1 \arrow[l] & \cdots \arrow[l]
\end{tikzcd}$$

Proposition 16. The following are also projective resolutions.

1. $Z(P_{p,\bullet}, d^h) \xrightarrow{\simeq} Z(A_p, d^A)$
2. $P_{p,\bullet} \xrightarrow{\simeq} A_p$

Proof.

1. We have essentially by definition the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & B(P_{p,\bullet}, d^h) \arrow[r, hook] & Z(P_{p,\bullet}, d^h) \arrow[r, two heads] & H(P_{p,\bullet}, d^h) \arrow[r] & 0
\end{tikzcd}$$

The outer terms are projective because they are the terms of projective resolutions, so $Z(P_{p,\bullet}, d^h)$ is also projective. Thus, $Z(P_{p,\bullet}, d^h)$ is at the very least a complex of projectives.

Further, we have the first few rows of the long exact sequence on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & Z(P_{p,1}, d^h) \arrow[r] & 0 \arrow[dll, "\delta"] \\
0 \arrow[r] & Z(P_{p,0}, d^h) \arrow[r] & 0 \arrow[dll, "\delta"] \\
B(A_p, d^A) \arrow[r] & Z(A_p, d^A) \arrow[r] & H(A_p, d^A) \arrow[r] & 0
\end{tikzcd}$$

This tells us that the homology of $Z(P_{p,\bullet})$ is

$$H_q(Z(P_{p,\bullet}, d^h)) = \begin{cases} Z(A_p, d^A), & q = 0 \\ 0, & q > 0 \end{cases},$$

i.e. that $Z(P_{p,\bullet}, d^h)$ is a projective resolution of $Z(A_p, d^A)$.

2. Exactly the same, but with the short exact sequence

$$0 \longrightarrow Z(P_{p,\bullet}, d^h) \hookrightarrow H(P_{p,\bullet}, d^h) \twoheadrightarrow B(P_{p-1,\bullet}, d^h) \longrightarrow 0$$

□

Proposition 17. Every chain complex $A_\bullet$ has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$.

Proof. We construct a Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ explicitly. Pick projective resolutions

$$P^B_{p,\bullet} \to B(A_p, d^h), \qquad B^H_{p,\bullet} \xrightarrow{\simeq} H(A_p, d^h).$$

By the horseshoe lemma, we can find a projective resolution $P^Z_{p,\bullet} \xrightarrow{\simeq} Z(A_p, d^h)$ fitting into the following exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & P^B_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^Z_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^H_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & B(A_p, d^h) \arrow[r, hook] & Z(A_p, d^h) \arrow[r, two heads] & H(A_p, d^h) \arrow[r] & 0
\end{tikzcd} \tag{1.3}$$

Playing the same game again, we find a projective resolution of A_p.

$$\begin{tikzcd}
0 \arrow[r] & P^Z_{p,\bullet} \arrow[r, hook] \arrow[d, "\simeq"] & P^A_{p,\bullet} \arrow[r, two heads] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & Z(A_p, d^h) \arrow[r, hook] & A_p \arrow[r, two heads] & B(A_{p-1}, d^h) \arrow[r] & 0
\end{tikzcd} \tag{1.4}$$

We then define our Cartan-Eilenberg resolution to be the double complex whose pth column is $(P^A_{p,\bullet}, (-1)^p d^{P^A})$, and whose horizontal differentials are given by the composition

$$\begin{tikzcd}
P^A_{p,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^B_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^Z_{p-1,\bullet} \arrow[r] \arrow[d, "\simeq"] & P^A_{p-1,\bullet} \arrow[d, "\simeq"] \\
A_p \arrow[r, "d^A"] & B_{p-1} \arrow[r, hook] & Z_{p-1} \arrow[r, hook] & A_{p-1}
\end{tikzcd} \tag{1.5}$$

In this case, all the criteria are satisfied by definition. □

Proposition 18. Let $A_\bullet$ and $A'_\bullet$ be bounded-below chain complexes, and let $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$ and $P'_{\bullet,\bullet} \xrightarrow{\simeq} A'_\bullet$ be Cartan-Eilenberg resolutions. Then any morphism of chain complexes $f\colon A_\bullet \to A'_\bullet$ can be lifted to a morphism $\tilde{f}\colon P^A_{\bullet,\bullet} \to P^B_{\bullet,\bullet}$ of double complex. Furthermore, this lift is unique up to a *homotopy of double complexes,* i.e. for any other lift $\tilde{\tilde{f}}$, there exist maps

$$s^h_{p,q}\colon P_{p,q} \to P'_{p+1,q}, \qquad s^v_{p,q}\colon P_{p,q} \to E_{p,q+1}$$

such that

Proof. We have the following lifts.

$$\begin{tikzcd}
 & & & Q^B_{p,\bullet} \arrow[dd] & & Q^Z_{p,\bullet} \arrow[dd] & & Q^H_{p,\bullet} \arrow[dd] & & \\
 & & P^B_{p,\bullet} \arrow[ur] \arrow[dd] & & P^Z_{p,\bullet} \arrow[dd] & & P^H_{p,\bullet} \arrow[ur] \arrow[dd] & & & \\
 & 0 \arrow[rr] & & B_p(B) \arrow[rr] & & Z_p(B) \arrow[rr] & & H_p(B) \arrow[rr] & & 0 \\
0 \arrow[rr] & & B_p(A) \arrow[rr] \arrow[ur] & & Z_p(A) \arrow[rr] \arrow[ur] & & H_p(A) \arrow[rr] \arrow[ur] & & 0 &
\end{tikzcd}$$

The horseshoe lemma for morphisms (mutatis mutandis) allows us to complete this diagram in a compatible way as follows.

$$\begin{tikzcd}
 & 0 \arrow[rr] & & Q^B_{p,\bullet} \arrow[rr] \arrow[dd] & & Q^Z_{p,\bullet} \arrow[rr] \arrow[dd] & & Q^H_{p,\bullet} \arrow[rr] \arrow[dd] & & 0 \\
0 \arrow[rr] & & P^B_{p,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] & & P^Z_{p,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] & & P^H_{p,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] & & 0 & \\
 & 0 \arrow[rr] & & B_p(B) \arrow[rr] & & Z_p(B) \arrow[rr] & & H_p(B) \arrow[rr] & & 0 \\
0 \arrow[rr] & & B_p(A) \arrow[rr] \arrow[ur] & & Z_p(A) \arrow[rr] \arrow[ur] & & H_p(A) \arrow[rr] \arrow[ur] & & 0 &
\end{tikzcd}$$

Using the lifts we constructed above and identical logic, we get the following.

$$
\begin{tikzcd}
& 0 \arrow[rr] && Q^Z_{p,\bullet} \arrow[rr] \arrow[dd] && Q_{p,\bullet} \arrow[rr] \arrow[dd] && Q^B_{p-1,\bullet} \arrow[rr] \arrow[dd] && 0 \\
0 \arrow[rr] && P^Z_{p,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] && P_{p,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] && P^B_{p-1,\bullet} \arrow[rr] \arrow[ur] \arrow[dd] && 0 \\
& 0 \arrow[rr] && Z_p(B) \arrow[rr] && B_p(A) \arrow[rr] && B_{p-1}(A) \arrow[rr] && 0 \\
0 \arrow[rr] && Z_p(A) \arrow[rr] \arrow[ur] && A_p \arrow[rr] \arrow[ur] && B_{p-1}(A) \arrow[rr] \arrow[ur] && 0
\end{tikzcd}
$$

Thus we have, for each $p \in \mathbb{Z}$, a morphism of complexes $P_{p,\bullet} \to Q_{p,\bullet}$. It remains only to show that each of the squares

$$
\begin{tikzcd}
P_{p,\bullet} \arrow[r] \arrow[d] & P_{p-1,\bullet} \arrow[d] \\
Q_{p,\bullet} \arrow[r] & Q_{p-1,\bullet}
\end{tikzcd}
$$

commutes. But we get these squares by pasting together squares stolen from the above diagrams as follows.

$$
\begin{tikzcd}
P_{p,\bullet} \arrow[r, "d^h"] \arrow[d] & P^Z_{p-1,\bullet} \arrow[r, hook] \arrow[d] & P^B_{p-1,\bullet} \arrow[r, hook] \arrow[d] & P_{p-1,\bullet} \arrow[d] \\
Q_{p,\bullet} \arrow[r, "d^h"] & Q^Z_{p-1,\bullet} \arrow[r, hook] & Q^B_{p-1,\bullet} \arrow[r, hook] & Q_{p-1,\bullet}
\end{tikzcd}
$$

□

Proposition 19. Let f, g be chain homotopic maps via the homotopy f.

Proposition 20. Let $P^A_{\bullet,\bullet}$ be a Cartan-Eilenberg resolution of $A_\bullet$. Then the canonical map

$$\mathrm{Tot}(P^A_{\bullet,\bullet}) \to A$$

is a quasi-isomorphism

Proof. Spectral sequence with vertical filtration. □

1.4.2 Hyper-derived functors

Definition 21 (left hyper-derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor, and assume $\mathcal{A}$ has enough projectives. The left hyper-derived functor $\mathbb{L}_i F$ of F is the functor

$$\mathbb{L}_i F\colon \mathbf{Ch}_+(\mathcal{A}) \to \mathcal{B}; \qquad \mathbb{L}_i F(A_\bullet) = H_i(\mathrm{Tot}(F(P^A_{\bullet,\bullet}))),$$

where $P^A_{\bullet,\bullet}$ is a Cartan-Eilenberg resolution of $A_\bullet$.

Hyper-derived functors are functorial by

Proposition 22. Let $A_\bullet$ be a bounded below chain complex in an abelian category $\mathcal{A}$. There are two spectral sequences

$$^{I}E^2_{p,q} = H_p(L_q F(A_\bullet)) \Longrightarrow \mathbb{L}_{p+q}(A_\bullet),$$

$$^{II}E^2_{p,q} = L_p F(H_q(A_\bullet)) \Longrightarrow \mathbb{L}_{p+q}(A_\bullet).$$

Proof. We use the spectral sequence for the double complex $F(P^A_{\bullet,\bullet})$. The horizontal filtration has no surprises. Using the vertical filtration, one must use that

$$H_p(F(P^A_{\bullet,\bullet})) = F(H_p(P^A_{\bullet,\bullet})),$$

which is true because the horizontal cycles and boundaries are □

This allows us to define the famous *Grothendieck spectral sequence,* which allows us to compute the composition of two derived functors.

Proposition 23. Let $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ be abelian categories, and assume that $\mathcal{A}$ and $\mathcal{B}$ have enough projectives. Let F and G as below be right exact, and assume that G sends projective objects to F-acyclic objects.[1] Then for all $A \in \mathcal{A}$, there exists a convergent spectral sequence

$$E^2_{p,q} = L_p F(L_q G(A)) \Longrightarrow L_{p+q}(F \circ G)(A).$$

Proof. Let $P_\bullet \xrightarrow{\simeq} A$ be a projective resolution. From Proposition 22, we find two spectral sequences. Applying the first to $G(P_\bullet)$, we find

$$H_p(L_q(F(G(P_\bullet)))) \Longrightarrow \mathbb{L}_{p+q}.$$

However, by assumption $G(P_\bullet)$ is a chain complex of F-acyclic objects, so

$$L_q F(G(P_\bullet)) = \begin{cases} F(G(P_\bullet)), & q = 0 \\ 0, & \text{otherwise} \end{cases}.$$

[1] An object $B \in \mathcal{B}$ is called *F-acyclic* if for all $q > 0$, $L_q F(B) = 0$. For example, projective objects are F-acyclic for all F.

Taking pth homology thus gives us, with $q = 0$,

$$\begin{aligned} H_p(L_0F(G(P_\bullet))) &= H_p(F \circ G(P_\bullet)) \\ &= L_p(F \circ G)(A). \end{aligned}$$

Thus, the spectral sequence collapses at the second page, and we have

$$\mathbb{L}_pF(G(P_\bullet)) \cong L_p(F \circ G)(A)$$

The second spectral sequence guarantees us that

$$L_pF(H_q(G(P_\bullet))) \Longrightarrow \mathbb{L}_{p+q}(G(P_\bullet)).$$

However, by definition

$$H_q(G(P_\bullet)) = L_qG(A),$$

giving us a spectral sequence

$$E^2_{p,q} = (L_pF \circ L_qG)(A) \Longrightarrow L_{p+q}(F \circ G)(A)$$

as required. □

Example 24 (Lyndon-Hochschild-Serre spectral sequence)**.** Let G be a group, and N a normal subgroup. This gives us the following commuting[2] diagram.

$$\begin{tikzcd} \mathbb{Z}G\text{-}\mathbf{Mod} \arrow[rr, "(-)_N"] \arrow[dr, "(-)_G"'] & & \mathbb{Z}G/N\text{-}\mathbf{Mod} \arrow[dl, "(-)_{G/N}"] \\ & \mathbf{Ab} & \end{tikzcd}$$

If we can show that $(-)_N$ takes projectives to $(-)_{G/N}$-acyclics, then we are entitled to apply Proposition 23. Note that $(-)_N$ takes free $\mathbb{Z}G$-modules to free $\mathbb{Z}G/N$ modules, simply by cutting away the basis elements on which N acts trivially. Since any projective is a direct summand of some free module and $(-)_N$ is additive, it thus takes projectives to projectives.

Proposition 23 now tells us that there exists a convergent spectral sequence

$$E^2_{p,q} = (L_p(-)_{G/N} \circ L_q(-)_N)(A) \Longrightarrow L_{p+q}(-)_G(A).$$

That is, a spectral sequence

$$E^2_{p,q} = H_p(G/N, H_q(N, A)) \Longrightarrow H_{p+q}(G, A).$$

[2]On the nose!

This is known as the *Lyndon-Hochschild-Serre spectral sequence.*