

1 Abelian categories

1.1 Basics

1.1.1 Building blocks

Ab-enriched categories

In this chapter we recall some basic results.

Definition 1 (Ab-enriched category). A category \mathcal{C} is called *Ab-enriched* if it is enriched over the symmetric monoidal category $(\mathbf{Ab}, \otimes_{\mathbb{Z}})$.

Lemma 2. Let \mathcal{C} be an Ab-enriched category with finite products. Then the product $X_1 \times \cdots \times X_n$ satisfies the universal property for coproducts. In particular, initial objects and terminal objects coincide.

Proof. First, we show that initial and terminal objects coincide. Let \emptyset be initial and $*$ terminal in \mathcal{C} .

Since $\text{Hom}_{\mathcal{C}}(\emptyset, \emptyset)$ has only one element, it must be both the identity morphism on \emptyset and the identity element of $\text{Hom}_{\mathcal{C}}(\emptyset, \emptyset)$ as an abelian group. Similarly, $\text{id}_* = 0$ in $\text{Hom}_{\mathcal{C}}(*, *)$.

Since \emptyset is initial and $*$ is final, $\text{Hom}_{\mathcal{C}}(\emptyset, *)$ has exactly one element f . We are thus guaranteed that $\text{Hom}_{\mathcal{C}}(*, \emptyset)$ has at least one element, g . Then $g \circ f = 0 = \text{id}$ and $f \circ g = 0 = \text{id}$, so $f: \emptyset \rightarrow *$ is an isomorphism. Thus, initial and terminal objects coincide, so \mathcal{C} has a zero object 0 .

We prove the lemma for binary products. The general case is precisely the same.

For each $X, Y \in \mathcal{C}$, we can define $i_X: X \rightarrow X \times Y$ using the universal property for products as follows.

$$\begin{array}{ccccc}
 & & X & & \\
 & \swarrow \text{id} & \downarrow i_X & \searrow 0 & \\
 X & \longleftarrow & X \times Y & \longrightarrow & Y
 \end{array}$$

Similarly, get $i_Y: Y \rightarrow X \times Y$.

The claim is that $X \times Y$, together with i_X and i_Y , satisfies the universal property for co-

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products; that is, that we can find a unique map ϕ making the below diagram commute.

$$\begin{array}{ccccc} X & \xrightarrow{i_X} & X \times Y & \xleftarrow{i_Y} & Y \\ & \searrow f & \downarrow \phi & \swarrow g & \\ & & Z & & \end{array}$$

In fact, $\phi = f \circ \pi_X + g \circ \pi_Y$ satisfies the universal property. It is clear that this makes the diagram commute since $\pi_X \circ i_X = \text{id}$, $\pi_X \circ i_Y = 0$, and similarly for Y .

Note that, by the universal property for products, there is a unique downward map. But both $\text{id}_{X \times Y}$ and $i_X \circ \pi_X + i_Y \circ \pi_Y$ make it commute, so they must be equal.

$$\begin{array}{ccccc} & & X \times Y & & \\ \pi_Y \swarrow & & \downarrow & \searrow \pi_X & \\ X & \xleftarrow{\pi_X} & X \times Y & \xrightarrow{\pi_Y} & Y \end{array}$$

Now we are ready to show that ϕ is unique. Suppose we have another candidate ψ which also makes the above diagram commute. Then

$$\psi = \psi \circ \text{id}_{X \times Y} = f \circ \pi_X + g \circ \pi_Y = \phi.$$

□

Additive categories

Definition 3 (additive category). A category \mathcal{C} is additive if it has products and is Ab-enriched.

Definition 4 (additive functor). Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between additive categories. We say that F is additive if for each $X, Y \in \text{Obj}(\mathcal{C})$ the map

$$\text{Hom}_{\mathcal{C}}(X, Y) \rightarrow \text{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups.

Lemma 5. Let A and B be objects in an additive category \mathcal{A} , and let X be an object in \mathcal{A} equipped with morphisms as follows,

$$\begin{array}{ccc} A & & A \\ & \searrow i_A & \nearrow \pi_A \\ & X & \\ & \nearrow i_B & \searrow \pi_A \\ B & & B \end{array}$$

such that the following equations hold.

$$\begin{aligned} i_A \circ \pi_A + i_B \circ \pi_B &= \text{id}_X \\ \pi_A \circ i_B &= 0 = \pi_B \circ i_A \\ \pi_A \circ i_A &= \text{id}_A \\ \pi_B \circ i_B &= \text{id}_B \end{aligned}$$

Then the π s and i s exhibit $X \cong A \oplus B$.

Proof. It suffices to show that X is a product of A and B . Let $f: Z \rightarrow A$ and $g: Z \rightarrow B$. We need to show that there exists a unique $\phi: Z \rightarrow X$ making the following diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & f \swarrow & \downarrow \phi & \searrow g & \\ A & \xleftarrow{\pi_A} & X & \xrightarrow{\pi_B} & B \end{array}$$

That is, we need

$$\pi_A \circ \phi = f, \quad \pi_B \circ \phi = g.$$

Thus, we need

$$\begin{aligned} i_A \circ f + \pi_B \circ g &= i_A \circ \pi_A \circ \phi + i_B \circ \pi_B \circ \phi \\ &= \phi. \end{aligned}$$

It is easy to check that ϕ defined in this way *does* make the diagram commute, so ϕ exists and is unique. \square

Corollary 6. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor between additive categories. Then F preserves direct sums in the sense that there is a natural isomorphism

$$F(A \oplus B) \simeq F(A) \oplus F(B).$$

Proof. Apply F to the data of [Lemma 5](#). \square

Pre-abelian categories

Definition 7 (kernel, cokernel). Let $f: A \rightarrow B$ be a morphism in an abelian category.

- A pair (K, ι) , where $\iota: K \rightarrow A$ is a morphism, is a kernel of f if $f \circ \iota = 0$ and for every object X and morphism $g: X \rightarrow A$ such that $f \circ g = 0$, there exists a unique

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morphism $\alpha: X \rightarrow K$ such that $f \circ \alpha = 0$.

$$\begin{array}{ccccc}
 & X & & & \\
 & \downarrow g & \searrow 0 & & \\
 K & \xrightarrow{\iota} & A & \xrightarrow{f} & B \\
 & \searrow 0 & & \nearrow & \\
 & & & &
 \end{array}$$

$\exists!$

- A pair (C, π) , where $\pi: B \rightarrow C$ is a morphism, is a cokernel of f if $\pi \circ f = 0$ and for every object Y and morphism $g: B \rightarrow Y$ such that $g \circ f = 0$, there exists a unique morphism $\beta: C \rightarrow Y$ such that $\beta \circ \pi = g$.

$$\begin{array}{ccccc}
 & & 0 & & \\
 & \searrow & & \nearrow & \\
 A & \xrightarrow{f} & B & \xrightarrow{\pi} & C \\
 & \searrow 0 & \downarrow g & \swarrow \exists! & \\
 & & Y & &
 \end{array}$$

Here are some equivalent ways of defining the kernel of a morphism $f: A \rightarrow B$.

- The equalizer of f with the zero map.

$$\ker f = \text{eq} \longrightarrow A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{0} \end{array} B$$

- The pullback along the zero morphism

$$\begin{array}{ccc}
 \ker f & \longrightarrow & 0 \\
 \downarrow & & \downarrow \\
 A & \xrightarrow{f} & B
 \end{array}$$

Definition 8 (pre-abelian category). A pre-abelian category is an additive category such that every morphism has a kernel and a cokernel.

Lemma 9. Pre-abelian categories have all finite limits and colimits.

Proof. In a pre-abelian category, the equalizer of f and g is the kernel of $f - g$, and the coequalizer is the cokernel of $f - g$. Thus, pre-abelian categories have finite products and equalizers, and finite coproducts and coequalizers. \square

Proposition 10. In a pre-abelian category, kernels are monic and cokernels are epic.

Proof. Let $f: A \rightarrow B$ be a morphism in an abelian category, and let $\iota: K \rightarrow A$ be a kernel

of f . Let $\alpha: X \rightarrow K$ be a morphism such that $\alpha \circ \iota = 0$.

$$\begin{array}{ccccc} & & X & & \\ & \swarrow \alpha & \downarrow 0 & \searrow 0 & \\ K & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

The universal property for kernels tells us that α is the *unique* morphism such that $0 = \iota \circ \alpha$. But setting $\alpha = 0$ certainly satisfies this equality, so we must have that $\alpha = 0$.

The cokernel case is dual. \square

Proposition 11. Let \mathcal{C} be a pre-abelian category, and let $f: A \rightarrow B$ be a morphism.

1. The morphism f is a monomorphism if and only if $\ker f = 0$.
2. The morphism f is an epimorphism if and only if $\operatorname{coker} f = 0$.

Proof.

1. Suppose f is a monomorphism, and consider a morphism g making the following diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & & \downarrow g & \searrow 0 & \\ \ker f & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

Because f is a monomorphism, we have $g = 0$. By the universal property for kernels, there exists a unique map $Z \rightarrow \ker f$ making the above diagram commute.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \exists! & \downarrow g & \searrow 0 & \\ \ker f & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

But this remains true if we replace $\ker f$ by 0 , so by definition we must have $\ker f = 0$.

Now suppose that $\ker f = 0$, and let $g: X \rightarrow A$ such that $f \circ g = 0$. The universal property guarantees us a morphism $\alpha: X \rightarrow K$ making the left-hand triangle below commute.

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow \exists! & \downarrow g & \searrow 0 & \\ \ker f = 0 & \xrightarrow{\iota} & A & \xrightarrow{f} & B \end{array}$$

However, the only way the left-hand triangle can commute is if $g = 0$. Thus, f is mono.

2. Dual.

□

1.1.2 Abelian categories

We have seen that in a pre-abelian category, kernels and cokernels more or less behave as they do in the category of modules over some ring R . However, the first isomorphism theorem tells us that every submodule is the kernel of some module homomorphism (namely the cokernel), and every quotient module of some module homomorphism (namely the kernel).

Definition 12 (abelian category). A pre-abelian category is said to be abelian if every monomorphism is the kernel of its cokernel, and every epimorphism is the cokernel of its kernel.

Proposition 13. Let \mathcal{A} be an abelian category, and let $f: A \rightarrow B$ be a morphism in \mathcal{A} . Then f is an isomorphism if and only if $\ker f = 0$ and $\operatorname{coker} f = 0$.

Proof. If f is an isomorphism, then it is mono and epi, and we are done by [Proposition 11](#).

Conversely, if $\ker f = 0$ and $\operatorname{coker} f = 0$, then f is mono and epi. We need only to show that any morphism in an abelian category which is monic and epic is an isomorphism.

Let f be monic and epic. Then f is the kernel of its cokernel and the cokernel of its kernel, and both of these are zero.

$$\begin{array}{ccccc}
 & & B & \xrightarrow{0} & 0 = \operatorname{coker} f \\
 & \swarrow \exists! f_R^{-1} & \parallel & \searrow & \\
 \ker f = 0 & \longrightarrow & A & \xrightarrow{f} & B \\
 & \nwarrow 0 & \parallel & \nearrow \exists! f_L^{-1} & \\
 & & A & &
 \end{array}$$

This gives us morphisms f_L^{-1} and f_R^{-1} such that

$$f \circ f_R^{-1} = \operatorname{id}_A, \quad f_L^{-1} \circ f = \operatorname{id}_B.$$

The usual trick shows that $f_L^{-1} = f_R^{-1}$. □

As the following proposition shows, additive functors from additive categories preserve direct sums.

Example 14. Once it is known that a category \mathcal{A} is abelian, a number of other categories are immediately known to be abelian.

- The category $\mathbf{Ch}(\mathcal{A})$ of chain complexes in \mathcal{A} is abelian, as we will see in
- For any small category I , the category $\mathbf{Fun}(I, \mathcal{A})$ of I -diagrams in \mathcal{A} is abelian.

1.2 Intermezzo: embedding theorems

In ordinary category theory, when manipulating a locally small category it is often helpful to pass through the Yoneda embedding, which gives a (fully) faithful rendition of the category under consideration in the category **Set**. One of the reasons that this is so useful is that the category **Set** has a lot of structure which can use to prove things about the subcategory of **Set** in which one lands. Having done this, one can then use the fully faithfulness to translate results to the category under consideration.

This sort of procedure, namely embedding a category which is difficult to work with into one with more desirable properties and then translating results back and forth, is very powerful. In the context of (small) Abelian categories one has essentially the best possible such embedding, known as the Freyd-Mitchell embedding theorem, which we will revisit at the end of this section. However, for now we will content ourselves with a simpler categorical embedding, one which we can work with easily.

In abelian categories, one can define kernels and cokernels slickly, as for example the equalizer along the zero morphism. However, in full generality, we can define both kernels and cokernels in any category with a zero object: f is a kernel for g if and only if the diagram below is a pullback, and a g is a cokernel of f if and only if the diagram below is a pullback.

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & C \end{array}$$

This means that kernels and cokernels are very general, as categories with zero objects are a dime a dozen. For example, the category **Set**_{*} of pointed sets has the singleton $\{*\}$ as a zero object. This means that one can speak of the kernel or the cokernel of a map between pointed sets, and talk about an exact sequence of such maps.

Given an abelian category \mathcal{A} , suppose one could find an embedding $\mathcal{H}: \mathcal{A} \rightarrow \mathbf{Set}_*$ which reflected exactness in the sense that if one started with a sequence $A \rightarrow B \rightarrow C$ in \mathcal{A} , mapped it into **Set**_{*} finding a sequence $\mathcal{H}(A) \rightarrow \mathcal{H}(B) \rightarrow \mathcal{H}(C)$, and found that this sequence was exact, one could be sure that $A \rightarrow B \rightarrow C$ had been exact to begin with. Then every time one wanted to check the exactness of a sequence, one could embed that sequence in **Set**_{*} using \mathcal{H} and check exactness there. Effectively, one could check exactness of a sequence in \mathcal{A} by manipulating the objects making up the sequence as if they had elements.

Or, suppose one could find an embedding as above as above which sent only zero morphisms to zero morphisms. Then one could check that a diagram commutes (equivalently, that the difference between any two different ways of getting between objects is equal to 0) by checking that the image of the diagram under our functor \mathcal{H} commutes.

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In fact, for \mathcal{A} small, we will find a functor which satisfies both of these and more. Then, just as we are feeling pretty good about ourselves, we will state the Freyd-Mitchell embedding theorem, which blows our pitiful result out of the water.

We now construct our functor to \mathbf{Set}_* .

Definition 15 (category of contravariant epimorphisms). Let \mathcal{A} be a small abelian category. Define a category \mathcal{A}_{\leftarrow} with $\text{Obj}(\mathcal{A}_{\leftarrow}) = \text{Obj}(\mathcal{A})$, and whose morphisms are defined by

$$\text{Hom}_{\mathcal{A}_{\leftarrow}}(X, Y) = \{f: Y \rightarrow X \text{ in } \mathcal{A} \mid f \text{ epimorphism}\}.$$

Note that this is indeed a category since the identity morphism is an epimorphism and epimorphisms are closed under composition.

Now for each $A \in \mathcal{A}$, we define a functor

$$\mathcal{H}_A: \mathcal{A}_{\leftarrow} \rightarrow \mathbf{Set}_*; \quad Z \mapsto \text{Hom}_{\mathcal{A}}(Z, A)$$

and sends a morphism $f: Z_1 \rightarrow Z_2$ in \mathcal{A}_{\leftarrow} (which is to say, an epimorphism $\tilde{f}: Z_2 \twoheadrightarrow Z_1$ in \mathcal{A}) to the map

$$\mathcal{H}_A(f): \text{Hom}_{\mathcal{A}}(Z_1, A) \rightarrow \text{Hom}_{\mathcal{A}}(Z_2, A); \quad (\alpha: Z_1 \rightarrow A) \mapsto (\alpha \circ \tilde{f}: Z_2 \rightarrow A).$$

Note that the distinguished point in the hom sets above is given by the zero morphism.

Definition 16 (member functor). Let \mathcal{A} be a small abelian category. We define a functor $\mathcal{M}: \mathcal{A} \rightarrow \mathbf{Set}_*$ on objects by

$$A \mapsto \mathcal{M}(A) = \text{colim } \mathcal{H}_A.$$

On morphisms, functoriality comes from the functoriality of the colimit and the co-Yoneda embedding.

Strictly speaking, we have finished our construction, but it doesn't do us much good as stated. It turns out that the sets $\mathcal{M}(A)$ have a much simpler interpretation.

Proposition 17. for any $A \in \mathcal{A}$, the value of the member functor $\mathcal{M}(A)$ is

$$\mathcal{M}(A) = \coprod_{X \in \mathcal{A}} \text{Hom}(X, A) / \sim,$$

where $g \sim g'$ if there exist epimorphisms f and f' making the below diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ f' \downarrow & & \downarrow g \\ X' & \xrightarrow{g'} & A \end{array}$$

Proof. The colimit can be computed using the following coequalizer.

$$\coprod_{\substack{f \in \text{Morph}(\mathcal{A}_{\leftarrow}) \\ \tilde{f}: Y \rightarrow X}} \text{Hom}_{\mathcal{A}}(X, A) \xrightarrow[\sim]{\text{id}} \coprod_{Z \in \text{Obj}(\mathcal{A}_{\leftarrow})} \text{Hom}(Z, A) \xrightarrow{\text{coeq}} \mathcal{M}(A)$$

On elements, we have the following.

$$\begin{array}{ccc} & \xrightarrow{\text{id}} & (g: X \rightarrow A) \\ (g: X \rightarrow A) & \searrow \scriptstyle - \circ \tilde{f} & \parallel \\ & \searrow \scriptstyle - \circ \tilde{f} & (g \circ \tilde{f}: Y \rightarrow A) \end{array}$$

Thus,

$$\mathcal{M}(A) = \coprod_{Z \in \text{Obj}(\mathcal{A}_{\leftarrow})} \text{Hom}(Z, A) / \sim,$$

where \sim is the equivalence relation generated by the relation

$$(g: X \rightarrow A) R (g': X' \rightarrow A) \iff \exists f: X' \twoheadrightarrow X \text{ such that } g' = g \circ f.$$

The above relation is reflexive and transitive, but not symmetric. The smallest equivalence relation containing it is the following.

$$(g: X \rightarrow A) \sim (g': X' \rightarrow A) \iff \exists f: Z \twoheadrightarrow X, f': Z \twoheadrightarrow X' \text{ such that } g' \circ f' = g \circ f.$$

That is, $g \sim g'$ if there exist epimorphisms making the below diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ f' \downarrow & & \downarrow g \\ X' & \xrightarrow{g'} & A \end{array}$$

□

In summary, the elements of $\mathcal{M}(A)$ are equivalence classes of morphisms into A modulo the above relation, and for a morphism $f: A \rightarrow B$ in \mathcal{A} , $\mathcal{M}(f)$ acts on an equivalence class $[g]$ by

$$\mathcal{M}(f): [g] \mapsto [g \circ f].$$

Lemma 18. Let $f: A \rightarrow B$ be a morphism in a small abelian category \mathcal{A} such that $f \sim 0$. Then $f = 0$.

Proof. We have that $f \sim 0$ if and only if there exists an object Z and epimorphisms

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making the following diagram commute.

$$\begin{array}{ccc} Z & \xrightarrow{g} & A \\ \downarrow & & \downarrow f \\ 0 & \longrightarrow & B \end{array}$$

But by the universal property for epimorphisms, $f \circ g = 0$ implies $f = 0$. \square

Now we introduce some load-lightening notation: we write $(\hat{-}) = \mathcal{M}(-)$.

Lemma 19. Let $f: A \rightarrow B$ be a morphism in a small abelian category \mathcal{A} . Then $f = 0$ if and only if $\hat{f}: \hat{A} \rightarrow \hat{B} = 0$.

Proof. Suppose that $f = 0$. Then for any $[g] \in \hat{A}$

$$\hat{f}([g]) = [g \circ 0] = [0].$$

Thus, $\hat{f}([g]) = [0]$ for all g , so $\hat{f} = 0$.

Conversely, suppose that $\hat{f} = 0$. Then in particular $\hat{f}([\text{id}_A]) = [0]$. But

$$\hat{f}([\text{id}_A]) = [\text{id}_A \circ f] = [f].$$

By Lemma 18, $[f] = [0]$ implies $f = 0$. \square

Corollary 20. A diagram commutes in \mathcal{A} if and only if its image in Set_* under \mathcal{M} commutes.

Proof. A diagram commutes in \mathcal{A} if and only if any two ways of going from one object to another agree, i.e. if the difference of any two

I actually don't see this right now. \square

Lemma 21. Let $f: A \rightarrow B$ be a morphism in a small abelian category.

- The morphism f is a monomorphism if and only if $\mathcal{M}(f)$ is an

1.3 Chain complexes

Definition 22 (category of chain complexes). Let \mathcal{A} be an abelian category. A chain complex C_\bullet in \mathcal{A} is a collection of objects $C_n \in \mathcal{A}$, $n \in \mathbb{Z}$, together with morphisms $d_n: C_n \rightarrow C_{n-1}$ such that $d_{n-1} \circ d_n = 0$.

The category of chain complexes in \mathcal{A} , denoted $\text{Ch}(\mathcal{A})$, is the category whose objects are chain complexes and whose morphisms are morphisms $C_\bullet \rightarrow D_\bullet$ of chain complexes,

i.e. for each $n \in \mathbb{Z}$ a map $f_n: C_n \rightarrow D_n$ such that all of the squares form commute.

$$\begin{array}{ccccccc} \cdots & \xrightarrow{d_{n+2}^C} & C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \xrightarrow{d_{n-1}^C} \cdots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} \\ \cdots & \xrightarrow{d_{n+2}^D} & D_{n+1} & \xrightarrow{d_{n+1}^D} & D_n & \xrightarrow{d_n^D} & D_{n-1} \xrightarrow{d_{n-1}^D} \cdots \end{array}$$

We also define $\mathbf{Ch}^+(\mathcal{A})$ to be the category of bounded-below chain complexes (i.e. the full subcategory consisting of chain complexes C_\bullet such that $C_n = 0$ for small enough n), $\mathbf{Ch}^-(\mathcal{A})$ to be the category of bounded-above chain complexes, and $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and $\mathbf{Ch}^{\leq 0}(\mathcal{A})$ similarly.

Theorem 23. The category $\mathbf{Ch}(\mathcal{A})$ is abelian.

Proof. We check each of the conditions.

The zero object is the zero chain complex, the \mathbf{Ab} -enrichment is given level-wise, and the product is given level-wise as the direct sum. That this satisfies the universal property follows almost immediately from the universal property in \mathcal{A} : given a diagram of the form

$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \swarrow f_\bullet & \downarrow \exists! \phi_\bullet & \searrow g_\bullet & \\ C_\bullet & \xleftarrow{p_1} & C_\bullet \times D_\bullet & \xrightarrow{p_2} & D_\bullet \end{array}$$

the only thing we need to check is that the map ϕ produced level-wise is really a chain map, i.e. that

$$\phi_{n-1} \circ d_n^Q = d_n^C \oplus d_n^D \circ \phi_n.$$

By the above work, the RHS can be re-written as

$$\begin{aligned} (d_n^C \oplus d_n^D) \circ (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) &= i_1 \circ d_n^C \circ f_n + i_2 \circ d_n^D \circ g_n \\ &= i_1 \circ f_{n-1} \circ d_n^Q + i_2 \circ g_{n-1} \circ d_n^Q \\ &= (i_1 \circ f_{n-1} + i_2 \circ g_{n-1}) \circ d_n^Q, \end{aligned}$$

which is equal to the LHS.

The kernel is also defined level-wise. The standard diagram chase shows that the induced morphisms between kernels make this into a chain complex; to see that it satisfies the universal property, we need only show that the map ϕ below is a chain map.

$$\begin{array}{ccccc} & & Q_\bullet & & \\ & \swarrow \phi_\bullet & \downarrow g_\bullet & \searrow 0 & \\ \ker(f_\bullet) & \xrightarrow{\iota_\bullet} & A_\bullet & \xrightarrow{f_\bullet} & B_\bullet \end{array}$$

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That is, we must have

$$\phi_{n-1} \circ d_n^Q = d_n^{\ker f} \circ \phi_n. \quad (1.1)$$

The composition

$$Q_n \xrightarrow{g_n} A_n \xrightarrow{f} B_n \xrightarrow{d_n^B} B_{n-1}$$

gives zero by assumption, giving us by the universal property for kernels a unique map

$$\psi: Q_n \rightarrow \ker f_{n-1}$$

such that

$$\iota_{n-1} \circ \psi = d_n^A \circ g_n.$$

We will be done if we can show that both sides of Equation 1.1 can play the role of ψ . Plugging in the LHS, we have

$$\begin{aligned} \iota_{n-1} \circ \phi_{n-1} \circ d_n^Q &= g_{n-1} \circ d_n^Q \\ &= d_n^A \circ g_n \end{aligned}$$

as we wanted. Plugging in the RHS we have

$$\iota_{n-1} \circ d_n^{\ker f} \circ \phi_n = d_n^A \circ g_n.$$

The case of cokernels is dual.

Since a chain map is a monomorphism (resp. epimorphism) if and only if it is a monomorphism (resp. epimorphism), we have immediately that monomorphisms are the kernels of their cokernels, and epimorphisms are the cokernels of their kernels. \square

The categories $\mathbf{Ch}^+(\mathcal{A})$, etc., are also abelian.

Definition 24 (image, coimage). Given a morphism $f: A \rightarrow B$, the image of f is $\text{im } f = \ker \text{coker } f$ and the coimage of f is $\text{coker } \ker f$.

Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be morphisms such that $g \circ f = 0$. We can build the following commutative diagram.

$$\begin{array}{ccccc} & & \text{im } f & & \\ & & \downarrow i & \searrow 0 & \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ & & \downarrow p & \nearrow & \\ & & \text{coker } f & & \end{array}$$

Since $g \circ i = 0$, i factors through $\ker g$, giving us a morphism $\phi: \operatorname{im} f \rightarrow \ker g$.

$$\begin{array}{ccccc}
 \operatorname{im} f & \xrightarrow{\phi} & \ker g & \longrightarrow & H_n \\
 & \searrow i & \downarrow & & \\
 A & \xrightarrow{f} & B & \xrightarrow{g} & C \\
 & & \downarrow p & \nearrow & \\
 & & \operatorname{coker} f & &
 \end{array}$$

Definition 25 (exact sequence). Let

$$A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

be objects and morphisms in an abelian category. We say that the above sequence is exact if ϕ_i is an isomorphism. We say that a complex (C_\bullet, d) is exact if ϕ is an isomorphism at all positions.

A chain complex (C_\bullet, d) is exact if it is exact at all positions.

Definition 26 (homology). Let (C_\bullet, d) be a complex. The n th homology of C_\bullet is the cokernel of the map $\phi: \operatorname{im} d_{n+1} \rightarrow \ker d_n$ described in Definition 25, denoted $H_n(C_\bullet)$.

Proposition 27. Homology extends to a family of additive functors

$$H_n: \mathbf{Ch}(\mathcal{A}) \rightarrow \mathcal{A}.$$

Proof. We need to define H_n on morphisms. To this end, let $f_\bullet: C_\bullet \rightarrow D_\bullet$ be a morphism of chain complexes.

$$\begin{array}{ccccc}
 C_{n+1} & \xrightarrow{d_{n+1}^C} & C_n & \xrightarrow{d_n^C} & C_{n-1} \\
 f_{n+1} \downarrow & & \downarrow f_n & & \downarrow f_{n-1} \\
 D_{n+1} & \xrightarrow{d_{n+1}^D} & D_n & \xrightarrow{d_n^D} & D_{n-1}
 \end{array}$$

We need a map $H_n(C_\bullet) \rightarrow H_n(D_\bullet)$. This will come from a map $\ker d_n^C \rightarrow \ker d_n^D$. In fact, f_n gives us such a map, essentially by restriction. We need to show that this descends to a map between cokernels. \square

We can also go the other way, by defining a map

$$\iota_n: \mathcal{A} \rightarrow \mathbf{Ch}(\mathcal{A})$$

which sends an object A to the chain complex

$$\cdots \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow \cdots$$

concentrated in degree n .

Example 28. In some situations, homology is easy to compute explicitly. Let

$$\cdots \longrightarrow C_1 \xrightarrow{f} C_0 \longrightarrow 0$$

be a chain complex in $\mathbf{Ch}_{\geq 0}(\mathcal{A})$. Then $H_0(C_\bullet) = \text{coker } f$.

Similarly, if

$$0 \longrightarrow C_0 \xrightarrow{f} C_{-1} \longrightarrow \cdots$$

is a chain complex in $\mathbf{Ch}_{\leq 0}(\mathcal{A})$, then $H_0(C_\bullet) = \ker f$.

Definition 29 (quasi-isomorphism). Let $f_\bullet: C_\bullet \rightarrow D_\bullet$ be a chain map. We say that f is a quasi-isomorphism if it induces isomorphisms on homology; that is, if $H_n(f)$ is an isomorphism for all n .

Quasi-isomorphisms are a fiddly concept. Since they induce isomorphisms on homology, and isomorphisms are invertible, one might naïvely hope that quasi-isomorphisms themselves were invertible. Unfortunately, we are not so lucky. Consider the following

By [Example 28](#), we

Definition 30 (resolution). Let A be an object in an abelian category. A resolution of A consists of a chain complex C_\bullet together with a quasi-isomorphism $C_\bullet \rightarrow \iota(A)$.

1.4 Diagram lemmas

1.4.1 The splitting lemma

Lemma 31 (Splitting lemma). Consider the following solid exact sequence in an abelian category \mathcal{A} .

$$0 \longrightarrow A \xrightarrow{i_A} B \xrightarrow{\pi_C} C \longrightarrow 0$$

$\nwarrow \pi_A \quad \nearrow i_C$

The following are equivalent.

- There exists a morphism $\pi_A: B \rightarrow A$ such that $\pi_A \circ i_A = \text{id}_A$
- There exists a morphism $i_C: C \rightarrow B$ such that $\pi_C \circ i_C = \text{id}_C$
- B is a direct sum $A \oplus C$ with the obvious canonical injections and projections.

1.4.2 The snake lemma

Lemma 32. Let $f: B \twoheadrightarrow C$ be an epimorphism, and let $g: D \rightarrow C$ be any morphism. Then the kernel of f functions as the kernel of the pullback of f along g , in the sense

that we have the following commuting diagram in which the right-hand square is a pullback square.

$$\begin{array}{ccccc}
 \ker f' & \xrightarrow{i} & S & \xrightarrow{f'} & D \\
 \parallel & & \downarrow g' & \lrcorner & \downarrow g \\
 \ker f & \xrightarrow{\iota} & B & \xrightarrow{f} & C
 \end{array}$$

Proof. Consider the following pullback square, where f is an epimorphism.

$$\begin{array}{ccc}
 S & \xrightarrow{f'} & D \\
 g' \downarrow & \lrcorner & \downarrow g \\
 B & \xrightarrow{f} & C
 \end{array}$$

Since we are in an abelian category, the pullback f' is also an epimorphism. Denote the kernel of f by K .

The claim is that K is also a kernel of f' . Of course, in order for this statement to make sense we need a map $i: K \rightarrow S$. This is given to us by the universal property of the pullback as follows.

$$\begin{array}{ccccc}
 K & & & & 0 \\
 & \searrow \exists! i & & & \searrow 0 \\
 & & S & \longrightarrow & D \\
 & & \downarrow & & \downarrow g \\
 & \searrow \iota & B & \xrightarrow{f} & C
 \end{array}$$

Next we need to verify that (K, i) is actually the kernel of f' , i.e. satisfies the universal property. To this end, let $v: Q \rightarrow S$ be a map such that $f' \circ v = 0$. We need to find a unique factorization of v through K .

$$\begin{array}{ccccc}
 & & Q & & \\
 & \swarrow \exists! v' & \downarrow v & \searrow 0 & \\
 K & \xrightarrow{i} & S & \xrightarrow{f'} & D \\
 & \searrow \iota & \downarrow g' & & \downarrow g \\
 & & B & \xrightarrow{f} & C
 \end{array}$$

By definition $f' \circ v = 0$. Thus,

$$\begin{aligned}
 f \circ g' \circ v &= g \circ f' \circ v \\
 &= g \circ 0 \\
 &= 0,
 \end{aligned}$$

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so $g' \circ v$ factors uniquely through K as $g' \circ v = \iota \circ v'$. It remains only to check that the triangle formed by v' commutes, i.e. $v = v' \circ i$. To see this, consider the following diagram, where the bottom right square is the pullback from before.

$$\begin{array}{ccccc}
 Q & & & 0 & \\
 & \searrow \exists! & & \searrow & \\
 & S & \xrightarrow{f'} & D & \\
 & \downarrow g' & & \downarrow g & \\
 & B & \xrightarrow{f} & C &
 \end{array}$$

$g' \circ v$ (curved arrow from Q to B)

By the universal property, there exists a unique map $Q \rightarrow S$ making this diagram commute. However, both v and $i \circ v'$ work, so $v = i \circ v'$.

Thus we have shown that, in a precise sense, the kernel of an epimorphism functions as the kernel of its pullback, and we have the following commutative diagram, where the right hand square is a pullback.

$$\begin{array}{ccccc}
 \ker f' & \xrightarrow{i} & S & \xrightarrow{f'} & D \\
 \parallel & & \downarrow g' & & \downarrow g \\
 \ker f & \xrightarrow{\iota} & B & \xrightarrow{f} & C
 \end{array}$$

□

At least in the case that g is mono, when phrased in terms of elements, this result is more or less obvious; we can imagine the diagram above as follows.

$$\begin{array}{ccccc}
 \left\{ \begin{array}{l} \text{elements of pullback} \\ \text{which map to } 0 \end{array} \right\} & \hookrightarrow & \left\{ \begin{array}{l} \text{elements of } B \\ \text{which map to } C \end{array} \right\} & \twoheadrightarrow & D \\
 \parallel & & \downarrow & & \downarrow \\
 \left\{ \begin{array}{l} \text{elements of } B \\ \text{which map to } 0 \end{array} \right\} & \hookrightarrow & B & \twoheadrightarrow & C
 \end{array}$$

Theorem 33 (snake lemma). Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A & \xrightarrow{m} & B & \xrightarrow{e} & C \longrightarrow 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h \\
 0 & \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' \longrightarrow 0
 \end{array}$$

This gives us an exact sequence

$$0 \rightarrow \ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h \rightarrow 0.$$

Proof. We provide running commentary on the diagram below.

$$\begin{array}{ccccccc}
 0 & \cdots \rightarrow & \ker f & \cdots \rightarrow & \ker g & \cdots \rightarrow & \ker h \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \rightarrow & A & \xrightarrow{m} & B & \xrightarrow{e} \twoheadrightarrow & C \rightarrow 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h \\
 0 & \rightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} \twoheadrightarrow & C' \rightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & \text{coker } f & \cdots \rightarrow & \text{coker } g & \cdots \rightarrow & \text{coker } h \cdots \rightarrow 0
 \end{array} \tag{1.2}$$

The dotted arrows come immediately from the universal property for kernels and cokernels, as do exactness at $\ker g$ and $\text{coker } g$. The argument for kernels appeared on the previous homework sheet, and the argument for cokernels is dual.

The only thing left is to define the dashed connecting homomorphism, and to prove exactness at $\ker h$ and $\text{coker } f$.

Extract from the data of [Diagram 1.2](#) the following diagram.

$$\begin{array}{ccccc}
 & & & & \ker h \\
 & & & & \downarrow \\
 A & \xrightarrow{m} & B & \xrightarrow{e} \twoheadrightarrow & C \\
 \downarrow f & & \downarrow g & & \downarrow h \\
 A' & \xrightarrow{m'} & B' & \xrightarrow{e'} \twoheadrightarrow & C' \\
 \downarrow & & & & \\
 & & \text{coker } f & &
 \end{array}$$

Take a pullback and a pushout, and using [Lemma 32](#) (and its dual), we find the following.

$$\begin{array}{ccccccc}
 \ker u & \xrightarrow{a} & S & \xrightarrow{u} \twoheadrightarrow & \ker h & & \\
 \parallel & & \downarrow r & & \downarrow & & \\
 A & \xrightarrow{m} & B & \xrightarrow{e} \twoheadrightarrow & C & & \\
 \downarrow f & & \downarrow g & & \downarrow h & & \\
 A' & \xrightarrow{m'} & B' & \xrightarrow{e'} \twoheadrightarrow & C' & & \\
 \downarrow & & \downarrow s & & \parallel & & \\
 \text{coker } f & \xrightarrow{v} & T & \xrightarrow{b} \twoheadrightarrow & \text{coker } v & &
 \end{array}$$

Consider the map

$$\delta_0 = S \xrightarrow{r} B \xrightarrow{g} B' \xrightarrow{s} T .$$

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By commutativity, $\delta \circ a = 0$, hence we get a map

$$\delta_1: \ker h \rightarrow T.$$

Composing this with b gives 0, hence we get a map

$$\delta: \operatorname{coker} f \rightarrow \ker h.$$

□

There is another version of the snake lemma which does not have the first and last zeroes.

Theorem 34 (snake lemma II). Given the following commutative diagram with exact rows,

$$\begin{array}{ccccccc} A & \xrightarrow{m} & B & \xrightarrow{e} & C & \longrightarrow & 0 \\ f \downarrow & & g \downarrow & & \downarrow h & & \\ 0 & \longrightarrow & A' & \xrightarrow{m'} & B' & \xrightarrow{e'} & C' \end{array}$$

we get a exact sequence

$$\ker f \rightarrow \ker g \rightarrow \ker h \rightarrow \operatorname{coker} f \rightarrow \operatorname{coker} g \rightarrow \operatorname{coker} h.$$

1.4.3 The long exact sequence on homology

Corollary 35. Given an exact sequence of complexes

$$0 \longrightarrow A_{\bullet} \xrightarrow{f_{\bullet}} B_{\bullet} \xrightarrow{g_{\bullet}} C_{\bullet} \longrightarrow 0$$

we get a long exact sequence on homology

$$\begin{array}{c} \dots \longrightarrow H_{n+1}(C) \\ \searrow \delta \nearrow \\ H_n(A) \xrightarrow{H_n(f)} H_n(B) \xrightarrow{H_n(g)} H_n(C) \\ \searrow \delta \nearrow \\ H_{n-1}(A) \xrightarrow{H_{n-1}(f)} H_{n-1}(B) \xrightarrow{H_{n-1}(g)} H_{n-1}(C) \\ \searrow \delta \nearrow \\ H_{n-2}(A) \longrightarrow \dots \end{array}$$

Proof. For each n , we have a diagram of the following form.

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} & \longrightarrow & C_{n+1} \longrightarrow 0 \\ & & \downarrow d_{n+1}^A & & \downarrow d_{n+1}^B & & \downarrow d_{n+1}^C \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n & \longrightarrow & C_n \longrightarrow 0 \end{array}$$

Applying the snake lemma ([Theorem 33](#)) gives us, for each n , two exact sequences as follows.

$$0 \longrightarrow \ker d_{n+1}^A \longrightarrow \ker d_{n+1}^B \longrightarrow \ker d_{n+1}^C$$

and

$$\operatorname{coker} d_n^A \longrightarrow \operatorname{coker} d_n^B \longrightarrow \operatorname{coker} d_n^C \longrightarrow 0$$

We can put these together in a diagram with exact rows as follows.

$$\begin{array}{ccccccc} \operatorname{coker} d_n^A & \longrightarrow & \operatorname{coker} d_n^B & \longrightarrow & \operatorname{coker} d_n^C & \longrightarrow & 0 \\ & & & & & & \\ 0 & \longrightarrow & \ker d_{n-1}^A & \longrightarrow & \ker d_{n-1}^B & \longrightarrow & \ker d_{n-1}^C \end{array}$$

Recall that for each n , we get a map

$$\phi_{n-1}^A: \operatorname{im} d_{n-1}^A \rightarrow \ker d_n^A,$$

and similarly for B_\bullet and C_\bullet .

Adding these in gives the following.

$$\begin{array}{ccccccc} \operatorname{coker} d_n^A & \longrightarrow & \operatorname{coker} d_n^B & \longrightarrow & \operatorname{coker} d_n^C & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \ker d_{n-1}^A & \longrightarrow & \ker d_{n-1}^B & \longrightarrow & \ker d_{n-1}^C \end{array}$$

A second application of the snake lemma then gives a long exact sequence

$$\ker \phi_f \rightarrow \ker \phi_g \rightarrow \ker \phi_h \rightarrow \operatorname{coker} \phi_f \rightarrow \operatorname{coker} \phi_g \rightarrow \operatorname{coker} \phi_h$$

□

Lemma 36. Let \mathcal{A} be an abelian category, and let f be a morphism of short exact se-

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quences in $\mathbf{Ch}(\mathcal{A})$ as below.

$$\begin{array}{ccccccc}
 0 & \longrightarrow & A'_\bullet & \xrightarrow{\alpha_\bullet} & A_\bullet & \xrightarrow{\alpha'_\bullet} & A''_\bullet \longrightarrow 0 \\
 & & \downarrow f'_\bullet & & \downarrow f_\bullet & & \downarrow f''_\bullet \\
 0 & \longrightarrow & B'_\bullet & \xrightarrow{\beta_\bullet} & B_\bullet & \xrightarrow{\beta'_\bullet} & B''_\bullet \longrightarrow 0
 \end{array}$$

This gives us the following morphism of long exact sequences on homology.

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & H_n(A) & \xrightarrow{H_n(\alpha')} & H_n(A'') & \xrightarrow{\delta} & H_{n-1}(A') \xrightarrow{H_{n-1}(\alpha)} H_{n-1}(A) \longrightarrow \cdots \\
 & & \downarrow H_n(f) & & \downarrow H_n(f'') & & \downarrow H_n(f') & & \downarrow H_n(f) \\
 \cdots & \longrightarrow & H_n(B) & \xrightarrow{H_n(\beta')} & H_n(B'') & \xrightarrow{\delta} & H_{n-1}(B') \xrightarrow{H_{n-1}(\beta)} H_{n-1}(B) \longrightarrow \cdots
 \end{array}$$

In particular, the connecting morphism δ is functorial.

1.4.4 The five lemma

Lemma 37 (four lemmas). Let

$$\begin{array}{ccccccc}
 B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
 \end{array}$$

be a commutative diagram with exact rows, with monomorphisms and epimorphisms as marked. Then

Theorem 38 (five lemma).

1.4.5 The nine lemma

Theorem 39 (nine lemma).