1 Homology

1.1 Basic definitions and examples

1.1.1 Ordinary homology

Singular homology is the study of topological spaces by associating to them chain complexes in the following way.

$$\mathbf{Top} \xrightarrow[\text{nerve}]{\text{singular}} \mathbf{Set}_\Delta \xrightarrow[\text{group}]{\text{free abelian}} \mathbf{Ab}_\Delta \xrightarrow[\text{correspondence}]{\text{Dold-Kan}} \mathbf{Ch}(\mathbf{Ab})$$

Definition 1 (singular complex, singular homology)**.** Let X be a topological space. The singular chain complex $S_\bullet(X)$ is defined level-wise by

$$S_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where $\mathcal{F}$ denotes the free group functor and M is the Moore functor (Definition **??**). That is, it has differentials

$$d_n \colon S_n \to S_{n-1}; \qquad (\alpha \colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(S_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f \colon X \to Y$ gives a chain map $S(f) \colon S(X) \to S(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 2. Denote by pt the one-point topological space. Then $\mathbf{Sing}(\mathrm{pt})_n = \{*\}$, and the chain complex $S_\bullet(\mathrm{pt})$ is at each level simply $\mathbb{Z}$.

The differential d_n is given by

$$d_n = \sum_{i=0}^{n} (-1)^i \partial^i; \qquad * \mapsto \sum_{i=0}^{n} (-1)^i * .$$

If n is odd, then there are as many summands with even sign as there are with odd sign, and they cancel each other out. If n is even, then there is one more term with positive sign than with negative sign, and one ends up with the identity. Thus $S(\mathrm{pt})_\bullet$ is given level-wise as follows.

$$\begin{tikzcd} \cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\ \cdots \arrow[r, "d_4"'] & S_3 \arrow[r, "d_3"'] & S_2 \arrow[r, "d_2"'] & S_1 \arrow[r, "d_1"'] & S_0 \arrow[r, "d_0"'] & S_{-1} \arrow[r, "d_{-1}"'] & \cdots \end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

1.1.2 Reduced homology

Recall that in addition to the *ordinary*, or *topologist's simplex category* Δ, there is the so-called *extended*, or *algebraist's simplex category* Δ_+, which includes the object $[-1] = \emptyset$. If we use this category to index our simplicial sets, our chain complexes have a term $\tilde{S}_{-1} = \mathbb{Z}$. It is traditional to denote the differential $d_0\colon \tilde{S}_0 \to \tilde{S}_{-1}$ by ε, and call it the *augmentation map*.

$$\cdots \longrightarrow \tilde{S}_2 \xrightarrow{d_2} \tilde{S}_1 \xrightarrow{d_1} \tilde{S}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Definition 3 (reduced singular chain complex)**.** The chain complex $\tilde{S}$ is called the reduced singular chain complex of X.

We see immediately that

$$S_i(X) = \tilde{S}_i(X), \qquad i \geq 0,$$

implying

$$H_i(X) = \tilde{H}_i(X), \qquad i \geq 1.$$

Clearly, there is not so much difference between singular and reduced homology. The difference is that, as one might suspect, ordinary homology is better-behaved from a topological point of view, and reduced homology is better behaved algebraically (for example, it turns wedge products into direct sums).

There is also another interpretation of reduced homology. Recall that we have used the name *augmentation map* before, when dealing with resolutions. There we reinterpreted the augmentation map ε as a chain map to a complex concentrated in degree zero. We can pull exactly the same trick here, viewing the reduced singular complex as a map of chain complexes.

$$\begin{tikzcd}
\cdots \arrow[r] & S_2 \arrow[r, "d_2"] \arrow[d] & S_1 \arrow[r, "d_1"] \arrow[d] & S_0 \arrow[r] \arrow[d, "\varepsilon"] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \cdots
\end{tikzcd}$$

Example 4. Let X be a nonempty path-connected topological space. Then $H_0(X) \cong \mathbb{Z}$. Specifically,

$$\varepsilon \colon H_0(X) \to \mathbb{Z}$$

is an isomorphism.

To see this, let $x \in S_0(X)$ be a point. Since $\varepsilon(nx) = n$ for any $n \in \mathbb{Z}$, we know that ε is surjective. If we can show that ε is injective, then we are done.

At face value, ε does not look injective; after all, there are as many elements of $S_0(X)$

consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

Suppose that X can be written as a disjoint union

$$X = \coprod_i X_i.$$

Then, since a map

$$\sigma \colon \Delta^n \to X$$

cannot hit more that one of the X_i, we have

$$\mathbf{Top}(\Delta^n, \coprod_i X_i) \cong \coprod_i \mathbf{Top}(\Delta^n, X_i).$$

Thus, since the free abelian group functor $\mathcal{F}$ and the Moore functor M are left adjoint, they preserve colimits and we have

$$C_n(\coprod_i X_i) \cong \bigoplus_i C_n(X_i).$$

1.2 The Hurewicz homomorphism

Theorem 5 (Hurewicz)**.** For any path-connected topological space X, there is an isomorphism

$$h_X \colon H_1(X) \cong \pi_1(X)_{\mathrm{ab}},$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism between the functors

$$h \colon H_1 \Rightarrow (\pi_1)_{\mathrm{ab}}.$$

Example 6. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\mathrm{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \qquad n > 1.$$

Example 7. Since

$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

1.3 Homotopy equivalence

Proposition 8. Let $f, g \colon X \to Y$ be continuous maps between topological spaces, and let $H \colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 9. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 10. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

1.4 Relative homology; the long exact sequence of a pair of spaces

Definition 11 (relative homology)**.** Let X be a topological space, and let $X \subset A$. The relative chain complex of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Denote by **Pair** the category whose objects are pairs (X, A), where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \to (Y, B)$ are maps $f: X \to Y$ such that $F(A) \subset B$.

Lemma 12. For each n, relative homology provides a functor H_n: **Pair** $\to$ **Ab**.

Proof. Consider the following diagram

$$\begin{tikzcd}
S(X)_\bullet \arrow[r, "f"] \arrow[d, two heads] & S(Y)_\bullet \arrow[d, two heads] \\
S(X)_\bullet/S(A)_\bullet \arrow[r, dashed] & S(Y)_\bullet/S(B)_\bullet
\end{tikzcd}$$

The dashed arrow is well-defined because of the assumption that $f(A) \subset B$. Functoriality now follows from the functoriality of H_n. ◻

Proposition 13. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{tikzcd}
& \cdots \arrow[r] & H_{j+1}(X, A) \arrow[dll, "\delta"] \\
H_j(A) \arrow[r] & H_j(X) \arrow[r] & H_j(X, A) \arrow[dll, "\delta"] \\
H_{j-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$\begin{tikzcd}
0 \arrow[r] & C_\bullet(A) \arrow[r, hook] & C_\bullet(X) \arrow[r, two heads] & C_\bullet(X, A) \arrow[r] & 0
\end{tikzcd}$$

◻

Example 14. Let $X = \mathbb{D}^n$, the n-disk, and $A = \mathbb{S}^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, out=-10, in=170, "\delta" description] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & H_j(\mathbb{D}^n) \arrow[r] & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, out=-10, in=170, "\delta" description] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

1.5 Barycentric subdivision

Fact 15. Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X. Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha \colon \Delta^n \to X$$

whose images are completely contained in some open set in the open cover $\mathfrak{U}$. That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$ induces a cochain structure on $S_\bullet^{\mathfrak{U}}(X)$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_2^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_1^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_0^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & 0 \\
\cdots \arrow[r] & S_2(X) \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & 0
\end{tikzcd}$$

In fact, this inclusion is homotopic to the identity, hence induces an isomorphism

$$H_n^{\mathfrak{U}}(X) := H_n(S^{\mathfrak{U}}(X)_\bullet) \equiv H_n(X).$$

This fact allows us almost immediately to read of two important theorems.

1.5.1 Excision

Theorem 16 (excision). Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{tikzcd} A \setminus W \arrow[r, hook, "i"] \arrow[d, hook] & A \arrow[d, hook] \\ X \setminus W \arrow[r, hook, "i"] & X \end{tikzcd}$$

induce an isomorphism

$$H_n(i)\colon H_n(X \setminus W, A \setminus W) \cong H_n(X, A).$$

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A.

1.5.2 The Mayer-Vietoris sequence

Theorem 17 (Mayer-Vietoris). Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X, i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, "\delta"] \\ H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n(X) \arrow[dll, "\delta"] \\ H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots & \end{tikzcd}$$

Proof. We can draw our inclusions as the following pushout.

$$\begin{tikzcd} & X_1 \arrow[dr, hook, "\kappa_1"] & \\ X_1 \cap X_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\ & X_2 \arrow[ur, hook, "\kappa_1"'] & \end{tikzcd}$$

We have, almost by definition, the following short exact sequence.

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xhookrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{\kappa_1 - \kappa_2} S_\bullet^{\mathfrak{U}}(X) \longrightarrow 0$$

This gives the following long exact sequence on homology.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H^{\mathfrak{U}}_{n+1}(X) \arrow[dll, "\delta"] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H^{\mathfrak{U}}_n(X) \arrow[dll, "\delta"] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}
$$

We have seen that $H^{\mathfrak{U}}(n)(X) \cong H_n(X)$; the result follows. □

Example 18 (Homology groups of spheres)**.** We can decompose $\mathbb{S}^n$ as

$$\mathbb{S}^n = (\mathbb{S}^n \smallsetminus N) \cup (\mathbb{S}^n \smallsetminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$
\begin{tikzcd}
 & \mathbb{S}^n \smallsetminus N \arrow[dr, hook] & \\
(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & \mathbb{S}^n \smallsetminus S \arrow[ur, hook] &
\end{tikzcd}
$$

The Mayer-Vietoris sequence is as follows.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_j((S^n \smallsetminus N) \cap (S^n \smallsetminus S)) \arrow[r] & H_j(S^n \smallsetminus N) \oplus H_j(S^n \smallsetminus S) \arrow[r] & H^{\mathfrak{U}}_j(X) \arrow[dll, "\delta"] \\
H_{j-1}((\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S)) \arrow[r] & \cdots &
\end{tikzcd}
$$

We know that

$$\mathbb{S}^n \smallsetminus N \cong \mathbb{S}^n \smallsetminus S \cong \mathbb{D}^n \simeq \mathrm{pt}$$

and that

$$(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$\cdots \longrightarrow H_{j+1}(\mathbb{S}^n) \xrightarrow{\delta} H_j(\mathbb{S}^{n-1}) \longrightarrow 0 \longrightarrow H_j(\mathbb{S}^n) \xrightarrow{\delta} H_{j-1}(\mathbb{S}^{n-1}) \longrightarrow \cdots$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X, $H_1(X) \cong \pi_1(X)_{\text{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\text{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$j = 3$	$\mathbb{Z}$	0		
$j = 2$	$\mathbb{Z}$	0		
$j = 1$	$\mathbb{Z}$	$\mathbb{Z}$		
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \qquad i > 1$$

allows us to fill in the above table as follows.

$j = 3$	$\mathbb{Z}$	0	0	$\mathbb{Z}$
$j = 2$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$j = 1$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

(1.1)

Example 19. Above, we used the Hurewicz homomorphism to see that

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed $\mathbb{S}^n$ as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$\begin{tikzcd} & X^+ \arrow[dr, hook] & \\ X^+ \cap X^- \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\ & X^- \arrow[ur, hook] & \end{tikzcd}$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^n) \arrow[dll, "\delta"] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0, i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r] & \cdots \end{tikzcd}$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^-, with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+, and $i_1(p)$ is a point of X^-. Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{tikzcd} \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[dll, "\delta"] \\ H_0(X^+ \cap X^-) \arrow[r, "H_0(i_0, i_1)"] & H_0(X^+) \oplus H_0(X^-) \arrow[r, "H_0(\kappa_1) - H_0(\kappa_2)"] & H_0(\mathbb{S}^1) \end{tikzcd}$$

We can immediately replace things we know, finding the following.

$$\begin{tikzcd}
 & & (a,b) \arrow[r, mapsto] & (a+b,a+b) & \\
0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[r, hook] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r, "f"] & \mathbb{Z}\oplus\mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & & & (c,d) \arrow[r, mapsto] & c-d
\end{tikzcd}$$

The kernel of f is the free group generated by (a,a). Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

1.5.3 The relative Mayer-Vietoris sequence

Theorem 20 (relative Mayer-Vietoris sequence)**.** Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X, A\cup B) \arrow[dll, "\delta"] \\
H_n(X, A\cap B) \arrow[r] & H_n(X,A)\oplus H_n(X,B) \arrow[r] & H_n(X, A\cup B) \arrow[dll, "\delta"] \\
H_{n-1}(X, A\cap B) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Consider the following chain complex of chain complexes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A\cap B) \arrow[r] \arrow[d] & S_n(A)\oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A\cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_n(X)\oplus S_n(X) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X, A\cap B) \arrow[r] \arrow[d] & S_n(X,A)\oplus S_n(X,B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A\cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem ??) implies that the last row is also exact.

Consider the following map of chain complexes; the first row is the last column of the above grid.

$$\begin{tikzcd}
0 \arrow[r] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r, hook] \arrow[d, "\phi"'] & S_n(X) \arrow[r, two heads] \arrow[d, equal] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & S_n(A \cup B) \arrow[r] & S_n(X) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0
\end{tikzcd}$$

This gives us, by Lemma ??, a morphism of long exact sequences on homology.

$$\begin{tikzcd}
H_n(S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\phi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_\bullet(X)/S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\psi)"] & H_{n-1}(S_\bullet\mathfrak{U}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\phi)"] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

We have seen (in Fact 15) that $H_i(\phi)$ is an isomorphism for all i. Thus, the five lemma (Theorem ??) tells us that $H_n(\psi)$ is an isomorphism. □

1.6 Reduced homology

It would be hard to argue that Table 1.1 is not pretty, but it would be much prettier were it not for the $\mathbb{Z}$s in the first column. We have to carry these around because every non-empty space has at least one connected component.

The solution is to define a new homology $\tilde{H}_n(X)$ which agrees with $H_n(X)$ in positive degrees, and is missing a copy of $\mathbb{Z}$ in the zeroth degree. There are three equivalent ways of doing this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Denoting the unique map $X \to \mathrm{pt}$ by ϵ, one can define relative homology by
$$\tilde{H}_n(X) = \ker H_n(\epsilon).$$

2. **In between:** One can replace homology $H_n(X)$ by relative homology
$$\tilde{H}_n(X) = H_n(X, x),$$
where $x \in X$ is any point of x.

3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of $\mathbb{Z}$ in degree -1, so that
$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$

Then one can define
$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the following. In constructing the singular chain complex of our space X, we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1)-simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through $\mathcal{F}$ thus gives a copy of $\mathbb{Z}$ as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- $(1 \Leftrightarrow 2)$: Since the composition

$$\mathrm{pt} \xhookrightarrow{x} X \xrightarrow{\epsilon} \mathrm{pt}$$

  is a weak retract, the sequence

$$0 \longrightarrow S_n(\mathrm{pt}) \longrightarrow S_n(X) \longrightarrow S_n(X, \{x\}) \longrightarrow 0$$

Proposition 21. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Proof. Trivial from algebraic definition. □

Many of our results for regular homology hold also for reduced homology.

Proposition 22. There is a long exact sequence for a pair of spaces

$$\begin{tikzcd}
 & \cdots \arrow[r] & \tilde{H}_{j+1}(X,A) \arrow[dll, "\delta"] \\
\tilde{H}_j(A) \arrow[r] & \tilde{H}_j(X) \arrow[r] & \tilde{H}_j(X,A) \arrow[dll, "\delta"] \\
\tilde{H}_{j-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proposition 23. We have a reduced Mayer-Vietoris sequence.

$$\begin{tikzcd} & \cdots \arrow[r] & \tilde{H}_{n+1}(X) \arrow[dll, "\delta"'] \\ \tilde{H}_n(X_1 \cap X_2) \arrow[r] & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \arrow[r] & \tilde{H}_n(X) \arrow[dll, "\delta"'] \\ \tilde{H}_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots & \end{tikzcd}$$

Proposition 24. Let $\{(X_i, x_i)\}_{i \in I}$, be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$[1] we have

$$\tilde{H}_n\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in E} \tilde{H}_n(X_i).$$

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of Proposition 23 tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 25 (good pair)**.** A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X.
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U.

$$A \hookrightarrow U \xrightarrow{r} A$$

Proposition 26. Let (X, A) be a good pair. Let $\pi : X \to X/A$ be the canonical projection. Then

$$\tilde{H}(X, A) \cong \tilde{H}_n(X/A) \qquad \text{for all } n > 0.$$

[1] This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Proof. □

Theorem 27 (suspension isomorphism)**.** Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

1.7 Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d\,\mu_n, \qquad \text{for some } d \in \mathbb{Z}.$$

Definition 28 (mapping degree)**.** We call $d \in \mathbb{Z}$ as above the mapping degree of f, and denote it by $\deg(f)$.

Example 29. Consider the map

$$\omega\colon [0,1] \longrightarrow \mathbb{S}^1; \qquad t \mapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism (Theorem 5), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1; \qquad x \mapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism (Theorem 5) tells us that the following diagram commutes.

$$\begin{tikzcd} \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[r, "\pi_1(f_n)_{\mathrm{ab}}"] \arrow[d, "h_{\mathbb{S}^1}"'] & \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[d, "h_{\mathbb{S}^1}"] \\ H_1(\mathbb{S}^1) \arrow[r, "H_1(f_n)"'] & H_1(\mathbb{S}^1) \end{tikzcd}$$

1.8 CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 30 (cell)**.** Let X be a topological space. We say that X is an n-cell if X is homeomorphic to $\mathbb{R}^n$. We call the number n the dimension of X.

Definition 31 (cell decomposition)**.** A cell decomposition of a topological space X is a decomposition

$$X = \coprod_{i \in I} X_i, \qquad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topologial spaces.

Definition 32 (CW complex)**.** A Hausdorff topological space is known as a CW complex[2] if it satisfies the following conditions.

(CW1) For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n - 1$.

(CW2) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X.

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ.
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map*.

Example 33. Consider the unit interval $I = [0, 1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n + 1$ 0-cells and n 1-cells. which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma^0_k = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \qquad \text{and} \qquad \sigma^0_\infty = 0$$

and one-cells

$$\sigma^1_k = \left(\frac{1}{k}, \frac{1}{k+1}\right), \qquad k \in \mathbb{N}^{\geq 1}.$$

[2] Axiom (CW2) is called the *closure-finiteness* condition. This is the ‘C’ in CW complex. Axiom (CW3) says that X carries the *weak topology* and is responsible for the ‘W’.

At first glance, this looks like a CW decomposition; it is certainly satisfies Axiom (CW1) and Axiom (CW2). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right),$$

is the midpoint of the interval σ_k^1. We have $A \cap \sigma_k^0 = \emptyset$ for all k, and $A \cap \sigma_k^1 = \{a_k\}$ for all k. In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i. However, the set A is not closed in I, since it does not contain its limit point $\lim_{n\to\infty} a_n = 0$.

Definition 34 (skeleton, dimension)**.** Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n-skeleton of X. If X is equal to its n-skeleton but not equal to its $(n-1)$-skeleton, we say that X is n-dimensional.

Note 35. Axiom (CW3) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 36 (subcomplex, CW pair)**.** Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$

We call such a pair (X, Y) a CW pair.

Fact 37. Let X and Y be CW complexes such that X is locally compact. Then $X \times Y$ is a CW complex.

Lemma 38. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 39. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact[3] if and only if it is locally finite.[4]

Proof. It is clear that 1. $\Rightarrow$ 2., since X is a subset of itself. Similarly, it is clear that 2. $\Rightarrow$ 3., since ☐

[3] I.e. every point of X has a compact neighborhood.
[4] I.e. if every point has a neighborhood which is contained in only finitely many cells.

Corollary 40. If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n.

$$\begin{tikzcd}
 & & & & K \arrow[d, "f"] \arrow[llld, dashed, "\exists \tilde{f}"'] \\
X^{n-1} \arrow[r, hook] & X^n \arrow[r, hook] & X^{n+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X
\end{tikzcd}$$

Proposition 41. Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0,1]$ is a strong deformation retract of $X \times [0,1]$.

Lemma 42. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborbood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.
- Every CW complex is paracompact, locally path-connected, and locally contractible.
- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 43. Let X be a CW complex. We have the following decompositions.

1. $$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$
2. $$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n-cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n-cell is homeomorphic to the open n-ball.
2. For every n-cell σ, the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$-skeleton.

□

1.9 Cellular homology

Lemma 44. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 42, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 26, and the second from Lemma 43. ◻

Lemma 45. Consider the inclusion $i_n \colon X^n \hookrightarrow X$.

- The induced map
$$H_n(i_n)\colon H_n(X^n) \to H_n(X)$$
is surjective.
- On the $(n+1)$-skeleton we get an isomorphism
$$H_n(i_{n+1})\colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n). The associated long exact sequence tells us that the sequence

$$\begin{tikzcd} H_n(X^n) \arrow[r] & H_n(X^{n+1}) \arrow[r] & H_n(X^{n+1}, X^n) \end{tikzcd}$$

is exact. But by Lemma 44,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n)\colon X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{tikzcd} & & H_{n+1}(X^{m+1}, X^m) \arrow[dll, out=-30, in=150, "\delta"] \\ H_n(X^m) \arrow[r] & H_n(X^{m+1}) \arrow[r] & H_n(X^{m+1}, X^m) \end{tikzcd}$$

But again by Lemma 44, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \to H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r] \arrow[drrr] & X^{n+1} \arrow[r] \arrow[drr] & X^{n+2} \arrow[r] \arrow[dr] & \cdots \\ & & & & X \end{tikzcd}$$

Taking nth singular homology, we find the following.

$$\cdots \longrightarrow H_n(X^n) \xrightarrow{\alpha_1} H_n(X^{n+1}) \overset{\alpha_2}{=\!=} H_n(X^{n+2}) \overset{\alpha_3}{=\!=} \cdots \longrightarrow H_n(X)$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \qquad \sigma_i \colon \Delta^n \to X.$$

Since the standard n-simplex Δ^n is compact, Corollary 40 implies that each σ_i factors through some X^{n_i}. Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \qquad \tilde{\sigma}_i \colon \Delta^n \to X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma \right]$$

□

Corollary 46. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q-cells, then $H_q(X) \cong 0$.
3. In particular, for an n-dimensional CW-complex X (Definition 34), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 45.
2.

□

Definition 47 (cellular chain complex)**.** Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}).

Theorem 48 (comparison of cellular and singular homology)**.** Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n \colon H_n(C_\bullet(X), d) \cong H_n(X).$$

Example 49 (complex projective space)**.** Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \mathrm{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots | x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{array}{ccccccccccc} 2n & & 2n-1 & & 2n-2 & & \cdots & & 1 & & 0 \\ \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \end{array}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i,\ 0 \le i \le n \\ 0, & \text{otherwise.} \end{cases}$$

Example 50 (real projective space)**.** As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{array}{ccccccccccc} n & & n-1 & & n-2 & & \cdots & & 1 & & 0 \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

1.10 Homology with coefficients

Definition 51 (homology with coefficients)**.** Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X;G) = S(X) \otimes_{\mathbb{Z}} G.$$

The nth singular homology of X with coefficients in G is the nth homology

$$H_n(X;G) = H_n(S(X;G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G.

Theorem 52 (topological universal coefficient thoerem for homology)**.** For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X;G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0 \ .$$

Furthermore, this sequence splits non-canonicaly, telling us that

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. REF to be updated with UCT. □

1.11 The topological Künneth formula

Recall that the Künneth formula (Section ??) tells us that the following sequence is short exact.

TODO

It turns out that we can relate $H_p(X) \otimes H_q(Y)$ and $H_{p+q}(X \times Y)$. We first define a so-called *homology cross product*

$$\times\colon S_p(X) \otimes S_q(Y) \longrightarrow S_{p+q}(X \times Y),$$

and then show that it descends to homology.

Our goal is to find a way of making a p-simplex in X and a q-simplex in Y into a $(p+q)$-simplex in $X \times Y$. At least in the case that one of p or q is equal to zero, it is clear what to do, since the Cartesian product of a p-simplex with a zero-simplex is again a p-simplex. In more general cases, however, we have to be clever. The strategy is to write down a list of properties that we would like our homology cross product to have, and then show that there exists a unique map satisfying them.

Lemma 53. We can define a homomorphism

$$\times\colon S_p(X) \otimes S_q(Y) \longrightarrow S_{p+q}(X \times Y), \qquad p, q \geq 0,$$

with the following properites.

1. For all points $x_0 \in X$ viewed as zero-chains in $S_0(X)$ and all $\beta\colon \Delta^q \to Y$, we have

   $$(x_0 \times \beta)(t_0, \ldots, t_q) = (x_0, \beta(t_0, \ldots, t_q));$$

   conversely, for $\alpha \in S_p(X)$ and $y_0 \in Y$, we have

   $$(\alpha \times y_0)(t_0, \ldots, t_p) = (\alpha(t_0, \ldots, t_q), y_0).$$

2. The map $\times$ is natural in the sense that the square

   $$\begin{tikzcd} S_p(X) \otimes S_q(Y) \arrow[r, "\times"] \arrow[d] & S_{p+q}(X \times Y) \arrow[d] \\ S_p(X') \otimes S_q(Y') \arrow[r, "\times"] & S_{p+q}(X' \times Y') \end{tikzcd}$$

   commutes.

3. The map $\times$ satisfies the Leibniz rule in the sense that

   $$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

Proof. As a warm up, let us work out some hypothetical consequences of our formulae, in the special case that $X = \Delta^p$ and $Y = \Delta^q$. In this case, we have

$$\mathrm{id}_{\Delta^p} \in S_p(\Delta^p), \qquad \mathrm{id}_{\Delta^q} \in S_q(\Delta^q),$$

so we can take the homology cross product $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} \in S_{p+q}(\Delta^p \times \Delta^q)$. By the Leibniz rule, we have

$$\partial(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) = \partial(\mathrm{id}_{\Delta^p}) \times \mathrm{id}_{\Delta^q} + (-1)^p \Delta^p \times \partial(\mathrm{id}_{\Delta^q}).$$

This is now a perfectly well-defined element of $S_{p+q-1}(\Delta^p \times \Delta^q)$, which we will give the nickname R.

A trivial computation shows that $\partial R = 0$, so there exists a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$. We choose some such c and define $\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q} = c$.

We now use the trick of expressing some $\alpha \colon \Delta^p \to X$ as $S_p(\alpha)(\mathrm{id}_{\Delta^p})$. Then

$$\alpha \times \beta = S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}).$$

But then the naturality forces our hand; chasing $\mathrm{id}_{\Delta^p} \otimes \mathrm{id}_{\Delta^q}$ around the naturality square

$$\begin{tikzcd}
S_p(\Delta^p) \otimes S_q(\Delta^q) \arrow[r, "\times"] \arrow[d, "S_p(\alpha)\otimes S_q(\beta)"'] & S_{p+q}(\Delta^p \times \Delta^q) \arrow[d, "S_{p+q}(\alpha,\beta)"] \\
S_p(X) \otimes S_p(Y) \arrow[r, "\times"] & S_{p+q}(X \times Y)
\end{tikzcd}$$

tells us that

$$\begin{aligned}
S_{p+q}(\alpha, \beta)(\mathrm{id}_{\Delta^p} \times \mathrm{id}_{\Delta^q}) &= S_p(\alpha)(\mathrm{id}_{\Delta^p}) \times S_q(\beta)(\mathrm{id}_{\Delta^q}) \\
S_{p+q}(\alpha, \beta)(c) &= \alpha \times \beta.
\end{aligned}$$

□

Theorem 54 (topological Künneth formula)**.** For any topological spaces X and Y, we have the following short exact sequence.

$$0 \longrightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \longrightarrow 0$$

1.12 The Eilenberg-Steenrod axioms

Denote by T the functor

$$T \colon \mathbf{Pair} \to \mathbf{Pair}; \qquad (X, A) \mapsto (A, \emptyset); \qquad (f \colon (X, A) \to (Y, B)) \mapsto f|_A.$$

Definition 55 (homology theory)**.** Let A be an abelian group. A homology theory with coefficients in A is a sequence of functors

$$H_n\colon \mathbf{Pair} \longrightarrow \mathbf{Ab}, \qquad n \geq 0$$

together with natural transformations

$$\delta\colon H_n \Rightarrow H_{n-1} \circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,

$$f \sim g \implies H_n(f) = H_n(g) \qquad \text{for all } f, g, n.$$

2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(i)\colon H_n(X \smallsetminus U, A \smallsetminus U) \cong H_n(X, A)$$

   for all n.

3. **Dimension:** We have

$$H_n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.

2 Cohomology

2.1 Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition 55) which govern cohomology theories.

Definition 56 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n\colon \mathbf{Pair}^{\mathrm{op}} \longrightarrow \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial\colon H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

   $$H^n(i)\colon (X, A) \cong H^n(X \setminus U, A \setminus U)$$

   for all n.
3. **Dimension:** We have

   $$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then

   $$H^n(X) = \prod_\alpha H^n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

2.2 Singular cohomology

Definition 57 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet, G)$$

We will denote the evaluation map $\mathrm{Hom}(A,G)\otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 58. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G)\otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet)\otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha\colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned}\langle \alpha, a + db\rangle &= \langle \alpha, a\rangle + \langle \alpha, db\rangle \\ &= \langle \alpha, a\rangle + \langle \delta\alpha, b\rangle \\ &= \langle \alpha, a\rangle.\end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned}\langle \alpha + \delta\beta, a\rangle &= \langle \alpha, a\rangle + \langle \delta\beta, a\rangle \\ &= \langle \alpha, a\rangle + \langle \beta, da\rangle \\ &= \langle \alpha, a\rangle\end{aligned}$$

□

Via $\otimes$-hom adjunction, we get a map

$$\kappa\colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 59 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space. There is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X;G) \twoheadrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

Example 60. We have seen (in Example 49) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \le k \le 2n,\ k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \le k \le n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(0, \mathbb{Z}) \oplus \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \le k \le n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \mathrm{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \mathrm{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

2.3 The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \longrightarrow \mathbb{Z},$$

allowing an n-cochain to eat an n-chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q-simplex, on which our cochain can act, and a singular $n-q$-simplex, which is along for the ride.

Definition 61 (front, rear face)**.** Let $a\colon \Delta^n \longrightarrow X$ be a singular n-simplex, and let $0 \le q \le n$.

- The (n − q)-dimensional front face of a is given by

  $$F^{n-q}(a) = \quad \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X\ ,$$

  where i is induced by the inclusion

  $$\{0, \ldots, n-q\} \hookrightarrow \{1, \ldots, n\}; \qquad k \mapsto k.$$

- The q-dimensional rear face of a is given by

  $$R^q(a) = \quad \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X\ ,$$

where r is the map

$$\{0, \dots q\} \longrightarrow \{0, \dots, n\}; \qquad k \mapsto n - q + k.$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \longrightarrow S_{n-q}(X); \qquad \alpha \otimes a \mapsto F^{n-q}(a)\langle a, R^q(a)\rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 62 (cap product)**.** The cap product is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \longrightarrow S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b)\rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b)\rangle = 0$ by definition of relative cohomology.

Lemma 63. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial\alpha \otimes r).$$

Proposition 64. The cup product descends to a map on homology; that is, we get a map

$$\frown : H^q(X, A; R) \otimes H_n(X, A; R) \longrightarrow H_{n-q}(X; R); \qquad [\alpha \otimes a \otimes r] \mapsto [F^{n-q}(a) \otimes \langle \alpha, R^q(a)\rangle r].$$