1 Triangulated categories

Definition 1 (triangulated category)**.** A triangulated category consists of the following data.

1. An additive category $\mathcal{T}$.
2. An equivalence
$$\mathcal{T} \to \mathcal{T}; \qquad X \mapsto X[1],$$
called the translation functor.
3. A collection of distinguished triangles
$$\left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1] \right\}.$$

These must satisfy the following four axioms.

(TR1) a) Every morphism f can be extended to a distinguished triangle as above.

b) The collection of distinguisned triangles is closed under isomorphism.

c) Given $X \in \mathcal{T}$, the diagram
$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$
is also distinguished.

(TR2) A diagram
$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$
if and only if the 'rotated' diagram
$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$
is distinguished.

(TR3) Given a solid commmutative diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d, "f[1]"] \\
X \arrow[r] & Y \arrow[r] & Z \arrow[r] & X[1]
\end{tikzcd}$$

with distinguished rows, there exists a dashed morphism making everything commute.

(TR4) Suppose we are given the following distinguished triangles.

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

Then there is a distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

making the following diagram commute.

$$\begin{tikzcd}
X \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & Y \arrow[r, "g"'] \arrow[d, "u"'] & Z \arrow[d, "v"] & & \\
& Y/X \arrow[r, "\phi"] & Z/X \arrow[r, "d''"] \arrow[d, "\psi"'] & X[1] \arrow[d, "f[1]"] & \\
& & Z/Y \arrow[r, "d'"] \arrow[rr, bend right, "\theta"'] & Y[1] \arrow[r, "u[1]"] & Y/X[1]
\end{tikzcd}$$