1 Homological algebra in abelian categories

1.1 Chain complexes

Definition 1 (category of chain complexes)**.** Let $\mathcal{A}$ be an abelian category. A chain complex $C_\bullet$ in $\mathcal{A}$ is a collection of objects $C_n \in \mathcal{A}$, $n \in \mathbb{Z}$, together with morphisms $d_n \colon C_n \to C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

The category of chain complexes in $\mathcal{A}$, denoted $\mathbf{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms $C_\bullet \to D_\bullet$ of chain complexes, i.e. for each $n \in \mathbb{Z}$ a map $f_n \colon C_n \to D_n$ such that all of the squares form commute.

$$\begin{tikzcd}
\cdots \arrow[r, "d^C_{n+2}"] & C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r, "d^C_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r, "d^D_{n+2}"] & D_{n+1} \arrow[r, "d^D_{n+1}"] & D_n \arrow[r, "d^D_n"] & D_{n-1} \arrow[r, "d^D_{n-1}"] & \cdots
\end{tikzcd}$$

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes (i.e. the full subcategory consisting of chain complexes $C_\bullet$ such that $C_n = 0$ for small enough n), $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Theorem 2. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the $\mathbf{Ab}$-enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in $\mathcal{A}$: given a diagram of the form

$$\begin{tikzcd}
& Q_\bullet \arrow[dl, "f_\bullet"'] \arrow[d, dashed, "\exists!\phi_\bullet"] \arrow[dr, "g_\bullet"] & \\
C_\bullet & C_\bullet \times D_\bullet \arrow[l, "p_1"] \arrow[r, "p_2"'] & D_\bullet
\end{tikzcd}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned}(d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q,\end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{tikzcd} & Q_\bullet \arrow[dl, dashed, "\phi_\bullet"'] \arrow[d, "g_\bullet"] \arrow[dr, "0"] & \\ \ker(f_\bullet) \arrow[r, "\iota_\bullet"] & A_\bullet \arrow[r, "f_\bullet"] & B_\bullet \end{tikzcd}$$

That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \tag{1.1}$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi \colon Q_n \to \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 1.1 can play the role of ψ. Plugging in the LHS, we have

$$\begin{aligned}\iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n\end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. □

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

1.2 Homology

We would like to replicate the notion of the homology of a chain complex $C_\bullet$ in the context of abelian categories.

$$\cdots \xrightarrow{d_{n+2}} C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n_1} \xrightarrow{d_{n-1}} \cdots$$

This will mean finding appropriate notions of every symbol appearing in the equation.

$$H_n(C_\bullet) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

We already have a notion of the kernel, which we can use to define a notion of the image in the following way. Given a map $f\colon A \to B$, we heuristically think of the cokernel of f as being $B/\operatorname{im} f$. Thus, taking the kernel of the cokernel should give us the image of f.

Definition 3 (image). Given a morphism $f\colon A \to B$, the <u>image</u> of f is $\operatorname{im} f = \ker \operatorname{coker} f$.

Now we need to figure out what it means to quotient the image of one map by the kernel of the next. Again, the cokernel saves our necks; we just need a map from $\operatorname{im} d_{n+1}$ into $\ker d_n$. The homology will be the cokernel of this map.

Definition 4 (homology). Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be morphisms such that $g \circ f = 0$. Using the universal properties of the kernel and cokernel, we can build the following commutative diagram.

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[d, hook, "i"] \arrow[dr, "0"] & \\
A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, two heads, "p"] & C \\
 & \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a monomorphism $\phi\colon \operatorname{im} f \to \ker g$.

$$\begin{tikzcd}
 & \operatorname{im} f \arrow[r, hook, "\phi"] \arrow[dr, hook, "i"] & \ker g \arrow[r, two heads] \arrow[d, hook] & H_n \\
A \arrow[rr, "f"] & & B \arrow[r, "g"] \arrow[d, two heads, "p"] & C \\
 & & \operatorname{coker} f \arrow[ur] &
\end{tikzcd}$$

The homology of the sequence $A \to B \to C$ at B is then $H_n = \operatorname{coker} \phi$.

Definition 5 (exact sequence)**.** Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

be objects and morphisms in an abelian category. We say that the above sequence is exact if ϕ_i is an isomorphism, i.e. if the homology is zero. We say that a complex $(C_\bullet, d)$ is exact if it is exact at all positions.

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Proposition 6. Homology extends to a family of additive functors

$$H_n\colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet\colon C_\bullet \to D_\bullet$ be a morphism of chain complexes.

$$\begin{tikzcd}
C_{n+1} \arrow[r, "d^C_{n+1}"] \arrow[d, "f_{n+1}"'] & C_n \arrow[r, "d^C_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[d, "f_{n-1}"] \\
D_{n+1} \arrow[r, "d^D_{n+1}"'] & D_n \arrow[r, "d^D_n"'] & D_{n-1}
\end{tikzcd}$$

We need a map $H_n(C_\bullet) \to H_n(D_\bullet)$. This will come from a map $\ker d^C_n \to \ker d^D_n$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. □

We can also go the other way, by defining a map

$$\iota_n\colon \mathcal{A} \to \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\begin{tikzcd} {} \arrow[r, dotted] & 0 \arrow[r] & A \arrow[r] & 0 \arrow[r, dotted] & {} \end{tikzcd}$$

concentrated in degree n.

Example 7. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then $H_0(C_\bullet) = \operatorname{coker} f$.

Similarly, if

$$0 \longrightarrow C_0 \xrightarrow{f} C_{-1} \longrightarrow \cdots$$

is a chain complex in $\mathbf{Ch}_{\leq 0}(\mathcal{A})$, then $H_0(C_\bullet) = \ker f$.

1.3 Diagram lemmas

1.3.1 The splitting lemma

Lemma 8 (Splitting lemma)**.** Consider the following solid short exact sequence in an abelian category $\mathcal{A}$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i_A"] & B \arrow[l, dashed, bend left, "\pi_A"] \arrow[r, "\pi_C"] & C \arrow[l, dashed, bend left, "i_C"] \arrow[r] & 0 \end{tikzcd}$$

The following are equivalent.

1. The sequence *splits from the left,* i.e. there exists a morphism $\pi_A \colon B \longrightarrow A$ such that $\pi_A \circ i_A = \mathrm{id}_A$
2. The sequence *splits from the right,* i.e. there exists a morphism $i_C \colon C \longrightarrow B$ such that $\pi_C \circ i_C = \mathrm{id}_C$
3. B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

Proof. That $3 \Rightarrow 1$ and $3 \Rightarrow 2$ is obvious. We now show that $1 \Rightarrow 2$.

By exactness, $i_A \colon A \longrightarrow B$ is a kernel of π_C. Consider the map $\phi = \mathrm{id}_B - i_C \circ \pi_C$. Since

$$\begin{aligned} \pi_C \circ \phi &= \pi_C - \pi_C \circ i_C \circ \pi_C \\ &= \pi_C - \mathrm{id} \circ \pi_C \\ &= 0, \end{aligned}$$

the universal property for the kernel guarantees us a map $\psi \colon B \longrightarrow A$ such that $i_A \circ \psi = \phi$.

Now we use the same universal property in a different situation. Consider the following diagram.

$$
\begin{tikzcd}
 & A \arrow[dl, dashed] \arrow[d, "i_A"] \arrow[dr, "0"] & \\
A \arrow[r, "i_A"] & B \arrow[r, "\pi_C"] & C
\end{tikzcd}
$$

The universal property guarantees us a unique dashed map making everything commute. But id_A and $i_A \circ \psi$ both do the job since

$$
\begin{aligned}
i_A \circ \psi \circ i_A &= i_A - i_C \circ \pi_C \circ i_A \\
&= i_A.
\end{aligned}
$$

□

1.3.2 The snake lemma

Lemma 9. Let $\mathcal{A}$ be an abelian category, and let the diagram

$$
\begin{tikzcd}
A \times_C B \arrow[r, "\tilde{f}"] \arrow[d, "\tilde{g}"'] & B \arrow[d, "g"] \\
A \arrow[r, two heads, "f"] & C
\end{tikzcd}
$$

be a pullback in $\mathcal{A}$. Then $\tilde{f}$ is an epimorphism.

Proof. The pullback is in particular a limit, hence is constructed with products and equalizers as follows.

$$
0 \longrightarrow A \times_C B \xhookrightarrow{(\tilde{f},\tilde{g})} A \oplus B \xrightarrow{f-g} C
$$

Because f is epic, $f - g$ is epic, so we have a bona fide short exact sequence

$$
0 \longrightarrow A \times_C B \xhookrightarrow{(\tilde{f},\tilde{g})} A \oplus B \xrightarrow{f-g} C \longrightarrow 0 \ .
$$

This means that the above square is also a pushout square. □

Lemma 10. Let $f\colon B \twoheadrightarrow C$ be an epimorphism, and let $g\colon D \to C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g, in the sense that we have the following commuting diagram in which the right-hand square is a pullback square.

$$
\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, "\ulcorner", very near start] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}
$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{tikzcd}
S \arrow[r, "f'"] \arrow[d, "g'"'] \arrow[dr, phantom, very near start, "\ulcorner"] & D \arrow[d, "g"] \\
B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

By Lemma 9 the pullback f' is also an epimorphism. Denote the kernel of f by K.

The claim is that K is also a kernel of f'. Of course, in order for this statement to make sense we need a map $i\colon K \to S$. This is given to us by the universal property of the pullback as follows.

$$\begin{tikzcd}
K \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists! i"] \arrow[ddr, bend right, "\iota"'] & & \\
& S \arrow[r] \arrow[d] & D \arrow[d, "g"] \\
& B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

Next we need to verify that (K, i) is actually the kernel of f', i.e. satisfies the universal property. To this end, let $v\colon Q \to S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K.

$$\begin{tikzcd}
& Q \arrow[dl, dashed, "\exists! v'"'] \arrow[d, "v"] \arrow[dr, "0"] & \\
K \arrow[r, "i"] \arrow[dr, "\iota"'] & S \arrow[r, two heads, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
f \circ g' \circ v &= g \circ f' \circ v \\
&= g \circ 0 \\
&= 0,
\end{aligned}$$

so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{tikzcd}
Q \arrow[drr, bend left, "0"] \arrow[dr, dashed, "\exists!"] \arrow[ddr, bend right, "g' \circ v"'] & & \\
& S \arrow[r, "f'"'] \arrow[d, "g'"] & D \arrow[d, "g"] \\
& B \arrow[r, "f"'] & C
\end{tikzcd}$$

By the universal property, there exists a unique map $Q \to S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{tikzcd}
\ker f' \arrow[r, "i"] \arrow[d, equal] & S \arrow[r, "f'"] \arrow[d, "g'"] & D \arrow[d, "g"] \\
\ker f \arrow[r, "\iota"'] & B \arrow[r, two heads, "f"'] & C
\end{tikzcd}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{tikzcd}
\left\{\begin{matrix}\text{elements of pullback}\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] \arrow[d, equal] & \left\{\begin{matrix}\text{elements of } B\\ \text{which map to } D\end{matrix}\right\} \arrow[r, two heads] \arrow[d, hook] & D \arrow[d, hook] \\
\left\{\begin{matrix}\text{elements of } B\\ \text{which map to } 0\end{matrix}\right\} \arrow[r, hook] & B \arrow[r, two heads] & C
\end{tikzcd}$$

Theorem 11 (snake lemma)**.** Consider the following commutative diagram with exact rows.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' \arrow[r] & 0
\end{tikzcd}$$

This gives us an exact sequence

$$0 \to \ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h \to 0.$$

Proof. We provide running commentary on the diagram below.

$$
\begin{tikzcd}
0 \arrow[r, dotted] & \ker f \arrow[r, dotted] \arrow[d, hook] & \ker g \arrow[r, dotted] \arrow[d, hook] & \ker h \arrow[d, hook] \arrow[dddll, dashed, bend left] & \\
0 \arrow[r] & A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] \arrow[d, two heads] & C' \arrow[r] \arrow[d, two heads] & 0 \\
 & \operatorname{coker} f \arrow[r, dotted] & \operatorname{coker} g \arrow[r, dotted] & \operatorname{coker} h \arrow[r, dotted] & 0
\end{tikzcd}
\tag{1.2}
$$

We will see why the dotted portions are exact later.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\operatorname{coker} f$.

Extract from the data of Diagram 1.2 the following diagram.

$$
\begin{tikzcd}
 & & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"'] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"'] & C' \\
\operatorname{coker} f & &
\end{tikzcd}
$$

Take a pullback and a pushout, and using Lemma 10 (and its dual), we find the following.

$$
\begin{tikzcd}
\ker u \arrow[r, hook, "a"] \arrow[d, equal] & S \arrow[r, two heads, "u"] \arrow[d, "r"] & \ker h \arrow[d, hook] \\
A \arrow[r, hook, "m"] \arrow[d, "f"'] & B \arrow[r, two heads, "e"] \arrow[d, "g"] & C \arrow[d, "h"] \\
A' \arrow[r, hook, "m'"] \arrow[d, two heads] & B' \arrow[r, two heads, "e'"] \arrow[d, "s"] & C' \arrow[d, equal] \\
\operatorname{coker} f \arrow[r, hook, "v"'] & T \arrow[r, two heads, "b"'] & \operatorname{coker} v
\end{tikzcd}
$$

Consider the map

$$
\delta_0 = \quad S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T \ .
$$

By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1 \colon \ker h \to T.$$

Composing this with b gives 0, hence we get a map

$$\delta \colon \operatorname{coker} f \to \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 12 (snake lemma II)**.** Given the following commutative diagram with exact rows,

$$\begin{tikzcd} & A \arrow[r, "m"] \arrow[d, "f"'] & B \arrow[r, "e"] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "m'"] & B' \arrow[r, "e'"] & C' & \end{tikzcd}$$

we get a exact sequence

$$\ker f \to \ker g \to \ker h \to \operatorname{coker} f \to \operatorname{coker} g \to \operatorname{coker} h.$$

1.3.3 The long exact sequence on homology

Corollary 13. Given an exact sequence of complexes

$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{tikzcd} & \cdots \arrow[r] & H_{n+1}(C) \arrow[dll, "\delta"] \\ H_n(A) \arrow[r, "H_n(f)"] & H_n(B) \arrow[r, "H_n(g)"] & H_n(C) \arrow[dll, "\delta"] \\ H_{n-1}(A) \arrow[r, "H_{n-1}(f)"] & H_{n-1}(B) \arrow[r, "H_{n-1}(g)"] & H_{n-1}(C) \arrow[dll, "\delta"] \\ H_{n-2}(A) \arrow[r] & \cdots & \end{tikzcd}$$

Proof. For each n, we have a diagram of the following form.

$$\begin{tikzcd}
0 \arrow[r] & A_{n+1} \arrow[r] \arrow[d, "d^A_{n+1}"'] & B_{n+1} \arrow[r] \arrow[d, "d^B_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "d^C_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r] & B_n \arrow[r] & C_n \arrow[r] & 0
\end{tikzcd}$$

Applying the snake lemma (Theorem 11) gives us, for each n, two exact sequences as follows.

$$0 \longrightarrow \ker d^A_{n+1} \longrightarrow \ker d^B_{n+1} \longrightarrow \ker d^C_{n+1}$$

and

$$\operatorname{coker} d^A_n \longrightarrow \operatorname{coker} d^B_n \longrightarrow \operatorname{coker} d^C_n \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{tikzcd}
& \operatorname{coker} d^A_n \arrow[r] & \operatorname{coker} d^B_n \arrow[r] & \operatorname{coker} d^C_n \arrow[r] & 0 \\
0 \arrow[r] & \ker d^A_{n-1} \arrow[r] & \ker d^B_{n-1} \arrow[r] & \ker d^C_{n-1} &
\end{tikzcd}$$

Recall that for each n, we get a map

$$\phi^A_{n-1} \colon \operatorname{im} d^A_{n-1} \to \ker d^A_n,$$

and similarly for $B_\bullet$ and $C_\bullet$.

Adding these in gives the following.

$$\begin{tikzcd}
& \operatorname{coker} d^A_n \arrow[r] \arrow[d] & \operatorname{coker} d^B_n \arrow[r] \arrow[d] & \operatorname{coker} d^C_n \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \ker d^A_{n-1} \arrow[r] & \ker d^B_{n-1} \arrow[r] & \ker d^C_{n-1} &
\end{tikzcd}$$

A second application of the snake lemma then gives a long exact sequence

$$\ker \phi_f \longrightarrow \ker \phi_g \longrightarrow \ker \phi_h \longrightarrow \operatorname{coker} \phi_f \longrightarrow \operatorname{coker} \phi_g \longrightarrow \operatorname{coker} \phi_h$$

□

Lemma 14. Let $\mathcal{A}$ be an abelian category, and let f be a morphism of short exact sequences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{tikzcd}
0 \arrow[r] & A'_\bullet \arrow[r, "\alpha_\bullet"] \arrow[d, "f'_\bullet"] & A_\bullet \arrow[r, "\alpha'_\bullet"] \arrow[d, "f_\bullet"] & A''_\bullet \arrow[r] \arrow[d, "f''_\bullet"] & 0 \\
0 \arrow[r] & B'_\bullet \arrow[r, "\beta_\bullet"'] & B_\bullet \arrow[r, "\beta'_\bullet"'] & B''_\bullet \arrow[r] & 0
\end{tikzcd}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{tikzcd}
\cdots \arrow[r] & H_n(A) \arrow[r, "H_n(\alpha')"] \arrow[d, "H_n(f)"'] & H_n(A'') \arrow[r, "\delta"] \arrow[d, "H_n(f'')"] & H_{n-1}(A') \arrow[r, "H_{n-1}(\alpha)"] \arrow[d, "H_n(f')"] & H_{n-1}(A) \arrow[r] \arrow[d, "H_n(f)"] & \cdots \\
\cdots \arrow[r] & H_n(B) \arrow[r, "H_n(\beta')"] & H_n(B'') \arrow[r, "\delta"] & H_{n-1}(B') \arrow[r, "H_{n-1}(\beta)"] & H_{n-1}(B) \arrow[r] & \cdots
\end{tikzcd}$$

In particular, the connecting morphism δ is functorial.

1.3.4 The five lemma

Lemma 15 (four lemmas).

1. Let
$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads, "b"'] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, two heads, "d"] & E \arrow[d, hook, "e"] \\
B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$
   be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is an epimorphism.
2. Let
$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads, "a"'] & B \arrow[r] \arrow[d, hook, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, hook, "d"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$
   be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then c is a monomorphism.

Proof.

1. Expand the diagram with image-kernel factorization as follows.

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[d, two heads, "b"] & I_1 \arrow[r, hook] \arrow[d, "i"] & C \arrow[r, two heads] \arrow[d, "c"] & I_2 \arrow[r, hook] \arrow[d, "j"] & D \arrow[r, two heads] \arrow[d, two heads, "d"] & I_3 \arrow[r, hook] \arrow[d, "k"] & E \arrow[d, hook, "e"] \\
B' \arrow[r, two heads] & I'_1 \arrow[r, hook] & C' \arrow[r, two heads] & I'_2 \arrow[r, hook] & D' \arrow[r, two heads] & I'_3 \arrow[r, hook] & E'
\end{tikzcd}$$

It is immediate that i is epi, and that k is mono. Now, consider the following morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_2 \arrow[r, hook] \arrow[d, "j"] & D \arrow[r, two heads] \arrow[d, two heads, "d"] & I_3 \arrow[r] \arrow[d, hook, "k"] & 0 \\
0 \arrow[r] & I_2' \arrow[r, hook] & D' \arrow[r, two heads] & I_3' \arrow[r] & 0
\end{tikzcd}$$

The snake lemma (Theorem 11) tells us that we have a short exact sequence

$$0 = \ker k \longrightarrow \operatorname{coker} j \longrightarrow \operatorname{coker} d = 0,$$

implying that $\operatorname{coker} j = 0$, i.e. that j is epi.

We have another morphism of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & I_1 \arrow[r, hook] \arrow[d, two heads, "i"] & C \arrow[r, two heads] \arrow[d, "c"] & I_2 \arrow[r] \arrow[d, two heads, "j"] & 0 \\
0 \arrow[r] & I_1' \arrow[r, hook] & C' \arrow[r, two heads] & I_2' \arrow[r] & 0
\end{tikzcd}$$

Using the information that i and j are epi, the snake lemma tells us that j is epi as required.

2. Dual.

□

Theorem 16 (five lemma)**.** Let

$$\begin{tikzcd}
A \arrow[r] \arrow[d, two heads] & B \arrow[r] \arrow[d, "\cong"] & C \arrow[r] \arrow[d, "c"] & D \arrow[r] \arrow[d, "\cong"] & E \arrow[d, hook] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

be a commutative diagram with exact rows, with monomorphisms, epimorphisms, and isomorphisms as marked. Then c is an isomorphism.

Proof. The first part of Lemma 15 tells us that c is epi, and the second part tells us that c is mono. □

1.3.5 The nine lemma

Theorem 17 (nine lemma). Given the following commuting diagram whose rows are exact sequences,

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A_2 \arrow[r, hook] \arrow[d] & B_2 \arrow[r, two heads, "p"] \arrow[d] & C_2 \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & A_1 \arrow[r, hook] \arrow[d] & B_1 \arrow[r, two heads] \arrow[d] & C_1 \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & A_0 \arrow[r, hook] \arrow[d] & B_0 \arrow[r, two heads] \arrow[d] & C_0 \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we have the following results.

1. If the first and the second columns $A_\bullet$ and $B_\bullet$ are exact sequences, then third column $C_\bullet$ is also an exact sequence.
2. If $B_\bullet$ and $C_\bullet$ are exact sequence, then $A_\bullet$ is an exact sequence.
3. If $A_\bullet$ and $C_\bullet$ are exact and $B_\bullet$ is a chain complex, then $B_\bullet$ is exact.

Proof.

1. First, we show that $C_\bullet$ is a chain complex, i.e. that $g \circ f = 0$. Since p is an epimorphism, it suffices to show that $g \circ f \circ p = 0$. But this is zero by commutativity of the diagram and the fact that $B_\bullet$ is a chain complex.

   Thus, the diagram above is a short exact sequence

$$0 \longrightarrow A_\bullet \hookrightarrow B_\bullet \twoheadrightarrow C_\bullet \longrightarrow 0$$

of chain complexes, and we get a long exact sequence on homology.

$$
\begin{tikzcd}
 & & 0 \arrow[dll, out=-10, in=170, "\delta"] \\
H_2(A_\bullet) \arrow[r] & H_2(B_\bullet) \arrow[r] & H_2(C_\bullet) \arrow[dll, out=-10, in=170, "\delta"] \\
H_1(A_\bullet) \arrow[r] & H_1(B_\bullet) \arrow[r] & H_1(C_\bullet) \arrow[dll, out=-10, in=170, "\delta"] \\
H_0(A_\bullet) \arrow[r] & H_0(B_\bullet) \arrow[r] & H_0(C_\bullet) \arrow[dll, out=-10, in=170, "\delta"] \\
0 & &
\end{tikzcd}
$$

If the first two columns are zeroes, then $H_n(C_\bullet) = 0$ by exactness; that is, $C_\bullet$ is exact.

2. Formally dual.
3. Identical.

□

1.4 Exact functors

Definition 18 (exact functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor, and let

$$
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0
$$

be an exact sequence. We use the following terminology.

- We call F left exact if

$$
0 \longrightarrow F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C)
$$

  is exact.
- We call F right exact if

$$
F(A) \xrightarrow{f} F(B) \xrightarrow{g} F(C) \longrightarrow 0
$$

  is exact.
- We call F exact if it is both left exact and right exact.

Example 19. Let
$$0 \longrightarrow A \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} C \longrightarrow 0$$
be an exact sequence. First, we show that $\mathrm{Hom}(M, -)$ is left exact, i.e. that
$$0 \longrightarrow \mathrm{Hom}(M, A) \xrightarrow{f_*} \mathrm{Hom}(M, B) \xrightarrow{g_*} \mathrm{Hom}(M, C)$$
is an exact sequence of abelian groups.

First, we show that f_* is injective. To see this, let $\alpha\colon M \to A$ such that $f_*(\alpha) = 0$. By definition, $f_*(\alpha) = f \circ \alpha$, and
$$f \circ \alpha = 0 \implies \alpha = 0$$
because f is a monomorphism.

Similarly, $\mathrm{im}\, f \subseteq \ker g$ because
$$(g_* \circ f_*)(\beta) = g \circ f \circ \beta = 0.$$

It remains only to check that $\ker g \subseteq \mathrm{im}\, f$. To this end, let $\gamma\colon M \to B$ such that $g_*\gamma = 0$.
$$\begin{tikzcd} & & M \arrow[dl, dashed, "\exists!\delta"'] \arrow[d, "\gamma"] \arrow[dr, "0"] & & \\ 0 \arrow[r] & A \arrow[r, hook, "f"] & B \arrow[r, two heads, "g"] & C \arrow[r] & 0 \end{tikzcd}$$
Then $\mathrm{im}\,\gamma \subset \ker g = \mathrm{im}\, f$, so γ factors through A; that is to say, there exists a $\delta\colon M \to A$ such that $f_*\delta = \gamma$.

One can also check that the functor $\mathrm{Hom}(-, B)$ is left exact via the same sort of arguments. However, we will see in the next section that one can make a much more general statement by relating left and right exactness to preservation of limits. The next theorem gives us a taste of this.

Lemma 20. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories which is either left or right exact. Then F is additive.

Proof. First, assume that F is left exact. By Corollary **??**, it suffices to show that F preserves direct sums. To this end, consider the image of a split exact sequence under a left-exact functor F.
$$0 \longrightarrow F(A) \xrightarrow{F(i_A)} F(A \oplus B) \xrightarrow{F(\pi_B)} F(B) \longrightarrow 0$$
Because we still have the identities
$$F(\pi_A) \circ F(i_A) = \mathrm{id}_{F(A)}, \qquad F(\pi_B) \circ F(i_B) = \mathrm{id}_{F(B)},$$

the morphism $F(\pi_B)$ is still epic; hence the splitting lemma (Theorem 8) implies that the sequence is still split exact. Thus, F preserves direct sums.

The case in which F is right exact is dual. □

1.4.1 Exactness and preservation of (co)limits

The above definitions of left and right exactness are evocative, but one can cast them in a more useful way. One starting point is to notice that they are related to the preservation of limits.

Proposition 21. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F preserves finite limits, then F is left exact.
- If F preserves finite colimits, then F is right exact.

Proof. Let $f\colon A \to B$ be a morphism in an abelian category. The universal property for the kernel of f is equivalent to the following: (K, ι) is a kernel of f if and only if the following diagram is a pullback.

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\iota"'] & 0 \arrow[d] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Any functor between abelian categories which preserves limits must in particular preserve pullbacks. Any such functor also sends initial objects to initial objects, and since initial objects are zero objects, such a functor preserves zero objects. Thus, any complete functor between abelian categories takes kernels to kernels.

Dually, any functor which preserves colimits preserves cokernels.

Next, note that the exactness of the sequence

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

is equivalent to the following three conditions.

1. Exactness at A means that f is mono, i.e. $0 \to A$ is a kernel of f.
2. Exactness at B means that $\operatorname{im} f = \ker g$.
   - If f is mono, this is equivalent to demanding that (A, f) is a kernel of g.
   - If g is epi, this is equivalent to demanding that (C, g) is a cokernel of f.
3. Exactness at C means that g is epi, i.e. $C \to 0$ is a cokernel of g.

□

As may now be clear, the converse is also true; any functor which is left exact preserves finite limits, and any functor which is right exact preserves finite colimits.

Proposition 22. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is left-exact, then F preserves finite limits.
- If F is right-exact, then F preserves finite colimits.

Proof. Suppose F is left exact. Then F preserves kernels and products, hence all finite limits. If F is right exact, then F preserves all cokernels and coproducts, hence all finite colimits. □

Our definition of homology now allows us to cleanly note the following.

Proposition 23. Exact functors preserve homology; that is, $H_n(F(C_\bullet)) = F(H_n(C_\bullet))$.

Proof. Exact functors preserve kernels and cokernels. □

Example 24. Consider the category $\mathcal{D}$ with objects and morphisms as follows.

$$\begin{tikzcd} & \star \arrow[d] \\ * \arrow[r] & \bullet \end{tikzcd}$$

Let $\mathcal{A}$ be an abelian category. By Example **??**, the category of functors $\mathcal{D} \to \mathcal{A}$ is an abelian category.

Consider a functor $\mathcal{D} \to \mathcal{A}$ which yields the following diagram in $\mathcal{A}$.

$$\begin{tikzcd} & 0 \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

Taking the pullback of the above diagram yields the kernel of f.

A natural transformation ϕ between functors $\mathcal{D} \to \mathcal{A}$ consists of the data of a commuting square as follows.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\phi_A"'] & B \arrow[d, "\phi_B"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd}$$

That is, we may view taking kernels as a functor from the category of morphisms in $\mathcal{A}$ to $\mathcal{A}$. Furthermore, because this functor is the result of a limiting procedure it preserves limits. Thus, given an exact sequence of morphisms in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] \arrow[d, "f"'] & B \arrow[r, two heads] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads] & C \arrow[r] & 0 \end{tikzcd}$$

one gets an exact sequence of kernels

$$0 \longrightarrow \ker f \hookrightarrow \ker g \longrightarrow \ker h \ .$$

Conversely, taking cokernels is cocomplete, hence right exact, so one has also an exact sequence of cokernels

$$\operatorname{coker} f \longrightarrow \operatorname{coker} g \twoheadrightarrow \operatorname{coker} h \longrightarrow 0$$

Note that the data of a kernel

1.4.2 Exactness of the hom functor and tensor products

We have already seen that the hom functor $\mathrm{Hom}_{\mathcal{A}}(-,-)$ is left exact in both slots. Now we understand precisely why this is true: the hom functor preserves limits in both slots![1]

Since any functor which is a right adjoint preserves limits, any right adjoint functor between abelian categories must be left exact. For example, there is an adjunction

$$- \otimes_R N : \mathbf{Mod}\text{-}R \longleftrightarrow \mathbf{Ab} : \mathrm{Hom}_{\mathbf{Ab}}(N,-)$$

which comes from replacing a map out of the tensor product by its corresponding bilinear map, and currying along the first entry. Thus, the map $- \otimes_R N$ is a left adjoint, hence preserves colimits, hence is right exact.

We can use this to check exactness (hence limit preservation) of other functors using the following beefed-up version of the Yoneda lemma.

Lemma 25. Let $\mathcal{A}$ be an abelian category, and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be objects and morphisms in $\mathcal{A}$. If for all X the abelian groups and homomorphisms

$$\mathrm{Hom}_{\mathcal{A}}(X,A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(X,B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(X,C)$$

form an exact sequence of abelian groups, then $A \to B \to C$ is exact.

[1] In the first slot, one must take the limits in the category $\mathcal{A}^{\mathrm{op}}$.

Proof. To see this, take $X = A$, giving the following sequence.

$$\mathrm{Hom}_{\mathcal{A}}(A, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(A, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(A, C)$$

Exactness implies that

$$0 = (g_* \circ f_*)(\mathrm{id}) = (g \circ f)(\mathrm{id}) = g \circ f,$$

so $\operatorname{im} f \subset \ker g$. Now take $X = \ker g$.

$$\mathrm{Hom}_{\mathcal{A}}(\ker g, A) \xrightarrow{f_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, B) \xrightarrow{g_*} \mathrm{Hom}_{\mathcal{A}}(\ker g, C)$$

The canonical inclusion $\iota\colon \ker g \to B$ is mapped to zero under g_*, hence is mapped to under f_* by some $\alpha\colon \ker g \to A$. That is, we have the following commuting triangle.

$$\begin{tikzcd} & A \arrow[dr, "f"] & \\ \ker g \arrow[ur, "\alpha"] \arrow[rr, hook, "\iota"'] & & B \end{tikzcd}$$

Thus $\operatorname{im} \iota = \ker g \subset \operatorname{im} f$. □

1.5 Quasi-isomorphism

One of the organizing principals of homological algebra is that the fundamental data contained in a chain complex is the level-wise homology of that chain complex, rather than the chain complex itself. We will see many examples (especially in Part **??**, on algebraic topology) in which one can perform a drastic simplification of a problem by replacing a chain complex by a better-behaved one, together with a homology-preserving map between them.

For this reason, one would like to understand morphisms of chain complexes $f_\bullet\colon C_\bullet \to D_\bullet$ which induce isomorphisms on homology. One even has a special name for such a morphism.

Definition 26 (quasi-isomorphism). Let $f_\bullet\colon C_\bullet \to D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n.

Quasi-isomorphisms are a surprisingly fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope that quasi-isomorphisms themselves were invertible; that is, that one could find a *quasi-inverse*. Unfortunately, we are not so lucky. Consider the following morphism of chain complexes.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

This map induces an isomorphism on homology, but there is no non-trivial map in the other direction.

1.6 Chain Homotopies

Chain homotopies are, at this point, best thought of as a technical tool which allow us to better understand quasi-isomorphisms. The interested reader should look forward to Part **??**, where we will reach an understanding of chain homotopies as an algebraic analog of a homotopy between maps of topological spaces.

Definition 27 (chain homotopy). Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be morphisms of chain complexes. A chain homotopy between $f_\bullet$ and $g_\bullet$ is a family of maps

$$h_n \colon C_n \to D_{n+1}$$

such that

$$d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n = f - g.$$

Chain homotopies will help us to understand the slippery notion of quasi-isomorphism because chain homotopic maps will turn out to induce the same maps on homology.

Proposition 28. Let $f, g \colon C_\bullet \to D_\bullet$ be chain homotopic maps via a homotopy h. Then for all n,

$$H_n(f) = H_n(g).$$

Proof. Since homology is additive, it suffices to show that any map which is homotopic to zero induces the zero map on homology, i.e. that if we can write

$$f = d^D_{n+1} \circ h_n + h_{n-1} \circ d^C_n,$$

then $H_n(f) = 0$ for all n. □

This is very useful, both theoretically and practically. We will often exploit this in order to prove that a complex is exact by showing that the identity morphism is homotopic to the zero morphism.

Lemma 29. Let $f_\bullet, g_\bullet \colon C_\bullet \to D_\bullet$ be a homotopic chain complexes via a homotopy h, and let F be an additive functor. Then $F(h)$ is a homotopy between $F(f_\bullet)$ and $F(g_\bullet)$.

Proof. We have
$$df + fd = h,$$
so
$$F(d)F(f) + F(f)F(d) = F(h).$$
□

Lemma 30. Homotopy of morphisms is an equivalence relation.

Proof.

- Reflexivity: the zero morphism provides a homotopy between $f \sim f$.
- Symmetry: If $f \overset{h}{\sim} g$, then $g \overset{-h}{\sim} f$
- Transitivity: If $f \overset{h}{\sim} f'$ and $f' \overset{h'}{\sim} f''$, then $f \overset{h+h'}{\sim} f''$.

□

Lemma 31. Homotopy respects composition. That is, let $f \overset{h}{\sim} g$ be homotopic morphisms, and let r be another morphism with appropriate domain and codomain.

- $f \circ r \overset{hr}{\sim} g \circ r$
- $r \circ f \overset{rh}{\sim} r \circ g$.

Proof. Obvious. □

1.7 Projectives and injectives

One of the drawbacks of diagrammatic reasoning in abelian categories (which is to say, reasoning on elements in R-**Mod**) is that not every maneuver one can perform on elements has a diagrammatic analog. One of the most egregious examples of this is the property of lifting against epimorphisms. Given an epimorphism $f : A \twoheadrightarrow B$ and an element of B, one can always find an element of A which maps to it under f. Unfortunately, it is *not* in general the case that given a map $X \to B$, one can lift it to a map $X \to A$ making the triangle formed commute.

$$\begin{tikzcd} & X \arrow[dl, dashed, "\nexists"'] \arrow[d] \\ A \arrow[r, two heads, "f"] & B \end{tikzcd}$$

Nowever, certain objects X do have this property. For example, if X is a free R-module, then we can find a lift by lifting generators; this is always possible. It will be interesting to find objects X which allow this. Such objects are called *projective.*

We will eventually be interested in projective objects because of the control they give us over quasi-isomorphisms.

1.7.1 Basic properties

Definition 32 (projective, injective). An object P in an abelian category $\mathcal{A}$ is projective if for every epimorphism $f: B \to C$ and every morphism $p: P \to C$, there exists a morphism $\tilde{p}: P \to B$ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\tilde{p}"'] \arrow[d, "p"] \\ B \arrow[r, two heads, "f"'] & C \end{tikzcd}$$

Dually, Q is injective if for every monomorphism $g: A \to B$ and every morphism $q: A \to Q$ there exists a morphism $\tilde{q}: B \to Q$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, hook, "g"] \arrow[d, "q"'] & B \arrow[dl, dashed, "\exists\tilde{q}"] \\ Q & \end{tikzcd}$$

Since the hom functor $\mathrm{Hom}(A, -)$ is left exact for every A, it is very easy to see simply by unwrapping the definitions that an object P in an abelian category is projective if the functor $\mathrm{Hom}(P, -)$ is exact, and injective if $\mathrm{Hom}(-, Q)$ is exact.

1.7.2 Enough projectives

A recurring theme of homological algebra is that projective and injective objects are extremely nice to work with, and whenever possible we would like to be able to trade in an object or chain complex for an object or chain complex which is (quasi-)isomorphic. We will go into more detail about precisely what we mean by this later. This will be possible when our abelian category has *enough projectives.* Dually, we can replace objects or chain complexes by injectives when we have *enough injectives.* As the abstract theory of projectives is dual to that of injectives, we will for the remainder of this chapter confine ourselves to the study of projectives.

Definition 33 (enough projectives). Let $\mathcal{A}$ be an abelian category. We say that $\mathcal{A}$ has enough projectives if for every object M there exists a projective object P and an epimorphism $P \twoheadrightarrow M$.

The category in which we have mainly been interested is R-**Mod**. We will now see that R-**Mod** has enough projectives; we need only prove the following lemma.

Lemma 34. Let

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

be a short exact sequence with P projective. Then the sequence splits.

Proof. We can add another copy of P artfully as follows.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"] & P \arrow[r] & 0 \end{tikzcd}$$

By definition, we get a morphism $P \to B$ making the triangle commute.

$$\begin{tikzcd} & & & P \arrow[d, "\mathrm{id}"] \arrow[dl, "\exists g"'] & \\ 0 \arrow[r] & A \arrow[r, hook] & B \arrow[r, two heads, "f"'] & P \arrow[r] & 0 \end{tikzcd}$$

But this says precisely that $f \circ g = \mathrm{id}_P$, i.e. the sequence splits from the right. The result follows from the splitting lemma (Lemma 8). □

Proposition 35. The category R-**Mod** has enough projectives.

We see this by proving the following characterization of projectives in R-**Mod**: P is projective if and only if it is a direct summand of a free module, i.e. if there exists an R-module M and a free R-module F such that

$$F \simeq P \oplus M.$$

This will be sufficient because any module can be written as a quotient of a free module.

First, suppose that P is projective. We can always express P in terms of a set of generators and relations. Denote by F the free R-module over the generators, and consider the following short exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & \ker \pi \arrow[r, hook] & F \arrow[r, two heads, "\pi"] & P \arrow[r] & 0 \end{tikzcd}$$

An easy consequence of the splitting lemma is that any short exact sequence with a projective in the final spot splits, so

$$F \simeq P \oplus \ker \pi$$

as required.

Conversely, suppose that $P \oplus M \simeq F$, for F free, and P and M arbitrary. Certainly, the following lifting problem has a solution (by lifting generators).

$$\begin{tikzcd} & P \oplus M \arrow[dl, dashed, "\exists (j,k)"'] \arrow[d, "(f,g)"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

In particular, this gives us the following solution to our lifting problem,

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists j"'] \arrow[d, "f"] \\ N \arrow[r, two heads] & M \end{tikzcd}$$

exhibiting P as projective.

1.7.3 Projective resolutions

We are now ready to enact a part of our goal of replacing an object or chain complex with a (quasi-)isomorphic chain complex of projectives: if we are working in an abelian category with enough projectives, we can replace an object A with a quasi-isomorphic chain complex of projectives.

More concretely, let $\mathcal{A}$ be an abelian category, and denote by ι the functor which takes an object of $\mathcal{A}$ and returns a chain complex with that object concentrated in degree zero. Clearly, the homology of $\iota(A)$ is

$$H_n(\iota(A)) = \begin{cases} A, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Our goal, namely replacing A by a chain complex which contains the same homological data, can certainly be achieved by finding a chain complex with the same homology, and a quasi-isomorphism from it to $\iota(A)$. Such an object is known as a *resolution.*

Definition 36 (resolution)**.** Let A be an object in an abelian category. A resolution of A consists of a chain complex $C_\bullet$ together with a quasi-isomorphism $C_\bullet \xrightarrow{\simeq} \iota(A)$.

What we want is a *projective resolution,* i.e. a resolution made out of projective objects.

Definition 37 (projective, injective resolution)**.** Let $\mathcal{A}$ be an abelian category, and let $A \in \mathcal{A}$. A projective resolution of A is a quasi-isomorphism $P_\bullet \xrightarrow{\simeq} \iota(A)$, where $P_\bullet$ is a complex of projectives. Similarly, an injective resolution is a quasi-isomorphism $\iota(A) \xrightarrow{\simeq} Q_\bullet$ where $Q_\bullet$ is a complex of injectives.

We said that our success in our goal would be predicated on the existence of enough projectives. Indeed, this is the case. In order to see this, we change our view on projective resolutions. Rather than a quasi-isomorphism between chain complexes whose homology is concentrated in degree zero, we should look at a projective resolution as a single, exact chain complex.

Lemma 38. Let $f_\bullet \colon P_\bullet \to \iota(A)$ be a chain map. Then f is a projective resolution if and only if the sequence

$$\cdots\cdots\cdots \; P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{f_0} A \longrightarrow 0$$

is exact.

Proposition 39. Let $\mathcal{A}$ be an abelian category with enough projectives. Then every object $A \in \mathcal{A}$ has a projective resolution.

Proof. Pick a projective which surjects onto A, and □

1.7.4 Lifting data to projective resolutions

We have seen that for every object in in abelian category $\mathcal{A}$ with enough projectives, we can find a *projective resolution*, i.e. a complex $P_\bullet$ of projectives and a quasi-isomorphism $P_\bullet \xrightarrow{\simeq} \iota(A)$. We would like to be able to extend this to a functor. This idea turns out not to work precisely as we would like, but it is nonetheless helpful to pursue it for a while.

The first thing we need is a way of lifting morphisms in $\mathcal{A}$ to morphisms between projective resolutions.

Proposition 40. Let $\mathcal{A}$ be an abelian category, let M and M' be objects of $\mathcal{A}$, and

$$P_\bullet \to M \qquad \text{and} \qquad P'_\bullet \to M'$$

be projective resolutions. Then for every morphism $f \colon M \to M'$ there exists a lift $\tilde{f} \colon P_\bullet \to P'_\bullet$ making the diagram

$$\begin{tikzcd} P_\bullet \arrow[r, "\tilde{f}_\bullet"] \arrow[d] & P'_\bullet \arrow[d] \\ \iota M \arrow[r, "\iota f"'] & \iota M' \end{tikzcd}$$

commute, which is unique up to homotopy.

Proof. We construct a lift inductively. □

Theorem 41 (horseshoe lemma)**.** Let $\mathcal{A}$ be an abelian category with enough projectives, and let $P'_\bullet \to M'$ and $P''_\bullet \to M''$ be projective resolutions. Then given an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & M' \arrow[r, hook, "f"] & M \arrow[r, two heads, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

there is a projective resolution $P_\bullet \to M$ and maps $\tilde{f}$ and $\tilde{g}$ such that the following diagram has exact rows and commutes.

$$\begin{tikzcd} 0 \arrow[r] & P'_\bullet \arrow[r, "\tilde{f}"] \arrow[d] & P_\bullet \arrow[r, "\tilde{g}"] \arrow[d] & P''_\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, hook, "f"'] & M \arrow[r, two heads, "\tilde{g}"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

Proof. We construct $P_\bullet$ inductively. We have specified data of the following form.

$$\begin{tikzcd} & \vdots \arrow[d, dotted, no head] & & \vdots \arrow[d, dotted, no head] & \\ 0 \arrow[r] & P'_1 \arrow[d] & & P''_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & P'_0 \arrow[d] & & P''_0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & M' \arrow[r, hook, "f"] & M \arrow[r, two heads, "g"] & M'' \arrow[r] & 0 \end{tikzcd}$$

We define $P_0 = P'_0 \oplus P''_0$. We get the maps to and from P_0 from the canonical injection and projection respectively.

$$\begin{tikzcd} 0 \arrow[r] & P'_0 \arrow[r, "\iota"] \arrow[d, "p'"'] \arrow[dr, dashed, "f\circ p'_0"] & P'_0 \oplus P''_0 \arrow[r, "\pi"] \arrow[d, dotted, "p"] & P''_0 \arrow[r] \arrow[d, "p''"] \arrow[dl, dashed, "\exists q"] & 0 \\ 0 \arrow[r] & M' \arrow[r, "f"'] & M \arrow[r, "g"'] & M'' \arrow[r] & 0 \end{tikzcd}$$

We get the (dashed) map $P'_0 \to M$ by composition, and the (dashed) map $P''_0 \to M$ by projectivity of P''_0 (since p'' is an epimorphism). From these the universal property for coproducts gives us the (dotted) map $p\colon P'_0 \oplus P''_0 \to M$.

At this point, the innocent reader may believe that we are in the clear, and indeed many books leave it at this. Not so! We don't know that the diagram formed in this way commutes. In fact it does not; there is nothing in the world that tells us that $q \circ \pi = p$.

However, this is but a small transgression, since the *squares* which are formed still commute. To see this, note that we can write

$$p = f \circ p'_0 \circ \pi_{P'_0} + q \circ \pi;$$

composing this with

The snake lemma guarantees that the sequence

$$\operatorname{coker} p' \simeq 0 \longrightarrow \operatorname{coker} p \longrightarrow 0 \simeq \operatorname{coker} p''$$

is exact, hence that p is an epimorphism.

One would hope that we could now repeat this process to build further levels of $P_\bullet$. Unfortunately, this doesn't work because we have no guarantee that $d_1^{P''}$ is an epimorphism, so we can't use the projectiveness of P_1'' to produce a lift. We have to be clever.

The trick is to add an auxiliary row of kernels; that is, to expand the relevant portion of our diagram as follows.

$$\begin{tikzcd}
0 \arrow[r] & P_1' \arrow[d, "p_1'"'] & & P_1'' \arrow[r] \arrow[d, "p_1''"] & 0 \\
0 \arrow[r] & \ker p' \arrow[r, dashed] \arrow[d] & \ker p \arrow[r, dashed] \arrow[d] & \ker p'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M' \arrow[r, hook] & M \arrow[r, two heads] & M'' \arrow[r] & 0
\end{tikzcd}$$

The maps p_1' and p_1'' come from the exactness of $P_\bullet'$ and $P_\bullet''$. In fact, they are epimorphisms, because they are really cokernel maps in disguise. We get the dashed map $\ker p' \to \ker p''$

That means that we are in the same situation as before, and are justified in saying "we proceed inductively". □

Theorem 42 (horseshoe lemma for morphisms)**.** Given a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, hook] \arrow[d, "f'"] & A \arrow[r, two heads] \arrow[d, "f"] & A'' \arrow[r] \arrow[d, "f''"] & 0 \\
0 \arrow[r] & B' \arrow[r, hook] & B \arrow[r, two heads] & B'' \arrow[r] & 0
\end{tikzcd}$$

Theorem 43. Let $\mathcal{A}$ be an abelian category with enough projectives, and let $C_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then there exists a

1.8 The homotopy category

In the previous section, we proved many theorems which were schematically of the following form:

Given some data in an abelian category, we can lift it to data between projective resolutions.

One might hope that this data would assemble itself functorially, but this is far too much to ask; we have no reason to believe, having picked lifts $\tilde{f}$ for our morphisms f, that $\widetilde{f \circ g} = \tilde{f} \circ \tilde{g}$.

In this, however, we are saved by the notion of homotopy! While $\widetilde{f \circ g}$ and $\tilde{f} \circ \tilde{g}$ may not be equal, they are certainly homotopic, thanks to Proposition 40. Thus, we will have a shot at functoriality, if we formally invert all homotopies.

And this we can do!

Definition 44 (homotopy category)**.** Let $\mathcal{A}$ be an abelian category. The homotopy category $\mathcal{K}(\mathcal{A})$, is the category whose objects are those of $\mathbf{Ch}(\mathcal{A})$, and whose morphisms are equivalence classes of morphisms in $\mathcal{A}$ up to homotopy.

This is well-defined by Lemma 30 and Lemma 31.

We will in general not be interested in the full homotopy category, but instead in the bounded-below homotopy category $\mathcal{K}^+(\mathcal{A})$ of *bounded-below* chain complexes up to homotopy.

Lemma 45. The homotopy category $\mathcal{K}^+(\mathcal{A})$ is an additive category, and the quotienting functor $\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{K}^+(\mathcal{A})$ is an additive functor.

Proof. Trivial. □

The upshot of the above discussion is that for any abelian category with enough projectives, there are projective resolution functors $\mathcal{A} \to \mathcal{K}^+(\mathcal{A})$ defined as follows.

1. Pick a projective resolution of every object in $\mathcal{A}$.
2. Pick a lift of every morphism of $\mathcal{A}$, and take equivalence classes.

This functor has

1.9 Mapping cones

Mapping cones are a technical tool, borrowd from algebraic topology. We will use them to great profit in the next section. For now,

Definition 46 (shift functor)**.** For any chain complex $C_\bullet$ and any $k \in \mathbb{Z}$, define the k-shifted chain complex $C_\bullet[k]$ by

$$C[k]_i = C_{k+i}; \qquad d_i^{C[k]} = (-1)^k d_{i+k}^C.$$

Definition 47 (mapping cone)**.** Let $f : C_\bullet \to D_\bullet$ be a chain map. Define a new complex $\operatorname{cone}(f)$ as follows.

- For each n, define
$$\mathrm{cone}(f)_n = C_{n-1} \oplus D_n.$$
- Define $d_n^{\mathrm{cone}(f)}$ by
$$d_n^{\mathrm{cone}(f)} = \begin{pmatrix} -d_{n-1}^C & 0 \\ -f_{n-1} & d_n^D \end{pmatrix},$$
which is shorthand for
$$d_n^{\mathrm{cone}(f)}(x_{n-1}, y_n) = (-d_{n-1}^C x_{n-1}, d_n^D y_n - f_{n-1}x_{n-1}).$$

Lemma 48. The complex $\mathrm{cone}(f)$ naturally fits into a short exact sequence
$$0 \longrightarrow D_\bullet \xrightarrow{\iota} \mathrm{cone}(f)_\bullet \xrightarrow{\pi} C[-1]_\bullet \longrightarrow 0 \ .$$

Proof. We need to specify ι and π. The morphism ι is the usual injection; the morphism π is given by minus the usual projection. □

Corollary 49. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Then f is a quasi-isomorphism if and only if $\mathrm{cone}(f)$ is an exact complex.

Proof. By Corollary 13, we get a long exact sequence
$$\cdots \to H_n(C) \xrightarrow{\delta} H_n(D) \to H_n(\mathrm{cone}(f)) \to H_{n-1}(C) \to \cdots .$$

We still need to check that $\delta = H_n(f)$. This is not hard to see; picking $x \in \ker d^X$, the zig-zag defining δ goes as follows.

$$\begin{tikzcd}
 & & x \arrow[d, mapsto] \\
 & (x,0) \arrow[r, mapsto] \arrow[d, mapsto] & -x \\
f(x) \arrow[r, mapsto] \arrow[d, mapsto] & (0, f(x)) & \\
{[f(x)]} & &
\end{tikzcd}$$

If $\mathrm{cone}(f)$ is exact, then we get a very short exact sequence
$$0 \longrightarrow H_n(X) \xrightarrow{H_n(f)} H_n(Y) \longrightarrow 0$$
implying that $H_n(f)$ must be an isomorphism. Conversely, if $H_n(f)$ is an isomorphism for all n, then the maps to and from $H_n(\mathrm{cone}(f))$ must be the zero maps, implying $H_n(\mathrm{cone}(f)) = 0$ by exactness. □

Proposition 50. Let $\mathcal{A}$ be an abelian category, and let $C_\bullet$ be a chain complex in $\mathcal{A}$. Let $P_\bullet$ be a bounded below chain complex of projectives. Then any quasi-isomorphism $g\colon P_\bullet \to C_\bullet$ has a quasi-inverse.

Proof. First we show that any morphism from a projective, bounded below complex into an exact complex is homotopic to zero. We do so by constructing a homotopy.

$$\begin{tikzcd}
\cdots\cdots & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] \arrow[dl, dashed, "h_1"'] & 0 \arrow[d] \arrow[dl, dashed, "h_0"'] & \cdots\cdots \\
\cdots\cdots & C_1 \arrow[r] & C_0 \arrow[r] & C_{-1} & \cdots\cdots
\end{tikzcd}$$

Consider the following solid commuting diagram, where K is the kernel of d_1^C and $r = f_1 - h_0 \circ d_1^P$ is *not* the map f_1.

$$\begin{tikzcd}
 & & P_1 \arrow[dll, dashed, bend right, "\exists h_1"'] \arrow[dl, dashed, "\exists !"] \arrow[dd, "r"] \arrow[ddr, "0"] & \\
 & K \arrow[dr, hook] & & \\
C_2 \arrow[ur, two heads] \arrow[rr, "d_2^C"'] & & C_1 \arrow[r, "d_1^C"'] & C_0
\end{tikzcd}$$

Because

$$\begin{aligned}
d_1^C \circ r &= d_1^C \circ f_1 - d_1^C \circ h_0 \circ d_1^P \\
&= d_1^C \circ f_1 - d_1^C \circ f_1 \\
&= 0,
\end{aligned}$$

the morphism r factors through K. This gives us a morphism from a projective onto the target of an epimorphism, which we can lift to the source. This is what we call h_1.

It remains to check that h_1 is a homotopy from f to 0. Plugging in, we find

$$\begin{aligned}
d_2^C \circ h_1 + h_0 \circ d_1^P &= r + h_0 \circ d_1^P \\
&= f_1
\end{aligned}$$

as required.

Iterating this process, we get a homotopy between 0 and f.

Now let $f\colon C_\bullet \to P_\bullet$ be a quasi-isomorphism, where $P_\bullet$ is a bounded-below projective complex. Since f is a quasi-isomorphism, $\operatorname{cone}(f)$ is exact, which means that $\iota\colon P \to$

cone(f) is homotopic to the zero morphism.

$$\begin{tikzcd}
P_{i+1} \arrow[r, "d^P_{i+1}"] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i+1}"] & P_i \arrow[r, "d^P_i"] \arrow[dl, dashed, "\tilde{h}_i"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_i"] & P_{i-1} \arrow[dl, dashed, "\tilde{h}_{i-1}"'] \arrow[d, bend right, "0"'] \arrow[d, bend left, "\iota_{i-1}"] \\
C_i \oplus P_{i+1} \arrow[r, "d^{\mathrm{cone}(f)}_{i+1}"'] & C_{i-1} \oplus P_i \arrow[r, "d^{\mathrm{cone}(f)}_i"'] & C_{i-2} \oplus P_{i-1}
\end{tikzcd}$$

That is, there exist $\tilde{h}_i \colon P_i \to \mathrm{cone}(f)_{i+1}$ such that

$$d^{\mathrm{cone}\, f}_{i+1} \circ \tilde{h}_i + \tilde{h}_{i-1} \circ d^P_i = \iota. \tag{1.3}$$

Writing $\tilde{h}_i$ in components as $\tilde{h}_i = (\beta_i, \gamma_i)$, where

$$\beta_i \colon P_i \to C_i \qquad \text{and} \qquad \gamma_i \colon P_{i-1} \to P_i,$$

we find what looks tantalizingly like a chain map $\beta_\bullet \colon P_\bullet \to C_\bullet$. Indeed, writing Equation 1.3 in components, we find the following.

$$\begin{pmatrix} -d^C_i & 0 \\ -f_i & d^P_{i+1} \end{pmatrix} \begin{pmatrix} \beta_i \\ \gamma_i \end{pmatrix} + \begin{pmatrix} \beta_{i-1} \\ \gamma_{i-1} \end{pmatrix} \begin{pmatrix} d^P_i \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$
$$\begin{pmatrix} -d^C_i \circ \beta_i + \beta_{i-1} \circ d^P_i \\ -f_i \circ \beta_i + d^P_{i+1} \circ \gamma_i + \gamma_{i-1} \circ d^P_i \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id}_{P_i} \end{pmatrix}$$

The first line tells us that β is a chain map. The second line tells us that[2]

$$d^P_{i+1} \circ \gamma_i + \gamma_{i-1} \circ d^P_i = \mathrm{id}_{P_i} - f_i \circ \beta_i,$$

i.e. that γ is a homotopy $\mathrm{id}_{P_i} \sim f_i \circ \beta_i$. But homology collapses homotopic maps, so

$$H_n(\mathrm{id}_{P_i}) = H_n(f_i) \circ H_n(\beta_i),$$

which imples that

$$H_n(\beta_i) = H_n(f_i)^{-1}.$$

□

1.10 Localization at weak equivalences: a love story

Hopefully, the reader will now agree that chain complexes are interesting enough to study, and that given a chain complex, the level-wise homology (and therefore the slippery notion of quasi-isomorphism) is an interesting invariant. Of course, one consid-

[2] Actually, it doesn't tell us this, but I suspect that it would if I were a little better at algebra.

ers the category of chain complexes, but this category has the annoying feature that quasi-isomorphisms do not necessarily have quasi-inverses. That is, it is difficult to view quasi-isomorphism as a notion of equivalence because is it is not invertible.

The way out of this is to replace the category $\mathbf{Ch}(\mathcal{A})$ by a category in which all quasi-isomorphisms have quasi-inverses. There is a fiddly and conceptually unclear way to do that, namely by forming the categorical localization.

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathbf{Ch}_{\geq 0}(\mathcal{A})[\{\text{quasi-isomorphisms}\}^{-1}].$$

Unfortunately, localization is not at all a trivial process, and one can get hurt if one is not careful. For that reason, actually constructing the above localization and then working with it is not a profitable approach to take.

However, note that we can get what we want by making a more draconian identification: collapsing all homotopy equivalences. Homotopy equivalence is friendlier than quasi-isomorphism in the sense that one can take the quotient by it; that is, there is a well-defined additive functor

$$\mathbf{Ch}_{\geq 0}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$$

which is the identity on objects and sends morphisms to their equivalence classes modulo homotopy; this sends precisely homotopy equivalences to isomorphisms. By

Definition 51 (derived functor)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between abelian categories.

- If F is right exact and $\mathcal{A}$ has enough projectives, we declare the left derived functor to be the homological δ-functor $\{L_n F\}$ defined by

$$L_n F = \mathcal{A} \xrightarrow{P} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A})) \xrightarrow{F} \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{B})) \xrightarrow{H_n} \mathcal{B}$$

  where $P\colon \mathcal{A} \to \mathcal{K}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ is a projective resolution functor.

- If F is left exact and $\mathcal{A}$ has enough injectives, we define the right derived functor to be the cohomological δ-functor

$$R^n F(X) = H^n \circ F \circ Q$$

  where Q is an injective resolution functor.

Definition 52 (homological δ-functor)**.** A homological δ-functor between abelian categories $\mathcal{A}$ and $\mathcal{B}$ consists of the following data.

1. For each $n \in \mathbb{Z}$, an additive functor

$$T_n\colon \mathcal{A} \to \mathcal{B}.$$

2. For every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

in $\mathcal{A}$ and for each $n \in \mathbb{Z}$, a morphism

$$\delta_n \colon T_n(C) \to T_{n-1}(A).$$

This data is subject to the following conditions.

1. For $n < 0$, we have $T_n = 0$.
2. For every short exact sequence as above, there is a long exact sequence

$$\begin{tikzcd}
 & & \cdots \arrow[r] & T_{n+1}(C) \arrow[dll, "\delta"] \\
T_n(A) \arrow[r, "T_n(f)"] & T_n(B) \arrow[r, "T_n(g)"] & T_n(C) \arrow[dll, "\delta"] \\
T_{n-1}(A) \arrow[r, "T_{n-1}(f)"] & T_{n-1}(B) \arrow[r, "T_{n-1}(g)"] & T_{n-1}(C) \arrow[dll, "\delta"] \\
T_{n-2}(A) \arrow[r] & \cdots
\end{tikzcd}$$

3. For every morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

and every n, the diagram

$$\begin{tikzcd}
T_n(C) \arrow[r, "\delta_n"] \arrow[d, "T_n(h)"'] & T_{n-1}(A) \arrow[d, "T_n(f)"] \\
T_n(C') \arrow[r, "\delta_n"'] & T_{n-1}(A')
\end{tikzcd}$$

commutes; that is, a morphism of short exact sequences leads to a morphism of long exact sequences.

A morphism $S \to T$ of δ-functors consists of, for each n, a natural transformation $S_n \to T_n$ compatible with the δ.

This seems like an inelegant definition, and indeed it is.

Example 53. The prototypical example of a homological δ-functor is homology: the collection $H_n \colon \mathbf{Ch}(\mathcal{A}) \to \mathcal{A}$, together with the collection of connecting homomorphisms, is a homological delta-functor. In fact, the most interesting homological delta functors called *derived functors,* come from homology.

Definition 54 (cohomological δ-functors)**.** Dual to Definition 52.

We still need to show that derived functors are in fact homological delta-functors.

Theorem 55. For any right exact functor $F \colon \mathcal{A} \to \mathcal{B}$ between abelian categories $\mathcal{A}$ and $\mathcal{B}$, where $\mathcal{A}$ has enough projectives, the left derived functor $\{L_n F\}$ is a homological δ-functor.

Proof. We check each condition separately.

1. $L_n F$ is computed by taking the nth homology of a chain complex concentrated in positive degree.
2. Start with a short exact sequence

   $$0 \longrightarrow A \hookrightarrow B \twoheadrightarrow C \longrightarrow 0$$

   in $\mathcal{A}$.

   We compute the image of this sequence under the derived functors $L_n F$ by using our chosen projective resolution functor to find projective resolutions and lift maps, and take homology. In general, we cannot be guaranteed that we can lift our exact sequence up to an exact sequence of projective resolutions

   $$0 \longrightarrow P^A_\bullet \hookrightarrow P^B_\bullet \twoheadrightarrow P^C_\bullet \longrightarrow 0\ .$$

   However, we have shown that, up to isomorphism, it doesn't matter which projective resolutions we choose, so we can choose those given by the horseshoe lemma (Theorem 41), giving us such an exact sequence. Applying F then gives us another exact sequence, and taking homology gives a long exact sequence.
3. We do the same thing, but now apply the horseshoe lemma on morphisms (Theorem 42).

□

Proposition 56. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories. We have that

$$L_0 F \simeq F.$$

Proof. Let $P_\bullet \to X$ be a projective resolution. By Lemma 38, the following sequence is exact.

$$P_1 \xrightarrow{f} P_0 \twoheadrightarrow X \longrightarrow 0$$

Since F is right exact, the following sequence is exact.

$$\begin{tikzcd} F(P_1) \arrow[r, "F(f)"] & F(P_0) \arrow[r, two heads] & F(X) \arrow[r] & 0 \end{tikzcd}$$

This tells us that $F(X) \simeq \operatorname{coker}(F(f))$. But

$$\operatorname{coker}(H(f)) \simeq H_0(P_\bullet) = L_0F(X).$$

□

Given a right exact functor F, the above theorem shows that, at the lowest level, the left derived functor gives F itself. It turns out that $\{L_nF\}$ is the *universal* homological δ-functor with this property:

Proposition 57. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a right exact functor between abelian categories where $\mathcal{A}$ has enough projectives, let $\{T_n\}$ be a homological δ-functor, and let $\phi_0\colon T_0 \Rightarrow F = L_0F$ be a natural transformation. Then there exists a unique lift of ϕ_0 to a morphism $\{\phi_i\colon T_i \to L_iF\}$.

Proof. First we construct $\phi_1\colon T_1 \Rightarrow L_1F$. For each object $A \in \mathcal{A}$, pick a projective which surjects onto it and take a kernel, giving the following short exact sequence.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook] & P \arrow[r, two heads] & A \arrow[r] & 0 \end{tikzcd}$$

Applying each of our homological δ-functors and using the fact that $L_iFP = 0$ for $i > 0$, we get the following data.

$$\begin{tikzcd} & T_1A \arrow[r, "\delta"] & T_0K \arrow[r] \arrow[d, "(\phi_0)_K"] & T_0P \arrow[r] \arrow[d, "(\phi_0)_P"] & T_0A \arrow[r] \arrow[d, "(\phi_0)_A"] & 0 \\ 0 \arrow[r] & L_1FA \arrow[r, hook, "\delta"] & FK \arrow[r] & FP \arrow[r] & FA \arrow[r] & 0 \end{tikzcd}$$

Since $L_1FA \to FK$ is a kernel of $T_0K \to T_0P$, we get a unique map $T_1A \to L_1FA$, which we declare to be $(\phi_1)_A$.

It remains to show that these are really the components of a natural transformation. Let $f\colon A \to A'$ be a morphism in $\mathcal{A}$. Consider any induced morphisms of short exact sequences.

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r, hook] \arrow[d] & P \arrow[r, two heads] \arrow[d] & A \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & K' \arrow[r, hook] & P' \arrow[r, two heads] & A' \arrow[r] & 0 \end{tikzcd}$$

Since $\{L_nF\}$ and $\{T_n\}$ are homological δ-functors, this gives us the following cube, where the components $(\phi_1)_A$ and $(\phi_1)_{A'}$ are dashed. All faces except the one on the left commute by definition.

$$\begin{tikzcd}
& T_1A' \arrow[rr] \arrow[dd, dashed, "(\phi_1)_{A'}"] & & T_0K' \arrow[dd] \\
T_1A \arrow[ur, "T_1f"] \arrow[rr] \arrow[dd, dashed, "(\phi_1)_A"'] & & T_0K \arrow[ur] \arrow[dd] & \\
& L_1FA' \arrow[rr, hook, "\delta"] & & FK' \\
L_1FA \arrow[ur, "L_1Ff"] \arrow[rr] & & FK \arrow[ur] &
\end{tikzcd}$$

In order to check that the $(\phi_1)_A$ form the components of a natural transformation, we need to check that the left face of the above cube commutes. Because $\delta\colon L_1FA' \to FK'$ is a monomorphism, it suffices to check that

$$\delta \circ (\phi_1)_{A'} \circ T_1F = \delta \circ L_1Ff \circ (\phi_1)_A.$$

But given the commutativity of the other faces of the cube, this is clear.

We now continue inductively, constructing ϕ_2, etc.

The last step is to show that the collection $\{\phi_n\}$ is really a morphism of homological δ-functors, i.e. that for any short exact sequence

$$0 \longrightarrow A'' \hookrightarrow A \twoheadrightarrow A' \longrightarrow 0$$

the squares

$$\begin{tikzcd}
T_nA'' \arrow[r, "\delta"] \arrow[d, "(\phi_n)_{A''}"'] & T_{n-1}A' \arrow[d, "(\phi_{n-1})_{A'}"] \\
L_nFA'' \arrow[r, "\delta"'] & L_{n-1}FA'
\end{tikzcd}$$

commute □

1.11 Double complexes

Given any abelian category $\mathcal{A}$, we have considered the category of chain complexes in $\mathcal{A}$, and shown (in Theorem 2) that this is an abelian category. It is therefore natural to consider chain complexes in the category of chain complexes.

$$\left(\cdots \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow M_{-1,\bullet} \longrightarrow \cdots \right) \in \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})),$$

Such a beast is a lattice of commuting squares.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots \arrow[r] & M_{i-1,j} \arrow[r] \arrow[d] & M_{i,j} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & M_{i-1,j-1} \arrow[r] \arrow[d] & M_{i,j-1} \arrow[r] \arrow[d] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

However, when manipulating these complexes it turns out to convenient to do something slightly different, namely work with complexes whose squares anticommute rather than commute. This is not a fundamental difference, as one can make from any lattice of commuting squares a lattice of anticommuting squares by multiplying the differentials of every other column by -1. This is known as the *sign trick.*

Definition 58 (double complex)**.** Let $\mathcal{A}$ be an abelian category. A double complex in $\mathcal{A}$ consists of an array $M_{i,j}$ of objects together with, for every i and $j \in \mathbb{Z}$, morphisms

$$d^h_{i,j} \colon M_{i,j} \to M_{i-1,j}, \qquad d^v_{i,j} \colon M_{i,j} \to M_{i,j-1}$$

subject to the following conditions.

- $(d^h)^2 = 0$
- $(d^v)^2 = 0$
- $d^h \circ d^v + d^v \circ d^h = 0$

We say that a double complex is $M_{\bullet,\bullet}$ is *first-quadrant* if $M_{i,j} = 0$ for $i, j < 0$.

Given any complex of complexes, one can construct a double complex by multiplying the differentials in every other row or column by -1.

Definition 59 (total complex)**.** Let $M_{\bullet,\bullet}$ be a first-quadrant[3] double complex. The total complex $\mathrm{Tot}(M)_\bullet$ is defined level-wise by

$$\mathrm{Tot}(M)_n = \bigoplus_{i+j=n} M_{i,j},$$

with differential given by $d_{\mathrm{Tot}} = d + \delta$.

[3] This restriction is not strictly necessary, but then one has to deal with infinite direct sums, and hence must decide whether one wants the direct sum or the direct product. We only need first-quadrant double complexes, so all of our sums will be finite.

Lemma 60. The total complex really is a complex, i.e.

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = 0.$$

Proof. Hand-wavily, we have

$$d_{\mathrm{Tot}} \circ d_{\mathrm{Tot}} = (d + \delta) \circ (d + \delta) = d^2 + d \circ \delta + \delta \circ d + \delta^2 = 0.$$

□

Example 61. Let $C_\bullet \in \mathbf{Ch}(\mathbf{Mod}\text{-}R)$ and $D_\bullet \in \mathbf{Ch}(R\text{-}\mathbf{Mod})$ be chain complexes. We can form a double complex by p

Theorem 62. Let $M_{i,j}$ be a first-quadrant double complex. Then if either the rows $M_{\bullet,j}$ or the columns $M_{i,\bullet}$ are exact, then the total complex $\mathrm{Tot}(M)_\bullet$ is exact.

Proof. Let $M_{i,j}$ be a first-quadrant double complex with exact rows.

$$\begin{tikzcd}
M_{3,0} \arrow[d, "d_{3,0}"'] & \iddots & & \\
M_{2,0} \arrow[d, "d_{2,0}"'] & M_{2,1} \arrow[l, "\delta_{2,1}"'] \arrow[d, "d_{2,1}"] & \iddots & \\
M_{1,0} \arrow[d, "d_{1,0}"'] & M_{1,1} \arrow[l, "\delta_{1,1}"'] \arrow[d, "d_{1,1}"] & M_{1,2} \arrow[l, "\delta_{1,2}"'] \arrow[d, "d_{1,2}"] & \iddots \\
M_{0,0} & M_{0,1} \arrow[l, "\delta_{0,1}"] & M_{0,2} \arrow[l, "\delta_{0,2}"] & M_{0,3} \arrow[l, "\delta_{0,3}"]
\end{tikzcd}$$

We want to show that the total complex

$$\begin{tikzcd}
M_{3,0} \oplus M_{2,1} \oplus M_{1,2} \oplus M_{0,3} \arrow[d] & = & \mathrm{Tot}(M)_3 \arrow[d] \\
M_{2,0} \oplus M_{1,1} \oplus M_{0,2} \arrow[d] & = & \mathrm{Tot}(M)_2 \arrow[d] \\
M_{1,0} \oplus M_{0,1} \arrow[d] & = & \mathrm{Tot}(M)_1 \arrow[d] \\
M_{0,0} \arrow[d] & = & \mathrm{Tot}(M)_0 \arrow[d] \\
0 & & 0
\end{tikzcd}$$

is exact.

The proof is inductive on the degree of the total complex. At level 0 there is nothing to show; the morphism d_1^{Tot} is manifestly surjective since $\delta_{0,1}$ is. Since numbers greater than 1 are for all intents and purposes interchangable, we give the inductive step for $n = 2$.

We have a triple $m = (m_{2,0}, m_{1,1}, m_{0,2}) \in \mathrm{Tot}(M)_2$ which maps to 0 under d_2^{Tot}; that is,

$$d_{2,0}m_{2,0} + \delta_{1,1}m_{1,1} = 0, \qquad d_{11}m_{1,1} + \delta_{0,2}m_{0,2} = 0.$$

Our goal is to find

$$n = (n_{3,0}, n_{2,1}, n_{1,2}, n_{0,3}) \in \mathrm{Tot}(M)_3$$

such that $d_3^{\mathrm{Tot}}n = m$.

It turns out that we can make our lives easier by choosing $n_{3,0} = 0$. By exactness, we can always $n_{2,1}$ such that $\delta n_{2,1} = m_{2,0}$. Thus, by anti-commutativity,

$$d\delta n_{2,1} = -\delta d n_{2,1}.$$

But $\delta n_{2,1} = m_{2,0} = -\delta m_{1,1}$, so

$$\delta(m_{1,1} - dn_{2,1}) = 0.$$

This means that $m_{1,1} - dn_{2,1} \in \ker\delta$, i.e. that there exists $n_{1,2}$ such that $\delta n_{1,2} = m_{1,1} - dn_{2,1}$. Thus

$$d\delta n_{1,2} = dm_{1,1} = -\delta m_{0,2}.$$

But

$$d\delta n_{1,2} = -\delta d n_{1,2},$$

so

$$\delta(m_{0,2} - dn_{1,2}) = 0.$$

Now we repeat this process, finding $n_{1,2}$ such that

$$\delta n_{1,2} = m_{1,1} - dn_{2,1},$$

and $n_{0,3}$ such that

$$\delta n_{0,3} = m_{0,2} - dn_{1,2}.$$

Then

$$\begin{aligned}
d^{\mathrm{Tot}}(n_{3,0} + n_{2,1} + n_{1,2} + n_{0,3}) &= 0 + (d+\delta)n_{2,1} + (d+\delta)n_{1,2} + (d+\delta)n_{0,3} \\
&= 0 + 0 + m_{2,0} + dn_{2,1} + (m_{1,1} - dn_{2,1}) + dn_{1,2} + (m_{0,2} - dn_{1,2}) + 0 \\
&= m_{2,0} + m_{1,1} + m_{0,2}
\end{aligned}$$

as required. □

Example 63. Let $f_\bullet \colon C_\bullet \to D_\bullet$ be a morphism of chain complexes. Form from this a double complex by multiplying the differential of $C_\bullet$ by -1 and padding by zeroes on either side.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 & D_3 \arrow[l] \arrow[d, "d_3^D"'] & C_3 \arrow[l, "f_3"'] \arrow[d, "-d_3^C"] & 0 \arrow[l] \\
0 & D_2 \arrow[l] \arrow[d, "d_2^D"'] & C_2 \arrow[l, "f_2"'] \arrow[d, "-d_2^C"] & 0 \arrow[l] \\
0 & D_1 \arrow[l] \arrow[d, "d_1^D"'] & C_1 \arrow[l, "f_1"'] \arrow[d, "-d_1^C"] & 0 \arrow[l] \\
0 & D_0 \arrow[l] \arrow[d, "d_0^D"'] & C_0 \arrow[l, "f_0"'] \arrow[d, "-d_0^C"] & 0 \arrow[l] \\
0 & D_{-1} \arrow[l] \arrow[d] & C_{-1} \arrow[l, "f_{-1}"'] \arrow[d] & 0 \arrow[l] \\
 & \vdots & \vdots &
\end{tikzcd}$$

The total complex of this is simply $\operatorname{cone}(f)$.

Now suppose that $f_\bullet$ is an isomorphism, so the rows are exact. Then by Theorem 62, $\operatorname{Tot}(f) = \operatorname{cone}(f)$ is exact. We have already seen this (in one direction) in Corollary 49.

Consider any morphism α in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ to a complex concentrated in degree 0. We can view this in two ways.

- As a chain

$$\cdots \longrightarrow C_{2,\bullet} \longrightarrow C_{1,\bullet} \longrightarrow C_{0,\bullet} \xrightarrow{\alpha} D_\bullet$$

  hence (by inserting appropriate minus signs) a double complex $C^D_{\bullet,\bullet}$;
- As a morphism between double complexes, hence between totalizations

$$\operatorname{Tot}(\alpha)_\bullet \colon \operatorname{Tot}(C)_\bullet \to \operatorname{Tot}(D)_\bullet.$$

Lemma 64. These two points of view agree in the sense that

$$\operatorname{Tot}(C^D) = \operatorname{Cone}(\operatorname{Tot}(\alpha)).$$

Proof. Write down the definitions. □

Theorem 65. Let $A_\bullet \in \mathbf{Ch}_{\geq 0}(\mathcal{A})$, and let

$$\begin{tikzcd} \cdots \arrow[r] & M_{2,\bullet} \arrow[r] \arrow[d] & M_{1,\bullet} \arrow[r] \arrow[d] & M_{0,\bullet} \arrow[d, "\alpha_\bullet"] \\ \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & A_\bullet \end{tikzcd}$$

be a resolution of $A_\bullet$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to A.

Proof. The sequence

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow A_\bullet \longrightarrow 0$$

is exact. By Theorem 62, the total complex $\mathrm{Tot}(M^A)$ is exact. But by Lemma 64 the total complex is equivalently the cone of $\alpha_\bullet$. However, we have seen (in Corollary 49) that exactness of $\mathrm{cone}(\alpha_\bullet)$ means that $\alpha_\bullet$ is a quasi-isomorphism. □

Corollary 66. Let

$$\cdots \longrightarrow M_{2,\bullet} \longrightarrow M_{1,\bullet} \longrightarrow M_{0,\bullet} \longrightarrow 0$$

be a complex in $\mathbf{Ch}_{\geq 0}(\mathbf{Ch}_{\geq 0}(\mathcal{A}))$ such that $H_0(M_{i,\bullet}) = 0$ for $i \neq 0$. Then $\mathrm{Tot}(M)$ is quasi-isomorphic to the sequence

$$\cdots \longrightarrow H_0(M_{2,\bullet}) \longrightarrow H_0(M_{1,\bullet}) \longrightarrow H_0(M_{0,\bullet}) \longrightarrow 0$$

Proof. For each i we can write

$$H_0(M_{i,\bullet}) = \operatorname{coker} d_1^{M_i}.$$

In particular, we have a double complex

$$
\begin{tikzcd}
M_{0,3} \arrow[d] & M_{1,3} \arrow[l] \arrow[d] & M_{2,3} \arrow[l] \arrow[d] & M_{3,3} \arrow[l] \arrow[d] \\
M_{0,2} \arrow[d] & M_{1,2} \arrow[l] \arrow[d] & M_{2,2} \arrow[l] \arrow[d] & M_{3,2} \arrow[l] \arrow[d] \\
M_{0,1} \arrow[d] & M_{1,1} \arrow[l] \arrow[d] & M_{2,1} \arrow[l] \arrow[d] & M_{3,1} \arrow[l] \arrow[d] \\
M_{0,0} \arrow[d] & M_{1,0} \arrow[l] \arrow[d] & M_{2,0} \arrow[l] \arrow[d] & M_{3,0} \arrow[l] \arrow[d] \\
H_0(M_{0,\bullet}) \arrow[d] & H_0(M_{1,\bullet}) \arrow[l] \arrow[d] & H_0(M_{2,\bullet}) \arrow[l] \arrow[d] & H_0(M_{3,\bullet}) \arrow[l] \arrow[d] \\
0 & 0 & 0 & 0
\end{tikzcd}
$$

Flipping along the main diagonal, we have the following resolution.

$$
M_{\bullet,2} \longrightarrow M_{\bullet,1} \longrightarrow M_{\bullet,0} \longrightarrow H_0(M_{\bullet,\bullet})
$$

Now Theorem 65 gives us the result we want. □

1.12 Tensor-hom adjunction for chain complexes

In a general abelian category, there is no notion of a tensor product. However, many interesting abelian categories carry tensor products, and we would like to be able to talk about them. In this section, we let $\mathcal{A}$ be an abelian category and let

$$
- \otimes - : \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{H},
$$

be an additive functor, where $\mathcal{H}$ is some abelian category equipped with an exact functor to **Ab**. Further suppose that

Definition 67 (tensor product of chain complexes)**.** Let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We define the tensor product of $C_\bullet$ and $D_\bullet$ by

$$
(C \otimes D)_\bullet = \mathrm{Tot}(C_\bullet \otimes D_\bullet),
$$

where $\mathrm{Tot}(C_\bullet \otimes D_\bullet)$ is the totalization of the double complex $C_\bullet \otimes D_\bullet$.

Definition 68 (internal hom)**.** Let $\mathcal{A}$ be an abelian category with an internal hom functor (for example, a category of modules over a commutative ring), and let $C_\bullet$, $D_\bullet$ be chain complexes in $\mathcal{A}$. We have

1.13 The Künneth formula

This section takes place in R-**Mod**, where R is a PID. For example, everything we are saying holds in **Ab**.

Lemma 69. Let $C_\bullet$ be a chain complex of free R-modules with trivial differential, and let $C'_\bullet$ be an arbitrary chain complex. Then there is an isomorphism

$$\lambda\colon \bigoplus_{p+q=n} H_p(C_\bullet) \otimes H_q(C'_\bullet) \cong H_n(C_\bullet \otimes C'_\bullet).$$

Proof. We write

$$Z_p = Z_p(C_\bullet), \qquad B_p = B_p(C_\bullet), \qquad Z'_p = Z_p(C'_\bullet), \qquad B'_p = B_p(C'_\bullet),$$

The sequence

$$0 \longrightarrow Z'_q \hookrightarrow C'_q \twoheadrightarrow B'_{q-1} \longrightarrow 0$$

is exact by definition, and since $Z_p = C_p$ is by assumption free, the sequence

By definition, $Z_p = C_p$ is free. Thus the sequence

$$0 \longrightarrow Z_p \otimes Z'_q \hookrightarrow Z_p \otimes C'_q \twoheadrightarrow Z_p \otimes B'_{q-1} \longrightarrow 0$$

is exact.

The differential of the tensor product complex $C_\bullet \otimes C'_\bullet$ is

$$\begin{aligned} d_n^{\text{tot}} &= \sum_{i=0}^{n} d_i^C \otimes \mathrm{id}_{C_{n-i}} + (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C'} \\ &= \sum_{i=0}^{n} (-1)^i \mathrm{id}_{C_i} \otimes d_{n-i}^{C}. \end{aligned}$$

Since C_i is by assumption free, tensoring with it doesn't change homology, and we have

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

and

$$Z_n(C_\bullet \otimes C'_\bullet) = \bigoplus_{p+q=n} Z_p \otimes Z'_q,$$

□