

Homological Algebra

With Applications to Algebraic Topology

Angus Rush

$$\begin{array}{ccccccc}
 0 & \cdots \rightarrow & \ker f & \cdots \rightarrow & \ker g & \cdots \rightarrow & \ker h \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \xrightarrow{m} & B & \xrightarrow{e} & C \longrightarrow 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h \\
 0 & \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{coker } f & \cdots \rightarrow & \text{coker } g & \cdots \rightarrow & \text{coker } h \cdots \rightarrow 0
 \end{array}$$

Dashed arrows indicate commutativity: $A \xrightarrow{f} A' \rightarrow \text{coker } f$, $B \xrightarrow{g} B' \rightarrow \text{coker } g$, $C \xrightarrow{h} C' \rightarrow \text{coker } h$.
 A dashed arrow also goes from $\ker h$ to $\text{coker } h$.
 A dashed arrow goes from A' to $\ker g$.
 A dashed arrow goes from B' to $\ker h$.

Contents

1. Homological algebra	7
1. Introduction	9
2. Abelian categories	11
2.1. Basics	11
2.1.1. Building blocks	11
2.1.2. Abelian categories	16
2.1.3. Abelian-ness is a property, not a structure	17
2.2. Basic properties of abelian categories	18
2.3. Embedding theorems	18
3. Homological algebra in abelian categories	23
3.1. Chain complexes	23
3.2. Homology	25
3.3. Diagram lemmas	27
3.3.1. The splitting lemma	27
3.3.2. The snake lemma	28
3.3.3. The long exact sequence on homology	33
3.3.4. The five lemma	34
3.3.5. The nine lemma	36
3.4. Exact functors	37
3.4.1. Exactness and preservation of (co)limits	39
3.4.2. Exactness of the hom functor and tensor products	41
3.5. Quasi-isomorphism	42
3.6. Chain Homotopies	43
3.7. Projectives and injectives	44
3.7.1. Basic properties	45
3.7.2. Enough projectives	45
3.7.3. Here be dragons	47
3.7.4. Projective resolutions	47
3.7.5. Lifting data to projective resolutions	48
3.8. The homotopy category	51
3.9. Mapping cones	52

3.10. Derived functors	55
3.10.1. Derived categories	55
3.10.2. Derived functors	56
3.11. Double complexes	60
3.11.1. Tensor-hom adjunction for chain complexes	66
3.11.2. The Künneth formula	68
4. Some important derived functors	71
4.1. The Tor functor	71
4.2. The Ext functor	73
4.2.1. Ext and extensions	74
4.2.2. The Baer sum	82
4.2.3. Higher extensions	84
4.3. Inverse and direct limits	84
4.3.1. Direct limits	84
4.3.2. Inverse limits	86
5. Basic applications	93
5.1. Syzygy theorem	93
5.2. Quiver representations	94
5.3. Group (co)homology	97
5.3.1. The Dold-Kan correspondence	97
5.3.2. The bar construction	100
5.3.3. Invariants and coinvariants	103
5.3.4. The bar construction for group rings	107
5.3.5. Computations in low degrees	111
5.3.6. Periodicity in group homology	117
6. Spectral sequences	119
6.1. Motivation	119
6.2. Homology spectral sequences	122
6.2.1. Notation and terminology	122
6.2.2. Stabilization and convergence	124
6.2.3. Exact couples	126
6.2.4. The spectral sequence of a filtered complex	126
6.3. The spectral sequence of a double complex	131
6.4. Hyper-derived functors	133
6.4.1. Cartan-Eilenberg resolutions	134
6.4.2. Hyper-derived functors	138
7. Triangulated categories	141

II. Algebraic topology	143
8. Homology	145
8.1. Basic definitions and examples	145
8.1.1. Ordinary homology	145
8.1.2. Reduced homology	146
8.2. The Hurewicz homomorphism	148
8.3. The method of acyclic models	150
8.4. Homotopy equivalence	152
8.4.1. Homotopy invariance: with acyclic models	153
8.4.2. Homotopy-invariance: without acyclic models	154
8.5. Relative homology	156
8.5.1. Relative homology	156
8.5.2. The long exact sequence on a pair of spaces	157
8.5.3. The braided monstrosity on a triple of spaces	160
8.6. Barycentric subdivision	161
8.6.1. Excision	164
8.6.2. The Mayer-Vietoris sequence	165
8.6.3. The relative Mayer-Vietoris sequence	168
8.7. Reduced homology	170
8.7.1. Deja vu all over again	172
8.8. Mapping degree	174
8.9. CW Complexes	175
8.10. Cellular homology	178
8.10.1. Euler characteristic	181
8.11. Homology with coefficients	182
8.12. The topological Künneth formula	183
8.12.1. The Eilenberg-Zilber theorem: with acyclic models	183
8.12.2. The Eilenberg-Zilber theorem: without acyclic models	184
8.12.3. The topological Künneth formula	186
8.13. The Eilenberg-Steenrod axioms	187
9. Cohomology	189
9.1. Axiomatic description of a cohomology theory	189
9.2. Singular cohomology	190
9.3. The cap product	191
9.4. The cup product	195
9.5. Manifold orientations	197
9.5.1. Basic definitions and results	197
9.5.2. The orientation cover	204
9.5.3. Degree	205
9.6. Cohomology with compact support	205

Contents

9.7. Poincaré duality	207
9.8. Alexander-Lefschetz duality	208
9.9. The Milnor sequence	208
9.10. Lens spaces	209

Part I.

Homological algebra

1. Introduction

Homological algebra is about chain complexes.

Definition 1 (chain complex). Let R be a ring.¹ A chain complex of R -modules, denoted (C_\bullet, d) , consists of the following data:

- For each $n \in \mathbb{Z}$, an R -module C_n , and
- R -linear maps $d_n: C_n \rightarrow C_{n-1}$, called the *differentials*,

such that the differentials satisfy the relation

$$d_{n-1} \circ d_n = 0.$$

¹Associative, with unit. Not necessarily commutative.

2. Abelian categories

2.1. Basics

Abelian categories provide a categorical axiomatization of the properties of the category $R\text{-Mod}$ of modules over a (not necessarily commutative) ring R ; in fact, one can show that any small abelian category is a full subcategory of $R\text{-Mod}$ for some R . Homological is the study of chain complexes of R -modules, and hence of the study of chain complexes in an abelian category. One of the benefits of working with abelian categories rather than working directly $R\text{-Mod}$ is that many things which are not obviously abelian categories turn out to be abelian categories; for example, the category of chain complexes in an abelian category is itself an abelian category, so when one is studying abelian categories one is already learning something about chain complexes in an abelian category.

2.1.1. Building blocks

Additive categories

Definition 2 (Ab-enriched category). A category \mathcal{C} is called **Ab-enriched** if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$.

Definition 3 (additive category). A category \mathcal{C} is additive if it has finite products (including an initial object) and is **Ab-enriched**.

Proposition 4. Let \mathcal{C} be an additive category. Then the initial object is also a terminal object (hence a zero object), and the product $A \times B$ satisfies the universal property for a coproduct $A \amalg B$.

Proof. Denote by \emptyset an initial object in \mathcal{C} . The unique morphism $\text{id}_{\emptyset}: \emptyset \rightarrow \emptyset$ must be the identity element $e_{\emptyset \rightarrow \emptyset} \in \text{Hom}(\emptyset, \emptyset)$. Let X be any other object in \mathcal{C} , and let $f: X \rightarrow \emptyset$. Then $f \circ e_{\emptyset \rightarrow \emptyset} = f$. However, since composition is bilinear, we also have $f \circ e = e_{X \rightarrow \emptyset}$. Thus, there is a unique map $X \rightarrow \emptyset$ for all X , so \emptyset is a terminal object (hence a zero object). From now on, we will write 0 for zero objects.

By the universal property for products, we get a unique map $i_A: A \rightarrow A \times B$ making the

2. Abelian categories

following diagram commute.

$$\begin{array}{ccccc}
 & & A & & \\
 & \swarrow \text{id}_A & \downarrow \exists! i_A & \searrow 0 & \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array}$$

Similarly, there exists a unique map $i_B: B \rightarrow A \times B$ making the diagram

$$\begin{array}{ccccc}
 & & B & & \\
 & \swarrow 0 & \downarrow \exists! i_B & \searrow \text{id}_B & \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array}$$

commute.

Consider the following diagram.

$$\begin{array}{ccccc}
 & & A \times B & & \\
 & \swarrow \pi_A & \downarrow \exists! & \searrow \pi_B & \\
 A & \xleftarrow{\pi_A} & A \times B & \xrightarrow{\pi_B} & B
 \end{array}$$

Obviously, the identity map $\text{id}_{A \times B}$ makes this diagram commute. However

$$\pi_A \circ (i_A \circ \pi_A + i_B \circ \pi_B) = \pi_A$$

and

$$\pi_B \circ (i_A \circ \pi_A + i_B \circ \pi_B) = \pi_B,$$

so the map $i_A \circ \pi_A + i_B \circ \pi_B$ also does. Thus,

$$\pi_A \circ i_A + \pi_B \circ i_B = \text{id}_{A \times B}.$$

We now claim that $A \times B$, together with the maps i_A and i_B , satisfy the universal property for coproducts. To see this, let X be an object and $f: A \rightarrow X$ and $g: B \rightarrow X$ be morphisms.

$$\begin{array}{ccccc}
 A & \xrightarrow{i_A} & A \times B & \xleftarrow{i_B} & B \\
 & \searrow f & \downarrow \phi & \swarrow g & \\
 & & X & &
 \end{array}$$

We need to show that there exists a unique morphism ϕ making the diagram commute. Note that, by logic identical to that above, taking

$$\phi = f \circ \pi_A + g \circ \pi_B$$

makes the above diagram commute. Now suppose that there exists another $\tilde{\phi}$ making

the diagram commute. Then

$$\begin{aligned}
 \tilde{\phi} &= \tilde{\phi} \circ \text{id}_{A \times B} \\
 &= \tilde{\phi} \circ (i_A \circ \pi_A + i_B \circ \pi_B) \\
 &= f \circ \pi_A + g \circ \pi_B \\
 &= \phi.
 \end{aligned}$$

□

Proposition 4 tells us that, in an additive category with products, we also have coproducts, and that these agree with the products. Dually, we could have defined an additive category to be a category with coproducts; the dual to **Proposition 4** would then have shown us that these agreed with products. Rather than preferencing either the notation for products or the notation for coproducts, we will use a new symbol.

Definition 5 (direct sum). Let \mathcal{C} be a category with a zero object 0 , and products and coproducts. If products and coproducts agree in the sense that the canonical map $A \amalg B$ Then we say that \mathcal{C} has direct sums, and denote the product/coproduct of A and B by $A \oplus B$.

Definition 6 (additive functor). Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between additive categories. We say that F is additive if for each $X, Y \in \text{Obj}(\mathcal{C})$ the map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Lemma 7. Let A and B be objects in an additive category \mathcal{A} , and let X be an object in \mathcal{A} equipped with morphisms as follows,

$$\begin{array}{ccccc}
 A & & & & A \\
 & \searrow i_A & & \nearrow \pi_A & \\
 & & X & & \\
 & \nearrow i_B & & \searrow \pi_B & \\
 B & & & & B
 \end{array}$$

such that the following equations hold.

$$\begin{aligned}
 i_A \circ \pi_A + i_B \circ \pi_B &= \text{id}_X \\
 \pi_A \circ i_B &= 0 = \pi_B \circ i_A \\
 \pi_A \circ i_A &= \text{id}_A \\
 \pi_B \circ i_B &= \text{id}_B
 \end{aligned}$$

2. Abelian categories

Then the π s and i s exhibit $X \cong A \oplus B$.

Proof. It suffices to show that X is a product of A and B . Let $f: Z \rightarrow A$ and $g: Z \rightarrow B$. We need to show that there exists a unique $\phi: Z \rightarrow X$ making the following diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow \phi & \searrow g & \\ A & \xleftarrow{\pi_A} & X & \xrightarrow{\pi_B} & B \end{array}$$

That is, we need

$$\pi_A \circ \phi = f, \quad \pi_B \circ \phi = g.$$

Thus, we need

$$\begin{aligned} i_A \circ f + \pi_B \circ g &= i_A \circ \pi_A \circ \phi + i_B \circ \pi_B \circ \phi \\ &= \phi. \end{aligned}$$

It is easy to check that ϕ defined in this way *does* make the diagram commute, so ϕ exists and is unique. \square

Corollary 8. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(A \oplus B) \simeq F(A) \oplus F(B).$$

Conversely, if F preserves products,

Proof. Apply F to the data of [Lemma 7](#). \square

Pre-abelian categories

Definition 9 (kernel, cokernel). Let $f: A \rightarrow B$ be a morphism in an abelian category.

- A pair (K, ι) , where $\iota: K \rightarrow A$ is a morphism, is a kernel of f if $f \circ \iota = 0$ and for every object X and morphism $g: X \rightarrow A$ such that $f \circ g = 0$, there exists a unique morphism $\alpha: X \rightarrow K$ such that $f \circ \alpha = 0$.

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \exists! & \downarrow g & \searrow 0 & \\ K & \xrightarrow{\iota} & A & \xrightarrow{f} & B \\ & \searrow & \downarrow 0 & \swarrow & \\ & & 0 & & \end{array}$$

- A pair (C, π) , where $\pi: B \rightarrow C$ is a morphism, is a cokernel of f if $\pi \circ f = 0$ and for every object Y and morphism $g: B \rightarrow Y$ such that $g \circ f = 0$, there exists a

unique morphism $\beta: C \rightarrow Y$ such that $\beta \circ f = 0$.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \curvearrowright & & \curvearrowright & \\
 A & \xrightarrow{f} & B & \xrightarrow{\pi} & C \\
 & \searrow & \downarrow g & \swarrow \exists! & \\
 & 0 & Y & &
 \end{array}$$

Here are some equivalent ways of defining the kernel of a morphism $f: A \rightarrow B$.

- The equalizer of f with the zero map.

$$\ker f = \text{eq} \longrightarrow A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} B$$

- The pullback along the zero morphism

$$\begin{array}{ccc}
 \ker f & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

Definition 10 (pre-abelian category). A pre-abelian category is an additive category such that every morphism has a kernel and a cokernel.

Lemma 11. Pre-abelian categories have all finite limits and colimits.

Proof. In a pre-abelian category, the equalizer of f and g is the kernel of $f - g$, and the coequalizer is the cokernel of $f - g$. Thus, pre-abelian categories have finite products and equalizers, and finite coproducts and coequalizers. \square

Proposition 12. In a pre-abelian category, kernels are monic and cokernels are epic.

Proof. The kernel of $f: A \rightarrow B$ is the pullback of the morphism $0 \rightarrow B$, which is monic. The pullback of a monic is a monic, which gives the result. \square

Proposition 13. Let \mathcal{C} be a pre-abelian category, and let $f: A \rightarrow B$ be a morphism.

1. The morphism f is a monomorphism if and only if $\ker f = 0$.
2. The morphism f is an epimorphism if and only if $\text{coker } f = 0$.

Proof.

1. Suppose f is a monomorphism, and consider a morphism g making the following

2. Abelian categories

diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & & \downarrow g & \searrow 0 & \\ \ker f & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

Because f is a monomorphism, we have $g = 0$. By the universal property for kernels, there exists a unique map $Z \rightarrow \ker f$ making the above diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \exists! & \downarrow g & \searrow 0 & \\ \ker f & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

But this remains true if we replace $\ker f$ by 0 , so by definition we must have $\ker f = 0$.

Now suppose that $\ker f = 0$, and let $g: X \rightarrow A$ such that $f \circ g = 0$. The universal property guarantees us a morphism $\alpha: X \rightarrow K$ making the left-hand triangle below commute.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \exists! & \downarrow g & \searrow 0 & \\ \ker f = 0 & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

However, the only way the left-hand triangle can commute is if $g = 0$. Thus, f is mono.

2. Dual.

□

2.1.2. Abelian categories

We have seen that in a pre-abelian category, kernels and cokernels more or less behave as they do in the category of modules over some ring R . However, the first isomorphism theorem tells us that every submodule is the kernel of some module homomorphism (namely the cokernel), and every quotient module of some module homomorphism (namely the kernel).

Definition 14 (abelian category). A pre-abelian category is said to be abelian if every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Proposition 15. Let \mathcal{A} be an abelian category, and let $f: A \rightarrow B$ be a morphism in \mathcal{A} . Then f is an isomorphism if and only if $\ker f = 0$ and $\operatorname{coker} f = 0$.

Proof. If f is an isomorphism, then it is mono and epi, and we are done by [Proposition 13](#).

Conversely, if $\ker f = 0$ and $\operatorname{coker} f = 0$, then f is mono and epi. We need only to show that any morphism in an abelian category which is monic and epic is an isomorphism.

Let f be monic and epic. Then f is the kernel of its cokernel and the cokernel of its kernel, and both of these are zero.

$$\begin{array}{ccccc}
 & & B & \xrightarrow{0} & 0 = \operatorname{coker} f \\
 & \swarrow \exists! f_R^{-1} & \parallel & & \\
 \ker f = 0 & \longrightarrow & A & \xrightarrow{f} & B \\
 & \nwarrow 0 & \parallel & \swarrow \exists! f_L^{-1} & \\
 & & A & &
 \end{array}$$

This gives us morphisms f_L^{-1} and f_R^{-1} such that

$$f \circ f_R^{-1} = \operatorname{id}_A, \quad f_L^{-1} \circ f = \operatorname{id}_B.$$

The usual trick shows that $f_L^{-1} = f_R^{-1}$. □

As the following proposition shows, additive functors from additive categories preserve direct sums.

Example 16. Once it is known that a category \mathcal{A} is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in \mathcal{A} is abelian, as we will see in
- For any small category I , the category $\mathbf{Fun}(I, \mathcal{A})$ of I -diagrams in \mathcal{A} is abelian. To see this, note the following.
 - Addition of natural transformations, defined pointwise, is an \mathbf{Ab} -enrichment.
 - Finite products exist because limits in functor categories are taken pointwise.
 - Every morphism has a kernel and cokernel; and every monomorphism is the kernel of its cokernel; and every epimorphism is the cokernel of its kernel; because limits and colimits in functor categories are taken pointwise.

2.1.3. Abelian-ness is a property, not a structure

Our definition of an abelian category, namely an \mathbf{Ab} -enriched category with products, kernels, and cokernels, such that every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel, makes it sound like ordinary categories could never aspire to be abelian categories. After all, in an ordinary category, one has to provide extra structure, namely an addition law on morphisms, to get an \mathbf{Ab} -enriched category.

2. Abelian categories

Not so! It turns out that demanding that finite products and coproducts exist and agree is sufficient to specify the addition law.

Proposition 17. Let \mathcal{C} be a category with finite direct sums¹ (Definition 5). Then \mathcal{C} has a canonical **Ab**-enrichment.

Proof. Let $A \in \mathcal{C}$ be an object. Denote the diagonal map $A \rightarrow A \oplus A$ by Δ , and the codiagonal map $A \oplus A \rightarrow A$ by ∇ . Let $f, g: A \rightarrow B$. We define

$$f + g = \nabla \circ (f \oplus g) \circ \Delta.$$

To see that this is an **Ab**-enrichment, note the following.

1. With 0 the zero morphism, we have

$$f + 0$$

□

In fact, in an abelian category, the **Ab**-enrichment is completely determined by the direct sum structure. Thus, we have the following.

Corollary 18. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories. Then F is additive if and only if it preserves direct sums.

2.2. Basic properties of abelian categories

2.3. Embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

¹Including 0-ary direct sums, i.e. a zero object.

In abelian categories, one can define kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero object: f is a kernel for g if and only if the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & C \end{array}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category \mathbf{Set}_* of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category \mathcal{A} , suppose one could find an embedding $\mathcal{H}: \mathcal{A} \rightarrow \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \rightarrow B \rightarrow C$ in \mathcal{A} , mapped it into \mathbf{Set}_* finding a sequence $\mathcal{H}(A) \rightarrow \mathcal{H}(B) \rightarrow \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \rightarrow B \rightarrow C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in \mathbf{Set}_* using \mathcal{H} and check exactness there. Effectively, one could check exactness of a sequence in \mathcal{A} by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking that the image of the diagram under our functor \mathcal{H} commutes.

In fact, for \mathcal{A} small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to \mathbf{Set}_* .

Definition 19 (category of contravariant epimorphisms). Let \mathcal{A} be a small abelian category. Define a category \mathcal{A}_{\leftarrow} with $\text{Obj}(\mathcal{A}_{\leftarrow}) = \text{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\text{Hom}_{\mathcal{A}_{\leftarrow}}(X, Y) = \{f: Y \rightarrow X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity morphism is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A: \mathcal{A}_{\leftarrow} \rightarrow \mathbf{Set}_*; \quad Z \mapsto \text{Hom}_{\mathcal{A}}(Z, A)$$

2. Abelian categories

and sends a morphism $f: Z_1 \rightarrow Z_2$ in \mathcal{A}_{\leftarrow} (which is to say, an epimorphism $\tilde{f}: Z_2 \twoheadrightarrow Z_1$ in \mathcal{A}) to the map

$$\mathcal{H}_A(f): \text{Hom}_{\mathcal{A}}(Z_1, A) \rightarrow \text{Hom}_{\mathcal{A}}(Z_2, A); \quad (\alpha: Z_1 \rightarrow A) \mapsto (\alpha \circ \tilde{f}: Z_2 \rightarrow A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 20 (member functor). Let \mathcal{A} be a small abelian category. We define a functor $\mathcal{M}: \mathcal{A} \rightarrow \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \text{colim } \mathcal{H}_A.$$

On morphisms, functoriality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(A)$ have a much simpler interpretation.

Proposition 21. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \text{Hom}(X, A) / \sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ f' \downarrow & & \downarrow g \\ X' & \xrightarrow{g'} & A \end{array}$$

Proof. The colimit can be computed using the following coequalizer.

$$\coprod_{\substack{f \in \text{Morph}(\mathcal{A}_{\leftarrow}) \\ \tilde{f}: Y \rightarrow X}} \text{Hom}_{\mathcal{A}}(X, A) \xrightarrow[\sim]{\text{id}} \coprod_{Z \in \text{Obj}(\mathcal{A}_{\leftarrow})} \text{Hom}(Z, A) \xrightarrow{\text{coeq}} \mathcal{M}(A)$$

On elements, we have the following.

$$\begin{array}{ccc} & & (g: X \rightarrow A) \\ & \swarrow \text{id} & \parallel \\ (g: X \rightarrow A) & & \\ & \searrow - \circ \tilde{f} & \\ & & (g \circ \tilde{f}: Y \rightarrow A) \end{array}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \text{Obj}(\mathcal{A}_{\leftarrow})} \text{Hom}(Z, A) / \sim,$$

where \sim is the equivalence relation generated by the relation

$$(g: X \rightarrow A)R(g': X' \rightarrow A) \iff \exists f: X' \twoheadrightarrow X \text{ such that } g' = g \circ \tilde{f}.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g: X \rightarrow A) \sim (g': X' \rightarrow A) \iff \exists f: Z \twoheadrightarrow X, f': Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ f' \downarrow & & \downarrow g \\ X' & \xrightarrow{g'} & A \end{array}$$

□

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphisms into A modulo the above relation, and for a morphism $f: A \rightarrow B$ in \mathcal{A} , $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f): [g] \mapsto [g \circ f].$$

Lemma 22. Let $f: A \rightarrow B$ be a morphism in a small abelian category \mathcal{A} such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms making the following diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{g} & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. □

Now we introduce some load-lightening notation: we write $\hat{(-)} = \mathcal{M}(-)$.

Lemma 23. Let $f: A \rightarrow B$ be a morphism in a small abelian category \mathcal{A} . Then $f = 0$ if and only if $\hat{f}: \hat{A} \rightarrow \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g , so $\hat{f} = 0$.

2. Abelian categories

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\text{id}_A]) = [0]$. But

$$\hat{f}([\text{id}_A]) = [\text{id}_A \circ f] = [f].$$

By [Lemma 22](#), $[f] = [0]$ implies $f = 0$. □

Corollary 24. A diagram commutes in \mathcal{A} if and only if its image in \mathbf{Set}_* under \mathcal{M} commutes.

Proof. A diagram commutes in \mathcal{A} if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. □

Lemma 25. Let $f: A \rightarrow B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

3. Homological algebra in abelian categories

3.1. Chain complexes

Definition 26 (category of chain complexes). Let \mathcal{A} be an abelian category. A chain complex C_\bullet in \mathcal{A} is a collection of objects $C_n \in \mathcal{A}$, $n \in \mathbb{Z}$, together with morphisms $d_n: C_n \rightarrow C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

The category of chain complexes in \mathcal{A} , denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms $C_\bullet \rightarrow D_\bullet$ of chain complexes, i.e. for each $n \in \mathbb{Z}$ a map $f_n: C_n \rightarrow D_n$ such that all of the squares form commute.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{n+2}^C} & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \xrightarrow{d_{n-1}^C} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \xrightarrow{d_{n+2}^D} & D_{n+1} & \xrightarrow{d_{n+1}^D} & D_n & \xrightarrow{d_n^D} & D_{n-1} \xrightarrow{d_{n-1}^D} \cdots \end{array}$$

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes (i.e. the full subcategory consisting of chain complexes C_\bullet such that $C_n = 0$ for small enough n), $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Theorem 27. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the **Ab**-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in \mathcal{A} : given a diagram of the form

$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \swarrow f_\bullet & \downarrow \exists! \phi_\bullet & \searrow g_\bullet & \\ C_\bullet & \xleftarrow{p_1} & C_\bullet \times D_\bullet & \xrightarrow{p_2} & D_\bullet \end{array}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain

3. Homological algebra in abelian categories

map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned} (d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q, \end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \swarrow \phi_\bullet & \downarrow g_\bullet & \searrow 0 & \\ \ker(f_\bullet) & \xrightarrow{\iota_\bullet} & A_\bullet & \xrightarrow{f_\bullet} & B_\bullet \end{array}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \quad (3.1)$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi: Q_n \rightarrow \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 3.1 can play the role of ψ . Plugging in the LHS, we have

$$\begin{aligned} \iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n \end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomor-

phism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. \square

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

3.2. Homology

We would like to replicate the notion of the homology of a chain complex C_\bullet in the context of abelian categories.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots$$

This will mean finding appropriate notions of every symbol appearing in the equation.

$$H_n(C_\bullet) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

We already have a notion of the kernel, which we can use to define a notion of the image in the following way. Given a map $f: A \rightarrow B$, we heuristically think of the cokernel of f as being $B/\operatorname{im} f$. Thus, taking the kernel of the cokernel should give us the image of f .

Definition 28 (image). Given a morphism $f: A \rightarrow B$, the image of f is $\operatorname{im} f = \ker \operatorname{coker} f$.

Now we need to figure out what it means to quotient the image of one map by the kernel of the next. Again, the cokernel saves our necks; we just need a map from $\operatorname{im} d_{n+1}$ into $\ker d_n$. The homology will be the cokernel of this map.

Definition 29 (homology). Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be morphisms such that $g \circ f = 0$. Using the universal properties of the kernel and cokernel, we can build the following commutative diagram.

$$\begin{array}{ccccc} & & \operatorname{im} f & & \\ & & \downarrow i & \searrow 0 & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & \downarrow p & \nearrow & \\ & & \operatorname{coker} f & & \end{array}$$

3. Homological algebra in abelian categories

Since $g \circ i = 0$, i factors through $\ker g$, giving us a monomorphism $\phi: \operatorname{im} f \rightarrow \ker g$.

$$\begin{array}{ccccc}
 \operatorname{im} f & \xrightarrow{\phi} & \ker g & \twoheadrightarrow & H_n \\
 & \searrow i & \downarrow & & \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & & \downarrow p & \nearrow & \\
 & & \operatorname{coker} f & &
 \end{array}$$

The homology of the sequence $A \rightarrow B \rightarrow C$ at B is then $H_n = \operatorname{coker} \phi$.

Definition 30 (exact sequence). Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

be objects and morphisms in an abelian category. We say that the above sequence is exact if ϕ_i is an isomorphism, i.e. if the homology is zero. We say that a complex (C_\bullet, d) is exact if it is exact at all positions.

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Proposition 31. Homology extends to a family of additive functors

$$H_n: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet: C_\bullet \rightarrow D_\bullet$ be a morphism of chain complexes.

$$\begin{array}{ccccc}
 C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \\
 f_{n+1} \downarrow & & \downarrow f_n & & \downarrow f_{n-1} \\
 D_{n+1} & \xrightarrow{d_{n+1}^D} & D_n & \xrightarrow{d_n^D} & D_{n-1}
 \end{array}$$

We need a map $H_n(C_\bullet) \rightarrow H_n(D_\bullet)$. This will come from a map $\ker d_n^C \rightarrow \ker d_n^D$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. \square

We can also go the other way, by defining a map

$$\iota_n: \mathcal{A} \rightarrow \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree n .

Example 32. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then $H_0(C_\bullet) = \text{coker } f$.

Similarly, if

$$0 \longrightarrow C_0 \xrightarrow{f} C_{-1} \longrightarrow \cdots$$

is a chain complex in $\mathbf{Ch}_{\leq 0}(\mathcal{A})$, then $H_0(C_\bullet) = \ker f$.

3.3. Diagram lemmas

3.3.1. The splitting lemma

Lemma 33 (Splitting lemma). Consider the following solid short exact sequence in an abelian category \mathcal{A} .

$$0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{\pi_C} C \longrightarrow 0$$

$\nwarrow \pi_A \quad \nearrow i_C$

The following are equivalent.

1. The sequence *splits from the left*, i.e. there exists a morphism $\pi_A: B \rightarrow A$ such that $\pi_A \circ i_A = \text{id}_A$
2. The sequence *splits from the right*, i.e. there exists a morphism $i_C: C \rightarrow B$ such that $\pi_C \circ i_C = \text{id}_C$
3. B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

Proof. That $3 \Rightarrow 1$ and $3 \Rightarrow 2$ is obvious. We now show that $1 \Rightarrow 2$.

By exactness, $i_A: A \rightarrow B$ is a kernel of π_C . Consider the map $\phi = \text{id}_B - i_C \circ \pi_C$. Since

$$\begin{aligned} \pi_C \circ \phi &= \pi_C - \pi_C \circ i_C \circ \pi_C \\ &= \pi_C - \text{id} \circ \pi_C \\ &= 0, \end{aligned}$$

the universal property for the kernel guarantees us a map $\psi: B \rightarrow A$ such that $i_A \circ \psi = \phi$.

3. Homological algebra in abelian categories

Now we use the same universal property in a different situation. Consider the following diagram.

$$\begin{array}{ccccc} & & A & & \\ & \swarrow & \downarrow i_A & \searrow 0 & \\ A & \xrightarrow{i_A} & B & \xrightarrow{\pi_C} & C \end{array}$$

The universal property guarantees us a unique dashed map making everything commute. But id_A and $i_A \circ \psi$ both do the job since

$$\begin{aligned} i_A \circ \psi \circ i_A &= i_A - i_C \circ \pi_C \circ i_A \\ &= i_A. \end{aligned}$$

□

3.3.2. The snake lemma

Lemma 34. Given a square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ g \downarrow & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

in any abelian category, consider the sequence

$$A \xrightarrow{(f, -g)} B \oplus C \xrightarrow{(h, k)} D .$$

We have the following results.

1. The composition $(h, k) \circ (f, -g) = 0$ if and only if the square commutes.
2. The morphism $(f, -g) = \ker(h, k)$ if and only if the square is a pullback.
3. The morphism $(h, k) = \text{coker}(f, -g)$ if and only if the square is a pushout.

Proof.

1. Obvious.
- 2.

□

Lemma 35. Let $f: A \rightarrow C$ be an epimorphism, and let $g: B \rightarrow C$ be any morphism. Then $(f, g): A \oplus B \rightarrow C$ is an epimorphism.

Proof. Let $\alpha: C \rightarrow Z$ such that $\alpha \circ (f, g) = 0$. Consider the following diagram.

$$\begin{array}{ccccc}
 A & \xrightarrow{i_A} & A \oplus B & \xleftarrow{i_B} & B \\
 & \searrow f & \downarrow (f, g) & \swarrow g & \\
 & & C & & \\
 \alpha \circ f \swarrow & & \downarrow \alpha & & \searrow \alpha \circ g \\
 & & Z & &
 \end{array}$$

We need to show that $\alpha = 0$. Certainly, we have that

$$\alpha \circ (f, g) \circ i_A = 0.$$

But this means that $\alpha \circ f = 0$; since f is an epimorphism this means that $\alpha = 0$. \square

Lemma 36. Let \mathcal{A} be an abelian category, and let the diagram

$$\begin{array}{ccc}
 A \times_C B & \xrightarrow{f} & B \\
 g \downarrow & & \downarrow h \\
 A & \xrightarrow{k} & C
 \end{array}$$

be a pullback in \mathcal{A} . Then f is an epimorphism.

Proof. Since the above square is a pullback, the morphism $(f, -g) = \ker(h, k)$. But (h, k) is an epimorphism, so it is the cokernel of its kernel, implying that the square is also a pushout square. Since pushouts reflect epimorphisms, f is an epimorphism. \square

Lemma 37. Let $f: B \twoheadrightarrow C$ be an epimorphism, and let $g: D \rightarrow C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g , in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{array}{ccccc}
 \ker f' & \xrightarrow{i} & S & \xrightarrow{f'} & D \\
 \parallel & & g' \downarrow & \lrcorner & \downarrow g \\
 \ker f & \xrightarrow{i} & B & \xrightarrow{f} & C
 \end{array}$$

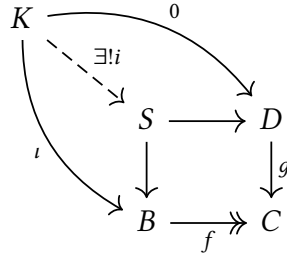
Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{array}{ccc}
 S & \xrightarrow{f'} & D \\
 g' \downarrow & \lrcorner & \downarrow g \\
 B & \xrightarrow{f} & C
 \end{array}$$

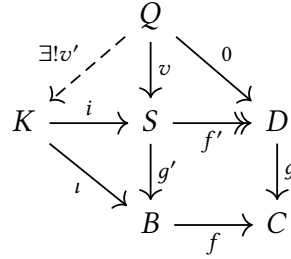
By Lemma 36 the pullback f' is also an epimorphism. Denote the kernel of f by K .

3. Homological algebra in abelian categories

The claim is that K is also a kernel of f' . Of course, in order for this statement to make sense we need a map $i: K \rightarrow S$. This is given to us by the universal property of the pullback as follows.



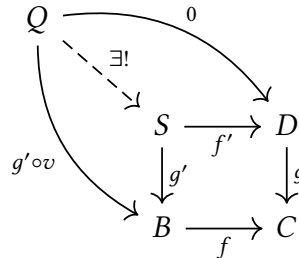
Next we need to verify that (K, i) is actually the kernel of f' , i.e. satisfies the universal property. To this end, let $v: Q \rightarrow S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K .



By definition $f' \circ v = 0$. Thus,

$$\begin{aligned} f \circ g' \circ v &= g \circ f' \circ v \\ &= g \circ 0 \\ &= 0, \end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.



By the universal property, there exists a unique map $Q \rightarrow S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the

right hand square is a pullback.

$$\begin{array}{ccccc}
 \ker f' & \xrightarrow{i} & S & \xrightarrow{f'} & D \\
 \parallel & & \downarrow g' & & \downarrow g \\
 \ker f & \xrightarrow{\iota} & B & \xrightarrow{f} & C
 \end{array}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{array}{ccccc}
 \left\{ \begin{array}{l} \text{elements of pullback} \\ \text{which map to 0} \end{array} \right\} & \hookrightarrow & \left\{ \begin{array}{l} \text{elements of } B \\ \text{which map to } D \end{array} \right\} & \twoheadrightarrow & D \\
 \parallel & & \downarrow & & \downarrow \\
 \left\{ \begin{array}{l} \text{elements of } B \\ \text{which map to 0} \end{array} \right\} & \hookrightarrow & B & \twoheadrightarrow & C
 \end{array}$$

Theorem 38 (snake lemma). Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{m} & B & \xrightarrow{e} & C \longrightarrow 0 \\
 & & f \downarrow & & g \downarrow & & \downarrow h \\
 0 & \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' \longrightarrow 0
 \end{array}$$

This gives us an exact sequence

$$0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h \rightarrow 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{array}{ccccccc}
 0 & \cdots \longrightarrow & \ker f & \cdots \longrightarrow & \ker g & \cdots \longrightarrow & \ker h \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & A & \xrightarrow{m} & B & \xrightarrow{e} & C \longrightarrow 0 \\
 & & f \downarrow & & g \downarrow & & \downarrow h \\
 0 & \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \operatorname{coker} f & \cdots \longrightarrow & \operatorname{coker} g & \cdots \longrightarrow & \operatorname{coker} h \cdots \longrightarrow 0
 \end{array}
 \tag{3.2}$$

We will see why the dotted portions are exact later.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

3. Homological algebra in abelian categories

Extract from the data of [Diagram 3.2](#) the following diagram.

$$\begin{array}{ccccc}
 & & & & \ker h \\
 & & & & \downarrow \\
 A & \xrightarrow{m} & B & \xrightarrow{e} \twoheadrightarrow & C \\
 f \downarrow & & \downarrow g & & \downarrow h \\
 A' & \xrightarrow{m'} & B' & \xrightarrow{e'} \twoheadrightarrow & C' \\
 \downarrow & & & & \\
 \text{coker } f & & & &
 \end{array}$$

Take a pullback and a pushout, and using [Lemma 37](#) (and its dual), we find the following.

$$\begin{array}{ccccc}
 \ker u & \xrightarrow{a} & S & \xrightarrow{u} \twoheadrightarrow & \ker h \\
 \parallel & & \downarrow r & & \downarrow \\
 A & \xrightarrow{m} & B & \xrightarrow{e} \twoheadrightarrow & C \\
 f \downarrow & & \downarrow g & & \downarrow h \\
 A' & \xrightarrow{m'} & B' & \xrightarrow{e'} \twoheadrightarrow & C' \\
 \downarrow & & \downarrow s & & \parallel \\
 \text{coker } f & \xrightarrow{v} & T & \xrightarrow{b} \twoheadrightarrow & \text{coker } v
 \end{array}$$

Consider the map

$$\delta_0 = S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T .$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1: \ker h \rightarrow T.$$

Composing this with b gives 0, hence we get a map

$$\delta: \text{coker } f \rightarrow \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 39 (snake lemma II). Given the following commutative diagram with exact rows,

$$\begin{array}{ccccccc}
 A & \xrightarrow{m} & B & \xrightarrow{e} & C & \longrightarrow & 0 \\
 f \downarrow & & \downarrow g & & \downarrow h & & \\
 0 \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' &
 \end{array}$$

we get a exact sequence

$$\ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h.$$

3.3.3. The long exact sequence on homology

Corollary 40. Given an exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{f_{\bullet}} B_{\bullet} \xrightarrow{g_{\bullet}} C_{\bullet} \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & H_{n+1}(C) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_n(A) & \xrightarrow{H_n(f)} & H_n(B) & \xrightarrow{H_n(g)} & H_n(C) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_{n-1}(A) & \xrightarrow{H_{n-1}(f)} & H_{n-1}(B) & \xrightarrow{H_{n-1}(g)} & H_{n-1}(C) \\ & & & & \searrow & \delta & \nearrow \\ & & & & H_{n-2}(A) & \longrightarrow & \cdots \end{array}$$

Proof. For each n , we have a diagram of the following form.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \longrightarrow 0 \\ & & \downarrow d_{n+1}^A & & \downarrow d_{n+1}^B & & \downarrow d_{n+1}^C \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n \longrightarrow 0 \end{array}$$

Applying the snake lemma ([Theorem 38](#)) gives us, for each n , two exact sequences as follows.

$$0 \longrightarrow \ker d_{n+1}^A \longrightarrow \ker d_{n+1}^B \longrightarrow \ker d_{n+1}^C$$

and

$$\operatorname{coker} d_n^A \longrightarrow \operatorname{coker} d_n^B \longrightarrow \operatorname{coker} d_n^C \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{array}{ccccccc} \operatorname{coker} d_n^A & \longrightarrow & \operatorname{coker} d_n^B & \longrightarrow & \operatorname{coker} d_n^C & \longrightarrow & 0 \\ & & & & & & \\ 0 & \longrightarrow & \ker d_{n-1}^A & \longrightarrow & \ker d_{n-1}^B & \longrightarrow & \ker d_{n-1}^C \end{array}$$

3. Homological algebra in abelian categories

Recall that for each n , we get a map

$$\phi_{n-1}^A: \operatorname{im} d_{n-1}^A \rightarrow \ker d_n^A,$$

and similarly for B_\bullet and C_\bullet .

Adding these in gives the following.

$$\begin{array}{ccccccc} \operatorname{coker} d_n^A & \longrightarrow & \operatorname{coker} d_n^B & \longrightarrow & \operatorname{coker} d_n^C & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_{n-1}^A & \longrightarrow & \ker d_{n-1}^B & \longrightarrow & \ker d_{n-1}^C \end{array}$$

A second application of the snake lemma then gives a long exact sequence

$$\ker \phi_f \rightarrow \ker \phi_g \rightarrow \ker \phi_h \rightarrow \operatorname{coker} \phi_f \rightarrow \operatorname{coker} \phi_g \rightarrow \operatorname{coker} \phi_h$$

□

Lemma 41. Let \mathcal{A} be an abelian category, and let f be a morphism of short exact sequences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A'_\bullet & \xrightarrow{\alpha_\bullet} & A_\bullet & \xrightarrow{\alpha'_\bullet} & A''_\bullet \longrightarrow 0 \\ & & \downarrow f'_\bullet & & \downarrow f_\bullet & & \downarrow f''_\bullet \\ 0 & \longrightarrow & B'_\bullet & \xrightarrow{\beta_\bullet} & B_\bullet & \xrightarrow{\beta'_\bullet} & B''_\bullet \longrightarrow 0 \end{array}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & H_n(A) & \xrightarrow{H_n(\alpha')} & H_n(A'') & \xrightarrow{\delta} & H_{n-1}(A') \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(A) \longrightarrow \cdots \\ & & \downarrow H_n(f) & & \downarrow H_n(f'') & & \downarrow H_n(f') & & \downarrow H_n(f) \\ \cdots & \longrightarrow & H_n(B) & \xrightarrow{H_n(\beta')} & H_n(B'') & \xrightarrow{\delta} & H_{n-1}(B') \xrightarrow{H_{n-1}(\beta)} H_{n-1}(B) \longrightarrow \cdots \end{array}$$

In particular, the connecting morphism δ is functorial.

3.3.4. The five lemma

Lemma 42 (four lemmas).

1. Let

$$\begin{array}{ccccccc} B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow b & & \downarrow c & & \downarrow d & & \downarrow e \\ B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is an epimorphism.

2. Let

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ a \downarrow & & \downarrow b & & \downarrow c & & \downarrow d \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is a monomorphism.

Proof.

1. Expand the diagram with image-kernel factorization as follows.

$$\begin{array}{ccccccccccc} B & \twoheadrightarrow & I_1 & \hookrightarrow & C & \twoheadrightarrow & I_2 & \hookrightarrow & D & \twoheadrightarrow & I_3 & \hookrightarrow & E \\ \downarrow b & & \downarrow i & & \downarrow c & & \downarrow j & & \downarrow d & & \downarrow k & & \downarrow e \\ B' & \twoheadrightarrow & I'_1 & \hookrightarrow & C' & \twoheadrightarrow & I'_2 & \hookrightarrow & D' & \twoheadrightarrow & I'_3 & \hookrightarrow & E' \end{array}$$

It is immediate that i is epi, and that k is mono. Now, consider the following morphism of short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_2 & \hookrightarrow & D & \twoheadrightarrow & I_3 \longrightarrow 0 \\ & & \downarrow j & & \downarrow d & & \downarrow k \\ 0 & \longrightarrow & I'_2 & \hookrightarrow & D' & \twoheadrightarrow & I'_3 \longrightarrow 0 \end{array}$$

The snake lemma ([Theorem 38](#)) tells us that we have a short exact sequence

$$0 = \ker k \longrightarrow \operatorname{coker} j \longrightarrow \operatorname{coker} d = 0,$$

implying that $\operatorname{coker} j = 0$, i.e. that j is epi.

We have another morphism of short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & I_1 & \hookrightarrow & C & \twoheadrightarrow & I_2 \longrightarrow 0 \\ & & \downarrow i & & \downarrow c & & \downarrow j \\ 0 & \longrightarrow & I'_1 & \hookrightarrow & C' & \twoheadrightarrow & I'_2 \longrightarrow 0 \end{array}$$

Using the information that i and j are epi, the snake lemma tells us that j is epi as required.

2. Dual.

□

3. Homological algebra in abelian categories

Theorem 43 (five lemma). Let

$$\begin{array}{ccccccccc}
 A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \downarrow & & \downarrow \cong & & \downarrow c & & \downarrow \cong & & \downarrow \\
 A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
 \end{array}$$

be a commutative diagram with exact rows, with monomorphisms, epimorphisms, and isomorphisms as marked. Then c is an isomorphism.

Proof. The first part of [Lemma 42](#) tells us that c is epi, and the second part tells us that c is mono. \square

3.3.5. The nine lemma

Theorem 44 (nine lemma). Given the following commuting diagram whose rows are exact sequences,

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & A_2 & \hookrightarrow & B_2 & \xrightarrow{p} & C_2 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow f \\
 0 & \longrightarrow & A_1 & \hookrightarrow & B_1 & \twoheadrightarrow & C_1 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow g \\
 0 & \longrightarrow & A_0 & \hookrightarrow & B_0 & \twoheadrightarrow & C_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

we have the following results. Note that we do not assume that the columns are chain complexes.

1. If the first and the second columns A_\bullet and B_\bullet are exact sequences, then third column C_\bullet is also an exact sequence.
2. If B_\bullet and C_\bullet are exact sequence, then A_\bullet is an exact sequence.
3. If A_\bullet and C_\bullet are exact and B_\bullet is a chain complex, then B_\bullet is exact.

Proof.

1. First, we show that C_\bullet is a chain complex, i.e. that $g \circ f = 0$. Since p is an epimorphism, it suffices to show that $g \circ f \circ p = 0$. But this is zero by commutativity of the diagram and the fact that B_\bullet is a chain complex.

Thus, the diagram above is a short exact sequence

$$0 \longrightarrow A_{\bullet} \hookrightarrow B_{\bullet} \twoheadrightarrow C_{\bullet} \longrightarrow 0$$

of chain complexes, and we get a long exact sequence on homology.

$$\begin{array}{ccccccc}
 & & & & & & 0 \\
 & & & & & \delta & \longrightarrow \\
 \hookrightarrow & & & & & & \\
 & H_2(A_{\bullet}) & \longrightarrow & H_2(B_{\bullet}) & \longrightarrow & H_2(C_{\bullet}) & \\
 & & & & & \delta & \longrightarrow \\
 \hookrightarrow & & & & & & \\
 & H_1(A_{\bullet}) & \longrightarrow & H_1(B_{\bullet}) & \longrightarrow & H_1(C_{\bullet}) & \\
 & & & & & \delta & \longrightarrow \\
 \hookrightarrow & & & & & & \\
 & H_0(A_{\bullet}) & \longrightarrow & H_0(B_{\bullet}) & \longrightarrow & H_0(C_{\bullet}) & \\
 & & & & & \delta & \longrightarrow \\
 \hookrightarrow & & & & & & \\
 & 0 & & & & &
 \end{array}$$

If the first two columns are zeroes, then $H_n(C_{\bullet}) = 0$ by exactness; that is, C_{\bullet} is exact.

2. Formally dual.
3. Identical.

□

3.4. Exact functors

Definition 45 (exact functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor, and let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)$$

is exact.

- We call F right exact if

$$F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0$$

3. Homological algebra in abelian categories

is exact.

- We call F exact if it is both left exact and right exact.

Example 46. Let

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be an exact sequence. First, we show that $\text{Hom}(M, -)$ is left exact, i.e. that

$$0 \longrightarrow \text{Hom}(M, A) \xrightarrow{f_*} \text{Hom}(M, B) \xrightarrow{g_*} \text{Hom}(M, C)$$

is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha: M \rightarrow A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and

$$f \circ \alpha = 0 \implies \alpha = 0$$

because f is a monomorphism.

Similarly, $\text{im } f \subseteq \ker g$ because

$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \text{im } f$. To this end, let $\gamma: M \rightarrow B$ such that $g_*\gamma = 0$.

$$\begin{array}{ccccccc} & & & M & & & \\ & & \exists! \delta \swarrow & \downarrow \gamma & \searrow 0 & & \\ 0 & \longrightarrow & A & \xrightarrow{f} & B & \xrightarrow{g} & C \longrightarrow 0 \end{array}$$

Then $\text{im } \gamma \subseteq \ker g = \text{im } f$, so γ factors through A ; that is to say, there exists a $\delta: M \rightarrow A$ such that $f_*\delta = \gamma$.

One can also check that the functor $\text{Hom}(-, B)$ is left exact via the same sort of arguments. However, we will see in the next section that one can make a much more general statement by relating left and right exactness to preservation of limits. The next theorem gives us a taste of this.

Lemma 47. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories which is either left or right exact. Then F is additive, i.e. preserves finite (co)products.

Proof. First, assume that F is left exact. By [Corollary 18](#), it suffices to show that F preserves direct sums. To this end, consider the image of a split exact sequence under a left-exact functor F .

$$0 \longrightarrow F(A) \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(\pi_B)} F(B) \longrightarrow 0$$

Because we still have the identities

$$F(\pi_A) \circ F(i_A) = \text{id}_{F(A)}, \quad F(\pi_B) \circ F(i_B) = \text{id}_{F(B)},$$

the morphism $F(\pi_B)$ is still epic; hence the splitting lemma ([Theorem 33](#)) implies that the sequence is still split exact. Thus, F preserves direct sums.

The case in which F is right exact is dual. □

3.4.1. Exactness and preservation of (co)limits

The above definitions of left and right exactness are evocative, but one can cast them in a more useful way. One starting point is to notice that they are related to the preservation of limits.

Proposition 48. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f: A \rightarrow B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{array}{ccc} K & \longrightarrow & 0 \\ \iota \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \rightarrow A$ is a kernel of f .
2. Exactness at B means that $\text{im } f = \ker g$.
 - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g .

3. Homological algebra in abelian categories

- If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f .
3. Exactness at C means that g is epi, i.e. $C \rightarrow 0$ is a cokernel of g .

□

As may now be clear, the converse is also true; any functor which is left exact preserves finite limits, and any functor which is right exact preserves finite colimits.

Proposition 49. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories.

- If F is left-exact, then F preserves finite limits.
- If F is right-exact, then F preserves finite colimits.

Proof. Suppose F is left exact. Then F preserves kernels and products, hence all finite limits. If F is right exact, then F preserves all cokernels and coproducts, hence all finite colimits. □

Our definition of homology now allows us to cleanly note the following.

Proposition 50. Exact functors preserve homology; that is, $H_n(F(C_\bullet)) = F(H_n(C_\bullet))$.

Proof. Exact functors preserve kernels and cokernels. □

Example 51. Consider the category \mathcal{D} with objects and morphisms as follows.

$$\begin{array}{ccc} & & \star \\ & & \downarrow \\ * & \longrightarrow & \bullet \end{array}$$

Let \mathcal{A} be an abelian category. By [Example 16](#), the category of functors $\mathcal{D} \rightarrow \mathcal{A}$ is an abelian category.

Consider a functor $\mathcal{D} \rightarrow \mathcal{A}$ which yields the following diagram in \mathcal{A} .

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Taking the pullback of the above diagram yields the kernel of f .

A natural transformation ϕ between functors $\mathcal{D} \rightarrow \mathcal{A}$ consists of the data of a commuting square as follows.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \phi_A \downarrow & & \downarrow \phi_B \\ A' & \xrightarrow{f'} & B' \end{array}$$

That is, we may view taking kernels as a functor from the category of morphisms in \mathcal{A} to \mathcal{A} . Furthermore, because this functor is the result of a limiting procedure it preserves limits. Thus, given an exact sequence of morphisms in \mathcal{A}

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \hookrightarrow & B & \twoheadrightarrow & C \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A & \hookrightarrow & B & \twoheadrightarrow & C \longrightarrow 0 \end{array}$$

one gets an exact sequence of kernels

$$0 \longrightarrow \ker f \hookrightarrow \ker g \longrightarrow \ker h .$$

Conversely, taking cokernels is cocomplete, hence right exact, so one has also an exact sequence of cokernels

$$\operatorname{coker} f \longrightarrow \operatorname{coker} g \twoheadrightarrow \operatorname{coker} h \longrightarrow 0$$

Note that the data of a kernel

Example 52. Let J be a filtered category, and let \mathcal{A} be an abelian category. Then the category \mathcal{A}^J is abelian (see [Example 16](#)).

3.4.2. Exactness of the hom functor and tensor products

We have already seen that the hom functor $\operatorname{Hom}_{\mathcal{A}}(-, -)$ is left exact in both slots. Now we understand precisely why this is true: the hom functor preserves limits in both slots!¹

Since any functor which is a right adjoint preserves limits, any right adjoint functor between abelian categories must be left exact. For example, there is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \operatorname{Hom}_{\mathbf{Ab}}(N, -)$$

which comes from replacing a map out of the tensor product by its corresponding bilinear map, and currying along the first entry. Thus, the map $- \otimes_R N$ is a left adjoint, hence preserves colimits, hence is right exact.

We can use this to check exactness (hence limit preservation) of other functors using the following beefed-up version of the Yoneda lemma.

Lemma 53. Let \mathcal{A} be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

¹In the first slot, one must take the limits in the category \mathcal{A}^{op} .

3. Homological algebra in abelian categories

be objects and morphisms in \mathcal{A} . If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X, C)$$

form an exact sequence of abelian groups, then $A \rightarrow B \rightarrow C$ is exact.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\mathrm{im} f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota: \ker g \rightarrow B$ is mapped to zero under g_* , hence is mapped to under f_* by some $\alpha: \ker g \rightarrow A$. That is, we have the following commuting triangle.

$$\begin{array}{ccc} & A & \\ \alpha \nearrow & & \searrow f \\ \ker g & \xrightarrow{\iota} & B \end{array}$$

Thus $\mathrm{im} \iota = \ker g \subset \mathrm{im} f$. □

3.5. Quasi-isomorphism

One of the organizing principles of homological algebra is that the fundamental data contained in a chain complex is the level-wise homology of that chain complex, rather than the chain complex itself. We will see many examples (especially in [Part II](#), on algebraic topology) in which one can perform a drastic simplification of a problem by replacing a chain complex by a better-behaved one, together with a homology-preserving map between them.

For this reason, one would like to understand morphisms of chain complexes $f_{\bullet}: C_{\bullet} \rightarrow D_{\bullet}$ which induce isomorphisms on homology. One even has a special name for such a morphism.

Definition 54 (quasi-isomorphism). Let $f_{\bullet}: C_{\bullet} \rightarrow D_{\bullet}$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n .

Quasi-isomorphisms are a surprisingly fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope that quasi-isomorphisms themselves were invertible; that is, that one could find a *quasi-inverse*. Unfortunately, we are not so lucky. Consider the following morphism of chain complexes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\times 2} & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow \text{id} & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 0 \end{array}$$

This map induces an isomorphism on homology, but there is no non-trivial map in the other direction.

3.6. Chain Homotopies

Chain homotopies are, at this point, best thought of as a technical tool which allow us to better understand quasi-isomorphisms. The interested reader should look forward to [Part II](#), where we will reach an understanding of chain homotopies as an algebraic analog of a homotopy between maps of topological spaces.

Definition 55 (chain homotopy). Let $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ be morphisms of chain complexes. A chain homotopy between f_\bullet and g_\bullet is a family of maps

$$h_n: C_n \rightarrow D_{n+1}$$

such that

$$d_{n+1}^D \circ h_n + h_{n-1} \circ d_n^C = f - g.$$

Chain homotopies will help us to understand the slippery notion of quasi-isomorphism because chain homotopic maps will turn out to induce the same maps on homology.

Proposition 56. Let $f, g: C_\bullet \rightarrow D_\bullet$ be chain homotopic maps via a homotopy h . Then for all n ,

$$H_n(f) = H_n(g).$$

Proof. Since homology is additive, it suffices to show that any map which is homotopic to zero induces the zero map on homology, i.e. that if we can write

$$f = d_{n+1}^D \circ h_n + h_{n-1} \circ d_n^C,$$

then $H_n(f) = 0$ for all n . □

This is very useful, both theoretically and practically. We will often exploit this in order to prove that a complex is exact by showing that the identity morphism is homotopic to the zero morphism.

3. Homological algebra in abelian categories

Lemma 57. Let $f_\bullet, g_\bullet: C_\bullet \rightarrow D_\bullet$ be a homotopic chain complexes via a homotopy h , and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have

$$df + fd = h,$$

so

$$F(d)F(f) + F(f)F(d) = F(h).$$

□

Lemma 58. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \stackrel{h}{\sim} g$, then $g \stackrel{-h}{\sim} f$
- Transitivity: If $f \stackrel{h}{\sim} f'$ and $f' \stackrel{h'}{\sim} f''$, then $f \stackrel{h+h'}{\sim} f''$.

□

Lemma 59. Homotopy respects composition. That is, let $f \stackrel{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \stackrel{hr}{\sim} g \circ r$
- $r \circ f \stackrel{rh}{\sim} r \circ g$.

Proof. Obvious.

□

3.7. Projectives and injectives

One of the drawbacks of diagrammatic reasoning in abelian categories (which is to say, reasoning on elements in $R\text{-Mod}$) is that not every maneuver one can perform on elements has a diagrammatic analog. One of the most egregious examples of this is the property of lifting against epimorphisms. Given an epimorphism $f: A \twoheadrightarrow B$ and an element of B , one can always find an element of A which maps to it under f . Unfortunately, it is *not* in general the case that given a map $X \rightarrow B$, one can lift it to a map $X \rightarrow A$ making the triangle formed commute.

$$\begin{array}{ccc} & & X \\ & \nwarrow \scriptstyle \# & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

However, certain objects X do have this property. For example, if X is a free R -module, then we can find a lift by lifting generators; this is always possible. It will be interesting to find objects X which allow this. Such objects are called *projective*.

We will eventually be interested in projective objects because of the control they give us over quasi-isomorphisms.

3.7.1. Basic properties

Definition 60 (projective, injective). An object P in an abelian category \mathcal{A} is projective if for every epimorphism $f: B \rightarrow C$ and every morphism $p: P \rightarrow C$, there exists a morphism $\tilde{p}: P \rightarrow B$ such that the following diagram commutes.

$$\begin{array}{ccc} & P & \\ \exists \tilde{p} \swarrow & \downarrow p & \\ B & \xrightarrow{f} & C \end{array}$$

Dually, Q is injective if for every monomorphism $g: A \rightarrow B$ and every morphism $q: A \rightarrow Q$ there exists a morphism $\tilde{q}: B \rightarrow Q$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ q \downarrow & \nwarrow \exists \tilde{q} & \\ Q & & \end{array}$$

Since the hom functor $\text{Hom}(A, -)$ is left exact for every A , it is very easy to see simply by unwrapping the definitions that an object P in an abelian category is projective if the functor $\text{Hom}(P, -)$ is exact, and injective if $\text{Hom}(-, Q)$ is exact.

3.7.2. Enough projectives

A recurring theme of homological algebra is that projective and injective objects are extremely nice to work with, and whenever possible we would like to be able to trade in an object or chain complex for an object or chain complex of projectives which is (quasi-)isomorphic. We will go into more detail about precisely what we mean by this later. This will be possible when our abelian category has *enough projectives*. Dually, we can replace objects or chain complexes by injectives when we have *enough injectives*. As the abstract theory of projectives is dual to that of injectives, we will for the remainder of this chapter confine ourselves to the study of projectives.

Definition 61 (enough projectives). Let \mathcal{A} be an abelian category. We say that \mathcal{A} has enough projectives if for every object M there exists a projective object P and an epi-

3. Homological algebra in abelian categories

morphism $P \twoheadrightarrow M$.

The category in which we have mainly been interested is $R\text{-Mod}$. We will now see that $R\text{-Mod}$ has enough projectives; we need only prove the following lemma.

Lemma 62. Let

$$0 \longrightarrow A \hookrightarrow B \xrightarrow{f} P \longrightarrow 0$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \downarrow \text{id} & & \\ 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{f} & P \longrightarrow 0 \end{array}$$

By definition, we get a morphism $P \rightarrow B$ making the triangle commute.

$$\begin{array}{ccccccc} & & & & P & & \\ & & & & \downarrow \text{id} & & \\ & & & \swarrow \exists g & & & \\ 0 & \longrightarrow & A & \hookrightarrow & B & \xrightarrow{f} & P \longrightarrow 0 \end{array}$$

But this says precisely that $f \circ g = \text{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma 33). \square

Proposition 63. The category $R\text{-Mod}$ has enough projectives.

We see this by proving the following characterization of projectives in $R\text{-Mod}$: P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R -module M and a free R -module F such that

$$F \simeq P \oplus M.$$

This will be sufficient because any module can be written as a quotient of a free module.

First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R -module over the generators, and consider the following short exact sequence.

$$0 \longrightarrow \ker \pi \hookrightarrow F \xrightarrow{\pi} P \longrightarrow 0$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker \pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{array}{ccc} & P \oplus M & \\ \exists(j,k) \swarrow & \downarrow (f,g) & \\ N & \longrightarrow & M \end{array}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{array}{ccc} & P & \\ \exists j \swarrow & \downarrow f & \\ N & \longrightarrow & M \end{array}$$

exhibiting P as projective.

3.7.3. Here be dragons

There are a few subtleties to the theory of projectives and injectives of which one should be aware.

- We have said that the abstract theory of injectives is dual to that of projectives. This is true, but does not imply that projectives and injectives are somehow on equal footing. For example, in $R\text{-Mod}$ we have given a complete classification of projective modules: they are direct summands of free modules. There is no such classification of injective modules. Despite this, $R\text{-Mod}$ does turn out to have enough injectives.
- We have so far mostly avoided the use of element-theoretic arguments, which are possible thanks to the Freyd-Mitchell embedding theorem. This is partially due to an aesthetic distaste, and partially because they can be tricky.

3.7.4. Projective resolutions

We are now ready to enact a part of our goal of replacing an object or chain complex with a (quasi-)isomorphic chain complex of projectives: if we are working in an abelian category with enough projectives, we can replace an object A with a quasi-isomorphic chain complex of projectives.

More concretely, let \mathcal{A} be an abelian category, and denote by ι the functor which takes an object of \mathcal{A} and returns a chain complex with that object concentrated in degree zero.

3. Homological algebra in abelian categories

Clearly, the homology of $\iota(A)$ is

$$H_n(\iota(A)) = \begin{cases} A, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Our goal, namely replacing A by a chain complex which contains the same homological data, can certainly be achieved by finding a chain complex with the same homology, and a quasi-isomorphism from it to $\iota(A)$. Such an object is known as a *resolution*.

Definition 64 (resolution). Let A be an object in an abelian category. A resolution of A consists of a chain complex C_\bullet together with a quasi-isomorphism $C_\bullet \xrightarrow{\cong} \iota(A)$.

What we want is a *projective resolution*, i.e. a resolution made out of projective objects.

Definition 65 (projective, injective resolution). Let \mathcal{A} be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \xrightarrow{\cong} \iota(A)$, where P_\bullet is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \xrightarrow{\cong} Q_\bullet$ where Q_\bullet is a complex of injectives.

We said that our success in our goal would be predicated on the existence of enough projectives. Indeed, this is the case. In order to see this, we change our view on projective resolutions. Rather than a quasi-isomorphism between chain complexes whose homology is concentrated in degree zero, we should look at a projective resolution as a single, exact chain complex.

Lemma 66. Let $f_\bullet: P_\bullet \rightarrow \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots \cdots P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Proposition 67. Let \mathcal{A} be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Proof. Pick a projective which surjects onto A , and □

3.7.5. Lifting data to projective resolutions

We have seen that for every object in an abelian category \mathcal{A} with enough projectives, we can find a *projective resolution*, i.e. a complex P_\bullet of projectives and a quasi-isomorphism $P_\bullet \xrightarrow{\cong} \iota(A)$. We would like to be able to extend this to a functor. This idea turns out not to work precisely as we would like, but it is nonetheless helpful to pursue it for a while.

The first thing we need is a way of lifting morphisms in \mathcal{A} to morphisms between projective resolutions.

Lemma 68. Let P_\bullet be a chain complex of projectives with $P_n = 0$ for $n \leq 0$, and let D_\bullet be a chain complex with $D_n = 0$ for $n \leq 0$ such that $H_n(D) = 0$ for $n > 0$. Let $f: P_\bullet \rightarrow D_\bullet$ such that $f_0 = 0$.

1. For any $f_0: P_0 \rightarrow D_0$, there exists a lift to a chain map $f: P \rightarrow D$.
2. Any such lifts $f, f': P \rightarrow D$ are chain homotopic.

Proof. 1. We construct such a lift inductively.

2. It suffices to show that any lift of the zero morphism is chain homotopic to zero.

□

Proposition 69. Let \mathcal{A} be an abelian category, let M and M' be objects of \mathcal{A} , and

$$P_\bullet \rightarrow M \quad \text{and} \quad P'_\bullet \rightarrow M'$$

be projective resolutions. Then for every morphism $f: M \rightarrow M'$ there exists a lift $\tilde{f}: P_\bullet \rightarrow P'_\bullet$ making the diagram

$$\begin{array}{ccc} P_\bullet & \xrightarrow{\tilde{f}} & P'_\bullet \\ \downarrow & & \downarrow \\ \iota M & \xrightarrow{\iota f} & \iota M' \end{array}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively.

□

Theorem 70 (horseshoe lemma). Let \mathcal{A} be an abelian category with enough projectives, and let $P'_\bullet \rightarrow M'$ and $P''_\bullet \rightarrow M''$ be projective resolutions. Then given an exact sequence

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0$$

there is a projective resolution $P_\bullet \rightarrow M$ and maps \tilde{f} and \tilde{g} such that the following diagram has exact rows and commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & P'_\bullet & \xrightarrow{\tilde{f}} & P_\bullet & \xrightarrow{\tilde{g}} & P''_\bullet \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0 \end{array}$$

3. Homological algebra in abelian categories

Proof. We construct P_\bullet inductively. We have specified data of the following form.

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & & \\
 0 & \longrightarrow & P'_1 & & P''_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & P'_0 & & P''_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0
 \end{array}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P'_0 & \xrightarrow{\iota} & P'_0 \oplus P''_0 & \xrightarrow{\pi} & P''_0 \longrightarrow 0 \\
 & & \downarrow p' & \searrow f \circ p'_0 & \downarrow p & \swarrow \exists q & \downarrow p'' \\
 0 & \longrightarrow & M' & \xrightarrow{f} & M & \xrightarrow{g} & M'' \longrightarrow 0
 \end{array}$$

We get the (dashed) map $P'_0 \rightarrow M$ by composition, and the (dashed) map $P''_0 \rightarrow M$ by projectivity of P''_0 (since p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p: P'_0 \oplus P''_0 \rightarrow M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\text{coker } p' \simeq 0 \longrightarrow \text{coker } p \longrightarrow 0 \simeq \text{coker } p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of P_\bullet . Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P''_1 to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of

our diagram as follows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P'_1 & & P''_1 & \longrightarrow & 0 \\
 & & \downarrow p'_1 & & \downarrow p''_1 & & \\
 0 & \longrightarrow & \ker p' & \dashrightarrow & \ker p & \dashrightarrow & \ker p'' \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & P'_0 & \longrightarrow & P_0 & \longrightarrow & P''_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M' & \hookrightarrow & M & \twoheadrightarrow & M'' \longrightarrow 0
 \end{array}$$

The maps p'_1 and p''_1 come from the exactness of P'_\bullet and P''_\bullet . In fact, they are epimorphisms, because they are really cokernel maps in disguise. We get the dashed map $\ker p' \rightarrow \ker p''$

That means that we are in the same situation as before, and are justified in saying “we proceed inductively”. \square

Theorem 71 (horseshoe lemma for morphisms). Given a morphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A' & \hookrightarrow & A & \twoheadrightarrow & A'' \longrightarrow 0 \\
 & & \downarrow f' & & \downarrow f & & \downarrow f'' \\
 0 & \longrightarrow & B' & \hookrightarrow & B & \twoheadrightarrow & B'' \longrightarrow 0
 \end{array}$$

Theorem 72. Let \mathcal{A} be an abelian category with enough projectives, and let $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then there exists a

3.8. The homotopy category

In the previous section, we proved many theorems which were schematically of the following form:

Given some data in an abelian category, we can lift it to data between projective resolutions.

One might hope that this data would assemble itself functorially, but this is far too much to ask; we have no reason to believe, having picked lifts \tilde{f} for our morphisms f , that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

In this, however, we are saved by the notion of homotopy! While $\widetilde{f \circ g}$ and $\tilde{f} \circ \tilde{g}$ may not be equal, they are certainly homotopic, thanks to [Proposition 69](#). Thus, we will have a shot at functoriality, if we formally invert all homotopies.

And this we can do!

3. Homological algebra in abelian categories

Definition 73 (homotopy category). Let \mathcal{A} be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in \mathcal{A} up to homotopy.

This is well-defined by [Lemma 58](#) and [Lemma 59](#).

We will in general not be interested in the full homotopy category, but instead in the bounded-below homotopy category $\mathcal{K}^+(\mathcal{A})$ of *bounded-below* chain complexes up to homotopy.

Lemma 74. The homotopy category $\mathcal{K}^+(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}_{\geq 0}(\mathcal{A}) \rightarrow \mathcal{K}^+(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

The upshot of the above discussion is that for any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \rightarrow \mathcal{K}^+(\mathcal{A})$ defined as follows.

1. Pick a projective resolution of every object in \mathcal{A} .
2. Pick a lift of every morphism of \mathcal{A} , and take equivalence classes.

This functor has

3.9. Mapping cones

Mapping cones are a technical tool, borrowed from algebraic topology. We will use them to great profit in the next section. For now,

Definition 75 (shift functor). For any chain complex C_\bullet and any $k \in \mathbb{Z}$, define the k -shifted chain complex $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \quad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 76 (mapping cone). Let $f: C_\bullet \rightarrow D_\bullet$ be a chain map. Define a new complex $\text{cone}(f)$ as follows.

- For each n , define

$$\text{cone}(f)_n = C_{n-1} \oplus D_n.$$

- Define $d_n^{\text{cone}(f)}$ by

$$d_n^{\text{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$

which is shorthand for

$$d_n^{\text{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1} x_{n-1}).$$

Lemma 77. The complex $\text{cone}(f)$ naturally fits into a short exact sequence

$$0 \longrightarrow D_{\bullet} \xrightarrow{\iota} \text{cone}(f)_{\bullet} \xrightarrow{\pi} C[-1]_{\bullet} \longrightarrow 0 .$$

Proof. We need to specify ι and π . The morphism ι is the usual injection; the morphism π is given by minus the usual projection. \square

Corollary 78. Let $f_{\bullet}: C_{\bullet} \rightarrow D_{\bullet}$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\text{cone}(f)$ is an exact complex.

Proof. By [Corollary 40](#), we get a long exact sequence

$$\cdots \rightarrow H_n(C) \xrightarrow{\delta} H_n(D) \rightarrow H_n(\text{cone}(f)) \rightarrow H_{n-1}(C) \rightarrow \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see; picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{array}{ccccc} & & & & x \\ & & & & \downarrow \\ & & (x, 0) & \xrightarrow{\quad} & -x \\ & & \downarrow & & \\ f(x) & \xrightarrow{\quad} & (0, f(x)) & & \\ \downarrow & & & & \\ [f(x)] & & & & \end{array}$$

If $\text{cone}(f)$ is exact, then we get a very short exact sequence

$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$

implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n , then the maps to and from $H_n(\text{cone}(f))$ must be the zero maps, implying $H_n(\text{cone}(f)) = 0$ by exactness. \square

Proposition 79. Let \mathcal{A} be an abelian category, and let C_{\bullet} be a chain complex in \mathcal{A} . Let P_{\bullet} be a bounded below chain complex of projectives. Then any quasi-isomorphism $g: P_{\bullet} \rightarrow C_{\bullet}$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into

3. Homological algebra in abelian categories

an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{array}{ccccccc}
 \cdots & P_1 & \xrightarrow{\quad} & P_0 & \xrightarrow{\quad} & 0 & \cdots \\
 & \downarrow & \swarrow h_1 & \downarrow & \swarrow h_0 & \downarrow & \\
 \cdots & C_1 & \xrightarrow{\quad} & C_0 & \xrightarrow{\quad} & C_{-1} & \cdots
 \end{array}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1 .

$$\begin{array}{ccccc}
 & & P_1 & & \\
 & \swarrow \exists h_1 & \downarrow r & \searrow 0 & \\
 & K & & & \\
 C_2 & \xrightarrow{d_2^C} & C_1 & \xrightarrow{d_1^C} & C_0
 \end{array}$$

Because

$$\begin{aligned}
 d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
 &= d_1^C \circ f_1 - d_1^C \circ f_1 \\
 &= 0,
 \end{aligned}$$

the morphism r factors through K . This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1 .

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
 d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
 &= f_1
 \end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f .

Now let $f: C_\bullet \rightarrow P_\bullet$ be a quasi-isomorphism, where P_\bullet is a bounded-below projective complex. Since f is a quasi-isomorphism, $\text{cone}(f)$ is exact, which means that $\iota: P \rightarrow \text{cone}(f)$ is homotopic to the zero morphism.

$$\begin{array}{ccccc}
 P_{i+1} & \xrightarrow{d_{i+1}^P} & P_i & \xrightarrow{d_i^P} & P_{i-1} \\
 \downarrow \iota_{i+1} & \swarrow \tilde{h}_i & \downarrow \iota_i & \swarrow \tilde{h}_{i-1} & \downarrow \iota_{i-1} \\
 C_i \oplus P_{i+1} & \xrightarrow{d_{i+1}^{\text{cone}(f)}} & C_{i-1} \oplus P_i & \xrightarrow{d_i^{\text{cone}(f)}} & C_{i-2} \oplus P_{i-1}
 \end{array}$$

That is, there exist $\tilde{h}_i: P_i \rightarrow \text{cone}(f)_{i+1}$ such that

$$d_{i+1}^{\text{cone } f} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d_i^P = \iota. \quad (3.3)$$

Writing \tilde{h}_i in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i: P_i \rightarrow C_i \quad \text{and} \quad \gamma_i: P_{i-1} \rightarrow P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet: P_\bullet \rightarrow C_\bullet$. Indeed, writing [Equation 3.3](#) in components, we find the following.

$$\begin{aligned} \begin{pmatrix} -d_i^C & 0 \\ -f_i & d_{i+1}^P \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d_i^P \end{pmatrix} &= \begin{pmatrix} 0 \\ \text{id}_{P_i} \end{pmatrix} \\ \begin{pmatrix} -d_i^C \circ \beta_i + \beta_{i-1} \circ d_i^P \\ -f_i \circ \beta_i + d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P \end{pmatrix} &= \begin{pmatrix} 0 \\ \text{id}_{P_i} \end{pmatrix} \end{aligned}$$

The first line tells us that β is a chain map. The second line tells us that

$$d_{i+1}^P \circ \gamma_i + \gamma_{i-1} \circ d_i^P = \text{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\text{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\text{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which implies that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

3.10. Derived functors

3.10.1. Derived categories

Hopefully, the reader will now agree that chain complexes are interesting enough to study, and that given a chain complex, the level-wise homology (and therefore the slippery notion of quasi-isomorphism) is an interesting invariant. Of course, one considers the category of chain complexes, but this category has the annoying feature that quasi-isomorphisms do not necessarily have quasi-inverses. That is, it is difficult to view quasi-isomorphism as a notion of equivalence because it is not invertible.

The way out of this is to replace the category $\mathbf{Ch}_{\geq 0}(\mathcal{A})$ by a category in which all quasi-isomorphisms have quasi-inverses. There is a fiddly and conceptually unclear way to

3. Homological algebra in abelian categories

do that, namely by forming the categorical localization.

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \rightarrow \mathbf{Ch}_{\geq 0}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \rightarrow \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. By

3.10.2. Derived functors

Definition 80 (derived functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and \mathcal{A} has enough projectives, we declare the left derived functor to be the homological δ -functor $\{L_n F\}$ defined by

$$L_n F = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

where $P: \mathcal{A} \rightarrow \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and \mathcal{A} has enough injectives, we define the right derived functor to be the cohomological δ -functor

$$R^n F(X) = H^n \circ F \circ Q$$

where Q is an injective resolution functor.

Definition 81 (homological δ -functor). A homological δ -functor between abelian categories \mathcal{A} and \mathcal{B} consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n: \mathcal{A} \rightarrow \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in \mathcal{A} and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n: T_n(C) \rightarrow T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & T_{n+1}(C) \\
 & & & & & & \uparrow \delta \\
 & & & & & & \searrow \\
 & & & & T_n(A) & \xrightarrow{T_n(f)} & T_n(B) & \xrightarrow{T_n(g)} & T_n(C) \\
 & & & & & & \uparrow \delta \\
 & & & & & & \searrow \\
 & & & & T_{n-1}(A) & \xrightarrow{T_{n-1}(f)} & T_{n-1}(B) & \xrightarrow{T_{n-1}(g)} & T_{n-1}(C) \\
 & & & & & & \uparrow \delta \\
 & & & & & & \searrow \\
 & & & & T_{n-2}(A) & \longrightarrow & \cdots
 \end{array}$$

3. For every morphism of short exact sequences

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & f \downarrow & & \downarrow g & & \downarrow h & & \\
 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0
 \end{array}$$

and every n , the diagram

$$\begin{array}{ccc}
 T_n(C) & \xrightarrow{\delta_n} & T_{n-1}(A) \\
 T_n(h) \downarrow & & \downarrow T_n(f) \\
 T_n(C') & \xrightarrow{\delta_n} & T_{n-1}(A')
 \end{array}$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

A morphism $S \rightarrow T$ of δ -functors consists of, for each n , a natural transformation $S_n \rightarrow T_n$ compatible with the δ .

This seems like an inelegant definition, and indeed it is.

Example 82. The prototypical example of a homological δ -functor is homology: the collection $H_n: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors*, come from homology.

3. Homological algebra in abelian categories

Definition 83 (cohomological δ -functors). Dual to [Definition 81](#).

We still need to show that derived functors are in fact homological delta-functors.

Theorem 84. For any right exact functor $F: \mathcal{A} \rightarrow \mathcal{B}$ between abelian categories \mathcal{A} and \mathcal{B} , where \mathcal{A} has enough projectives, the left derived functor $\{L_n F\}$ is a homological δ -functor.

Proof. We check each condition separately.

1. $L_n F$ is computed by taking the n th homology of a chain complex concentrated in positive degree.
2. Start with a short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

in \mathcal{A} .

We compute the image of this sequence under the derived functors $L_n F$ by using our chosen projective resolution functor to find projective resolutions and lift maps, and take homology. In general, we cannot be guaranteed that we can lift our exact sequence up to an exact sequence of projective resolutions

$$0 \longrightarrow P_\bullet^A \hookrightarrow P_\bullet^B \twoheadrightarrow P_\bullet^C \longrightarrow 0 .$$

However, we have shown that, up to isomorphism, it doesn't matter which projective resolutions we choose, so we can choose those given by the horseshoe lemma ([Theorem 70](#)), giving us such an exact sequence. Applying F then gives us another exact sequence, and taking homology gives a long exact sequence.

3. We do the same thing, but now apply the horseshoe lemma on morphisms ([Theorem 71](#)).

□

Proposition 85. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0 F \simeq F.$$

Proof. Let $P_\bullet \rightarrow X$ be a projective resolution. By [Lemma 66](#), the following sequence is exact.

$$P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$F(P_1) \xrightarrow{F(f)} F(P_0) \twoheadrightarrow F(X) \longrightarrow 0$$

This tells us that $F(X) \simeq \text{coker}(F(f))$. But

$$\text{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$

□

Given a right exact functor F , the above theorem shows that, at the lowest level, the left derived functor gives F itself. It turns out that $\{L_nF\}$ is the *universal* homological δ -functor with this property:

Proposition 86. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor between abelian categories where \mathcal{A} has enough projectives, let $\{T_n\}$ be a homological δ -functor, and let $\phi_0: T_0 \Rightarrow F = L_0F$ be a natural transformation. Then there exists a unique lift of ϕ_0 to a morphism $\{\phi_i: T_i \rightarrow L_iF\}$.

Proof. First we construct $\phi_1: T_1 \Rightarrow L_1F$. For each object $A \in \mathcal{A}$, pick a projective which surjects onto it and take a kernel, giving the following short exact sequence.

$$0 \longrightarrow K \hookrightarrow P \twoheadrightarrow A \longrightarrow 0$$

Applying each of our homological δ -functors and using the fact that $L_iFP = 0$ for $i > 0$, we get the following data.

$$\begin{array}{ccccccccc} T_1A & \xrightarrow{\delta} & T_0K & \longrightarrow & T_0P & \longrightarrow & T_0A & \longrightarrow & 0 \\ & & \downarrow (\phi_0)_K & & \downarrow (\phi_0)_P & & \downarrow (\phi_0)_A & & \\ 0 & \longrightarrow & L_1FA & \xhookrightarrow{\delta} & FK & \longrightarrow & FP & \longrightarrow & FA \longrightarrow 0 \end{array}$$

Since $L_1FA \rightarrow FK$ is a kernel of $T_0K \rightarrow T_0P$, we get a unique map $T_1A \rightarrow L_1FA$, which we declare to be $(\phi_1)_A$.

It remains to show that these are really the components of a natural transformation. Let $f: A \rightarrow A'$ be a morphism in \mathcal{A} . Consider any induced morphisms of short exact sequences.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \hookrightarrow & P & \twoheadrightarrow & A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & K' & \hookrightarrow & P' & \twoheadrightarrow & A' & \longrightarrow & 0 \end{array}$$

Since $\{L_nF\}$ and $\{T_n\}$ are homological δ -functors, this gives us the following cube, where the components $(\phi_1)_A$ and $(\phi_1)_{A'}$ are dashed. All faces except the one on the left commute

3. Homological algebra in abelian categories

by definition.

$$\begin{array}{ccccc}
 & & T_1 A' & \xrightarrow{\quad} & T_0 K' \\
 & \nearrow T_1 f & \downarrow (\phi_1)_{A'} & \nearrow & \downarrow \\
 T_1 A & \xrightarrow{\quad} & T_0 K & \xrightarrow{\quad} & T_0 K' \\
 \downarrow (\phi_1)_A & \searrow L_1 F f & \downarrow & \searrow \delta & \downarrow \\
 & L_1 F A & \xrightarrow{\quad} & FK & \xrightarrow{\quad} FK'
 \end{array}$$

In order to check that the $(\phi_1)_A$ form the components of a natural transformation, we need to check that the left face of the above cube commutes. Because $\delta: L_1 F A' \rightarrow FK'$ is a monomorphism, it suffices to check that

$$\delta \circ (\phi_1)_{A'} \circ T_1 f = \delta \circ L_1 F f \circ (\phi_1)_A.$$

But given the commutativity of the other faces of the cube, this is clear.

We now continue inductively, constructing ϕ_2 , etc.

The last step is to show that the collection $\{\phi_n\}$ is really a morphism of homological δ -functors, i.e. that for any short exact sequence

$$0 \longrightarrow A'' \hookrightarrow A \twoheadrightarrow A' \longrightarrow 0$$

the squares

$$\begin{array}{ccc}
 T_n A'' & \xrightarrow{\delta} & T_{n-1} A' \\
 (\phi_n)_{A''} \downarrow & & \downarrow (\phi_{n-1})_{A'} \\
 L_n F A'' & \xrightarrow{\delta} & L_{n-1} F A'
 \end{array}$$

commute

□

3.11. Double complexes

Given any abelian category \mathcal{A} , we have considered the category of chain complexes in \mathcal{A} , and shown (in [Theorem 27](#)) that this is an abelian category. It is therefore natural to consider chain complexes in the category of chain complexes.

$$\left(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \right) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

Such a beast is a lattice of commuting squares.

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & M_{i-1,j} & \longrightarrow & M_{i,j} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & M_{i-1,j-1} & \longrightarrow & M_{i,j-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & &
 \end{array}$$

However, when manipulating these complexes it turns out to be convenient to do something slightly different, namely work with complexes whose squares anticommute rather than commute. This is not a fundamental difference, as one can make from any lattice of commuting squares a lattice of anticommuting squares by multiplying the differentials of every other column by -1 . This is known as the *sign trick*.

Definition 87 (double complex). Let \mathcal{A} be an abelian category. A double complex in \mathcal{A} consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d_{i,j}^h: M_{i,j} \rightarrow M_{i-1,j}, \quad d_{i,j}^v: M_{i,j} \rightarrow M_{i,j-1}$$

subject to the following conditions.

- $(d^h)^2 = 0$
- $(d^v)^2 = 0$
- $d^h \circ d^v + d^v \circ d^h = 0$

We say that a double complex is $M_{\bullet,\bullet}$ is *first-quadrant* if $M_{i,j} = 0$ for $i, j < 0$.

Given any complex of complexes, one can construct a double complex by multiplying the differentials in every other row or column by -1 .

Definition 88 (total complex). Let $M_{\bullet,\bullet}$ be a first-quadrant² double complex. The total complex $\text{Tot}(M)_{\bullet}$ is defined level-wise by

$$\text{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\text{Tot}} = d + \delta$.

²This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

3. Homological algebra in abelian categories

Lemma 89. The total complex really is a complex, i.e.

$$d_{\text{Tot}} \circ d_{\text{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\text{Tot}} \circ d_{\text{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Example 90. Let $C_\bullet \in \mathbf{Ch}(\mathbf{Mod}\text{-}R)$ and $D_\bullet \in \mathbf{Ch}(R\text{-}\mathbf{Mod})$ be chain complexes. We can form a double complex by sticking C_\bullet into the first slot of the tensor product, D_\bullet into the second, and using the sign trick to make the squares anticommute. The total complex of this double complex is known as the *tensor product* of C_\bullet and D_\bullet .

In general, computing the homology of a double complex is very difficult; we will explore some techniques for this in [Chapter 6](#). However, under some special conditions, we can say something.

Theorem 91. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\text{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex with exact rows.

$$\begin{array}{ccccccc}
 & & M_{3,0} & & \cdots & & \\
 & & \downarrow d_{3,0} & & & & \\
 & & M_{2,0} & \xleftarrow{\delta_{2,1}} & M_{2,1} & & \cdots \\
 & & \downarrow d_{2,0} & & \downarrow d_{2,1} & & \\
 & & M_{1,0} & \xleftarrow{\delta_{1,1}} & M_{1,1} & \xleftarrow{\delta_{1,2}} & M_{1,2} & \cdots \\
 & & \downarrow d_{1,0} & & \downarrow d_{1,1} & & \downarrow d_{1,2} & \\
 & & M_{0,0} & \xleftarrow{\delta_{0,1}} & M_{0,1} & \xleftarrow{\delta_{0,2}} & M_{0,2} & \xleftarrow{\delta_{0,3}} & M_{0,3}
 \end{array}$$

We want to show that the total complex

$$\begin{array}{ccc}
 M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} & = & \text{Tot}(M)_3 \\
 \downarrow & & \downarrow \\
 M_{2,0} \oplus M_{1,1} \oplus M_{0,2} & = & \text{Tot}(M)_2 \\
 \downarrow & & \downarrow \\
 M_{1,0} \oplus M_{0,1} & = & \text{Tot}(M)_1 \\
 \downarrow & & \downarrow \\
 M_{0,0} & = & \text{Tot}(M)_0 \\
 \downarrow & & \downarrow \\
 0 & & 0
 \end{array}$$

is exact.

The proof is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangeable, we give the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \text{Tot}(M)_2$ which maps to 0 under d_2^{Tot} ; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \quad d_{1,1}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \text{Tot}(M)_3$$

such that $d_3^{\text{Tot}}n = m$.

It turns out that we can make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta dn_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker \delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta dn_{1,2},$$

3. Homological algebra in abelian categories

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned} d^{\text{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d + \delta)n_{2,1} + (d + \delta)n_{1,2} + (d + \delta)n_{0,3} \\ &= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\ &= m_{2,0} + m_{1,1} + m_{0,2} \end{aligned}$$

as required. \square

Example 92. Let $f_\bullet: C_\bullet \rightarrow D_\bullet$ be a morphism of chain complexes. Form from this a double complex by multiplying the differential of C_\bullet by -1 and padding by zeroes on either side.

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & D_3 & \xleftarrow{f_3} & C_3 & \longleftarrow & 0 \\ & & d_3^D \downarrow & & \downarrow -d_3^C & & \\ 0 & \longleftarrow & D_2 & \xleftarrow{f_2} & C_2 & \longleftarrow & 0 \\ & & d_2^D \downarrow & & \downarrow -d_2^C & & \\ 0 & \longleftarrow & D_1 & \xleftarrow{f_1} & C_1 & \longleftarrow & 0 \\ & & d_1^D \downarrow & & \downarrow -d_1^C & & \\ 0 & \longleftarrow & D_0 & \xleftarrow{f_0} & C_0 & \longleftarrow & 0 \\ & & d_0^D \downarrow & & \downarrow -d_0^C & & \\ 0 & \longleftarrow & D_{-1} & \xleftarrow{f_{-1}} & C_{-1} & \longleftarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & \vdots & & \vdots & & \end{array}$$

The total complex of this is simply $\text{cone}(f)$.

Now suppose that f_\bullet is an isomorphism, so the rows are exact. Then by [Theorem 91](#), $\text{Tot}(f) = \text{cone}(f)$ is exact. We have already seen this (in one direction) in [Corollary 78](#).

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We

can view this in two ways.

- As a chain

$$\cdots \longrightarrow C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_{\bullet}$$

hence (by inserting appropriate minus signs) a double complex $C_{\bullet,\bullet}^D$;

- As a morphism between double complexes, hence between totalizations

$$\mathrm{Tot}(\alpha)_{\bullet} : \mathrm{Tot}(C)_{\bullet} \rightarrow \mathrm{Tot}(D)_{\bullet}.$$

Lemma 93. These two points of view agree in the sense that

$$\mathrm{Tot}(C^D) = \mathrm{Cone}(\mathrm{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 94. Let $A_{\bullet} \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{array}{ccccccc} \cdots & \longrightarrow & M_{2,\bullet} & \longrightarrow & M_{1,\bullet} & \longrightarrow & M_{0,\bullet} \\ & & \downarrow & & \downarrow & & \downarrow \alpha_{\bullet} \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & A_{\bullet} \end{array}$$

be a resolution of A_{\bullet} . Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A .

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_{\bullet} \longrightarrow 0$$

is exact. By [Theorem 91](#), the total complex $\mathrm{Tot}(M^A)$ is exact. But by [Lemma 93](#) the total complex is equivalently the cone of α_{\bullet} . However, we have seen (in [Corollary 78](#)) that exactness of $\mathrm{cone}(\alpha_{\bullet})$ means that α_{\bullet} is a quasi-isomorphism. □

Corollary 95. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \mathrm{coker} \, d_1^{M_i}.$$

3. Homological algebra in abelian categories

In particular, we have a double complex

$$\begin{array}{ccccccc}
 M_{0,3} & \longleftarrow & M_{1,3} & \longleftarrow & M_{2,3} & \longleftarrow & M_{3,3} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,2} & \longleftarrow & M_{1,2} & \longleftarrow & M_{2,2} & \longleftarrow & M_{3,2} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,1} & \longleftarrow & M_{1,1} & \longleftarrow & M_{2,1} & \longleftarrow & M_{3,1} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 M_{0,0} & \longleftarrow & M_{1,0} & \longleftarrow & M_{2,0} & \longleftarrow & M_{3,0} \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 H_0(M_{0,\bullet}) & \longleftarrow & H_0(M_{1,\bullet}) & \longleftarrow & H_0(M_{2,\bullet}) & \longleftarrow & H_0(M_{3,\bullet}) \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & & 0 & & 0 & & 0
 \end{array}$$

Flipping along the main diagonal, we have the following resolution.

$$M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})$$

Now [Theorem 94](#) gives us the result we want. \square

3.11.1. Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, in the category $R\text{-Mod}$, there is a canonical tensor product, and we would like to be able to use homological algebra to understand it.

One of the fundamental properties of the tensor product (indeed, the property that justifies giving a universal construction the name tensor product) is that there is an adjunction between the tensor product and the hom functor. In this section, we will define that the tensor product on $R\text{-Mod}$ extends to a tensor product on $\text{Ch}(R\text{-Mod})$, and that this is left adjoint to an internal hom functor on $\text{Ch}(R\text{-Mod})$.

Definition 96 (tensor product of chain complexes). Let C_\bullet, D_\bullet be chain complexes in \mathcal{A} . We define the tensor product of C_\bullet and D_\bullet by

$$(C \otimes D)_\bullet = \text{Tot}(C_\bullet \otimes D_\bullet),$$

where $\text{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

The next step is to introduce the notion of the internal hom. The way to do this is the

obvious one: we should have

$$\mathbf{Hom}(C_\bullet, D_\bullet) = \text{Tot}(\text{Hom}(C_\bullet, D_\bullet)).$$

Unfortunately, even if C_\bullet and D_\bullet are bounded below, the double complex above will be second-quadrant rather than first, so our definition [Definition 88](#) of the total complex will not do. The reason for this mismatch is that, for double complexes which are not first quadrant, one must choose between the product and the coproduct, which no longer agree for infinite indexing sets.

In our case, the product will be the correct choice. For this reason, we make the following definition.

Definition 97 (internal hom). Let C_\bullet, D_\bullet be chain complexes in \mathcal{A} . The internal hom functor on $\mathbf{Ch}(\mathcal{A})$ is the functor

$$\mathbf{Hom}_{\mathcal{A}} : \mathbf{Ch}(\mathcal{A})^{\text{op}} \times \mathbf{Ch}(\mathcal{A}) \rightarrow \mathbf{Ch}(\mathcal{A})$$

is defined by

$$(\mathbf{Hom}_{\mathcal{A}})_n = \prod_{p+q=n} \text{Hom}(C_{-p}, D_q),$$

and

$$d = d^C + (-1)^p d^D,$$

where the factor of $(-1)^p$ implements the sign trick.

Theorem 98. There is an isomorphism

$$\text{Hom}_{\mathcal{A}}(C_\bullet \otimes_R D_\bullet, E_\bullet) \cong \text{Hom}_{\mathcal{A}}(C_\bullet, \mathbf{Hom}(D_\bullet, E_\bullet)),$$

natural in everything.

Proof. This is clear level-wise, since

$$\begin{aligned} \text{Hom}((C_\bullet \otimes_R D_\bullet)_{-n}, E_m) &\cong \text{Hom}\left(\coprod_{p+q=-n} C_p \otimes_R D_q, E_m\right) \\ &\cong \prod_{p+q=-n} \text{Hom}(C_p \otimes_R D_q, E_m) \\ &\cong \prod_{p+q=-n} \text{Hom}(C_p, \text{Hom}(D_q, E_m)) \end{aligned}$$

□

3.11.2. The Künneth formula

This section takes place in \mathbf{Ab} . The same result holds in $R\text{-}\mathbf{Mod}$, for R a PID, but we will only need the \mathbf{Ab} case.

Theorem 99 (Künneth). Let $C_\bullet, D_\bullet \in \mathbf{Ch}(\mathbf{Ab})$, and suppose that C_\bullet is free, that is, that C_n is free for all n . Then there is a short exact sequence

$$0 \rightarrow \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(D_\bullet) \hookrightarrow H_n(C_\bullet \otimes D_\bullet) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}_1(H_p(C_\bullet), H_q(D_\bullet)) \rightarrow 0 .$$

Proof. Denote by Z_\bullet^C the chain complex given level-wise by $Z_n^C = Z_n(C_\bullet)$, and with trivial differentials. Define B_\bullet^C and H_\bullet^C similarly. Then, essentially by definition, we have the following short exact sequences of chain complexes.

$$0 \longrightarrow Z_\bullet^C \hookrightarrow C_\bullet \twoheadrightarrow B^C[-1]_\bullet \longrightarrow 0 \quad (3.4)$$

$$0 \longrightarrow B_\bullet^C \hookrightarrow Z_\bullet^C \twoheadrightarrow H_\bullet^C \longrightarrow 0 \quad (3.5)$$

We happily note that [Sequence 3.5](#), taken level-wise, gives a free resolution of H_\bullet^C .

Since [Sequence 3.4](#) is level-wise split, applying $- \otimes D$ takes the rows to short exact sequences. Since a short exact sequence of chain complexes is simply a sequence of chain complexes whose rows are short exact, the following sequence of chain complexes is short exact.

$$0 \longrightarrow (Z^C \otimes D)_\bullet \hookrightarrow (C \otimes D)_\bullet \twoheadrightarrow (B^C[-1] \otimes D)_\bullet \longrightarrow 0 .$$

Thus, we get a long exact sequence on homology.

$$\begin{array}{ccccccc} & & & \dots & \longrightarrow & H_{n+1}(B^C[-1] \otimes D) & \\ & & & \searrow & & \delta & \nearrow \\ & & & H_n(Z^C \otimes D) & \longrightarrow & H_n(C \otimes D) & \longrightarrow & H_n(B^C[-1] \otimes D) \\ & & & \searrow & & \delta & \nearrow \\ & & & H_{n-1}(Z^C \otimes D) & \longrightarrow & \dots & \end{array}$$

Consider several terms of the chain complex $(B^C[-1] \otimes D)_\bullet$.

$$\dots \longrightarrow \bigoplus_{p+q=n+1} B_{p-1}^C \otimes D_q \longrightarrow \bigoplus_{p+q=n} B_{p-1}^C \otimes D_q \longrightarrow \dots$$

Because of the triviality of the differential of $B^C[-1]$, the above morphisms are a direct

sum of morphisms

$$B_{p-1}^C \otimes D_q \xrightarrow{\text{id}_B \otimes d^D} B_{p-1}^C \otimes D_{q-1} \ .$$

Hence, because B_{p-1}^C is free, the homology of $(B^C[-1] \otimes D)_\bullet$ is

$$\begin{aligned} H_n(B[-1] \otimes D) &= \bigoplus_{p+q=n} B_{p-1} \otimes H_n(D) \\ &= \bigoplus_{p+q=n-1} B_p \otimes H_n(D) \\ &= (B^C \otimes H^D)_{n-1}, \end{aligned}$$

where we have defined H_\bullet^D in analogy with H_\bullet^C .

Exactly the same reasoning as above shows that

$$H_n(Z \otimes D) = (Z \otimes H^D)_n.$$

Making these replacements in our long exact sequence gives us the following.

$$\begin{array}{ccccccc} & & & & \cdots & \longrightarrow & (B^C \otimes H^D)_n \\ & & & & & \searrow \delta & \nearrow \\ & & & & (Z^C \otimes H^D)_n & \longrightarrow & H_n(C \otimes D) \longrightarrow (B^C \otimes H^D)_{n-1} \\ & & & & & \searrow \delta & \nearrow \\ & & & & (Z^C \otimes H^D)_{n-1} & \longrightarrow & \cdots \end{array}$$

A quick computation shows that

$$\delta = \bigoplus_{p+q=n} (B_p^C \hookrightarrow Z_p^C) \otimes \text{id}_{H_q^D}.$$

The kernel and cokernel of the connecting homomorphism are refreshingly familiar. The kernel corresponds precisely to our definition of Tor coming from the projective resolution from [Sequence \(3.5\)](#), and the cokernel is the definition of n th homology.

$$\begin{array}{ccccccc} \bigoplus_{p+q=n} \text{Tor}_1^{\mathbb{Z}}(H_p(C), H_q^D) & & & & \bigoplus_{p+q=n} H_p(C) \otimes H_q^D & & \\ \parallel & & & & \parallel & & \\ \ker \delta \hookrightarrow \bigoplus_{p+q=n} B_p^C \otimes H_q^D & \xrightarrow{\delta} & \bigoplus_{p+q=n} Z_p^C \otimes H_q^D & \twoheadrightarrow & \text{coker } \delta & & \end{array} \ .$$

3. Homological algebra in abelian categories

We can decorate our long exact sequence with these to find the following solid short exact sequence.

$$\begin{array}{ccccccc}
 & 0 & \searrow & & & & \\
 & & \bigoplus_{p+q=n} H_p(C) \otimes H_q^D & \hookrightarrow & H_n(C \otimes D) & \dashrightarrow & (B^C \otimes H^D)_{n-1} \rightarrow \dots \\
 & \nearrow & & \searrow & \searrow & \nearrow & \\
 \dots \rightarrow & (Z^C \otimes H^D)_n & \dashrightarrow & H_n(C \otimes D) & \twoheadrightarrow & \bigoplus_{p+q=n-1} \mathrm{Tor}_1^{\mathbb{Z}}(H_p(C), H_q^D) & \rightarrow 0 \\
 & & & & & & \nearrow \\
 & & & & & & (B^C \otimes H^D)_{n-1} \rightarrow \dots
 \end{array}$$

Replacing H_n^C by $H_n(C)$ and H_n^D by $H_n(D)$, we find precisely the sequence we were looking for. \square

Proposition 100. The short exact sequence in the Künneth formula splits, but not canonically.

Corollary 101 (universal coefficient theorem). Let C_\bullet be a chain complex of free abelian groups, and let A be an abelian group. Then there is a short exact sequence

$$0 \longrightarrow H_n(C) \otimes A \hookrightarrow H_n(C \otimes A) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(C), A) \longrightarrow 0 .$$

4. Some important derived functors

4.1. The Tor functor

Let R be a ring, and N a right R -module. There is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \leftrightarrow \mathbf{Ab} : \mathrm{Hom}(N, -).$$

Thus, the functor $- \otimes_R N$ preserves colimits, hence by [Proposition 48](#) is right exact. Thus, we may form the left derived functor.

Definition 102 (Tor functor). The left derived functor of $- \otimes_R N$, is called the Tor functor and denoted

$$\mathrm{Tor}_i^R(-, N).$$

Example 103. Let R be a ring, and let $r \in R$. Assume that left multiplication by r is injective, and consider the right R -module R/rR . We have the short exact sequence

$$0 \longrightarrow R \xrightarrow{r \cdot} R \xrightarrow{\pi} R/rR \longrightarrow 0$$

exhibiting

$$0 \longrightarrow R \xrightarrow{r \cdot} R \longrightarrow 0$$

as a free (hence projective) resolution of R/rR . Thus we can calculate

$$\mathrm{Tor}_i^R(R/rR, N) \simeq H_i \left(0 \longrightarrow R \otimes_R N \xrightarrow{r \cdot} R \otimes_R N \longrightarrow 0 \right)$$

That is, we have

$$\mathrm{Tor}_i^R(R/rR, N) \simeq \begin{cases} N/rN, & i = 0 \\ rN, & i = 1 \\ 0, & \text{otherwise.} \end{cases}$$

In particular, for any abelian group D , $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, D) = {}_n D$, the n -torsion elements of D .

There is a worrying asymmetry to our definition of Tor. The tensor product is, morally speaking, symmetric; that is, it should not matter whether we derive the first factor or the second. Pleasingly, Tor respects this.

4. Some important derived functors

Proposition 104. The functor Tor is *balanced*; that is,

$$L_i \text{Hom}(M, -)(N) \simeq L_i \text{Hom}(-, N)(M).$$

Proof. Let $P_\bullet \rightarrow M$ and $P'_\bullet \rightarrow N$ be projective resolutions. We need to show that

$$H_i(P_\bullet \otimes_R N)_\bullet \simeq H_i(M \otimes_R P'_\bullet)_\bullet.$$

To see this, consider the following double complex (with factors of -1 added as necessary to actually make it a double complex).

$$\begin{array}{ccccc}
 & & & & \\
 & & & & \\
 & & & & \\
 M \otimes_R P'_1 & \xleftarrow{\quad} & P_0 \otimes_R P'_1 & \xleftarrow{\quad} & P_1 \otimes_R P'_1 \\
 \downarrow & & \downarrow & & \downarrow \\
 M \otimes_R P'_0 & \xleftarrow{\quad} & P_0 \otimes_R P'_0 & \xleftarrow{\quad} & P_1 \otimes_R P'_0 \\
 & & \downarrow & & \downarrow \\
 & & P_0 \otimes_R N & \xleftarrow{\quad} & P_1 \otimes_R N \\
 & & & & \\
 & & & &
 \end{array}$$

$$\cdots \longrightarrow P_\bullet \otimes_R P'_2 \longrightarrow P_\bullet \otimes_R P'_1 \longrightarrow P_\bullet \otimes_R P'_0 \longrightarrow 0$$

Since each P'_i is projective, $-\otimes_R P'_i$ is exact, so the rows above the dotted line are resolutions of $M \otimes_R P'_j$. Similarly, the columns to the right of the dotted line are resolutions of $P_i \otimes_R N$.

This means that the part of the double complex above the dotted line is a double complex whose rows are resolutions; thus,

$$\text{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq M \otimes_R P'_\bullet.$$

Similarly, the part of the double complex to the right of the dotted line has resolutions as its columns, implying that

$$\text{Tot}(P_\bullet \otimes_R P'_\bullet) \simeq P_\bullet \otimes_R N.$$

Thus, taking homology, we have

$$H_i(P_\bullet \otimes_R N) \simeq H_i(\text{Tot}(P_\bullet \otimes_R P'_\bullet)) \simeq H_i(M \otimes_R P'_\bullet).$$

□

Example 105. Let N be an R -module. The Tor functor controls flatness in the sense that the following are equivalent:

1. The module N is flat
2. For all R -modules M , we have $\mathrm{Tor}_1^R(M, N) = 0$.

To see that 1. \Rightarrow 2., suppose that N is flat and pick a projective resolution $P_\bullet \xrightarrow{\sim} M$. Then

$$\begin{aligned} \mathrm{Tor}_1(M, N) &= H_1(P_\bullet \otimes_R N) \\ &= H_1(P_\bullet) \otimes_R N \\ &= 0. \end{aligned}$$

Conversely, suppose that for all M , $\mathrm{Tor}_1(M, N) = 0$. Then for any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0 ,$$

we have a long exact sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \longrightarrow C \otimes_R N \longrightarrow \mathrm{Tor}_1(A, N) \longrightarrow \cdots .$$

But $\mathrm{Tor}_1(A, N) = 0$, so the sequence

$$0 \longrightarrow A \otimes_R N \hookrightarrow B \otimes_R N \twoheadrightarrow C \otimes_R N \longrightarrow 0 .$$

is short exact. Thus, $-\otimes_R N$ is exact, which means by definition that N is flat.

4.2. The Ext functor

We know that $\mathrm{Hom}(N, -): R\text{-Mod} \rightarrow \mathbf{Ab}$ is left exact, so we may form right derived functors.

Definition 106 (Ext functor). Let R be a ring, and $N \in R\text{-Mod}$ an R -module. The right derived functors

$$R^i \mathrm{Hom}(N, -): R\text{-Mod} \rightarrow \mathbf{Ab}$$

are called the Ext functors, and denoted

$$R^i \mathrm{Hom}(N, -) = \mathrm{Ext}_R^i(N, -).$$

An argument very similar to that given for Tor in [Proposition 104](#) shows that Ext is balanced; that is, when computing $\mathrm{Ext}_R^i(M, N)$, it doesn't matter whether we take an

4. Some important derived functors

projective resolution of N or an injective resolution of M ; we will get the same result either way. Rather than spelling this out in detail, we will wait for [Chapter 6](#), where a spectral sequences argument gives the result almost immediately.

Example 107. Consider $R = \mathbb{Z}$, so $R\text{-Mod} = \mathbf{Ab}$. Let us calculate

$$\text{Ext}^*(\mathbb{Z}/(p), \mathbb{Z}).$$

We can take either an injective resolution of \mathbb{Z} or a projective resolution of $\mathbb{Z}/(p)$. Each $\mathbb{Z}/(p)$ has a canonical projective resolution

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot p} \mathbb{Z} \twoheadrightarrow \mathbb{Z}/(p) \longrightarrow 0 ,$$

which we might as well take advantage of. Doing so, we find

$$\text{Ext}^* = H^* \left(0 \longrightarrow \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{\cdot p} \text{Hom}(\mathbb{Z}, \mathbb{Z}) \longrightarrow 0 \right).$$

We can immediately calculate

$$\text{Ext}^0(\mathbb{Z}/(p), \mathbb{Z}) \cong \ker \cdot p \cong 0, \quad \text{Ext}^1(\mathbb{Z}/(p), \mathbb{Z}) \cong \text{coker } \cdot p \cong \mathbb{Z}/(p),$$

and $\text{Ext}^i(\mathbb{Z}/(p), \mathbb{Z}) = 0$ for $i > 1$.

Note that balancedness immediately

4.2.1. Ext and extensions

There turns out to be a deep connection between the Ext groups and extensions.

Definition 108 (extension). Let A and B be R -modules. An extension of A by B is a short exact sequence

$$\xi: \quad 0 \longrightarrow B \hookrightarrow X \twoheadrightarrow A \longrightarrow 0 .$$

Two extensions ξ and ξ' are said to be equivalent if there is a morphism of short exact sequences

$$\begin{array}{ccccccc} \xi: & 0 & \longrightarrow & B & \hookrightarrow & X & \twoheadrightarrow A \longrightarrow 0 \\ & & & \downarrow \text{id}_B & & \downarrow \varphi & \downarrow \text{id}_A \\ \xi': & 0 & \longrightarrow & B & \hookrightarrow & X' & \twoheadrightarrow A \longrightarrow 0 \end{array} .$$

The five lemma ([Theorem 43](#)) implies that φ is an isomorphism.

Let ξ be an extension of A by B . Applying the Ext functor we find an associated long exact sequence on cohomology.

$$\begin{array}{ccccccc} \text{Ext}^1(A, B) & \longrightarrow & \text{Ext}^1(X, B) & \longrightarrow & \cdots \\ \uparrow & & \delta & & \uparrow \\ \text{Hom}(A, B) & \longrightarrow & \text{Hom}(X, B) & \longrightarrow & \text{Hom}(B, B) \end{array}$$

Definition 109 (obstruction class). The image of id_B under the connecting homomorphism $\delta(\text{id}_B) \in \text{Ext}^1(A, B)$ is known as the obstruction class of ξ . For any extension ξ , we denote the obstruction class of ξ by $\Theta(\xi)$.

Proposition 110. An extension ξ is split if and only if the corresponding obstruction class is zero.

Proof. We are given an extension

$$\xi: \quad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0 .$$

First suppose that $\Theta(\xi) = 0$. Consider the following portion of the exact sequence on cohomology.

$$\text{Hom}(X, B) \longrightarrow \text{Hom}(B, B) \longrightarrow \text{Ext}^1(A, B)$$

If $\delta(\text{id}_B) = 0$, then there exists $a \in \text{Hom}(X, B)$ such that $a \circ f = \text{id}_B$. Thus, by the splitting lemma (Lemma 33), the sequence splits.

Now suppose that the sequence ξ splits; that is, that we have a morphism $a: A \rightarrow X$ such that $a \circ f = \text{id}_B$.

$$0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} A \longrightarrow 0 .$$

$\nwarrow \scriptstyle a$

Since $\text{Ext}^1(-, B)$ is additive, the sequence

$$0 \longrightarrow \text{Ext}^1(A, B) \xrightarrow{\text{Ext}^1(g, B)} \text{Ext}^1(X, B) \xrightarrow{\text{Ext}^1(f, B)} \text{Ext}^1(B, B) \longrightarrow 0$$

$\nwarrow \scriptstyle \text{Ext}^1(a, B)$

is split exact. Thus, $\text{Ext}^1(g, B)$ is a monomorphism, so it has trivial kernel.

$$\begin{array}{ccccccc} \text{Ext}^1(A, B) & \xrightarrow{\text{Ext}^1(g, B)} & \cdots \\ \uparrow & & \delta & & \uparrow \\ \text{Hom}(A, B) & \longrightarrow & \text{Hom}(X, B) & \longrightarrow & \text{Hom}(B, B) \end{array}$$

4. Some important derived functors

Exactness then forces $\delta = 0$. □

Proposition 110 explains the name *obstruction class*: $\Theta(\xi)$ is the obstruction to the splitness of ξ . The first Ext group contains some information about when a sequence can split, or fail to split.

In fact, the group $\text{Ext}^1(A, B)$ parametrizes extensions of A by B .

Theorem 111. There is a natural bijection

$$\{\text{Extension classes of } \xi \text{ extensions of } A \text{ by } B\} \xrightarrow{\cong} \text{Ext}^1(A, B); \quad [\xi] \mapsto \Theta(\xi).$$

Proof. We first show well-definedness. Let ξ and ξ' be equivalent extensions. By naturality of δ , the following diagram commutes.

$$\begin{array}{ccccc} \text{Hom}(X, B) & \longrightarrow & \text{Hom}(B, B) & \xrightarrow{\delta} & \text{Ext}^1(A, B) \\ \text{id} \uparrow & & \text{id} \uparrow & & \uparrow \text{id} \\ \text{Hom}(X, B) & \longrightarrow & \text{Hom}(B, B) & \xrightarrow{\delta} & \text{Ext}^1(A, B) \end{array}$$

Thus, $\Theta(\xi) = \Theta(\xi')$.

We now construct an explicit inverse Ψ to Φ . Since $R\text{-Mod}$ has enough projectives, we can find a projective P which surjects onto A . Fix such a P and take the kernel of the surjection, giving the following short exact sequence.

$$0 \longrightarrow K \xrightarrow{\psi} P \xrightarrow{\gamma} A \longrightarrow 0 \quad (4.1)$$

We construct from such an exact sequence a map which takes an element of $\text{Ext}^1(A, B)$ and gives an extension of A by B . First applying $\text{Ext}^*(-, B)$ and taking the long exact sequence on cohomology we find the following exact fragment.

$$\text{Hom}(K, B) \xrightarrow{\delta} \text{Ext}^1(A, B) \longrightarrow \text{Ext}^1(P, B)$$

Since P is projective, the group $\text{Ext}^1(P, B)$ vanishes, so $\delta: \text{Hom}(K, B) \rightarrow \text{Ext}^1(A, B)$ is surjective. Thus, given any $e \in \text{Ext}^1(A, B)$, we can find a (not unique!) lift to $\phi \in \text{Hom}(K, B)$. Take the pushout in the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\psi} & P & \xrightarrow{\gamma} & A \longrightarrow 0 \\ & & \phi \downarrow & & \downarrow i_P & & \\ & & B & \xrightarrow{i_B} & B \amalg_K^\phi P & & \end{array}$$

By the dual to [Lemma 37](#), the cokernel of i_B is equal to the cokernel of ψ , so we get an

induced short exact sequence as follows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xhookrightarrow{\psi} & P & \xrightarrow{\gamma} \twoheadrightarrow & A \longrightarrow 0 \\
 & & \downarrow \phi & & \downarrow i_P & & \parallel \\
 0 & \longrightarrow & B & \xhookrightarrow{i_B} & B \amalg_K^\phi P & \xrightarrow{\pi} \twoheadrightarrow & A \longrightarrow 0
 \end{array}$$

It remains to show that Ψ as defined is independent of the choice of ϕ , and that it really is an inverse to Φ . To this end, pick a different $\tilde{\phi} \in \text{Hom}(N, B)$ such that $\delta(\tilde{\phi}) = e$.

This gives us a new morphism of exact sequences as follows

$$\begin{array}{ccccccc}
 0 & \longrightarrow & K & \xhookrightarrow{\psi} & P & \xrightarrow{\gamma} \twoheadrightarrow & A \longrightarrow 0 \\
 & & \downarrow \tilde{\phi} & & \downarrow \tilde{i}_P & & \parallel \\
 0 & \longrightarrow & B & \xhookrightarrow{\tilde{i}_B} & B \amalg_K^{\tilde{\phi}} P & \xrightarrow{\tilde{\pi}} \twoheadrightarrow & A \longrightarrow 0
 \end{array} \tag{4.2}$$

Because of the way we chose $\tilde{\phi}$, there exists some $\rho \in \text{Hom}(P, B)$ so that $\rho \circ \psi = \tilde{\phi} - \phi$. Consider the following outer square.

$$\begin{array}{ccc}
 K & \xhookrightarrow{\psi} & P \\
 \downarrow \phi & & \downarrow i_P \\
 B & \xhookrightarrow{i_B} & B \amalg_N^\phi P
 \end{array}
 \begin{array}{c}
 \searrow \tilde{i}_P - \tilde{i}_B \circ \rho \\
 \downarrow \exists! \alpha \\
 B \amalg_N^{\tilde{\phi}} P
 \end{array}
 \tag{4.3}$$

Using the commutativity of the left-hand square in [Diagram 4.2](#), we find

$$\begin{aligned}
 (\tilde{i}_P - \tilde{i}_B \circ \rho) \circ \psi &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \rho \circ \psi \\
 &= \tilde{i}_P \circ \psi - \tilde{i}_B \circ \tilde{\phi} + \tilde{i}_B \circ \phi \\
 &= \tilde{i}_B \circ \phi,
 \end{aligned}$$

so this outer square commutes, giving us the unique morphism α .

The claim that our two extensions are equivalent unwraps to the claim that the diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & B & \xhookrightarrow{i_B} & B \amalg_K^\phi P & \xrightarrow{\pi} \twoheadrightarrow & A \longrightarrow 0 \\
 & & \parallel & & \downarrow \alpha & & \parallel \\
 0 & \longrightarrow & B & \xhookrightarrow{\tilde{i}_B} & B \amalg_K^{\tilde{\phi}} P & \xrightarrow{\tilde{\pi}} \twoheadrightarrow & A \longrightarrow 0
 \end{array}$$

4. Some important derived functors

commutes. The left-hand square is the lower commuting triangle in [Diagram 4.3](#).

Reminding ourselves of the proof of [Lemma 37](#), we recall that the morphism π is defined by a pushout: it is the unique map $P \amalg_K^\phi P \rightarrow A$ such that $\pi \circ i_B = 0$ and $\pi \circ i_P = \gamma$. If we can show that $\tilde{\pi} \circ \alpha$ also satisfies these equations, then we can content ourselves that the diagram commutes. Indeed, we have

$$\begin{aligned} (\tilde{\pi} \circ \alpha) \circ i_B &= \tilde{\pi} \circ \tilde{i}_B \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} (\tilde{\pi} \circ \alpha) \circ i_P &= \tilde{\pi} \circ (\tilde{i}_P - \tilde{i}_B \circ \rho) \\ &= \gamma. \end{aligned}$$

Our last job, which was really the whole point of the endeavor, is to check that the maps Φ and Ψ are inverse to each other. To that end, fix some A and B , and pick $e \in \text{Ext}^1(A, B)$. By construction of Ψ , we get a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \xrightarrow{\psi} & P & \xrightarrow{\gamma} \twoheadrightarrow & A \longrightarrow 0 \\ & & \phi \downarrow & & \downarrow i_P & & \parallel \\ 0 & \longrightarrow & B & \xrightarrow{i_B} & B \amalg_K^\phi P & \xrightarrow{\pi} \twoheadrightarrow & A \longrightarrow 0 \end{array} \quad (4.4)$$

where ϕ is an element of $\text{Hom}(K, A)$ which is mapped to by e under δ . The next step in computing $\Phi(\Psi(e))$ would now be to stick $\Psi(e)$ into the hom functor $\text{Hom}(-, B)$, and look at the image of id_B under the connecting homomorphism. However, using the fact that Ext is a homological δ -functor, we can stick the whole morphism of exact sequences [Equation 4.4](#) in and get a morphism of long exact sequences out. Picking out the square containing the relevant boundary map, we find the following.

$$\begin{array}{ccc} \text{Hom}(B, B) & \xrightarrow{\delta} & \text{Ext}^1(A, B) \\ \text{Hom}(\phi, B) \downarrow & & \parallel \\ \text{Hom}(K, B) & \xrightarrow{\delta'} & \text{Ext}^1(A, B) \end{array}$$

Chasing the identity around in both directions tells us that

$$\delta(\text{id}_B) = \delta'(\phi) = e.$$

Next, we start with an extension

$$\xi: \quad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} \twoheadrightarrow A \longrightarrow 0.$$

Using the connecting homomorphism, we get a representative for the obstruction class $e = \delta(\text{id}_B) \in \text{Ext}^1(A, B)$. Using our short exact sequence from Equation 4.1, we can find the fo □

Example 112. Let's do a sort of capstone computation, which pulls in just about everything we've done so far.

Consider the ring $R = \mathbb{Z}[x]$. We will compute the obstruction class of the extension

$$\xi: \quad 0 \longrightarrow \mathbb{Z} \xrightarrow{\cdot x} \mathbb{Z}[x]/(x^2) \twoheadrightarrow \mathbb{Z} \longrightarrow 0 ,$$

where we have written $\mathbb{Z} = \mathbb{Z}[x]/(x)$.

At a very high level of abstraction, we compute the obstruction class of ξ by taking the long exact sequence associated to it under the right derived functor of $\text{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z})$, namely

$$\begin{array}{ccccccc} \text{Ext}_{\mathbb{Z}[x]}^1(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & \cdots & & & & \\ \uparrow & & & \delta & & & \\ \text{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x]/(x^2), \mathbb{Z}) & \longrightarrow & \text{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}, \mathbb{Z}) & & \end{array}$$

and finding the image of the identity $\text{id}_{\mathbb{Z}}$ under δ .

In order to find this long exact sequence, we need to lift ξ to a short exact sequence of projective resolutions. If we can find a projective resolution of \mathbb{Z} , the horseshoe lemma will do the rest of the work for us.

Our work on syzygies will help us here. We note that \mathbb{Z} has a single generator, namely 1, so we begin our resolution

$$\mathbb{Z}[x] \xrightarrow{\pi} \mathbb{Z} .$$

The kernel of this is the ideal (x) .

$$\begin{array}{ccc} (x) & & \\ \searrow & & \\ \mathbb{Z}[x] & \xrightarrow{\pi} & \mathbb{Z} \end{array} .$$

This again has a single generator, namely x .

$$\begin{array}{ccccc} & (x) & & & \\ & \swarrow \quad \searrow & & & \\ \mathbb{Z}[x] & \xrightarrow{\cdot x} & \mathbb{Z}[x] & \xrightarrow{\pi} & \mathbb{Z} \end{array} .$$

The composition gives us our next differential. Note that multiplication by x is injective,

4. Some important derived functors

so our projective resolution terminates.

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{Z}[x] \xrightarrow{\cdot x} \mathbb{Z}[x] \longrightarrow \cdots$$

Now we apply the horseshoe lemma. We first build our horseshoe.

$$\begin{array}{ccccccc} & & \mathbb{Z}[x] & & \mathbb{Z}[x] & & \\ & & \downarrow & & \downarrow & & \\ & & \mathbb{Z}[x] & & \mathbb{Z}[x] & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot x} & \mathbb{Z}[x]/(x^2) & \xrightarrow{\pi} & \mathbb{Z} \longrightarrow 0 \end{array}$$

The center column is then given by the direct sum of the outer columns, and the horizontal maps are the canonical injection and projection respectively.

$$\begin{array}{ccccccc} & & \mathbb{Z}[x] & \hookrightarrow & \mathbb{Z}[x]^2 & \twoheadrightarrow & \mathbb{Z}[x] \\ & & \downarrow & & \downarrow \alpha & & \downarrow \\ & & \mathbb{Z}[x] & \hookrightarrow & \mathbb{Z}[x]^2 & \twoheadrightarrow & \mathbb{Z}[x] \\ & & \downarrow & & \downarrow \beta & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\cdot x} & \mathbb{Z}[x]/(x^2) & \xrightarrow{\pi} & \mathbb{Z} \longrightarrow 0 \end{array}$$

(3) (4)

(1) (2)

The horseshoe lemma guarantees the existence of maps α and β above such that all the squares commute and the middle column forms a projective resolution, but being assured of existence is of no help to us; we must find them explicitly. We are helped tremendously that what we have found are free resolutions, so we can write the maps as matrices. We first figure out what we can about the morphisms α and β from the commutativity of the squares labelled (1)-(4).

(1) Taking the low road maps

$$1 \mapsto 1 \mapsto x,$$

while taking the high road maps

$$1 \mapsto (1, 0) \mapsto \beta(1, 0).$$

Thus,

$$\beta: (1, 0) \mapsto x$$

(2) The high road maps

$$(0, 1) \mapsto 1 \mapsto 1,$$

and the low road maps

$$(0, 1) \mapsto \beta(0, 1) \mapsto \beta(0, 1).$$

Thus,

$$\beta: (0, 1) \mapsto 1.$$

Already, this is enough to tell us that

$$\beta(a, b) = ax + b.$$

We make the ansatz

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

(3) Taking the high road maps

$$1 \mapsto (1, 0) \mapsto (a, c),$$

while taking the low road maps

$$1 \mapsto x \mapsto (x, 0),$$

so $a = x$ and $c = 0$.

(4) Taking the high road maps

$$(0, 1) \mapsto 1 \mapsto x,$$

and taking the low road maps

$$(0, 1) \mapsto (b, d) \mapsto d$$

so we must have $d = x$.

Thus, we have

$$\alpha = \begin{pmatrix} a & b \\ 0 & x \end{pmatrix},$$

where we have not yet fixed b . However, we have not yet used that the middle column must be a chain complex, i.e. that $\beta \circ \alpha$ must be equal to zero. Multiplying out, we have

$$\beta \circ \alpha = \begin{pmatrix} x & 1 \end{pmatrix} \begin{pmatrix} x & b \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & x + bx \end{pmatrix}.$$

Thus, we must take $b = -1$, fixing

$$\alpha = \begin{pmatrix} x & -1 \\ 0 & x \end{pmatrix}.$$

4. Some important derived functors

Applying $\text{Hom}_{\mathbb{Z}[x]}(-, \mathbb{Z}[x])$, we find the following diagram.

At last, we can calculate $\Theta(\xi)$ as the image of $\text{id}_{\mathbb{Z}} \in \text{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x], \mathbb{Z}[x])$ under the connecting homomorphism.

4.2.2. The Baer sum

By definition, $\text{Ext}^1(A, B)$ has an abelian group structure. We have provided a bijection between equivalence classes of extensions of B by A and $\text{Ext}^1(A, B)$, which allows us to transport our addition law from $\text{Ext}^1(A, B)$ to the set of extension classes. However, it is not at all clear how to interpret the sum of two classes of extensions.

The group operation on the set of extension classes is known as the *Baer sum*. Let

$$\xi: \quad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} \twoheadrightarrow A \longrightarrow 0$$

and

$$\xi': \quad 0 \longrightarrow B \xrightarrow{f'} X' \xrightarrow{g'} \twoheadrightarrow A \longrightarrow 0$$

be extensions. We shall form a new extension from ξ and ξ' called the *Baer sum*, which will turn out to be in the class of $\Psi(\Phi(\xi + \xi'))$.

A word of apology: things are about to get very element-theoretic. Of course, this is not a problem thanks to the Freyd-Mitchell embedding theorem, but it's still a bit sad.

From the above extensions above, take the following pushout, and form $X'' = \ker \alpha$.

$$\begin{array}{ccccc}
 B & \xrightarrow{f'} & X' & & \\
 \downarrow f & \searrow \exists! f'' & \downarrow & \searrow -g' & \\
 & & X'' & & A \\
 & & \downarrow & & \\
 X & \xrightarrow{\quad} & X \amalg_B X' & \xrightarrow{\exists! \alpha} & A \\
 & \searrow g & & & \\
 & & & &
 \end{array}$$

We have immediately the following morphism.

$$0 \longrightarrow B \xrightarrow{f''} X'' \longrightarrow A \longrightarrow 0$$

We would like to fill in morphisms so that we have an exact sequence. The obvious choice of morphism $X'' \rightarrow A$, namely the composition $\alpha \circ \ker \alpha$, won't work because it's zero, so we have to define something else.

Warning: brutal element-wise computation ahead.

We have an explicit description of X'' as

$$X \amalg_B X' = X \oplus X' / \langle f(b) = f'(b) \rangle.$$

The kernel X'' is therefore the part of this which goes to zero under α , namely $(x, x') \in X \amalg_B X'$ such that

$$\alpha(x, x') = g(x) - g'(x') = 0,$$

i.e.

$$X'' = \{(x, x') \in X \amalg_B X' \mid g(x) = g'(x')\}.$$

The map f'' then simply sends $b \mapsto (f(b), 0)$, which by definition is equivalent to $(0, f'(b))$.

Note that f' is clearly injective.

Next, we define a map $g'': X'' \rightarrow A$ by sending

$$(x, x') \mapsto g(x) = g'(x').$$

This is clearly surjective since g and g' are. The kernel of g'' consists precisely of those (x, x') such that $g(x) = 0$, which is to say, such that $x = f(b)$ by exactness of ξ . Thus, we have constructed a bona fide short exact sequence

$$\xi'': \quad 0 \longrightarrow B \xrightarrow{f''} X'' \xrightarrow{g''} A \longrightarrow 0. \quad (4.5)$$

Definition 113 (Baer sum). The extension ξ'' from Equation 4.5 is called the Baer sum of ξ and ξ' .

Lemma 114. The Baer sum is compatible with equivalences; that is, if $\xi \sim \zeta$, then $\xi + \xi' \sim \zeta + \xi'$.

Proof. Note that equivalence implies in particular \square

This means that the Baer sum descends to an addition law on equivalence classes.

Lemma 115. Let

$$\eta: \quad 0 \longrightarrow B \xrightarrow{i} B \oplus A \xrightarrow{p} A \longrightarrow 0$$

be a split extension. Then for any extension ξ , $\eta + \xi \sim \xi$.

Proof. \square

4. Some important derived functors

Lemma 116. Let

$$\xi: \quad 0 \longrightarrow B \xrightarrow{f} X \xrightarrow{g} \twoheadrightarrow A \longrightarrow 0 .$$

be an extension. Then

$$-\xi: \quad 0 \longrightarrow B \xrightarrow{-f} X \xrightarrow{g} \twoheadrightarrow A \longrightarrow 0$$

is the inverse under the Baer sum, i.e. $\xi + (-\xi) \sim \eta$.

4.2.3. Higher extensions

We can also interpret higher Ext groups in terms of extensions. Let

$$\xi: \quad 0 \longrightarrow B \hookrightarrow X_0 \longrightarrow \cdots \longrightarrow X_{n-1} \twoheadrightarrow A \longrightarrow 0$$

be an extension. We define the obstruction class $\Theta(\xi)$ as follows: first, find a projective resolution $P_\bullet \rightarrow A$, and a lift of the identity id_A as follows.

$$\begin{array}{ccccccccc} P_{n+1} & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & A \\ \downarrow & & \downarrow f & & \downarrow & & & & \downarrow & & \parallel \\ 0 & \longrightarrow & B & \longrightarrow & X_0 & \longrightarrow & \cdots & \longrightarrow & X_{n-1} & \longrightarrow & A \end{array}$$

Now apply $\text{Hom}(-, B)$, and take n th homology. The morphism $f \in \text{Hom}(P_n, B)$ descends to an element of $\text{Ext}^n(A, B)$ (since we can write it as $f = f^* \text{id}_B$, implying that it is a cycle). This is what we call $\Theta(\xi)$.

4.3. Inverse and direct limits

4.3.1. Direct limits

Definition 117 (filtered poset). A poset (I, \leq) is called filtered if every finite subset of I has an upper bound; that is, thinking about a poset as a category, if for every $i, j \in I$, there exists $k \in I$ and arrows $i \rightarrow k, j \rightarrow k$.

$$\begin{array}{ccc} i & & \\ & \searrow & \\ & & k \\ & \nearrow & \\ j & & \end{array}$$

Example 118. Let X be a set, and consider the poset of finite subsets $\mathcal{P}_{\text{fin}}(X)$. This is

a filtered poset since every finite collection $X_i \in \mathcal{P}_{\text{fin}}(X)$ is contained in the set $\bigcup_i X_i$, which is again finite.

Let I be a filtered poset, and let $F: I \rightarrow R\text{-Mod}$ be a functor. Using the description of a colimit as a coequalizer, we can calculate that

$$\text{colim } F \cong \left(\coprod_{i \in I} F(i) \right) / E,$$

where E is the subgroup generated by elements of the form $x - x'$, where $x \in F(i)$ and $x' \in F(i')$, such that there exists $j \in I$, $y \in F(j)$, and maps $u: i \rightarrow j$ and $u': i' \rightarrow j$ such that $F(u)(x) = y$ and $F(u')(x') = y$.

$$\begin{array}{ccc} F(i) & \xrightarrow{F(u)} & F(j) \\ & \nearrow F(u') & \\ F(i') & & \end{array}, \quad \begin{array}{ccc} x & \searrow & y \\ & \nearrow & \\ x' & & \end{array}$$

That is, two elements of $\coprod_{i \in I} F(i)$ are identified if they ‘eventually’ become equal.

Lemma 119. Let I be a filtered poset, and let $A: I \rightarrow R\text{-Mod}$ be a functor.

1. Every element $a \in \text{colim } A$ is the image of some $a_i \in A_i$ (for some $i \in I$) under the canonical map $A_i \rightarrow \text{colim } A$.
2. For every i , the kernel of the canonical map $A_i \rightarrow \text{colim } A$ is the union of the kernels of the maps $A(\phi): A_i \rightarrow A_j$ (for $\phi: i \rightarrow j$ a map in I).

Proof.

1. Denote the canonical map $A_i \rightarrow \text{colim } A$ by λ_i . A priori, we are only guaranteed that $a \in \text{colim } A$ can be written as an equivalence class

$$\left[\sum_{i \in J} \lambda_i(a_i) \right], \quad a_i \in A_i,$$

for some finite subset $J \subset I$. Since $J = \{i_1, \dots, i_n\}$ is finite, we can find an upper bound $j \geq i_\ell$ for all ℓ . In the quotient, we are justified in replacing $\lambda_i(a_i)$ by $\lambda_j(\hat{a}_i)$, where \hat{a}_i is the image of a_i in A_j . Thus, we can write our element a as

$$a = \lambda_j \left(\sum_{i \in I} \hat{a}_i \right),$$

where the sum is over elements of A_j , and hence itself belongs to A_j .

4. Some important derived functors

2.

□

Theorem 120. Let I be a filtered poset. The colimit functor

$$\operatorname{colim}: R\text{-}\mathbf{Mod}^I \rightarrow R\text{-}\mathbf{Mod}$$

is exact.

Proof. By general abstract nonsense, we know that colimits commute with colimits, and thus that the colimit functor preserves colimits, hence is right exact. We need only show that colim preserves monics.

To this end, let $t \in \operatorname{Hom}_{R\text{-}\mathbf{Mod}^I}(A, B)$ be monic, and choose some $a \in \operatorname{colim} A$ such that $t(a) = 0$ in $\operatorname{colim} B$. By [Lemma 119](#), there exists some $a_i \in A_i$ which maps to a under the canonical map $A_i \rightarrow \operatorname{colim} A$. Since each $t_i: A_i \rightarrow B_i$ is monic this maps to some $t_i(a_i) \neq 0$ in B_i . The element $t_i(a_i)$ vanishes in $\operatorname{colim} B$, so there exists some $j \in I$ and $\phi: i \rightarrow j$ such that

$$0 = B(\phi)(t_i(a_i)) = t_j(A(\phi)(a_i)) \quad \text{in } B_j.$$

However, t_j is monic, so $A(\phi)(a_i) = 0$ in A_j . Thus a maps to zero in $\operatorname{colim} A$. □

Corollary 121. The functor

$$\operatorname{colim}: \mathbf{Ch}(R\text{-}\mathbf{Mod})^I \rightarrow \mathbf{Ch}(R\text{-}\mathbf{Mod})$$

is exact.

Proof. Colimits are computed level-wise. □

Since direct limits are exact, there is no need to take derived functors. However, the situation for limits is not so pretty.

4.3.2. Inverse limits

Let I be a small category, and let \mathcal{A} be an abelian category such that for every set-indexed collection $\{A_s\}_{s \in S}$, the product $\prod_{s \in S} A_s$ exists. Then \mathcal{A}^I is an abelian category (which we have seen in REF) and has enough injectives (which we will not show unless we have time).

The limit functor

$$\lim: \mathcal{A}^I \rightarrow \mathcal{A}$$

commutes with limits, hence is left exact. We can form the right derived functors

$$\lim^i := R^i \lim: \mathcal{A}^I \rightarrow I.$$

For us it will be sufficient to consider the case in which $I = \mathbb{N}$ and $\mathcal{A} = \mathbf{Ab}$, so a functor $I \rightarrow \mathcal{A}$ is a sequence of abelian groups and morphisms as follows.

$$\cdots \rightarrow A_2 \rightarrow A_1 \rightarrow A_0$$

We will call functors $\mathbb{N} \rightarrow \mathbf{Ab}$ *towers*, and denote $A: \mathbb{N} \rightarrow \mathbf{Ab}$ by $\{A_i\}$.

Lemma 122. Let $\{A_i\}$ be a tower in \mathbf{Ab} , and denote $A(j \rightarrow i) = \lambda_{j,i}$. We can express the limit $\lim_i A_i$ as a kernel

$$\ker \Delta \hookrightarrow \prod_{i=0} A_i \xrightarrow{\Delta} \prod_{i=0} A_i$$

where Δ is the map

$$(\dots, a_n, \dots) \mapsto (\dots, a_n - \lambda_{n+1,n} a_{n+1}, \dots).$$

Proof. We can express any limit as a kernel

$$\ker \xi \hookrightarrow \prod_{a \in \mathbb{N}} A_a \xrightarrow{\Delta} \prod_{(m,n) \in D} A_m$$

where ξ is defined on components by

$$\xi: \prod_{a \in \mathbb{N}} A_a \rightarrow A_{\text{cod}(m,n)=n}; \quad (\dots, a_n, \dots) \mapsto a_m - \lambda_{n,m} a_n$$

where $D = \{(n, m) \in \mathbb{N} \times \mathbb{N} \mid m < n\}$ indexes morphisms in \mathbb{N} . An element

$$a = (\dots, a_n, \dots, a_1, a_0)$$

is in the kernel of ξ if and only if

$$a_m - \lambda_{n,m}(a_n) = 0$$

for all $(m, n) \in D$. But because $\lambda_{k,j} \circ \lambda_{j,i} = \lambda_{k,i}$, this is true if and only

$$a_m - \lambda_{m+1,m}(a_{m+1}) = 0$$

for all m . But we can view this as a function from $\prod_{a \in \mathbb{N}} A_a$ to itself, giving us the required formula. \square

Let

$$0 \longrightarrow \{A_i\} \hookrightarrow \{B_i\} \twoheadrightarrow \{C_i\} \longrightarrow 0$$

be a short exact sequence of towers, and consider the following morphism of short exact

4. Some important derived functors

sequences.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \prod_i A_i & \hookrightarrow & \prod_i B_i & \twoheadrightarrow & \prod_i C_i \longrightarrow 0 \\
 & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\
 0 & \longrightarrow & \prod_i A_i & \hookrightarrow & \prod_i B_i & \twoheadrightarrow & \prod_i C_i \longrightarrow 0
 \end{array}$$

The snake lemma gives us a long exact sequence

$$0 \rightarrow \ker \Delta_A \rightarrow \ker \Delta_B \rightarrow \ker \Delta_C \rightarrow \operatorname{coker} \Delta_A \rightarrow \operatorname{coker} \Delta_B \rightarrow \operatorname{coker} \Delta_C \rightarrow 0.$$

We have already seen that $\ker \Delta_A = \lim\{A_i\}$. This tells us that

$$\lim^i = \begin{cases} \lim, & i = 0 \\ \operatorname{coker} \Delta & i = 1 \\ 0, & \text{otherwise} \end{cases}$$

is a homological δ -functor. In fact, it turns out to be universal, hence the derived functor of \lim .

Lemma 123. Suppose all the maps $\lambda_{i,i+1}: A_{i+1} \rightarrow A_i$ are epimorphisms. Then $\lim^1 A_i = 0$.

We need to show that Δ is an epimorphism. To that end, let

$$(\cdots, a_i, \cdots) \in \prod_i A_i.$$

Let $b_0 = a_0$. Define b_{i+1} inductively as a lift of $b_i - a_i$. Then Δ maps (\cdots, b_i, \cdots) to

$$(\cdots, b_i - (b_i - a_i), \cdots) = (\cdots, a_i, \cdots).$$

Definition 124 (Mittag-Leffler condition). Let $\{A_i\}$ be a tower of abelian groups. We say that $\{A_i\}$ satisfies the Mittag-Leffler condition if for each $n \geq 0$ there exists a $N = N(n) > n$ such that for all $m \geq N$, the image of $A_m \rightarrow A_n$ is equal to the image of $A_N \rightarrow A_n$.

We say that $\{A_i\}$ satisfies the trivial Mittag-Leffler condition if for all $n \geq 0$ there exists $N(n) > n$ such that $\operatorname{im}(A_{N(n)} \rightarrow A_n) = 0$.

Example 125. If all the maps $A_n \rightarrow A_{n-1}$ are epimorphisms, then $\{A_i\}$ satisfies the Mittag-Leffler condition with $N(n) = n + 1$.

Proposition 126. Suppose $\{A_i\}$ is a tower of abelian groups satisfying the Mittag-Leffler condition. Then

$$\lim^1 A_i = 0.$$

Proof. First, suppose that A_i satisfies the trivial Mittag-Leffler condition. We need to show that $\lim^1 A_i = 0$, i.e. that Δ is epi.

To that end, choose some $a_i \in A_i$, $i \geq 0$. We need to find $b_i \in A_i$, $i \geq 0$ such that

$$\Delta: (\dots, b_1, b_0) \mapsto (\dots, a_1, a_0).$$

Define

$$b_n = \sum_{m \geq n} \lambda_{n,m} a_m.$$

Note that this is a finite sum because $\lambda_{n,m} = 0$ for large enough m . Then

$$\begin{aligned} \Delta(\dots, b_n, \dots) &= \left(\dots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \lambda_{n,n+1} \sum_{m \geq n+1} \lambda_{m,n+1} a_m \right) \\ &= \left(\dots, \sum_{m' \geq n} \lambda_{n,m'} a_{m'} - \sum_{m \geq n+1} \lambda_{n,m} a_m \right) \\ &= (\dots, a_n, \dots). \end{aligned}$$

Now suppose that A_i satisfies the ordinary Mittag-Leffler condition. Without loss of generality we may assume that $N(n)$ is nondecreasing. Define

$$B_i = \lambda_{i,N(i)}(A_{N(i)}).$$

Consider the following short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_{i+1} & \hookrightarrow & A_{i+1} & \twoheadrightarrow & A_{i+1}/B_{i+1} \longrightarrow 0 \\ & & & & \downarrow \lambda_{i,i+1} & & \\ 0 & \longrightarrow & B_i & \hookrightarrow & A_i & \twoheadrightarrow & A_i/B_i \longrightarrow 0 \end{array}$$

Because N is nondecreasing, the λ s give us a map $B_{i+1} \rightarrow B_i$ making the diagram commute. The universal property for cokernels gives us an induced map $A_{i+1}/B_{i+1} \rightarrow A_i/B_i$ making the diagram commute.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_{i+1} & \hookrightarrow & A_{i+1} & \twoheadrightarrow & A_{i+1}/B_{i+1} \longrightarrow 0 \\ & & \downarrow & & \downarrow \lambda_{i,i+1} & & \downarrow \\ 0 & \longrightarrow & B_i & \hookrightarrow & A_i & \twoheadrightarrow & A_i/B_i \longrightarrow 0 \end{array}$$

This defines a tower $\{C_i\}$ with $C_i = A_i/B_i$. By definition, this satisfies the trivial Mittag-Leffler condition.

4. Some important derived functors

Note that by definition, the maps of the tower $\{B_i\}$ are onto, so

$$\lim^1 B_i = 0.$$

The long exact sequence associated to the short exact sequence

$$0 \longrightarrow B_i \hookrightarrow A_i \twoheadrightarrow A_i/B_i \longrightarrow 0$$

thus tells us that $\lim^1 A_i = 0$, as it is sandwiched between two things which are zero. \square

Example 127. Consider the tower

$$\cdots \longrightarrow \mathbb{Z}/(p^3) \longrightarrow \mathbb{Z}/(p^2) \longrightarrow \mathbb{Z}/(p)$$

Theorem 128 (Milnor sequence). Let

$$\cdots \longrightarrow C_1 \longrightarrow C_0$$

be a tower of cochain complexes of abelian groups such that the Mittag-Leffler condition is satisfied (i.e. satisfied level-wise). Let $C = \lim C_i$. Then for each q there is an exact sequence

$$0 \longrightarrow \lim^1 H^{q-1}(C_i) \hookrightarrow H^q(C) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0 .$$

Proof. We first fix notation. Let

$$B_i \subset Z_i \subset C_i$$

be the subcomplexes of C_i on boundaries and cycles, with trivial differentials; then

$$H^\bullet(C_i) = \text{coker}(B_i \rightarrow Z_i).$$

Let B denote the level-wise coboundaries of C , and Z the level-wise cocycles, again with trivial differentials.

- Consider the exact sequence

$$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \longrightarrow \{C_i[1]\} .$$

Applying the left exact functor \lim , we find that

$$\lim Z_i \cong \ker(d: C \rightarrow C[1]),$$

i.e. that $\lim Z_i$ is the subcomplex of cycles in C ; that is, that

$$Z = \lim Z_i.$$

- Consider the short exact sequence

$$0 \longrightarrow \{Z_i\} \hookrightarrow \{C_i\} \twoheadrightarrow \{B_i[1]\} \longrightarrow 0 .$$

The long exact sequence corresponding to \lim^i is

$$\begin{array}{ccccccc} 0 & \longrightarrow & Z & \longrightarrow & C & \longrightarrow & \lim B_i[1] \\ & & & & & \searrow \delta & \\ & & \lim^1 Z_i & \longrightarrow & \lim^1 C_i & \longrightarrow & \lim^1 B_i[1] \longrightarrow 0 \end{array}$$

But since C_i satisfies the Mittag-Leffler condition, $\lim^1 C_i = 0$, implying that

$$\lim^1 B_i = 0.$$

Furthermore, $B[1] \cong \text{coker}(Z \hookrightarrow C)$, so image-kernel factorization gives us

$$0 \longrightarrow B[1] \hookrightarrow \lim B_i[1] \twoheadrightarrow \lim^1(Z_i) \longrightarrow 0$$

- The long exact sequence on homology corresponding to

$$0 \longrightarrow \{B^i\} \hookrightarrow \{Z^i\} \twoheadrightarrow \{H^i\} \longrightarrow 0$$

is

$$\begin{array}{ccccccc} 0 & \longrightarrow & \lim B_i & \longrightarrow & Z & \longrightarrow & \lim H_i \\ & & & & & \searrow \delta & \\ & & 0 & \longrightarrow & \lim^1 Z^i & \longrightarrow & \lim^1 H^i \longrightarrow 0 \end{array}$$

This tells us both that

$$0 \longrightarrow \lim B_i \hookrightarrow Z \twoheadrightarrow \lim H_i \longrightarrow 0$$

is short exact, *and* that

$$\lim^1 Z^i \cong \lim^1 H_i.$$

The exact sequences we have discovered give us the chain of inclusions

$$0 \subset B \subset \lim B_i \subset Z \subset C.$$

The third isomorphism theorem or one of the axioms for triangulated categories or

4. Some important derived functors

whatever gives that the sequence

$$0 \longrightarrow \frac{\lim B_i}{B} \hookrightarrow \frac{Z}{B} \twoheadrightarrow \frac{Z}{\lim B_i} \longrightarrow 0$$

is short exact. However,

$$\frac{\lim B_i}{B} \cong \lim^1 Z_i \cong \lim^1 H_i \quad \frac{Z}{B} \cong H, \quad \frac{Z}{\lim B_i} \cong \lim H_i.$$

As every chain complex we are working with has trivial differentials, this amounts to the statement needed. \square

5. Basic applications

You know how it is when someone asks you to ride in a terrific sports car, and then you wish you hadn't?

John Adams

5.1. Syzygy theorem

Let $R = \mathbb{C}[x_1, \dots, x_k]$. Following Hilbert, we consider a finitely generated¹ R -module M .

By Hilbert's basis theorem, M is Noetherian, hence has finitely many generators a_1, \dots, a_k .

Denote by R^k the free R -module with k generators e_1, \dots, e_k , and consider

$$\phi: R^k \rightarrow M; \quad e_i \mapsto a_i.$$

This map is surjective, but it may have a kernel, called the *module of first Syzygies*, denoted $\text{Syz}^1(M)$. This kernel consists of those x such that

$$x = \sum_{i=1}^k \lambda_i e_i \xrightarrow{\phi} \sum_{i=1}^k \lambda_i a_i = 0.$$

This kernel is again a module, the module of relations.

By Hilbert's basis theorem (since polynomial rings over a field are Noetherian), $\text{Syz}^1(M)$ is again a finitely generated R -module. Hence, pick finite set of generators b_i , $1 \leq i \leq \ell$, and look at

$$R^\ell \rightarrow \text{Syz}^1(M); \quad e_i \mapsto b_i.$$

In general, there is non-trivial kernel, denoted $\text{Syz}^2(M)$. This consists of relation between relations.

¹i.e. expressible as an R -linear combination of finitely many generators.

5. Basic applications

We can keep going. This gives us a sequence $\text{Syz}^\bullet(M)$ corresponding to relations between relations between...between relations.

$$\begin{array}{ccccccc}
 & & \text{Syz}^2(M) & & & & \\
 & \nearrow & & \searrow & & & \\
 \cdots & \nearrow & R^m & \xrightarrow{d} & R^l & \xrightarrow{d} & R^k \longrightarrow M \\
 & \nwarrow & & & \searrow & & \nearrow \\
 & & \text{Syz}^3 & & \text{Syz}^1(M) & &
 \end{array}$$

From this we build a chain complex (C_\bullet, d) , where C_n corresponds to the n th copy of R^k , known as the *free resolution* of M .

Hilbert's Syzygy theorem tells us that every finitely generated $\mathbb{C}[x_1, \dots, x_n]$ -module admits a free resolution of length at most n .

Definition 129 (Koszul complex). Let R be a commutative ring, and let $x \in R$ be an element of R such that left multiplication by x is injective on first homology. The Koszul complex of x is the complex

$$K(x) = \cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\cdot x} R \longrightarrow 0 \longrightarrow \cdots,$$

concentrated in degrees 0 and 1. For elements x_1, \dots, x_n , we define the Koszul complex of x_1, \dots, x_n to be

$$K(x_1, \dots, x_n) = K(x_1) \otimes_R \cdots \otimes_R K(x_n).$$

Lemma 130. The Koszul complex $K(x)$ provides a free resolution of $R/(x)$.

Proof. There is an easy spectral sequence argument. □

5.2. Quiver representations

In the remainder of this chapter, we will apply homological algebra to the study of various classes of objects. Our main goal will be to implement the following cockamamie scheme in two contexts: quiver representations and groups.

Cockamamie Scheme 131. Given a category of algebraic objects:

1. Find a way of turning the objects in question into rings.
2. Study modules over those rings using homological algebra.
3. ???
4. Profit.

The first problem to which we will apply [Cockamamie Scheme 131](#) is the study of *quiver representations*. Unfortunately, due to time constraints, we will only get as far as step 2.

Definition 132 (quiver). A quiver Q consists of the following data.

- A finite² set Q_0 of *vertices*
- A finite set Q_1 of *arrows*
- A pair of maps

$$s, t: Q_1 \rightarrow Q_0$$

called the *source* and *target* maps respectively.

Definition 133 (quiver representation). Let Q be a quiver, and let k be a field. A k -linear representation V of Q consists of the following data.

- For every vertex $x \in Q_0$, a vector space V_x .
- For every arrow $\rho \in Q_1$, a k -linear map $V_{s(\rho)} \rightarrow V_{t(\rho)}$.

A morphism f between k -linear representations V and W consists of, for each vertex $x \in Q_0$ a linear map $f_x: V_x \rightarrow W_x$, such that for edge $s \in Q_1$, the square

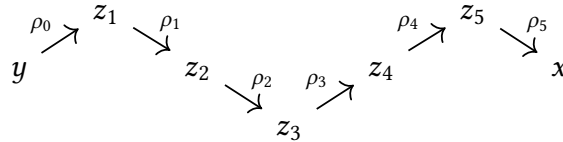
$$\begin{array}{ccc} V_{s(\rho)} & \longrightarrow & V_{t(\rho)} \\ \downarrow & & \downarrow \\ W_{s(\rho)} & \longrightarrow & W_{t(\rho)} \end{array}$$

commutes.

This allows us to define a category $\text{Rep}_k(Q)$ of k -linear representations of Q .

Following our grand plan, we first find a way of turning a quiver representation into a ring.

First, we form the free category over the quiver Q , which we denote by $\mathcal{F}(Q)$; that is, the objects of the category $\mathcal{F}(Q)$ are the vertices of Q , and the morphisms $x \rightarrow y$ consist of finite chains of arrows starting at y and ending at x .



We will denote this morphism by

$$\rho_5 \rho_4 \rho_3 \rho_2 \rho_1 \rho_0.$$

²There is no fundamental reason for this finiteness condition, except that it will make the following analysis more convenient.

5. Basic applications

We call such a finite non-empty chain a *non-trivial path*. Note that in addition to the non-trivial paths, we must formally adjoin identity arrows e_x with $s(e_x) = t(e_x) = x$; these are the *trivial paths*. Note that the notation introduced above is a good one, since in the category $\mathcal{F}(Q)$ composition is given by

$$(\rho_n \cdots \rho_1) \circ (\rho'_m \cdots \rho'_1) = \rho_n \cdots \rho_1 \rho'_m \cdots \rho'_1.$$

Definition 134 (path algebra). Let k be a field and Q a quiver. The path algebra of Q over k is, as a vector space, the free vector space over the set $\text{Mor}(\mathcal{F}(Q))$. The multiplication is given by composition

$$\rho \cdot \sigma = \begin{cases} \rho\sigma, & s(\rho) = t(\sigma) \\ 0, & s(\rho) \neq t(\sigma). \end{cases}$$

We will denote the path algebra of Q over k by kQ .

Another way of expressing the above composition rule is as follows: the multiplication of two paths ρ, σ in kQ is given by the composition $\rho \circ \sigma$ in the category $\mathcal{F}(Q)$ if the morphisms ρ and σ are composable (in the sense that ρ begins where σ ends) and zero otherwise.

From now on, we fix (arbitrarily) a bijection of Q with $\{1, 2, \dots, n\}$. It is clear that

$$\text{id}_{kQ} = e_1 + \cdots + e_n.$$

Example 135. Consider the following quiver.



Then paths are in correspondence with natural numbers, with the empty path being e_1 . The path algebra is therefore

$$kQ \cong ke_1 \oplus k\rho \oplus k\rho\rho \oplus k\rho\rho\rho \oplus \cdots \cong k[\rho].$$

Example 136. Let Q be any quiver, and denote by $A = kQ$ its path algebra. For any vertex i and any path ρ , the multiplication law for the path algebra tells us that

$$\rho e_i = \begin{cases} \rho, & s(\rho) = i \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the left ideal Ae_i consists of those paths which start at i . Similarly, the right ideal $e_i A$ consists of those paths which terminate at i .

Since any path either starts at i or doesn't, we have

$$A \cong Ae_i \oplus A(1 - e_i).$$

This means (by [Example 63](#)) that each Ae_i is a projective module. The same argument tells us that $e_i A$ is projective for all i .

Proposition 137. Let Q be a quiver and k its path algebra. There is an equivalence of categories

$$\text{Rep}_k(Q) \xrightarrow{\simeq} kQ\text{-Mod}.$$

Proof. First, we construct out of any k -linear representation V of Q a kQ -module. Our underlying space will be the vector space $\bigoplus_{i \in Q_0} V_i$, which we will also denote by V .

Now let $\rho: i \rightarrow j \in Q_0$, corresponding under the representation V to the map $f: V_i \rightarrow V_j$. We send this to the map $V \rightarrow V$ given by the matrix with f in the (i, j) th position and zeroes elsewhere. In particular, we send e_i to the matrix with a 1 in the i th place along the diagonal, and zeroes elsewhere.

It is easy to see that this assignment respects addition and composition, hence defines our functor on objects. On objects, we send $\alpha_i: V_i \rightarrow W_i$ to

$$\text{diag}(0, \dots, 0, \alpha_x^i, 0, \dots, 0).$$

It is painfully clear that this is the right thing to do.

We now construct a functor in the opposite direction. □

Proposition 138. Let M be a left A -module. We have an exact sequence of left A -modules

$$0 \longrightarrow \bigoplus_{\rho \in Q_1} Ae_{t(\rho)} \otimes_k e_{s(\rho)} M \hookrightarrow \bigoplus_{i \in Q_0} Ae_i \otimes_k e_i M \twoheadrightarrow M \longrightarrow 0,$$

exhibiting a resolution of M by projective A -modules.

Corollary 139. Let M, N be left A -modules. Then

$$\text{Ext}^i(M, N) = 0, \quad i \geq 2.$$

5.3. Group (co)homology

Next, we will apply [Cockamamie Scheme 131](#) to the study of groups. In order to do this, we need some theory.

5.3.1. The Dold-Kan correspondence

The Dold-Kan correspondence, and its stronger, better-looking cousin the Dold-Puppe correspondence, tell us roughly that studying bounded-below chain complexes is the same as studying simplicial objects.

5. Basic applications

Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group, and define

$$NA_n = \bigcap_{i=0}^{n-1} \ker(d_i) \subset A_n.$$

This becomes a chain complex when given the differential $(-1)^n d_n$. This is easy to see; we have

$$d_{n-1} \circ d_n = d_{n-1} \circ d_{n-1},$$

and the domain of this map is contained in $\ker d_{n-1}$.

Definition 140 (normalized chain complex). Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group. The normalized chain complex of A is the following chain complex.

$$\cdots \xrightarrow{(-1)^{n+2} d_{n+2}} NA_{n+1} \xrightarrow{(-1)^{n+1} d_{n+1}} NA_n \xrightarrow{(-1)^n d_n} NA_{n-1} \xrightarrow{(-1)^{n-1} d_{n-1}} \cdots$$

Definition 141 (Moore complex). Let $A: \Delta^{\text{op}} \rightarrow \mathbf{Ab}$ be a simplicial abelian group. The Moore complex of A is the chain complex with n -chains A_n and differential

$$\partial = \sum_{i=0}^n (-1)^i d_i.$$

This is a bona fide chain complex due to the following calculation.

$$\begin{aligned} \partial^2 &= \left(\sum_{j=0}^{n-1} (-1)^j d_j \right) \circ \left(\sum_{i=0}^n (-1)^i d_i \right) \\ &= \sum \sum (-1)^{i+j} d_j \circ d_i \\ &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_j \circ d_i + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\ &= \sum_{0 \leq j < i \leq n} (-1)^{i+j} d_{i-1} \circ d_j + \sum_{0 \leq i \leq j \leq n-1} (-1)^{i+j} d_j \circ d_i \\ &= 0. \end{aligned}$$

Following Goerss-Jardine, we denote the Moore complex of A simply by A , unless this is confusing. In this case, we will denote it by MA .

Definition 142 (alternating face maps chain modulo degeneracies). Denote by DA_n the subgroup of A_n generated by degenerate simplices. Since $\partial: A_n \rightarrow A_{n-1}$ takes degenerate simplices to linear combinations of degenerate simplices, it descends to a chain map.

$$\cdots \xrightarrow{[\partial]} A_{n+1}/DA_{n+1} \xrightarrow{[\partial]} A_n/DA_n \xrightarrow{[\partial]} A_{n-1}/DA_{n-1} \xrightarrow{[\partial]} \cdots$$

We denote this chain complex by $A/D(A)$, and call it the alternating face maps chain

modulo degeneracies.³

Lemma 143. We have the following map of chain complexes, where i is inclusion and p is projection.

$$NA \xhookrightarrow{i} A \xrightarrow{p} A/D(A)$$

Proof. We need only show that the following diagram commutes.

$$\begin{array}{ccccc} NA_n & \hookrightarrow & A_n & \twoheadrightarrow & A_n/DA_n \\ (-1)^n d_n \downarrow & & \downarrow \partial & & \downarrow [\partial] \\ NA_{n-1} & \hookrightarrow & A_{n-1} & \twoheadrightarrow & A_{n-1}/DA_{n-1} \end{array}$$

The left-hand square commutes because all differentials except d_n vanish on everything in NA_n , and the right-hand square commutes trivially. \square

Theorem 144. The composite

$$p \circ i: NA \rightarrow A/D(A)$$

is an isomorphism of chain complexes.

Let $f: [m] \rightarrow [n]$ be a morphism in the simplex category Δ . By functoriality, f induces a map

$$N\Delta^n \rightarrow N\Delta^m.$$

In fact, this map is rather simple. Take, for example, $f = d_0: [n] \rightarrow [n-1]$. By definition, this map gives

We define a simplicial object

$$\mathbb{Z}[-]: \Delta \rightarrow \mathbf{Ch}_+(\mathbf{Ab})$$

on objects by

$$[n] \mapsto N\mathcal{F}(\Delta^n),$$

and on morphisms by

$$f: [m] \rightarrow [n] \mapsto$$

which takes each object $[n]$ to the corresponding normalized complex on the free abelian group on the corresponding simplicial set.

We then get a nerve and realization

$$N: \mathbf{Ab}_\Delta \longleftrightarrow \mathbf{Ch}_+(\mathbf{Ab}) : \Gamma. \quad (5.1)$$

³The nLab is responsible for this terminology.

5. Basic applications

Theorem 145 (Dold-Kan). The adjunction in Equation 5.1 is an equivalence of categories

Proof. later, if I have time. □

Also later if time, Dold-Puppe correspondence: this works not only for \mathbf{Ab} , but for any abelian category.

5.3.2. The bar construction

This section assumes a basic knowledge of simplicial sets. We will denote by Δ_+ the extended simplex category, i.e. the simplex category which includes $[-1] = \emptyset$.

Let \mathcal{C} and \mathcal{D} be categories, and

$$L : \mathcal{C} \leftrightarrow \mathcal{D} : R$$

an adjunction with unit $\eta : \text{id}_{\mathcal{C}} \rightarrow RL$ and counit $\varepsilon : LR \rightarrow \text{id}_{\mathcal{D}}$.

Recall that this data gives a comonad in \mathcal{D} , i.e. a comonoid internal to the category $\text{End}(\mathcal{D})$, or equivalently, a monoid LR internal to the category $\text{End}(\mathcal{D})^{\text{op}}$.

$$LRLR \xleftarrow{L\eta R} LR \xrightarrow{\varepsilon} \text{id}_{\mathcal{D}}$$

The category Δ_+ is the free monoidal category on a monoid; that is, whenever we are given a monoidal category \mathcal{C} and a monoid $M \in \mathcal{C}$, it extends to a monoidal functor $\tilde{M} : \Delta \rightarrow \mathcal{C}$.

$$\begin{array}{ccc} \{*\} & \xrightarrow{[0]} & \Delta_+ \\ & \searrow M & \downarrow \exists! \tilde{M} \\ & & \mathcal{C} \end{array}$$

In particular, with $\mathcal{C} = \text{End}(\mathcal{D})^{\text{op}}$, the monoid LR extends to a monoidal functor $\Delta_+ \rightarrow \text{End}(\mathcal{D})^{\text{op}}$.

$$\begin{array}{ccc} \{*\} & \xrightarrow{[0]} & \Delta_+ \\ & \searrow M & \downarrow \exists! \\ & & \text{End}(\mathcal{D})^{\text{op}} \end{array}$$

Equivalently, this gives a functor

$$B : \Delta_+^{\text{op}} \rightarrow \text{End}(\mathcal{D}).$$

For each $d \in \mathcal{D}$, there is an evaluation map

$$\text{ev}_d: \text{End}(\mathcal{D}) \rightarrow \mathcal{D}.$$

Composing this with the above functor gives, for each object $d \in \mathcal{D}$, a simplicial object in \mathcal{D} .

$$S_d = \Delta_+^{\text{op}} \xrightarrow{B} \text{End}(\mathcal{D}) \xrightarrow{\text{ev}_d} \mathcal{D}$$

Definition 146 (bar construction). The composition $\text{ev}_d \circ B$ is called the bar construction.

Example 147. I believe this example comes from John Baez, although I can't find the exact source at the moment.

Consider the free-forgetful adjunction

$$F: \mathbf{Set} \longleftrightarrow \mathbf{Ab}: G.$$

The unit $\eta: \text{id}_{\mathbf{Set}} \rightarrow U \circ F$ has components

$$\eta_S: S \rightarrow \mathbb{Z}[S]$$

which assigns any set S to the set underlying the free abelian group on it by mapping elements to their corresponding generators. The counit $\varepsilon: F \circ U \rightarrow \text{id}_{\mathbf{Ab}} \rightarrow U \circ F$ has components

$$\varepsilon_A: \mathbb{Z}[A] \rightarrow A$$

which takes formal linear combinations of group elements to actual linear combinations of group elements.

Denoting $F \circ U$ by Z , we find the following comonad in \mathbf{Ab} , where $\nabla = F\eta U$.

$$\begin{array}{c} Z \circ Z \\ \uparrow \nabla \\ Z \\ \downarrow \varepsilon \\ \text{id}_{\mathbf{Ab}} \end{array}$$

This gives us a simplicial object in $\text{End}(\mathbf{Ab})$.

Post-composing with the evaluation functor $\text{ev}_{\mathbb{Z}}$ gives the following simplicial object in \mathbf{Ab} .

$$\begin{array}{ccccccc} & & \xrightarrow{(ZZ\varepsilon)_{\mathbb{Z}}} & & \xrightarrow{(Z\varepsilon)_{\mathbb{Z}}} & & \\ & & \xleftarrow{\quad} & & \xleftarrow{\nabla_{\mathbb{Z}}} & & \\ \cdots & \mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]] & \xleftarrow{\quad} & \mathbb{Z}[\mathbb{Z}[\mathbb{Z}]] & \xleftarrow{\quad} & \mathbb{Z}[\mathbb{Z}] & \xrightarrow{\varepsilon_{\mathbb{Z}}} \mathbb{Z} \\ & & \xrightarrow{\quad} & & \xrightarrow{(\varepsilon Z)_{\mathbb{Z}}} & & \\ & & \xleftarrow{(\varepsilon ZZ)_{\mathbb{Z}}} & & & & \end{array}$$

5. Basic applications

Let's examine some of these objects in detail.

- The first object is the friendly and reassuring \mathbb{Z} , whose elements are integers. An example of an element of \mathbb{Z} is

$$17.$$

- The next object, $\mathbb{Z}[\mathbb{Z}]$, consists of formal linear combinations of elements of \mathbb{Z} , i.e. formal sums of the form

$$3 \times (5) + 2 \times (-1).$$

The funny notation of enclosing the elements of (the set underlying) \mathbb{Z} in parentheses will come in handy shortly.

- The next object, $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, consists of formal linear combinations of elements of $\mathbb{Z}[\mathbb{Z}]$, i.e. formal linear combinations of formal linear combinations of integers. An element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$ looks like this.

$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$

- I wonder if you can figure out for yourself what the elements of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]]$ are.

Next, we should understand the face maps in some low degrees.

Consider some element of $\mathbb{Z}[\mathbb{Z}]$, say

$$3 \times (5) + 2 \times (-1)$$

The map $\varepsilon_{\mathbb{Z}} = d_0$ collapses the formal multiplication into actual multiplication:

$$\varepsilon_{\mathbb{Z}}: 3 \times (5) + 2 \times (-1) \mapsto 15 - 2 = 13.$$

Consider now some element of $\mathbb{Z}[\mathbb{Z}[\mathbb{Z}]]$, say

$$3 \times (2 \times (1) + 3 \times (5)) - 4 \times (1 \times (3)).$$

The map $\mathbb{Z}\varepsilon_{\mathbb{Z}} = d_0$ sends this to

$$3 \times (17) - 4 \times (3);$$

that is, it strips off the inner-most parentheses. The map $(\varepsilon Z)_{\mathbb{Z}} = d_1$ sends it to

$$6 \times (1) + 9 \times (5) - 4 \times (3);$$

that is, it strips off the outer-most parentheses.

Note that, as the simplicial identities promised us, $\epsilon_{\mathbb{Z}}$ sends both of these to

$$3 \times 17 - 4 \times 3 = 6 \times 1 + 9 \times 5 - 4 \times 3 = 39.$$

In the case that the category \mathcal{D} is abelian, we can form from a simplicial object the chain complex of alternating face maps modulo degeneracies.

Definition 148 (bar complex). Let

$$F : \mathcal{A} \longleftrightarrow \mathcal{B} : G$$

be an adjunction between abelian categories, let $b \in \mathcal{B}$, and denote by $S_b : \Delta^{\text{op}} \rightarrow \mathbf{Set}$ the bar construction of b .

The bar complex Bar_b of b is the chain complex of face maps modulo degeneracies (Definition 142) of S_b .

We will see an example of the bar complex in the next chapter.

5.3.3. Invariants and coinvariants

Now we are ready to apply [Cockamamie Scheme 131](#) to the study of groups. Here a somewhat popular method exists for turning a group G into a ring, namely by constructing the group ring $\mathbb{Z}G$. As is tradition, we will study modules over this ring.

In [Section 5.2](#), we had a nice interpretation of modules over our rings, as corresponding to k -linear representations of a quiver. We will work backwards, finding another perspective on modules over the group ring $\mathbb{Z}G$ of a group G as G -modules.

Given a group G , a G -module is a functor $\mathbf{BG} \rightarrow \mathbf{Ab}$. More generally, the category of such G -modules is the category $\mathbf{Fun}(\mathbf{BG}, \mathbf{Ab})$. We will denote this category by $G\text{-Mod}$.

More explicitly, a G -module consists of an abelian group M , and a left G -action on M . However, since we are in \mathbf{Ab} , we have more structure immediately available to us: we can define for $n \in \mathbb{Z}$, $g \in G$ and $a \in M$,

$$(ng)a = n(ga),$$

and, extending by linearity, a $\mathbb{Z}G$ -module structure on M .

Because of the obvious equivalence $G\text{-Mod} \simeq \mathbb{Z}G\text{-Mod}$, we know that $G\text{-Mod}$ has enough projectives.

Note that there is a canonical functor $\text{triv} : \mathbf{Ab} \rightarrow \mathbf{Fun}(\mathbf{BG}, \mathbf{Ab})$ which takes an abelian group to the associated constant functor.

5. Basic applications

Proposition 149. The functor triv has left adjoint

$$(-)_G: M \mapsto M_G = M / \langle g \cdot m - m \mid g \in G, m \in M \rangle$$

and right adjoint

$$(-)^G: M \mapsto M^G = \{m \in M \mid g \cdot m = m\}.$$

That is, there is an adjoint triple

$$(-)_G \longleftrightarrow \text{triv} \longleftrightarrow (-)^G$$

Proof. In each case, we exhibit a hom-set adjunction. In the first case, we need a natural isomorphism

$$\text{Hom}_{\text{Ab}}(M_G, A) \equiv \text{Hom}_{G\text{-Mod}}(M, \text{triv } A).$$

Starting on the left with a homomorphism $\alpha: M_G \rightarrow A$, the universal property for quotients allows us to replace it by a homomorphism $\hat{\alpha}: M \rightarrow A$ such that $\hat{\alpha}(g \cdot m - m) = 0$ for all $m \in M$ and $g \in G$; that is to say, a homomorphism $\hat{\alpha}: M \rightarrow A$ such that

$$\hat{\alpha}(g \cdot m) = \hat{\alpha}(m). \quad (5.2)$$

However, in this form it is clear that we may view $\hat{\alpha}$ as a G -linear map $M \rightarrow \text{triv } A$. In fact, G -linear maps $M \rightarrow \text{triv } A$ are precisely those satisfying [Equation 5.2](#), showing that this is really an isomorphism.

The other case is similar. □

Definition 150 (invariants, coinvariants). For a G -module M , we call M^G the invariants of M and M_G the coinvariants.

Clearly, the functors taking invariants and coinvariants are interesting things that deserve to be studied in their own right, and in keeping with [Cockamamie Scheme 131](#) we could procede by deriving them. However, now something of a miracle occurs: these functors are really special cases of the hom and tensor product!

Lemma 151. We have the formulae

$$(-)^G \simeq \text{Hom}_{\mathbb{Z}G\text{-Mod}}(\mathbb{Z}, -)$$

and

$$(-)_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} -,$$

where in the first formula \mathbb{Z} is taken to be a trivial left $\mathbb{Z}G$ -module, and in the second a trivial right $\mathbb{Z}G$ -module.

Proof. A $\mathbb{Z}G$ -linear map $\mathbb{Z} \rightarrow M$ picks out an element of M on which G acts trivially.

Consider the element

$$1 \otimes (m - g \cdot m) \in \mathbb{Z} \otimes_{\mathbb{Z}G} M.$$

By $\mathbb{Z}G$ -linearity, this is equal to $1 \otimes m - 1 \otimes m = 0$. \square

Lemma 151 means that we already have a good handle on the derived functors of $(-)_G$ and $(-)^G$: they are given by the Ext and Tor functors we have already seen. More specifically,

$$L_i(-)_G = \text{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \quad R^i(-)^G = \text{Ext}_{\mathbb{Z}G}^i(\mathbb{Z}, A).$$

This is a stroke of luck. Had we proceeded naïvely by forming the right and left derived functors of invariants and coinvariants using [Definition 150](#), we would have been stuck picking resolutions for each $\mathbb{Z}G$ -module whose value under the derived functors we wanted to compute, which would have been messy. By the balancedness of Tor and Ext, we can simply pick a resolution of \mathbb{Z} as a $\mathbb{Z}G$ -module and get on with our lives.⁴

The story so far is as follows.

1. We fixed a group G and considered the category $G\text{-Mod}$ of functors $\mathbf{BG} \rightarrow \mathbf{Ab}$.
2. We noticed that picking out the constant functor gave us a canonical functor $\mathbf{Ab} \rightarrow G\text{-Mod}$, and that taking left- and right adjoints to this gave us interesting things.
 - The right adjoint $(-)^G$ applied to a G -module A gave us the subgroup of A stabilized by G .
 - The left adjoint $(-)_G$ applied to a G -module A gave us the A modulo the stabilized subgroup.
3. Due to adjointness, these functors have interesting exactness properties, leading us to derive them.
 - Since $(-)_G$ is left adjoint, it is right exact, and we can take the left derived functors $L_i(-)_G$.
 - Since $(-)^G$ is right adjoint, it is left exact, and we can take the right derived functors $R^i(-)^G$.

4. We denote

$$L_i(-)_G = H_i(G, -), \quad R^i(-)^G = H^i(G, -),$$

and call them *group homology* and *group cohomology* respectively.

5. We notice that, taking \mathbb{Z} as a trivial $\mathbb{Z}G$ -module, we have

$$A_G \equiv \mathbb{Z} \otimes_{\mathbb{Z}G} A, \quad A^G \equiv \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A),$$

⁴In the case of invariants (which correspond to Ext), we need to pick a resolution of \mathbb{Z} as a *left* $\mathbb{Z}G$ -module, and in the case of coinvariants (corresponding to Tor), we need to pick a resolution of \mathbb{Z} as a *right* $\mathbb{Z}G$ -module.

5. Basic applications

and thus that

$$H_i(G, A) = \operatorname{Tor}_i^{\mathbb{Z}G}(\mathbb{Z}, A), \quad H^i(G, A) = \operatorname{Ext}_{\mathbb{Z}G}^i(\mathbb{Z}, A).$$

This means that (by balancedness of Tor and Ext) we can compute group homology and cohomology by taking a resolution of \mathbb{Z} as a free $\mathbb{Z}G$ -module, rather than having to take resolutions of A for all A . The bar complex gave us such a resolution.

Example 152. Let $G = \mathbb{Z}$, and denote the generator by t . Suppose we want to compute $H_*(G; A)$ and $H^*(G; A)$ for some $\mathbb{Z}G$ -module A .

Thanks to [Lemma 151](#), instead of computing a projective resolution of some specific A , we may compute a projective resolution of \mathbb{Z} as a $\mathbb{Z}G$ -module.

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \twoheadrightarrow \mathbb{Z} \longrightarrow 0$$

This gives us a free resolution $F_\bullet \xrightarrow{\sim} \mathbb{Z}$.

By definition,

$$H_i(G; A) = H_i(F_\bullet \otimes_{\mathbb{Z}G} A) = H_i \left(0 \longrightarrow \mathbb{Z}G \otimes_{\mathbb{Z}G} A \xrightarrow{f} \mathbb{Z}G \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right).$$

The map f takes $t \otimes a \mapsto a - t \otimes a$; the homology $H_0(G; A)$ is thus

$$\operatorname{coker} f = A / \langle t \cdot a - a \mid a \in A \rangle = A_G.$$

Similarly, we have

$$H_1(G; A) = \ker f = A^G.$$

For $n > 1$, we have $H_n(G; A) = 0$.

Now turning our attention to group cohomology, we find by definition

$$H^n(G; A) = H^n \left(0 \longrightarrow \operatorname{Hom}(\mathbb{Z}G, A) \xrightarrow{g} \operatorname{Hom}(\mathbb{Z}G, A) \longrightarrow 0 \right),$$

where the map g sends

$$g: (f: t \mapsto f(t)) \mapsto (gf: t \mapsto f(t-1)).$$

Note that under the correspondence $f \mapsto f(1)$, we have

$$\operatorname{Hom}(\mathbb{Z}G; A) \cong A.$$

Having made this identification, g simply sends $f \mapsto tf - f$. Thus, similarly to before,

we have

$$H^0(G; A) = \ker(g) = A^G, \quad H^1(G; A) = \operatorname{coker}(g) = A_G,$$

and $H^n(G; A) = 0$ for $n > 1$.

5.3.4. The bar construction for group rings

In the last section, we saw that we could compute group homology and cohomology of a group G by finding a projective resolution of \mathbb{Z} as a trivial $\mathbb{Z}G$ -module. This is nice, but computing projective resolutions is in general rather difficult.

It turns out that the bar construction (explored in [Subsection 5.3.2](#)) computes a free resolution of \mathbb{Z} as a $\mathbb{Z}G$ -module for any group G .

Consider the functor

$$U: \mathbb{Z}G\text{-}\mathbf{Mod} \rightarrow \mathbf{Ab}$$

which forgets multiplication. This has left adjoint

$$\mathbb{Z}G \otimes_{\mathbb{Z}} -: \mathbf{Ab} \rightarrow \mathbb{Z}G\text{-}\mathbf{Mod},$$

which assigns to an abelian group A the left $\mathbb{Z}G$ -module $\mathbb{Z}G \otimes_{\mathbb{Z}} A$.

The unit $\eta: \operatorname{id}_{\mathbf{Ab}} \Rightarrow U(\mathbb{Z}G \otimes_{\mathbb{Z}} -)$ and counit $\varepsilon: \mathbb{Z}G \otimes_{\mathbb{Z}} U(-) \Rightarrow \operatorname{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}}$ of this adjunction have components

$$\eta_A: A \rightarrow U(\mathbb{Z}G \otimes_{\mathbb{Z}} A); \quad a \mapsto 1 \otimes a$$

and

$$\varepsilon_M: \mathbb{Z}G \otimes_{\mathbb{Z}} U(M) \rightarrow M; \quad g \otimes m \mapsto gm.$$

Denoting $\mathbb{Z}G \otimes_{\mathbb{Z}} (-) \circ U$ by Z , we find the following comonad in $\mathbb{Z}G\text{-}\mathbf{Mod}$, where $\nabla = \mathbb{Z}G \otimes_{\mathbb{Z}} (-)\eta U$.

$$\begin{array}{c} Z \circ Z \\ \uparrow \nabla \\ Z \\ \downarrow \varepsilon \\ \operatorname{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \end{array}$$

This gives us a simplicial object in $\mathbf{End}(\mathbb{Z}G\text{-}\mathbf{Mod})$.

$$\begin{array}{ccccccc} & & \xrightarrow{\mathbb{Z}Z\varepsilon} & & & & \\ & & \xleftarrow{\quad} & & & & \\ & & \xrightarrow{\quad} & & & & \\ \cdots & Z \circ Z \circ Z & \xleftarrow{\quad} & Z \circ Z & \xleftarrow{\nabla} & Z & \xrightarrow{\quad} \operatorname{id}_{\mathbb{Z}G\text{-}\mathbf{Mod}} \\ & & \xrightarrow{\quad} & & & & \\ & & \xleftarrow{\quad} & & & & \\ & & \xrightarrow{\varepsilon Z Z} & & & & \end{array}$$

5. Basic applications

Evaluating on a specific $\mathbb{Z}G$ -module M then gives a simplicial object in $\mathbb{Z}G\text{-Mod}$. We continue in the special case $M = \mathbb{Z}$ taken as a trivial $\mathbb{Z}G$ -module.

$$\begin{array}{ccccccc} & & & \xrightarrow{d_0} & & & \\ & & & \xleftarrow{\quad} & & & \\ & & & \xrightarrow{\quad} & & & \\ \cdots & \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} & \xleftarrow{\quad} & \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} & \xleftarrow{\quad} & \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z} & \longrightarrow \mathbb{Z} \\ & & & \xrightarrow{\quad} & & & \\ & & & \xleftarrow{d_2} & & & \\ & & & \xrightarrow{d_1} & & & \end{array}$$

Because we are tensoring over \mathbb{Z} , we are justified in notationally suppressing the last copy of \mathbb{Z} in our tensor products.

The face map d_i removes the i th copy of Z ; on a generator, we have the action of d_i given by

$$d_i: x \otimes x_1 \otimes \cdots \otimes x_n \mapsto \begin{cases} xx_1 \otimes \cdots \otimes x_n, & i = 0 \\ x \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n, & 0 < i < n + 1 \\ x \otimes \cdots \otimes x_{n-1}, & i = n + 1. \end{cases}$$

and the degeneracy maps s_i send

$$x \otimes x_1 \otimes \cdots \otimes x_{n-1} \mapsto x \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes 1 \otimes x_i \otimes \cdots \otimes x_{n-1}.$$

It is traditional to use the notation

$$x \otimes x_1 \otimes \cdots \otimes x_n = x[x_1 | \cdots | x_n].$$

This notation is the origin of the name *bar construction*.

We now form the bar complex; the objects B_n are given by

$$B_n = \mathbb{Z}G^{\otimes n} / \{\text{degenerate simplices}\};$$

that is, whenever we see something of the form

$$x[x_1 | \cdots | 1 | \cdots | x_n],$$

we set it to zero.

The elements of B_n are simply formal \mathbb{Z} -linear combinations of the $x[x_1 | \cdots | x_n]$. Therefore, we can also write

$$B_n = \bigoplus_{(g_i) \in (G \setminus \{e\})^n} \mathbb{Z}G[g_1 | \cdots | g_n], \quad n \geq 0.$$

That is, we can write the bar complex as follows.

$$\cdots \longrightarrow \bigoplus_{(g_1, g_2) \in (G \setminus \{1\})^2} \mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g \in G \setminus \{1\}} \mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow \mathbb{Z}$$

To summarize, we have the following definition.

Definition 153 (bar complex). Let G be a group. The bar complex of G is the chain complex defined level-wise to be the free $\mathbb{Z}G$ -module

$$B_n = \bigoplus_{(g_i) \in (G \setminus \{e\})^n} \mathbb{Z}G[g_1 | \cdots | g_n],$$

where $[g_1 | \cdots | g_n]$ is simply a symbol denoting to the basis element corresponding to (g_1, \dots, g_n) . The differential is defined by

$$d([g_1 | \cdots | g_n]) = g_1[g_2 | \cdots | g_n] + \sum_{i=1}^{n-1} (-1)^i [g_1 | \cdots | g_i g_{i+1} | \cdots | g_n] + (-1)^n [g_1 | \cdots | g_{n-1}].$$

Proposition 154. For any group G , the bar construction gives a free resolution of \mathbb{Z} as a (left) $\mathbb{Z}G$ -module.

Proof. We need to show that the sequence

$$\cdots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0$$

is exact, where ϵ is the augmentation map.

We do this by considering the above as a sequence of \mathbb{Z} -modules, and providing a \mathbb{Z} -linear homotopy between the identity and the zero map.

$$\begin{array}{ccccccc} B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \\ \downarrow \text{id} & \swarrow h_1 & \downarrow \text{id} & \swarrow h_0 & \downarrow \text{id} & \swarrow h_{-1} & \downarrow \text{id} \\ B_2 & \longrightarrow & B_1 & \longrightarrow & B_0 & \xrightarrow{\epsilon} & \mathbb{Z} \longrightarrow 0 \end{array}$$

We define

$$h_{-1}: \mathbb{Z} \rightarrow B_0; \quad n \mapsto n[\cdot].$$

and, for $n \geq 0$,

$$h_n: B_n \mapsto B_{n+1}; \quad g[g_1 | \cdots | g_n] \mapsto [g|g_1 | \cdots | g_n].$$

We then have

$$\epsilon \circ h_{-1}: n \mapsto n[\cdot] \mapsto n,$$

5. Basic applications

and doing the butterfly

$$\begin{array}{ccc}
 B_n & \xrightarrow{d_n} & B_{n-1} \\
 & \nwarrow h_{n-1} & \\
 B_n & &
 \end{array}
 +
 \begin{array}{ccc}
 & & B_n \\
 & \nwarrow h_n & \\
 B_{n+1} & \xrightarrow{d_{n+1}} & B_n
 \end{array}$$

gives us in one direction

$$\begin{array}{ccc}
 g[g_1 | \cdots | g_n] & \longmapsto & gg_1[g_2 | \cdots | g_n] \\
 & & + \sum_{i=1}^{n-1} (-1)^i g[g_1 | \cdots | g_i g_{i+1} | \cdots | g_n] \\
 & & + (-1)^n g[g_1 | \cdots | g_{n-1}] \\
 & \nwarrow & \\
 [gg_1 | \cdots | g_n] & & \\
 + \sum_{i=1}^{n-1} (-1)^i [gg_1 | \cdots | g_i g_{i+1} | \cdots | g_n] & & \\
 + (-1)^n [gg_1 | \cdots | g_n] & &
 \end{array}$$

and in the other

$$\begin{array}{ccc}
 & & g[g_1 | \cdots | g_n] \\
 & \nwarrow & \\
 [g[g_1 | \cdots | g_n]] & \longmapsto & - \sum_{i=1}^{n-1} [gg_1 | \cdots | g_i g_{i+1} | \cdots | g_n] \\
 & & - (-1)^n [gg_1 | \cdots | g_{n-1}] \\
 & & - [gg_1 | g_2 | \cdots | g_n] \quad .
 \end{array}$$

The sum of these is simply $g[g_1 | \cdots | g_n]$, i.e. we have

$$h \circ d + d \circ h = \text{id}.$$

This proves exactness as desired. \square

Note that we immediately get a free resolution of $\mathbb{Z}G$ as a *right* $\mathbb{Z}G$ by mirroring the construction above.

Thus, we have the following general formulae:

$$H^i(G, A) = H^i \left(0 \longrightarrow \text{Hom}_{\mathbb{Z}G}(B_0, A) \xrightarrow{(d_1)^*} \text{Hom}_{\mathbb{Z}G}(B_1, A) \xrightarrow{(d_2)^*} \cdots \right)$$

and

$$H_i(G, A) = H_i \left(\cdots \longrightarrow B_1 \otimes_{\mathbb{Z}G} A \longrightarrow B_0 \otimes_{\mathbb{Z}G} A \longrightarrow 0 \right)$$

5.3.5. Computations in low degrees

It turns out that one can find compelling interpretations of low homology classes. In the course of this section we will find the following interpretations.

- Taking \mathbb{Z} as a trivial G -module, the first homology $H_1(G, \mathbb{Z})$ simply coincides with G_{ab} .
- For any G -module A , the first homology $H^1(G; A)$ corresponds to *crossed homomorphisms* $G \rightarrow A$ modulo *principal crossed homomorphisms* $G \rightarrow A$. In particular, if A is a trivial G -module, then $H^1(G, A)$ corresponds to $\text{Hom}_{\text{Ab}}(G_{\text{ab}}, A)$.
- Second cohomology $H^2(A; G)$ controls extensions of G by A , i.e. short exact sequences of groups

$$0 \longrightarrow A \hookrightarrow E \twoheadrightarrow G \longrightarrow 0 .$$

First homology and abelianizations

Let G be a group, and consider \mathbb{Z} as a trivial $\mathbb{Z}G$ -module. Let us compute $H_1(G, \mathbb{Z})$.

To compute $H_1(G, \mathbb{Z})$, consider the first few terms in the bar complex for G .

$$\cdots \longrightarrow \bigoplus_{(g_1, g_2) \in (G \setminus \{1\})^2} \mathbb{Z}G[g_1|g_2] \longrightarrow \bigoplus_{g \in G \setminus \{1\}} \mathbb{Z}G[g] \longrightarrow \mathbb{Z}G[\cdot] \longrightarrow 0$$

Tensoring over $\mathbb{Z}G$ with \mathbb{Z} is the same as trivializing the action, giving us the following complex.

$$\cdots \longrightarrow \bigoplus_{(g_1, g_2) \in (G \setminus \{1\})^2} \mathbb{Z}[g_1|g_2] \longrightarrow \bigoplus_{g \in G \setminus \{1\}} \mathbb{Z}[g] \longrightarrow \mathbb{Z}[\cdot] \longrightarrow 0$$

The differential $d_1 \otimes \text{id}_{\mathbb{Z}}$ acts on a generator $[g]$ by sending it to

$$[\cdot]g - [\cdot] = [\cdot] - [\cdot] = 0,$$

implying that everything is a 1-cycle. The differential d_2 sends a generator $[g|h]$ to

$$[g]h - [gh] + [h] = [g] + [h] - [gh].$$

Thus, taking cycles modulo boundaries is the same as

$$\{[g] \in G\} / \langle [gh] - [g] - [h] \rangle \cong G_{\text{ab}}.$$

5. Basic applications

Thus, $H_1(G, \mathbb{Z})$ is simply the abelianization of G .

First cohomology and abelianizations

To compute $H^1(G, A)$, consider the sequence

$$0 \longrightarrow \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[\cdot], A) \xrightarrow{(d_1)^*} \text{Hom}_{\mathbb{Z}G}(\bigoplus_{g \in G \setminus \{1\}} \mathbb{Z}G[g], A) \xrightarrow{(d_2)^*} \cdots$$

Notice immediately that since the hom functor preserves direct sums in the first slot, we have

$$\text{Hom}_{\mathbb{Z}G} \left(\bigoplus_{(g_i) \in (G \setminus \{1\})^n} \mathbb{Z}G[g_1 | \cdots | g_n], A \right) \cong \bigoplus_{(g_i) \in (G \setminus \{1\})^n} \text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1 | \cdots | g_n], A)$$

An element of $\text{Hom}_{\mathbb{Z}G}(\mathbb{Z}G[g_1 | \cdots | g_n])$ is determined completely by where it sends the generator of $\mathbb{Z}G$. We can therefore re-package the RHS as consisting of functions

$$\phi: (G \setminus \{1\})^n \rightarrow A,$$

or equivalently as functions

$$\phi: G^n \rightarrow A, \quad \phi(\dots, 1, \dots) = 0.$$

The differential $d^n = (d_n)_*$ sends $\phi \mapsto d^n \phi = \phi \circ d_n$, where

$$\begin{aligned} \phi \circ d_n: (g_1, \dots, g_{n+1}) &\mapsto g_1 \phi(g_2, \dots, g_{n+1}) \\ &+ \sum_{i=1}^n (-1)^i \phi(g_1, \dots, g_i g_{i+1}, \dots, g_{n+1}) \\ &+ (-1)^{n+1} \phi(g_1, \dots, g_n). \end{aligned}$$

We will denote the n th term in [Equation 5.3.5](#) by C^n ; that is, we have

$$C^n \cong \{\phi: G^n \rightarrow A \mid \phi(g_1, \dots, 1, \dots, g_n) = 0\}.$$

Let us explicitly compute $H^1(G; A)$.

The set of 1-coboundaries corresponds to functions $\psi: G \rightarrow A$ with

$$d\psi: G^2 \rightarrow A; \quad d\psi(g, h) = g\psi(h) - \psi(gh) + \psi(g) = 0,$$

i.e. functions

$$\phi: G \rightarrow A; \quad \psi(g, h) = \psi(g) + g\psi(h).$$

We will call these *crossed homomorphisms*.

The set of 1-cocycles corresponds to the image $d^0(C^0)$, i.e. those functions

$$d\phi: G \rightarrow A, \quad d\phi([g]) = g\phi([\cdot]) - \phi([\cdot]).$$

We will call these *principal crossed homomorphisms*.

Thus, in this case we have

$$H^1(G; A) = \frac{\{\text{crossed homomorphisms from } G \text{ to } A\}}{\{\text{principal crossed homomorphisms from } G \text{ to } A\}}.$$

Now suppose that A is a trivial G -module. In that case, crossed homomorphisms are simply homomorphisms $G \rightarrow A$, and principal crossed homomorphisms are zero. Thus, $H^1(G; A)$ consists of homomorphisms $G \rightarrow A$, i.e.

$$H^1(G; A) = \text{Hom}_{\mathbf{Ab}}(G_{\text{ab}}, A).$$

First cohomology and semidirect products

Definition 155 (semidirect product). Let G be a group, and A a G -module. The semidirect product of A and G is the group whose underlying set is $A \times G$, and whose group operation is given by

$$(a, g) \cdot (a', g') = (a + ga', gg').$$

We denote the semidirect product of A and G by $A \rtimes G$.

Note that we have a short exact sequence of groups⁵

$$0 \longrightarrow A \hookrightarrow A \rtimes G \twoheadrightarrow G \longrightarrow 1.$$

Let $\sigma: A \rtimes G \rightarrow A \rtimes G$ be a group homomorphism. We say that σ *stabilizes* A and G if the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \hookrightarrow & A \rtimes G & \twoheadrightarrow & G \longrightarrow 1 \\ & & \parallel & & \downarrow \sigma & & \parallel \\ 0 & \longrightarrow & A & \hookrightarrow & A \rtimes G & \twoheadrightarrow & G \longrightarrow 1 \end{array}$$

Given any crossed homomorphism ϕ , we get a group homomorphism σ_ϕ by the formula

$$\sigma_\phi: (a, g) \mapsto (a + \phi(g), g).$$

⁵We are cheating a bit. Since \mathbf{Grp} is not an abelian category, we have no notion of an exact sequence of groups. We trust that the reader will not be confused.

5. Basic applications

This is an easy check; we have

$$\begin{aligned}
 \sigma_\phi((a, g)(a', g')) &= \sigma_\phi(a + ga', gg') \\
 &= (a + ga' + \phi(gg'), gg') \\
 &= (a + \phi(g) + g(a' + \phi(g')), gg') \\
 &= (a + \phi(g), g)(a' + \phi(g'), g') \\
 &= \sigma_\phi(a, g)\sigma_\phi(a', g').
 \end{aligned}$$

In fact, σ_ϕ trivially stabilizes A and G . This gives us a way of turning a cocycle in $Z^1(G, A)$ into an automorphism of $A \rtimes G$ which stabilizes A and G .

Proposition 156. The map

$$\psi: Z^1(G; A) \rightarrow \text{Aut}(A \rtimes G); \quad \phi \mapsto \sigma_\phi$$

is an isomorphism onto the subgroup of $\text{Aut}(A \rtimes G)$

Proof. The fact that ψ is a homomorphism is trivial. We show that we have an inverse given by □

This is miserable.

Second cohomology and extensions

Definition 157 (extension of a group). Let G be a group and A an abelian group. An extension of G by A is a short exact sequence of groups

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0.$$

Two such extensions are said to be equivalent if there is a morphism of short exact sequences

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xhookrightarrow{i} & E & \xrightarrow{\pi} & G \longrightarrow 0 \\
 & & \parallel & & \downarrow \sigma & & \parallel \\
 0 & \longrightarrow & A & \xhookrightarrow{i'} & E & \xrightarrow{\pi'} & G \longrightarrow 0
 \end{array}$$

We are no longer justified in using the snake lemma to show that σ is an isomorphism, but the logic in [Example 51](#) does carry over to our current situation.

Definition 158 (split extension). An extension

$$0 \longrightarrow A \xhookrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0$$

$\xleftarrow[s]{}$

is said to be *split* if there exists a group homomorphism $s: G \rightarrow E$ such that $\pi \circ s = \text{id}_G$.

Proposition 159. An extension of E of G by A is equivalent to the semidirect product $A \rtimes G$ if and only if it is split.

Proof. The semidirect product $A \rtimes G$ admits the extension $g \mapsto (0, g)$. Given a split extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0 ,$$

we can define a map

$$E \rightarrow A \rtimes G; \quad e \mapsto (e(s\pi(e)^{-1}), \pi(e)).$$

□

As the last proposition shows, in general we cannot find a section of an extension. However, since π is surjective, given any extension,

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow{\pi} G \longrightarrow 0 ,$$

we can find at least a *set-theoretic extension* of G by A , i.e. a set-function $G \rightarrow E$ such that $\pi \circ s = e_G$.

Note that E can act on A via conjugation; if $a \in A$ and $e \in E$, then $ea e^{-1} \in A$, because

$$\begin{aligned} \pi(eae^{-1}) &= \pi(e)\pi(a)\pi(e^{-1}) \\ &= \pi(e)\pi(e)^{-1} \\ &= e_G. \end{aligned}$$

In fact, since A is abelian, it acts trivially on itself via conjugation, so it descends to an action of $G \cong E/A$ on A .

This gives a section of E relative to G . Given such a splitting, we can define for each pair $g, h \in G$ an element $f(g, h) \in A$ in the fiber over the identity $e_G \in G$ which corresponds to how far...

$$f(g, h) = s(g)s(h)s(gh)^{-1}.$$

The function $f: G \times G \rightarrow A$ is called the *factor set* of G . If s were a group homomorphism, the factor set would vanish; in general, the factor set measures how far apart $s(g)s(h)$ and $s(gh)$ are.

Lemma 160. Given an extension

$$0 \longrightarrow A \xrightarrow{i} E \xrightarrow[\pi]{s} G \longrightarrow 0 ,$$

we can completely recover the group law on E from the injection i and the factor set f .

5. Basic applications

Proof. As a set, E is in bijection with $A \times G$ via

$$e \mapsto (e(s\pi(e)^{-1}), \pi(e)).$$

It is easy to check that this has inverse

$$(a, g) \mapsto i(a)s(g).$$

Let $e, e' \in E$ corresponding to (a, g) and (a', g') respectively. Then

$$\begin{aligned} ee' &= i(a)s(g)i(a')s(g') \\ &= i(a)s(g)i(a')s(g)^{-1}s(g)s(h) \\ &= i(a)i(g \cdot b)s(g)s(h) \\ &= i(a + g \cdot b)i(f(g, h))s(gh) \\ &= i(a + g \cdot b + f(g, h))s(gh) \end{aligned}$$

so under the bijection $E \cong A \times G$, ee' corresponds to $(a + g \cdot b + f(g, h), gh)$. \square

Associativity now immediately implies that

$$((a, g)(b, h))(c, k) \stackrel{!}{=} (a, g)((b, h)(c, k));$$

expanding, we find that

$$(a + g \cdot b + f(g, h) + (gh) \cdot c + f(gh, k), ghk)$$

must be the same as

$$(a + g \cdot b + (gh) \cdot c + gf(h, k) + f(g, hk), ghk).$$

That means that

$$g \cdot f(h, k) - f(gh, k) + f(g, hk) - f(g, h) \stackrel{!}{=} 0.$$

Thus, the factor set f must be a 2-cocycle in $C^\bullet(G; A)$.

Our construction depends on the choice of section s . supposed we had picked a different section s' . Then the 'difference' $s'(g)s(g)^{-1}$ can be written $i(\phi(a))$ for some $a \in A$. We thus obtain a map $\phi: G \rightarrow A$.

To warm up, we do a computation.

$$s'(g)s'(h) = \dots$$

In the end, we find

$$f_{s'}(g, h) - f_s(g, h) = g \cdot \phi(h) - \phi(gh) + \phi(g).$$

Theorem 161. Let G be a group, and A a G -module. Then there is a natural bijection

$$\left\{ \begin{array}{l} \text{Extension classes of } G \text{ by } A \\ \text{such that the action of } G \text{ on } A \\ \text{agrees with the module structure} \end{array} \right\} \xrightarrow{\cong} H^2(G; A).$$

5.3.6. Periodicity in group homology

In this section, we will prove a combinatorial version of the following result.

Fact 162. Let G be a group. Suppose G acts freely on a sphere of dimension $2k - 1$. Then G has $2k$ -periodic group homology.

Simplicial complexes

Definition 163 (simplicial complex). Let I be a finite set. A simplicial complex K on I is a collection of subsets $K \subset \mathcal{P}(I)$ such that for $\sigma \in K$ and $\tau \subset \sigma$, $\tau \in K$.

Let K be a simplicial complex on a set I , and let K' be a simplicial complex on a set I' . A morphism $K \rightarrow K'$ is a function $f: I \rightarrow I'$ such that for every $\sigma \in K$, $f(\sigma) \in K'$.

We define a cosimplicial object in the category of simplicial complexes by

$$[n] \mapsto \mathcal{P}(\{0, \dots, n\}).$$

Taking nerves gives us, for every simplicial complex K , a simplicial set \tilde{K} .

Note that for each $\sigma \in K_n$, there are $n!$ simplices in \tilde{K}_n , one for each permutation of the elements of σ . In **Set** there is not much we can do about this; however, applying the free abelian group functor \mathcal{F} brings us to **Ab**, where we can mod out by this ambiguity.

We thus define a simplicial complex $\tilde{C}(K)$ to be the alternating face maps complex of the simplicial object $\mathcal{F} \circ \tilde{K}$.

A more concrete description of what we have just done is as follows. We have

$$\tilde{C}(K)_n = \bigoplus_{\{x_0, \dots, x_n\} \in K_n} \mathbb{Z}e_{(x_0, \dots, x_n)},$$

and the differential is

$$\sum_{i=0}^n (-1)^i d_i,$$

5. Basic applications

where the face maps d_i are the \mathbb{Z} -linear extensions of the maps

$$d_i: e_{(x_0, \dots, x_n)} \mapsto e_{(x_0, \dots, \widehat{x_i}, \dots, x_n)}.$$

Now that we are in **Ab**, we can get rid of the extraneous simplices by defining

$$C(K)_n = \widetilde{C}(K)_n / \langle e_{(x_1, \dots, x_n)} - \text{sign}(\sigma) e_{(x_{\sigma(1)}, \dots, x_{\sigma(n)})} \mid \sigma \in S_n \rangle.$$

It is not hard to see that the differential on $\widetilde{C}(K)$ descends to $C(K)$.

Example 164. Let $K = \mathcal{P}(\{0, 1\})$. Then $e_{(0,1)} = -e_{(1,0)}$ in $C(K)_1$.

Let $f: K \rightarrow K$ be an automorphism of simplicial complexes K . This descends to an automorphism of chain complexes $C_\bullet(K) \rightarrow C_\bullet(K)$ defined on generators by

$$e_{(x_0, \dots, x_n)} \mapsto e_{(f(x_0), \dots, f(x_n))}.$$

Now suppose G acts trivially on

6. Spectral sequences

6.1. Motivation

Let $E_{\bullet,\bullet}$ be a first-quadrant double complex. As we saw in [Section 3.11](#), computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the building blocks of the homology of the total complex of such a double complex.

They do this via a series of successive approximations. One starts with a first-quadrant double complex $E_{\bullet,\bullet}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
E_{0,3} & \xleftarrow{d^h} & E_{1,3} & \xleftarrow{d^h} & E_{2,3} & \xleftarrow{d^h} & E_{3,3} \xleftarrow{\quad} \cdots \\
d^v \downarrow & & d^v \downarrow & & d^v \downarrow & & d^v \downarrow \\
E_{0,2} & \xleftarrow{d^h} & E_{1,2} & \xleftarrow{d^h} & E_{2,2} & \xleftarrow{d^h} & E_{3,2} \xleftarrow{\quad} \cdots \\
d^v \downarrow & & d^v \downarrow & & d^v \downarrow & & d^v \downarrow \\
E_{0,1} & \xleftarrow{d^h} & E_{1,1} & \xleftarrow{d^h} & E_{2,1} & \xleftarrow{d^h} & E_{3,1} \xleftarrow{\quad} \cdots \\
d^v \downarrow & & d^v \downarrow & & d^v \downarrow & & d^v \downarrow \\
E_{0,0} & \xleftarrow{d^h} & E_{1,0} & \xleftarrow{d^h} & E_{2,0} & \xleftarrow{d^h} & E_{3,0} \xleftarrow{\quad} \cdots
\end{array}$$

One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

6. Spectral sequences

$$E_{\bullet,\bullet}^0$$

$$\begin{array}{ccccccc}
 & & \vdots & \vdots & \vdots & \vdots & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 \nearrow^q & & E_{0,3}^0 & E_{1,3}^0 & E_{2,3}^0 & E_{3,3}^0 & \\
 & & \downarrow d^v & \downarrow d^v & \downarrow d^v & \downarrow d^v & \\
 & & E_{0,2}^0 & E_{1,2}^0 & E_{2,2}^0 & E_{3,2}^0 & \\
 & & \downarrow d^v & \downarrow d^v & \downarrow d^v & \downarrow d^v & \\
 & & E_{0,1}^0 & E_{1,1}^0 & E_{2,1}^0 & E_{3,1}^0 & \\
 & & \downarrow d^v & \downarrow d^v & \downarrow d^v & \downarrow d^v & \\
 & & E_{0,0}^0 & E_{1,0}^0 & E_{2,0}^0 & E_{3,0}^0 & \\
 & & & & & & \searrow^p
 \end{array} \tag{6.1}$$

One then takes homology of the corresponding vertical complexes, replacing $E_{p,q}^0$ by the homology $H_q(E_{p,\bullet}^0)$. To ease notation, one writes

$$H_q(E_{p,\bullet}^0) = E_{p,q}^1.$$

The functoriality of H_q means that the maps $H_q(d^h)$ act as differentials for the rows.

$$\begin{array}{ccccccc}
 & & \vdots & \vdots & \vdots & \vdots & \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \\
 \nearrow^q & & E_{0,3}^1 & E_{1,3}^1 & E_{2,3}^1 & E_{3,3}^1 & \leftarrow \dots \\
 & & \xleftarrow{H_3(d^h)} & \xleftarrow{H_3(d^h)} & \xleftarrow{H_3(d^h)} & & \\
 & & E_{0,2}^1 & E_{1,2}^1 & E_{2,2}^1 & E_{3,2}^1 & \leftarrow \dots \\
 & & \xleftarrow{H_2(d^h)} & \xleftarrow{H_2(d^h)} & \xleftarrow{H_2(d^h)} & & \\
 & & E_{0,1}^1 & E_{1,1}^1 & E_{2,1}^1 & E_{3,1}^1 & \leftarrow \dots \\
 & & \xleftarrow{H_1(d^h)} & \xleftarrow{H_1(d^h)} & \xleftarrow{H_1(d^h)} & & \\
 & & E_{0,0}^1 & E_{1,0}^1 & E_{2,0}^1 & E_{3,0}^1 & \leftarrow \dots \\
 & & \xleftarrow{H_0(d^h)} & \xleftarrow{H_0(d^h)} & \xleftarrow{H_0(d^h)} & & \\
 & & & & & & \searrow^p
 \end{array} \tag{6.2}$$

We now write $E_{p,q}^2$ for the horizontal homology $H_p(E_{\bullet,q}^1)$.

The claim is that the terms $E_{p,q}^r$ are pieces of a successive approximation of the homology of the total complex $\text{Tot}(E)$. We will prove this claim much later. However, in the particularly simple case that $E_{\bullet,\bullet}$ consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

Example 165. Let E be a double complex where all but two adjacent columns are zero; that is, let E be a diagram consisting of two chain complexes $E_{0,\bullet}$ and $E_{1,\bullet}$ and a morphism $f: E_{1,\bullet} \rightarrow E_{0,\bullet}$, padded by zeroes on either side.

$$\begin{array}{ccc}
 \vdots & & \vdots \\
 \downarrow & & \downarrow \\
 E_{0,3} & \xleftarrow{f_3} & E_{1,3} \\
 d_3^0 \downarrow & & \downarrow -d_3^1 \\
 E_{0,2} & \xleftarrow{f_2} & E_{1,2} \\
 d_2^0 \downarrow & & \downarrow -d_2^1 \\
 E_{0,1} & \xleftarrow{f_1} & E_{1,1} \\
 d_1^0 \downarrow & & \downarrow -d_1^1 \\
 E_{0,0} & \xleftarrow{f_0} & E_{1,0} \\
 d_0^0 \downarrow & & \downarrow -d_0^1 \\
 E_{0,-1} & \xleftarrow{f_{-1}} & E_{1,-1} \\
 \downarrow & & \downarrow \\
 \vdots & & \vdots
 \end{array}$$

Note that we have multiplied the differentials of $E_{1,\bullet}$ by -1 so that we have a double complex. This does not enter into the discussion.

Fix some $n \in \mathbb{Z}$, and let $q = n - p$. Further, let $T = \text{Tot}(E)$; that is, let $T = \text{Cone}(f)$. Recall that $\text{Cone}(f)$ fits into the following short exact sequence,

$$0 \longrightarrow E_{\bullet,1} \hookrightarrow \text{cone}(f)_{\bullet} \twoheadrightarrow E_{\bullet,0}[-1] \longrightarrow 0$$

6. Spectral sequences

and that this gives us the following long exact sequence on homology.

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & H_n(E_{\bullet,0}) \\
 & & & & & & \uparrow \delta \\
 & & & & H_n(E_{\bullet,1}) & \longrightarrow & H_n(\text{Cone}(f)_{\bullet}) \longrightarrow H_{n-1}(E_{\bullet,0}) \\
 & & & & & & \uparrow \delta \\
 & & & & H_{n-1}(E_{\bullet,1}) & \longrightarrow & \cdots
 \end{array}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \text{coker}(H_n(f)) \hookrightarrow H_n(\text{Cone}(f)) \twoheadrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation; that is, we have a short exact sequence

$$0 \longrightarrow E_{p-1,q+1}^2 \hookrightarrow H_{p+q}(T) \twoheadrightarrow E_{p,q}^2 \longrightarrow 0 .$$

6.2. Homology spectral sequences

6.2.1. Notation and terminology

Definition 166 (homology spectral sequence). Let \mathcal{A} be an abelian category. A homology spectral sequence starting at E^a in \mathcal{A} consists of the following data.

1. A family $E_{p,q}^r$ of objects of \mathcal{A} , defined for all integers p, q , and $r \geq a$.
2. Maps

$$d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r,q+r+1}^r$$

that are differentials in the sense that $d^r \circ d^r = 0$.

3. Isomorphisms

$$E_{p,q}^{r+1} \cong \ker(d_{p,q}^r) / \text{im}(d_{p+r,q-r+1}^r)$$

between $E_{p,q}^{r+1}$ and the homology at the corresponding position of the E^r

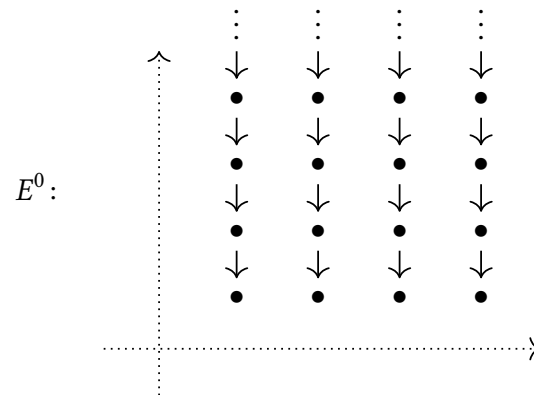
We denote such a spectral sequence by E . A morphism $E' \rightarrow E$ is a family of maps

$$f_{p,q}^r: E_{p,q}^{r'} \rightarrow E_{p,q}^r$$

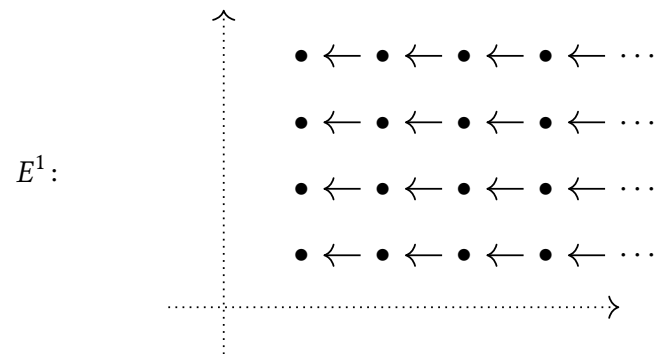
such that $d^r f^r = f^r d^r$ such that $f_{p,q}^{r+1}$ is induced on homology by $f_{p,q}^r$.

For each r , a spectral sequence E has a two-dimensional array of objects $E_{p,q}^r$. For a given r , one calls the objects $E_{p,q}^r$ the r th *page* of the spectral sequence E .

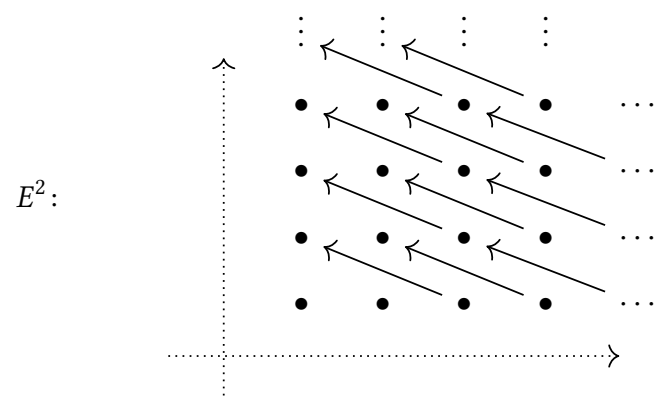
We have already seen an example of the first two pages of a spectral sequence (in [Diagram 6.1](#) and [Diagram 6.2](#)). Schematically, the differentials on the zeroth page point down.



The differentials on the first page point to the left.

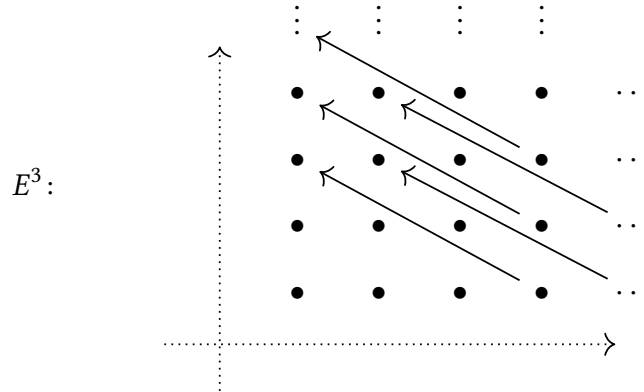


As r increases, the arrows rotate clockwise. The second page looks like this.



6. Spectral sequences

The third page looks like this.



6.2.2. Stabilization and convergence

We have so far placed no conditions on $E_{p,q}^r$. However, we will confine our study of certain classes of spectral sequences.

Definition 167 (bounded, first quadrant). Let E be a spectral sequence starting at E^a .

- We say that E is bounded if each $n \in \mathbb{Z}$, there are only finitely many terms of total degree n on the a th page, i.e. terms of the form $E_{p,q}^a$ for $n = p + q$.
- We say that E is first-quadrant if $E_{p,q}^a = 0$ for $p < 0$ and $q < 0$.

Clearly, every first-quadrant spectral sequence is bounded.

Note that if a spectral sequence is bounded, then

We get $E_{p,q}^{r+1}$ by taking the homology of a complex on the r th page. If E is a first-quadrant spectral sequence, then for $r > \max(p, q + 1)$, the differential entering $E_{p,q}^r$ comes from the fourth quadrant (hence from a zero object), and the differential leaving it lands in the second quadrant (also a zero object). Thus, $E_{p,q}^{r+1} = E_{p,q}^r$.

Definition 168 (stabilize). A spectral sequence starting at E^a is said to stabilize at position (p, q) if there exists some $r \geq a$ such that $E_{p,q}^{r'} = E_{p,q}^r$ for all $r' \geq r$. In this case, we write $E_{p,q}^\infty$ for the stable value at position (p, q) .

By the argument above, each position in a first-quadrant spectral sequence eventually stabilizes. We will mostly be interested in first-quadrant spectral sequences. However, boundedness is (clearly!) sufficient to guarantee stabilization.

Suppose one has been given a bounded spectral sequence, which we now know eventually stabilizes at each position. We would like to get some useful information out of such a spectral sequence.

Definition 169 (filtration). Let C_\bullet be a chain complex. A filtration F of C_\bullet is a chain of inclusions of subcomplexes $F_p C_\bullet$ of C_\bullet .

$$\cdots \subset F_{p-1} C_\bullet \subset F_p C_\bullet \subset F_{p+1} C_\bullet \subset \cdots \subset C_\bullet$$

Definition 170 (bounded convergence). Let E be a bounded spectral sequence starting at E^a in an abelian category \mathcal{A} , and let $\{H_i\}_{i \in \mathbb{Z}}$ be a collection of objects in \mathcal{A} , each equipped with a finite filtration

$$0 = F_s H_n \subset \cdots \subset F_{p-1} H_n \subset F_p H_n \subset \cdots \subset F_t H_n = H_n.$$

We say that E converges to $\{H_i\}_{i \in \mathbb{Z}}$ if there exist isomorphisms

$$E_{p,q}^\infty \cong F_p H_{p+q} / F_{p-1} H_{p+q}.$$

We express this convergence by writing

$$E_{p,q}^a \Rightarrow H_{p+q}.$$

Example 171. Consider the spectral sequence E starting at E^0 from [Example 165](#). This is certainly bounded, and stabilizes at each position (p, q) after the second page; that is, we have

$$E_{p,q}^\infty = E_{p,q}^2.$$

The exact sequence

$$0 \longrightarrow E_{p-1,q+1}^2 \hookrightarrow H_{p+q}(T) \twoheadrightarrow E_{p,q}^2 \longrightarrow 0$$

gives us, in a wholly unsatisfying and trivial way, a filtration

$$\begin{array}{ccccc} F_{-1} H_n & & F_0 H_n & & F_1 H_n \\ \parallel & & \parallel & & \parallel \\ 0 & \hookrightarrow & E_{0,n}^2 & \hookrightarrow & H_n \end{array}$$

of H_n . The claim is that

$$E_{p,q}^0 \Rightarrow H_{p+q}.$$

In order to check this, we have to check that

$$E_{p,q}^2 \cong F_p H_{p+q} / F_{p-1} H_{p+q}$$

6. Spectral sequences

for all p, q . The only non-trivial cases are when $p = 0$ or 1 . When $p = 0$, we have

$$\begin{aligned} E_{0,n}^\infty &\stackrel{!}{\cong} F_0 H_n / F_{-1} H_n \\ &\cong E_{0,n}^2 / 0 \\ &\cong E_{0,n}^2, \end{aligned}$$

and when $p = 1$ we have

$$\begin{aligned} E_{1,n}^\infty &\stackrel{!}{\cong} F_1 H_{n+1} / F_0 H_{n+1} \\ &\cong H_{n+1} / E_{0,n+1}^2 \\ &\cong E_{1,n}^2 \end{aligned}$$

as required.

6.2.3. Exact couples

We now take a detour to the much simpler, more elegant world of exact couples.

Definition 172 (exact couple). Let \mathcal{A} be an abelian category. An exact couple in \mathcal{A} consists of objects and morphisms

$$\begin{array}{ccc} & E & \\ \gamma \nearrow & & \searrow \alpha \\ A & \xleftarrow{\beta} & E \end{array}$$

such that $\text{im } \alpha = \ker \beta$

6.2.4. The spectral sequence of a filtered complex

Theorem 173. A filtration F of a chain complex C_\bullet naturally determines a spectral sequence starting with

$$E_{p,q}^0 = F_p C_{p+q} / F_{p-1} C_{p+q}, \quad E_{p,q}^1 = H_{p+q}(E_{p,\bullet}^0)$$

Proof. We construct explicitly the spectral sequence above. Let C_\bullet be a chain complex, and F a filtration. We will use the following notation.

- We will denote by η_p the quotient

$$\eta_p: F_p C \twoheadrightarrow F_p C / F_{p-1} C.$$

coming from the short exact sequence

$$0 \longrightarrow F_{p-1}C \hookrightarrow F_pC \twoheadrightarrow F_pC/F_{p-1}C \longrightarrow 0$$

- We will denote by A_p^r the subobject

$$A_p^r = \{c \in F_pC : dc \in F_{p-r}C\} \subset F_pC$$

of those elements whose differentials survive r many levels of the grading.

- We will denote by Z_p^r the image

$$Z_p^r = \eta_p(A_p^r).$$

- We will denote by B_p^r the image

$$B_p^r = \eta_p(d(a_{p+r-1}^{r-1})).$$

Examining the definitions, we see that we have the following inclusion.

$$\begin{array}{ccc} dA_{p+r-1}^{r-1} & \longequal{\quad} & \{dc \mid c \in F_{p+r-1}, dc \in F_pC\} \\ \downarrow & & \downarrow \\ dA_{p+r}^r & \longequal{\quad} & \{dc \mid c \in F_{p+r}, dc \in F_pC\} \end{array}$$

Both of these are subobjects of F_pC ; by applying η_p and taking a sharp look at the definition of the B s, we find an inclusion

$$B_p^r \subset B_p^{r+1}.$$

Working inductively, we are left with the following sequence of inclusions.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^r \subset \cdots$$

Defining $B_p^\infty = \bigcup_r B_p^r$, we can crown our sequence of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty.$$

6. Spectral sequences

Again examining definitions, we find the following inclusion.

$$\begin{array}{ccc} A_p^r & \xlongequal{\quad} & \{c \in F_p C \mid dc \in F_{p-r} C\} \\ \downarrow & & \downarrow \\ A_p^{r+1} & \xlongequal{\quad} & \{c \in F_p C \mid dc \in F_{p-r-1} C\} \end{array}$$

As before, applying η_p and working inductively gives us a chain

$$\cdots \subset Z_p^r \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0$$

and defining $Z_p^\infty = \bigcap_r Z_p^r$ gives us

$$Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0,$$

But we should not rest on our laurels. Instead, note that for each r, r' , we have an inclusion

$$\begin{array}{ccc} dA_{p+r-1}^{r-1} & \xlongequal{\quad} & \{dc \mid c \in F_{p+r-1}, dc \in F_p C\} \\ \downarrow & & \downarrow \\ A_p^{r'} & \xlongequal{\quad} & \{c \in F_p C \mid dc \in F_{p-r'} C\} \end{array}$$

since $d^2 = 0$. Thus, we can graft our chains of inclusions as follows.

$$0 = B_p^0 \subset B_p^1 \subset \cdots \subset B_p^\infty \subset Z_p^\infty \subset \cdots \subset Z_p^1 \subset Z_p^0 = E_p^0.$$

We defined

$$Z_p^r = \eta_p(A_p^r).$$

Recall that η_p was defined to be the canonical projection corresponding to a short exact sequence. Taking subobjects (where the left-hand square is a pullback), we find the following monomorphism of short exact sequences.

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_{p-1}C \cap A_r^p & \hookrightarrow & A_r^p & \longrightarrow & Z_r^p \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & F_{p-1}C & \hookrightarrow & F_p C & \twoheadrightarrow & F_p C / F_{p-1}C \longrightarrow 0 \end{array}$$

Rewriting the definitions, we find that

$$F_{p-1}C \cap A_r^p = \{c \in F_{p-1}C \mid dc \in F_{p-r}C\} = A_{p-1}^{r-1},$$

so

$$Z_p^r \cong A_p^r / A_{p-1}^{r-1}.$$

We now define

$$E_p^r = \frac{Z_p^r}{B_p^r}.$$

We can write

$$\frac{Z_p^r}{B_p^r} \cong \frac{A_p^r + F_{p-1}C}{d(A_{p+r-1}^{r-1}) + F_{p-1}C} \cong \frac{A_p^r}{d(A_{p+r-1}^{r-1}) + A_{p-1}^{r-1}}.$$

I don't understand this step, and I don't have time to do the rest. \square

Theorem 174. Let C_\bullet be a chain complex, and F a bounded filtration on C_\bullet . Then the spectral sequence constructed in [Theorem 173](#) converges to the homology of the homology of C_\bullet ; that is

$$E_{p,q}^1 = H_{p+q}(F_p C_\bullet / F_{p-1} C_\bullet) \implies H_{j+q}(\text{Tot}(C_\bullet)).$$

Example 175. Let E be the spectral sequence associated to a filtered complex C_\bullet , and suppose that

$$E_{p,q}^r = \begin{cases} A_p, & q = 0 \\ 0, & q \neq 0 \end{cases}, \quad r > 0.$$

Then certainly the spectral sequence collapses.

The claim is that $A_p = H_p(C_\bullet)$; that is, the spectral sequence converges to A_p . To see this, we need to find filtrations

$$0 = F_{s-1}A_p \subset F_s A_p \subset F_{s+1}A_p \subset \cdots \subset F_t A_p \subset F_{t+1}A_p = A_p$$

such that, for each p and q , we have

$$E_{p,q}^\infty = F_p A_{p+q} / F_{p-1} A_{p+q}.$$

This requires that

$$\begin{aligned} 0 &= F_s A_p / F_{s-1} A_p \\ &\vdots \\ 0 &= F_{p-1} A_p / F_{p-2} A_p \\ A_p &= F_p A_p / F_{p-1} A_p \\ 0 &= F_{p+1} A_p / F_p A_p \\ &\vdots \\ 0 &= F_{t+1} A_p / F_t A_p. \end{aligned}$$

6. Spectral sequences

Starting at $F_{s-1}A_p = 0$, these equations tell us that

$$0 = F_{s-1}A_p = F_sA_p = F_{s+1}A_p = \cdots = F_{p-1}A_p.$$

We then have that

$$A_p = F_pA_p/0 = F_pA_p.$$

Again, our equations tell us that

$$F_pA_p = F_{p+1}A_p = \cdots = F_tA_p.$$

This tells us that $H_p(C_\bullet) = A_p$.

Note that we could have taken our filtration to be the one-step filtration

$$0 = F_{p-1}A_p \subset F_pA_p = A_p.$$

Example 176. Suppose instead of collapsing to a single row, our spectral sequence stabilizes to two rows:

$$E_{p,q}^\infty = \begin{cases} A_p, & q = 0 \\ B_p, & q = 1 \\ 0, & \text{otherwise.} \end{cases}$$

Identical logic to that above tells us that

$$\begin{aligned} F_{p-1}H_p/F_{p-2}H_p &= 0 \\ F_pH_p/F_{p-1}H_p &= A_p \\ F_{p+1}H_p/F_pH_p &= B_{p-1} \\ F_{p+2}H_p/F_{p+1}H_p &= 0, \end{aligned}$$

i.e. that we have a two-step filtration

$$0 = F_{p-1}H_p \subset F_pH_p \subset F_{p+1}H_p = H_p,$$

with $F_pH_p = A_p$ and $F_{p+1}H_p/A_p = B_{p-1}$. This says precisely that we have a short exact sequence

$$0 \longrightarrow A_p \hookrightarrow H_p \twoheadrightarrow B_{p-1} \longrightarrow 0.$$

That is, we know our homology up to an extension problem. This is exactly what happened in [Example 165](#).

6.3. The spectral sequence of a double complex

A special case of the spectral sequence of a filtered complex is the spectral sequence of a double complex. Given a double complex $C_{\bullet,\bullet}$, we define a new double complex

$$\left({}^I\tau_{\leq n}C\right)_{p,q} = \begin{cases} C_{p,q} & p \leq n \\ 0, & p > n \end{cases}.$$

That is, the $({}^I\tau_{\leq n}C)$ is the double complex obtained by setting all the entries to the right of the column $C_{p,\bullet}$ to zero.

This leads to a filtered complex

$$0 \subset \cdots \subset \text{Tot}({}^I\tau_{\leq 0}C)_{p,q} \subset \text{Tot}({}^I\tau_{\leq 1}C)_{p,q} \subset \cdots \subset \text{Tot}(C)_{p,q};$$

that is, with

$$F_p C = \text{Tot}({}^I\tau_{\leq p}C).$$

This gives rise to a spectral sequence via [Theorem 173](#).

Specifically, we have

$$\begin{aligned} {}^I E_{p,q}^0 &= \frac{F_p C_{p+q}}{F_{p-1} C_{p+q}} \\ &= \frac{\text{Tot}({}^I\tau_{\leq p}C)_{p+q}}{\text{Tot}({}^I\tau_{\leq p-1}C)_{p+q}} \\ &= \frac{\bigoplus_{i+j=p+q} C_{i,j}}{\bigoplus_{\substack{i+j=p+q \\ i \leq p-1}} C_{i,j}} \\ &= \frac{C_{0,p+q} \oplus \cdots \oplus C_{p,q}}{C_{0,p+q} \oplus \cdots \oplus C_{p-1,q+1}} \\ &= C_{p,q}. \end{aligned}$$

The differentials $C_{p,q} \rightarrow C_{p,q-1}$ are simply the vertical differentials $d_{p,q}^v$.

Of course, we can also transpose our double complex before performing the above procedure; this gives a different spectral sequence ${}^II E_{p,q}^0$.

By [Theorem 173](#), both of these converge to the homology of the total complex $H_{p+q}(\text{Tot}(C))$.

Definition 177 (spectral sequences associated to a double complex).

- The vertically filtered spectral sequence associated to a double complex is the spectral sequence ${}^I E_{p,q}^0 \Rightarrow H_{p+q}(\text{Tot}(C))$.
- The horizontally filtered spectral sequence associated to a double complex is the spectral sequence ${}^II E_{p,q}^0 \Rightarrow H_{p+q}(\text{Tot}(C))$.

The vertically filtered spectral sequence associated to a double complex is the one given in [Section 6.1](#).

$$M_{p,q} = P_p \otimes_R Q_q.$$

Let us work out explicitly the vertical filtration. The zeroth page is as follows.

$$\begin{array}{ccc}
 \vdots & \vdots & \vdots \\
 \downarrow & \downarrow & \downarrow \\
 P_0 \otimes_R Q_2 & P_1 \otimes_R Q_2 & P_2 \otimes_R Q_2 \\
 \text{id} \otimes d^Q \downarrow & \text{id} \otimes d^Q \downarrow & \text{id} \otimes d^Q \downarrow \\
 P_0 \otimes_R Q_1 & P_1 \otimes_R Q_1 & P_2 \otimes_R Q_1 \\
 \text{id} \otimes d^Q \downarrow & \text{id} \otimes d^Q \downarrow & \text{id} \otimes d^Q \downarrow \\
 P_0 \otimes_R Q_0 & P_1 \otimes_R Q_0 & P_2 \otimes_R Q_0
 \end{array}$$

$$\begin{aligned} H_q(P_p \otimes_R Q_\bullet) &= P_p \otimes_R H_q(Q_\bullet) \\ &= \begin{cases} P_p \otimes_R N, & q = 0 \\ 0, & q \neq 0 \end{cases}. \end{aligned}$$

132

page.

$$\begin{array}{ccccccc}
 & & & & & & \nearrow^q \\
 & & 0 & & 0 & & 0 & & \cdots \\
 & & 0 & & 0 & & 0 & & \cdots \\
 & & & & & & & & \\
 & & P_0 \otimes_R N & \longleftarrow & P_1 \otimes_R N & \longleftarrow & P_2 \otimes_R N & \longleftarrow & \cdots \\
 & & & & & & & & \\
 & & & & & & & & \searrow_p
 \end{array}$$

We now have

$$\begin{aligned}
 {}^I E_{p,q}^2 &= H_p(P_\bullet \otimes_R N) \\
 &= \operatorname{Tor}_p(M, N).
 \end{aligned}$$

At this point the spectral sequence collapses, telling us that

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(P_\bullet \otimes_R N) = \operatorname{Tor}_p(M, N).$$

The other filtration gives

$$H_p(P_\bullet \otimes_R Q_\bullet) = H_p(M \otimes_R Q_\bullet).$$

Thus,

$$H_p(M \otimes_R Q_\bullet) = \operatorname{Tor}_p(M, N),$$

which says that Tor is balanced.

Example 179 (universal coefficient theorem). Let X_\bullet be a chain complex of free abelian groups, and let A be an abelian group. Then there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes A \hookrightarrow H_n(X \otimes A) \twoheadrightarrow \operatorname{Tor}_1(H_{n-1}(X), A) \longrightarrow 0.$$

6.4. Hyper-derived functors

Consider abelian categories and right exact functors as follows.

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{E}$$

$$2. P_{p,\bullet} \xrightarrow{\cong} A_p$$

Proof.

1. We have essentially by definition the following short exact sequence.

$$0 \longrightarrow B(P_{p,\bullet}, d^h) \hookrightarrow Z(P_{p,\bullet}, d^h) \twoheadrightarrow H(P_{p,\bullet}, d^h) \longrightarrow 0$$

The outer terms are projective because they are the terms of projective resolutions, so $Z(P_{p,\bullet}, d^h)$ is also projective. Thus, $Z(P_{p,\bullet}, d^h)$ is at the very least a complex of projectives.

Further, we have the first few rows of the long exact sequence on homology.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & Z(P_{p,1}, d^h) & \longrightarrow & 0 & & \\ & & \searrow \delta & & \nearrow & & \\ & 0 & \longrightarrow & Z(P_{p,0}, d^h) & \longrightarrow & 0 & \\ & & \searrow \delta & & \nearrow & & \\ & B(A_p, d^A) & \longrightarrow & Z(A_p, d^A) & \longrightarrow & H(A_p, d^A) & \longrightarrow 0 \end{array}$$

This tells us that the homology of $Z(P_{p,\bullet})$ is

$$H_q(Z(P_{p,\bullet}, d^h)) = \begin{cases} Z(A_p, d^A), & q = 0 \\ 0, & q > 0 \end{cases},$$

i.e. that $Z(P_{p,\bullet}, d^h)$ is a projective resolution of $Z(A_p, d^A)$.

2. Exactly the same, but with the short exact sequence

$$0 \longrightarrow Z(P_{p,\bullet}, d^h) \hookrightarrow H(P_{p,\bullet}, d^h) \twoheadrightarrow B(P_{p-1,\bullet}, d^h) \longrightarrow 0$$

□

Proposition 182. Every chain complex A_\bullet has a Cartan-Eilenberg resolution $P_{\bullet,\bullet} \xrightarrow{\cong} A_\bullet$.

Proof. We construct a Cartan-Eilenberg resolution $P_{\bullet,\bullet}$ explicitly. Pick projective resolutions

$$P_{p,\bullet}^B \rightarrow B(A_p, d^h), \quad B_{p,\bullet}^H \xrightarrow{\cong} H(A_p, d^h).$$

By the horseshoe lemma, we can find a projective resolution $P_{p,\bullet}^Z \xrightarrow{\cong} Z(A_p, d^h)$ fitting

6. Spectral sequences

into the following exact sequence.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_{p,\bullet}^B & \hookrightarrow & P_{p,\bullet}^Z & \twoheadrightarrow & P_{p,\bullet}^H \longrightarrow 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & B(A_p, d^h) & \hookrightarrow & Z(A_p, d^h) & \twoheadrightarrow & H(A_p, d^h) \longrightarrow 0
 \end{array} \tag{6.3}$$

Playing the same game again, we find a projective resolution of A_p .

$$\begin{array}{ccccccc}
 0 & \longrightarrow & P_{p,\bullet}^Z & \hookrightarrow & P_{p,\bullet}^A & \twoheadrightarrow & P_{p-1,\bullet}^B \longrightarrow 0 \\
 & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 0 & \longrightarrow & Z(A_p, d^h) & \hookrightarrow & A_p & \twoheadrightarrow & B(A_{p-1}, d^h) \longrightarrow 0
 \end{array} \tag{6.4}$$

We then define our Cartan-Eilenberg resolution to be the double complex whose p th column is $(P_{p,\bullet}^A, (-1)^p d^{p^A})$, and whose horizontal differentials are given by the composition

$$\begin{array}{ccccccc}
 P_{p,\bullet}^A & \longrightarrow & P_{p-1,\bullet}^B & \longrightarrow & P_{p-1,\bullet}^Z & \longrightarrow & P_{p-1,\bullet}^A \\
 \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\
 A_p & \xrightarrow{d^A} & B_{p-1} & \hookrightarrow & Z_{p-1} & \hookrightarrow & A_{p-1}
 \end{array} \tag{6.5}$$

In this case, all the criteria are satisfied by definition. \square

Proposition 183. Let A_\bullet and A'_\bullet be bounded-below chain complexes, and let $P_{\bullet,\bullet} \xrightarrow{\simeq} A_\bullet$ and $P'_{\bullet,\bullet} \xrightarrow{\simeq} A'_\bullet$ be Cartan-Eilenberg resolutions. Then any morphism of chain complexes $f: A_\bullet \rightarrow A'_\bullet$ can be lifted to a morphism $\tilde{f}: P_{\bullet,\bullet}^A \rightarrow P_{\bullet,\bullet}^B$ of double complex. Furthermore, this lift is unique up to a *homotopy of double complexes*, i.e. for any other lift $\tilde{\tilde{f}}$, there exist maps

$$s_{p,q}^h: P_{p,q} \rightarrow P'_{p+1,q}, \quad s_{p,q}^v: P_{p,q} \rightarrow E_{p,q+1}$$

such that

Proof. We have the following lifts.

$$\begin{array}{ccccccc}
 & & Q_{p,\bullet}^B & & Q_{p,\bullet}^Z & & Q_{p,\bullet}^H \\
 & \nearrow & \downarrow & & \downarrow & & \nearrow \\
 P_{p,\bullet}^B & & & & P_{p,\bullet}^Z & & P_{p,\bullet}^H \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 \longrightarrow & B_p(B) & \longrightarrow & Z_p(B) & \longrightarrow & H_p(B) & \longrightarrow 0 \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
 0 \longrightarrow & B_p(A) & \longrightarrow & Z_p(A) & \longrightarrow & H_p(A) & \longrightarrow 0
 \end{array}$$

The horseshoe lemma for morphisms (*mutatis mutandis*) allows us to complete this diagram in a compatible way as follows.

$$\begin{array}{ccccccc}
 0 \longrightarrow & Q_{p,\bullet}^B & \longrightarrow & Q_{p,\bullet}^Z & \longrightarrow & Q_{p,\bullet}^H & \longrightarrow 0 \\
 \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\
 0 \longrightarrow & P_{p,\bullet}^B & \longrightarrow & P_{p,\bullet}^Z & \longrightarrow & P_{p,\bullet}^H & \longrightarrow 0 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 0 \longrightarrow & B_p(B) & \longrightarrow & Z_p(B) & \longrightarrow & H_p(B) & \longrightarrow 0 \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
 0 \longrightarrow & B_p(A) & \longrightarrow & Z_p(A) & \longrightarrow & H_p(A) & \longrightarrow 0
 \end{array}$$

Using the lifts we constructed above and identical logic, we get the following.

$$\begin{array}{ccccccc}
 0 \longrightarrow & Q_{p,\bullet}^Z & \longrightarrow & Q_{p,\bullet} & \longrightarrow & Q_{p-1,\bullet}^B & \longrightarrow 0 \\
 \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \\
 0 \longrightarrow & P_{p,\bullet}^Z & \longrightarrow & P_{p,\bullet} & \longrightarrow & P_{p-1,\bullet}^B & \longrightarrow 0 \\
 \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
 0 \longrightarrow & Z_p(B) & \longrightarrow & B_p(A) & \longrightarrow & B_{p-1}(A) & \longrightarrow 0 \\
 \downarrow & \nearrow & \downarrow & \nearrow & \downarrow & \nearrow & \\
 0 \longrightarrow & Z_p(A) & \longrightarrow & A_p & \longrightarrow & B_{p-1}(A) & \longrightarrow 0
 \end{array}$$

Thus we have, for each $p \in \mathbb{Z}$, a morphism of complexes $P_{p,\bullet} \rightarrow Q_{p,\bullet}$. It remains only to

6. Spectral sequences

show that each of the squares

$$\begin{array}{ccc} P_{p,\bullet} & \longrightarrow & P_{p-1,\bullet} \\ \downarrow & & \downarrow \\ Q_{p,\bullet} & \longrightarrow & Q_{p-1,\bullet} \end{array}$$

commutes. But we get these squares by pasting together squares stolen from the above diagrams as follows.

$$\begin{array}{ccccccc} P_{p,\bullet} & \xrightarrow{d^h} & P_{p-1,\bullet}^Z & \hookrightarrow & P_{p-1,\bullet}^B & \hookrightarrow & P_{p-1,\bullet} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Q_{p,\bullet} & \xrightarrow{d^h} & Q_{p-1,\bullet}^Z & \hookrightarrow & Q_{p-1,\bullet}^B & \hookrightarrow & Q_{p-1,\bullet} \end{array}$$

□

Proposition 184. Let f, g be chain homotopic maps via the homotopy f .

Proposition 185. Let $P_{\bullet,\bullet}^A$ be a Cartan-Eilenberg resolution of A_\bullet . Then the canonical map

$$\mathrm{Tot}(P_{\bullet,\bullet}^A) \rightarrow A$$

is a quasi-isomorphism

Proof. Spectral sequence with vertical filtration. □

6.4.2. Hyper-derived functors

Definition 186 (left hyper-derived functor). Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a right exact functor, and assume \mathcal{A} has enough projectives. The left hyper-derived functor $\mathbb{L}_i F$ of F is the functor

$$\mathbb{L}_i F: \mathrm{Ch}_+(\mathcal{A}) \rightarrow \mathcal{B}; \quad \mathbb{L}_i F(A_\bullet) = H_i(\mathrm{Tot}(F(P_{\bullet,\bullet}^A))),$$

where $P_{\bullet,\bullet}^A$ is a Cartan-Eilenberg resolution of A_\bullet .

Hyper-derived functors are functorial by

Proposition 187. Let A_\bullet be a bounded below chain complex in an abelian category \mathcal{A} . There are two spectral sequences

$${}^I E_{p,q}^2 = H_p(L_q F(A_\bullet)) \implies \mathbb{L}_{p+q} F(A_\bullet),$$

$${}^II E_{p,q}^2 = L_p F(H_q(A_\bullet)) \implies \mathbb{L}_{p+q} F(A_\bullet).$$

Proof. We use the spectral sequence for the double complex $F(P_{\bullet,\bullet}^A)$. The horizontal filtration has no surprises. Using the vertical filtration, one must use that

$$H_p(F(P_{\bullet,\bullet}^A)) = F(H_p(P_{\bullet,\bullet}^A)),$$

which is true because the horizontal cycles and boundaries are □

This allows us to define the famous *Grothendieck spectral sequence*, which allows us to compute the composition of two derived functors.

Proposition 188. Let \mathcal{A} , \mathcal{B} , and \mathcal{C} be abelian categories, and assume that \mathcal{A} and \mathcal{B} have enough projectives. Let F and G as below be right exact, and assume that G sends projective objects to F -acyclic objects.¹ Then for all $A \in \mathcal{A}$, there exists a convergent spectral sequence

$$E_{p,q}^2 = L_p F(L_q G(A)) \implies L_{p+q}(F \circ G)(A).$$

Proof. Let $P_\bullet \xrightarrow{\sim} A$ be a projective resolution. From [Proposition 187](#), we find two spectral sequences. Applying the first to $G(P_\bullet)$, we find

$$H_p(L_q(F(G(P_\bullet)))) \implies \mathbb{L}_{p+q}.$$

However, by assumption $G(P_\bullet)$ is a chain complex of F -acyclic objects, so

$$L_q F(G(P_\bullet)) = \begin{cases} F(G(P_\bullet)), & q = 0 \\ 0, & \text{otherwise} \end{cases}.$$

Taking p th homology thus gives us, with $q = 0$,

$$\begin{aligned} H_p(L_0 F(G(P_\bullet))) &= H_p(F \circ G(P_\bullet)) \\ &= L_p(F \circ G)(A). \end{aligned}$$

Thus, the spectral sequence collapses at the second page, and we have

$$\mathbb{L}_p F(G(P_\bullet)) \cong L_p(F \circ G)(A)$$

The second spectral sequence guarantees us that

$$L_p F(H_q(G(P_\bullet))) \implies \mathbb{L}_{p+q}(G(P_\bullet)).$$

However, by definition

$$H_q(G(P_\bullet)) = L_q G(A),$$

¹An object $B \in \mathcal{B}$ is called *F-acyclic* if for all $q > 0$, $L_q F(B) = 0$. For example, projective objects are F -acyclic for all F .

6. Spectral sequences

giving us a spectral sequence

$$E_{p,q}^2 = (L_p F \circ L_q G)(A) \implies L_{p+q}(F \circ G)(A)$$

as required. \square

Example 189 (Lyndon-Hochschild-Serre spectral sequence). Let G be a group, and N a normal subgroup. This gives us the following commuting² diagram.

$$\begin{array}{ccc} \mathbb{Z}G\text{-Mod} & \xrightarrow{(-)_N} & \mathbb{Z}G/N\text{-Mod} \\ & \searrow (-)_G & \swarrow (-)_{G/N} \\ & \text{Ab} & \end{array}$$

If we can show that $(-)_N$ takes projectives to $(-)_{G/N}$ -acyclics, then we are entitled to apply [Proposition 188](#). Note that $(-)_N$ takes free $\mathbb{Z}G$ -modules to free $\mathbb{Z}G/N$ modules, simply by cutting away the basis elements on which N acts trivially. Since any projective is a direct summand of some free module and $(-)_N$ is additive, it thus takes projectives to projectives.

[Proposition 188](#) now tells us that there exists a convergent spectral sequence

$$E_{p,q}^2 = (L_p(-)_{G/N} \circ L_q(-)_N)(A) \implies L_{p+q}(-)_G(A).$$

That is, a spectral sequence

$$E_{p,q}^2 = H_p(G/N, H_q(N, A)) \implies H_{p+q}(G, A).$$

This is known as the *Lyndon-Hochschild-Serre spectral sequence*.

²On the nose!

7. Triangulated categories

Definition 190 (triangulated category). A triangulated category consists of the following data.

1. An additive category \mathcal{T} .
2. An equivalence

$$\mathcal{T} \rightarrow \mathcal{T}; \quad X \mapsto X[1],$$

called the translation functor.

3. A collection of distinguished triangles

$$\left\{ X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[-1] \right\}.$$

These must satisfy the following four axioms.

- (TR1) a) Every morphism f can be extended to a distinguished triangle as above.
b) The collection of distinguished triangles is closed under isomorphism.
c) Given $X \in \mathcal{T}$, the diagram

$$X \xrightarrow{\text{id}} X \longrightarrow 0 \longrightarrow X[1]$$

is also distinguished.

- (TR2) A diagram

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$$

if and only if the ‘rotated’ diagram

$$Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1]$$

is distinguished.

7. Triangulated categories

(TR3) Given a solid commutative diagram

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \\ f \downarrow & & \downarrow & & \downarrow & & \downarrow f[1] \\ X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1] \end{array}$$

with distinguished rows, there exists a dashed morphism making everything commute.

(TR4) Suppose we are given the following distinguished triangles.

$$X \xrightarrow{f} Y \xrightarrow{u} Y/X \xrightarrow{d} X[1]$$

$$Y \xrightarrow{g} Z \xrightarrow{v} Z/Y \xrightarrow{d'} Y[1]$$

$$X \xrightarrow{g \circ f} Z \xrightarrow{w} Z/X \xrightarrow{d''} X[1]$$

Then there is a distinguished triangle

$$Y/X \xrightarrow{\phi} Z/X \xrightarrow{\psi} Z/Y \xrightarrow{\theta} Y/X[1]$$

making the following diagram commute.

$$\begin{array}{ccccccc} & & g \circ f & & & & \\ & \searrow & & \nearrow & & & \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & & \\ & \searrow u & & \nearrow v & & & \\ & Y/X & \xrightarrow{\phi} & Z/X & \xrightarrow{d''} & X[1] & \\ & & \searrow \psi & & \nearrow f[1] & & \\ & & Z/Y & \xrightarrow{d'} & Y[1] & \xrightarrow{u[1]} & Y/X[1] \\ & & & \searrow \theta & & \nearrow & \end{array}$$

Part II.

Algebraic topology

8. Homology

8.1. Basic definitions and examples

8.1.1. Ordinary homology

Singular homology is the study of topological spaces by associating to them chain complexes in the following way.

$$\mathbf{Top} \xrightarrow{\text{singular nerve}} \mathbf{Set}_\Delta \xrightarrow{\text{free abelian group}} \mathbf{Ab}_\Delta \xrightarrow{\text{Dold-Kan correspondence}} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

Definition 191 (singular complex, singular homology). Let X be a topological space. The singular chain complex $S_\bullet(X)$ is defined level-wise by

$$S_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where \mathcal{F} denotes the free group functor and M is the Moore functor ([Definition 141](#)). That is, it has differentials

$$d_n: S_n \rightarrow S_{n-1}; \quad (\alpha: \Delta^n \rightarrow X) \mapsto \sum_{i=0}^n \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(S_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f: X \rightarrow Y$ gives a chain map $S(f): S(X) \rightarrow S(Y)$. This immediately implies that the n th homology of a space X is invariant under homeomorphism.

Example 192. Denote by pt the one-point topological space. Then $\mathbf{Sing}(\text{pt})_n = \{*\}$, and the chain complex $S_\bullet(\text{pt})$ is at each level simply \mathbb{Z} .

The differential d_n is given by

$$d_n = \sum_{i=0}^n (-1)^i \partial^i; \quad * \mapsto \sum_{i=0}^n (-1)^i *.$$

8. Homology

If n is odd, then there are as many summands with even sign as there are with odd sign, and they cancel each other out. If n is even, then there is one more term with positive sign than with negative sign, and one ends up with the identity. Thus $S(\text{pt})_\bullet$ is given level-wise as follows.

$$\begin{array}{ccccccccccccccc} \cdots & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \xrightarrow{0} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \parallel & & \parallel & & \parallel & & \parallel & & \parallel & & \\ \cdots & \xrightarrow{d_4} & S_3 & \xrightarrow{d_3} & S_2 & \xrightarrow{d_2} & S_1 & \xrightarrow{d_1} & S_0 & \xrightarrow{d_0} & S_{-1} & \xrightarrow{d_{-1}} & \cdots \end{array}$$

Thus, the n th homology of the point is

$$H_n(\text{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

8.1.2. Reduced homology

Recall that in addition to the *ordinary*, or *topologist's simplex category* Δ , there is the so-called *extended*, or *algebraist's simplex category* Δ_+ , which includes the object $[-1] = \emptyset$. If we use this category to index our simplicial sets, our chain complexes have a term $\tilde{S}_{-1} = \mathbb{Z}$. It is traditional to denote the differential $d_0: \tilde{S}_0 \rightarrow \tilde{S}_{-1}$ by ε , and call it the *augmentation map*.

$$\cdots \longrightarrow \tilde{S}_2 \xrightarrow{d_2} \tilde{S}_1 \xrightarrow{d_1} \tilde{S}_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

Definition 193 (reduced singular chain complex). The chain complex \tilde{S} is called the reduced singular chain complex of X .

We see immediately that

$$S_i(X) = \tilde{S}_i(X), \quad i \geq 0,$$

implying

$$H_i(X) = \tilde{H}_i(X), \quad i \geq 1.$$

The difference between ordinary and reduced homology is that, as one might suspect, ordinary homology is better-behaved from a topological point of view, and reduced homology is better behaved algebraically (for example, as a functor reduced homology turns wedge products into direct sums).

There is also another interpretation of reduced homology. Recall that we have used the name *augmentation map* before, when dealing with resolutions. There we reinterpreted the augmentation map ε as a chain map to a complex concentrated in degree zero. We can pull exactly the same trick here, viewing the augmentation map as a map of chain

complexes

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_2 & \xrightarrow{d_2} & S_1 & \xrightarrow{d_1} & S_0 \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \varepsilon \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}.$$

Example 194. We can immediately read off that the reduced homology of the point is

$$\tilde{H}_n(\text{pt}) = 0 \quad \text{for all } n.$$

Example 195. Let X be a nonempty path-connected topological space. Then $H_0(X) \cong \mathbb{Z}$. Specifically,

$$\varepsilon: H_0(X) \rightarrow \mathbb{Z}$$

is an isomorphism.

To see this, let $x \in S_0(X)$ be a point. Since $\varepsilon(nx) = n$ for any $n \in \mathbb{Z}$, we know that ε is surjective. If we can show that ε is injective, then we are done.

At face value, ε does not look injective; after all, there are as many elements of $S_0(X)$

consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation “there is a path from x to y .” However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

Example 196. More generally, for any (not necessarily path connected) space X ,

$$H_0(X) = \mathbb{Z}^{\pi_0(X)}.$$

Suppose that X can be written as a disjoint union

$$X = \bigsqcup_i X_i.$$

Then, since a map

$$\sigma: \Delta^n \rightarrow X$$

cannot hit more than one of the X_i , we have

$$\mathbf{Top}\left(\Delta^n, \bigsqcup_i X_i\right) \cong \bigsqcup_i \mathbf{Top}(\Delta^n, X_i).$$

Thus, since the free abelian group functor \mathcal{F} and the Moore functor M are both left adjoints, they preserve colimits and we have

$$S_n\left(\bigsqcup_i X_i\right) \cong \bigoplus_i S_n(X_i).$$

8. Homology

We will come back to reduced homology in [Section 8.7](#), when we have the tools to understand it properly.

8.2. The Hurewicz homomorphism

Let X be a topological space, and let $x \in X$. Let γ be a loop in X which starts and ends at x , i.e.

$$\gamma: \Delta^1 \rightarrow X; \quad \gamma(0) = x = \gamma(1).$$

We can view γ either as a singular 1-simplex in X , or as a representative of a homotopy class in $\pi_1(X, x)$. It turns out that these points of view are compatible.

Lemma 197. The assignment $\gamma \mapsto [\gamma]_{H_1}$ respects homotopy, and thus descends to a map of sets out of $\pi_1(X, x)$.

$$\begin{array}{ccc} \left\{ \begin{array}{l} \text{Loops starting and} \\ \text{ending at } x \end{array} \right\} & \xrightarrow{\text{homology class}} & H_1(X) \\ \text{homotopy class} \downarrow & \nearrow \exists! \tilde{h}_X & \\ \pi_1(X, x) & & \end{array}$$

Proof. Let γ_1 and γ_2 be loops based at x .

First, suppose that $\gamma_1 \stackrel{H}{\sim} \gamma_2$, i.e.

In order to see that □

The map \tilde{h}_X is very well-behaved.

Lemma 198. For a topological space X and point $x \in X$, the map

$$\tilde{h}_X: \pi_1(X, x) \rightarrow H_1(X)$$

is a group homomorphism.

Proof. We need to show that it sends the identity to the identity and respects composition. □

Because $H_1(X)$ is abelian, \tilde{h}_X descends to a map out of $\pi_1(X, x)_{\text{ab}}$.

Definition 199 (Hurewicz homomorphism). Let X be a path-connected topological space, and let $x \in X$. The Hurewicz homomorphism is the map

$$h_X: \pi_1(X, x)_{\text{ab}} \rightarrow H_1(X).$$

Theorem 200 (Hurewicz). For any path-connected topological space X , the Hurewicz homomorphism is an isomorphism

$$h_X: \pi_1(X, x_0)_{\text{ab}} \cong H_1(X),$$

where $(-)_{\text{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism.

$$\begin{array}{ccc} & \xrightarrow{(\pi_1)_{\text{ab}}} & \\ \text{Top}_* & \Downarrow h & \text{Ab} \\ & \xleftarrow{H_1} & \end{array}$$

Proof. We construct an inverse explicitly.

For each point $x \in X$, pick a path

$$\gamma_x: \Delta^1 \rightarrow X; \quad \gamma_x(0) = x_0, \quad \gamma_x(1) = x$$

connecting x_0 to x . For $x = x_0$, choose γ_{x_0} to be the constant path.

For each generator $\alpha: \Delta^1 \rightarrow X$ in $H_1(X)$, the concatenation

$$\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}$$

is a path starting and ending at x . We may thus define a map

$$\phi: S_1(X) \rightarrow \pi_1(X, x)_{\text{ab}}$$

on generators by

$$\alpha \mapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}],$$

and addition via

$$\alpha + \beta \mapsto [\gamma_{\alpha(0)} * \alpha * \overline{\gamma_{\alpha(1)}}][\gamma_{\beta(0)} * \beta * \overline{\gamma_{\beta(1)}}].$$

In order to check that defining this homomorphism on generators descends to a map on homology, we have to check that it sends boundaries $\partial\sigma$ to zero. \square

Example 201. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\text{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \quad n > 1.$$

Example 202. Denote by Σ_g the two-dimensional surface of genus g . We know that

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_{2g}, b_{2g} \mid \prod_{i=1}^g [a_i, b_i] \rangle.$$

8. Homology

The abelianization of this is simply \mathbb{Z}^{2g} , so

$$H_1(\Sigma_g) = \mathbb{Z}^{2g}.$$

Example 203. Since

$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

8.3. The method of acyclic models

We need to break the flow here for a theorem which will show up several times, called the *method of acyclic models*.

Definition 204 (category with models). A category with models is a pair $(\mathcal{C}, \mathcal{M})$, where \mathcal{C} is a category and $\mathcal{M} \subset \text{Obj}(\mathcal{C})$ is a set of objects of \mathcal{C} .

Definition 205 (free, acyclic functor). Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let $F: \mathcal{C} \rightarrow \mathbf{Ch}_{\geq 0}(R\text{-}\mathbf{Mod})$.

1. We say that F is acyclic on \mathcal{M} if for each $M \in \mathcal{M}$, $F(M)$ is acyclic in positive degree, i.e. if $H_n(F(M)) = 0$ for $n > 0$.
2. Let J be a set, and $\mathcal{M}_J \subset \mathcal{M}$ a J -indexed set of objects in \mathcal{M} . Note that we allow the possibility that each $M \in \mathcal{M}$ can appear more than once in \mathcal{M}_J , or not at all.

Let $F_*: \mathcal{C} \rightarrow R\text{-}\mathbf{Mod}$ be a functor. An \mathcal{M}_J -basis for F_* is, for each $j \in J$ an element $m_j \in F_*(M_j)$ (forming an indexed collection $\{m_j \in F_*(M_j)\}_{j \in J}$) such that for any $X \in \mathcal{C}$ the indexed collection

$$\{F_*(f)(m_j)\}_{j \in J, f \in \text{Hom}(M_j, X)}$$

is a basis for $F_*(X)$ as a free R -module; that is, that we can write

$$F_*(X) = \mathbb{Z}\{F_*(f)(m_j)\}_{j \in J, f \in \text{Hom}(M_j, X)}.$$

We say that F is free on \mathcal{M} if for each $q \geq 0$ there exists some set J_q and indexed set \mathcal{M}_{J_q} such that each F_q has an \mathcal{M}_{J_q} -basis.

Example 206. Consider the category \mathbf{Top} with models $\{\Delta^n\}_{n=0,1,\dots}$. The functor $S: \mathbf{Top} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$ is both free and acyclic. Acyclicity is clear since Δ^n is contractible for all n .

To see freeness, note that the singleton

$$\{\text{id}_{\Delta^n} : \Delta^n \rightarrow \Delta^n\}$$

forms a basis for F_n because the abelian group $S_n(X)$ is free with generating set

$$\{S(\alpha)(\text{id}_{\Delta^n}) = \alpha\}_{\alpha \in \text{Hom}(\Delta^n, X)}.$$

The freeness condition can be interpreted as telling us that in order to know how the functor F behaves on any object, it is enough to know how it behaves on the M_α . In fact, this is sufficiently precise that it can be made into a theorem.

Proposition 207. Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let $F, G : \mathcal{C} \rightarrow \mathbf{Ab}$ be functors, with F \mathcal{M} -free. Then specifying a natural transformation $\eta : F \Rightarrow G$ is equivalent to specifying only the values

$$\eta_{M_i}(m_i),$$

Theorem 208 (acyclic model theorem). Let $(\mathcal{C}, \mathcal{M})$ be a category with models, and let F, G be functors $\mathcal{C} \rightarrow \mathbf{Ch}_{\geq 0}(R\text{-Mod})$ such that F is free on \mathcal{M} and G is acyclic on \mathcal{M} .

1. Any natural transformation $\bar{\tau}_0 : H_0(F) \rightarrow H_0(G)$ can be extended to a natural chain map $\tau : F \rightarrow G$.
2. This extension is unique up to homotopy in the sense that two natural chain maps $\tau, \tau' : F \rightarrow G$ inducing the same natural transformation $H_0(G) \rightarrow H_0(G')$ are naturally chain homotopic.

Proof.

1. Let $\tau : F \Rightarrow G$ be a natural transformation. First, we collect some results about the natural transformation $\tau_q : F_q \Rightarrow G_q$. Note that, since

$$F_q(X) \cong R\{F_q(f)(m_j)\}_{j \in J_q, f \in \text{Hom}(M_j, X)}$$

we can specify $\tau_q(X)$ (the component of τ_q at X) completely by specifying how it acts on those elements of $F_q(X)$ of the form $F_q(f)(m_j)$, and that we are free in choosing this action.

8. Homology

Consider the following naturality square for some map $f: M_j \rightarrow X$.

$$\begin{array}{ccc}
 F_q(M_j) & \xrightarrow{\tau_q(m_j)} & G_q(M_j) \\
 \downarrow F_q(f) & & \downarrow G_q(f) \\
 F_q(X) & \xrightarrow{\tau_q(X)} & G_q(X)
 \end{array}$$

$m_j \mapsto \tau_q(M_j)(m_j)$
 $\downarrow \qquad \qquad \downarrow$
 $F_q(f)(m_j) \mapsto \star$

Naturality requires that

$$\tau_q(X)(F_q(f)(m_j)) = G_q(f)(\tau_q(M_j)(m_j)).$$

Thus, in order to define $\tau_q(X)$ for all X , we need only specify how $\tau_q(M_j)$ behaves on $m_j \in F_q(M_j)$. Defining it in this way and extending it by naturality will ensure that the maps we construct form the components of a natural transformation.

For $q = 0$, this is easy, since we have a natural transformation $\bar{\tau}_0$. Since everything in $H_0(F)$ is a cycle, we can define $\tau_0(M_j)(m_j)$ to be any representative of the equivalence class

$$\bar{\tau}_0(M_j)[m_j] \in H_0(G(M_j)).$$

Now suppose we have defined natural transformations τ_{q-1} , for $q > 0$. For $j \in J_q$, we define $\tau_q(M_j(m_j))$ by

$$\partial \tau_q(M_j)(m_j) = \tau_{q-1}(M_j(\partial m_j)).$$

This is well-defined precisely because

$$\partial \tau_{q-1}(M_j)(\partial m_j) = \tau_{q-2}(\partial^2 m_j) = 0$$

since τ_{q-1} is by assumption a chain map and G is by assumption acyclic on \mathcal{M} .

2. One defines a chain homotopy inductively using the same trick.

□

8.4. Homotopy equivalence

We would like that two maps which are homotopic induce the same maps on homology. As alluded to in [Section 3.6](#), we will see that a homotopy H between maps $f, g: X \rightarrow Y$

induces a chain homotopy h between $S_n(f)$ and $S_n(g)$.

8.4.1. Homotopy invariance: with acyclic models

Proposition 209. Let $f, g: X \rightarrow Y$ be continuous maps between topological spaces, and let $H: X \times [0, 1] \rightarrow Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$.

Proof. Suppose we are given a homotopy $f \stackrel{H}{\sim} g$.

$$\begin{array}{ccc}
 X & & \\
 \text{id} \times \{0\} \downarrow & \searrow f & \\
 X \times I & \xrightarrow{H} & X \\
 \text{id} \times \{1\} \uparrow & \nearrow g & \\
 X & &
 \end{array}$$

Consider the functors

$$F = S_\bullet(-), \quad G = S_\bullet(I \times -): \mathbf{Top} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab}).$$

Take \mathbf{Top} with models $\mathcal{M} = \{\Delta^q \mid q = 0, 1, \dots\}$, $J_n = \{*\}$, $\mathcal{M}_{J_n} = \{\Delta^n\}$ and $m_* = \text{id}_{\Delta^n}$. For all n , we have that F is free because

$$S_n(X) \cong \mathbb{Z}\{S_n(\alpha)(\text{id}_{\Delta^n}) \mid \alpha \in \text{Hom}(\Delta^n, X)\},$$

and that G is acyclic because $\Delta^n \times I$ is contractible for all n .

Denote by ι_X^0 the map $X \rightarrow X \times I$, which sends $x \mapsto (x, 0)$. Define ι_X^1 analogously.

Consider the natural transformations

$$i^0, i^1: F \Rightarrow G$$

with components

$$i_X^0 = S_\bullet(\iota_X^0), \quad i_X^1 = S_\bullet(\iota_X^1).$$

These clearly agree on H_0 . Thus, by [Theorem 208](#), i^0 and i^1 are chain homotopic via some chain homotopy D .

If D is homotopy between i^0 and i^1 , then $S(H) \circ D$ is a homotopy between $S(H) \circ i^0$ and $S(H) \circ i^1$. But $S(H) \circ i^0 = S(f)$, and $S(H) \circ i^1 = S(g)$. \square

Corollary 210. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 211. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

8.4.2. Homotopy-invariance: without acyclic models

Such a homotopy H is a map

$$H: X \times [0, 1] \rightarrow Y.$$

It seems reasonable that we would get the chain homotopy in question by relating the singular homology of X and the singular homology of $X \times [0, 1]$.

Here is the game plan.

- We notice that there are n obvious ways of mapping

$$p_i: \Delta^{n+1} \rightarrow \Delta^n \times [0, 1],$$

and $\Delta^n \times [0, 1]$ is the union of the images, which overlap only along their boundaries.

- Given an n -simplex $\alpha: \Delta^n \rightarrow X$, we can produce an $(n + 1)$ -simplex in $X \times [0, 1]$ by pulling back:

$$P_i: \Delta^{n+1} \rightarrow \Delta^n \times [0, 1] \rightarrow X \times [0, 1].$$

- Given a homotopy

$$H: X \times [0, 1] \rightarrow Y$$

between f and g , we can pull back by the P_i , giving us an $(n + 1)$ -simplex

$$P_i(\alpha): \Delta^{n+1} \rightarrow \Delta^n \times [0, 1] \rightarrow X \times [0, 1] \rightarrow Y.$$

The association

$$\alpha \mapsto P_i \alpha$$

is a homomorphism.

- If we take a sum of the $P_i \alpha$, we get a cellular decomposition of the image of α . However, by taking an alternating sum

$$P = \sum_{i=0}^n P_i$$

we can get (upon passing to boundaries) the interior walls to cancel each other

out. Thus, the composition

$$d \circ P$$

gives a cellular decomposition of the boundary of the image of the cylinder. The composition $P \circ d$ gives you a cellular decomposition of the sides of the cylinder with the opposite sign. Thus, the sum $d \circ P + P \circ d$ gives you the (difference of) the top and the bottom of the cylinder.

- This tells us that *homotopic maps between topological spaces induce the same map between homotopy groups*.

Definition 212 (prism maps). The i th prism map is the map

$$p_i: \Delta^{n+1} \rightarrow \Delta^n \times I; \quad (t_0, \dots, t_{n+1}) \mapsto \left((t_0, \dots, t_i + t_{i+1}, \dots, t_n), \sum_{j=i+1}^n t_j \right)$$

The i th prism map sends the vertices $\{0, \dots, n+1\} \in \Delta^n$ to the vertices of the prism as follows.

$$\begin{array}{ccccccc} 0 & 1 & 2 & \cdots & i & i+1 & \cdots & n+1 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\ (0, 0) & (1, 0) & (2, 0) & \cdots & (i, 0) & (i, 1) & \cdots & (n, 1) \end{array}$$

We will denote the vertex $(j, 0)$ by a_j and $(j, 1)$ by b_j . We can thus summarize the situation by saying that

$$p_i: (0, \dots, n+1) \rightarrow (a_1, \dots, a_i, b_i, \dots, b_n),$$

where the i th element of the last tuple corresponds to the location of the i th vertex in the prism $\Delta^n \times I$.

Definition 213 (prism operator). We define maps

$$P_i: S_n(X) \rightarrow S_{n+1}(X \times I)$$

which send $\alpha: \Delta^n \rightarrow X$ to the composition

$$\Delta^{n+1} \xrightarrow{p_i} \Delta^n \times I \xrightarrow{\alpha \times \text{id}_I} X \times I.$$

The prism operator is the alternating sum

$$\sum_{i=0}^n P_i: S_n(X) \rightarrow S_{n+1}(X \times I).$$

Given an n -simplex σ , the prism operator P_i creates an $n+1$ -simplex $(\alpha \times \text{id}) \circ p_i$. To

8. Homology

keep track of the indices of the $n + 1$ -simplex we are using, we notate this by

$$(\sigma \times \text{id})|_{[a_0, \dots, a_i, b_i, \dots, b_n]}.$$

Proposition 214. Denote by j_i , for $i = 0, 1$, the map

$$j_i: X \rightarrow X \times I; \quad x \mapsto (x, i).$$

The prism operator is a chain homotopy between $S_\bullet(j_0)$ and $S_\bullet(j_1)$.

Proof. We □

8.5. Relative homology

8.5.1. Relative homology

Denote by **Pair** the category whose objects are pairs (X, A) , where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \rightarrow (Y, B)$ are maps $f: X \rightarrow Y$ such that $f(A) \subset B$. That is, the category **Pair** is the full subcategory on the category $\mathbf{Fun}(I, \mathbf{Top})$ on monomorphisms; the objects are diagrams

$$A \hookrightarrow X$$

and the morphisms are commuting squares

$$\begin{array}{ccc} A & \hookrightarrow & X \\ \downarrow & & \downarrow \\ B & \hookrightarrow & Y \end{array}$$

Definition 215 (relative homology). Let (X, A) be a pair of spaces. The inclusion $i: A \hookrightarrow X$ induces a map of chain complexes $A \hookrightarrow X$. The relative chain complex of (X, A) is the cokernel of $C_\bullet(i)$.

Since colimits are computed level-wise, the relative chain complex is given by the level-wise quotient

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Lemma 216. For each n , relative homology provides a functor $H_n: \mathbf{Pair} \rightarrow \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{array}{ccc} S(X)_\bullet & \xrightarrow{f} & S(Y)_\bullet \\ \Downarrow & & \Downarrow \\ S(X)_\bullet/S(A)_\bullet & \dashrightarrow & S(Y)_\bullet/S(B)_\bullet \end{array}$$

The dashed arrow is uniquely well-defined because of the assumption that $f(A) \subset B$, meaning that the relative homology construction is well-defined on the level of the relative chain complex. Functoriality of the relative homology now follows from the functoriality of H_n . \square

8.5.2. The long exact sequence on a pair of spaces

Proposition 217. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & H_{j+1}(X, A) & & \\ & \searrow & & \delta & \nearrow & & \\ & H_j(A) & \longrightarrow & H_j(X) & \longrightarrow & H_j(X, A) & \\ & \searrow & & \delta & \nearrow & & \\ & H_{j-1}(A) & \longrightarrow & \cdots & & & \end{array}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X, A) \longrightarrow 0$$

\square

Example 218. Let $X = \mathbb{D}^n$, the n -disk, and $A = \mathbb{S}^{n-1}$ its boundary n -sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) & & \\ & \searrow & & \delta & \nearrow & & \\ & H_j(\mathbb{S}^{n-1}) & \longrightarrow & H_j(\mathbb{D}^n) & \longrightarrow & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) & \\ & \searrow & & \delta & \nearrow & & \\ & H_{j-1}(\mathbb{S}^{n-1}) & \longrightarrow & \cdots & & & \end{array}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible. Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

8. Homology

for $j > 1$ and $n \geq 1$.

Proposition 219. Suppose $i: A \hookrightarrow X$ is a weak retract, i.e. that there is an $r: X \rightarrow A$ such that $r \circ i = \text{id}_A$.

$$\begin{array}{ccccc} A & \xrightarrow{i} & X & \xrightarrow{r} & A \\ & \searrow & \text{id}_A & \nearrow & \\ & & & & \end{array}$$

Then

$$H_n(X) \cong H_n(A) \oplus H_n(X, A).$$

Proof. Applying the functor H_n to the diagram above, we find that

$$H_n(r) \circ H_n(i) = \text{id}_{H_n(A)},$$

implying that $H_n(i)$ is injective. Consider the long exact sequence on the pair (X, A) .

$$\begin{array}{ccccccc} & & & \cdots & \longrightarrow & H_{n+1}(X, A) & \\ & & \swarrow \delta & & & \searrow \delta & \\ & H_n(A) & \xrightarrow{H_n(i)} & H_n(X) & \longrightarrow & H_n(X, A) & \\ & \swarrow \delta & & \searrow \delta & & & \\ & H_{n-1}(A) & \xrightarrow{H_{n-1}(i)} & \cdots & & & \end{array}$$

The injectivity of $H_n(i)$ implies that the connecting homomorphisms are zero, meaning that the following sequence is short exact for all n .

$$0 \longrightarrow H_n(A) \xrightarrow{H_n(i)} H_n(X) \longrightarrow H_n(X, A) \longrightarrow 0$$

As $H_n(r)$ is a left inverse to $H_n(i)$, the sequence above splits from the left, implying by the splitting lemma (Lemma 33) the result. \square

Proposition 220. Let $i: A \rightarrow X$ be a deformation retract,¹ i.e. that there is a homotopy

$$R: X \times [0, 1] \rightarrow X$$

such that the following conditions are satisfied.

1. $R(x, 0) = x$ for all $x \in X$
2. $R(x, 1) \in A$ for all $x \in X$
3. $R(a, 1) = a$ for all $a \in A$

Then $H_n(i): H_n(A) \rightarrow H_n(X)$ is an isomorphism.

¹I believe we only need i to be a homotopy equivalence; a deformation retract is a homotopy equivalence in which one of the two homotopies is an identity.

Proof. Let

$$r = R(-, 1): X \rightarrow A.$$

Then 3. implies that $r \circ i = \text{id}_A$.

Furthermore, $R(x, -)$ provides a homotopy between $i \circ r$ and id_X . Thus, X and A are homotopy equivalent, and $H_n(i)$ is an isomorphism. \square

Corollary 221. If $i: X \rightarrow A$ is a deformation retract, then $H_n(X, A) = 0$ for all n .

These results may not seem like much, but we are now in a position to show a truly fundamental result: invariance of dimension.

Example 222. Let $A \subset \mathbb{R}^m$ and $B \subset \mathbb{R}^n$ be open subsets, and let $f: A \rightarrow B$ be a homeomorphism. Then $m = n$.

To see this, let $a \in A$. Then

$$f|_{A \setminus \{a\}} = \tilde{f}: A \setminus \{a\} \rightarrow B \setminus \{f(a)\}$$

is a homeomorphism, so there is an isomorphism of pairs

$$(A, A \setminus \{a\}) \cong (B, B \setminus \{f(a)\}).$$

Thus

$$H_q(A, A \setminus \{a\}) \cong H_q(B, B \setminus \{f(a)\}).$$

By excision, we have

$$H_q(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) \cong H_q(\mathbb{R}^n, \mathbb{R}^n \setminus \{f(a)\}).$$

But we have a homotopy equivalence of pairs

$$(\mathbb{R}^m, \mathbb{R}^m \setminus \{a\}) \cong (\mathbb{D}^m, \mathbb{S}^{m-1}), \quad (\mathbb{R}^n, \mathbb{R}^n \setminus \{f(a)\}) \cong (\mathbb{D}^n, \mathbb{S}^{n-1}),$$

so we have

$$H_q(A, A \setminus \{a\}) \cong \begin{cases} \mathbb{Z}, & q = 0, m \\ 0, & \text{otherwise} \end{cases}$$

and

$$H_q(B, B \setminus \{f(a)\}) \cong \begin{cases} \mathbb{Z}, & q = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

For these homologies to agree for all q we must have $m = n$.

8.5.3. The braided monstrosity on a triple of spaces

Definition 223 (triple of spaces). Let X be a topological space, and let $B \subset A \subset X$ be subspaces. We call (X, A, B) a triple.

A triple of spaces is in particular three pairs of spaces: (X, A) , (X, B) , and (A, B) . All of these have associated long exact sequences. There is also a fourth long exact sequence.

Proposition 224. There is a long exact sequence

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & H_{j+1}(A, B) & \\
 & & \swarrow & & \searrow & \delta & \\
 & & H_j(A, B) & \longrightarrow & H_j(X, B) & \longrightarrow & H_j(X, A) \\
 & & \swarrow & & \searrow & \delta & \\
 & & H_{j-1}(A, B) & \longrightarrow & \cdots & &
 \end{array}$$

Proof. This comes from the short exact sequence

$$0 \longrightarrow S_n(A)/S_n(B) \hookrightarrow S_n(X)/S_n(B) \twoheadrightarrow S_n(X)/S_n(A) \longrightarrow 0 .$$

□

We can put all four of these together in the following handsome commutative diagram.

$$\begin{array}{ccccccc}
 & & & \cdots & & & \cdots \\
 & \searrow & & \delta & \searrow & & \searrow \\
 & H_{n+1}(X, A) & & H_n(A, B) & & H_{n-1}(B) & \\
 & \nearrow & \searrow & \delta & \nearrow & \searrow & \nearrow \\
 \cdots & & H_n(A) & & H_n(X, B) & & \cdots \\
 & \searrow & \nearrow & \delta & \searrow & \nearrow & \searrow \\
 & H_n(B) & & H_n(X) & & H_n(X, A) & \\
 & \nearrow & \searrow & \delta & \nearrow & \searrow & \nearrow \\
 & \cdots & & \delta & \cdots & & \cdots
 \end{array}$$

8.6. Barycentric subdivision

Definition 225 (cone maps). Let $v \in \Delta^p$ be a point, and let $\alpha: \Delta^n \rightarrow \Delta^p$ be a singular n -simplex. The cone of α with respect to v is

$$K_v(\alpha): \Delta^{n+1} \rightarrow \Delta^p$$

defined by sending

$$(t_0, \dots, t_n) \mapsto \begin{cases} (1 - t_{n+1}) \alpha\left(\frac{t_0}{1-t_{n+1}}, \dots, \frac{t_n}{1-t_{n+1}}\right) + t_{n+1}v, & t_{n+1} < 1 \\ v, & t_{n+1} = 1. \end{cases}$$

This is well-defined and continuous.

We extend K_v linearly to $S_\bullet(\Delta^p)$.

Lemma 226. The map $K_v: S_n(\Delta^p) \rightarrow S_{n+1}(\Delta^p)$ satisfies the following identities.

1. For a zero-simplex $c \in S_0(\Delta^p)$, we have

$$\partial K_v(c) = \epsilon(c)\kappa_v - c,$$

where $\kappa_v(e_0) = v$, and $\epsilon: S_0(\Delta^p) \rightarrow \mathbb{Z}$ is the augmentation map.

2. For $n > 0$, we have the equality

$$\partial \circ K_v - K_v \circ \partial = (-1)^{n+1} \text{id}$$

Proof.

1. It suffices to show the claim on generators, i.e. singular 0-simplices $\Delta^0 \rightarrow \Delta^p$. Let $\alpha: \Delta^0 \rightarrow \Delta^p$. Then $\epsilon(\alpha) = 1$, and

$$\begin{aligned} \partial K_v(\alpha)(e_0) &= \partial_0 K_v(\alpha)(e_0) - \partial_1 K_v(\alpha)(e_0) \\ &= \kappa_v(\alpha)(d_0 e_0) - \kappa_v(\alpha)(d_1 e_0) \\ &= \kappa_v(\alpha)(e_1) - \kappa_v(\alpha)(e_0) \\ &= v - \alpha(e_0) \end{aligned}$$

2. An easy calculation shows that

$$\partial_i K_v(\alpha) = \kappa_v(\partial_i \alpha), \quad i < n+1,$$

and

$$\partial_{n+1} K_v(\alpha) = \alpha.$$

8. Homology

Thus,

$$\partial K_v - K_v \partial = (-1)^{n+1} \text{id}$$

as required. □

We first define *Barycentric subdivision* on models $\{\Delta^n\}$: it is a chain map

$$B: S_\bullet(\Delta^p) \rightarrow S_\bullet(\Delta^p)$$

defined on an 0-simplex $\alpha: \Delta^0 \rightarrow \Delta^p$ by

$$B(\alpha) = \alpha,$$

and for $\sigma: \Delta^n \rightarrow \Delta^p$ inductively by

$$B(\sigma) = (-1)^n (K_v)(B(\partial \sigma)).$$

This is a bona-fide chain map because...

Since the functor S_\bullet is $\{\Delta^n\}$ -free, we can extend B to a natural chain map $S_\bullet \rightarrow S_\bullet$ via

$$B_X(\alpha) = S_n(\alpha)(B_{\Delta^n}(\text{id}_{\Delta^n})), \quad \alpha: \Delta^n \rightarrow X.$$

Theorem 227. The chain map $B: S_\bullet(X)$ is chain homotopic to the identity.

Proof. For a zero-simplex $\alpha: \Delta^0 \rightarrow X$

$$\begin{aligned} B_X(\alpha) &= S_0(\alpha)(B_{\Delta^0}(\text{id}_{\Delta^0})) \\ &= S_0(\alpha)(\text{id}_{\Delta^0}) \\ &= \alpha, \end{aligned}$$

so B and id_{S_\bullet} agree on $H_0(S_\bullet)$. The claim follows from [Theorem 208](#). □

Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X . Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha: \Delta^n \rightarrow X$$

whose images are completely contained in some open set in the open cover \mathfrak{U} . That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$

induces a chain structure on $S_{\bullet}^{\mathfrak{U}}(X)$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_2^{\mathfrak{U}}(X) & \longrightarrow & S_1^{\mathfrak{U}}(X) & \longrightarrow & S_0^{\mathfrak{U}}(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & S_2(X) & \longrightarrow & S_1(X) & \longrightarrow & S_0(X) \longrightarrow 0 \end{array}$$

Fact 228. For any open cover \mathfrak{U} of a topological space X , for every simplex $\alpha: \Delta^n \rightarrow X$, there exists some $n \geq 0$ such that $B^n \alpha \in S^{\mathfrak{U}}(X)$.

Sketch of proof. This is really just some messing with Lebesgue numbers. One first shows that for any *affine simplex*² $\alpha: \Delta^n \rightarrow \Delta^p$,

$$\max_{\sigma} \text{diam}(\sigma) \leq \left(\frac{n}{n+1} \right)^k \text{diam}(\alpha),$$

where σ runs over all simplices in $B^k \alpha$. Thus, for any n ,

$$\lim_{k \rightarrow \infty} \max \text{diam}(B^k \alpha) = 0.$$

Lebesgue's number lemma tells us that for any compact metric space X and any open cover $\mathfrak{U} = \{U_i\}$ of X , there is some $\delta > 0$ such that any subset $S \subset X$ with $\text{diam}(S) < \delta$ is completely contained in some U_i .

Now let X be any topological space, and let $\alpha: \Delta^n \rightarrow X$. Let \mathfrak{U} be an open cover of X . Then the preimages of the open cover under α form an open cover of Δ^n . Since Δ^n is a compact metric space, Lebesgue's number lemma tells us that there exists $k \geq 0$ such that every simplex in $B^k \text{id}_{\Delta^n}$ is completely contained in the preimage of at least one of the $U_i \in \mathfrak{U}$. Mapping everything forward with $S_{\bullet}(\alpha)$ gives us the result we want. \square

We have proved the following.

Corollary 229. Every chain in $S_n(X)$ is homologous to a chain in $S^{\mathfrak{U}}(X)$

Proposition 230. There is a natural homotopy equivalence $S_{\bullet}^{\mathfrak{U}} \cong S_{\bullet}$.

Corollary 231.

$$H_n(X) \cong H_n^{\mathfrak{U}}(X).$$

This fact allows us almost immediately to read of two important theorems.

²I.e. a singular simplex which comes from an affine map $\mathbb{R}^{n+1} \rightarrow \mathbb{R}^{p+1}$

8.6.1. Excision

Theorem 232 (excision). Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{array}{ccc} A \setminus W & \xhookrightarrow{i} & A \\ \downarrow & & \downarrow \\ X \setminus W & \xhookrightarrow{i} & X \end{array}$$

induce an isomorphism

$$H_n(i): H_n(X \setminus W, A \setminus W) \cong H_n(X, A).$$

Proof. Consider the open cover $\mathfrak{U} = (\mathring{A}, X \setminus \bar{W})$ of X .

First, we show that $H_n(i)$ is surjective. Let $c \in S_n(X, A)$ be a relative cycle. By [Corollary 229](#), we can replace (not uniquely, but who cares) c by a homologous sum $c' + c''$ with $c' \in S_n(\mathring{A})$ and $c'' \in S_n(X \setminus \bar{W})$.

NOTE: THIS IS WRONG. NEED TO FIX.

But $c' \sim 0$ in $H_n(X, A)$, so we have found $c \sim c''$, with $c'' \in S_n(X \setminus \bar{W})$.

Now we compute a boundary:

$$\partial c \sim \partial c''.$$

By definition,

$$\partial c'' \in S_{n-1}(X \setminus \bar{W}) \subset S_{n-1}(X \setminus W).$$

However, $c'' \in S_{n-1}(A)$, so

$$c'' \in S_{n-1}(X \setminus W) \cap S_{n-1}(A) = S_{n-1}(A \setminus W).$$

Therefore, c is homologous to a relative cycle in $S_n(X \setminus W, A \setminus W)$.

Now we show that $H_n(i)$ is injective. Suppose that $c \in S_n(X \setminus W)$ such that $\partial c \in S_n(A \setminus W)$. Further suppose that $H_n(i)[c] = 0$, i.e. that we can write

$$i(c) = \partial b + a'$$

with $b \in S_{n+1}(X)$ and $a' \in S_n(A)$. Then $i(c)$ is homologous to \square

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A .

8.6.2. The Mayer-Vietoris sequence

Theorem 233 (Mayer-Vietoris). Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X , i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$\begin{array}{ccccccc}
 & & & & \dots & \longrightarrow & H_{n+1}(X) \\
 & & & & & & \uparrow \delta \\
 & & & & H_n(X_1 \cap X_2) & \longrightarrow & H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \\
 & & & & & & \uparrow \delta \\
 & & & & H_{n-1}(X_1 \cap X_2) & \longrightarrow & \dots
 \end{array}$$

Proof. We can draw our inclusions as the following pushout.

$$\begin{array}{ccccc}
 & & X_1 & & \\
 & \nearrow i_1 & & \searrow \kappa_1 & \\
 X_1 \cap X_2 & & & & X \\
 & \searrow i_2 & & \nearrow \kappa_2 & \\
 & & X_2 & &
 \end{array}$$

We have, almost by definition, the following short exact sequence.

$$0 \longrightarrow S_\bullet(X_1 \cap X_2) \xrightarrow{(i_1, i_2)} S_\bullet(X_1) \oplus S_\bullet(X_2) \xrightarrow{\kappa_1 - \kappa_2} S_\bullet^{\mathfrak{U}}(X) \longrightarrow 0$$

This gives the following long exact sequence on homology.

$$\begin{array}{ccccccc}
 & & & & \dots & \longrightarrow & H_{n+1}^{\mathfrak{U}}(X) \\
 & & & & & & \uparrow \delta \\
 & & & & H_n(X_1 \cap X_2) & \longrightarrow & H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n^{\mathfrak{U}}(X) \\
 & & & & & & \uparrow \delta \\
 & & & & H_{n-1}(X_1 \cap X_2) & \longrightarrow & \dots
 \end{array}$$

We have seen that $H_n^{\mathfrak{U}}(X) \cong H_n(X)$; the result follows. \square

Example 234 (Homology groups of spheres). We can decompose \mathbb{S}^n as

$$\mathbb{S}^n = (\mathbb{S}^n \setminus N) \cup (\mathbb{S}^n \setminus S),$$

where N and S are the North and South pole respectively. This gives us the following

8. Homology

pushout.

$$\begin{array}{ccccc}
 & & \mathbb{S}^n \setminus N & & \\
 & \nearrow & & \searrow & \\
 (\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) & & & & \mathbb{S}^n \\
 & \searrow & & \nearrow & \\
 & & \mathbb{S}^n \setminus S & &
 \end{array}$$

The Mayer-Vietoris sequence is as follows.

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & H_{j+1}(\mathbb{S}^n) \\
 & & & & \searrow & \delta & \nearrow \\
 & & H_j((\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S)) & \longrightarrow & H_j(\mathbb{S}^n \setminus N) \oplus H_j(\mathbb{S}^n \setminus S) & \longrightarrow & H_j^{\text{all}}(X) \\
 & & \searrow & \delta & \nearrow \\
 & & H_{j-1}((\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S)) & \longrightarrow & \cdots
 \end{array}$$

We know that

$$\mathbb{S}^n \setminus N \cong \mathbb{S}^n \setminus S \cong \mathbb{D}^n \simeq \text{pt}$$

and that

$$(\mathbb{S}^n \setminus N) \cap (\mathbb{S}^n \setminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$\begin{array}{ccccccc}
 & & \cdots & \longrightarrow & H_{j+1}(\mathbb{S}^n) & & \\
 & & \searrow & \delta & \nearrow & & \\
 & & H_j(\mathbb{S}^{n-1}) & \longrightarrow & 0 & \longrightarrow & H_j(\mathbb{S}^n) \\
 & & \searrow & \delta & \nearrow & & \\
 & & H_{j-1}(\mathbb{S}^{n-1}) & \longrightarrow & \cdots & &
 \end{array}$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X , $H_1(X) \cong \pi_1(X)_{\text{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\text{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$j = 3$	\mathbb{Z}	0		
$j = 2$	\mathbb{Z}	0		
$j = 1$	\mathbb{Z}	\mathbb{Z}		
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \quad i > 1$$

allows us to fill in the above table as follows.

$j = 3$	\mathbb{Z}	0	0	\mathbb{Z}
$j = 2$	\mathbb{Z}	0	\mathbb{Z}	0
$j = 1$	\mathbb{Z}	\mathbb{Z}	0	0
$j = 0$	$\mathbb{Z} \oplus \mathbb{Z}$	0	0	0
$H_i(\mathbb{S}^j)$	$i = 0$	$i = 1$	$i = 2$	$i = 3$

(8.1)

Example 235. Above, we used the Hurewicz homomorphism to see that

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed \mathbb{S}^n as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$\begin{array}{ccc}
 & X^+ & \\
 \nearrow & & \searrow \\
 X^+ \cap X^- & & \mathbb{S}^n \\
 \searrow & & \nearrow \\
 & X^- &
 \end{array}$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

8. Homology

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{S}^n) & & \\
 & & & \searrow \delta & \swarrow & & \\
 & & H_0(X^+ \cap X^-) & \xrightarrow{H_0(i_0, i_1)} & H_0(X^+) \oplus H_0(X^-) & \longrightarrow & \dots
 \end{array}$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^- , with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+ , and $i_1(p)$ is a point of X^- . Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{array}{ccccccc}
 \dots & \longrightarrow & 0 & \longrightarrow & H_1(\mathbb{S}^1) & & \\
 & & & \searrow \delta & \swarrow & & \\
 & & H_0(X^+ \cap X^-) & \xrightarrow{H_0(i_0, i_1)} & H_0(X^+) \oplus H_0(X^-) & \xrightarrow{H_0(\kappa_1) - H_0(\kappa_2)} & H_0(\mathbb{S}^1)
 \end{array}$$

We can immediately replace things we know, finding the following.

$$\begin{array}{c}
 (a, b) \longmapsto (a + b, a + b) \\
 \\
 0 \longrightarrow H_1(\mathbb{S}^1) \hookrightarrow \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{f} \mathbb{Z} \oplus \mathbb{Z} \longrightarrow \mathbb{Z} \\
 \\
 (c, d) \longmapsto c - d
 \end{array}$$

The kernel of f is the free group generated by (a, a) . Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

8.6.3. The relative Mayer-Vietoris sequence

Theorem 236 (relative Mayer-Vietoris sequence). Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathcal{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{array}{ccccccc}
 & & & & \cdots & \longrightarrow & H_{n+1}(X, A \cup B) \\
 & \swarrow & & \searrow & \delta & & \swarrow \\
 & H_n(X, A \cap B) & \longrightarrow & H_n(X, A) \oplus H_n(X, B) & \longrightarrow & H_n(X, A \cup B) \\
 & \swarrow & & \searrow & \delta & & \swarrow \\
 & H_{n-1}(X, A \cap B) & \longrightarrow & \cdots & & &
 \end{array}$$

Proof. Consider the following commuting diagram.

$$\begin{array}{ccccccc}
 & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & \\
 0 & \longrightarrow & S_n(A \cap B) & \longrightarrow & S_n(A) \oplus S_n(B) & \longrightarrow & S_n^{\mathcal{U}}(A \cup B) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_n(X) & \longrightarrow & S_n(X) \oplus S_n(X) & \longrightarrow & S_n(X) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_n(X, A \cap B) & \longrightarrow & S_n(X, A) \oplus S_n(X, B) & \longrightarrow & S_n(X)/S_n^{\mathcal{U}}(A \cup B) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma ([Theorem 44](#)) implies that the last row is also exact.

Consider the following map of short exact sequences; the first row is the last column of the above grid.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & S_n^{\mathcal{U}}(A \cup B) & \hookrightarrow & S_n(X) & \twoheadrightarrow & S_n(X)/S_n^{\mathcal{U}}(A \cup B) \longrightarrow 0 \\
 & & \phi \downarrow & & \parallel & & \downarrow \psi \\
 0 & \longrightarrow & S_n(A \cup B) & \longrightarrow & S_n(X) & \twoheadrightarrow & S_n(X, A \cup B) \longrightarrow 0
 \end{array}$$

This gives us, by [Lemma 41](#), a morphism of long exact sequences on homology.

$$\begin{array}{ccccccc}
 H_n(S_{\bullet}^{\mathcal{U}}(A \cup B)) & \longrightarrow & H_n(X) & \longrightarrow & H_n(S_{\bullet}(X)/S_{\bullet}^{\mathcal{U}}(A \cup B)) & \longrightarrow & H_{n-1}(S_{\bullet}^{\mathcal{U}}(A \cup B)) \longrightarrow H_{n-1}(X) \\
 H_n(\phi) \downarrow & & \parallel & & \downarrow H_n(\psi) & & \downarrow H_{n-1}(\phi) & \parallel \\
 H_n(A \cup B) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X, A \cup B) & \longrightarrow & H_{n-1}(A \cup B) & \longrightarrow H_{n-1}(X)
 \end{array}$$

We have seen (in [Fact 228](#)) that $H_i(\phi)$ is an isomorphism for all i . Thus, the five lemma ([Theorem 43](#)) tells us that $H_n(\psi)$ is an isomorphism. Thus, the bottom row of our grid

8. Homology

can be written

$$0 \longrightarrow S_n(X, A \cap B) \hookrightarrow S_n(X, A) \oplus S_n(X, B) \twoheadrightarrow S_n(X, A \cup B) \longrightarrow 0,$$

and taking the long exact sequence on homology gives the result. \square

8.7. Reduced homology

We have seen that reduced homology $\tilde{H}_n(X)$ agrees with $H_n(X)$ in positive degrees, and is missing a copy of \mathbb{Z} in the zeroth degree. There are three equivalent ways of understanding this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Picking any point $x \in X$, one can define the reduced homology of X by

$$\tilde{H}_n(X) = H_n(X, x).$$

2. **In between:** One can define

$$\tilde{H}_n(X) = \ker(H_n(X) \rightarrow H_n(\text{pt})).$$

3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of \mathbb{Z} in degree -1 , so that

$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$

Then one can define

$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the one we have taken so far. In constructing the singular chain complex of our space X , we used the following composition.

$$\mathbf{Top} \xrightarrow{\text{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1) -simplices:

$$\mathbf{Sing}(X)_{-1} = \text{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \text{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through \mathcal{F} thus gives a copy of \mathbb{Z} as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- (1 \Leftrightarrow 2): Since $H_n(P)$ is free for all n , the sequence

$$0 \longrightarrow \ker H_n(\epsilon) \hookrightarrow H_n(X) \twoheadrightarrow H_n(\{x\}) \longrightarrow 0$$

splits, telling us that

$$H_n(X) \cong \ker H_n(\epsilon) \oplus H_n(\{x\}).$$

Applying the functor $H_n \circ S$ to the diagram

$$\begin{array}{ccccc} \{x\} & \xhookrightarrow{i} & X & \xrightarrow{\epsilon} & \{x\} \\ & & \searrow & \nearrow & \\ & & \text{id}_{\{x\}} & & \end{array}$$

one finds that $H_n(i)$ is an injection. Therefore, the connecting homomorphisms for the long exact sequence on the pair (X, x)

$$\cdots \longrightarrow H_{n+1}(X, x) \xrightarrow{\delta} H_n(\{x\}) \xhookrightarrow{i} H_n(X) \longrightarrow \cdots$$

must be zero, so the sequence

$$0 \longrightarrow H_n(\{x\}) \xhookrightarrow{i} H_n(X) \twoheadrightarrow H_n(X, x) \longrightarrow 0$$

is exact. Thus, $H_n(X, x) \cong H_n(X) \oplus H_n(X, x)$. But

$$H_n(X) \oplus H_n(X, x) \cong H_n(X) \oplus \ker H_n(\epsilon),$$

so $H_n(X, x) \cong \ker H_n(\epsilon)$.

- (2 \Leftrightarrow 3): Consider the chain map $S_\bullet(X) \rightarrow S_\bullet(P)$.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & S_2(X) & \xrightarrow{d_2} & S_1(X) & \xrightarrow{d_1} & S_0(X) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \epsilon \\ \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \longrightarrow 0 \end{array}.$$

Considering this as a chain complex

$$\cdots \longrightarrow S_2 \longrightarrow S_1 \longrightarrow S_0 \twoheadrightarrow \mathbb{Z} \longrightarrow 0,$$

the homology $H_0(S_\bullet(X))$ is given by $\ker \epsilon / \text{im } d_1$. But d_1 is epi, so $H_0(\tilde{C}_\bullet) \cong \ker \epsilon$.

Proposition 237. Relative homology agrees with ordinary homology in degrees greater

8. Homology

than 0, and in degree zero we have the relation

$$H_0(X) \cong \tilde{H}_0(X) \oplus \mathbb{Z},$$

although this isomorphism is not canonical.

Proof. Trivial from algebraic definition. □

8.7.1. Deja vu all over again

Many of our results for regular homology hold also for reduced homology.

Proposition 238. There is a long exact sequence for a pair of spaces

$$\begin{array}{ccccccc} & & \cdots & \longrightarrow & \tilde{H}_{j+1}(X, A) & & \\ & \swarrow & & & \searrow \delta & & \\ & \tilde{H}_j(A) & \longrightarrow & \tilde{H}_j(X) & \longrightarrow & \tilde{H}_j(X, A) & \\ & \swarrow & & & \searrow \delta & & \\ & \tilde{H}_{j-1}(A) & \longrightarrow & \cdots & & & \end{array}$$

We're now ready to prove quite a nice theorem.

Proposition 239 (Brouwer). For any $n \geq 1$, any continuous bijective map $f: \mathbb{D}^n \rightarrow \mathbb{D}^n$ has at least one fixed point.

Proof. Suppose we have such a map. Then x and $f(x)$ are different for all x , and we can draw a ray starting at $f(x)$ and passing through $f(x)$. This ray intersects $\partial\mathbb{D}^n \cong \mathbb{S}^{n-1}$ at exactly one place $r(x)$, giving us a map

$$r: \mathbb{D}^n \rightarrow \mathbb{S}^{n-1}; \quad x \mapsto r(x).$$

This is clearly continuous, and it fixes $\partial\mathbb{D}^n \subset \mathbb{D}^n$. Thus, we have the following commutative diagram.

$$\begin{array}{ccccc} \mathbb{S}^{n-1} & \xhookrightarrow{\text{incl.}} & \mathbb{D}^n & \xrightarrow{r} & \mathbb{S}^{n-1} \\ & & \searrow & \nearrow & \\ & & \text{id}_{\mathbb{S}^{n-1}} & & \end{array}$$

Applying \tilde{H}_n gives the following commutative diagram.

$$\begin{array}{ccccc} \tilde{H}_{n-1}(\mathbb{S}^{n-1}) & \xhookrightarrow{\tilde{H}_{n-1}(\text{incl.})} & \tilde{H}_{n-1}(\mathbb{D}^n) & \xrightarrow{\tilde{H}_{n-1}(r)} & \tilde{H}_{n-1}(\mathbb{S}^{n-1}) \\ & & \searrow & \nearrow & \\ & & \text{id}_{\tilde{H}_n(\mathbb{S}^{n-1})} & & \end{array}$$

But $\tilde{H}_{n-1}(\mathbb{S}^{n-1}) \cong \mathbb{Z}$ and $\tilde{H}_{n-1}(\mathbb{D}^n) \cong 0$ for all $n \geq 1$, so $\tilde{H}_{n-1}(r)$ cannot be surjective. \square

Proposition 240. We have a reduced Mayer-Vietoris sequence.

$$\begin{array}{ccccccc}
 & & & \cdots & \longrightarrow & \tilde{H}_{n+1}(X) & \\
 & \swarrow & & \delta & \searrow & & \\
 \tilde{H}_n(X_1 \cap X_2) & \longrightarrow & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) & \longrightarrow & \tilde{H}_n(X) & & \\
 \swarrow & & \delta & \searrow & & & \\
 \tilde{H}_{n-1}(X_1 \cap X_2) & \longrightarrow & \cdots & & & &
 \end{array}$$

Proposition 241. Let $\{(X_i, x_i)\}_{i \in I}$ be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I^3$ we have

$$\tilde{H}_n\left(\bigvee_{i \in E} X_i\right) \cong \bigoplus_{i \in E} \tilde{H}_n(X_i).$$

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of [Proposition 240](#) tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

\square

Definition 242 (good pair). A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X .
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U .

Proposition 243. Let (X, A) be a good pair. Let $\pi: X \rightarrow X/A$ be the canonical projection. Then

$$H(X, A) \cong \tilde{H}_n(X/A) \quad \text{for all } n \geq 0.$$

³This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

8. Homology

Proof. First, note that since $X \setminus A \cong (X/A) \setminus \{b\}$ and $U \setminus A \cong (U/A) \setminus \{b\}$, we have an isomorphism of pairs

$$(X \setminus A, U \setminus A) \cong ((X/A) \setminus \{b\}, (U/A) \setminus \{b\}). \quad (8.2)$$

Thus, we have the following chain of isomorphisms.

$$\begin{aligned} H_n(X, A) &\cong H_n(X, U) && \left(\begin{array}{l} A \text{ deformation} \\ \text{retract of } U \end{array} \right) \\ &\cong H_n(X \setminus A, U \setminus A) && (\text{excision}) \\ &\cong H_n((X/A) \setminus \{b\}, (U/A) \setminus \{b\}) && (\text{Equation 8.2}) \\ &\cong H_n(X/A, U/A) && (\text{excision}) \\ &\cong H_n(X/A, \{b\}) && \left(\begin{array}{l} \{b\} \text{ deformation} \\ \text{retract of } U/A \end{array} \right) \\ &\cong \tilde{H}_n(X/A) \end{aligned}$$

□

Theorem 244 (suspension isomorphism). Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \quad \text{for all } n > 0.$$

8.8. Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n . Let $f: \mathbb{S}^n \rightarrow \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d \mu_n, \quad \text{for some } d \in \mathbb{Z}.$$

Definition 245 (mapping degree). We call $d \in \mathbb{Z}$ as above the mapping degree of f , and denote it by $\deg(f)$.

Example 246. Consider the map

$$\omega: [0, 1] \rightarrow \mathbb{S}^1; \quad t \mapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism ([Theorem 200](#)), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n: \mathbb{S}^1 \rightarrow \mathbb{S}^1; \quad x \mapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism ([Theorem 200](#)) tells us that the following diagram commutes.

$$\begin{array}{ccc} \pi_1(\mathbb{S}^1)_{\text{ab}} & \xrightarrow{\pi_1(f_n)_{\text{ab}}} & \pi_1(\mathbb{S}^1)_{\text{ab}} \\ h_{\mathbb{S}^1} \downarrow & & \downarrow h_{\mathbb{S}^1} \\ H_1(\mathbb{S}^1) & \xrightarrow{H_1(f_n)} & H_1(\mathbb{S}^1) \end{array}$$

8.9. CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 247 (cell). Let X be a topological space. We say that X is an n -cell if X is homeomorphic to \mathbb{R}^n . We call the number n the dimension of X .

Definition 248 (cell decomposition). A cell decomposition of a topological space X is a decomposition

$$X = \bigsqcup_{i \in I} X_i, \quad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topological spaces.

Definition 249 (CW complex). A Hausdorff topological space X , together with a cell decomposition, is known as a CW complex⁴ if it satisfies the following conditions.

(CW1) For every n -cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma: \mathbb{D}^n \rightarrow X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n - 1$.

(CW2) For every n -cell σ , the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X .

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ .

⁴[Axiom \(CW2\)](#) is called the *closure-finiteness* condition. This is the ‘C’ in CW complex. [Axiom \(CW3\)](#) says that X carries the *weak topology* and is responsible for the ‘W’.

8. Homology

- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map*.

Example 250. Consider the unit interval $I = [0, 1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has a CW structure with $n + 1$ 0-cells and n 1-cells, which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma_k^0 = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \quad \text{and} \quad \sigma_\infty^0 = 0$$

and one-cells

$$\sigma_k^1 = \left(\frac{1}{k}, \frac{1}{k+1} \right), \quad k \in \mathbb{N}^{\geq 1}.$$

At first glance, this looks like a CW decomposition; it certainly satisfies [Axiom \(CW1\)](#) and [Axiom \(CW2\)](#). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2} \left(\frac{1}{k} + \frac{1}{k+1} \right)$$

is the midpoint of the interval σ_k^1 . We have $A \cap \sigma_k^0 = \emptyset$ for all k , and $A \cap \sigma_k^1 = \{a_k\}$ for all k . In each case, $A \cap \bar{\sigma}_j^i$ is closed in σ_j^i . However, the set A is not closed in I , since it does not contain its limit point $\lim_{n \rightarrow \infty} a_n = 0$.

Definition 251 (skeleton, dimension). Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n -skeleton of X . If X is equal to its n -skeleton but not equal to its $(n-1)$ -skeleton, we say that X is n -dimensional.

Note 252. [Axiom \(CW3\)](#) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 253 (subcomplex, CW pair). Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$.

We call such a pair (X, Y) a CW pair.

Fact 254. Let X and Y be CW complexes such that X is locally compact.⁵ Then $X \times Y$ is a CW complex.

⁵I.e. if for every point x there is an open neighborhood U containing x and a compact set K containing U .

Lemma 255. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 256. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact if and only if it is locally finite.⁶

Proof. It is clear that 1. \Rightarrow 2., since X is a subset of itself. Similarly, it is clear that 2. \Rightarrow 3., since \square

Corollary 257. If $f: K \rightarrow X$ is a continuous map from a compact space K to a CW complex X , then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n .

$$\begin{array}{ccccccc}
 & & & & & & K \\
 & & & & & \nearrow \exists \tilde{f} & \downarrow f \\
 X^{n-1} & \hookrightarrow & X^n & \hookrightarrow & X^{n+1} & \hookrightarrow & \dots \hookrightarrow X
 \end{array}$$

Proposition 258. Let A be a subcomplex of a CW complex X . Then $X \times \{0\} \cup A \times [0, 1]$ is a strong deformation retract of $X \times [0, 1]$.

Lemma 259. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborhood U of A in X together with a strong deformation retract to A . In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U .
- Every CW complex is paracompact, locally path-connected, and locally contractible.
- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 260. Let X be a CW complex. We have the following decompositions.

1.

$$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2.

$$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

⁶I.e. if every point has a neighborhood which is contained in only finitely many cells.

8. Homology

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n -cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n -cell is homeomorphic to the open n -ball.
2. For every n -cell σ , the characteristic map Φ_σ sends $\partial\Delta^n$ to the $(n-1)$ -skeleton.

□

8.10. Cellular homology

Lemma 261. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n/X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By [Lemma 259](#), (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from [Proposition 243](#), and the second from [Lemma 260](#). □

Lemma 262. Consider the inclusion $i_n: X^n \hookrightarrow X$.

- The induced map

$$H_n(i_n): H_n(X^n) \rightarrow H_n(X)$$

is surjective.

- On the $(n+1)$ -skeleton we get an isomorphism

$$H_n(i_{n+1}): H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n) . The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by [Lemma 261](#),

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n): X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the follow-

ing sequence is exact.

$$\begin{array}{c}
 \xrightarrow{\quad \delta \quad} H_{n+1}(X^{m+1}, X^m) \\
 \curvearrowright \\
 H_n(X^m) \longrightarrow H_n(X^{m+1}) \longrightarrow H_n(X^{m+1}, X^m)
 \end{array}$$

But again by [Lemma 261](#), both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \rightarrow H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & X^n & \longrightarrow & X^{n+1} & \longrightarrow & X^{n+2} \longrightarrow \cdots \\
 & & & & \searrow & & \searrow \\
 & & & & & & \searrow \\
 & & & & & & X
 \end{array}$$

Taking n th singular homology, we find the following.

$$\cdots \longrightarrow H_n(X^n) \xrightarrow{\alpha_1} H_n(X^{n+1}) \xrightarrow{\alpha_2} H_n(X^{n+2}) \xrightarrow{\alpha_3} \cdots \longrightarrow H_n(X)$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \quad \sigma_i: \Delta^n \rightarrow X.$$

Since the standard n -simplex Δ^n is compact, [Corollary 257](#) implies that each σ_i factors through some X^{n_i} . Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \quad \tilde{\sigma}_i: \Delta^n \rightarrow X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma \right]$$

□

Corollary 263. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q -cells, then $H_q(X) \cong 0$.

8. Homology

3. In particular, for an n -dimensional CW-complex X (Definition 251), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 262.
- 2.

□

Definition 264 (cellular chain complex). Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \rightarrow S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}) .

Theorem 265 (comparison of cellular and singular homology). Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n: H_n(C_\bullet(X), d) \cong H_n(X).$$

Example 266 (complex projective space). Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \text{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots : x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{array}{ccccccc} 2n & & 2n-1 & & 2n-2 & & \cdots & & 1 & & 0 \\ \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} \end{array}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i, 0 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Example 267 (real projective space). As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{array}{ccccccc} n & & n-1 & & n-2 & & \cdots & & 1 & & 0 \\ \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

8.10.1. Euler characteristic

Here's one last application of CW stuff: euler characteristic.

Definition 268 (Euler characteristic). Let X be a finite CW complex. The Euler characteristic of X is defined to be

$$\chi(X) = \sum_{n \geq 0} (-1)^n \operatorname{rk}(H_n(X; \mathbb{Z}))$$

where for an abelian group A , $\operatorname{rk}(A)$ denotes the number of free summands.

Note that the above sum must be finite since there is some simplex of highest degree,

8. Homology

say k , so $H_{k'}(X; \mathbb{Z}) \cong 0$ for all $k' > k$.

Proposition 269. For X a finite CW complex, denote by $c_n(X)$ the number of n -cells. Then

$$\chi(X) = \sum_{n \geq 0} (-1)^n c_n(X).$$

Proof. We will use the following fact. For any short exact sequence

$$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0 ,$$

we have the equality

$$\text{rk}(B) = \text{rk}(A) + \text{rk}(C).$$

Consider the short exact sequences

$$0 \longrightarrow Z_n \hookrightarrow C_n \twoheadrightarrow B_{n-1} \longrightarrow 0$$

and

$$0 \longrightarrow B_n \hookrightarrow Z_n \twoheadrightarrow H_n \longrightarrow 0 .$$

Applying our rank formula to both and summing leads to the equation

$$\text{rk}(C_n) = \text{rk}(B_{n-1}) + \text{rk}(B_n) + \text{rk}(H_n).$$

Substituting $C_n(X)$ by the above and summing results in the B_i terms cancelling each other in the alternating sum, giving us the result we want. \square

Proposition 270. For finite CW complexes X and Y , we have

$$\chi(X \times Y) \cong \chi(X) \times \chi(Y).$$

Proof. First, note that for \mathbb{S}^n \square

8.11. Homology with coefficients

Definition 271 (homology with coefficients). Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X; G) = S(X) \otimes_{\mathbb{Z}} G.$$

The n th singular homology of X with coefficients in G is the n th homology

$$H_n(X; G) = H_n(S(X; G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G .

Theorem 272 (topological universal coefficient theorem for homology). For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X; G) \twoheadrightarrow \operatorname{Tor}(H_{n-1}(X), G) \longrightarrow 0 .$$

Furthermore, this sequence splits non-canonically, telling us that

$$H_n(X; G) \cong (H_n(X) \otimes G) \oplus \operatorname{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. [Theorem 272](#). □

8.12. The topological Künneth formula

Let X and Y be topological spaces. Plugging $C = S(X)$ and $D = S(Y)$ into [Theorem 99](#) tells us that the following sequence is split exact.

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \hookrightarrow H_n(S(X)_\bullet \otimes S(Y)_\bullet) \twoheadrightarrow \bigoplus_{p+q=n-1} \operatorname{Tor}_1(H_p(X), H_q(Y)) \rightarrow 0 .$$

It turns out that we can relate $H_n(S(X)_\bullet \otimes S(Y)_\bullet)$ and $H_n(X \times Y)$. In fact, they turn out to be isomorphic; this is known as the *Eilenberg-Zilber theorem*.

8.12.1. The Eilenberg-Zilber theorem: with acyclic models

This turns out to be a direct consequence of the acyclic model theorem ([Theorem 208](#)).

Lemma 273. The functors

$$F = S(-) \otimes S(-): \mathbf{Top} \times \mathbf{Top} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

and

$$G = S(- \times -): \mathbf{Top} \times \mathbf{Top} \rightarrow \mathbf{Ch}_{\geq 0}(\mathbf{Ab})$$

are both free and acyclic ([Definition 205](#)) on

$$\mathcal{M} = \{(\Delta^p, \Delta^q)\}_{p,q=0,1,\dots}$$

Proof. Clear, basically. □

Proposition 274. There is a chain homotopy equivalence between F and G as defined in [Lemma 273](#).

8. Homology

Proof. Consider the following diagram.

$$\begin{array}{ccc}
 & f_0: x \otimes y \mapsto (x, y) & \\
 S_0(X) \otimes S_0(Y) & \xrightarrow{\quad} & S_0(X \times Y) \\
 & g_0: (x, y) \mapsto x \otimes y &
 \end{array}$$

We can find lifts...

□

8.12.2. The Eilenberg-Zilber theorem: without acyclic models

We first define a so-called *homology cross product*

$$\times: S_p(X) \otimes S_q(Y) \rightarrow S_{p+q}(X \times Y),$$

and then show that it descends to homology.

Our goal is to find a way of making a p -simplex in X and a q -simplex in Y into a $(p+q)$ -simplex in $X \times Y$, called the *homology cross product*. At least in the case that one of p or q is equal to zero, it is clear what to do, since the Cartesian product of a p -simplex with a zero-simplex is in an obvious way a p -simplex. In more general cases, however, we have to be clever. The strategy is to write down a list of properties that we would like our homology cross product to have, and then show that there exists a unique map satisfying them.

Lemma 275. We can define a homomorphism

$$\times: S_p(X) \otimes S_q(Y) \rightarrow S_{p+q}(X \times Y), \quad p, q \geq 0,$$

with the following properties.

1. For all points $x_0 \in X$ viewed as zero-chains in $S_0(X)$ and all $\beta: \Delta^q \rightarrow Y$, we have

$$(x_0 \times \beta)(t_0, \dots, t_q) = (x_0, \beta(t_0, \dots, t_q));$$

conversely, for $\alpha \in S_p(X)$ and $y_0 \in Y$, we have

$$(\alpha \times y_0)(t_0, \dots, t_p) = (\alpha(t_0, \dots, t_p), y_0).$$

2. The map \times is natural in the sense that the square

$$\begin{array}{ccc}
 S_p(X) \otimes S_q(Y) & \xrightarrow{\times} & S_{p+q}(X \times Y) \\
 \downarrow & & \downarrow \\
 S_p(X') \otimes S_q(Y') & \xrightarrow{\times} & S_{p+q}(X' \times Y')
 \end{array}$$

commutes.

3. The map \times satisfies the Leibniz rule in the sense that

$$\partial(\alpha \times \beta) = \partial(\alpha) \times \beta + (-1)^p \alpha \times \partial(\beta).$$

Proof. As a warm up, let us work out some hypothetical consequences of our formulae, in the special case that $X = \Delta^p$ and $Y = \Delta^q$. In this case, we have

$$\text{id}_{\Delta^p} \in S_p(\Delta^p), \quad \text{id}_{\Delta^q} \in S_q(\Delta^q),$$

so we can take the homology cross product $\text{id}_{\Delta^p} \times \text{id}_{\Delta^q} \in S_{p+q}(\Delta^p \times \Delta^q)$. By the Leibniz rule, we have

$$\partial(\text{id}_{\Delta^p} \times \text{id}_{\Delta^q}) = \partial(\text{id}_{\Delta^p}) \times \text{id}_{\Delta^q} + (-1)^p \text{id}_{\Delta^p} \times \partial(\text{id}_{\Delta^q}).$$

This is now a perfectly well-defined element of $S_{p+q-1}(\Delta^p \times \Delta^q)$, which we will give the nickname R .⁷

A trivial computation shows that $\partial R = 0$, so there exists a $c \in S_{p+q}(\Delta^p \times \Delta^q)$ with $\partial c = R$. We choose some such c and define $\text{id}_{\Delta^p} \times \text{id}_{\Delta^q} = c$.

We now use the trick of expressing some $\alpha: \Delta^p \rightarrow X$ as $S_p(\alpha)(\text{id}_{\Delta^p})$. Then

$$\alpha \times \beta = S_p(\alpha)(\text{id}_{\Delta^p}) \times S_q(\beta)(\text{id}_{\Delta^q}).$$

But then the naturality forces our hand; chasing $\text{id}_{\Delta^p} \otimes \text{id}_{\Delta^q}$ around the naturality square

$$\begin{array}{ccc} S_p(\Delta^p) \otimes S_q(\Delta^q) & \xrightarrow{\times} & S_{p+q}(\Delta^p \times \Delta^q) \\ S_p(\alpha) \otimes S_q(\beta) \downarrow & & \downarrow S_{p+q}(\alpha, \beta) \\ S_p(X) \otimes S_q(Y) & \xrightarrow{\times} & S_{p+q}(X \times Y) \end{array}$$

tells us that

$$\begin{aligned} S_{p+q}(\alpha, \beta)(\text{id}_{\Delta^p} \times \text{id}_{\Delta^q}) &= S_p(\alpha)(\text{id}_{\Delta^p}) \times S_q(\beta)(\text{id}_{\Delta^q}) \\ S_{p+q}(\alpha, \beta)(c) &= \alpha \times \beta. \end{aligned}$$

Thus, we can define

$$\alpha \times \beta = S_{p+q}(\alpha, \beta)(c).$$

This satisfies all the desired properties by construction. \square

Proposition 276. Any natural transformations f, g with components

$$f_{X,Y}, g_{X,Y}: (S(X) \otimes S(Y))_{\bullet} \rightarrow S_{\bullet}(X \times Y)$$

⁷I think there's an induction argument hidden here.

8. Homology

which agree in degree zero and send $x_0 \otimes y_0 \mapsto (x_0, y_0)$ are chain homotopic.

Proof. First, suppose that $X = \Delta^p$ and $Y = \Delta^q$. Then $(S(\Delta^p) \otimes S(\Delta^q))_\bullet$ is free, hence certainly projective, and $S(\Delta^p \times \Delta^q)$ is acyclic, so the result follows from [Lemma 68](#): that is, we get a chain homotopy H with components

$$H_n: (S_\bullet(\Delta^p) \otimes S_\bullet(\Delta^q))_n \rightarrow S_{n+1}(\Delta^p \times \Delta^q)$$

such that

$$\partial H_n + H_{n-1} \partial = f_n - g_n.$$

□

8.12.3. The topological Künneth formula

Theorem 277 (topological Künneth formula). For any topological spaces X and Y , we have the following short exact sequence, natural in X and Y .

$$0 \rightarrow \bigoplus_{p+q=n} H_p(X) \otimes H_q(Y) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(X), H_q(Y)) \rightarrow 0$$

This sequence splits, but not canonically, and the splitting is not natural.

Example 278. Consider the torus $T^2 \cong \mathbb{S}^1 \times \mathbb{S}^1$. We have that the following sequence is exact.

$$0 \rightarrow \bigoplus_{p+q=n} H_p(\mathbb{S}^1) \otimes H_q(\mathbb{S}^1) \rightarrow H_n(\mathbb{S}^1 \times \mathbb{S}^1) \rightarrow \bigoplus_{p+q=n-1} \text{Tor}(H_p(\mathbb{S}^1), H_q(\mathbb{S}^1)) \rightarrow 0$$

Because $H_i(\mathbb{S})$ is either 0 or \mathbb{Z} , hence certainly projective, we get that

$$H_n(\mathbb{S}^1 \times \mathbb{S}^1) \cong \bigoplus_{p+q=n} H_p(\mathbb{S}) \otimes H_q(\mathbb{S}) \cong \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^2 & n = 1 \\ \mathbb{Z} & n = 2 \\ 0 & \text{otherwise.} \end{cases}$$

By induction, $H_n(\mathbb{S}^k) = \mathbb{Z}^{\binom{k}{n}}$.

Example 279. For a space of the form $X \times \mathbb{S}^n$, we get

$$H_n(X \times \mathbb{S}^n) \cong H_n(X) \oplus H_{n-n}(X).$$

Example 280 (products of Moore spaces). Let A be an abelian group. For $n \geq 1$, the

n th Moore space of A is the space $M(A, n)$ such that

$$H_r(M(A, n)) \cong \begin{cases} \mathbb{Z}, & r = 0 \\ A, & r = n \\ 0, & \text{otherwise} \end{cases}.$$

For example, $M(\mathbb{Z}, n) \cong \mathbb{S}^n$.

The topological Künneth formula allows us to compute products of Moore spaces. More specifically, it tells us that that we can express $H_r(M(A, n) \times M(B, m))$ as a direct sum

$$\left(\bigoplus_{p+q=r} H_p(M(A, n)) \otimes H_q(M(B, m)) \right) \oplus \left(\bigoplus_{p+q=r-1} \text{Tor}_1(H_p(M(A, n)), M(B, m)) \right).$$

First, assume that $m \neq n$. Then

$$H_0(M(A, m) \times M(B, n)) \cong H_0(M(A, n)) \otimes H_0(M(B, m)) \cong \mathbb{Z}.$$

For $1 \leq r < \min(m, n)$, we have no non-trivial terms. Taking $m < n$ without loss of generality, the only non-trivial

Therefore, for $m \neq n$, we have that...

Thus,

$$H_r(M(A, m) \times M(B, n)) \cong \begin{cases} \mathbb{Z}, & r = 0 \\ A, & r = m \\ B, & r = n \\ A \otimes B, & r = n + m \\ \text{Tor}_1(A, B), & r = n + m + 1 \\ 0, & \text{otherwise.} \end{cases}$$

For example,

8.13. The Eilenberg-Steenrod axioms

Denote by T the functor

$$T: \mathbf{Pair} \rightarrow \mathbf{Pair}; \quad (X, A) \mapsto (A, \emptyset); \quad (f: (X, A) \rightarrow (Y, B)) \mapsto f|_A.$$

Definition 281 (homology theory). Let A be an abelian group. A homology theory

8. Homology

with coefficients in A is a sequence of functors

$$H_n: \mathbf{Pair} \rightarrow \mathbf{Ab}, \quad n \geq 0$$

together with natural transformations

$$\delta: H_n \Rightarrow H_{n-1} \circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,

$$f \sim g \implies H_n(f) = H_n(g) \quad \text{for all } f, g, n.$$

2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(i): H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$$

for all n .

3. **Dimension:** We have

$$H_n(\text{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_{\alpha} X_{\alpha}$ is a disjoint union of topological spaces, then

$$H_n(X) = \bigoplus_{\alpha} H_n(X_{\alpha}).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.

9. Cohomology

9.1. Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in [Definition 281](#)) which govern cohomology theories.

Definition 282 (cohomology theory). Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n: \mathbf{Pair}^{\text{op}} \rightarrow \mathbf{Ab}; \quad n \geq 0$$

together with natural transformations

$$\partial: H^n \circ T^{\text{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n .
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i: (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H^n(i): H^n(X, A) \cong H^n(X \setminus U, A \setminus U)$$

for all n .

3. **Dimension:** We have

$$H^n(\text{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_{\alpha} X_{\alpha}$ is a disjoint union of topological spaces, then

$$H^n(X) = \prod_{\alpha} H^n(X_{\alpha}).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

9.2. Singular cohomology

Definition 283 (singular cohomology). Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X; G) = \text{Hom}(S_\bullet(X), G).$$

We also define relative homology:

$$S^\bullet(X, A; G) = \text{Hom}(S_\bullet(X, A), G).$$

We will denote the evaluation map $\text{Hom}(A, G) \otimes A \rightarrow G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 284. Let C_\bullet be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle : C^n(X; G) \otimes C_n(X) \rightarrow G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle : H^n(C^\bullet) \otimes H_n(C_\bullet) \rightarrow G.$$

Proof. Let $\alpha : C_n \rightarrow G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned} \langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\ &= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\ &= \langle \alpha, a \rangle. \end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned} \langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\ &= \langle \alpha, a \rangle + \langle \beta, da \rangle \\ &= \langle \alpha, a \rangle \end{aligned}$$

□

Via \otimes -hom adjunction, we get a map

$$\kappa : H^n(C^\bullet) \rightarrow \text{Hom}(H_n(C_\bullet), G).$$

Theorem 285 (universal coefficient theorem for singular cohomology). Let X be a topo-

logical space, and G an abelian group. There is a split exact sequence

$$0 \longrightarrow \text{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X; G) \twoheadrightarrow \text{Hom}(H_n(X), G) \longrightarrow 0$$

Proof. We now □

Example 286. We have seen (in [Example 266](#)) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \leq k \leq 2n, k \text{ even}, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, for $0 \leq k \leq n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \text{Ext}^1(0, \mathbb{Z}) \oplus \text{Hom}(\mathbb{Z}, \mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \leq k \leq n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n; \mathbb{Z}) &\cong \text{Ext}^1(\mathbb{Z}, \mathbb{Z}) \oplus \text{Hom}(0, \mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n; \mathbb{Z}) = 0$.

9.3. The cap product

The Kroenecker pairing gives us a natural evaluation map

$$S^n(X) \otimes S_n(X) \rightarrow \mathbb{Z},$$

allowing an n -cochain to eat an n -chain. A cochain of order q cannot directly act on a chain of degree $n > q$, but we can split such a chain up into a singular q -simplex, on which our cochain can act, and a singular $n - q$ -simplex, which is along for the ride.

Definition 287 (front, rear face). Let $a: \Delta^n \rightarrow X$ be a singular n -simplex, and let $0 \leq q \leq n$.

- The $(n - q)$ -dimensional front face of a is given by

$$F^{n-q}(a) = \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X,$$

where i is induced by the inclusion

$$\{0, \dots, n - q\} \hookrightarrow \{1, \dots, n\}; \quad k \mapsto k.$$

9. Cohomology

- The q -dimensional rear face of a is given by

$$R^q(a) = \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X ,$$

where r is the map

$$\{0, \dots, q\} \rightarrow \{0, \dots, n\}; \quad k \mapsto n - q + k.$$

Note that for $a \in S_n(X)$, we can write

$$F^{n-q}(a) = (\partial_{n-q+1} \circ \partial_{n-q+2} \circ \dots \circ \partial_n)(a)$$

and

$$R^q(a) = \overbrace{(\partial_0 \circ \partial_0 \circ \dots \circ \partial_0)}^{q \text{ times}}(a).$$

The cap product will be a map

$$\frown : S^q(X) \otimes S_n(X) \rightarrow S_{n-q}(X); \quad \alpha \otimes a \mapsto F^{n-q}(a) \langle a, R^q(a) \rangle.$$

However, we will want to generalize slightly, to arbitrary coefficient systems and relative homology.

Definition 288 (cap product). The cap product is the map

$$\frown : S^q(X, A; R) \otimes S_n(X, A; R) \rightarrow S_{n-q}(X; R); \quad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a) \rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b) \rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b) \rangle = 0$ by definition of relative cohomology.

Lemma 289. The cap product obeys the Leibniz rule, in the sense that

$$\partial(\alpha \frown (a \otimes r)) = (\delta\alpha) \frown (a \otimes r) + (-1)^q \alpha \frown (\partial a \otimes r).$$

Proof. We do three calculations.

$$\begin{aligned}\partial(\alpha \frown a) &= \partial(F^{n-q}(\alpha) \otimes \langle \alpha, R^q(a) \rangle) \\ &= \partial(F^{n-q}(a)) \otimes \langle \alpha, R^q(a) \rangle \\ &= \sum_{i=0}^{n-q} (-1)^i \partial_i(\partial_{n-q+1} \circ \cdots \circ \partial_n)(a) \otimes \langle \alpha, R^q(a) \rangle\end{aligned}$$

$$\begin{aligned}(\delta\alpha) \frown a &= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a) \rangle \\ &= F^{n-q}(a) \otimes \langle \delta\alpha, R^q(a) \rangle \\ &= F^{n-q}(a) \otimes \langle \alpha, \partial R^q(a) \rangle \\ &= \sum_{i=0}^q (-1)^i F^{n-q}(a) \otimes \langle \alpha, \partial_i \circ \partial_0^q a \rangle\end{aligned}$$

$$\alpha \frown (\partial a) =$$

□

Proposition 290. The cap product descends to a map on homology

$$\frown: H^q(X, A; R) \otimes H_n(X, A; R) \rightarrow H^{n-q}(X, A; R)$$

via the formula

$$[\alpha] \cap [a] := [\alpha \frown a].$$

Proof. Let us first collect some general results. Consider the formula

$$\partial(\alpha \frown a) = (\delta\alpha) \frown a + (-1)^q \alpha \frown \partial a.$$

1. Let α be a cocycle and a be a cycle. Then each term on the right-hand side is zero, so

$$\text{cocycle} \frown \text{cycle} = \text{cycle}.$$

2. Let α be arbitrary, so $\delta\alpha$ is a coboundary, and let a be a cycle. Then the above formula, read backwards, tells us that

$$\text{coboundary} \frown \text{cycle} = \text{boundary}.$$

3. Let a be arbitrary, so ∂a is a cycle, and let α be a coboundary. Then we get

$$\text{cocycle} \frown \text{boundary} = \text{boundary}.$$

9. Cohomology

Point 1. tells us that the formula is well-defined to begin with: $\alpha \cap a$ is really a cycle. Now suppose we had picked different representatives $\alpha + \delta\beta$ and $a + \partial b$. Then

$$\begin{aligned} [\alpha + \delta\beta] \cap [a + \partial b] &= [(\alpha + \delta\beta) \cap (a + \partial b)] \\ &= [\alpha \cap a] + [\delta\beta \cap a] + [\alpha \cap \partial b] + [\delta\beta \cap \partial b] \\ &= [\alpha \cap a] \\ &= [\alpha] \cap [a], \end{aligned}$$

where the the second term is zero because of 2. and the third term is zero because of 3. \square

Proposition 291. The cap product is natural in the sense that for any $f: X \rightarrow X$ inducing a map of pairs $(X, A) \rightarrow (X, B)$, the diagram

$$\begin{array}{ccc} H^q(X, A; R) \otimes H^n(X, A; R) & \xrightarrow{\quad \cap \quad} & H_{n-q}(X; R) \\ \downarrow & & \downarrow \\ \begin{array}{ccc} (f^*\beta, a) & \longmapsto & F(a) \otimes \langle f^*\beta, R(a) \rangle \\ \downarrow & & \downarrow \\ (\beta, f_*a) & \longmapsto & \star \end{array} & & \\ H^q(X, B; R) \otimes H^n(X, B; R) & \xrightarrow{\quad \cap \quad} & H_{n-q}(X; R) \end{array}$$

commutes, i.e.

$$f_*(F(a) \otimes \langle f^*\beta, R(a) \rangle) = F(f_*a) \otimes \langle \beta, R(f_*a) \rangle.$$

Proof.

$$\begin{aligned} f_*(F(a) \otimes \langle f^*\beta, R(a) \rangle) &= f_*(F(a) \otimes \langle f^*\beta, R(\alpha) \rangle) \\ &= f_*(F(a) \otimes \langle \beta, f_*(R(a)) \rangle) \\ &= f_*(F(a) \otimes \langle \beta, R(f_*a) \rangle) \\ &= F(f_*a) \otimes \langle \beta, R(f_*a) \rangle. \end{aligned}$$

\square

We can write this in a prettier way as

$$f_*(f^*\beta \cap a) = \beta \cap f_*(a).$$

Example 292. Suppose X is simply connected, and let

9.4. The cup product

This section takes place over a commutative ring with unit R . We will notationally suppress R .

Recall that we have constructed an Eilenberg-Zilber map

$$S_{\bullet}(X \times Y) \rightarrow S_{\bullet}(X) \otimes S_{\bullet}(Y).$$

in [Subsection 8.12.1](#). There exist generalizations, which we will not construct explicitly.

Definition 293 (Eilenberg-Zilber map). An Eilenberg-Zilber map is a natural chain map

$$S_{\bullet}(X \times Y, X \times B \cup A \times Y) \rightarrow S_{\bullet}(X, A) \otimes S_{\bullet}(Y, B).$$

Definition 294 (cohomology cross product). Let EZ be any Eilenberg-Zilber map. For any $\alpha \in S^p(X, A)$ and $\beta \in S^q(Y, B)$, a cohomology cross product of α and β is the $(p+q)$ -cochain $\alpha \times \beta$ defined by making the below diagram commute.

$$\begin{array}{ccc}
 S_n(X \times Y; X \times B \cup A \times Y) & & \\
 \text{EZ} \downarrow & \searrow \alpha \times \beta & \\
 \bigoplus_{p+q=n} S_p(X) \otimes S_q(Y) & & \\
 \text{proj} \downarrow & & \\
 S_p(X, A) \otimes S_q(Y, B) & \xrightarrow{\alpha \otimes \beta} R \otimes R & \xrightarrow{\mu} R
 \end{array}$$

Fact 295. Any homology cross product extends to a natural chain map

$$S^{\bullet}(X, A) \otimes S^{\bullet}(Y, B) \rightarrow S^{\bullet}(X \times Y, X \times B \cup A \times Y).$$

Note that since the Eilenberg-Zilber map is natural by assumption, projection is natural for any product, and evaluation is natural by definition of the tensor product, the homology cross product is natural in that the diagram

$$\begin{array}{ccc}
 S^p(X) \otimes S^q(Y) & \longrightarrow & S^{p+q}(X \times Y) \\
 \downarrow & & \downarrow \\
 S^p(X') \otimes S^q(Y') & \longrightarrow & S^{p+q}(X' \times Y')
 \end{array}$$

commutes; that is

$$(f, g)^*(\alpha \times \beta) = (f^* \alpha) \times (g^* \beta).$$

Fact 296. The homology cross product has the following properties.

9. Cohomology

1. The Leibniz rule holds, i.e.

$$\delta(\alpha \times \beta) = (\delta\alpha) \times \beta + (-1)^{|\alpha|} \alpha \times (\delta\beta).$$

2. For the Kroenecker pairing we have for homology classes a, b , and cohomology classes α, β of corresponding degree

$$\langle \alpha \times \beta, a \times b \rangle = \langle \alpha, a \rangle \langle \beta, b \rangle.$$

3. For $1 \in R$ and thus $1 \in S^0(X, A)$, we have

$$1 \times \beta = p_2^*(\beta), \quad \alpha \times 1 = p_1^*(\alpha),$$

where p_i denotes the projection onto the i th factor of $X \times Y$.

4. The cohomology cross product is associative.
5. The cohomology cross product satisfies a graded version of commutativity: the twist map $\tau: X \times Y \rightarrow Y \times X$ yields on homology

$$\alpha \times \beta = (-1)^{|\alpha||\beta|} \tau^*(\beta \times \alpha).$$

Definition 297 (cup product). Denote by $\Delta: X \rightarrow X \times X$ the diagonal map. The cup product is the composition

$$H^p(X, A) \otimes H^q(X, B) \rightarrow H^{p+q}(X \times X, X \times B \cup A \times X) \rightarrow H^{p+q}(X, A \cup B),$$

where the first map is the cross product, and the second map is the pullback along the diagonal. That is,

$$\alpha \cup \beta = \Delta^*(\alpha \times \beta).$$

Proposition 298. The cup product satisfies the following identities.

1. $\alpha \cup (\beta \cup \gamma) = (\alpha \cup \beta) \cup \gamma$
2. $\alpha \cup \beta = (-1)^{|\alpha||\beta|} \beta \cup \alpha$.
3. For $\partial: H^*(A) \rightarrow H^{*+1}(X, A)$, $\alpha \in H^*(A)$, $\beta \in H^*(X)$, we have

$$\partial(\alpha \cup i^* \beta) = (\partial\alpha) \cup \beta \in H^*(X, A).$$

4. For $f: X \rightarrow Y$, we have

$$f^*(\alpha \cup \beta) = f^*\alpha \cup f^*\beta.$$

5. We can express the cohomology cross product via the cup product using the for-

mula

$$\alpha \times \beta = p_1^*(\alpha) \cup p_2^*(\beta).$$

Proof.

1. Consequence of associativity of the cross product.
- 2.
- 3.
- 4.
- 5.

□

Definition 299 (diagonal approximation). A diagonal approximation is a natural chain map

$$D: S_\bullet(X) \rightarrow S_\bullet(X) \otimes S_\bullet(X)$$

with $D(x) = x \otimes x$ for $x \in S_0(X)$.

Theorem 208 tells us immediately that any two diagonal approximations are naturally homotopic.

Example 300 (Alexander-Whitney map). Consider the map

$$AW: S_n(X) \rightarrow (S_\bullet(X) \times S_\bullet(X))_n, \quad a \mapsto \sum_{p+q=n} F^p(A) \otimes R^q(a).$$

This is a chain map because

9.5. Manifold orientations

9.5.1. Basic definitions and results

Let M be an n -dimensional topological manifold. Without loss of generality, we may choose an atlas for M such that each point $x \in M$ has a chart (U_x, ϕ_x) such that $U_x \cong \mathbb{R}^n$ and $\phi_x(x) = 0 \in \mathbb{R}^n$.

Let $x \in M$. Then

$$\begin{aligned} H_n(M, M \setminus \{x\}) &\cong H_n(M \setminus (M \setminus U_x), M \setminus \{x\} \setminus (M \setminus U_x)) && \text{(excision)} \\ &\cong H_n(U_x, U_x \setminus \{x\}) \\ &\cong H_n(\mathbb{D}^n, \mathbb{S}^{n-1}) && \text{(homotopy)} \\ &\cong H_{n-1}(\mathbb{S}^{n-1}) && \text{(Example 218)} \\ &\cong \mathbb{Z}. \end{aligned}$$

9. Cohomology

We call a choice of generator of $H_n(X, X \setminus \{x\})$ a *local orientation* on M at x . For any other point $y \in U_x$, a local orientation at x gives a local orientation at y via the composition

$$H_n(X, X \setminus \{x\}) \cong H_n(X, X \setminus U_x) \cong H_n(X, X \setminus \{y\}).$$

We will use the following notation: for a pair X, A , we will write

$$H_n(X, A \setminus X) = H_n(X | A).$$

Furthermore, if $A = \{x\}$, we will write $H_n(X | x)$ instead of $H_n(X | \{x\})$.

We think of $H_n(X | A)$ as the *local homology* of A in X , since it depends (by excision) only on the closure of A in X .

Definition 301 (orientation cover). Let M be an n -dimensional manifold. The orientation cover of M is, as a set,

$$\tilde{M} = \{\mu_x \mid x \in M, \mu_x \text{ is a local orientation of } M \text{ at } x\}.$$

There is a clear surjection from \tilde{M} to M as follows.

$$\begin{array}{ccc} \tilde{M} & & \mu_x \\ \downarrow & & \downarrow \\ M & & x \end{array}$$

The set \tilde{M} turns out to be a topological manifold, and the surjection $\tilde{M} \rightarrow M$ turns out to be a double cover. We will show this later.

Definition 302 (orientation). Let M be an n -dimensional manifold. An orientation on M is a set-section σ of the bundle $\tilde{M} \rightarrow M$ satisfying the following condition: for any $x \in M$ and any $y \in U_x$, the induced local orientation at y is equal to $\sigma(y)$.

For a triple $B \subset A \subset X$ denote the canonical map on pairs $(X, X \setminus A) \rightarrow (X, X \setminus B)$ by $\rho_{B,A}$.

Lemma 303. Let M be a connected, orientable topological manifold of dimension m , and let $K \subset M$ be a compact subset.

1. For all $q > m$, $H_q(M | K) = 0$.
2. If $a \in H_m(M | K)$, then $a = 0$ if and only if $(\rho_{x,K})_* a = 0$ for all $x \in X$.

Sketch of proof. The proof uses manifold induction, which consists of the following steps.

1. Prove the result the special case that our manifold M is simply \mathbb{R}^n and K is a convex compact subset.

2. Using Mayer-Vietoris, show that the result holds when K is a union of two compact, convex subsets.
3. The case of $M = \mathbb{R}^n$ K a union of n compact, convex subsets follows by induction.
4. For K an arbitrary compact subset of \mathbb{R}^m , we cover K by finitely many closed balls B_i , then show that every cycle on $H_q(\mathbb{R}^m \mid K)$ comes from a cycle on $H_q(\mathbb{R}^m \mid \bigcup_i B_i)$. Since $\bigcup_i B_i$ is a finite union of compact, convex subsets, there are no non-trivial cycles, implying the claim.
5. For an arbitrary compact subset contained in a single chart, the claim reduces to the \mathbb{R}^m case above.
6. For an arbitrary compact subset contained in two charts, the result follows from the Mayer-Vietoris sequence. The general claim follows by induction.

□

Proof.

1. First take $M = \mathbb{R}^m$ and $K \subset M$ compact and convex, so that any $x \in K$ is a deformation retract of K . Then, since K is closed and bounded, we can find an open ball $\mathring{\mathbb{D}}^m$ with $K \subset \mathring{\mathbb{D}}^m$. Then certainly

$$\overline{M \setminus \mathring{\mathbb{D}}^m} = M \setminus \mathbb{D}^m \subset (M \setminus K) = M \setminus K,$$

so by excision and deformation retractness we have

$$\begin{aligned} H_q(M, M \setminus K) &\cong H_q(M \setminus (M \setminus \mathring{\mathbb{D}}^m), (M \setminus K) \setminus (M \setminus \mathring{\mathbb{D}}^m)) \\ &= H_q(\mathbb{D}^m, \mathbb{D}^m \setminus K) \\ &\cong H_q(\mathbb{D}^m, \mathbb{D}^m \setminus \{x\}) \\ &\cong H_q(\mathbb{D}^m, \mathbb{S}^{m-1}) \\ &\cong H_{q-1}(\mathbb{S}^{m-1}). \end{aligned}$$

For $q > m$, this is 0, and for $q = m$ we note that similar reasoning provides an isomorphism

$$H_q(M, M \setminus K) \cong H_q(M, M \setminus \{x\}) \text{ for all } x \in K.$$

2. Now let $M = \mathbb{R}^n$ and $K = K_1 \cup K_2$, where both K_i are compact and convex. Note that $K_1 \cap K_2$ is also compact and convex. Consider the following relative Mayer-

9. Cohomology

Vietoris sequence.

$$\begin{array}{ccccccc}
 & & \dots & \longrightarrow & H_{q+1}(M \mid K_1 \cap K_2) & & \\
 & \swarrow & & & \searrow & & \\
 & & & \delta & & & \\
 H_q(M \mid K_1 \cup K_2) & \longrightarrow & H_q(M \mid K_1) \oplus H_q(M \mid K_2) & \longrightarrow & H_q(M \mid K_1 \cap K_2) & & \\
 & \swarrow & & & \searrow & & \\
 & & & \delta & & & \\
 H_{q-1}(M \mid K_1 \cup K_2) & \longrightarrow & \dots & & & &
 \end{array}$$

For $q > n$, Part 1. implies that

$$H_{q+1}(M \mid K_1 \cap K_2) = H_q(M \mid K_1) = H_q(M \mid K_2) = 0,$$

so by exactness $H_q(M \mid K_1 \cup K_2) = 0$.

3. For $M = \mathbb{R}^n$, $K = \bigcup_i K_i$ for K_i compact and contractible, we get the result by induction.
4. Let $M = \mathbb{R}^n$, and let K be an arbitrary compact subset.

Choose some $a \in H_q(\mathbb{R}^n \mid K)$, and choose $\psi \in S_q(\mathbb{R}^m)$ representing a . Then because ψ is a relative cycle, $\partial\psi \in S_{q-1}(\mathbb{R}^n \setminus K)$.

This means that we can express $\partial\psi$ as

$$\sum_{j=1}^{\ell} \lambda_j \tau_j, \quad \tau_j: \Delta^{q-1} \rightarrow \mathbb{R}^m \setminus K.$$

Since Δ^{q-1} is compact, the images $\tau(\Delta^{q-1})$ are compact, and their union

$$T = \bigcup_{j=1}^{\ell} \tau_j(\Delta^{q-1})$$

is compact.

There exists an open set $U \subset \mathbb{R}^m$ with $K \subset U$ and $T \cap U = \emptyset$. Thus, we know that

$$\partial\psi \in \mathbb{R}^m \setminus U \implies \psi \in Z_q(\mathbb{R}^m \mid U).$$

Denote the representative of ψ in $H_q(\mathbb{R}^m \mid U)$ by $[\psi] = a'$.

Due to general topology stuff, we can always find finitely many closed balls B_i satisfying the following conditions.

- a) $B_i \subset U$ for all i
- b) $B_i \cap K \neq \emptyset$

c) $K \subset \bigcup_i B_i$

Then $K \subset \bigcup_i B_i \subset U$, and we have restriction maps as follows.

$$S_q(\mathbb{R}^m | U) \xrightarrow{(\rho_{\bigcup_i U_i, U})_*} S_q(\mathbb{R}^m | \bigcup_i B_i) \xrightarrow{(\rho_{K, \bigcup_i U_i})_*} S_q(\mathbb{R}^m | K)$$

$$a' \longmapsto (\rho_{\bigcup_i B_i, U})_*(a') = a'' \longmapsto a$$

Suppose $q > m$. We have already showed that $H_q(\mathbb{R}^m | \bigcup_i B_i) = 0$, so $a'' = 0$. Hence $a = 0$.

Now suppose $q = m$. We will provide running commentary on the following commutative diagram.

$$\begin{array}{ccccc} H_m(X | U) & \longrightarrow & H_m(X | \bigcup_i B_i) & \longrightarrow & H_m(X | K) \\ & & \downarrow & & \\ & & H_m(X | B_i) & & \\ & \swarrow & & \searrow & \\ H_m(X | y) & & & & H_m(X | x) \end{array}$$

We have shown that given $a \in H_m(X | K)$, one can find $a' \in H_m(X | U)$ and $a'' \in H_m(X | \bigcup_i B_i)$ which map to each other as follows. By assumption, a'' maps to zero in $H_m(X | x)$.

$$\begin{array}{ccccc} a' & \longmapsto & a'' & \longmapsto & a \\ & & \searrow & & \\ & & 0 & & \end{array}$$

Pick some B_i containing x . Since $(\rho_{x, B_i})_*$ is an isomorphism, we have that $(\rho_{B_i, \bigcup_i B_i})_*(a'') = 0$.

$$\begin{array}{ccccc} a' & \longmapsto & a'' & \longmapsto & a \\ & & \downarrow & \searrow & \\ & & 0 & & 0 \end{array}$$

9. Cohomology

For any $y \in B_i$, we know that $(\rho_{y,B_i})_*$ is an isomorphism.

$$\begin{array}{ccccc}
 a' & \xrightarrow{\quad} & a'' & \xrightarrow{\quad} & a \\
 & \searrow & \downarrow & \swarrow & \\
 & & 0 & & \\
 & \swarrow & & \searrow & \\
 0 & & & & 0
 \end{array}$$

Thus $(\rho_{y,\bigcup_i B_i})_*(a'') = 0$ for all $y \in \bigcup_i B_i$, so the result is implied by work above.

5. Mayer-Vietoris and induction.

□

Proposition 304. Let $K \subset M$ be compact, and assume that M is connected and oriented via an orientation $x \mapsto o_x \in H_m(X \mid x)$. Then there exists a unique orientation class $o_K \in H_m(M|K)$ such that $(\rho_{x,K})_* o_K = o_x$ for all $x \in K$.

Proof. First we show uniqueness, as we will need it later. Suppose that there exist two orientation classes o_K, o'_K such that

$$(\rho_{x,K})_* o_K = (\rho_{x,K})_* o'_K = o_x.$$

Taking a difference, we find

$$(\rho_{x,K})_*(o_K - o'_K) = 0,$$

so by [Lemma 303](#), we get uniqueness.

As a first step towards proving existence, suppose that K is contained in the domain $U_\alpha \cong \mathbb{R}^n$ of a single chart. Pick some $x \in K$, and consider the following morphisms in **Pair**

$$\begin{array}{ccc}
 M \setminus U_\alpha & \hookrightarrow & M \\
 \downarrow & & \parallel \\
 M \setminus K & \hookrightarrow & M \\
 \downarrow & & \parallel \\
 M \setminus \{x\} & \hookrightarrow & M
 \end{array}$$

which correspond to the commuting diagram.

$$\begin{array}{ccc}
 H_m(M \mid U_\alpha) & \xlongequal{\quad} & H_m(M \mid x) \\
 \phi \downarrow & \nearrow & \\
 H_m(M \mid K) & &
 \end{array}$$

Pulling $o_x \in H_m(M | x)$ back to $H_m(M | U_\alpha)$, then mapping it to $H_m(M|K)$ with ϕ gives an orientation class $o_K \in H_m(M | K)$. The commutativity of the diagram ensures that $(\rho_{x,K})_* o_K = o_x$.

Now suppose that K is not contained in a single chart. We can find an open cover of K by charts, and since K is compact, there exists a finite subcover.

Suppose K is contained in two charts U_1 and U_2 ; the general case follows by induction. Then we can express K as a union of compacta $K = K_1 \cup K_2$, where $K_i \subset U_i$. Consider the relative Mayer-Vietoris sequence

$$0 \rightarrow H_m(M | K_1 \cup K_2) \xrightarrow{i} H_m(M | K_1) \oplus H_m(M | K_2) \xrightarrow{\kappa} H_m(M | K_1 \cap K_2) \rightarrow \dots$$

By our above work, we get unique orientation classes $o_{K_i} \in H_m(M|K_i)$ such that $(\rho_{x,K_i})_* = o_x$ for all $x \in K_i$, $i = 1, 2$. Applying κ , we find

$$\kappa(o_{K_1}, o_{K_2}) = \rho_{K_1 \cap K_2, K_1} o_{K_1} - \rho_{K_1 \cap K_2, K_2} o_{K_2}.$$

But by uniqueness, these must be equal, so there exists a unique o_K in $H_m(M | K)$ as in the theorem. \square

In order to give \tilde{M} a manifold structure, we define a topology on \tilde{M} as follows. Let $B \subset \mathbb{R}^n \subset M$ be a ball of finite radius, and choose a generator $\mu_B \in H_n(M | B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \rightarrow M$ is two-sheeted covering. This immediately implies that \tilde{M} is a manifold. In fact, \tilde{M} is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} | \mu_b) \cong H_n(U(\mu_B) | \mu_b) \cong H_n(B | x),$$

where

Definition 305 (orientation class). Let M be a connected, orientable, compact manifold of dimension m . An orientation class on M is an element $o_m \in H_m(M; \mathbb{Z})$ such that $o_x = (\rho_{x,M})_* o_m$ is an orientation.

Theorem 306. Let M be a compact, connected manifold of dimension m . The following are equivalent.

1. M is (\mathbb{Z}) -orientable.
2. There exists an orientation class $o_M \in H_m(M; \mathbb{Z})$.
3. $H_m(M; \mathbb{Z}) \cong \mathbb{Z}$.

Proof.

9. Cohomology

(1 \Rightarrow 2) Consequence of [Proposition 304](#).

(2 \Rightarrow 3) Let $a \in H_m(M)$, and let $x \in M$. Then $(\rho_{x,M})_* a = k_x o_x$ for some $k_x \in \mathbb{Z}$. This defines a function $M \rightarrow \mathbb{Z}$, $x \mapsto k_x$.

We aim to show that this function is constant. Choose a chart U_x around x , and let $y \in U_x$. Then the outside of the following diagram commutes by functoriality of H_m , and the lower triangle commutes by definition of an orientation class, so the upper triangle commutes.

$$\begin{array}{ccc}
 & H_n(M) & \\
 \swarrow & & \searrow \\
 H_m(M | x) & \xrightarrow{\phi} & H_m(M | y) \\
 \searrow & & \swarrow \\
 & H_m(X | U_x) &
 \end{array}$$

This means in particular that

$$\phi(k_x o_x) = k_x \phi(o_x) = k_x o_y$$

must be equal to $k_y o_y$, so $k_x = k_y$. Thus, the function $x \mapsto k_x$ is locally constant, hence constant. Call this constant k .

Thus, $ka - o_M$ is zero when restricted to any $x \in M$, implying by [Proposition 304](#) that $a = ko_M$. o_M generates $H_m(M)$.

We know that o_M is non-trivial because $(\rho_{x,M})_* o_M$ generates $H_m(M|x)$, and that it has infinite order because if $\ell o_M = 0$ for some $\ell \neq 0 \in \mathbb{Z}$ $H_m(M)$, then $\ell(\rho_{x,M})_* o_M = \ell o_x = 0$. This shows that $H_m(M)$ has a single generator of infinite order, i.e. is isomorphic to \mathbb{Z} .

(2 \Rightarrow 3) Pick a generator $o_M \in \mathbb{Z}$, and define an orientation by

$$o_x = (\rho_{x,M})_* o_M.$$

□

From now, we will sometimes denote an orientation class on M by $[M]$.

9.5.2. The orientation cover

Let M be an n -dimensional manifold, not necessarily orientable.

In order to give \tilde{M} a manifold structure, we define a topology on \tilde{M} as follows. Let $D: \mathbb{D}^m \hookrightarrow M$ be a compact subset of M homeomorphic to the closed ball, and pick

$\mu_D \in H_m(M \mid \mathring{\mathbb{D}})$. We define a subset $U_B \subset \tilde{M}$ by

Let $\mathbb{D}^n \subset \mathbb{R}^n \subset M$ be a closed ball of finite radius and, and choose a generator $\mu_B \in H_n(M \mid B)$. Let $U(\mu_B)$ be the subset consisting of elements $\mu_x \in \tilde{M}$ such that $x \in B$ and $(\rho_{x,B})_*(\mu_B) = \mu_x$. These form a basis for a topology; to see this note that

Relative to this topology, the projection $\tilde{M} \rightarrow M$ is two-sheeted covering. This immediately implies that \tilde{M} is a manifold. In fact, \tilde{M} is an oriented manifold. To see this, note that we have isomorphisms

$$H_n(\tilde{M} \mid \mu_b) \cong H_n(U(\mu_B) \mid \mu_b) \cong H_n(B \mid x),$$

where

9.5.3. Degree

Definition 307 (degree). Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $g: M \rightarrow N$ be continuous. The degree of f is the integer d such that

$$f(o_M) = do_N.$$

Proposition 308. Let M and N be oriented, connected, compact manifolds of the same dimension $m \geq 1$, and let $f, g: M \rightarrow N$ be continuous.

1. The degree is multiplicative, i.e.

$$\deg(f \circ g) = \deg(f) \circ \deg(g)$$

2. If \bar{M} is the same manifold as M but with the opposite orientation, then

$$\deg(f: M \rightarrow N) = -\deg(f: \bar{M} \rightarrow N).$$

3. If the degree of f is not trivial, then f is surjective.

Proof. All obvious. □

9.6. Cohomology with compact support

For this section, let R be a commutative ring with unit 1_R . For most of this section, we will work with cohomology over R , but notationally suppress this.

Definition 309 (singular cochains with compact support). Let X be a topological space, and let R be a commutative ring with unit 1_R . The singular n -cochains with compact

9. Cohomology

support are the set

$$S_c^n(X; R) = \left\{ \phi: S_n(X) \rightarrow R \mid \begin{array}{l} \text{there exists } K_\phi \subset X \text{ compact such that} \\ \phi(\sigma) = 0 \text{ for all } \sigma: \Delta^n \rightarrow X \text{ with } \sigma(\Delta^n) \cap K_\phi = \emptyset \end{array} \right\}.$$

That is, singular cochains with compact support are those ϕ which ignore simplices which do not hit some compact subset K_ϕ .

Clearly $S_c^n(X) \subset S^n(X)$, and since the boundary of any simplex with compact support clearly has compact support, there is an induced chain complex structure

$$S_c^*(X).$$

However, there is a more categorical description of this chain complex coming from relative cohomology.

Let X be a topological space. Denote by \mathcal{K} the set of compact subsets of X . This is clearly a poset under

$$K \leq K' \iff K \subseteq K',$$

and it is filtered because for a finite set I , $\bigcup_i K_i$ is an upper bound for $\{K_i\}_{i \in I}$.

Fixing some X , consider the functor

$$\mathcal{K}^{\text{op}} \rightarrow \mathbf{Pair}; \quad K \mapsto (X, X \setminus K).$$

We get a functor $\mathcal{K} \rightarrow \mathbf{Coch}_{\geq 0}(R\text{-Mod})$ via the composition

$$\mathcal{K} \longrightarrow \mathbf{Pair}^{\text{op}} \xrightarrow{S^\bullet} \mathbf{Coch}_{\geq 0}(R\text{-Mod}).$$

For obvious reasons, we call this functor $S^\bullet(X \mid -)$

Proposition 310. The colimit

$$\operatorname{colim}_K S^\bullet(X \mid K)$$

agrees with cohomology with compact support $S_c^\bullet(X)$.

Proof. Let $f \in S^n(X \mid K)$. Then by definition, f is a map $S_n(X \mid K) \rightarrow R$, i.e. a map $S_n(X) \rightarrow R$ such that for any $\sigma: \Delta^n \rightarrow X$ such that $\sigma(\Delta^n) \cap K = \emptyset$, $f(\sigma) = 0$. Thus, $f \in S_c^n(X)$.

This inclusion gives a chain map $S^\bullet(X \mid K) \hookrightarrow S_c^\bullet(X)$, and these together induce an inclusion¹ $\operatorname{colim}_K S^\bullet(X \mid K) \rightarrow S_c^\bullet(X)$. In fact, this is a level-wise epimorphism since any $g \in S_c^\bullet(X)$ is of the above form. \square

¹This is a monomorphism because the components are monomorphisms, and by [Corollary 121](#) filtered colimits are exact, hence preserve monomorphisms.

Corollary 311. We have an isomorphism

$$H_c^n(X) \cong \operatorname{colim}_K H^n(X | K).$$

Proof. Filtered colimits are exact by [Corollary 121](#). □

9.7. Poincaré duality

This section takes place over a ring R with unit 1_R . We will usually notationally suppress R .

Let M be a connected, m -dimensional manifold with R -orientation $x \mapsto o_x$, and let $K \subset M$ be a compact subset. By [Proposition 304](#), there exists an orientation class $o_K \in H_m(M | K)$ compatible with the orientation o_x .

Consider the cap product

$$H^q(M | K) \otimes H_m(M | K) \xrightarrow{\cap} H_{m-q}(M).$$

Sticking $o_K \in H_m(M | K)$ into the second slot of the cap product and setting $p = n - q$, we find a map

$$H^{m-p}(M | K) \xrightarrow{(-) \frown o_K} H_p(M)$$

Proposition 312. The maps

$$(-) \frown o_K: H^{m-p}(M | K) \rightarrow H_p(M)$$

descend to a map

$$H_c^{m-p}(M) \rightarrow H_p(M).$$

Proof. By [Proposition 291](#), the identity $\operatorname{id}_M: M \rightarrow M$ makes the following diagram commute.

$$\begin{array}{ccc} H^{m-p}(M | L) \otimes H_m(M | L) & \xrightarrow{(-) \frown (-)} & H_p(M) \\ \downarrow & & \downarrow (\operatorname{id}_M)_* \\ H^{m-p}(M | K) \otimes H_m(M | K) & \xrightarrow{(-) \frown (-)} & H_p(M) \end{array}$$

Since

$$(\operatorname{id}_M)_* = \operatorname{id}_{H_p(M)},$$

9. Cohomology

this in particular means that the triangles

$$\begin{array}{ccc}
 H^{m-p}(M \mid L) & & \\
 \downarrow & \searrow^{(-) \frown o_L} & \\
 H^{m-p}(M \mid K) & \nearrow_{(-) \frown o_K} & H_p(M)
 \end{array}$$

commute. But this says that the maps $(-) \frown o_K$ form a cocone under $H^{m-p}(M \mid -)$. \square

Definition 313 (Poincaré duality). The map defined in [Proposition 312](#) is called the Poincaré Duality map, and denoted

$$\text{PD}_M: H_c^{m-q}(M) \rightarrow H_m(M).$$

Theorem 314 (Poincaré duality). Let M be a connected manifold of dimension m with an R -orientation $x \mapsto o_x$. Then

$$\text{PD}_M: H_c^{m-p}(M; R) \rightarrow H_p(M; R)$$

is an isomorphism for all $p \in \mathbb{Z}$.

Proof. Manifold induction! \square

9.8. Alexander-Lefschetz duality

9.9. The Milnor sequence

In [Subsection 4.3.2](#), we showed that for any tower of cochain complexes of abelian groups $\{C_i\}$ satisfying the Mittag-Leffler condition² and any $q \in \mathbb{Z}$, we have a short exact sequence

$$0 \longrightarrow \lim^1 H^{q-1}(C_i) \hookrightarrow H^q(\lim_i C_i) \twoheadrightarrow \lim H_q(C_i) \longrightarrow 0 .$$

Theorem 315 (Milnor sequence for CW complexes). Let X be a CW complex with filtration

$$X_0 \subset X_1 \subset \cdots, \quad \text{so} \quad X \cong \bigcup_{i \geq 0} X_i.$$

Then for any abelian group G there is a short exact sequence

$$0 \longrightarrow \lim^1 H^{m-1}(X_n; G) \hookrightarrow H^m(X; G) \twoheadrightarrow \lim H^m(X_n; G) \longrightarrow 0 .$$

²i.e. satisfying it level-wise

Proof. The tower of chain complexes $C_n^\bullet = \text{Hom}(S_\bullet(X_n), G)$ satisfies the Mittag-Leffler condition (Definition 124) because the maps $C_n^\bullet \rightarrow C_{n-1}^\bullet$ dual to the inclusions $S_\bullet(X_{n-1}) \hookrightarrow S_\bullet(X_n)$ are epimorphisms (because the exact sequence on the pair splits).

Plugging this into in Theorem 128, we find that the sequence

$$0 \longrightarrow \lim^1 H^{q-1}(X_i) \hookrightarrow H^q(\lim \text{Hom}(S_\bullet(X_i), G)) \twoheadrightarrow \lim H_q(X_i) \longrightarrow 0 .$$

It remains only to show that

$$H^q(\lim \text{Hom}(S_\bullet(X_i), G)) \cong H^q(X; G),$$

i.e. that

$$\lim \text{Hom}(S_\bullet(X_i), G) \cong \text{Hom}(S_\bullet(X); G).$$

But this is true because Hom turns limits into colimits in the first slot, and S_\bullet commutes with colimits because reasons. \square

Example 316. Consider the tower (in Top!)

$$\mathbb{C}P^0 \hookrightarrow \mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \hookrightarrow \dots .$$

Denote

$$\mathbb{C}P^\infty := \lim_i \mathbb{C}P^i.$$

The above tower is the CW filtration of $\mathbb{C}P^\infty$, including only even levels. Because the sequence

$$0 \longrightarrow S_\bullet(\mathbb{C}P^{n-1}) \hookrightarrow S_\bullet(\mathbb{C}P^n) \twoheadrightarrow S_\bullet(\mathbb{C}P^n, \mathbb{C}P^{n-1}) \longrightarrow 0$$

splits and Hom and H_q are additive, we get that the maps

$$H_q(\mathbb{C}P^n) \rightarrow H_q(\mathbb{C}P^{n-1})$$

are epimorphisms. Thus, the tower $H^{q-1}(\mathbb{C}P^n)$ also satisfies the Mittag-Leffler condition, and we get that

$$H^q(\mathbb{C}P^\infty) \cong \lim_i H^q(\mathbb{C}P^i) \cong \begin{cases} \mathbb{Z}, & q \text{ even} \\ 0, & q \text{ odd.} \end{cases}$$

9.10. Lens spaces