1 Homology

1.1 Basic definitions and examples

We assume a basic knowledge of simplicial sets just to get the ball rolling.

Definition 1 (singular complex, singular homology)**.** Let X be a topological space. The singular chain complex $C_\bullet(X)$ is defined level-wise by

$$C_n(X) = M(\mathcal{F}(\mathbf{Sing}(X)))$$

where $\mathcal{F}$ denotes the free group functor and M is the Moore functor (Definition **??**). That is, it has differentials

$$d_n\colon C_n \to C_{n-1}; \qquad (\alpha\colon \Delta^n \to X) \mapsto \sum_{i=0}^{n} \partial^i \alpha.$$

The singular homology of X is the homology

$$H_n(X) = H_n(C_\bullet(X)).$$

Note that the singular chain complex construction is functorial: any map $f\colon X \to Y$ gives a chain map $C(f)\colon C(X) \to C(Y)$. This immediately implies that the nth homology of a space X is invariant under homeomorphism.

Example 2. Denote by pt the one-point topological space. Then $C(\mathrm{pt})_\bullet$ is given level-wise as follows.

$$\begin{tikzcd}
\cdots \arrow[r, "\mathrm{id}"] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "0"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, equal] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\cdots \arrow[r, "d_4"'] & C_3 \arrow[r, "d_3"'] & C_2 \arrow[r, "d_2"'] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "d_0"'] & C_{-1} \arrow[r, "d_{-1}"'] & \cdots
\end{tikzcd}$$

Thus, the nth homology of the point is

$$H_n(\mathrm{pt}) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}.$$

Example 3. Let X be a path-connected topological space. Then $H_0(X) = \mathbb{Z}$. To see this, consider a general element of $H_0(X)$. Because d_0 is the zero map, $H_n(X)$ is simply the free group generated by the collection of points of X modulo the relation "there is a path from x to y." However, path-connectedness implies that every two points of X are connected by a path, so every point of X is equivalent to any other. Thus, $H_0(X)$ has only one generator.

More generally,

$$H_n(X) = \mathbb{Z}^{\pi_0(X)}.$$

1.2 The Hurewicz homomorphism

Theorem 4 (Hurewicz)**.** For any path-connected topological space X, there is an isomorphism

$$h_X \colon H_1(X) \cong \pi_1(X)_{\mathrm{ab}},$$

where $(-)_{\mathrm{ab}}$ denotes the abelianization. Furthermore, the maps h_X form the components of a natural isomorphism between the functors

$$h \colon H_1 \Rightarrow (\pi_1)_{\mathrm{ab}}.$$

Example 5. We can now confidently say that

$$H_1(\mathbb{S}^1) = \pi_1(\mathbb{S}^1)_{\mathrm{ab}} = \mathbb{Z},$$

and that

$$H_1(\mathbb{S}^n) = 0, \qquad n > 1.$$

Example 6. Since

$$\pi_1(X \times Y) \equiv \pi_1(X) \times \pi_1(Y)$$

and

$$\pi_1(X \vee Y) \equiv \pi_1(X) * \pi_1(Y),$$

we have that

$$H_1(X \times Y) \cong H_1(X) \times H_1(Y) \cong H_1(X \vee Y).$$

1.3 Homotopy equivalence

Proposition 7. Let $f, g \colon X \to Y$ be continuous maps between topological spaces, and let $H \colon X \times [0,1] \to Y$ be a homotopy between them. Then H induces a homotopy between $C(f)$ and $C(g)$. In particular, f and g agree on homology.

Corollary 8. Any two topological spaces which are homotopy equivalent have the same homology groups.

Example 9. Any contractible space is homotopy equivalent to the one point space pt. Thus, for any contractible space X we have

$$H_n(X) = \begin{cases} \mathbb{Z}, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

1.4 Relative homology; the long exact sequence of a pair of spaces

Definition 10 (relative homology)**.** Let X be a topological space, and let $X \subset A$. The relative chain complex of (X, A) is

$$S_\bullet(X, A) = S_\bullet(X)/S_\bullet(A).$$

The relative homology of (X, A) is

$$H_n(X, A) = H_n(S_\bullet(X, A)).$$

Denote by **Pair** the category whose objects are pairs (X, A), where X is a topological space and $A \hookrightarrow X$ is a subspace, and whose morphisms $(X, A) \to (Y, B)$ are maps $f\colon X \to Y$ such that $F(A) \subset B$.

Lemma 11. For each n, relative homology provides a functor $H_n\colon \mathbf{Pair} \to \mathbf{Ab}$.

Proof. Consider the following diagram

$$\begin{tikzcd} S(X)_\bullet \arrow[r, "f"] \arrow[d, two heads] & S(Y)_\bullet \arrow[d, two heads] \\ S(X)_\bullet/S(A)_\bullet \arrow[r, dashed] & S(Y)_\bullet/S(B)_\bullet \end{tikzcd}$$

The dashed arrow is well-defined because of the assumption that $f(A) \subset B$. Functoriality now follows from the functoriality of H_n. □

Proposition 12. Let (X, A) be a pair of spaces. There is the following long exact sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(X,A) \arrow[dll, "\delta"] \\
H_j(A) \arrow[r] & H_j(X) \arrow[r] & H_j(X,A) \arrow[dll, "\delta"] \\
H_{j-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. This is the long exact sequence associated to the following short exact sequence.

$$0 \longrightarrow C_\bullet(A) \hookrightarrow C_\bullet(X) \twoheadrightarrow C_\bullet(X,A) \longrightarrow 0$$

□

Example 13. Let $X = \mathbb{D}^n$, the n-disk, and $A = \mathbb{S}^{n-1}$ its boundary n-sphere.

Consider the long exact sequence on the pair $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & H_j(\mathbb{D}^n) \arrow[r] & H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \arrow[dll, "\delta"] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that $H_j(\mathbb{D}^n) = 0$ for $n > 0$ because it is contractible.

Thus, exactness forces

$$H_j(\mathbb{D}^n, \mathbb{S}^{n-1}) \cong H_{j-1}(\mathbb{S}^{n-1})$$

for $j > 1$ and $n \geq 1$.

1.5 Barycentric subdivision

Fact 14. Let X be a topological space, and let $\mathfrak{U} = \{U_i \mid i \in I\}$ be an open cover of X. Denote by

$$S_n^{\mathfrak{U}}(X)$$

the free group generated by those continuous functions

$$\alpha \colon \Delta^n \to X$$

whose images are completely contained in some open set in the open cover $\mathfrak{U}$. That is, such that there exists some i such that $\alpha(\Delta^n) \subset U_i$. The inclusion $S_n^{\mathfrak{U}}(X) \hookrightarrow S_n(X)$

induces a cochain structure on $S_\bullet^{\mathfrak{U}}(X)$.

$$\begin{tikzcd}
\cdots \arrow[r] & S_2^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_1^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & S_0^{\mathfrak{U}}(X) \arrow[r] \arrow[d, hook] & 0 \\
\cdots \arrow[r] & S_2(X) \arrow[r] & S_1(X) \arrow[r] & S_0(X) \arrow[r] & 0
\end{tikzcd}$$

In fact, this inclusion is homotopic to the identity, hence induces an isomorphism

$$H_n^{\mathfrak{U}}(X) := H_n(S^{\mathfrak{U}}(X)_\bullet) \equiv H_n(X).$$

This fact allows us almost immediately to read of two important theorems.

1.5.1 Excision

Theorem 15 (excision)**.** Let $W \subset A \subset X$ be a triple of topological spaces such that $\bar{W} \subset \mathring{A}$. Then the right-facing inclusions

$$\begin{tikzcd}
A \smallsetminus W \arrow[r, hook, "i"] \arrow[d, hook] & A \arrow[d, hook] \\
X \smallsetminus W \arrow[r, hook, "i"] & X
\end{tikzcd}$$

induce an isomorphism

$$H_n(i)\colon H_n(X \smallsetminus W, A \smallsetminus W) \cong H_n(X, A).$$

That is, when considering relative homology $H_n(X, A)$, we may cut away a subspace from the interior of A without harming anything. This gives us a hint as to the interpretation of relative homology: $H_n(X, A)$ can be interpreted the part of $H_n(X)$ which does not come from A.

1.5.2 The Mayer-Vietoris sequence

Theorem 16 (Mayer-Vietoris)**.** Let X be a topological space, and let $\mathfrak{U} = \{X_1, X_2\}$ be an open cover of X, i.e. let $X = X_1 \cup X_2$. Then we have the following long exact sequence.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X) \arrow[dll, out=-10, in=170, "\delta"] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H_n(X) \arrow[dll, out=-10, in=170, "\delta"] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}
$$

Proof. We can draw our inclusions as the following pushout.

$$
\begin{tikzcd}
 & X_1 \arrow[dr, hook, "\kappa_1"] & \\
X_1 \cap X_2 \arrow[ur, hook, "i_1"] \arrow[dr, hook, "i_2"'] & & X \\
 & X_2 \arrow[ur, hook, "\kappa_1"'] &
\end{tikzcd}
$$

We have, almost by definition, the following short exact sequence.

$$
\begin{tikzcd}
0 \arrow[r] & S_\bullet(X_1 \cap X_2) \arrow[r, hook, "{(i_1, i_2)}"] & S_\bullet(X_1) \oplus S_\bullet(X_2) \arrow[r, two heads, "\kappa_1 - \kappa_2"] & S_\bullet^{\mathfrak{U}}(X) \arrow[r] & 0
\end{tikzcd}
$$

This gives the following long exact sequence on homology.

$$
\begin{tikzcd}
 & \cdots \arrow[r] & H^{\mathfrak{U}}_{n+1}(X) \arrow[dll, out=-10, in=170, "\delta"] \\
H_n(X_1 \cap X_2) \arrow[r] & H_n(X_1) \oplus H_n(X_2) \arrow[r] & H^{\mathfrak{U}}_n(X) \arrow[dll, out=-10, in=170, "\delta"] \\
H_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}
$$

We have seen that $H^{\mathfrak{U}}(n)(X) \cong H_n(X)$; the result follows. □

Example 17 (Homology groups of spheres)**.** We can decompose $\mathbb{S}^n$ as

$$\mathbb{S}^n = (\mathbb{S}^n \smallsetminus N) \cup (\mathbb{S}^n \smallsetminus S),$$

where N and S are the North and South pole respectively. This gives us the following pushout.

$$\begin{tikzcd}
 & \mathbb{S}^n \smallsetminus N \arrow[dr, hook] & \\
(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & \mathbb{S}^n \smallsetminus S \arrow[ur, hook] &
\end{tikzcd}$$

The Mayer-Vietoris sequence is as follows.

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_j((S^n \smallsetminus N) \cap (S^n \smallsetminus S)) \arrow[r] & H_j(S^n \smallsetminus N) \oplus H_j(S^n \smallsetminus S) \arrow[r] & H_j^{\mathfrak{U}}(X) \arrow[dll, "\delta"] \\
H_{j-1}((\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S)) \arrow[r] & \cdots &
\end{tikzcd}$$

We know that

$$\mathbb{S}^n \smallsetminus N \cong \mathbb{S}^n \smallsetminus S \cong \mathbb{D}^n \simeq \mathrm{pt}$$

and that

$$(\mathbb{S}^n \smallsetminus N) \cap (\mathbb{S}^n \smallsetminus S) \cong I \times \mathbb{S}^{n-1} \simeq \mathbb{S}^{n-1},$$

so using the fact that homology respects homotopy, the above exact sequence reduces (for $j > 1$) to

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{j+1}(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_j(\mathbb{S}^{n-1}) \arrow[r] & 0 \arrow[r] & H_j(\mathbb{S}^n) \arrow[dll, "\delta"] \\
H_{j-1}(\mathbb{S}^{n-1}) \arrow[r] & \cdots &
\end{tikzcd}$$

Thus, for $i > 1$, we have

$$H_i(\mathbb{S}^j) \equiv H_{i-1}(\mathbb{S}^{j-1}).$$

We have already noted the following facts.

- H_0 counts the number of connected components, so

$$H_0(\mathbb{S}^j) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & j = 0 \\ \mathbb{Z}, & j > 0 \end{cases}$$

- For path connected X, $H_1(X) \cong \pi_1(X)_{\mathrm{ab}}$, so

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 0 \\ 0, & \text{otherwise} \end{cases}$$

- For $i > 0$, $H_i(\mathrm{pt}) = 0$, so $H_i(\mathbb{S}^0) = 0$.

This gives us the following table.

$$\begin{array}{c|cccc}
j = 3 & \mathbb{Z} & 0 & & \\
j = 2 & \mathbb{Z} & 0 & & \\
j = 1 & \mathbb{Z} & \mathbb{Z} & & \\
j = 0 & \mathbb{Z} \oplus \mathbb{Z} & 0 & 0 & 0 \\
\hline
H_i(\mathbb{S}^j) & i = 0 & i = 1 & i = 2 & i = 3
\end{array}$$

The relation

$$H_i(\mathbb{S}^j) \cong H_{i-1}(\mathbb{S}^{j-1}), \qquad i > 1$$

allows us to fill in the above table as follows.

$$\begin{array}{c|cccc}
j = 3 & \mathbb{Z} & 0 & 0 & \mathbb{Z} \\
j = 2 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
j = 1 & \mathbb{Z} & \mathbb{Z} & 0 & 0 \\
j = 0 & \mathbb{Z} \oplus \mathbb{Z} & 0 & 0 & 0 \\
\hline
H_i(\mathbb{S}^j) & i = 0 & i = 1 & i = 2 & i = 3
\end{array} \tag{1.1}$$

Example 18. Above, we used the Hurewicz homomorphism to see that

$$H_1(\mathbb{S}^j) = \begin{cases} \mathbb{Z}, & j = 1 \\ 0, & \text{otherwise} \end{cases}.$$

We can also see this directly from the Mayer-Vietoris sequence. Recall that we expressed $\mathbb{S}^n$ as the following pushout, with $X^+ \cong X^- \simeq \mathbb{D}^n$.

$$\begin{tikzcd}
 & X^+ \arrow[dr, hook] & \\
X^+ \cap X^- \arrow[ur, hook] \arrow[dr, hook] & & \mathbb{S}^n \\
 & X^- \arrow[ur, hook] &
\end{tikzcd}$$

Also recall that with this setup, we had $X^+ \cap X^- \simeq \mathbb{S}^{n-1}$.

First, fix $n > 1$, and consider the following part of the Mayer-Vietoris sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^n) \arrow[dlll, "\delta"] \\
H_0(X^+ \cap X^-) \arrow[rr, "H_0(i_0, i_1)"] & & H_0(X^+) \oplus H_0(X^-) \arrow[r] & \cdots
\end{tikzcd}$$

If we can verify that the morphism $H_0(i_0, i_1)$ is injective, then we are done, because exactness will force $H_1(\mathbb{S}^n) \cong 0$.

The elements of $H_0(X^+ \cap X^-)$ are equivalence classes of points of X^+ and X^-, with one equivalence class per connected component. Let $p \in X^+ \cap X^-$. Then $i_0(p)$ is a point of X^+, and $i_1(p)$ is a point of X^-. Each of these is a generator for the corresponding zeroth homology, so (i_0, i_1) sends the generator $[p]$ to a the pair $([i_0(p)], [i_1(p)])$. This is clearly injective.

Now let $n = 1$, and consider the following portion of the Mayer-Vietoris sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & 0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[dlll, "\delta"] \\
H_0(X^+ \cap X^-) \arrow[rr, "H_0(i_0, i_1)"] & & H_0(X^+) \oplus H_0(X^-) \arrow[r, "H_0(\kappa_1) - H_0(\kappa_2)"] & H_0(\mathbb{S}^1)
\end{tikzcd}$$

We can immediately replace things we know, finding the following.

$$\begin{tikzcd}
 & & (a, b) \arrow[r, mapsto] & (a + b, a + b) & \\
0 \arrow[r] & H_1(\mathbb{S}^1) \arrow[r, hook] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r, "f"] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \\
 & & & (c, d) \arrow[r, mapsto] & c - d
\end{tikzcd}$$

The kernel of f is the free group generated by (a, a). Thus, $H_1(\mathbb{S}^1) \cong \mathbb{Z}$.

1.5.3 The relative Mayer-Vietoris sequence

Theorem 19 (relative Mayer-Vietoris sequence)**.** Let X be a topological space, and let $A, B \subset X$ open in $A \cup B$. Denote $\mathfrak{U} = \{A, B\}$.

Then there is a long exact sequence

$$\begin{tikzcd}
 & \cdots \arrow[r] & H_{n+1}(X, A \cup B) \arrow[dll, "\delta"] \\
H_n(X, A \cap B) \arrow[r] & H_n(X, A) \oplus H_n(X, B) \arrow[r] & H_n(X, A \cup B) \arrow[dll, "\delta"] \\
H_{n-1}(X, A \cap B) \arrow[r] & \cdots &
\end{tikzcd}$$

Proof. Consider the following chain complex of chain complexes.

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & S_n(A \cap B) \arrow[r] \arrow[d] & S_n(A) \oplus S_n(B) \arrow[r] \arrow[d] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X) \arrow[r] \arrow[d] & S_n(X) \oplus S_n(X) \arrow[r] \arrow[d] & S_n(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_n(X, A \cap B) \arrow[r] \arrow[d] & S_n(X, A) \oplus S_n(X, B) \arrow[r] \arrow[d] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

All columns are trivially short exact sequences, as are the first two rows. Thus, the nine lemma (Theorem **??**) implies that the last row is also exact.

Consider the following map of chain complexes; the first row is the last column of the above grid.

$$\begin{tikzcd}
0 \arrow[r] & S_n^{\mathfrak{U}}(A \cup B) \arrow[r, hook] \arrow[d, "\phi"'] & S_n(X) \arrow[r, two heads] \arrow[d, equal] & S_n(X)/S_n^{\mathfrak{U}}(A \cup B) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & S_n(A \cup B) \arrow[r] & S_n(X) \arrow[r, two heads] & S_n(X, A \cup B) \arrow[r] & 0
\end{tikzcd}$$

This gives us, by Lemma **??**, a morphism of long exact sequences on homology.

$$\begin{tikzcd}
H_n(S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\phi)"'] & H_n(X) \arrow[r] \arrow[d, equal] & H_n(S_\bullet(X)/S_\bullet^{\mathfrak{U}}(A \cup B)) \arrow[r] \arrow[d, "H_n(\psi)"] & H_{n-1}(S_\bullet\mathfrak{U}(A \cup B)) \arrow[r] \arrow[d, "H_{n-1}(\phi)"] & H_{n-1}(X) \arrow[d, equal] \\
H_n(A \cup B) \arrow[r] & H_n(X) \arrow[r] & H_n(X, A \cup B) \arrow[r] & H_{n-1}(A \cup B) \arrow[r] & H_{n-1}(X)
\end{tikzcd}$$

We have seen (in Fact 14) that $H_i(\phi)$ is an isomorphism for all i. Thus, the five lemma (Theorem **??**) tells us that $H_n(\psi)$ is an isomorphism. □

1.6 Reduced homology

It would be hard to argue that Table 1.1 is not pretty, but it would be much prettier were it not for the $\mathbb{Z}$s in the first column. We have to carry these around because every non-empty space has at least one connected component.

The solution is to define a new homology $\tilde{H}_n(X)$ which agrees with $H_n(X)$ in positive degrees, and is missing a copy of $\mathbb{Z}$ in the zeroth degree. There are three equivalent ways of doing this: one geometric, one algebraic, and one somewhere in between.

1. **Geometric:** Denoting the unique map $X \to \mathrm{pt}$ by ϵ, one can define relative homology by
$$\tilde{H}_n(X) = \ker H_n(\epsilon).$$
2. **In between:** One can replace homology $H_n(X)$ by relative homology
$$\tilde{H}_n(X) = H_n(X, x),$$
where $x \in X$ is any point of x.
3. **Algebraic:** One can augment the singular chain complex $C_\bullet(X)$ by adding a copy of $\mathbb{Z}$ in degree -1, so that
$$\tilde{C}_n(X) = \begin{cases} C_n(X), & n \neq -1 \\ \mathbb{Z}, & n = -1. \end{cases}$$
Then one can define
$$\tilde{H}_n(X) = H_n(\tilde{C}_\bullet).$$

There is a more modern point of view, which is the following. In constructing the singular chain complex of our space X, we used the following composition.

$$\mathbf{Top} \xrightarrow{\mathbf{Sing}} \mathbf{Set}_\Delta \xrightarrow{\mathcal{F}} \mathbf{Ab}_\Delta \xrightarrow{N} \mathbf{Ch}(\mathbf{Ab})$$

For many purposes, there is a more natural category than Δ to use: the category $\bar{\Delta}$, which includes the empty simplex $[-1]$. The functor **Sing** now has a component corresponding to (-1)-simplices:

$$\mathbf{Sing}(X)_{-1} = \mathrm{Hom}_{\mathbf{Top}}(\rho([-1]), X) = \mathrm{Hom}_{\mathbf{Top}}(\emptyset, X) = \{*\},$$

since the empty topological space is initial in **Top**. Passing through $\mathcal{F}$ thus gives a copy of $\mathbb{Z}$ as required. Thus, using $\bar{\Delta}$ instead of Δ gives the augmented singular chain complex.

These all have the desired effect, and which method one uses is a matter of preference. To see this, note the following.

- (1 ⇔ 2): Since the composition

$$\mathrm{pt} \overset{x}{\hookrightarrow} X \overset{\epsilon}{\twoheadrightarrow} \mathrm{pt}$$

is a weak retract, the sequence

$$0 \longrightarrow S_n(\mathrm{pt}) \longrightarrow S_n(X) \longrightarrow S_n(X, \{x\}) \longrightarrow 0$$

Proposition 20. Relative homology agrees with ordinary homology in degrees greater than 0, and in degree zero we have the relation

$$H_0(X) = \tilde{H}_0(X) \oplus \mathbb{Z}.$$

Proof. Trivial from algebraic definition. □

Many of our results for regular homology hold also for reduced homology.

Proposition 21. There is a long exact sequence for a pair of spaces

$$\begin{tikzcd}
 & \cdots \arrow[r] & \tilde{H}_{j+1}(X, A) \arrow[dll, "\delta"] \\
\tilde{H}_j(A) \arrow[r] & \tilde{H}_j(X) \arrow[r] & \tilde{H}_j(X, A) \arrow[dll, "\delta"] \\
\tilde{H}_{j-1}(A) \arrow[r] & \cdots &
\end{tikzcd}$$

Proposition 22. We have a reduced Mayer-Vietoris sequence.

$$\begin{tikzcd}
 & \cdots \arrow[r] & \tilde{H}_{n+1}(X) \arrow[dll, "\delta"] \\
\tilde{H}_n(X_1 \cap X_2) \arrow[r] & \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \arrow[r] & \tilde{H}_n(X) \arrow[dll, "\delta"] \\
\tilde{H}_{n-1}(X_1 \cap X_2) \arrow[r] & \cdots &
\end{tikzcd}$$

Proposition 23. Let $\{(X_i, x_i)\}_{i \in I}$, be a set of pointed topological spaces such that each x_i has an open neighborhood $U_i \subset X_i$ of which it is a deformation retract. Then for any finite $E \subset I$[1] we have

$$\tilde{H}_n\left(\bigvee_{i \in I} X_i\right) \cong \bigoplus_{i \in E} \tilde{H}_n(X_i).$$

[1] This finiteness condition is not actually necessary, but giving it here avoids a colimit argument.

Proof. We prove the case of two bouquet summands; the rest follows by induction. We know that

$$X_1 \vee X_2 = (X_1 \vee U_2) \cup (U_1 \vee X_2)$$

is an open cover. Thus, the reduced Mayer-Vietoris sequence of Proposition 22 tells us that the following sequence is exact.

$$0 \longrightarrow \tilde{H}_n(X_1) \oplus \tilde{H}_n(X_2) \longrightarrow \tilde{H}_n(X) \longrightarrow 0$$

In particular, for $n > 0$, we find that the corresponding sequence on non-reduced homology is exact.

$$0 \longrightarrow H_n(X_1) \oplus H_n(X_2) \longrightarrow H_n(X) \longrightarrow 0$$

□

Definition 24 (good pair)**.** A pair of spaces (X, A) is said to be a good pair if the following conditions are satisfied.

1. A is closed inside X.
2. There exists an open set U with $A \subset U$ such that A is a deformation retract of U.

$$A \hookrightarrow U \xrightarrow{r} A$$

Proposition 25. Let (X, A) be a good pair. Let $\pi : X \longrightarrow X/A$ be the canonical projection. Then

$$\tilde{H}(X, A) \cong \tilde{H}_n(X/A) \qquad \text{for all } n > 0.$$

Proof. □

Theorem 26 (suspension isomorphism)**.** Let (X, A) be a good pair. Then

$$H_n(\Sigma X, \Sigma A) \cong \tilde{H}_{n-1}(X, A), \qquad \text{for all } n > 0.$$

1.7 Mapping degree

We have shown that

$$\tilde{H}_n(\mathbb{S}^m) \cong \begin{cases} \mathbb{Z}, & n = m \\ 0, & n \neq m \end{cases}.$$

Thus, we may pick in each $H_n(\mathbb{S}^n)$ a generator μ_n. Let $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be a continuous map. Then

$$H_n(f)(\mu_n) = d\,\mu_n, \qquad \text{for some } d \in \mathbb{Z}.$$

Definition 27 (mapping degree)**.** We call $d \in \mathbb{Z}$ as above the mapping degree of f, and denote it by $\deg(f)$.

Example 28. Consider the map

$$\omega\colon [0,1] \longrightarrow \mathbb{S}^1; \qquad t \longmapsto e^{2\pi i t}.$$

The 1-simplex ω generates the fundamental group $\pi_1(\mathbb{S}^1)$, so by the Hurewicz homomorphism (Theorem 4), the class $[\omega]$ generates $H_1(\mathbb{S}^1)$. We can think of $[\omega]$ as $1 \in \mathbb{Z}$.

Now consider the map

$$f_n\colon \mathbb{S}^1 \longrightarrow \mathbb{S}^1; \qquad x \longmapsto x^n.$$

We have

$$\begin{aligned} H_1(f_n)(\omega) &= [f_n \circ \omega] \\ &= [e^{2\pi i n t}]. \end{aligned}$$

The naturality of the Hurewicz isomorphism (Theorem 4) tells us that the following diagram commutes.

$$\begin{tikzcd} \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[r, "\pi_1(f_n)_{\mathrm{ab}}"] \arrow[d, "h_{\mathbb{S}^1}"'] & \pi_1(\mathbb{S}^1)_{\mathrm{ab}} \arrow[d, "h_{\mathbb{S}^1}"] \\ H_1(\mathbb{S}^1) \arrow[r, "H_1(f_n)"'] & H_1(\mathbb{S}^1) \end{tikzcd}$$

1.8 CW Complexes

CW complexes are a class of particularly nicely-behaved topological spaces.

Definition 29 (cell)**.** Let X be a topological space. We say that X is an n-cell if X is homeomorphic to $\mathbb{R}^n$. We call the number n the dimension of X.

Definition 30 (cell decomposition)**.** A cell decomposition of a topological space X is a decomposition

$$X = \coprod_{i \in I} X_i, \qquad X_i \cong \mathbb{R}^{n_i}$$

where the disjoint union is of sets rather than topologial spaces.

Definition 31 (CW complex)**.** A Hausdorff topological space is known as a CW complex[2] if it satisfies the following conditions.

[2] Axiom (CW2) is called the *closure-finiteness* condition. This is the 'C' in CW complex. Axiom (CW3) says that X carries the *weak topology* and is responsible for the 'W'.

(CW1) For every n-cell $\sigma \subset X$, there is a continuous map $\Phi_\sigma \colon \mathbb{D}^n \to X$ such that the restriction of Φ_σ to $\mathring{\mathbb{D}}^n$ is a homeomorphism

$$\Phi_\sigma|_{\mathring{\mathbb{D}}^n} \cong \sigma,$$

and Φ_σ maps $\mathbb{S}^{n-1} = \partial\mathbb{D}^n$ to the union of cells of dimension of at most $n-1$.

(CW2) For every n-cell σ, the closure $\bar{\sigma} \subset X$ has a non-trivial intersection with at most finitely many cells of X.

(CW3) A subset $A \subset X$ is closed if and only if $A \cap \bar{\sigma}$ is closed for all cells $\sigma \in X$.

At this point, we define some terminology.

- The map Φ_σ is called the *characteristic map* of the cell σ.
- Its restriction $\Phi_\sigma|_{\mathbb{S}^{n-1}}$ is called the *attaching map.*

Example 32. Consider the unit interval $I = [0,1]$. This has an obvious CW structure with two 0-cells and one 1-cell. It also has an CW structure with $n+1$ 0-cells and n 1-cells. which looks like n intervals glued together at their endpoints.

However, we must be careful. Consider the cell decomposition of the interval with zero-cells

$$\sigma^0_k = \frac{1}{k} \text{ for } k \in \mathbb{N}^{\geq 1}, \qquad \text{and} \qquad \sigma^0_\infty = 0$$

and one-cells

$$\sigma^1_k = \left(\frac{1}{k}, \frac{1}{k+1}\right), \qquad k \in \mathbb{N}^{\geq 1}.$$

At first glance, this looks like a CW decomposition; it is certainly satisfies Axiom (CW1) and Axiom (CW2). However, consider the set

$$A = \{a_k \mid k \in \mathbb{N}^{\geq 1}\},$$

where

$$a_k = \frac{1}{2}\left(\frac{1}{k} + \frac{1}{k+1}\right),$$

is the midpoint of the interval σ^1_k. We have $A \cap \sigma^0_k = \emptyset$ for all k, and $A \cap \sigma^1_k = \{a_k\}$ for all k. In each case, $A \cap \bar{\sigma}^i_j$ is closed in σ^i_j. However, the set A is not closed in I, since it does not contain its limit point $\lim_{n\to\infty} a_n = 0$.

Definition 33 (skeleton, dimension)**.** Let X be a CW complex, and let

$$X^n = \bigcup_{\substack{\sigma \in X \\ \dim(\sigma) \leq n}} \sigma.$$

We call X^n the n-skeleton of X. If X is equal to its n-skeleton but not equal to its $(n-1)$-skeleton, we say that X is n-dimensional.

Note 34. Axiom (CW3) implies that X carries the direct limit topology, i.e. that

$$X \cong \varinjlim X^n.$$

Definition 35 (subcomplex, CW pair)**.** Let X be a CW complex. A subspace $Y \subset X$ is a subcomplex if it has a cell decomposition given by cells of X such that for each $\sigma \subset Y$, we also have that $\bar{\sigma} \subset Y$

We call such a pair (X, Y) a CW pair.

Fact 36. Let X and Y be CW complexes such that X is locally compact. Then $X \times Y$ is a CW complex.

Lemma 37. Let D be a subset of a CW complex such that for each cell $\sigma \subset X$, $D \cap \sigma$ consists of at most one point. Then D is discrete.

Corollary 38. Let X be a CW complex.

1. Every compact subset $K \subset X$ is contained in a finite union of cells.
2. The space X is compact if and only if it is a finite CW complex.
3. The space X is locally compact[3] if and only if it is locally finite.[4]

Proof. It is clear that 1. $\Rightarrow$ 2., since X is a subset of itself. Similarly, it is clear that 2. $\Rightarrow$ 3., since □

Corollary 39. If $f\colon K \to X$ is a continuous map from a compact space K to a CW complex X, then the image of K under f is contained in a finite skeleton. That is to say, f factors through some X^n.

$$\begin{tikzcd} & & & & K \arrow[d, "f"] \arrow[llld, dashed, "\exists \tilde{f}"'] \\ X^{n-1} \arrow[r, hook] & X^n \arrow[r, hook] & X^{n+1} \arrow[r, hook] & \cdots \arrow[r, hook] & X \end{tikzcd}$$

Proposition 40. Let A be a subcomplex of a CW complex X. Then $X \times \{0\} \cup A \times [0, 1]$ is a strong deformation retract of $X \times [0, 1]$.

Lemma 41. Let X be a CW complex.

- For any subcomplex $A \subset X$, there is an open neighborbood U of A in X together with a strong deformation retract to A. In particular, for each skeleton X^n there is an open neighborhood U in X (as well as in X^{n+1}) of X^n such that X^n is a strong deformation retract of U.
- Every CW complex is paracompact, locally path-connected, and locally contractible.

[3] I.e. every point of X has a compact neighborhood.
[4] I.e. if every point has a neighborhood which is contained in only finitely many cells.

- Every CW complex is semi-locally 1-connected, hence possesses a univesal covering space.

Lemma 42. Let X be a CW complex. We have the following decompositions.

1. $$X^n \setminus X^{n-1} = \coprod_{\sigma \text{ an } n\text{-cell}} \sigma \cong \coprod_{\sigma \text{ an } n\text{-cell}} \mathring{\mathbb{D}}^n.$$

2. $$X^n / X^{n-1} \cong \bigvee_{\sigma \text{ an } n\text{-cell}} \mathbb{S}^n$$

Proof.

1. Since $X^n \setminus X^{n-1}$ is simply the union of all n-cells (which must by definition be disjoint), we have the first equality. The homeomorphism is simply because each n-cell is homeomorphic to the open n-ball.
2. For every n-cell σ, the characteristic map Φ_σ sends $\partial \Delta^n$ to the $(n-1)$-skeleton.

□

1.9 Cellular homology

Lemma 43. For X a CW complex, we always have

$$H_q(X^n, X^{n-1}) \cong \tilde{H}_q(X^n / X^{n-1}) \cong \bigoplus_{\sigma \text{ an } n\text{-cell}} \tilde{H}_q(\mathbb{S}^n).$$

Proof. By Lemma 41, (X^n, X^{n-1}) is a good pair. The first isomorphism then follows from Proposition 25, and the second from Lemma 42. □

Lemma 44. Consider the inclusion $i_n \colon X^n \hookrightarrow X$.

- The induced map
$$H_n(i_n) \colon H_n(X^n) \to H_n(X)$$
is surjective.
- On the $(n+1)$-skeleton we get an isomorphism
$$H_n(i_{n+1}) \colon H_n(X^{n+1}) \cong H_n(X).$$

Proof. Consider the pair of spaces (X^{n+1}, X^n). The associated long exact sequence tells us that the sequence

$$H_n(X^n) \longrightarrow H_n(X^{n+1}) \longrightarrow H_n(X^{n+1}, X^n)$$

is exact. But by Lemma 43,

$$H_n(X^{n+1}, X^n) \cong \bigoplus_{\sigma \text{ an } (n+1)\text{-cell}} \tilde{H}_n(\mathbb{S}^{n+1}) \cong 0,$$

so $H_n(i_n)\colon X^n \hookrightarrow X^{n+1}$ is surjective.

Now let $m > n$. The long exact sequence on the pair (X^{m+1}, X^m) tells us that the following sequence is exact.

$$\begin{tikzcd} & & H_{n+1}(X^{m+1}, X^m) \arrow[dll, "\delta", rounded corners, to path={ -- ([yshift=-2ex]\tikztostart.south) -| ([xshift=-2ex, yshift=2ex]\tikztotarget.north west) -- (\tikztotarget)}] \\ H_n(X^m) \arrow[r] & H_n(X^{m+1}) \arrow[r] & H_n(X^{m+1}, X^m) \end{tikzcd}$$

But again by Lemma 43, both $H_{n+1}(X^{m+1}, X^m)$ and $H_n(X^{m+1}, X^m)$ are trivial, so

$$H_n(X^m) \to H_n(X^{m+1})$$

is an isomorphism.

Now consider X expressed as a colimit of its skeleta.

$$\begin{tikzcd} \cdots \arrow[r] & X^n \arrow[r] \arrow[drrr] & X^{n+1} \arrow[r] \arrow[drr] & X^{n+2} \arrow[r] \arrow[dr] & \cdots \\ & & & & X \end{tikzcd}$$

Taking nth singular homology, we find the following.

$$\begin{tikzcd} \cdots \arrow[r] & H_n(X^n) \arrow[r, two heads, "\alpha_1"] & H_n(X^{n+1}) \arrow[r, equal, "\alpha_2"] & H_n(X^{n+2}) \arrow[r, equal, "\alpha_3"] & \cdots \arrow[r] & H_n(X) \end{tikzcd}$$

Let $[\alpha] \in H_n(X^n)$, with

$$\alpha = \sum_i \alpha^i \sigma_i, \qquad \sigma_i\colon \Delta^n \to X.$$

Since the standard n-simplex Δ^n is compact, Corollary 39 implies that each σ_i factors through some X^{n_i}. Therefore, each σ_i factors through X^N with $N = \max_i n_i$, and we can write

$$\sigma_i = i_N \circ \tilde{\sigma}_i, \qquad \tilde{\sigma}_i\colon \Delta^n \to X^N.$$

Now consider

$$\tilde{\alpha} = \sum_i \alpha^i \tilde{\sigma}_i \in S_n(X^N).$$

Thus,

$$[\alpha] = \left[\sum_i \alpha^i i_n \circ \sigma\right]$$

□

Corollary 45. Let X and Y be CW complexes.

1. If $X^n \cong Y^n$, then $H_q(X) \cong H_q(Y)$ for all $q < n$.
2. If X has no q-cells, then $H_q(X) \cong 0$.
3. In particular, for an n-dimensional CW-complex X (Definition 33), $H_q(X) = 0$ for $q > n$.

Proof.

1. This follows immediately from Lemma 44.
2.

□

Definition 46 (cellular chain complex)**.** Let X be a CW complex. The cellular chain complex of X is defined level-wise by

$$C_n(X) = H_n(X^n, X^{n-1}),$$

with boundary operator d_n given by the following composition

$$H_n(X^n, X^{n-1}) \xrightarrow{\delta} H_{n-1}(X^{n-1}) \xrightarrow{\varrho} H_{n-1}(X^{n-1}, X^{n-2})$$

where ϱ is induced by the projection

$$S_{n-1}(X^{n-1}) \to S_{n-1}(X^{n-1}, X^{n-2}).$$

This is a bona fide differential, since

$$d^2 = \varrho \circ \delta \circ \varrho \circ \delta,$$

and $\delta \circ \varrho$ is a composition in the long exact sequence on the pair (X^n, X^{n-1}).

Theorem 47 (comparison of cellular and singular homology)**.** Let X be a CW complex. Then there is an isomorphism

$$\Upsilon_n \colon H_n(C_\bullet(X), d) \cong H_n(X).$$

Example 48 (complex projective space)**.** Consider the complex projective space $\mathbb{C}P^n$. We know that $\mathbb{C}P^0 = \mathrm{pt}$, and from the homogeneous coordinates

$$[x_0 : \cdots | x_n]$$

on $\mathbb{C}P^n$, we have a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}P^{n-2}.$$

Inductively, we find a decomposition

$$\mathbb{C}P^n \cong \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C}^0,$$

giving us a cell decomposition

$$\mathbb{C}P^{2n} \cong \mathbb{R}^{2n} \sqcup \mathbb{R}^{2n-2} \sqcup \cdots \sqcup \mathbb{R}^0.$$

This is a CW complex because

The cellular chain complex is as follows.

$$\begin{array}{cccccc} 2n & 2n-1 & 2n-2 & \cdots & 1 & 0 \\ \mathbb{Z} \longrightarrow & 0 \longrightarrow & \mathbb{Z} \longrightarrow & \cdots \longrightarrow & 0 \longrightarrow & \mathbb{Z} \end{array}$$

The differentials are all zero. Thus, we have

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & k = 2i,\ 0 \leq i \leq n \\ 0, & \text{otherwise.} \end{cases}$$

Example 49 (real projective space)**.** As in the complex case, appealing to homogeneous coordinates gives a cell decomposition

$$\mathbb{R}P^n \cong \mathbb{R}^n \sqcup \mathbb{R}^{n-1} \sqcup \cdots \sqcup \mathbb{R}^0.$$

The cellular chain complex is thus as follows.

$$\begin{array}{cccccc} n & n-1 & n-2 & \cdots & 1 & 0 \\ \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & \mathbb{Z} \longrightarrow & \cdots \longrightarrow & \mathbb{Z} \longrightarrow & \mathbb{Z} \end{array}$$

Unlike the complex case, we don't know how the differentials behave, so we can't calculate the homology directly.

1.10 Homology with coefficients

Definition 50 (homology with coefficients)**.** Let G be an abelian group, and X a topological space. The singular chain complex of X with coefficients in G is the chain complex

$$S(X;G) = S(X) \otimes_{\mathbb{Z}} G.$$

The nth singular homology of X with coefficients in G is the nth homology

$$H_n(X;G) = H_n(S(X;G)).$$

We can relate homology with integral coefficients (i.e. standard homology) and homology with coefficients in G.

Theorem 51 (topological universal coefficient thoerem for homology)**.** For every topological space X there is a short exact sequence

$$0 \longrightarrow H_n(X) \otimes G \hookrightarrow H_n(X;G) \twoheadrightarrow \mathrm{Tor}(H_{n-1}(X), G) \longrightarrow 0 \ .$$

Furthermore, this sequence splits non-canonicaly, telling us that

$$H_n(X;G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{n-1}(X), G).$$

Proof. REF to be updated with UCT. ◻

1.11 The topological Künneth formula

Theorem 52 (topological Künneth formula)**.** For any topological spaces X and Y, we have the following short exact sequence.

$$0 \to \bigoplus_{p+q=n} H_p(X) \otimes H_q(X) \hookrightarrow H_n(X \times Y) \twoheadrightarrow \bigoplus_{p+q=n-1} \mathrm{Tor}(H_p(X), H_q(Y)) \to 0$$

1.12 The Eilenberg-Steenrod axioms

Denote by T the functor

$$T\colon \mathbf{Pair} \to \mathbf{Pair}; \qquad (X,A) \mapsto (A, \emptyset); \qquad (f\colon (X,A) \to (Y,B)) \mapsto f|_A.$$

Definition 53 (homology theory)**.** Let A be an abelian group. A homology theory with coefficients in A is a sequence of functors

$$H_n\colon \mathbf{Pair} \longrightarrow \mathbf{Ab}, \qquad n \geq 0$$

together with natural transformations

$$\delta\colon H_n \Rightarrow H_{n-1} \circ T$$

satisfying the following conditions.

1. **Homotopy:** Homotopic maps induce the same maps on homology; that is,

$$f \sim g \implies H_n(f) = H_n(g) \qquad \text{for all } f, g, n.$$

2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i\colon (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism

$$H_n(i)\colon H_n(X \setminus U, A \setminus U) \cong H_n(X, A)$$

for all n.

3. **Dimension:** We have

$$H_n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then

$$H_n(X) = \bigoplus_\alpha H_n(X_\alpha).$$

5. **Exactness:** Each pair (X, A) induces a long exact sequence on homology.

We have seen that singular homology satisfies all of these axioms, and hence is a homology theory.

2 Cohomology

2.1 Axiomatic description of a cohomology theory

There are dual axioms to the Eilenberg-Steenrod axioms (introduced in Definition 53) which govern cohomology theories.

Definition 54 (cohomology theory)**.** Let A be an abelian group. A cohomology theory with coefficients in A is a series of functors

$$H^n : \mathbf{Pair}^{\mathrm{op}} \longrightarrow \mathbf{Ab}; \qquad n \geq 0$$

together with natural transformations

$$\partial : H^n \circ T^{\mathrm{op}} \Rightarrow H^{n+1}$$

satisfying the following conditions.

1. **Homotopy:** If f and g are homotopic maps of pairs, then $H^n(f) = H^n(g)$ for all n.
2. **Excision:** If (X, A) is a pair with $U \subset X$ such that $\bar{U} \subset \mathring{A}$, then the inclusion $i : (X \smallsetminus U, A \smallsetminus U) \hookrightarrow (X, A)$ induces an isomorphism
$$H^n(i) : (X, A) \cong H^n(X \smallsetminus U, A \smallsetminus U)$$
for all n.
3. **Dimension:** We have
$$H^n(\mathrm{pt}) = \begin{cases} A, & n = 0 \\ 0, & \text{otherwise.} \end{cases}$$
4. **Additivity:** If $X = \coprod_\alpha X_\alpha$ is a disjoint union of topological spaces, then
$$H^n(X) = \prod_\alpha H^n(X_\alpha).$$
5. **Exactness:** Each pair (X, A) induces a long exact sequence on cohomology.

2.2 Singular cohomology

Definition 55 (singular cohomology)**.** Let G be an abelian group. The singular cochain complex of X with coefficients in G is the cochain complex

$$S^\bullet(X;G) = \mathrm{Hom}(S_\bullet, G)$$

We will denote the evaluation map $\mathrm{Hom}(A,G) \otimes A \to G$ using angle brackets $\langle \cdot, \cdot \rangle$. In this form, it is usually called the *Kroenecker pairing*.

Lemma 56. Let $C_\bullet$ be a chain complex. The evaluation map (also known as the *Kroenecker pairing*)

$$\langle \cdot, \cdot \rangle \colon C^n(X;G) \otimes C_n(X) \to G$$

descends to a map on homology

$$\langle \cdot, \cdot \rangle \colon H^n(C^\bullet) \otimes H_n(C_\bullet) \to G.$$

Proof. Let $\alpha \colon C_n \to G \in C^n$ be a cocycle and $a \in C_n$ a cycle, and let $db \in C_n$ be a boundary. Then

$$\begin{aligned} \langle \alpha, a + db \rangle &= \langle \alpha, a \rangle + \langle \alpha, db \rangle \\ &= \langle \alpha, a \rangle + \langle \delta\alpha, b \rangle \\ &= \langle \alpha, a \rangle. \end{aligned}$$

Furthermore, if $\delta\beta \in C^n$ is a cocycle, then

$$\begin{aligned} \langle \alpha + \delta\beta, a \rangle &= \langle \alpha, a \rangle + \langle \delta\beta, a \rangle \\ &= \langle \alpha, a \rangle + \langle \beta, da \rangle \\ &= \langle \alpha, a \rangle \end{aligned}$$

□

Via $\otimes$-hom adjunction, we get a map

$$\kappa \colon H^n(C^\bullet) \to \mathrm{Hom}(H_n(C_\bullet), G).$$

Theorem 57 (universal coefficient theorem for singular cohomology)**.** Let X be a topological space. There is a split exact sequence

$$0 \longrightarrow \mathrm{Ext}(H_{n-1}(X), G) \hookrightarrow H^n(X;G) \twoheadrightarrow \mathrm{Hom}(H_n(X), G) \longrightarrow 0$$

Example 58. We have seen (in Example 48) that the homology of $\mathbb{C}P^n$ is

$$H_k(\mathbb{C}P^n) = \begin{cases} \mathbb{Z}, & 0 \le k \le 2n,\ k \text{ even}, \\ 0, & \text{otherwise}. \end{cases}$$

Thus, for $0 \le k \le n$, k even, we find

$$\begin{aligned} H^k(\mathbb{C}P^n;\mathbb{Z}) &\cong \operatorname{Ext}^1(0,\mathbb{Z}) \oplus \operatorname{Hom}(\mathbb{Z},\mathbb{Z}) \\ &\cong \mathbb{Z}. \end{aligned}$$

For k odd with $0 \le k \le n$, we have

$$\begin{aligned} H^k(\mathbb{C}P^n;\mathbb{Z}) &\cong \operatorname{Ext}^1(\mathbb{Z},\mathbb{Z}) \oplus \operatorname{Hom}(0,\mathbb{Z}) \\ &\cong 0. \end{aligned}$$

For $k > n$, we get $H^k(\mathbb{C}P^n;\mathbb{Z}) = 0$.

2.3 The cap product

Definition 59 (front, rear face)**.** Let $a\colon \Delta^n \to X$ be a singular n-simplex, and let $0 \le q \le n$.

- The $(n-q)$-dimensional front face of a is given by

$$F^{n-q}(a) = \quad \Delta^{n-q} \xrightarrow{i} \Delta^{tn} \xrightarrow{a} X \ ,$$

  where i is induced by the inclusion

$$\{0,\dots,n-q\} \hookrightarrow \{1,\dots,n\}; \qquad k \mapsto k.$$

- The q-dimensional rear face of a is given by

$$R^q(a) = \quad \Delta^q \xrightarrow{r} \Delta^n \xrightarrow{a} X \ ,$$

  where r is the map

$$\{0,\dots q\} \to \{0,\dots,n\}; \qquad k \mapsto n-q+k.$$

The cap product is a map

$$\frown\colon S^q(X,A;R) \otimes S_n(X,A;R) \to S_{n-q}(X;R).$$

The idea is simple. An element $\alpha \in S^q(X, A; R)$ is simply a map $\alpha\colon S_q(X, A) \to R$. While this cannot act directly on singular n-simplices, we can split an n-simplex up into its $(n-q)$-dimensional front face and its q-dimensional rear face and have α act on only the rear face.

Definition 60 (cap product)**.** The cap product is the map

$$\frown\colon S^q(X, A; R) \otimes S_n(X, A; R) \to S_{n-q}(X; R); \qquad \alpha \otimes a \otimes r \mapsto F^{n-q}(a) \otimes \langle \alpha, R^q(a) \rangle r.$$

Of course, we need to check that this is well-defined, i.e. that for $b \in S_n(A) \subset S_n(X)$, we have $\alpha \frown b = 0$. We compute

$$\alpha \frown b = F^{n-1}(b) \otimes \langle \alpha, R^q(b) \rangle.$$

Since $b \in S_n(A)$, we certainly have that $R^q(b) \in S_n(A)$. But then $\langle \alpha, R^q(b) \rangle = 0$ by definition of relative cohomology.