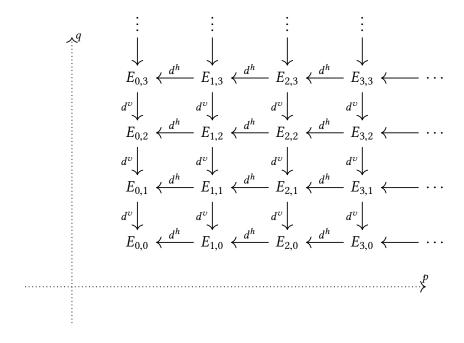
## 1 Spectral sequences

## 1.1 Motivation

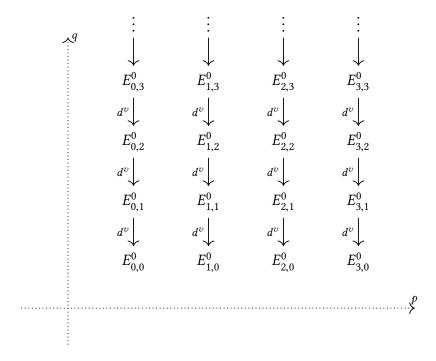
Let  $E_{\bullet,\bullet}$  be a first-quadrant double complex. As we saw in Section ??, computing the homology of the total complex of such a double complex is in general difficult with no further information, for example exactness of rows or columns. Spectral sequences provide a means of calculating, among other things, the homology of the total complex such a double complex.

They do this via a series of successive 'approximations'. One starts with a first-quadrant double complex  $E_{\bullet,\bullet}$ .



One first forgets the data of the horizontal differentials (or equivalently, sets them to zero). To avoid confusion, one adds a subscript, denoting this new double complex by

 $E^0_{\bullet,\bullet}$ 



One then takes homology of the corresponding vertical complexes, replacing  $E_{p,q}^0$  by the homology  $H_q(E_{p,\bullet}^0)$ . To ease notation, one writes

$$H_q(E_{p,\bullet}^0) = E_{p,q}^1$$

The functoriality of  $H_q$  means that the maps  $H_q(d^h)$  act as differentials for the rows.

$$E_{0,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{1,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{2,3}^{1} \stackrel{H_{3}(d^{h})}{\longleftarrow} E_{3,3}^{1} \longleftarrow \cdots$$

$$E_{0,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{1,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{2,2}^{1} \stackrel{H_{2}(d^{h})}{\longleftarrow} E_{3,2}^{1} \longleftarrow \cdots$$

$$E_{0,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{1,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{2,1}^{1} \stackrel{H_{1}(d^{h})}{\longleftarrow} E_{3,1}^{1} \longleftarrow \cdots$$

$$E_{0,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{1,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{2,0}^{1} \stackrel{H_{0}(d^{h})}{\longleftarrow} E_{3,0}^{1} \longleftarrow \cdots$$

We now write  $E_{p,q}^2$  for the horizontal homology  $H_p(E_{\bullet,q}^1)$ .

The claim is that the terms  $E_{p,q}^r$  are pieces of a successive approximation of the homology of the total complex Tot(E). In the particularly simple case that  $E_{\bullet,\bullet}$  consists of only two nonzero adjacent columns, we have already succeeded in computing the total homology, at least up to an extension problem.

**Proposition 1.** Let *E* be a double complex where all but two adjacent columns are zero; that is, let *E* be a diagram consisting of two chain complexes and a morphism between them, padded by zeroes on either side.

$$\begin{array}{c}
\vdots \\
\downarrow \\
D_3 \leftarrow f_3 \qquad \downarrow \\
C_3
\end{array}$$

$$\begin{array}{c}
d_3^D \downarrow \\
D_2 \leftarrow f_2 \qquad \downarrow \\
C_2
\end{array}$$

$$\begin{array}{c}
d_2^D \downarrow \\
D_1 \leftarrow f_1 \qquad \downarrow \\
C_1
\end{array}$$

$$\begin{array}{c}
d_1^D \downarrow \\
D_0 \leftarrow C_0
\end{array}$$

$$\begin{array}{c}
f_0 \\
C_0
\end{array}$$

$$\begin{array}{c}
d_0^D \downarrow \\
D_{-1} \leftarrow C_{-1}
\end{array}$$

$$\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\vdots
\end{array}$$

$$\vdots \qquad \vdots$$

Note that we have multiplied the differentials of C by -1 so that we have a double complex.

Fix some  $n \in \mathbb{Z}$ , and let q = n - p.

Let T = Tot(E); that is, let T = Cone(f). Then we have a short exact sequence

$$0 \longrightarrow E_{p-1,q+1}^2 \hookrightarrow H_{p+q}(T) \longrightarrow E_{p,q}^2 \longrightarrow 0$$

*Proof.* Recall that Cone(f) fits into the following short exact sequence,

$$0 \longrightarrow C_{\bullet} \hookrightarrow D_{\bullet} \longrightarrow Cone(f)_{\bullet} \longrightarrow 0$$

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and that this gives us the following long exact sequence on homology.

$$\begin{array}{cccc}
& \cdots & \longrightarrow & H_{n+1}(\operatorname{Cone}(f)) \\
& & \delta & \longrightarrow \\
& & H_n(C_{\bullet}) & \xrightarrow{H_n(f)} & H_n(D_{\bullet}) & \xrightarrow{H_n(p)} & H_n(\operatorname{Cone}(f)_{\bullet}) \\
& & \delta & \longrightarrow \\
& & H_{n-1}(C_{\bullet}) & \longrightarrow & \cdots
\end{array}$$

Through image-kernel factorization, we get the following short exact sequence.

$$0 \longrightarrow \operatorname{coker}(H_n(f)) \hookrightarrow H_n(\operatorname{Cone}(f)) \longrightarrow \ker(H_{n-1}(f)) \longrightarrow 0$$

This is precisely the second page of the above computation.

Of course, in a general situation we will not be so lucky. Let us examine the terms

$$E_{p,q}^2 = H_p(E_{\bullet,q}^1).$$

First, some notation. We will define in the obvious way  $\mathbb{Z}_{p,q}^r$  and  $\mathbb{B}_{p,q}^r$  so that

$$E_{p,q}^r = Z_{p,q}^r / B_{p,q}^r;$$

that is, we have

$$Z_{p,q}^r \subset E_{p,q}^{r-1}, \qquad B_{p,q}^r \subset E_{p,q}^{r-1}.$$

**Proposition 2.** There is an isomorphism

$$E_{p,q}^2 \cong U/V,$$

where

$$U = \{(a, b) \subset E_{p-1, q+1} \times E_{p, q} \mid d^{\upsilon}a + d^{h}b = d^{\upsilon}b = 0\}$$

and

$$V = \begin{pmatrix} (a,0) \colon d^{v}a = 0 \\ (d^{h}x, d^{v}x) \colon x \in E_{p,q+1} \\ (0, d^{h}c) \colon c \in E_{p+1,q}, d^{v}c = 0 \end{pmatrix}.$$

*Proof.* First we construct a homomorphism

$$\phi\colon U\to E_{p,q}^2.$$

Define

$$\phi^0 \colon U \to E^0_{p,q}; \qquad (a,b) \mapsto b.$$

Since by assumption  $d^vb=0$  we know that  $\operatorname{im}\phi^0\subset\ker d^v$ , so  $\phi^0$  descends to  $\ker d^v$ , and therefore to a map

$$\phi^1 \colon U \to E^1_{p,q}; \qquad (a,b) \mapsto b + B^1_{p,q}.$$

Since

$$H_q(d^h)(b+B_{p,q}^1)=d^v(-a)+B_{p-1,q}^1=0+B_{p-1,q}^1,$$

we have that im  $\phi^1 \subset \ker H_q(d^h)$ , so the map  $\phi^1$  descends to  $\ker H_q(d^h)$ , and therefore to a map

$$\phi\colon U\to E_{p,q}^2;\qquad (a,b)\mapsto (b+B_{p,q}^1)+B_{p,q}^2.$$

Next, we show that  $\phi$  is surjective. Let

$$\underbrace{(x + B_{p,q}^{1})}_{\in Z_{p,q}^{2}} + B_{p,q}^{2} \in E_{p,q}^{2}.$$

- The fact that  $x \in \mathbb{Z}_{p,q}^1$  means that  $d^v x = 0$ .
- The fact that  $x+B^1_{p,q}\in Z^2_{p,q}$  means that  $H_q(d^h)x=[0],$  i.e. that  $d^hx\in B^1_{p-1,q}.$